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Professor Omolehin Joseph Olorunju (FAC)
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Ulam-Hayers Stability Analysis via Quadrupled Fixed Point for Mixed Monotone Operators

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ABSTRACT
This study deals with quadrupled fixed point for mixed monotone operators. The approach is primarily based on Perov-type fixed point theorem for contractions in metric spaces provided by the vector-value metric. This research covers Ulam-Hyers stability results for quadruple operators with fixed point contractive-type on a complete metric space as a continuation to recent research in literature.

Keywords*: Metric space, Quadrupled fixed point, Single-valued operator, Perov type contraction*

1.0 INTRODUCTION

In contemporary times, contraction principle has become a very famous and indispensable device in modern analysis, especially in nonlinear analysis which includes its functions to differential and integral equations, variational inequality theory, equilibrium problems, minimization problems, and many others. Perov exibit that Banach contraction principle was extended to single-valued contraction on spaces endowed with vector-valued metrics (Gupta, 2016). Moreover, one of the most famous generalizations of fixed point theorems is coupled fixed point theorem for continuous and discontinuous operators introduced in (Guo & Lakshmikantham, 1987) in relation with coupled quasi solutions of an initial value problem for ordinary differential equations. (Bhaskar & Lakshmikantham, 2006) introduced the concept of mixed monotone property in partially ordered metric space. Afterward, (Lakshmikantham & Ciric, 2009) extended the consequences by giving the definition of the g-monotone property.

Recently, Rauf and Aniki, 2021 extended the work of Berinde and Borcut, 2011 to quadrupled fixed point theorems for contractive type mappings in partially ordered Cauchy spaces. Recently, Gupta (2016) presented a research on Ulam-Hyers stability theorem by tripled fixed point theorem, whose outcomes shaped the foundation for this work. For more details on coupled and tripled fixed point results, check (Bhaskar & Lakshmikantham, 2006; Cho *et al*., 2006; Gupta, 2014; Gupta *et al*., 2014; Aniki & Rauf, 2019) and cited therein.

2.0 MATERIALS AND METHODS

In the following, we summarize some of the concepts and results in (Gupta, 2016; Aniki & Rauf, 2020, Rauf & Aniki, 2021) with a view to generalize them.

Definition 2.1 Let X be a non-empty set. A map $d: X^2 \to \mathbb{R}^m$ is referred to as a vector-valued metric on X if the following properties are satisfied:

i. $d(u,v) \geq 0$ for all $u, v \in X$,
ii. $d(u,v) = 0$ if and only $u = v$,
iii. $d(u,v) = d(v,u)$ for all $u, v \in X$,
iv. $d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w \in X$,

If $u, v \in \mathbb{R}^m, u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_m)$, then, by definition: $u \leq v$ if and only if $u_i \leq v_i$ for $i \in \{1,2, \dots, m\}$.
A set endowed with a vector-valued metric d is referred to as generalized metric space.
We denote with the aid of $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and I the identity matrix.

Theorem 2.1 Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalent,

i. A is convergent towards zero,
ii. $A^n \to 0$ as $n \to \infty$
iii. The eigenvalues of A are in the open

unit disc, i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$,

iv. The matrix $(I - A)$ is nonsingular and $(I - A)^{-1} = I + A + \cdots + A^n + \cdots$ (1)

v. The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative element.

vi. $A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$ for every $q \in \mathbb{R}^m$.

Theorem 2.2 Let (X, d) be a complete generalized metric space and the operator $f: X \to X$ with the property that there exists a matrix $A \in M_{mm}(\mathbb{R})$ such that $d\big(f(u), f(v)\big) \leq Ad(u, v)$ for all $u, v \in X$. If A is a matrix convergent towards zero, then:

i. $Fix(f) = \{u^*\}$

ii. The sequence of successive approximations $(u_n)_{n\in\mathbb{N}}, u_n = f^n(u_0)$ is convergent and has limit u^*, for all $u_0 \in X$,

iii. One has the following estimation

$$d(u_n, u^*) \leq A^n(I - A)^{-1}d(u_0, u_1), \quad (2)$$

iv. If $g: X \to X$ is an operator such that there exists $v^* \in Fix(g)$ and $\epsilon \in (\mathbb{R}_m^+)$ with $d\big(f(u), g(u)\big) <\in$, for each $u \in X$, then $d(u^*, v^*) \leq (I - A)^{-1}\epsilon$,

v. If $g: X \to X$ is an operator such that there exists $\epsilon \in (\mathbb{R}_m^+)^*$ such that $\big(f(u), g(u)\big) \leq \epsilon$, for all $u \in X$, then for the sequence $v_n = g^n(x_0)$, we have the following estimation

$$d(v_n, u^*) \leq (I - A)^{-1}\epsilon + A^n(I - A)^{-1}d(v_0, v_1). \quad (3)$$

Definition 2.2 Let (X, d) be a metric space. The system of operational equations

$$u = T(u, v, w, x)$$
$$v = T(v, u, v, x)$$
$$w = T(w, u, v, w)$$
$$x = T(x, w, v, u)$$

where $T: X^4 \to X$ be a mapping.

Then, the solution $(u, v, w, x) \in X^4$ of the system is known as quadrupled fixed point for T.

Definition 2.3 Let (X, d) be a generalized metric space with the operator $f: X \to X$. Then, the fixed point equation

$$u = f(u) \quad (4)$$

is said to be generalized Ulam-Hyers stable if there exists an increasing function $\psi: \mathbb{R}_+^m \to \mathbb{R}_+^m$, continuous at 0 with $\psi(0) = 0$, such that for any $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ with $\epsilon_i > 0$ for $i \in \{1, \dots, m\}$ and any solution $u^* \in X$ of the inequality

$$d\big(v, f(v)\big) \leq \epsilon \quad (5)$$

there exists a solutionu^* of (4) such that

$$d(u_n, u^*) \leq \psi(\epsilon) \quad (6)$$

In particular, if $\psi(t) = ct,\ t \in \mathbb{R}_+^m$, (where $c \in M_{mm}(\mathbb{R}_+)$), then the fixed point equation (4) is known as Ulam-Hyers Stable.

Theorem 2.3 Let (X, d) be a generalized metric space and the operator $f: X \to X$ with the property that there exists a matrix $A \in M_{mm}(\mathbb{R})$ such that A converges to zero and $d\big(f(u), f(v)\big) \leq Ad(u, v)$ for all $u, v \in X$. Then, the fixed point equation $u = f(u),\ u \in X$ is Ulam-Hyers Stable.

Proof

From Perov's fixed point theorem, it was proven that $Fix(f) = \{u^*\}$. Let $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ with $\epsilon_i > 0$ for $i \in \{1, \dots, m\}$ and let $v^* \in X$ be the solution of the inequality

$$d\big(v, f(v)\big) \leq \epsilon.$$

Then, we successively have that

$$d(u^*, v^*) = d(f(u^*), v^*)$$
$$\leq d\big(f(u^*), f(v^*)\big) + d(f(v^*), v^*)$$
$$\leq Ad(u^*, v^*) + \epsilon$$

Thus, using Theorem 2.2,

$$d(u^*, v^*) \leq (I - A)^{-1}\epsilon.$$

Definition 2.4 Let (X, d) be a metric space and let $T: X^4 \to X$ be an operator. Then, the system of operational equations

$$u = T(u, v, w, x)$$
$$v = T(v, u, v, x) \quad (7)$$
$$w = T(w, u, v, w)$$
$$x = T(x, w, v, u)$$

is stated to be Ulam-Hyers stable if there exists $c_1, c_2, \dots, c_{15}, c_{16} > 0$ such that for each $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0$ and every quadruple $(u^*, v^*, w^*, x^*) \in X^4$ such that

$$d\big(p^*, T(p, q, r, s)\big) \leq \epsilon_1$$
$$d\big(q^*, T(q, p, q, s)\big) \leq \epsilon_2$$
$$d\big(r^*, T(r, p, q, r)\big) \leq \epsilon_3 \quad (8)$$
$$d\big(s^*, T(s, r, q, p)\big) \leq \epsilon_4$$

there exists a solution $(u^*, v^*, w^*, x^*) \in X^4$ of (8) such that

$$d(p^*, u^*) \leq c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 + c_4\epsilon_4$$
$$d(q, v^*) \leq c_5\epsilon_1 + c_6\epsilon_2 + c_7\epsilon_3 + c_8\epsilon_4$$
$$d(r^*, w^*) \leq c_9\epsilon_1 + c_{10}\epsilon_2 + c_{11}\epsilon_3 + c_{12}\epsilon_4$$
$$d(s^{\cdot}, x^{\cdot}) \leq c_{13}\epsilon_1 + c_{14}\epsilon_2 + c_{15}\epsilon_3 + c_{16}\epsilon_4$$

3.0 RESULT AND DISCUSSION

The main result is the following existence, uniqueness, data dependence and Ulam-Hyers

stability theorem for the quadrupled fixed point with operator T.

Theorem 3.1 Let (X,d) be a complete metric space, $T: X^4 \to X$ be a mapping such that

$$\begin{aligned} d\big(T(u,v,w,x),T(p,q,r,s)\big) &\le k_1 d(u,p)+k_2 d(v,q)\\ &\quad + k_3 d(w,r)+k_4 d(x,s)\\ d\big(T(v,u,v,x),T(q,p,q,s)\big) &\le k_5 d(v,q)+k_6 d(u,p)\\ &\quad + k_7 d(v,q)+k_8 d(x,s)\\ d\big(T(w,u,v,w),T(r,p,q,r)\big) &\le k_9 d(w,r)+k_{10} d(u,p)\\ &\quad + k_{11} d(v,q)+k_{12} d(w,r)\\ d\big(T(x,w,v,u),T(s,r,q,p)\big) &\le k_{13} d(x,s)+k_{14} d(w,r)\\ &\quad + k_{15} d(v,q)+k_{16} d(u,p) \end{aligned}$$

for all $(u,v,w,x),(p,q,r,s) \in X^4$. Because of the mixed monotone property, consider that

$$A=\begin{pmatrix} k_1 & k_2 & k_3 & k_4\\ k_6 & k_5+k_7 & 0 & k_8\\ k_{10} & k_{11} & k_9+k_{12} & 0\\ k_{16} & k_{15} & k_{14} & k_{13}\end{pmatrix}$$

converges to zero. Then,

i. There exists a unique element of $(u^*,v^*,w^*,x^*) \in X^4$ such that

$$\begin{aligned} u^* &= T(u^*,v^*,w^*,x^*)\\ v^* &= T(v^*,u^*,v^*,x^*)\\ w^* &= T(w^*,u^*,v^*,w^*)\\ x^* &= T(x^*,w^*,v^*,u^*) \end{aligned} \tag{9}$$

ii. The sequence

$$\begin{pmatrix} T^n(u,v,w,x), T^n(v,u,v,x), T^n(w,u,v,w),\\ T^n(x,w,v,u)\end{pmatrix}, \quad n \in \mathbb{N}$$

converges to (u^*,v^*,w^*,x^*) as $n \to \infty$, where

$$T^{n+1}(u,v,w,x)=T\begin{pmatrix} T^n(u,v,w,x),\\ T^n(v,u,v,x),\\ T^n(w,u,v,w),\\ T^n(x,w,v,u)\end{pmatrix}$$

$$T^{n+1}(v,u,v,x)=T\begin{pmatrix} T^n(v,u,v,x),\\ T^n(u,v,w,x),\\ T^n(v,u,v,x),\\ T^n(x,w,v,u)\end{pmatrix}$$

$$T^{n+1}(w,u,v,w)=T\begin{pmatrix} T^n(w,u,v,w),\\ T^n(u,v,w,x),\\ T^n(v,u,v,x),\\ T^n(w,u,v,w)\end{pmatrix}$$

$$T^{n+1}(x,w,v,u)=T\begin{pmatrix} T^n(x,w,v,u),\\ T^n(w,u,v,w),\\ T^n(v,u,v,x),\\ T^n(u,v,w,x)\end{pmatrix}$$

iii. For all $n \in \mathbb{N}$, the following estimation is obtained,

$$\begin{pmatrix} d\big(u^*,T^n(u_0,v_0,w_0,x_0)\big)\\ d\big(v^*,T^n(v_0,u_0,v_0,x_0)\big)\\ d\big(w^*,T^n(w_0,u_0,v_0,w_0)\big)\\ d\big(x^*,T^n(x_0,w_0,v_0,u_0)\big)\end{pmatrix} \le A^n(I-A)^{-1}\begin{pmatrix} d\big(u_0,T(u_0,v_0,w_0,x_0)\big)\\ d\big(v_0,T(v_0,u_0,v_0,x_0)\big)\\ d\big(w_0,T(w_0,u_0,v_0,w_0)\big)\\ d\big(x_0,T(x_0,w_0,v_0,u_0)\big)\end{pmatrix}$$

iv. Let $F: X^4 \to X$ be a mapping such that, there exists $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 > 0$ with

$$\begin{aligned} d\big(T(u,v,w,x),F(u,v,w,x)\big) &\le \epsilon_1\\ d\big(T(v,u,v,x),F(v,u,v,x)\big) &\le \epsilon_2\\ d\big(T(w,u,v,w),F(w,u,v,w)\big) &\le \epsilon_3\\ d\big(T(x,w,v,u),F(x,w,v,u)\big) &\le \epsilon_4 \end{aligned}$$

for all $(u,v,w,x) \in X^4$. If $(a^*,b^*,c^*,d^*) \in X^4$ is such that

$$\begin{aligned} a^* &= F(a,b,c,d)\\ b^* &= F(b,a,b,d)\\ c^* &= F(c,a,b,c)\\ d^* &= F(d,c,b,a) \end{aligned}$$

then

$$\begin{pmatrix} d(a^*,u^*)\\ d(b^*,v^*)\\ d(c^*,w^*)\\ d(d^*,x^*)\end{pmatrix} \le (I-A)^{-1}\epsilon \tag{10}$$

where $\epsilon=\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\end{pmatrix}$.

v. Let $F: X^4 \to X$ be a mapping such that, there exists $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 > 0$ with

$$\begin{aligned} d\big(T(u,v,w,x),F(u,v,w,x)\big) &\le \epsilon_1\\ d\big(T(v,u,v,x),F(v,u,v,x)\big) &\le \epsilon_2\\ d\big(T(w,u,v,w),F(w,u,v,w)\big) &\le \epsilon_3\\ d\big(T(x,w,v,u),F(x,w,v,u)\big) &\le \epsilon_4 \end{aligned}$$

for all $(u,v,w,x) \in X^4$. Considering the sequence

$$\begin{pmatrix} F^n(u,v,w,x), F^n(v,u,v,x), F^n(w,u,v,w),\\ F^n(x,w,v,u)\end{pmatrix}, \quad n \in \mathbb{N}$$

converges to (u^*,v^*,w^*,x^*) as $n \to \infty$, where

$$F^{n+1}(u,v,w,x)=F\begin{pmatrix} F^n(u,v,w,x),\\ F^n(v,u,v,x),\\ F^n(w,u,v,w),\\ F^n(x,w,v,u)\end{pmatrix}$$

$$F^{n+1}(v,u,v,x)=F\begin{pmatrix} F^n(v,u,v,x),\\ F^n(u,v,w,x),\\ F^n(v,u,v,x),\\ F^n(x,w,v,u)\end{pmatrix}$$

$$F^{n+1}(w,u,v,w) = F\begin{pmatrix} F^n(w,u,v,w), \\ F^n(u,v,w,x), \\ F^n(v,u,v,x), \\ F^n(w,u,v,w) \end{pmatrix}$$

$$F^{n+1}(x,w,v,u) = F\begin{pmatrix} F^n(x,w,v,u), \\ F^n(w,u,v,w), \\ F^n(v,u,v,x), \\ F^n(u,v,w,x) \end{pmatrix}$$

then, for all $n \in \mathbb{N}$, the following estimation is obtained,

$$\begin{pmatrix} d\big(u^*, F^n(u_0,v_0,w_0,x_0)\big) \\ d\big(v^*, F^n(v_0,u_0,v_0,x_0)\big) \\ d\big(w^*, F^n(w_0,u_0,v_0,w_0)\big) \\ d\big(x^*, F^n(x_0,w_0,v_0,u_0)\big) \end{pmatrix}$$

$$\leq A^n(I-A)^{-1}\begin{pmatrix} d\big(u_0, F(u_0,v_0,w_0,x_0)\big) \\ d\big(v_0, F(v_0,u_0,v_0,x_0)\big) \\ d\big(w_0, F(w_0,u_0,v_0,w_0)\big) \\ d\big(x_0, F(x_0,w_0,v_0,u_0)\big) \end{pmatrix}$$

vi. The system of operational equations

$$\begin{aligned} u &= T(u,v,w,x) \\ v &= T(v,u,v,x) \\ w &= T(w,u,v,w) \\ x &= T(x,w,v,u) \end{aligned} \qquad (11)$$

is Ulam-Hyers stable.

