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A 5-STEP BLOCK METHOD FOR SPECIAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

Y.A YAHAYA[1] AND U. MOHAMMED

ABSTRACT. Various schemes exist to solve the initial/boundary value problems of the ordinary differential equation $y'' = f(x,y)$.Parts of the inadequacies of conventional linear multistep procedures for direct solution of this class of problem is the requirement of a starting value and this paper addressed that. Five specific schemes of orders (6,6,6,6,6)T have been derived and presented in a way to avoid computational burden and computer time wastage involved in the usual reduction of second order problem into system of first order equations. Furthermore, a stability analysis of the 5-step block method and efficiency of the method is tested on problems whose solutions are, oscillatory or nearly periodic, stiff and non-linear ivp and bvp ODES

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1. INTRODUCTION

Let us consider the numerical solution of the second order differential equations of the form.

$$y'' = f(x,y),\ a \leq x \leq b \tag{1.1}$$

With associated initial or boundary conditions. Solutions to the initial value problem (IVP) of type (1.1) according to Fatunla [5] are often highly oscillatory in nature and thus severely restricts the mesh

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[1] Corresponding Author

size of the conventional linear multistep method. Such system often occur in mechanical systems without dissipation, satellite tracking and celestial mechanics (cf: Henrici [7], p. 289).

Lambert [8] and several other authors, have written on conventional LMM.

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j},\ k \geq 2, \tag{1.2}$$

or compactly in the form

$$\rho(E) y_n = h^2 \delta(E) f_n, \tag{1.3}$$

where E is the shift operator given by

$$\rho(r) \sum_{j=0}^{k} \alpha_{jr} j, \delta(r) = \sum_{j=0}^{k} \beta_j r^j, \tag{1.4}$$

y_n is the numerical approximation to the theoretical solution $y_n(x_n)$ and $f_n = f(x_n, y_n)$.

The application of the conventional LMM (1.2) to (1.1) results in the adoption of intolerably small mesh size in the integration interval $[a, b]$. This indeed will result in very long computing time and accumulation of round-off errors. This however, does not becloud their usefulness. In fact, some of these methods posses non vanishing stability region, especially when they are implicit and are easily implemented. Their implementation requires the generation of $k-1$ initial starting values $y(x_{n+1}) = y_{n+j} j = 1(1)k-1$ using a starting method which is most often RKM. (cf. Fatunla etal, [6] p. 2). These initial starting-values requirement are eliminated in the new proposed block methods described in section 3-5. In the present consideration, our motivations for the study of this approach is a further advancement in efficiency, i.e obtaining the most accuracy per unit of computational effort, that can be secured with the group of methods proposed in section 3 of this paper over Awoyemi [2] and Aladeselu [1].

Definition 1.1 A-Stable (Dahlquist [3]).

A numerical method is said to be A-stable if its region of absolute stability contains, the whole of the left-hand half-plane $\text{Re}h\lambda < 0$.

Definition 1.2 $A(\alpha)$ stable (Wundluid [10]. A numerical method is said to be $A(\alpha)$-stable, $\alpha \in (0, \frac{\pi}{2})$, if its region of absolute stability contains the infinite wedge $W_\alpha = [h\lambda] - \alpha < \pi - \text{arg}h\lambda$; it is said to be $A(0)$- stable if it is $A(\alpha)$-stable for some (sufficiently small)$\alpha \in (0, \frac{\pi}{2})$.

Definition 1.3 A block method is zero-stable provided the roof $\lambda_j, j = 1(1)s$ of the first characteristics polynomial $\rho(\lambda)$ specified as $\rho(\lambda) = \det|\sum i = 0^s A^I \lambda^(s - i)| = 0$ satisfies $|\lambda_j| \leq 1$, and for those roots with $|\lambda_j| = 1$ the multiplicity must not exceed two. The principal root of $\rho(\lambda)$ is denoted by $\lambda_1 = \lambda_2 = 1$.

2. The Multistep Collocation Method

In the spirit of Onumayi etal [9] and Yahaya [11] we consider the construction of multistep collocation method of constant step size h, though h can be variable and give continuous expression for the coefficient. The values of K and M are arbitrary except for collocation at the mesh points where $0 < m \leq k + 1$.

Let y_{n+j} be approximation to Y_{n+j} where $Y_{n+j} = Y(x_{n+j}),\ n = 0, \ldots, k - 1$.