Proof

For (i)-(ii), define $T: X^4 \to X$ by

$$T(u,v,w,x) = \begin{pmatrix} T(u,v,w,x) \\ T(v,u,v,x) \\ T(w,u,v,w) \\ T(x,w,v,u) \end{pmatrix}$$

Consider $\tilde{d}: X^4 \times X^4 \to \mathbb{R}_+^4$,

$$\tilde{d}\big((w,u,v,w),(p,q,r,s)\big) = \begin{pmatrix} d(u,p) \\ d(v,q) \\ d(w,r) \\ d(x,s) \end{pmatrix}$$

Then,

$$\tilde{d}\big((u,v,w,x),(p,q,r,s)\big)$$

$$= \tilde{d}\left(\begin{pmatrix} T(u,v,w,x) \\ T(v,u,v,x) \\ T(w,u,v,w) \\ T(x,w,v,u) \end{pmatrix}, \begin{pmatrix} T(p,q,r,s) \\ T(q,p,q,s) \\ T(r,p,q,r) \\ T(s,r,q,p) \end{pmatrix}\right)$$

$$= \begin{pmatrix} d\big(T(u,v,w,x),T(p,q,r,s)\big) \\ d\big(T(v,u,v,x),T(q,p,q,s)\big) \\ d\big(T(w,u,v,w),T(r,p,q,r)\big) \\ d\big(T(x,w,v,u),T(s,r,q,p)\big) \end{pmatrix}$$

$$\leq \begin{pmatrix} k_1 d(u,p) + k_2 d(v,q) + k_3 d(w,r) + k_4 d(x,s) \\ k_5 d(v,q) + k_6 d(u,p) + k_7 d(v,q) + k_8 d(x,s) \\ k_9 d(w,r) + k_{10} d(u,p) + k_{11} d(v,q) + k_{12} d(w,r) \\ k_{13} d(x,s) + k_{14} d(w,r) + k_{15} d(v,q) + k_{16} d(u,p) \end{pmatrix}$$

$$= \begin{pmatrix} k_1 & k_2 & k_3 & k_4 \\ k_6 & k_5 + k_7 & 0 & k_8 \\ k_{10} & k_{11} & k_9 + k_{12} & 0 \\ k_{16} & k_{15} & k_{14} & k_{13} \end{pmatrix}\begin{pmatrix} d(u,p) \\ d(v,q) \\ d(w,r) \\ d(x,s) \end{pmatrix}$$

$$= A\tilde{d}\big((u,v,w,x),(p,q,r,s)\big) \qquad (12)$$

If we denote $(u,v,w,x) = \alpha$, $(p,q,r,s) = \beta$, then

$$\tilde{d}\big(T(\alpha),T(\beta)\big) \leq A\tilde{d}(\alpha,\beta).$$

Applying Perov's fixed point Theorem 2.1, then there exists a unique element $(u^*,v^*,w^*,x^*) \in X^4$ such that

$$(u^*,v^*,w^*,x^*) = T(u^*,v^*,w^*,x^*)$$

which is equivalent to

$$\begin{aligned} u^* &= T(u^*,v^*,w^*,x^*) \\ v^* &= T(v^*,u^*,v^*,x^*) \\ w^* &= T(w^*,u^*,v^*,w^*) \\ x^* &= T(x^*,w^*,v^*,u^*) \end{aligned}$$

Moreover, for each $\alpha \in X^4$, then $T(\alpha) \to \alpha^*$ as $n \to \infty$,

where

$$T^0(\alpha) = \alpha, T^1(\alpha) = (u,v,w,x) = \begin{pmatrix} T(u,v,w,x), T(v,u,v,x), \\ T(w,u,v,w), T(x,w,v,u) \end{pmatrix}$$

$$T^2(\alpha) = T\big(T^1(\alpha)\big) = T\begin{pmatrix} T(u,v,w,x), T(v,u,v,x), \\ T(w,u,v,w), T(x,w,v,u) \end{pmatrix} = \begin{pmatrix} T^2(u,v,w,x), T^2(v,u,v,x), \\ T^2(w,u,v,w), T^2(x,w,v,u) \end{pmatrix}$$

And generally,

$$T^{n+1}(\alpha) = T^n\begin{pmatrix} T(u,v,w,x), T(v,u,v,x), \\ T(w,u,v,w), T(x,w,v,u) \end{pmatrix}$$

Now, it has been obtained that $T(\alpha) \to \alpha^* = (u^*,v^*,w^*,x^*)$ as $n \to \infty$, for all $\alpha = (u,v,w,x) \in X^4$. So, for all $(u,v,w,x) \in X^4$, then

$$\begin{aligned} T(u,v,w,x) &\to u^* \text{ as } n \to \infty \\ T(v,u,v,x) &\to v^* \text{ as } n \to \infty \\ T(w,u,v,w) &\to w^* \text{ as } n \to \infty \\ T(x,w,v,u) &\to x^* \text{ as } n \to \infty \end{aligned}$$

iii. By Perov's theorem, then

$$\begin{pmatrix} (T^n(u_0,v_0,w_0,x_0),u^*) \\ (T^n(v_0,u_0,v_0,x_0),v^*) \\ (T^n(w_0,u_0,v_0,w_0),w^*) \\ (T^n(x_0,w_0,v_0,u_0),x^*) \end{pmatrix}$$

$$= \tilde{d}\big(T^n(u_0,v_0,w_0,x_0),(u^*,v^*,w^*,x^*)\big)$$

$$\leq A^n(I-A)^{-1}.\tilde{d}\begin{pmatrix} (u_0,v_0,w_0,x_0), \\ T(u_0,v_0,w_0,x_0), \\ T(v_0,u_0,v_0,x_0), \\ T(w_0,u_0,v_0,w_0), \\ T(x_0,w_0,v_0,u_0) \end{pmatrix}$$

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$$\leq A^n(I-A)^{-1}.\begin{pmatrix} d\big(u_0,F(u_0,v_0,w_0,x_0)\big) \\ d\big(v_0,F(v_0,u_0,v_0,x_0)\big) \\ d\big(w_0,F(w_0,u_0,v_0,w_0)\big) \\ d\big(x_0,F(x_0,w_0,v_0,u_0)\big) \end{pmatrix}$$

iv. Now, considering $F: X^4 \to X^4$ such that

$$T(u,v,w,x) = \begin{pmatrix} F(u,v,w,x) \\ F(v,u,v,x) \\ F(w,u,v,w) \\ F(x,w,v,u) \end{pmatrix}$$

consider $\tilde{d}: X^4 \times X^4 \to \mathbb{R}_+^4$,

$$\tilde{d}\big((u,v,w,x),(p,q,r,s)\big) = \begin{pmatrix} d(u,p) \\ d(v,q) \\ d(w,r) \\ d(x,s) \end{pmatrix}$$

Then,

$$\begin{aligned} &\tilde{d}\big((u,v,w,x),(p,q,r,s)\big) \\ &= \tilde{d}\left(\begin{pmatrix} F(u,v,w,x) \\ F(v,u,v,x) \\ F(w,u,v,w) \\ F(x,w,v,u) \end{pmatrix}, \begin{pmatrix} F(p,q,r,s) \\ F(q,p,q,s) \\ F(r,p,q,r) \\ F(s,r,q,p) \end{pmatrix}\right) \\ &= \begin{pmatrix} d\big(F(u,v,w,x),F(p,q,r,s)\big) \\ d\big(F(v,u,v,x),F(q,p,q,s)\big) \\ d\big(F(w,u,v,w),F(r,p,q,r)\big) \\ d\big(F(x,w,v,u),F(s,r,q,p)\big) \end{pmatrix} \\ &\leq \epsilon \qquad (13) \end{aligned}$$

then, applying Perov's fixed point Theorem 2.2 gives

$$\begin{aligned} &d\big((u^*,v^*,w^*,x^*),(a^*,b^*,c^*,d^*)\big) \\ &\leq (I-A)^{-1}\epsilon \qquad (14) \end{aligned}$$

v. By (13) the following is obtained

$$\tilde{d}\big(T(u,v,w,x),F(u,v,w,x)\big) \leq \epsilon.$$

Since, $F^n(u,v,w,x) = F\big(F^{n-1}(u,v,w,x)\big)$, for all $(u,v,w,x) \in X^4$.

Using the assertion (iii) of this theorem, it can successively be written as:

$$\begin{aligned} &\tilde{d}\big(F^n(u_0,v_0,w_0,x_0),(u^*,v^*,w^*,x^*)\big) \\ &\leq \tilde{d}\big(F^n(u_0,v_0,w_0,x_0),T^n(u_0,v_0,w_0,x_0)\big) \\ &+ \tilde{d}\big(T^n(u_0,v_0,w_0,x_0),(u^*,v^*,w^*,x^*)\big) \\ &\leq \tilde{d}\big(F^n(u_0,v_0,w_0,x_0),T^n(u_0,v_0,w_0,x_0)\big) \\ &+ A^n(I \\ &- A)^{-1}\tilde{d}\big(T(u_0,v_0,w_0,x_0),(u_0,v_0,w_0,x_0)\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\tilde{d}\big(F^n(u_0,v_0,w_0,x_0),T^n(u_0,v_0,w_0,x_0)\big) \\ &= \tilde{d}\begin{pmatrix} F\big(F^{n-1}(u_0,v_0,w_0,x_0)\big), \\ T\big(T^{n-1}(u_0,v_0,w_0,x_0)\big) \end{pmatrix} \\ &\leq \tilde{d}\begin{pmatrix} F\big(F^{n-1}(u_0,v_0,w_0,x_0)\big), \\ T\big(F^{n-1}(u_0,v_0,w_0,x_0)\big) \end{pmatrix} \\ &+ \tilde{d}\begin{pmatrix} T\big(F^{n-1}(u_0,v_0,w_0,x_0)\big), \\ T\big(T^{n-1}(u_0,v_0,w_0,x_0)\big) \end{pmatrix} \\ &\leq \epsilon + A\tilde{d}\begin{pmatrix} F^{n-1}(u_0,v_0,w_0,x_0), \\ T^{n-1}(u_0,v_0,w_0,x_0) \end{pmatrix} \\ &\leq \epsilon + A\left[\epsilon + A\tilde{d}\begin{pmatrix} F^{n-2}(u_0,v_0,w_0,x_0), \\ T^{n-2}(u_0,v_0,w_0,x_0) \end{pmatrix}\right] \\ &\leq \cdots \leq \epsilon(I + A + A^2 + \cdots + A^n + \cdots) \\ &= \epsilon(I-A)^{-1}. \qquad (15) \end{aligned}$$

Thus, finally

$$\begin{aligned} &\tilde{d}\begin{pmatrix} F^n(u_0,v_0,w_0,x_0), \\ (u^*,v^*,w^*,x^*) \end{pmatrix} \\ &\leq \epsilon(I-A)^{-1} \\ &+ A^n(I \\ &- A)^{-1}\tilde{d}\begin{pmatrix} T(u_0,v_0,w_0,x_0), \\ (u_0,v_0,w_0,x_0) \end{pmatrix}. \end{aligned}$$

vi. By (i) and (ii) there exists a unique element $(u^*,v^*,w^*,x^*) \in X^4$ such that (u^*,v^*,w^*,x^*) is a solution for (11) and the sequence

$$\begin{pmatrix} T^n(u,v,w,x), \\ T^n(v,u,v,x), \\ T^n(w,u,v,w), \\ T^n(x,w,v,u) \end{pmatrix} \to (u^*,v^*,w^*,x^*)$$

as $n \to \infty$.

Let $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 > 0$ and $(u^*,v^*,w^*,x^*) \in X^4$ such that

$$\begin{aligned} d\big(p^*,T(p,q,r,s)\big) &\leq \epsilon_1 \\ d\big(q^*,T(q,p,q,s)\big) &\leq \epsilon_2 \\ d\big(r^*,T(r,p,q,r)\big) &\leq \epsilon_3 \\ d\big(s^*,T(s,r,q,p)\big) &\leq \epsilon_4 \end{aligned}$$

Then,

$$\begin{aligned} &\tilde{d}\big((p^*,q^*,r^*,s^*),(u^*,v^*,w^*,x^*)\big) \\ &\leq \tilde{d}\left((p^*,q^*,r^*,s^*),\begin{pmatrix} T(p^*,q^*,r^*,s^*), \\ T(q^*,p^*,q^*,s^*), \\ T(r^*,p^*,q^*,r^*), \\ T(s^*,r^*,q^*,p^*) \end{pmatrix}\right) \\ &+ \tilde{d}\left(\begin{pmatrix} T(p^*,q^*,r^*,s^*), \\ T(q^*,p^*,q^*,s^*), \\ T(r^*,p^*,q^*,r^*), \\ T(s^*,r^*,q^*,p^*) \end{pmatrix},(u^*,v^*,w^*,x^*)\right) \end{aligned}$$

$$= \tilde{d}\left((p^*,q^*,r^*,s^*),\begin{pmatrix}T(p^*,q^*,r^*,s^*),\\T(q^*,p^*,q^*,s^*),\\T(r^*,p^*,q^*,r^*),\\T(s^*,r^*,q^*,p^*)\end{pmatrix}\right)$$

$$+\tilde{d}\left(\begin{pmatrix}T(p^*,q^*,r^*,s^*)\\T(q^*,p^*,q^*,s^*)\\T(r^*,p^*,q^*,r^*)\\T(s^*,r^*,q^*,p^*)\end{pmatrix},\begin{pmatrix}T(u^*,v^*,w^*,x^*)\\T(v^*,u^*,v^*,x^*)\\T(w^*,u^*,v^*,w^*)\\T(x^*,w^*,v^*,u^*)\end{pmatrix}\right)$$

$$=\begin{pmatrix}d\big(p^*,T(p^*,q^*,r^*,s^*)\big)\\d\big(q^*,T(q^*,p^*,q^*,s^*)\big)\\d\big(r^*,T(r^*,p^*,q^*,r^*)\big)\\d\big(s^*,T(s^*,r^*,q^*,p^*)\big)\end{pmatrix}$$

$$+\begin{pmatrix}d\big(T(p^*,q^*,r^*,s^*),T(u^*,v^*,w^*,x^*)\big)\\d\big(T(q^*,p^*,q^*,s^*),T(v^*,u^*,v^*,x^*)\big)\\d\big(T(r^*,p^*,q^*,r^*),T(w^*,u^*,v^*,w^*)\big)\\d\big(T(s^*,r^*,q^*,p^*),T(x^*,w^*,v^*,u^*)\big)\end{pmatrix}$$

$$\leq\begin{pmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\\\epsilon_4\end{pmatrix}$$

$$+\tilde{d}\big(T(p^*,q^*,r^*,s^*),T(u^*,v^*,w^*,x^*)\big)$$

$$\leq \epsilon + A\tilde{d}\big(T(p^*,q^*,r^*,s^*),T(u^*,v^*,w^*,x^*)\big)$$

since $(I-A)$ is invertible and $(I-A)^{-1}$ has positive elements.
then

$$\tilde{d}\big(T(p^*,q^*,r^*,s^*),T(u^*,v^*,w^*,x^*)\big) \leq (I-A)^{-1}\epsilon$$

or equivalently

$$\begin{pmatrix}d(p^*,u^*)\\d(q^*,v^*)\\d(r^*,w^*)\\d(s^*,x^*)\end{pmatrix}=(I-A)^{-1}\epsilon.$$

Denote

$$(I-A)^{-1}=\begin{pmatrix}c_1&c_2&c_3&c_4\\c_5&c_6&c_7&c_8\\c_9&c_{10}&c_{11}&c_{12}\\c_{13}&c_{14}&c_{15}&c_{16}\end{pmatrix}$$

then

$$d(p^*,u^*)\leq c_1\epsilon_1+c_2\epsilon_2+c_3\epsilon_3+c_4\epsilon_4$$
$$d(q,v^*)\leq c_5\epsilon_1+c_6\epsilon_2+c_7\epsilon_3+c_8\epsilon_4$$
$$d(r^*,w^*)\leq c_9\epsilon_1+c_{10}\epsilon_2+c_{11}\epsilon_3+c_{12}\epsilon_4$$
$$d(s^*,x^*)\leq c_{13}\epsilon_1+c_{14}\epsilon_2+c_{15}\epsilon_3+c_{16}\epsilon_4$$

This proves that the operational system (11) is Ulam-Hyers stable.

4.0 CONCLUSION

This work extended the concept of Ulam-Hayers stability from tripled fixed point to quadrupled fixed point. It opens up opportunities for further extension and generalization of Ulam-Hayers stability analysis for mixed monotone operators.

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Some Fixed Point Theorems for Contractive Type Conditions in C^* −algebra b −metric Spaces

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ABSTRACT
Fixed point theory has significantly improved the understanding of mathematical analysis especially the area of linear and nonlinear analysis. The concept of tripled fixed point theorem is an improvement on the ideology of coupled fixed point. This work establishes some tripled fixed point theorems for mappings which satisfies unique type contractive conditions and on the basis of C^* −algebra b-metric spaces. This work also serves as an extension to other relevant researches in literature.

Keywords*: Fixed point, Tripled fixed point, C^*-algebra-valued, b-metric Spaces, C^*-algebra-valued b-metric*

1.0 INTRODUCTION

The approach of fixed point theorem has significantly improve on the concepts of mastering the connection between the field of mathematics with specification to mathematical analysis and other areas of sciences and its applications. Bahkitin (1989) introduced b-metric spaces as a generalization of metric space. Since then, more other generalized b-metric spaces like, quasi-b-metric spaces (Czerwick, 1993) and quasi-b-metric-like spaces (Czerwick, 1998) were introduced.

Ma *et al*., (2014) initially introduced the concept of a C^*-algebra-valued b-metric space which generalized the concept of b-metric spaces, and established certain basic fixed point theorems for self-map with contractive condition in this new setting. Aydi *et al.,* (2015) also introduced the concept of C^*-algebra-valued b-metric space and generalized the Banach contraction principle on such spaces.

The idea of coupled fixed point was introduced by Guo and Lakshmikanthan, (1987). Since then, many researchers investigated coupled fixed point theorems in ordered metric spaces and gave some applications (Petre, 2014; Roshan *et al.* 2014; Batul & Kamran, 2015; Kamran *et al.,* 2016).
Motivated by the work of Kamran *et al.* (2016), this work establish existence and uniqueness of tripled fixed point satisfying certain contractive condition and based on the concept of C^*algebra-valued b^*-metric space, as an extension of coupled fixed point to tripled fixed point.

2.0 MATERIALS AND METHODS

We recall some basic definitions, notations, and results of C^*-algebra which can be found in (Batul & Kamran, 2015; Kamran *et al.,* 2016).

Definition 2.1 Let $\mathbb{A}$ be an algebra. An involution on $\mathbb{A}$ is a conjugate linear map a $\mapsto$ a* such that $(a^*)^* = a$ and $(ab)^* = a^*b^*$ for all a, b $\in$ $\mathbb{A}$. The pair $(\mathbb{A},*)$ is called a $*$-algebra. If $\mathbb{A}$ contains the identity element $1_{\mathbb{A}}$, then $(\mathbb{A},*)$ is called a united $*$-algebra. A $*$-algebra $\mathbb{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ is said to be a Banach $*$-algebra, then $\mathbb{A}$ is known as C^*-algebra.

Definition 2.2 Let $\mathbb{A}$ be a C^*-algebra and X be a non empty set. Let $\mathbb{A}'_+$ be such that $\|b\| \geq 1$. A mapping $d_b: X^2 \to \mathbb{A}_+$ is said to be a C^*-algebra-valued b-metric on X if the subsequent conditions hold for all $x, y, z \in \mathbb{A}$:
C$_1$. $d_b(x, y) = 0_{\mathbb{A}}$ if and only if $x = y$
C$_2$. $d_b(x, y) = d_b(y, x)$;
C$_2$. $d_b(x, y) \preccurlyeq b[d_b(x, z) + d_b(z, y)]$
The triplet $(X, \mathbb{A}, d_b)$ is termed a C^*-algebra-valued b-metric space with coefficient b.

Definition 2.3 Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric, $x \in X$ and $\{x_n\}$ a sequence in

X. Then:

i. $\{x_n\}$ converges to x with reference to $\mathbb{A}$ whenever for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $\|d_b(x_n, x)\| < \varepsilon$ for all $n > N$
ii. $\{x_n\}$ is a Cauchy sequence with reference to $\mathbb{A}$ if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $\|d_b(x_n, x_m)\| < \varepsilon$ for all $m, n > N$
iii. $(X, \mathbb{A}, d_b)$ is complete if every Cauchy sequence in X is convergent with respect to $\mathbb{A}$.

Lemma 2.1 Assume that $\mathbb{A}$ is a unital C^*-algebra with a unit $1_{\mathbb{A}}$.

i. For any $x \in \mathbb{A}_+$, we have $x \preccurlyeq 1_{\mathbb{A}} \Leftrightarrow \|x\| \leq 1$;
ii. If $a \in \mathbb{A}_+$, with $\|a\| \leq \frac{1}{2}$, then $1_{\mathbb{A}} - a$ is inevitable and $\|a(1_{\mathbb{A}} - a)^{-1}\| < 1$;
iii. Assume that $a, b \in \mathbb{A}$ with $a, b \succcurlyeq 0_{\mathbb{A}}$, and $ab = ba$, then $ab \succcurlyeq 0_{\mathbb{A}}$;
iv. Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succcurlyeq c \succcurlyeq 0_{\mathbb{A}}$, and $1_{\mathbb{A}} - a \in \mathbb{A}'_+$ is an invertible operator, then
v. $(1_{\mathbb{A}} - a)^{-1} b \succcurlyeq (1_{\mathbb{A}} - a)^{-1} c$;
vi. If $a, b, c \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $a \in \mathbb{A}$, then $b \preccurlyeq c \Longrightarrow a^* b a \preccurlyeq a^* c a$;
vii. If $0_{\mathbb{A}} \preccurlyeq a \preccurlyeq b$, then $\|a\| \leq \|b\|$.

Lemma 2.2 The sum of two positive elements in a C^*-algebra is a positive element.

Definition 2.4 Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space. An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $T: X^2 \to X$ if $T(x, y) = x$ and $T(y, x) = y$.

Definition 2.5 Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space. An element $(x, y, z) \in X^3$ is said to be a tripled fixed point of the mapping $T: X^3 \to X$ if $T(x, y, z) = x$, $T(y, x, z) = y$, and $T(z, y, x) = x$.

3.0 RESULTS

These main results prove tripled fixed point theorem for mappings with two contractive conditions in the setting of C^*-algebra-valued b -metric space.

Theorem 3.1 Let $(X, \mathbb{A}, d_b)$ be a C^*-algebra-valued b-metric space. Assume that the mapping $T: X^3 \longrightarrow X$ satisfies the following conditions:

$$d_b\big(T(u,v,w), T(p,q,r)\big) \preccurlyeq \alpha^* d_b(u,p)\alpha + \alpha^* d_b(v,q)\alpha + \alpha^* d_b(w,r)\alpha, \quad \forall\, u,v,w,p,q,r \in X, \tag{1}$$

where $\alpha \in \mathbb{A}$ with $2\|\alpha\|^2\|b\| < 1$. Then T has a unique tripled fixed point in X. Moreover, T has a unique fixed point in X.

Proof

Let $u_0, v_0, w_0 \in X$. Define three sequences $\{u_n\}, \{v_n\}$ and $\{w_n\}$ in X by the iterative procedure as

$u_{n+1} = T(u_n, v_n, w_n)$, $v_{n+1} = T(v_n, u_n, v_n)$ and $w_{n+1} = T(w_n, v_n, u_n)$

Applying condition (1), for $n \in \mathbb{N}$, we have

$$\begin{aligned} & d_b(u_n, u_{n+1}) \\ &= d_b\big(T(u_{n-1}, v_{n-1}, w_{n-1}), T(u_n, v_n, w_n)\big) \\ &\preccurlyeq \alpha^* d_b(u_{n-1}, u_n)\alpha + \alpha^* d_b(v_{n-1}, v_n)\alpha \\ &+ \alpha^* d_b(w_{n-1}, w_n)\alpha = \alpha^* M_n \alpha, \end{aligned} \tag{2}$$

Where

$$M_n = d_b(u_{n-1}, u_n) + d_b(v_{n-1}, v_n) + d_b(w_{n-1}, w_n) \tag{3}$$

Similarly,

$$\begin{aligned} & d_b(v_n, v_{n+1}) \\ &= d_b\big(T(v_{n-1}, u_{n-1}, v_{n-1}), T(v_n, u_n, v_n)\big) \\ &\preccurlyeq \alpha^* d_b(v_{n-1}, v_n)\alpha + \alpha^* d_b(u_{n-1}, u_n)\alpha \\ &\quad + \alpha^* d_b(v_{n-1}, v_n)\alpha \\ &= \alpha^* M_n \alpha, \end{aligned} \tag{4}$$

Because of the mixed monotone property,

$$\begin{aligned} & d_b(u_{n-1}, u_n) + d_b(v_{n-1}, v_n) + d_b(w_{n-1}, w_n) \\ &= d_b(u_{n-1}, u_n) \\ &\quad + 2d_b(v_{n-1}, v_n) = M_n \end{aligned} \tag{5}$$

$$d_b(v_{n-1}, v_n) = d_b(w_{n-1}, w_n) \tag{6}$$

Similarly, we get

$$\begin{aligned} & d_b(w_n, w_{n+1}) \\ &= d_b\big(T(w_{n-1}, v_{n-1}, u_{n-1}), T(w_n, v_n, u_n)\big) \\ &\preccurlyeq \alpha^* M_n \alpha, \end{aligned} \tag{7}$$

By (2)-(7), we have

$$\begin{aligned} M_{n+1} &= d_b(u_n, u_{n+1}) + d_b(v_n, v_{n+1}) \\ &\quad + d_b(w_n, w_{n+1}) \\ &\preccurlyeq 3\alpha^*[d_b(u_{n-1}, u_n) \\ &\quad + 2d_b(v_{n-1}, v_n)]\alpha \\ &\preccurlyeq \left(\sqrt{3}\alpha\right)^* M_n \left(\sqrt{3}\alpha\right) \end{aligned} \tag{8}$$

From (8) and the conditions of Lemma 2.1, we have

$$\begin{aligned} 0_{\mathbb{A}} \preccurlyeq M_{n+1} &\preccurlyeq \left(\sqrt{3}a\right)^* M_n \left(\sqrt{3}a\right) \preccurlyeq \cdots \\ &\preccurlyeq \left[\left(\sqrt{3}a\right)^*\right]^n M_1 \left(\sqrt{3}a\right)^n. \end{aligned}$$

If $M_1 = 0_{\mathbb{A}}$, it can easily be shown that from Definition 2.4 (u_0, v_0, w_0) is a tripled fixed point of the mixed monotone mapping T. Now, let $0_{\mathbb{A}} \preccurlyeq M_1$ and $n, m \in \mathbb{N}$ with $m > n$, by using Definition 2.2, it follows that

$$d_b(u_n, u_m) \preccurlyeq b[d_b(u_n, u_{n+1}) + d_b(u_{n+1}, u_m)]$$

$$\begin{aligned}&\preccurlyeq bd_b(u_n,u_{n+1})+b^2[d_b(u_{n+1},u_{n+2})+\cdots\\&\quad+b^{m-n-1}d_b(u_{m-2},u_{m-1})\\&\quad+b^{m-n-1}d_b(u_{m-1},u_m)]\end{aligned}$$

Similarly,

$$\begin{aligned}d_b(v_n,v_m)&\preccurlyeq bd_b(v_n,v_{n+1})\\&\quad+b^2d_b(v_{n+1},v_{n+2})+\cdots\\&\quad+b^{m-n-1}d_b(v_{m-2},v_{m-1})\\&\quad+b^{m-n-1}d_b(v_{m-1},v_m).\end{aligned}$$

and from (6), because of the mixed monotone property,

$$d_b(v_n,v_m)\preccurlyeq d_b(w_n,w_m).$$

Hence,

$$\begin{aligned}&d_b(u_n,u_m)+d_b(v_n,v_m)+d_b(w_n,w_m)\\&\quad=b[d_b(u_n,u_{n+1})\\&\quad+2d_b(v_n,v_{n+1})]\\&\quad+b^2[d_b(u_{n+1},u_{n+2})\\&\quad+2d_b(w_n,w_{n+1})]+\cdots\\&\quad+b^{m-n-1}[d_b(u_{m-2},u_{m-1})\\&\quad+2d_b(v_{m-2},v_{m-1})]\\&\quad+b^{m-n-1}[d_b(u_{m-1},u_m)\\&\quad+2d_b(w_{m-1},w_m)]\\&\preccurlyeq bM_{n+1}+b^2M_{n+2}+\cdots+b^{m-n-1}M_{m-1}\\&\quad+b^{m-n-1}M_m\\&\preccurlyeq b\left(\sqrt{3}\alpha\right)^*M_n\left(\sqrt{3}\alpha\right)+b^2\left(\sqrt{3}\alpha\right)^*M_{n+1}\left(\sqrt{3}\alpha\right)\\&\quad+\cdots\\&\quad+b^{m-n-1}\left(\sqrt{3}\alpha\right)^*M_{n-2}\left(\sqrt{3}\alpha\right)\\&\quad+b^{m-n-1}\left(\sqrt{3}\alpha\right)^*M_{n-1}\left(\sqrt{3}\alpha\right)\\&=b\sum_{j=n}^{m-2}b^{j-n}\left|M_1^{\frac{1}{2}}\left(\sqrt{3}\alpha\right)^j\right|^2\\&\quad+b^{m-n-1}\left|M_1^{\frac{1}{2}}\left(\sqrt{3}\alpha\right)^{m-1}\right|^2\\&\preccurlyeq\left\|b\sum_{j=n}^{m-2}b^{j-n}\left|M_1^{\frac{1}{2}}\left(\sqrt{3}\alpha\right)^j\right|^2\right\|1_{\mathbb{A}}\\&\quad+\left\|b^{m-n-1}\left|M_1^{\frac{1}{2}}\left(\sqrt{3}\alpha\right)^{m-1}\right|^2\right\|1_{\mathbb{A}}\\&\preccurlyeq\|b\|^{1-n}\left\|M_1^{\frac{1}{2}}\right\|^2\sum_{i=n}^{m-2}\|b\|^j\left\|\left(\sqrt{3}\alpha\right)^2\right\|^i1_{\mathbb{A}}\\&\quad+\|b\|^{-n}\left\|M_1^{\frac{1}{2}}\right\|^2\|b\|^{m-1}\left\|\left(\sqrt{3}\alpha\right)^2\right\|^{m-1}1_{\mathbb{A}}\\&=\|b\|^{1-n}\left\|M_1^{\frac{1}{2}}\right\|^2\sum_{j=n}^{m-2}(3\|\alpha\|^2\|b\|)^j1_{\mathbb{A}}\\&\quad+\|b\|^{-n}\left\|M_1^{\frac{1}{2}}\right\|^2(3\|\alpha\|^2\|b\|)^{m-1}1_{\mathbb{A}}\\&\longrightarrow 0_{\mathbb{A}}\ (\text{as } m,n\to\infty)\qquad(9)\end{aligned}$$

by the condition $3\|\alpha\|^2\|b\|<1$ and $\|b\|\geq 1$.

Hence $\{u_n\},\{v_n\}$ and $\{w_n\}$ are Cauchy sequences in X. By the completeness of $(X,\mathbb{A},d)$, shows that there exists $u^*,v^*,w^*\in X$ such that $u_n\to u^*$, $v_n\to v^*$ and $w_n\to w^*$ as $n\to\infty$. We now show that $T(u^*,v^*,w^*)=u^*$, $T(v^*,u^*,v^*)=v^*$ and $T(w^*,v^*,u^*)=w^*$. From Definition 2.2 and by condition (1), we get

$$\begin{aligned}0_{\mathbb{A}}&\preccurlyeq d_b(T(u^*,v^*,w^*),u^*)\\&\preccurlyeq b[d_b(T(u^*,v^*,w^*),u_{n+1})\\&\quad+d_b(u_{n+1},u^*)]\\&\preccurlyeq b\alpha^*d_b(u^*,u_n)\alpha+2b\alpha^*d_b(v^*,v_n)\alpha\\&\quad+bd_b(u_{n+1},u^*)\\&\longrightarrow 0_{\mathbb{A}}\ (\text{as } n\to\infty)\qquad(10)\end{aligned}$$

So, $T(u^*,v^*,w^*)=u^*$. Similarly, $T(v^*,u^*,v^*)=v^*$ and $T(w^*,v^*,u^*)=w^*$. Thus, (u^*,v^*,w^*) is a tripled fixed point of T. If there exists another tripled fixed point (p,q,r) of T, then

$$\begin{aligned}0_{\mathbb{A}}\preccurlyeq d_b(u^*,p)&=d_b(T(u^*,v^*,w^*),\mathrm{T}(p,q,r))\\&\preccurlyeq\alpha^*d_b(u^*,p)\alpha\\&\quad+\alpha^*d_b(v^*,q)\alpha\\&\quad+\alpha^*d_b(w^*,r)\alpha,\\0_{\mathbb{A}}\preccurlyeq d_b(v^*,q)&=d_b(T(v^*,u^*,v^*),\mathrm{T}(q,p,q))\\&\preccurlyeq\alpha^*d_b(u^*,p)\alpha\\&\quad+2\alpha^*d_b(v^*,q)\alpha,\\0_{\mathbb{A}}\preccurlyeq d_b(w^*,r)&=d_b(T(w^*,v^*,u^*),\mathrm{T}(r,q,p))\\&\preccurlyeq\alpha^*d_b(w^*,r)\alpha\\&\quad+\alpha^*d_b(v^*,q)\alpha\\&\quad+\alpha^*d_b(u^*,p)\alpha,\end{aligned}$$

which implies that

$$\begin{aligned}0_{\mathbb{A}}&\preccurlyeq d_b(u^*,p)+d_b(v^*,q)+d_b(w^*,r)\\&\preccurlyeq\left(\sqrt{3}\alpha\right)^*\left(d_b(u^*,p)\right.\\&\quad+d_b(v^*,q)\\&\quad\left.+d_b(w^*,r)\right)\left(\sqrt{3}\alpha\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}0_{\mathbb{A}}&\preccurlyeq\|d_b(u^*,p)+d_b(v^*,q)+d_b(w^*,r)\|\\&\preccurlyeq\left\|\sqrt{3}\alpha\right\|^2\|d_b(u^*,p)+d_b(v^*,q)+d_b(w^*,r)\|\\&<\frac{1}{\|b\|}\|d_b(u^*,p)+d_b(v^*,q)+d_b(w^*,r)\|\\&\leq\|d_b(u^*,p)+d_b(v^*,q)+d_b(w^*,r)\|\end{aligned}$$

which is a contraction. Thus, $(p,q,r)=(u^*,v^*,w^*)$, implies that the tripled fixed point is unique.

Finally, we will show that T has a unique fixed point.

Since $u,v,w\in X$ are comparable, then

$$\begin{aligned}d(u,v)=d(v,w)=d(u,w)\\0_{\mathbb{A}}&\preccurlyeq d_b(u^*,v^*)\\&=d_b\left(T(u^*,v^*,w^*),T(v^*,u^*,v^*)\right)\\&\preccurlyeq\alpha^*d_b(u^*,v^*)\alpha+\alpha^*d_b(v^*,u^*)\alpha\\&\quad+\alpha^*d_b(w^*,v^*)\alpha\\&\preccurlyeq 2\alpha^*d_b(u^*,v^*)\alpha+\alpha^*d_b(w^*,v^*)\alpha\end{aligned}$$

we have

$$\begin{aligned}\|d_b(u^*,v^*)\|&\leq\|1_{\mathbb{A}}\\&\quad-2\alpha^*\alpha\|^{-1}\|\alpha\|^2d_b(w^*,v^*).\end{aligned}$$

It follows from the condition $\|1_{\mathbb{A}}-2\alpha^*\alpha\|^{-1}\|\alpha\|^2\leq 1$ that $\|d_b(w^*,v^*)\|=0$

Hence,

$$v^* = w^* \qquad (11)$$

Similarly,

$$\begin{aligned} 0_{\mathbb{A}} &\preccurlyeq d_b(u^*, w^*) \\ &= d_b\big(T(u^*, v^*, w^*), T(w^*, v^*, u^*)\big) \\ &\preccurlyeq \alpha^* d_b(u^*, w^*)\alpha + \alpha^* d_b(v^*, v^*)\alpha \\ &\quad + \alpha^* d_b(w^*, u^*)\alpha \end{aligned}$$

we have

$$\|d_b(u^*, w^*)\| \leq 2\|\alpha\|^2 \|d_b(u^*, w^*)\|,$$

It follows from the condition $2\|\alpha\|^2 \leq \frac{1}{\|b\|} < 1$ that $\|d_b(u^*, w^*)\| = 0$.