Then a k-step multistep collocation method is constructed as follows. We find a polynomial $y(x)$ of degree $P = t + m - 1,\ t > 0,\ m > 0$ and such that it satisfies the conditions:

$$y(x_{n+j}) = y_{n+j} j \in \{0, \ldots, k\} \tag{2.1}$$

$$y''(\bar{x}_{n+j}) = f(x_{n+j}, y(\bar{x}_n + j);\ j = 0, \ldots, m - 1, \tag{2.2}$$

Where $\bar{x}_1, \ldots, \bar{x}_{m-1}$ are free collocations points, we then take as an approximation to $y_{n+k}, Y_{n+k} = Y(x_{n+k})$. Let

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{m-1} \beta_j(x) f(\bar{x}_{j+1}, y(\bar{x}_{j+1})), \qquad (2.3a)$$

Where α_j and β_j are assumed polynomial of the form

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_j, i+1x^j; h^2\beta_j(x) = \sum_{i=0}^{t+m-1} h^2\beta_j, i+1x^i \qquad (2.3b)$$

and the collocation point $\bar{x}_{j+1}$ in (2.3a) belong to the extended set

$$Q = \{x_n, \ldots, x_{n+k}\} \cup (x_{n+k-1}, x_{n+k}). \qquad (2.3c)$$

From the interpolation conditions (2.1) and the expression for $y(x)$ in (2.3a) the following conditions are imposed on $\alpha_j(x)$ and $\beta_j(x)$:

$$\left.\begin{array}{l} \alpha_j(x_{m+i}) = \delta_{ij},\ j = 0, \ldots, t-1,\ i = 0, \ldots, t-1 \\ h^2\beta_j(x_{n+i} = 0,\ j = 0, \ldots, m-1;\ i = 0, \ldots, t-1 \end{array}\right\} \qquad (2.3d)$$

and

$$\left.\begin{array}{l} \alpha''_j(\bar{x}_{n+i}) = 0\ j = 0, \ldots, t-1,\ i = 0, \ldots, m-1 \\ h^2\beta_j(x_{n+i} = \delta_{ij},\ j = 0, \ldots, m-1;\ i = 0, \ldots, m-1 \end{array}\right\}. \qquad (2.3e)$$

Next we write (2.3d)–(2.3e) in a matrix equation of the form:

$$DC = I, \qquad (2.3f)$$

where I is the identity matrix of dimension $(t+m)$.
The matrices D and C are both of dimensions $(t+m) \times (t+m)$. It follows from (2.3f) that the columns of $C = D^{-1}$ give the continuous coefficient $\alpha_j(x)$ and $\beta_j(x)$

3. Derivation of the Present Method

We propose an approximate solution to (1.1)in the form

$$y(x) = \sum_{j=0}^{m+t-1} a_j x^i,\ i = 0, (1)(m+t-1), \qquad (3.1)$$

$$y''(x) = \sum_{j=0}^{m+t-1} i(i-1)a_j x^{i-2},\ i = 2, 3, \ldots, (m+t-1), \qquad (3.2)$$

with $m = 5,\ t = 2,\ p = m + t - 1$ and $\alpha_j,\ \beta_j,\ j = 0, 1, (m + t - 1)$ are parameters to be determined. p is the degree of the polynomial interpolant of our choice. Specifically, we collocate equation (3.2) at $\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}\}$ and interpolate equation (3.1) at $\{x_{n+3}, x_{n+4}\}$ using the method described in section 2 of this paper; we obtain a continuous form for the solution

$$\sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} f(\bar{x}_{j+1}, y(\bar{x}_{j+1})) \qquad (3.3a)$$

This gives rise to the system of equations put in the matrix form below.

$$\begin{pmatrix} 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 \\ 0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+4}^4 & 42x_{n+5}^5 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{pmatrix} = \begin{pmatrix} y_{n+3} \\ y_{n+4} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} \qquad (3.3b)$$

Note,

$$\underline{\mathrm{D}}\ \underline{\mathrm{a}} = \underline{\mathrm{F}}$$

$$\underline{\mathrm{a}} = \underline{\underline{D}}^{-1}\underline{\mathrm{F}}$$

where $\underline{\mathrm{F}} = (y_1, y_2, \ldots, y_r, f_1, f_2, \ldots, f_s)^T$

Matrix D in equation (3.3b), which when solved either by matrix inversion techniques or Gaussian elimination method to obtain the values of the parameters $\alpha_j,\ j = 0, 1, m+t-1$ and then substituting them into equation (3.1) give a scheme expressed in the form.

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k-2} \beta_j(x) f_{n+j} \qquad (3.4)$$

If we now let $k = 5$, after some manipulations we obtain a continuous form of solution

$$\begin{aligned}
y(x) \quad = \quad & \left[\frac{(4h-(x-x_n))}{h}\right] y_{n+3} + \left[\frac{(-3h+(x-x_n))}{h}\right] y_{n+4} \\
& \frac{1}{5040h^5}\left[-(x-x_n)^7 + 21(x-x_n)^6 h - 179(x-x_n)^5 h^2\right. \\
& +789(x-x_n)^4 h^3 + -1918(x-x_n)^3 h^4 \\
& \left.+2520(x-x_n)^2 h^5 - 1588(x-x_n)h^6 + 336h^7\right] f_n \\
& +\frac{1}{10080h^5}\left[10(x-x_n)^7 - 196(x-x_n)^6 h + 1491(x-x_n)^5 h^2\right. \\
& -5390(x-x_n)^4 h^3 + 8400(x-x_n)^3 h^4 - 14059(x-x_n)h^6 \\
& \left.+10668h^7\right] f_{n+1} \\
& +\frac{1}{504h^2}\left[-(x-x_n)^7 + 18(x-x_n)^6 h - 124(x-x_n)^5 h^2\right. \\
& +375(x-x_n)^4 h^3 - 420(x-x_n)^3 h^4 - 314(x-x_n)h^6 \\
& \left.+974h^7\right] f_{n+2} \\
& +\frac{1}{504h^2}\left[10(x-x_n)^7 - 168(x-x_n)^6 h + 1029(x-x_n)^5 h^2\right. \\
& -2730(x-x_n)^4 h^3 + 2800(x-x_n)^3 h^4 - 5813(x-x_n)h^6 \\
& \left.+13524h^7\right] f_{n+3} \\
& +\frac{1}{10080h^5}\left[-(x-x_n)^7 + 15(x-x_n)^6 h - 86(x-x_n)^5 h^2\right. \\
& +214(x-x_n)^4 h^3 - 210(x-x_n)^3 h^4 - 3(x-x_n)h^6 \\
& \left.+269h^7\right] f_{n+4} \\
& \frac{1}{10080h^5}\left[2(x-x_n)^7 - 28(x-x_n)^6 h + 147(x-x_n)^5 h^2\right. \\
& -350(x-x_n)^4 h^3 + 336(x-x_n)^3 h^4 - 107(x-x_n)h^6 \\
& \left.-84h^7\right] f_{n+5} \qquad (3.5)
\end{aligned}$$

The continuous form (3.5) was evaluated at some selected points it yielded the following discrete schemes:

$$\begin{aligned}
\text{a.}\quad & y_{n+5} + y_{n+3} - 2y_{n+4} = \frac{h^2}{240}\left[f_n - 6f_{n+1} + 14f_{n+2}\right.\\
& \left.+4f_{n+3} + 209f_{n+4} + 18f_{n+5}\right]\\
\text{b.}\quad & y_{n+4} - \frac{4}{3}y_{n+3} + \frac{1}{3}y_n = \frac{h^2}{360}\left[8f_n + 127f_{n+1} + 232f_{n+2}+\right.\\
& \left.+232f_{n+3} + 32f_{n+4} - f_{n+5}\right]\\
\text{c.}\quad & y_{n+2} - 2y_{n+3} + y_{n+4} = \frac{h^2}{240}\left[-f_{n+1} + 24f_{n+2}\right.\\
& \left.+194f_{n+3} + 24f_{n+4} - f_{n+5}\right]\\
\text{d.}\quad & y_{n+1} - 3y_{n+3} + 2y_{n+4} = \frac{h^2}{240}\left[-f_n + 22f_{n+1}+\right.\\
& \left.+242f_{n+2} + 412f_{n+3} + 47f_{n+4} - 2f_{n+5}\right] \qquad (3.6)
\end{aligned}$$