Hence,

$$u^* = w^* \qquad (12)$$

It follows from (11) and (12) that since $u^* = w^*$ and $v^* = w^*$, it implies that $u^* = v^* = w^*$.

which concludes the proof.

4.0 CONCLUSION

In this research, tripled fixed point results in C^*-algebra b-cauchy spaces in the presence of a unique contractive-type condition was obtained. The findings improve the existing results on tripled fixed points in literature. By implication, further studies should be carried out by taking this work as a basis for improvement and applicability.

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Mathematical Model for the Vaccination and Treatment Strategy to Eradicate Tuberculosis with Migration and Permanent Immunity Effect

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ABSTRACT
In this study, we proposed a mathematical model for the vaccination and treatment strategy to eradicate tuberculosis with migration and permanent immunity effect, where we modified Kalu and Inyama's work by incorporating the migration effect, efficacy of vaccination, treatment and new babies were considered 100% vaccinated. Existence and uniqueness of the modified model was carried out and it shows that the solution exists and it is unique. The stability analysis of the disease free equilibrium shows that the disease-free equilibrium (DFE) is locally asymptotically stable. The effective reproductive number was found to be 0.1741. Our study of R_e, and the stability of the DFE show that tuberculosis can be eliminated through vaccination and treatment. The analysis also suggests that if vaccines can confer permanent immunity, then a more stable equilibrium can be achieved. In other words, the disease can die out quicker. The results show that mycobacterium tuberculosis can be eradicated if massive vaccination and treatment of both latent and infectious individuals is given with high efficacy. This implies that every effort should be made to begin appropriate treatment and to ensure completion of the entire course of treatment. Also, infected immigrants need to be restricted from entering the population to ensure fast eradication of mycobacterium tuberculosis.

Keywords: *Vaccination, Treatment, Tuberculosis, Migration and Permanent Immunity*

1.0 INTRODUCTION
Tuberculosis is an airborne infection caused by Mycobacterium Tuberculosis (MTB). Kumar et al (2007) opined that MTB or TB (short for tubercle bacillus) is a common, and in many cases lethal infectious disease caused by various strains of mycobacteria, usually mycobacterium tuberculosis. TB typically attacks the lungs, but can also affect other parts of the body. It is spread through the air when people who have active TB infection cough, sneeze, laugh or sing or otherwise propel their saliva into the air (Konstantinos, 2010). Most infections are asymptomatic and latent, but about one in ten infections eventually progress to active disease which, if left untreated, kills more than 50% of those infected (WHO, 2007). Mycobacterium tuberculosis is a rod-shaped, slow-growing bacterium. MTB cell wall has acid contents which makes it hydrophobic, resistant to oral fluids (Emmanuel, 2013). The cell wall of the mycobacteria absorbs a certain dye used in the preparation of slides under the microscope and maintains this red color despite attempt of decolourisation, hence the name acid-fast bacilli (Emmanuel, 2013).

In 2016, WHO estimated that there were 10.4 million incidence cases of TB worldwide and 1.7 million deaths due to TB including 0.4 million among people with HIV. Globally, the TB mortality rate fell by 37% between 2000 and 2016 (WHO, 2017). Once infected, an individual enters a period of latency during which he exhibits no symptoms of the disease and is not infectious to others. Such a person is said to have latent TB infection. This latent period can be of extremely variable length of time. A great majority of those infected may live with the disease for as long as possible without it degenerating or progressing into Active TB. However, a small proportion of those infected will develop active TB, falling ill within months after infection (Colijn et al, 2006). Africa alone had 8.8 new TB infections which results in 1.7 million deaths in 2003 (WHO, 2007). Of all African countries, Nigeria has the highest TB burden and is ranked 4th

among the 22 burden countries in the world (WHO, 2012). According to estimates of World Health Organization, Nigeria has 311 TB cases for every 100,000 population. Research have shown that more people in the developing world would contract TB because their immune systems are more likely to be compromised due to higher exposure to immune suppressive drugs, substance abuse or AIDS (Egbetade et al, 2014).

TB is believed to have been present in humans for thousands of years. Skeletal remains show that pre-historic humans (4000BCE) had TB and tubercular decay has been found in the spines of Egyptian mummies dating from 3000-2400BCE (Herzog, 1998). Due to its variety of symptoms, it was not identified as a single disease until the 1920's. In 1834, Johann Lukas Schonlein gave the disease name 'tuberculosis' (Herzog, 1998). Mycobacterium tuberculosis, the bacteria that cause the tuberculosis was identified by Nobel Laureate Robert Koch in March 1882 and in the 1900's, Albert Calmette and Camille Guerin achieved the first genuine success in immunizing against tuberculosis using attenuated bovin-strain tuberculosis called 'BCG' (Bacillus of Calmette and Guerin) (Herzog, 1998). Physicians in ancient Greece called this illness phthisis to reflect its wasting character.

During the 17th and 18th centuries, TB caused up to 25% of all deaths in Europe. In more recent times, tuberculosis has been called consumption. It was not clear how TB transmitted until Robert Koch's brilliant discovery of the tubercle bacillus in 1882 and established TB as an infectious disease (Daniel, 2000). In the 19th century, patients were isolated in Sonatonia and given treatments such as injecting air into the chest cavity. Attempts were made to decrease lungs size by surgery called thoracoplasty. During the first half of the 20th century, no effective treatment was available. It was not until 1946 with the development of Streptomycin that effective treatment and cure became possible and isoniazid (Laniazid, Nydrazid originally) and antidepressant medication became available in 1952 (Daniel, 2000). However, hopes of completely eliminating the disease were dashed following the rise of drug-resistant strains in the 1980's (Herzog, 1998). Tuberculosis models are either deterministic or stochastic.

The World Health Organization has achieved some successes with improved treatment regimens, and a small decrease in case numbers (Lawn &Zumla, 2011). Tuberculosis is treated by killing the bacterial using antibiotics. The treatment usually last at least six months in duration and sometimes longer, up to twenty-four months. It involves different antibiotics to increase effectiveness while preventing the bacteria from becoming resistant to the medicine (Bhunu et al, 2008). Not everyone infected with TB becomes sick. As a result, two TB-related conditions exist: latent TB infection and active TB disease. Both latent TB infection and active TB disease can be treated (CDC, 2018). People with latent TB are usually treated using a single antibiotic to prevent them from progressing to active TB disease later in life. According to Bhunu et al (2007), the medications used to treat latent TB infection include: Isoniazid (INH),Rifampicin (RIF),Rifapentine (RPT).Certain groups of people (such as people with weakened immune systems) are at very high risk of developing TB disease once infected with the bacteria. Every effort should be made to begin appropriate treatment and to ensure completion of the entire course of treatment for latent TB infections (CDC, 2007). Treatment of latent TB infections is essential for controlling and eliminating TB. The most common medications used for treatment of active tuberculosis as approved by the WHO's Stop TB Department include: Isoniazid (INH), Rifampin (RIF), Ethambutol (EMB),Pyrazinamidel (PZA) (Bhunu et al, 2008). Regimens for treatment of TB disease have an initial phase of 2 months followed by a choice of several options for the continuation phase of 4 to 7 months (total of 6 to 9 months) for treatment (Emmanuel, 2013).

The success of TB treatment is largely dependent on the compliance of the patient. It is very important that people who have active TB disease finish the medication, taking it exactly as prescribed. If they stop taking the drug too soon, they can become sick again, if they do not take the drugs correctly, the TB bacteria that are still alive may become resistant to those drugs (CDC, 2007). Treatment completion is determined by the number of doses ingested over a given period of time. Although basic TB regimens are broadly applicable, there are modifications that should be made under special circumstances (such as people with HIV infection, drug resistance, pregnant women or treatment of children) (Emmanuel, 2013). Not only was TB one of the first disease for which the causative agent MTB was identified, but also one of the first for which

a vaccine was developed. The first breakthrough in TB vaccine development was the use of *mycobacterium bovis*, bacillus Calmette-Guerin commonly known as BCG (Doherty & Andersen, 2005). BCG is a neonatal pre-exposure vaccine that is widely used around the globe with about 3 billion people, that is approximately half of the world's population having received it, mostly in areas where TB is endemic (Garba et al, 2008). As of 2004, the vaccine was given to about 100 million children per year globally (WHO, 2004).

Pre-exposure vaccines, also known as pre-infection vaccines, are given before infection with the pathogen, usually at birth as neonatal vaccines. Pre-exposure vaccines speed up the development of immune response, therefore preventing further infections from becoming symptomatic (Doherty & Andersen, 2005). Post exposure vaccines are those give to individuals after evidence of infection. Although these vaccines cannot prevent the initial acute infection, their purpose is to strengthen the immune surveillance to prevent reactivation of latent infection (Doherty & Andersen, 2005). A multiphase vaccine effective both as a pre-exposure and post-exposure vaccine would be a great achievement in fighting MTB, but so far, there are none available for use. In theory, multiphase vaccine will not only inhibit the infection from becoming symptomatic, but will also prevent later activation. Despite some successes associated with using vaccines such as BCG (though immunity due to vaccination is not lifelong), and some TB treatment therapies, the overall global incidence of TB is rising as a result of the emergence of drug resistant TB strains and the deadly combination of HIV (Human Immune-deficiency Virus) and TB epidemics (Porco et al, 2011).

Once infected, an individual may recover without treatment, may be cured with antibiotics or may die of the disease (actively infected only). Recovered individuals may relapse to the disease or may be re-infected. According to Sanga, (2008), the highest risk group to contract TB when exposed to it are the children under five years old, persons who are immune compromised (i.e. have weakened immunity), especially those who are HIV-positive, persons who have diabetes or kidney failure. Tuberculosis usually attacks the lungs but can also attack other parts of the body like kidney, spine, brain, bones and joints (Kalu et al, 2012). The chronic symptoms of TB of the lungs are a chronic cough, which may result in blood-tinged sputum, fever, night sweats, loss of appetite and weight loss. Infection of other organs causes a wide range of symptoms. Pneumonia, lung collapse and enlarge lymphnode may also occur (WHO, 2007). From the foregoing, there is no doubt that Mycobacterium Tuberculosis is an infectious disease. The prevalence level of TB in African society in general and Nigeria in particular cannot be overestimated. Although vaccines are available for immunization against the disease and antibiotics for the treatment of infected persons, its wide spread still persist. The models operate by defining states for individuals within a population, essentially assigning individuals to subpopulation groups based on the characteristics such as 'infected with' or 'immune to', tuberculosis (Trottier & Philippe, 2001). Stochastic means being or having random variable. A stochastic model is a tool for estimating probability distribution of potential outcomes by allowing for random variations in one or more inputs over time. Stochastic models depend on the chance variation risk of exposure, disease and other illness dynamics.

They are used when these fluctuations are important, as in small populations (Trottier&Philppe, 2001). In deterministic models, individuals in the population are assigned to different subgroups or classes, each representing a specific stage of the epidemic. Letters such as M, S, E, I and R are often used to represent different stages (Emmanuel, 2013). Tuberculosis prevention and control efforts primarily rely on the vaccination of infants and the appropriate detection and treatment latent and active cases. It is against this backdrop that this paper ''Mathematical Model for the Vaccination and Treatment Strategy to Eradicate Tuberculosis with Migration and Permanent Immunity Effect" becomes necessary. Despite many decades of study, the wide spread availability of vaccines, an arsenal of anti-microbial drugs for treatment and more recently, a highly visible World Health Organization (WHO) effort to promote a unified global control strategy, Tuberculosis remains a leading cause of infectious mortality in the world. The World Health Organization declared TB a global emergency in 1993. Cohent et al (2004) as cited in Kalu(2012) notes that about one-third of the world's population is infected with MTB. Nigeria is not exempted from this epidemic.

According to World Health Organization (WHO, 2010), the number of people falling ill with TB

each year is declining. However, this downward trend is threatened by the number of TB case in the immigrant especially in countries that have substantial level of immigration from areas with high prevalence of the disease (Jia et al, 2008). As an airborne disease it is impossible for any country to isolate itself from TB because of migration. In the long term, the best defense against TB is to bring the disease under control worldwide. Since the 1960's, simple mathematical models have been used to understand tuberculosis transmission dynamics and to predict the effect of different intervention strategies (Waaler et al, 1962). Recently, there has been renewed interest in using mathematical models to study tuberculosis epidemics (Blower et al, 1996). KAlu and Inyama (2012) in their work "Mathematical Model of the Role of Vaccination and Treatment on the Transmission Dynamics of Tuberculosis" studied the role of vaccination of newborn babies against tuberculosis and treatment of latently and actively infected individuals in controlling the spread of tuberculosis based on the standard SEIR model. Among other assumptions, the researchers assumed that: A proportion of the population of newborns is vaccinated against TB infections through vaccination. There are no immigrants and emigrants. The only way of entry into the population is through new births and the only way of exit is through deaths from natural causes or deaths due to TB-related causes. In reality, migration cannot be totally ruled out in any population (such as a country like Nigeria).

People quest for businesses, job opportunities and better living standard have made migration inevitable. Thus the assumption in Kalu and Inyama's model is unrealistic. Bhunu et al (2008), report that the success of tuberculosis treatment is largely dependent on the compliance of the patients. In most cases, proper treatment with appropriate antibiotics will cure tuberculosis (Bhunu et al, 2008). This report suggests that not all treatments are effective. The exclusion of the efficacy of vaccination and treatment is one shortcoming in many TB epidemic models. This paper is a modification of Kalu and Inyama's work by incorporating migration effects. Also the study assumes 100% vaccination of newborns and includes the efficacy of vaccination and treatment as a parameter against its negligence in many TB models.

2.0 METHODS

Mathematical models have played a key role in the formulation of TB control strategies and the establishment of interim goals for intervention programs. Many types of epidemic models exist. They include: the stochastic models, the deterministic (compartmental) models such as the SIR, SIS, SIRS, SEIS, SEIR, MSEIR, models, (Where S=Susceptible class; I=Infective class; M=passively immune class; E=Exposed class; and R=Recovered class) .

Our model is a deterministic MSEIR type model where the population is partitioned into components or classes based on the epidemiological state of the individuals, and it is assumed that the epidemic process is deterministic.

2.1 The Modified Model

We modified the work of Kalu and Inyama (2012) by incorporating migration effect and assume 100% vaccination for the new-births. Modified model based on the following assumptions: That the individuals that make up the population can be grouped into different compartments according to their epidemiological state, the population size in a compartment changes with time, all new-births are immunized against TB infection and enter the vaccinated class, M, and there is migration in the population. That is, there are immigrants and emigrants. and there is no vertical transmission of TB. That is, no transmission from mother to new-born, hence all new-births are previously uninfected. That the immunity conferred on individuals by vaccination wanes after some time at a given rate, the population mixes homogeneously. That infection does not confer permanent immunity on the individuals. That a susceptible individual once infected develops latent infection. Those latently infected individuals are treated and are recovered or the infection develops to active TB. That every individual can die a natural death. That all immigrants are either vaccinated and are immune or they are unvaccinated and are susceptible. Latently infected and infectious individuals are restricted from entering the population.

2.2 Model Variables And Parameters

The following variables and parameters shall be used in this model:

M(t): the number of individuals who are immunized/vaccinated against TB at time t.

S(t): the number of susceptible

individuals. That is, the individuals who can catch the disease because they have no immunity to the infectious agent so might become infected if exposed.

L(t): the number of latently infected individuals at time t.

I(t): the number of infectious individuals at time t.

R(t): the number of individuals who have been treated and have recovered from the infection at time t.

β: the rate of new-births in the population

f: the efficacy of the vaccine in preventing initial infection.

ϒ: Average immigration rate into the population

K: the rate at which susceptible individuals develop latent infection

q: the rate of expiration of vaccine (rate at which immunity wanes)

ψ: the rate at which active TB is treated.

ε: recovery rate of latent infection due to treatment

α: average emigration rate

e: efficacy of treatment in curing infected persons

m: the rate at which latently infected become actively infected

π: the rate at which recovered individuals become susceptible to TB again

μ: natural mortality rate

μ_τ: TB induced deaths

η: Proportion of immigrants vaccinated and are immune

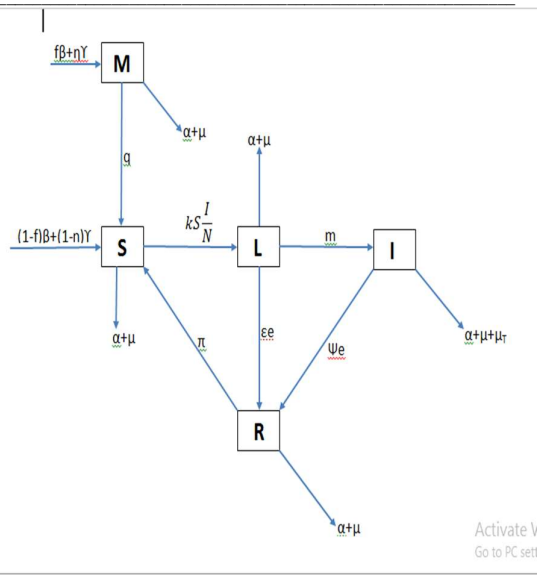

Figure 1: Schematic presentation of the modified model

2.3 The Modified Model Equation

This section presents the modified model equations by a system of differential equations thus:

$$\frac{dM}{dt} = f\beta + n\Upsilon - \alpha M - \mu M - qM$$
$$= f\beta + n\Upsilon - (\alpha + \mu + q)M \quad (1)$$

$$\frac{dS}{dt} = (1-f)\beta + (1-n)\Upsilon - \mu S - \alpha S + qM + \pi R - kS\frac{I}{N}$$
$$= (1-f)\beta + (1-n)\Upsilon - (\mu + \alpha)S + qM + \pi R - kS\frac{I}{N} \quad (2)$$

$$\frac{dL}{dt} = kS\frac{I}{N} - \alpha L - \mu L - mL - \varepsilon L$$
$$= kS\frac{I}{N} - (\alpha + \mu + m + \varepsilon)L \quad (3)$$

$$\frac{dI}{dt} = mL - \alpha I - \psi eI - (\mu + \mu_\tau)I$$
$$= mL - (\alpha + \psi e + \mu + \mu_\tau)I \quad (4)$$

$$\frac{dR}{dt} = \psi eI + \varepsilon L - \alpha R - \mu R - \pi R$$

$$= \psi eI + \varepsilon L - (\alpha + \mu + \pi)R \quad (5)$$

$$N(t) = M(t) + S(t) + L(t) + I(t) + R(t) \quad (6)$$

$$M(0) \geq 0, S(0) \geq 0, L(0) \geq 0, I(0) \geq 0, R(0) \geq 0$$

The system of equations (1) to (5) are the deterministic model equations which will be used to determine the existence and uniqueness of solution, the Disease-Free Equilibrium (DFE) for the disease as well as calculate the effective reproductive number R_e which determines whether the disease can be eliminated or not.