Taking the first derivative of equation (3.5), thereafter, evaluate the resulting continuous polynomial solution at $x = x_0$ yields

$$\begin{aligned}
& y_{n+3} - y_{n+4} + hz_0 = \frac{h^2}{10080}\left[-3176f_n - 14059f_{n+1} - 6280f_{n+2}\right.\\
& \left.-11626f_{n+3} - 32f_{n+4} - 107f_{n+5}\right]. \qquad (3.7)
\end{aligned}$$

Equation (3.6) and (3.7) constitute the member of a zero-stable block integrators of order (6,6,6,6,6)T with $C_8 = \left(\frac{-221}{60480}, \frac{-11}{10080}, \frac{31}{60480}, \frac{31}{20160}, \frac{-1313}{12}\right)$. The application of the block integrators with $n = 0$ give the accurate values of unknown as shown in tables 1-2 of section 5.

To start the IVP integration on the sub interval $[X_0, X_5]$, we combine (3.6) and (3.7), when $n = 0$ i.e the 1-block 5-point method as given in equation (3.8). Thus produces simultaneously values for along with without recourse to any predictor like Aladeselu [1] and Awoyemi [2] to provide y_1 and y_2 in the main method. Hence this is an improvement over these reported works. Though, this does not becloud the contribution of these authors.

4. Stability Analysis

Recall, that, it is a desirable property for a numerical integrator to produce solution that behave similar to the theoretical solution to a problem at all times. Thus several definitions, which call for the method to possess some "adequate" region of absolute stability, can be found in several literatures. See Lambert [8], Fatunla [4,5] etc. Following Fatunla [4,5], the five integrator proposed in this report in equations (3.6) and (3.7) are put in the matrix-equation form and for easy analysis the result was normalized to obtain:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h^2$$

$$\left\{ \begin{pmatrix} \frac{-23803}{10080} & \frac{-761}{2520} & \frac{-941}{5040} & \frac{-341}{5040} & \frac{107}{10080} \\ \frac{-9707}{2520} & \frac{-37}{63} & \frac{136}{315} & \frac{-101}{630} & \frac{8}{315} \\ \frac{-3523}{672} & \frac{-9}{140} & \frac{87}{112} & \frac{-9}{35} & \frac{9}{224} \\ \frac{-8363}{1260} & \frac{176}{315} & \frac{608}{315} & \frac{-16}{63} & \frac{16}{315} \\ \frac{-16243}{2016} & \frac{625}{504} & \frac{3125}{1008} & \frac{625}{1008} & \frac{275}{2016} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+1} \\ f_{n+5} \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1231}{5040} \\ 0 & 0 & 0 & 0 & \frac{71}{126} \\ 0 & 0 & 0 & 0 & \frac{123}{140} \\ 0 & 0 & 0 & 0 & \frac{376}{315} \\ 0 & 0 & 0 & 0 & \frac{152}{1008} \end{pmatrix} \begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n+1} \\ f_n \end{pmatrix} \right\} \tag{3.8}$$

Equation (3.8) is the 1-block 5 point method. The first characteristic polynomial of the proposed 1-block 5-step method is

$$P(R) = \det |RA^{(0)} - A^{(1)}|$$

$$\begin{aligned} P(R) &= \det \left[R \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \\ &= \det \begin{pmatrix} R & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & R-1 \end{pmatrix} \\ &= \left[R^4 (R-1) \right]. \end{aligned} \tag{3.9}$$

$P(R) = R^4(R-1)$, and this implies $R_1 = R_2 = R_3 = R_4 = 0$ or $R_5 = 1$.

From definition (1.3) and the equation (3.9), the 1 block 5-point is zero stable and is also consistent as its order $(6, 6, 6, 6, 6)^T > 1$, thus, it is convergent, following Henrici [7].

5. Numerical Experiment

In what follows, we present some numerical results on some problems.