2.4 Methods Of Solution And Analysis

2.4.1exixtence And Uniqueness Of Solution

To prove the existence and uniqueness of solution of the system of equations in section 2.3, we shall use the method described by Egbetade& Ibrahim (2012).

Consider the system of equations below

$$\left.\begin{aligned} x_1' &= f_1(t, x_1, x_2, \dots, x_n),\ x_1(t_0) = x_{10} \\ x_2' &= f_2(t, x_1, x_2, \dots, x_n),\ x_2(t_0) = x_{20} \\ &\vdots \\ x_n' &= f_n(t, x_1, x_2, \dots, x_n),\ x_n(t_0) = x_{n0} \end{aligned}\right\} \quad (7)$$

We may write (7) in compact form as

$$x' = f(t, x), x(t_0) = x_0 \quad (8)$$

Theorem1:

Let D denotes the region $|t - t_0| \leq a, ||x - x_0|| \leq b, x = (x_1, x_2 \dots, x_n)$ (9)

Suppose that $f(t, x)$ satisfies the Lipschitz condition

$\|f(t, x_1) - f(t, x_2)\| \leq k\|x_1 - x_2\|$, where the pairs$(t, x_1), (t, x_2) \in D$, k is a positive constant. Then, there is a constant $\delta > 0$ such that there exist a unique continuous vector solution $\underline{x}(t)$ of the system (7) in the interval$|t - t_0| \leq \delta$.

It is important to note that Lipschitz condition is satisfied by the requirement that $\frac{\partial f_i}{\partial x_j}$, $i, j = 1,2, \dots, n$ are continuous and bounded in D.

2.4.2 Equilibrium And Stability Analysis For The Existing Model

We shall use the formulation of Disease Free Equilibrium (DFE) and stability analysis presented in Ayodeji (2016) to find the DFE for the formulated model and carryout stability analysis.

Consider the equation (7).

Definition 1: An equilibrium solution or fixed point, or steady-state solution of the system (7) is a constant solution x of the equation (Ayodeji, 2016).

At the equilibrium point, the derivatives in the equations (1) to (5) are equal to zero. That is, $M' = S' = L' = I' = R' = 0$. In the absence of any infections (DFE), $L = I = 0$.

To determine the stability of the model, we shall evaluate the DFE of the system.

Theorem 2: Suppose that x^* is an equilibrium solution of (7), i.e. $f(x^*) = 0$, then $x^*$is locally asymptotically stable (LAS) if all the eigenvalues of J_{x^*} have negative real parts. If at least one eigenvalue has positive real part then x^* is unstable. The eigenvalues are the roots of the characteristic equation of the Jacobian matrix, J, where $J = \left[\frac{\partial f_i}{\partial x_j}\right], i, j = 1, 2, \dots, n.$

3.0 RESULTS

The modified model of section 2.3 was consider in details by carrying out the existence and stability analysis of the disease-free equilibrium (DFE) state and determines the basic reproductive number R_0 for the formulated model.

3.1 Existence and Uniqueness of Solution

We shall prove the existence and uniqueness of solution or otherwise of model equations (1) to (5) using the ***Theorem 1*** presented in ***Section 2.4.1***of this work.

Proof:

Let

$$f_1 = f\beta + n\Upsilon - (\alpha + \mu + q)M$$

$$f_2 = (1 - f)\beta + (1 - n)\Upsilon - (\mu + \alpha)S + qM + \pi R - kS\frac{I}{N}$$

$$f_3 = kS\frac{I}{N} - (\alpha + \mu + m + \varepsilon e)L$$

$$f_4 = mL - (\alpha + \psi e + \mu + \mu_\tau)I$$

$$f_5 = \psi eI + \varepsilon eL - (\alpha + \mu + \pi)R.$$

It suffices to show that $\frac{\partial f_i}{\partial x_j}$, $i, j = 1,2, \dots, n$ are continuous and bounded in the region D defined by equation (9).

Consider the partial derivatives below:

$$\left|\frac{\partial f_1}{\partial M}\right| = |-(\alpha + \mu + q)| < \infty$$

$$\left|\frac{\partial f_1}{\partial S}\right| = \left|\frac{\partial f_1}{\partial L}\right| = \left|\frac{\partial f_1}{\partial I}\right| = \left|\frac{\partial f_1}{\partial R}\right| = 0 < \infty$$

$$\left|\frac{\partial f_2}{\partial M}\right| = |q| < \infty$$

$$\left|\frac{\partial f_2}{\partial S}\right| = \left|-(\alpha + \mu) - k\frac{I}{N}\right| < \infty$$

$$\left|\frac{\partial f_2}{\partial L}\right| = 0 < \infty$$
$$\left|\frac{\partial f_2}{\partial I}\right| = |\pi| < \infty$$
$$\left|\frac{\partial f_2}{\partial R}\right| = |-kS| < \infty$$
$$\left|\frac{\partial f_3}{\partial M}\right| = \left|\frac{\partial f_3}{\partial R}\right| = 0 < \infty$$
$$\left|\frac{\partial f_3}{\partial S}\right| = \left|k\frac{I}{N}\right| < \infty$$
$$\left|\frac{\partial f_3}{\partial I}\right| = |kS| < \infty$$
$$\left|\frac{\partial f_3}{\partial L}\right| = |-(\alpha+\mu+m+\varepsilon e)| < \infty$$
$$\left|\frac{\partial f_4}{\partial M}\right| = \left|\frac{\partial f_4}{\partial R}\right| = \left|\frac{\partial f_4}{\partial S}\right| = 0 < \infty$$
$$\left|\frac{\partial f_4}{\partial L}\right| = |m| < \infty$$
$$\left|\frac{\partial f_4}{\partial I}\right| = |-(\alpha+\psi e+\mu+\mu_T)| < \infty$$
$$\left|\frac{\partial f_5}{\partial M}\right| = \left|\frac{\partial f_5}{\partial S}\right| = 0 < \infty$$
$$\left|\frac{\partial f_5}{\partial L}\right| = |\varepsilon e| < \infty$$
$$\left|\frac{\partial f_5}{\partial I}\right| = |\psi e| < \infty$$
$$\left|\frac{\partial f_5}{\partial R}\right| = |-(\alpha+\mu+\pi)| < \infty$$

Clearly, all these partial derivatives are continuous and bounded.

Hence by ***Theorem 1*** there exists a unique solution of the model equation (1)-(5) in the region D

3.2 Existence And Stability Of Disease-Free Equilibrium State Of The Modified Model

The researcher investigates for the existence and stability of the DFE state of the modified model.

3.2.1 Equilibrium Solution

Let $E(M,S,L,I,R)$ be the equilibrium point of the system described by the equations (1) to (5). At the equilibrium state we have:

$$\frac{dM}{dt} = \frac{dS}{dt} = \frac{dL}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$$

That is,

$$f\beta + n\Upsilon - (\alpha+\mu+q)M = 0 \quad (10)$$
$$(1-f)\beta + (1-n)\Upsilon - (\mu+\alpha)S + qM + \pi R - kS\frac{I}{N} = 0 \quad (11)$$
$$kS\frac{I}{N} - (\alpha+\mu+m+\varepsilon)L = 0 \quad (12)$$
$$mL - (\alpha+\psi e+\mu+\mu_\tau)I = 0 \quad (13)$$
$$\psi eI + \varepsilon L - (\alpha+\mu+\pi)R = 0 \quad (14)$$

In order to obtain the disease-free equilibrium state, we solve the system of equations (10) to (14) simultaneously.

3.2.2 Existence Of Trivial Equilibrium State

Let $E_0(M_0,S_0,L_0,I_0,R_0)$ be the trivial equilibrium state for the model. That is, when $M = S = L = I = R = 0$.

so that$E_0(M_0,S_0,L_0,I_0,R_0) = (0,0,0,0,0)$. But no such equilibrium exists for the model since the population cannot go extinct so long as new babies are born into the population and there is migration into the population. In other words,

so long as the recruitment terms $f\beta$ and $(1-f)\beta$ are not both zero and also $\eta\Upsilon$ and $(1-\eta)\Upsilon$ cannot be both zero, the population will never go extinct; and so $E_0(M_0,S_0,L_0,I_0,R_0) \neq (0,0,0,0,0)$.

3.2.3 The Diseease-Free Equilibrium State

The disease-free equilibrium state is the state of total eradication of the disease.

Let $E^0(M^0,S^0,L^0,I^0,R^0)$ be the DFE state for the model. For disease-free equilibrium state, the disease states of the model must be zero. That is, the infectious class, I and the latently infected class, L must be zero. Mathematically, for the DFE state $L^0 = I^0 = 0$.

Now, substituting $L^0 = I^0 = 0$ into the system of equations (10) to (14) we obtain the following:

From (10),

$$f\beta + n\Upsilon - (\alpha+\mu+q)M = 0$$
$$\Rightarrow \quad (\alpha+\mu+q)M = f\beta + n\Upsilon$$
$$\Rightarrow \quad M = \frac{f\beta+n\Upsilon}{\alpha+\mu+q},$$

That is,

$$M^0 = \frac{f\beta+n\Upsilon}{\alpha+\mu+q} \quad (15a)$$

From (11),

$$(1-f)\beta + (1-n)\Upsilon - (\mu+\alpha)S + qM + \pi R - kS\frac{I}{N} = 0$$

For $I = 0$, we have:

$$(1-f)\beta + (1-n)\Upsilon - (\mu+\alpha)S + qM + \pi R = 0 \quad (15b)$$

But $M^0 = \frac{f\beta+n\Upsilon}{\alpha+\mu+q}$, substituting into (*) yields:

$$(1-f)\beta + (1-n)\Upsilon - (\mu+\alpha)S + \frac{q(f\beta+\Upsilon)}{\alpha+\mu+q} + \pi R = 0 \quad (16a)$$

From (12),

$$kS\frac{I}{N} - (\alpha+\mu+m+\varepsilon e)L = 0$$

For $I = L = 0$,the equation vanishes.

Similarly, equation (13) vanishes on substituting$I = L = 0$,

since it depends entirely on I and L only.
From equation (14),

$$\psi eI + \varepsilon eL - (\alpha + \mu + \pi)R = 0$$

$\Rightarrow$ $0 + 0 - (\alpha + \mu + \pi)R = 0$ (since I=L=0)

$\Rightarrow$ $(\alpha + \mu + \pi)R = 0$ (16b)

Either $\alpha + \mu + \pi = 0$ Or $R = 0$

But $\alpha + \mu + \pi$ cannot be zero since α, μ, π are positive constants. i.e. $(\alpha + \mu + \pi) \neq 0$

$\Rightarrow$ For (16b) to be true, then necessarily, $R = 0$.

Therefore, $R^0 = 0$ (17)

If $R = 0$, equation (16a) becomes

$$(1-f)\beta + (1-n)\Upsilon - (\mu + \alpha)S + \frac{q(f\beta + n\Upsilon)}{\alpha + \mu + q} + \pi(0) = 0$$

$$\Rightarrow \quad S = \frac{\{(1-f)\beta + (1-n)\Upsilon\}(\alpha+\mu+q) + q(f\beta + n\Upsilon)}{(\alpha+\mu)(\alpha+\mu+q)}$$

$$\Rightarrow \quad S^0 = \frac{q(\beta+\Upsilon) + (\alpha+\mu)\{(1-f)\beta + (1-\eta)\Upsilon\}}{(\alpha+\mu)(\alpha+\mu+q)} \quad (18)$$

Therefore, the DFE state for the model is

$E_0(M_0, S_0, L_0, I_0, R_0)$

$= (\frac{f\beta + n\Upsilon}{\alpha+\mu+q}, \frac{q(\beta+\Upsilon) + (\alpha+\mu)\{(1-f)\beta + (1-\eta)\Upsilon\}}{(\alpha+\mu)(\alpha+\mu+q)}, 0, 0, 0)$

We now linearize the system of equations, to get the Jacobian matrix, J as;

$$J = \begin{bmatrix} -(\alpha+\mu+q) & 0 & 0 & 0 & 0 \\ q & -(\alpha+\mu) - k\frac{I}{N} & 0 & -kS & \pi \\ 0 & kI & -(\alpha+\mu+m+\varepsilon) & kS & 0 \\ 0 & 0 & m & -(\alpha+\psi e+\mu+\mu_\tau) & 0 \\ 0 & 0 & \varepsilon & \psi e & -(\alpha+\mu+\pi)R \end{bmatrix}$$

At the disease-free equilibrium

$E^0(M^0, S^0, L^0, I^0, R^0) = (\frac{f\beta+n\Upsilon}{\alpha+\mu+q}, \frac{q(\beta+\Upsilon)+(\alpha+\mu)\{(1-f)\beta+(1-\eta)\Upsilon\}}{(\alpha+\mu)(\alpha+\mu+q)}, 0, 0, 0)$,

The Jacobian Matrix above becomes

$$J_{E^0} = \begin{bmatrix} -(\alpha+\mu+q) & 0 & 0 & 0 & 0 \\ q & -(\alpha+\mu) & 0 & \frac{-k\{q(\beta+\Upsilon)+(\alpha+\mu)[(1-f)\beta+(1-\eta)\Upsilon]\}}{(\alpha+\mu)(\alpha+\mu+q)} & \pi \\ 0 & 0 & -(\alpha+\mu+m+\varepsilon) & \frac{k\{q(\beta+\Upsilon)+(\alpha+\mu)[(1-f)\beta+(1-\eta)\Upsilon]\}}{(\alpha+\mu)(\alpha+\mu+q)} & 0 \\ 0 & 0 & m & -(\alpha+\psi e+\mu+\mu_T) & 0 \\ 0 & 0 & \varepsilon & \psi e & 0 \end{bmatrix}$$

3.2.4 Stability Analysis Of the Disease-Free Equilibrium state

To determine the stability or otherwise of the disease-free equilibrium state E^0, we examine the behavior of the model equations near this equilibrium solution. Here we examine the condition(s) that must be met for the disease-free equilibrium state to be stable. In other words, we determine the conditions that must be met if the disease is to be totally eradicated from the population.

Recall that the system of equations in this model at equilibrium state is:

$$f\beta + n\Upsilon - (\alpha+\mu+q)V = 0$$

$$(1-f)\beta + (1-n)\Upsilon - (\mu+\alpha)S + qV + \pi R - kS\frac{I}{N} = 0$$

$$kS\frac{I}{N} - (\alpha+\mu+m+\varepsilon)L = 0$$

$$mL - (\alpha+\psi e+\mu+\mu_\tau)I = 0$$

$$\psi eI + \varepsilon L - (\alpha+\mu+\pi)R = 0$$

Where the quantity $\frac{q(\beta+\Upsilon)+(\alpha+\mu)[(1-f)\beta+(1-\eta)\Upsilon]}{(\alpha+\mu)(\alpha+\mu+q)} = s^o$

The Eigen values are calculated from the characteristics equation $|J_{E^o} - \lambda I| = 0$ where I is a5×5 identity matrix. That is,

$$|J_{E^o} - \lambda I| = \begin{vmatrix} -(\alpha+\mu+q)-\lambda & 0 & 0 & 0 & 0 \\ q & -(\alpha+\mu)-\lambda & 0 & -ks^o & \pi \\ 0 & 0 & -(\alpha+\mu+m+\varepsilon)-\lambda & ks^o & 0 \\ 0 & 0 & m & -(\alpha+\psi e+\mu+\mu_\tau)-\lambda & 0 \\ 0 & 0 & \varepsilon & \psi e & -(\alpha+\mu+\pi)-\lambda \end{vmatrix} = 0$$

For simplicity of appearance and computational advantage, we let $c = \alpha + \mu$. Then we obtain the following

$$|J_{E^o} - \lambda I| = (-(c+q)-\lambda)\begin{vmatrix} -c-\lambda & 0 & -ks^o & \pi \\ 0 & -(c+m+\varepsilon)-\lambda & ks^o & 0 \\ 0 & m & -(c+\psi e+\mu_\tau)-\lambda & 0 \\ 0 & \varepsilon & \psi e & -(c+\pi)-\lambda \end{vmatrix} = 0$$

$$(-(c+q)-\lambda)(-c-\lambda)\begin{vmatrix} -(c+m+\varepsilon)-\lambda & ks^o & 0 \\ m & -(c+\psi e+\mu_\tau)-\lambda & 0 \\ \varepsilon & \psi e & -(c+\pi)-\lambda \end{vmatrix} = 0$$

$$(-(c+q)-\lambda)(-c-\lambda)(-(c+\pi)-\lambda)\begin{vmatrix} -(c+m+\varepsilon)-\lambda & ks^o \\ m & -(c+\psi e+\mu_\tau)-\lambda \end{vmatrix} = 0 \quad (19)$$

From equation (19),

Either $(-(c+q)-\lambda)(-c-\lambda)(-(c+\pi)-\lambda) = 0$ (20)

or

$$\begin{vmatrix} -(c+m+\varepsilon)-\lambda & ks^o \\ m & -(c+\psi e+\mu_\tau)-\lambda \end{vmatrix} = 0 \quad (21)$$

From equation (20)

$$\left.\begin{aligned} \lambda_1 &= -(c+q) \\ \lambda_2 &= -c \\ \lambda_3 &= -(c+\pi) \end{aligned}\right\} (22)$$

From equation (14), we see that the first three (3) Eigen values $\lambda_1, \lambda_2 \; and \; \lambda_3$ are all negative.

Using ***theorem 2,*** we see that the DFE of this model will be asymptotically stable iff the remaining Eigen values, $\lambda_4 \; and \; \lambda_5$ are also negative.

Now, we consider equation (13). For local asymptotic stability (LAS) of the DFE, we require the remaining two eigenvalues $\lambda_4 \; and \; \lambda_5$ to be negative.

Theorem 3

Let A be a n$n \times n$ matrix. Then:

i. The matrix A has n eigenvalues (including each according to its multiplicity).
ii. The sum of the n eigenvalues of A is the same as the trace of A.
iii. The product of the n eigenvalues of A is equal to the determinant of A.

Using t***heorem 3***, we shall prove that $\lambda_4 \; and \; \lambda_5$ are both negative or otherwise.

Let A$= \begin{pmatrix} -(c+m+\varepsilon) & ks^o \\ m & -(c+\psi e+\mu_\tau) \end{pmatrix}$

$$Trace(A) = -(c+m+\varepsilon) - (c+\psi e+\mu_\tau) \quad (23a)$$

$$Det(A) = (c+m+\varepsilon)(c+\psi e+\mu_\tau) - kms^o$$

(23b)

If λ_4 *and* λ_5 are both negative, then we have $\lambda_4 + \lambda_5 < 0$, it implies that $Trace(A) < 0$

i.e.$-(c + m + \varepsilon) - (c + \psi e + \mu_\tau) < 0$

It is clear that$Trace(A) < 0$ since all the parameters are positive constant.

Also, $\lambda_4 . \lambda_5 > 0$, it implies that $Det(A) > 0$

i.e.$(c + m + \varepsilon)(c + \psi e + \mu_\tau) - kms^o > 0$

$$\Rightarrow \quad kms^o < (c + m + \varepsilon)(c + \psi e + \mu_\tau) \qquad (23c)$$

Dividing both sides of inequality (23c) by $(c + m + \varepsilon)(c + \psi e + \mu_\tau)$ yields:

$$\frac{kms^o}{(c+m+\varepsilon)(c+\psi e+\mu_\tau)} < 1 \text{ , where } c = \alpha + \mu \qquad (24)$$

The inequality (24) determines the threshold under which the disease can be eliminated or brought under control. It is the necessary and sufficient condition for the disease free equilibrium of the model to be stable.

3.3 The Effective Reproduction Number, R_e

We determine the basic reproduction number, R_e for the model equations (10) to (14). This will be calculated using the next generation matrix method as described by Hefferman et al (2005).

Consider the next generation matrix G, which is made up of two parts: F and V^{-1}, where

$$F = \left[\frac{\partial F_i(E^o)}{\partial x_j}\right]$$

And

$$V = \left[\frac{\partial V_i(E^o)}{\partial x_j}\right]$$

The F_i's are the new infections while the V_i's shows the transfer of infections from one compartment to another. Here E^o is the disease-free equilibrium state. The basic reproduction number is the dominant Eigen value of the matrix G.

In this model, there are two disease states i.e. the latent class, L and the infectious class, I.

Recall that

$$\frac{dL}{dt} = kS\frac{I}{N} - (\alpha + \mu + m + \varepsilon)L$$

$$\frac{dI}{dt} = mL - (\alpha + \psi e + \mu + \mu_\tau)I$$

The vector F_x, of the rates of new infections in compartments L and I is given by

$$F_x = \begin{bmatrix} kS\dfrac{I}{N} \\ 0 \end{bmatrix}$$

Also the remaining transfer terms in compartments L and I is given by

$$V_x = \begin{bmatrix} ((\alpha + \mu + m + \varepsilon)L \\ (\alpha + \psi e + \mu + \mu_\tau)I - mL \end{bmatrix}$$

Now we compute the matrix of partial derivatives of F_x at the disease-free equilibrium state $E^o = (V^o, S^o, 0,0,0,)$. Thus,

$$F_x(E^o) = \begin{pmatrix} 0 & kS^o \\ 0 & 0 \end{pmatrix} \text{Where} s^o = \frac{q(\beta+\Upsilon)+(\alpha+\mu)\{(1-f)\beta+(1-\eta)\Upsilon\}}{(\alpha+\mu)(\alpha+\mu+q)}$$

And the matrix of the partial derivatives of V_x at the disease-free equilibrium state $E^o = (V^o, S^o, 0,0,0,)$ is:

$$V_x(E^o) = \begin{pmatrix} \alpha + \mu + m + \varepsilon & 0 \\ -m & \alpha + \psi e + \mu + \mu_\tau \end{pmatrix}$$

R_0 is the dominant Eigen value of the next generation matrix G.

$G = F_x(E^o)V_x^{-1}$.

Using the software, Mapple, we have:

$$V_x^{-1} = \begin{pmatrix} \dfrac{1}{\alpha + \mu + m + \varepsilon} & 0 \\ \dfrac{m}{(\alpha + \mu + m + \varepsilon)(\alpha + \psi e + \mu + \mu_\tau)} & \dfrac{1}{\alpha + \psi e + \mu + \mu_\tau} \end{pmatrix}$$

So that

$$G = \begin{pmatrix} 0 & kS^o \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} \dfrac{1}{\alpha + \mu + m + \varepsilon} & 0 \\ \dfrac{m}{(\alpha + \mu + m + \varepsilon)(\alpha + \psi e + \mu + \mu_\tau)} & \dfrac{1}{\alpha + \psi e + \mu + \mu_\tau} \end{pmatrix}$$

$$G = \begin{pmatrix} \dfrac{kmS^o}{(\alpha + \mu + m + \varepsilon)(\alpha + \psi e + \mu + \mu_\tau)} & \dfrac{kS^o}{\alpha + \psi e + \mu + \mu_\tau} \\ 0 & 0 \end{pmatrix}$$

By definition, R_o is the dominant or the leading Eigen value of G. So,

$$R_e = \frac{kmS^o}{(\alpha + \mu + m + \varepsilon)(\alpha + \psi e + \mu + \mu_\tau)}$$

But $S^o = \frac{q(\beta+\Upsilon)+(\alpha+\mu)\{(1-f)\beta+(1-\eta)\Upsilon\}}{(\alpha+\mu)(\alpha+\mu+q)}$

Therefore,

$$R_e = \frac{kmq(\beta + \Upsilon) + km(\alpha + \mu)\{(1 - f)\beta + (1 - \eta)\Upsilon\}}{(\alpha + \mu + m + \varepsilon)(\alpha + \psi e + \mu + \mu_\tau)(\alpha + \mu)(\alpha + \mu + q)}$$

We now use the parameter values presented in

Table 1 to find the numerical value of R_e which determines whether the disease can be eliminated or not.

Let $R_e = \frac{Num}{Den}$

The table below gives the values of R_e under different conditions.

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Table 1: Computed Effective Reproductive Number, (R_e) and Basic Reproductive Number, (R_0) of the modified model.

Population	R_e: Treatment and vaccination	R_e: Treatment but No Vaccination	R_e: Vaccination but No Treatment	R_0: Without Vaccination and without Treatment
With Migrant	0.1741	0.1811	11.0201	12.0731

3. 4 Graphical Simulation

In this section, the numerical simulations for the model systems under different conditions are presented. This we shall achieve by using the parameter values given in **Table 1.**

Table 2: ***Model parameters and their interpretations***

S/N	PARAMETER	SYMBOL	VALUE(per year)	SOURCE
01	The rate of new births	β	0.0369	Nigerian Demographic Profile 2018.
02	The rate of Latent infection	k	0.2380	Egbetade & Ibrahim (2014)
03	The rate of Expiration of vaccine	q	0.3700	Egbetade & Ibrahim (2014)
04	Treatment rate for active TB	ψ	0.5500	Nadhirah BT A. H. (2013
05	Treatment rate of latent TB	ε	0.7000	Nadhirah BT A. H. (2013)
06	The rate at which latent becomes infectious	m	0.1300	Nadhirah BT A. H. (2013
07	The natural mortality rate	μ	0.0124	Nigerian Demographic Profile 2018.
08	TB induced death	μ_τ	0.0240	Nadhirah BT A. H. (2013)
09	Efficacy of vaccine	f	0.9500	Egbetade & Ibrahim (2014)
10	Efficacy of treatment	e	0.8000	Assumed
11	The rate at which recovered become Susceptible	π	0.0001	Nadhirah BT A. H. (2013
12	Average Immigration rate	γ	0.0049	Nigerian Demographic Profile 2017.
13	Average Emigration rate	α	0.0051	Nigerian Demographic Profile 2017.
14	Proportion of vaccinated immigrants	η	0.1400	Egbetade & Ibrahim (2014)

Initial conditions are taken as follows:
S(0)=11 000, L(0)=3 500, I(0)=500, R(0)=0 (Nadhirah,2013).

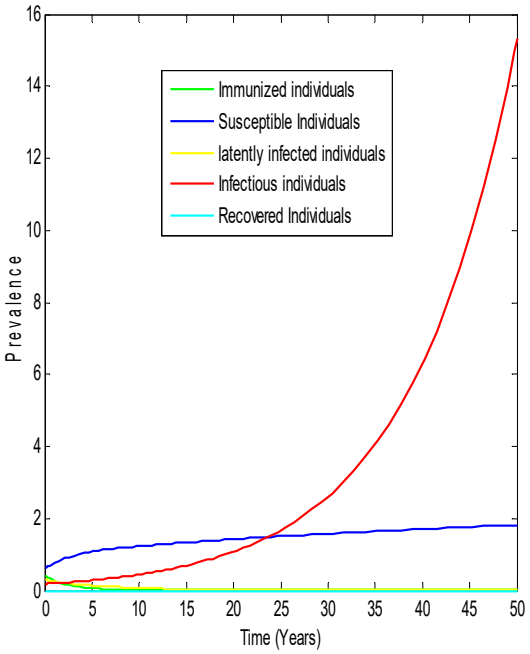

Figure 1: Graph showing the prevalence of each class in the absence of vaccination and treatment.

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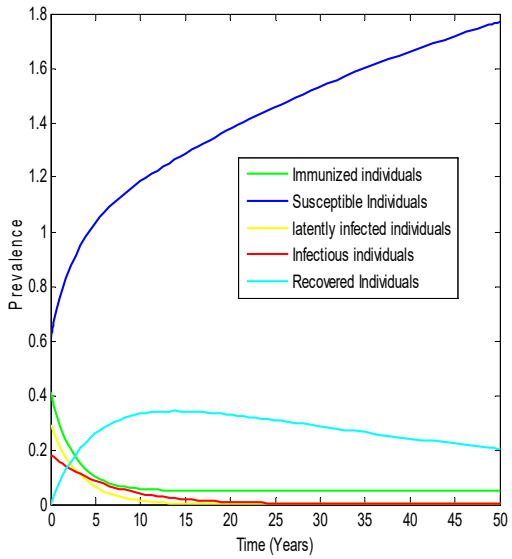

Figure 2: Prevalence level when treatment and vaccination set in at low rates

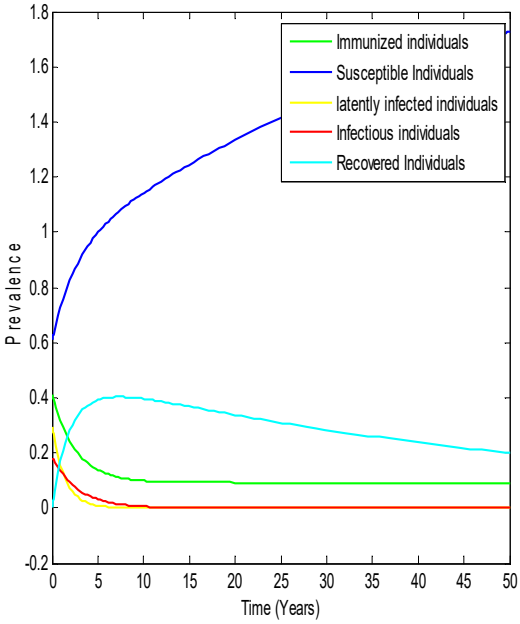

Figure 3: Prevalence level when there is treatment and vaccination at high rates

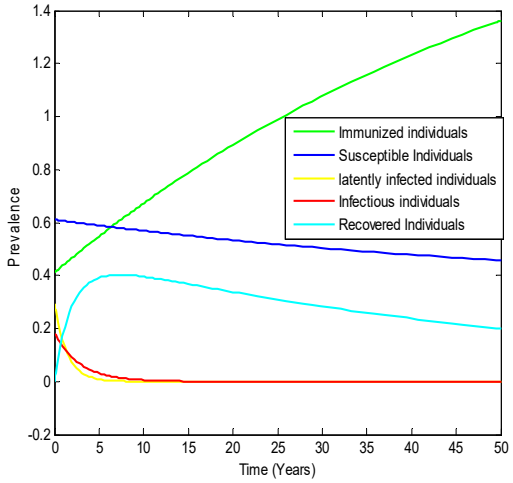

Figure 4: Graph showing the prevalence of each class when vaccination confers permanent immunity

4.0 DISCUSSION

In this paper a mathematical model for the vaccination and treatment strategy to eradicate tuberculosis with migration and permanent immunity effect. The population was partitioned into five compartments namely the Immunized, the Susceptible, the Latent infected, the Infectious and the Recovered compartments or classes. It was assumed that the entrants into the population are new-births and immigration. However the immigration of latent and infectious individuals was assumed prohibited.

Under this section, we shall discuss our results based on the existence and uniqueness of solution, the stability of the DFE, the effective reproduction number and the numerical/graphical simulations.

4.1 Numerical Simulations

Graphical simulations were carried out using the ODE solver, $ode45$ inbuilt on MatLab. Figures1 to 4 show the trend on the Immunized, Susceptible, Latent, Infectious and Recovered under different conditions over 50 years' time period.

Fig.1 reviews what will happen if we remove vaccination and treatment of both latently infected and infectious individuals. We see that the immune class and the recovered both remains zero throughout the time. Everyone becomes susceptible in the population. In less than five years most of the latent infections developed to infectious state. This increases the contact rates. Thus the disease becomes endemic as the proportion of the infectious class grows without bounds.

In Fig.2, once vaccination and treatment set in (though at a low rate), we notice the latent and the infectious classes reducing. After 10 years, they are both at a very low level but still traceable. At about the 20th year, the latent has gone completely, leaving only small traces of infectiousness. By the 25th year, all of the cases have disappeared completely. During the same period when latent and infectious classes are decreasing, the recovered class rises due to the treatment but later decreases as they enter the susceptible class again. The decrease in the immunized class is due to loss of immunity with time.

Massive treatment and high vaccination rates can curb the disease in no distant time. This is illustrated in Fig.3. We see that the latent infections die out between the 6th and the 7th year of massive treatment while the infectious persist until after the 10th year before it finally dies out. The Susceptible continue to rise due to waning immunity.

In a situation where vaccination confers permanent immunity and high treatment rate is still maintained, Fig.4 reveals that the proportion of susceptible individuals reduces considerably while the proportion of immune individuals rises steadily. In this case too, the disease dies out between the ten to fifteen years.

5.0 CONCLUSION

We focused on vaccination and treatment of both latent and infectious individuals. Our study of R_e, and the stability of the DFE show that tuberculosis can be eliminated through vaccination and treatment. The analysis also suggests that if vaccines can confer permanent immunity, then a more stable equilibrium can be achieved. In other words, the disease can die out quicker. Lastly, our numerical simulations in Fig.1 to 4 show that vaccination and treatment has significant impact in bringing the disease under control.

Conclusively, tuberculosis can be eradicated if massive vaccination and treatment of both latent and infectious individuals is given with high efficacy. This implies that every effort should be made to begin appropriate treatment and to ensure completion of the entire course of treatment.

6.0 RECOMMENDATION

Based on the findings of this research, we recommend the following:

i. That, the Government of countries should embark on massive vaccination programmes against tuberculosis as well as massive diagnosis and treatment of both latent and infectious individuals. Vaccination and treatment must be properly carried out and monitored to ensure high efficacy.
ii. That, policies prohibiting latent and infectious immigrants be enacted and strictly enforced.
iii. That, new vaccines should be designed and targeted at conferring permanent immunity or bringing mycobacterium tuberculosis to sterility.
iv. That, further research to build a model that takes in to account interventions that take involves quarantine of infectious individuals be carried out.

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Application of Thermomechanics Procedure to Linear Theory of Elasticity with Isotropic Solid Material as Case Study.

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Abstract
In this paper, thermomechanical approach is used to develop a constitutive theory for linear thermoelasticity. This approach is based on the first Piola-Kirchhoff stress tensor, the referential heat flux, the free energy and entropy as material responses for which Constitutive equations are given. The deformation gradient, the temperature and the referential temperature gradient are used as independent variables. Linear approximation of series expansion about a natural reference state are invoked to define the elasticity tensor, the stress temperature tensor, the thermal tensor, the latent heat tensor, the heat conductivity tensor, and the specific heat for isotopic solid material are obtained as our main results. Lastly, constitutive equations and linear thermoelasticity properties of isotropic solids were also obtained.

Keywords: Thermomechanics, Elasticity, Isotropic, Tensor, Temperature, Deformation.