Problem 1: From Awoyemi [2], we consider the problem
$y'' = 2y^3, y(1) = 1, y'(1) = -1$. The Exact solution is $y(x) = \frac{1}{x}$

Table of Results 1a

N	x	Exact value	Approx value	Error
0	1	1	1	0
1	1.1	0.909090109	0.9090915345	1.4255000000E-06
2	1.2	0.833333333	0.8333348725	1.5395000000E-06
3	1.3	0.769230769	0.7692331795	2.4105000000E-06
4	1.4	0.714285714	0.7142889753	3.2613000001E-06
5	1.5	0.656666667	0.6666709495	1.0004282500E-02
6	1.6	0.625	0.6250043096	4.3096000000E-06
7	1.7	0.588235294	0.5882396428	4.3488000000E-06
8	1.8	0.555555556	0.5555599414	4.3854000000E-06
9	1.9	0.526315789	0.5263202115	4.4225000000E-06
10	2.0	0.5	0.5000044666	4.4666000000E-06

Table of Results 1b

N	x	Exact value	Awoyemi[2] error	Present Error
1	1.1	0.909090109	2.8483722D-03	1.4255000000E-06
2	1.2	0.833333333	2.26883436D-01	1.5395000000E-06
3	1.3	0.769230769	7.3968630D+00	2.4105000000E-06
4	1.4	0.714285714	2.1168783D+01	3.2613000001E-06
5	1.5	0.656666667	3.3156524D+01	1.0004282500E-02
6	1.6	0.625	4.3968593D+01	4.3096000000E-06
7	1.7	0.588235294	5.3903097D+01	4.3488000000E-06
8	1.8	0.555555556	6.3121827D+01	4.3854000000E-06
9	1.9	0.526315789	7.1723621D+01	4.4225000000E-06
10	2.0	0.5	7.9776590D+01	4.4666000000E-06

Problem 2 From Aladeselu [1] we consider the problem $y'' = -100y, \;\; y(0) = 1, \;\; y'(0) = 10$

Table 2a

$h = 0.001$

X	**Exact Value**	**Approx Value**	**Present Error**
0	1	1	0
0.001	1.00994983	1.00994983	3.00000E-09
0.002	1.01979867	1.01979867	3.00000E-09
0.003	1.02954553	1.02954553	4.00000E-09
0.004	1.03918944	1.03918944	0
0.005	1.04872943	1.04872943	0

$h = 0.0025$

X	**Exact Value**	**Approx Value**	**Present Error**
0	1.00000000	1.00000000	0
0.0025	1.02468491	1.02468491	3.00000E-09
0.005	1.04872943	1.04872943	1.00000E-09
0.0075	1.07211853	1.07211853	5.00000E-09
0.01	1.09483758	1.09483758	2.00000E-09
0.0125	1.11687240	1.11687240	0

h=0.005

X	**Exact Value**	**Approx Value**	**Present Error**
0	1.00000000	1.00000000	0
0.005	1.04872943	1.04872943	1.00000E-09
0.01	1.09483758	1.09483758	3.00000E-09
0.015	1.13820921	1.13820921	1.00000E-09
0.02	1.17873591	1.17873591	1.00000E-09
0.025	1.21631638	1.21631638	2.00000E-09

Table 2b

H	**Aladeselu (2007)**	**Present Method**
0.001	-2121925354003096E-05	3.00000E-09
	-5.066394805908203E-05	3.00000E-09
	-9.107589721679688E-05	4.00000E-09
		0.00000E+00
		0.00000E+00
0.0025	-1.30891799926758E-04	3.00000E-09
	-3.131628036499023E-04	1.00000E-09
	-5.636215209960938E-04	5.00000E-09
		2.00000E-09
		0.00000E+00
0.005	-5.2449431610107742E-04	1.00000E-09
	-1.250743865966797E-03	3.00000E-09
	-2.251148223876953E-03	1.00000E-09
		1.00000E-09
		2.00000E-09

REMARKS: Aladeselu [1] proposes 1-block 3-point method, while the present method is a 1-block 5-points that produces simultaneously, $y_1,\ y_2,\ y_3,\ y_4$ and y_5 (see Table 2b)

Problem 3 (Boundary Value Problem)

$$y'' = 3x + 4y, \ \ y(0) = 0, \ \ y(1) = 1.$$

We solve the equation in the range $[0, 1]$ using $h = 0.2$

The exact solution is $y = \frac{7\left(e^{2x} - e^{-2x}\right)}{4(e^2 - e^{-2})} - \frac{3}{4}x$

Table 3

X	**Exact Value**	**Approx Value**	**Absolute Error**
0	0.00000000000	0.00000000000	0.00000E+00
0.2	0.04819251100	0.04800965479	1.82856E-04
0.4	0.12852089500	0.12812778320	3.93112E-04
0.6	0.27833169000	0.27766443290	6.67257E-04
0.8	0.54623764000	0.54518670770	1.05093E-03
1.0	1.00000000000	1.00000000000	0.00000E+00