1.0 INTRODUCTION

Thermoelastic properties are of primary interest in material science, solid mechanics, biomechanics and engineering. Examples are the coefficient of thermal expansion, the heat conductivity tensor and the specific heats. Although experimental papers dealing with these properties are formal, a more detailed view into the literature reveals that the concept of thermoelastic properties itself and its underlying theoretical framework linear thermoelasticity is often not fully understood by material scientists and engineers.

The reason is that the exact treatments of the subject are mostly of a mathematical complexity that makes them in accessible for a larger part of the audience in the material science and engineering community [9].

Three objectives were used as guidelines here first, this treatment should be as brief and concise as possible but should contain in a unified framework all purely thermoelastic properties arising in the linear theory.

Secondly, the number of variables introduced should be kept to a minimum but sufficient to allow for materials of arbitrary symmetry. Thirdly, a pragmatic compromise was sought between correctness of the statements and the attempt to explain the physical meaning of these statements in term familiar to others without requiring special knowledge in mathematics. Several researchers have worked in the field of thermoelasticity such as the Silhavy's [13], Truesdell and Noll's [16]. The present work is meant to serve as an easy to grasp and self contain introduction into the large work on thermoelasticity in general. It has been tried wherever possible to be in accord with the Silhavy's authoritative treatment [13].

2.0 MATHEMATICAL FORMULATION

The fundamental functions describing motion of material a body consisting of the so-called deformation function is . This vector function x, uniquely determines the actual position x of a material particle to the deformed configuration at time t, which has been at referential position X in the initial configuration at an arbitrary chosen reference time.

$x = x(X, t)$.

For fixed X(i.e. selecting a certain material particle x determine a trajectory for fixed). Imagine the whole body at a certain moment x, determines a configuration (placement).

The referential gradient (i.e. the derivative with respect to the referential position) of the deformed function is a second-order tensor called deformation gradient by F and defined as:

$F = Grad\ x = \frac{\partial x(X,t)}{\partial x}$

The function x contains complete information about the motion of a material body (translation, rotation and change of volume and shape). While the deformation gradient F is a local measure of

motion and contains information only on rotation and change of volume and shape.

Here the material is perceived as compressible materials that can be modelled by deformation gradient in equation (2.2) only (i.e. without invoking deformation gradient of higher order) are called simple materials [16].

In the theory of linear elasticity, it is useful to introduce the displacement vector functions.

$u(X,t) = \mathcal{X}(X,t) - X$

Using equation (3), the deformation gradient can be written as

$F = Grad\ \mathcal{X} = \frac{\partial x(X,t)}{\partial x} = \frac{\partial [X+u(X,t)]}{\partial x} = I + \nabla u$

Where ∇u is the displacement gradient and I, the second-order unit tensor. Applying only the change of volume and shape [8] of the deformation gradient F, made possible after eliminating the rotational part of F through Cauchy's polar decomposition theorem [16, 10].

$F = RU = VR$

Here R is an orthogonal rotation tensor, whereas U and V are symmetric positive definite matrices called the right and left Cauchy stretch tensors respectively.

The Strain measures frequently applied in theories of non-linear elasticity is the Green Lagrange strain tensor.

$G = \frac{1}{2}\lfloor \nabla u + \nabla u^T + \nabla u^T \nabla u \rfloor$ (2.6)

This is a symmetric second-order tensor. Since we are dealing with linear theory of elasticity we now set the quadratic term in equation (2.6) to be zero. i.e.

$\nabla u^T \nabla u = 0$ (2.7) and obtain

$\frac{1}{2}\lfloor \nabla u + \nabla u^T \rfloor$ (2.8)

Called the small strain tensor. In addition to the vector field in equation (2.1), continuum thermomechanics introduces an additional scalar field, the temperature (small thermal term) [7],

$\theta = \theta(x,t)$ (2.9)

And it's (referential) gradient

$\nabla\theta = \frac{\partial \theta}{\partial x}(X,t)$ (2.10)

Equation (2.1) now takes the following form

$x = x(X,t) + \theta(x,t) + \nabla\theta(x,t)$

To the constitutive equations. The material response of a simple thermoelastic material is assumed to be a function of the deformation gradient F, the temperature θ and its gradient $\nabla\theta$. Invoking Truesdell's equipresence rule [16, 15], there is no prior reason to exclude a certain independent variables from any member of the set of constitutive equation. They are as follows:

$P = p(F,\theta,\nabla\theta), h = h(F,\theta,\nabla\theta), \psi = \psi(F,\theta,\nabla\theta), \eta = \eta(F,\theta,\nabla\theta)$ (2.12)

Where P is the first Piola-Kirchhoff stress tensor, h is the referential heat flux vector, $\psi\ u - \theta_\eta$ is specific free energy (Helmholtz energy), u is the specific internal energy

η = specific entropy

The total force in the deformed configuration is defined as

$dP = \sigma_{ij} n_j ds$ (2.3)

Where dP is the total force, σ_{ij} is the Cauchy stress, n_j is the normal vector to the surface of the deformed configuration, and ds is the surface area in the deformed configuration.

The same force developed by the new stress tensor T, in the reference configuration with normal N and area ds we defined

$dP = T_{ij} N_j ds'$ (2.14)

Where we denote the new stress tensor as the first Piola-Kirchhoff stress tensor, with the mapping between reference and deformed configuration surface area, we can write the definition of the force in the deformed configuration as

$n_j ds = J N_K F_{Kj}^{-1} ds'$ (2.5)

But $\sigma_{ij} n_j ds = \sigma_{ij} J N_K F_{Kj}^{-1}$

From the relationship established in equations (2.15) and (2.16), hence subtracting equation (2.13) from equation (2.14) and setting the result equal to zero

$(T_{ij} - \sigma_{ik} J F_{jk}^{-1})\ \ N_j ds' = 0$

Since the term in the parenthesis must be zero, we obtain the relationship between the first Piola-Kirchhoff stress and Cauchy stress as

$T_{ij} = \sigma_{ik} J F_{jk}^{-1}$ (2.18)

Comparing the terms in equation (2.18)

$P = T_{ij} \quad T = \sigma_{ij} F_{jk}^{-1} = F^{-1}$

So that the first Piola-Kirchhoff stress tensor P is related to the Cauchy's stress tensor T [10] by

$P = JTF^{-1}$ (2.19)

Similarly, the referential heat flux vector h is related to the spatial (actual) heat flux vector the relation

$h = JF^{-1}q$ (2.20)

From equation (19), $F^{-1} = (F^{-1})^T$ is the transpose of the inverse deformation gradient and J, the Jacobian determinant. $Jdet \neq 0$ (Non-singular) and $J > 0$. (2.11)

3.0 BALANCE LAWS

In referential description, where all variables are referential fields i.e. functions of time t and the referential position X, the balance laws of rational thermomechanics valid for uniform bodies(Isotropic material inclusive) are given (in

differential form).

3.1 Balance of Mass

$\rho_0 = \rho|J|$ (3.1)

Where ρ_0 is the referential density and ρ the actual density of the material. Since $J \equiv \det F > 0$and by compressibility, $J \equiv \det F = 1$ Since $J > 0$. It follows that $\rho = 0$ if and only if $\rho_0 = 0$.

3.2 Balance of Linear Momentum

$\rho_0 \dot{v} = divP + \rho_0 b$

Where div is the referential divergence operator, $divP = tr(GadP)$ where tr denotes the trace, a superimposed dot ($\dot{V}$), a material time derivative v the velocity and b the specific body force.

3.3 Balance of Angular Momentum

$PF^T = FP^T$ (3.3)

For the second Piola-Kirchhoff stress tensors, the angular momentum balance reduces to the symmetry statement

$S = S^T$

Because of its symmetry, equation (3.4) is more adequate to develop constitutive theory of non-linear elasticity or thermoelasticity[6], for linear theory of elasticity the first Piola-Kirchhoff stress tensor is adopted here [15].

3.4 Balance of Energy

$\rho_0 \dot{U} = P.F - divh + \rho_0 Q$ (3.5)

Where U is the specific internal energy, Q is the specific heat supply and the inner product $P.F = tr(PF)$ is a scalar quantity describing the dissipation of mechanical energy.

In thermoelastic materials this latter part of dissipation is usually weak [13].

3.5 Entropy Inequality (Clausius Duhem Inequality)

$\rho_0 \eta \geq -div\frac{h}{\theta} + \rho_0 \frac{Q}{\theta}$

Where η is the specific entropy combining equation (3.5) and (3.6) we obtain the dissipation inequality

$\rho_0(\psi + \eta\theta) - P.F. - \frac{h}{\theta} \nabla\theta \leq 0$

According to Coleman and Noll's interpretation of the Second Law of thermodynamics [3, 4]. We now insert the material time derivative of the free energy $\psi(F, \theta, \nabla\theta)$.

$\dot{\psi} = \left(\frac{\partial\psi}{\partial F}\right).\dot{F} + \left(\frac{\partial\psi}{\partial\theta}\right).\dot{\theta} + \left(\frac{\partial\psi}{\partial\nabla\theta}\right).\frac{\cdot}{\nabla\theta}$ (3.8)

Where $\frac{\cdot}{\nabla\theta}$ is the material time derivative of the referential temperature gradient, into the dissipation inequality of equation (3.7) gives.

$\rho_0\left[\eta + \left(\frac{\partial\psi}{\partial\theta}\right)\right].\dot{\theta} - \left[P - \rho_0\left(\frac{\partial\psi}{\partial F}\right)\right].\dot{F} - \rho_0\left(\frac{\partial\psi}{\partial\nabla\theta}\right).\frac{\cdot}{\nabla\theta} + \frac{h}{\theta}\nabla\theta \leq 0$ (3.9)

Thus inequality must be valid for an arbitrary thermomechanical processes i.e. for arbitrary values of time rate $F, \theta, and\ \nabla\theta$. This corresponds to the Coleman Noll interpretation of the second law which states roughly that the constitutive equation must be such that the entropy inequality in equation (3.6) is satisfied in all thermomechanical process or precisely, in every smooth admissible processes (it is evident that $A = B = C = 0\ and\ D \geq 0$)must hold if the inequality $Ax + By + Cz + D \geq 0$ is to hold for arbitrary values of x, y, z [15].

There are three direct consequence of this application of the dissipation inequality.

1. Thermostatic relations. For a thermoelastic material, the entropy inequality satisfy Clausius Duhem inequality and so

   $\rho_0\left(\frac{\partial\psi}{\partial\theta} - \eta\right)\dot{\theta} + \left(\rho\frac{\partial\psi}{\partial F}.F^{-T}\right):\dot{F} + \frac{q\nabla\theta}{\theta} \leq 0$ (3.10) Here, we make some constitutive assumptions.

   a. Like the internal energy, P and η are also functions of F and θ, thus
   b. $P = P(F, \theta), \eta = \eta(F, \theta)$
   c. The heat flux q satisfies the thermal conductivity inequality and if q is independent of $\theta\ and\ \dot{F}$, we have

      $q.\nabla\theta \leq 0 \Longrightarrow -(k\nabla\theta).\nabla\theta \leq 0 \Longrightarrow k \geq 0$ (3.12)

      Therefore, the entropy inequality in (3.11) may be written as

      $\rho_0\left(\frac{\partial\psi}{\partial\theta} - \eta\right)\dot{\theta} + \left(\rho_0\frac{\partial\psi}{\partial F}\ P.F^{-T}\right);\ \dot{F} \leq 0$ (3.6)

      Since $\dot{\theta} and\ \dot{F}$ are arbitrary, the entropy inequality will be satisfied if and only if (3.7)

      $\frac{\partial\psi}{\partial\theta} - \eta = 0 \Rightarrow \eta = \frac{\partial\psi}{\partial\theta}$

      And

      $\rho_0\frac{\partial\psi}{\partial F}\ P.F^{-T} = 0 \Rightarrow P = \rho_0\frac{\partial\psi}{\partial F}\ F$ (3.15)

      Since the energy is independent of temperature gradient

      $\frac{\partial\psi}{\partial\nabla\theta} = 0$

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2. Similarly from equation (3.6) we obtain the following
   (a) $\rho_0\theta\eta = P.\dot{F} - \rho_0\eta\dot{\theta}$ (3.17)
   (b) $\rho_0\theta\eta = \rho_0\dot{U} - P.F$ (3.18)

3. Thirdly, heat conduction inequality are obtain as follows
   $\frac{h}{\theta}\nabla\theta \leq 0 \Rightarrow h\nabla\theta \leq 0$

Since ψ is independent of $\nabla\theta$ and is also potential for P and η, also the latter are independent of $\nabla\theta$ thus as a consequence of the dissipation inequality (3.6) the constitutive equations of thermoelastic materials are

$P = P(F,\theta), h = h(F,\theta,\nabla\theta), \psi = \psi(F,\theta), \eta = \eta(\mathrm{F},\theta)$ (3.20)

In particular, as a consequence of the Coleman-Noll interpretation of the second law of thermodynamics [1,3], the stress cannot depend on the temperature gradient. Material models for which additionally $h = 0$ i.e. material models which do not admit heat conduction may be called adiabatic material. In the case of adiabatic elasticity, only shock waves can be responsible for the energy dissipation [13]. In order to obtain a linear theory of thermoelasticity we consider the series expansion about a reference state (F_0, θ_0). The linear approximation of the first Piola-Kirchhoff stress tensor for small deformation gradients and small temperature difference is

$$P(F,\theta) = P_0\frac{\partial P}{\partial F}(F - F_0) + \frac{\partial P}{\partial\theta}(\theta - \theta_0) + \cdots$$

The linear approximation of the heat flux is also taken about the thermal equilibrium state with $\nabla\theta = \theta\ i.e.\ h_0 = h(F_0,\theta_0,0)$

$$h(\nabla\theta) = h(F_0,\theta_0,\nabla\theta) = h_0 + \frac{\partial h}{\partial\nabla\theta}.\nabla\theta + \cdots$$

and the linear approximation of the entropy is a scalar analogue of that of the stress tensor.

$$\eta(F,\theta) = \eta_0 + \frac{\partial\eta}{\partial F}(F - F_0) + \frac{\partial\eta}{\partial\theta}(\theta - \theta_0) + \cdots$$

Note that in order to achieve consistency with the thermostatic relations above, equations (3.14) and (3.15), and energy must contain quadratic terms

$$\nabla\theta = \theta - \theta_0, F = F - F_0$$

$$\psi(F,\theta) = \psi_0 + \frac{\partial\psi}{\partial F}(F - F_0)(\theta - \theta_0) + \frac{1}{2}(F - F_0)^T\frac{\partial^2\psi}{\partial F^2}(F - F_0) + \frac{\partial^2\psi}{\partial F\partial\theta}\ (F - F_0)(\theta - \theta_0) + \frac{1}{2}\frac{\partial^2\psi}{\partial\theta^2}(\theta - \theta_2)^2 + \cdots$$ (3.24)

In equation (3.24) set $\theta = \theta_0$, we obtain

$$\psi(F,\theta) = \psi_0 + \frac{\partial\psi}{\partial F}(F - F_0) + \frac{1}{2}(F - F_0)^T\frac{\partial^2\psi}{\partial F^2}(F - F_0) + \cdots$$ (3.24a)

Also $F = F_0$ gives

$$\psi(F,0) = \psi_0 + \frac{\partial\psi}{\partial\theta}(\theta - \theta_0) + \frac{1}{2}\frac{\partial^2\psi}{\partial\theta^2}(\theta - \theta_0) + \cdots$$ (3.24b)

Based on the approximations of equations (3.21 - 3.24) we obtain the following thermoelastic properties as the coefficient occurring in linear terms of the approximation. (3.19)

1. $C = \frac{\partial P}{\partial F} = \rho_0\frac{\partial^2\psi}{\partial F^2}$ (3.25)
   Called Elasticity or stiffness tensor.
   C denote spatial and the Cauchy-Hooke tensor of isothermal elasticity. C is symmetric fourth-order tensor.
2. Stress-temperature tensor also called thermal coefficient of stress is given by
   $M = \frac{\partial P}{\partial\theta} = \rho_0\frac{\partial^2\psi}{\partial F\partial\theta}$ (3.26)
   M is connected to latent heat (L_F) a second order tensor for fixed temperature by
   $L_F = -\frac{\theta_0}{\rho_0}MF^T = -\theta\frac{\partial^2\psi}{\partial F\partial\theta}F^T$
   M is related to the thermal tensor A via the relation $A = C^{-1}M$ (3.28)
   Where C^{-1} is the fourth-order compliance tensor (3.11, 3.13) with respect to practical measurability, the thermal tensor is best in describing thermoelastic response of solids.
3. Heat conductivity tensor (referential heat conductivity tensor) (3.21)
   $K = -\frac{\partial h}{\partial\nabla\theta}$
4. Specific heat at constant deformation
   $C_F = \theta_0\frac{\partial\eta}{\partial\theta} = \theta_0\frac{\partial^2\psi}{\partial\theta^2}$ (3.22)
   Although the Clausius-Duhem inequality (3.6) says nothing about the sign of C_F, the positivity of C_F can be derived from the requirement (3.23) of thermodynamics stability of the body [4,9].

4.0 LINEAR THERMOELASTICITY APPLIED TO ISOTROPIC SOLID MATERIALS

The general theories and formulations will applied to isotropic solid materials as required with discussion and conclusion drawn.

For isotropic materials the following implication results so far from thermomechanics derivations and formulations for linear theory of elasticity[6,7]. The generalized Hooke's law or

linear elasticity coefficient

$T = CG$

With its fourth-order elasticity tensor C (with 21independent components in the case of mono crystals solids) can be replaced by the Cauchy-Hooke's law.