Problem 4

$$y'' = -100y + 99\sin x, \quad y(0) = 1, \quad y'(0) = 11.$$

We solve the equation in the range $[0, 2\pi]$ using $h = \frac{\pi}{2^5}$
The exact solution is $y(x) = \cos(10x) + \sin(10x) + \sin x$

Table 4

x	Exact Solution	Numerical Solution	Exact Error
	1	1	0
	1.484987182	1.486674194	1.68701200E-03
	1.707107000	1.716007839	8.90083900E-03
	0	-0.007809376051	7.80937605E-03
	-0.29289300	-0.2957147581	2.82175810E-03
	1	1.012907416	1.29074160E-02
	0.2928930000	0.2968513269	3.95832690E-03
	-2	-2.054221848	5.42218480E-02
	-1.707107000	-1.710563310	3.45631000E-03
2	1	0.9945228202	5.47717980E-03

Problem 5

$$y'' = -y + 0.001e^{ix}, \quad y(0) = 1, \quad y'(0) = 0.9995i.$$

We solve the equation in the range $[0, 2\pi]$ using $h = \frac{\pi}{2^3}$ The exact solution is $y(x) = u(x) + iv(x)$, where
$i^2 = -1, \quad u(x) = \cos x + 0.0005x\sin x$ and
$v(x) = \sin x - 0.0005x\cos x.$

To compute the error we use

$$\text{Error}\,(y) \equiv |y\,(x) - y| = \left[(u\,(x) - u)^2 + (v\,(x) - v)^2\right]^{\frac{1}{2}}$$

Table 5

n	EXACT SOLUTION		NUMERICAL SOLUTION		ERROR
	U(x)	V(x)	U(x)	V(x)	
1	0.923954672	0.382502029	0.923957552	0.382505370	4.410591536E-06
2	0.707384461	0.706829101	0.707391100	0.706836742	8.339167678E-06
3	0.383227642	0.923654113	0.383236832	0.923664694	1.401475590E-05
4	0.000785398	1.000000000	0.000795633	1.000011652	1.550895536E-05
5	-.3817764158	0.924255231	-0.381765839	0.924267960	1.655019240E-05
6	-.7062737406	0.707939822	-0.706268156	0.707952968	1.428323238E-05
7	-.9233535545	0.383953256	-0.923355672	0.383965476	1.240205369E-05
8	-1	0.001570796	-1.000009240	0.001580122	1.312825906E-05
9	-.9245557900	-.3810508025	-0.924570467	-0.381045791	1.550873066E-05
10	-.7084951821	-.7057183803	-0.708515033	-0.000554345	1.985075033E-05
11	-.3846788689	-.9230529956	-0.384696891	-0.923057354	1.854121268E-05
12	-0.002356194490	-1	-0.002369543	-1.000010067	1.671867759E-05
13	0.380325189	-.9248563488	0.380318548	-0.924870314	1.546355230E-05
14	0.705163020	-.7090505425	0.705163991	-0.709066021	1.550883303E-05
15	0.922752437	-.3854044822	0.922761523	-0.385420982	1.883636632E-05
16	1.000000000	-0.003141592654	1.000012339	-0.003154864	1.812089929E-05

6. Conclusion

Onumanyi etal [9], and Awoyemi [2] discussed in some detail theoretical and practical aspects of collocation with piecewise polynomial functions. Roughly, their results particularly Awoyemi [2] indicate that the solution of a second order non linear problem can be approximated with linear multistep methods. In this paper we developed a uniform order 1-block 5-point integrators of orders $(6, 6, 6, 6, 6)$ and the resultant numerical integrators posses the following desirable properties.

I. zero- stability i.e. stability at the origin
II. cheap and reliable error estimates
III. Facility to generate the solution at five point simultaneously.
IV. It is a convergent scheme.

Hence our work is an improvement over other cited works.

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Math's/Computer Science Department, Federal University of Technology, Minna, Niger State, Nigeria

E-mail address: `yusuphyahaya@yahoo.com, digitalumar@yahoo.com`

E-mail address: `???`

Math's/Computer Science Department, Federal University of Technology, Minna, Niger State, Nigeria