$T = \lambda(trg)I + 2\mu g$

Where the scalar coefficients $\lambda\, and\, \mu$ are isothermal Lame constant [10, 12]

(a) $M = mI$ (b) $A = \alpha I$ (c) $K = KI$ (4.3 a,b,c)

Where K is the heat conductivity, α= linear thermal expansion

$(m = -\alpha(3\lambda + 2\mu)$

The constitutive equation for the stress tensor of isotropic solid material under consideration (monocrystal) within the completely linearized theory of thermoelasticity using Duhamel-Newman Law is

$\theta = \lambda(trg)I + 2\mu g + m(\theta - \theta_0)$

$i.e. \quad T = m(\theta - \theta_0)I$

Converse when the stress vanishes ($\theta = 0$) strain equals omnidirectional dilation or hydrostatic compression in an isotropic body is

$G = \alpha(\theta - \theta_0)I$

This is possible if inverse Duhamel-Neuman Law exist. The remaining constitutive equations are

$q = -K\nabla\theta$

For heat flux

$\eta = \frac{C_F}{\theta_0}(\theta - \theta_0) - \frac{M}{\rho_0}(trg)$

For entropy

$\psi = \frac{1}{2\rho}\lambda(trg)^2 + \mu tr(g)^2 + m(trg)(\theta - \theta_0) - \frac{1}{2}\frac{C_F}{\theta_0}(\theta - \theta_0)^2$ (4.10)

for free energy.

5.0 **CONCLUSION**

Following modern treatment of rational thermomechanics, the physically linearized theory has been derived in referential formulation, using the first Piola-Kirchhoff stress tensor P, the referential heat flux h, the free energy $\psi\, and\, the\, entropy\, \eta$ as material responses for which constitutive equations are given. The deformation gradient F, the temperature θ and the referential temperature gradient $\nabla\theta$ have been used as independent variables. Linear approximation of the series expansion about a national reference state (F_0, θ_0) and thermal equilibrium have been invoked to define the fourth-order elasticity tensor C. The stress-temperature tensor M, the thermal expansion tensor A, the latent heat tensor L_F and L_T, the heat conductivity tensor K and specific heat C_F and C_T, for solids. For the completely linearized theory, the constitutive equations and linear thermoelastic properties of isotropic solids, including scalar coefficients of thermal expansion and heat conductivity, have been derived in order to demonstrate the scope of this theory. The assumptions underlying each steps of simplification and the physical meaning of the coefficients have been emphasized throughout. Heterogeneous solid materials and bones as a case study of how thermomechanics can be used in theory of thermoelasticity are being considered for further investigation.

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Volumetric Growth Model of Thermoelastic Materials (Using Hard Tissues as a Case Study)

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Abstract
In this paper the volumetric growth theory for thermoelastic materials are developed by means of stress-induced volumetric growth in biological tissue. The developed theoretical formula of the stress induced volumetric material growth model with temperature term is applied to hard tissue (bone) as a case study. The resulting equations are solved to obtain the stress strain relations. Practical examples are obtained in terms of crack in bone deformation with graphical illustration to show the result.

Keywords: Growth, Thermo elasticity, Temperature, Stress, Strain, Biomaterial.

1.0 INTRODUCTION

In the last years, bioengineering has become one of the most active research areas. A large amount of research work related with bio-materials, tissue engineering regeneration is done and is increasing and also in continuous expansion. The development of FEM software (Finite Element Method) and CAD tools (Computer Aided Design) to generate geometric bio-forms with an approximation to physical real forms of hard tissues (bones) allows for examples estimate hip prosthesis behaviour inside femoral bone medullar canal or produce stresses behaviour at bone level and its effect, in the periodontal ligament when teeth are subjected to loads when using orthodontic appliances as an examples of FEM use in applications for hard tissues (Momoh, 2014). It is possible to know the stresses, strains and also displacement for biological systems or systems formed by an organs and implants when submitted to loads with a very good approximation, but up to this point, it has not been possible to predict if tissues grows or reabsorb when loaded. If tissue growth or reabsorption will occur, because of the success of the implant depends on growth. The formulation of the constitutive theory of finite deformation thermo elasticity was presented by Klish et al (2001), Lubarda (2004), the intermediate configuration is introduced here by biconseptual isothermal distressing of the current material configuration to zero stress. The total deformation gradient is then decomposed into the product of purely elastic and thermal parts. Such an approach was also used by Stojanovic et al (1976) in the constitutive study of non-polar and polar thermo elastic materials. Cowin and Hegedus (1976) presented a model for living bone in which bone modeling, remodeling and morphogenesis were well treated. An early work passing through the kinematic treatment to describe surface growth based on growth velocities was described by Skalak (1981), the extension of this work including the effect of incompatible growth was presented by Rodriguez et al (1994). Casey and Krishnaswamy (1998) presented a theory of thermo elastic materials which were summarized in its Eulerian form stated mathematically as let B be a thermoelastic body with a fixed reference configuration $k_0(B)$ and a configuration $k(B)$ at time t. A particle of B occupies positions $X \in k_\theta(B)$ and $x \in k(B)$. Let k > 0 denote the absolute temperature of B in $k(B)$ and take k_θ to be the reference temperature in $k_\theta(B)$. The motion and temperature of B are defined by smooth mapping.

$$x = \mathcal{X}(X,t), \vartheta = \theta(X,t) \qquad (1.1)$$

Let V, F, L and D denoting the velocity, deformation gradient tensor, velocity gradient tensor and the rate of deformation tensor respectively. It is assumed that the determinant of F satisfies

$$J = detF > 0 \qquad (1.2)$$

The balance equation for mass, linear momentum, angular momentum and energy take the form

$$\rho + \rho divv = 0, divT + \rho b = \rho v, T^T = T, \ \rho \in = \rho r divq + T.D \qquad (1.3)$$

On $k(B)$, where ρ is the mass density, T is the Cauchy stress tensor, b is the external body, E is the internal energy, r is the external heat supply and q is the heat flux vector.

A thermoelastic material m has constitutive equations

$$E = \bar{E}(F,\theta),\ T = \bar{T}(F,\theta),\ q = \bar{q}(F,\theta,G) \quad (1.4)$$

Where $g = grad\theta, \bar{E}(0,\theta_0) and\ \bar{Q}(F,\theta,0) = 0,$ due to invariance requirement eqn. (1.4) may be written as

$$E = \bar{E}(C,\theta) \quad (1.5)$$

Where

$$C = F^T F \quad (1.6)$$

Growth describes the physical process by which a material or solid body increases its size by addition of mass. A clear distinction is generally made between growths in general, remodeling that change of properties and morphogenesis (Shapes changes) is a classification suggested by Taber (1995). The advantages and drawbacks of the existing growth models are exposed in the recent contribution of Kuhl, (2014). Momoh (2012) use finite element method to model the growth in finite strain thermoelasticity in which the thermal term was reflected. O' Connor et al (2010) wrote on the stress induced finite growth in human tissues (bone) and also gave examples of growth in cardiovascular system. This paper presented a theory for thermoelastic material with growth especially volume increase and its treatment in the context of continuum mechanics. Vignes and Papadopoulos (2010) worked on materials growth in thermoelastic continua. Mahmoud et al (2015) worked on volume and heat for one-dimensional anisotropic thermoelasticity. A first class of models is the kinematic models describing an evolution towards an homeostatic state that rely on the kinematic decomposition of the transformation gradient into a generally incompatible mapping and an elastic mapping; they were historically introduced by Rodriguez et al. (1994). The growth transformation evolves in time as a function of the difference between a stress measure and a corresponding measure associated to the surmised homeostatic state Taber (1998), Kuhl (2014), Vignes and Papadopoulos (2010).

The aim of this work is to present the thermoelastic theory for volumetric materials growth (using hard tissue as a case study) with particular reference to volume increase and thermal differential terms and its treatment in the continuum mechanic process taking into context, the kinematic treatment to describe surface growth of the hard tissue based on growth velocities and thermoelastic inclusion of growth model.

2.0 MATHEMATICAL FORMULATION OF THE PROBLEM.

A theory for growing thermoelastic material is developed, the development combine the theories of thermoelastic materials using hard tissue (bone) as a case study. Assumed that a homothermal process is path-independent or reversible, we identify materials that cannot experience reversible growth. Thus the statement of the second law of thermodynamics is modified, which state that "in all energy exchanges, if no energy enters or leave the system, the potential energy of the state will always be less than that of the initial state".

Constructing a family of growing thermoelastic materials that inherit their thermomechanical response functions from a single generating material (Mahmoud et al, 2015). This approach is motivated by the idea that, at any time during a continuous growth process, the growth process may be stopped. When the growth process is stopped, the growing materials behave as a thermoelastic material, for which homothermal process are reversible. This thermoelastic material is called the generating materials. Thus, each growing thermoelastic material is associated with a single generating material.

2.1 Growing Elastic Materials

The theory of Klisch et al (2001) for growing compressible elastic materials is summarized. The structure of that theory was motivated by how it may be applied in practice. More specifically, tissue explants may be harvested at different stages of the growth process and the tissues compositional, geometric, and material properties can be experimentally characterized. In order to characterize these properties, the experimental data must be defined relative to a pre-determined reference configuration. Therefore, we introduce a fixed reference configuration that can be identified with an experimental configuration of the material, and can be used as a reference configuration for the growth boundary-value problem.

Let B be a growing body with a fixed reference configuration $l_0 B$ and a configuration $l(B)$ at time t. For simplicity, let the body be unloaded and stress-free in$l_0(B)$. The configuration $l(B)$ represents the time-dependent loaded configuration of B during a continuous growth process. A particle of B occupies positions $X \in l_0(B) and\ X \in l(B)$.The motion for B is defined as in (1.1) and is

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assumed to be invertible. Any material point that is added during the growth process is associated with a unique material point in $l_0(B)$ through the inverse mapping of (1.1).The deformation gradient of the mapping from $l_0(B)$ to $l(B)$ is assumed to obey the decomposition (Lubarda, 2004) of the form

$$F = F_t M_e M_g \quad (2.1)$$

The tensor $M_e M_g$ describes the total deformation due to growth relative to $l_0(B)$, whereas the deformation gradient F_t represents a superposed elastic deformation caused by applied loads. In the decomposition (2.1), the amount and orientation of mass deposition are described by M_g Furthermore, it is assumed that the mass density, the free energy density, and the stress functions are independent of M_gThis latter assumption establishes the existence of the tensors that appear in (2.1). More specifically, F_t can be measured by removing the external loads acting on $l(B)$. Then, $M_e M_g$ can be calculated by measuring the total deformation gradient tensor for $l(B)$ relative to$l(B)$. Finally, M_e and, consequently, $M_g$can be measured by performing a series of destructive experiments on both $l_0(B)$ and $l(B)$ that are designed to relieve the residual stress field in the tissue.

In this theory, the tensors M_e and M_g are introduced relative to a fixed reference configuration and, consequently, lack a clear physical interpretation.

However, when interpreted in terms of the continuously changing current configuration of a material during a growth process, these tensors have clear physical meanings. The governing equations for a small increment of growth were derived in order to provide an intuitive description of the quantities that describe growth. For an increment of growth, where M_a is equivalent to the incremental growth tensor and M_e is equivalent to the elastic accommodation tensor that ensures compatibility of $l(B)$.Note that arbitrary, orthogonal tensors Q_1 and Q_2 may appear in (2.1) such that

$$F_l M_e M_g = F_l Q_1^T Q_1 M_e Q_1^T Q_1 M_g \quad (2.2)$$

Defining

$$L_l = F_l F_l^{-1}, \quad L_e = M_e M_e^{-1}, \quad L_g = M_g M_g^{-1} \quad (2.3)$$

The velocity tensor associated with F is

$$L = FF^{-1} = L_l + F_l L_e F^{-1} + F_l M_e L_g M_e^{-1} F_l^{-1} \quad (2.4)$$

The effective elastic deformation gradient tensor F_* and the effective right Cauchy Green tensor C_* are defined as

$$F_* = F_l M_e, \quad C_* = F_*^T F_* \quad (2.5)$$

To state the balance equations, we recall the assumption that the material deposited during growth has the same mechanical properties as the original material. This assumption has two implications. First, the mass density, linear momentum, angular momentum, internal energy, and kinetic energy of the deposited material are the same as that of the original material at a point. Second, the mechanical response functions of the deposited material are the same as those of the original material. Also, we introduce two scalar parameters. The mass growth function c is the rate of mass deposition per unit current mass. The growth energy term β is the rate of growth energy per unit current mass that is required in addition to that needed to create material with the same internal and kinetic energy as the existing material. Then, the balance equations for mass, linear momentum, angular momentum, and work-energy on $l(B)$ take the form

$$\rho + \rho div v = \rho c, \quad div T + \rho b = \rho v, \quad T = T^T, \quad \rho E = T.D + \rho\beta \quad (2.6)$$

2.2 Kinematics and Balance Law

Let B be a growing thermoelastic body with a fixed reference configuration $y_0(B)$ and a configuration $y(B)$ at time t.

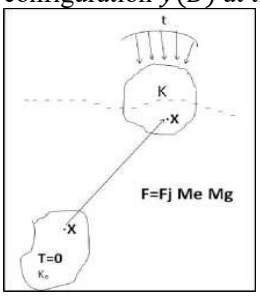

Figure 1: Schematic of the decomposition of the deformation gradient tensor F. From mathematics and mechanics of solids 8: 377-402, 2003

The temperature history for B is defined as in (1.2). The kinematics for the growing thermoelastic material is the same as those presented above for the growing elastic material. In particular, relations hold Motivated by earlier work; it is assumed that the material deposited during growth has the same thermomechanical properties as the existing material at a point. Then, the balance equations for mass, linear momentum and angular momentum are the same

The reduce balance of mass equation

$$\rho J = \rho_0 \quad (2.7)$$

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And the growth equation

$$detMg = \exp\left[\int_{\tau = t_0}^{t} cdr\right]$$

(2.8)

the balance of energy on $y(B)$ is derived by adding appropriate heating terms to the work-energy balance

$$T = T^T$$

(2.9)

The resulting equation is

$$\rho \in = \rho r - divq + T.D + \rho\beta$$

(2.10)

Recalling the expression for the velocity gradient tensor

$$L = FF^{-1} = L_l + F_l L_e F^{-1} + F_l M_e L_g M_e^{-1} F_l^{-1}$$

The energy equation (2.10) may be expressed as

$$\rho \in -\rho\beta - T.\left(L_l + F_l L_e F^{-1} + F_l M_e L_g M_e^{-1} F_l^{-1}\right) - \rho r + divq = 0 \quad (2.11)$$

2.3 Growth Function and its Response on Hard Tissue (bone)

The growth law for the time-rate of change of M_g takes the form

$$M_g = \xi(M)$$

where the list of stimuli M may include thermal variables, such as temperature or temperature gradient. Also, a growth response function is needed for the growth energy supply β. Here, we consider growth response functions with the general forms

$$M_g = \xi(M). \quad \beta = \beta(M) \quad (2.12)$$

where $\xi(M)$, $\beta = \beta(M)$ are experimentally determined functions of thermomechanical stimuli M. It will be seen that the growth response functions (2.12) must obey a constitutive restriction derived from an entropy inequality. As such (2.12) must satisfy appropriate invariance requirements.

2.4 Stress Induced Finite Growth in Human Tissues

A growth displacement field is specified in an unloaded cylindrical tube and residual stresses resulting from growth fields shows how circumferential growth gives rise to residual stresses that would cause the cylinder to change shape when cut as the opening angle experimental tests. The point P in B has coordinates $(\Re, \theta, Z)$. F_g maps B in B_g where P has the coordinates $(\rho, \varphi, \epsilon)$ as shown in the figure below. The term $k(\Re)$ is the circumferential growth strecth ratio which depends on the radius and is assumed constant, when $k_\theta > 1$ growth occurs and when $k_\theta < 1$ reabsorption occurs

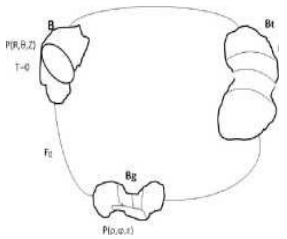

Figure 3: Multiplicative decomposition of growth deformation gradient

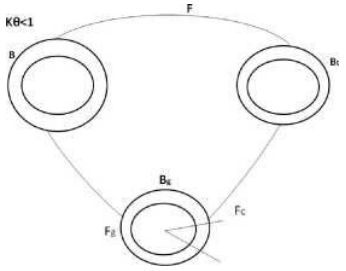

Figure 4: Cylindrical models after circumferential volume growth in tissue.

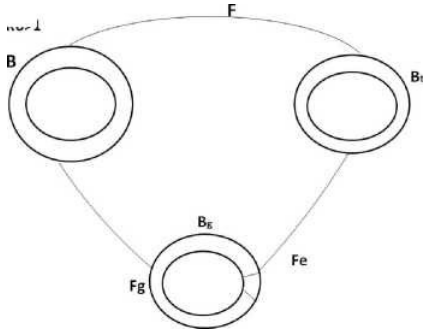

Figure 5: adapted from (Rodrigues et al 1994)

To obtain growth deformation gradient the following displacement field is prescribed as

$$\rho = R, \varphi = k_\theta(R)\theta, \epsilon = Z$$

(2.13)

And the field that maps B_g in B_t where ρ has coordinates (r, θ, Z) is

$$r = r(\rho), \theta = \eta_\theta(\rho)\varphi, \quad Z = \epsilon Z$$

(2.14)

Then F_g and F_e in cylindrical results in

$$F_g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\rho}{R}k_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_e = \begin{pmatrix} \frac{dr\rho}{d\rho} & 0 & 0 \\ 0 & \frac{r}{\rho}\eta_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2.15)

$\eta_\theta(\rho)$ allows total deformation gradient to be compatible for k_θ constant, $\eta_\theta(\rho) = \frac{1}{k_\theta}$.

Incompressibility constraints is only applied F_e, so the third quadrant of right Cauchy – Green tensor unity

$$F_gF_e => \begin{pmatrix} \frac{dr(\rho)}{d\rho} & 0 & 0 \\ 0 & \frac{r}{\rho}\eta_\theta & 0 \\ 0 & 0 & \epsilon \end{pmatrix}\begin{pmatrix} \frac{dr(\rho)}{d\rho} & 0 & 0 \\ 0 & \frac{r}{\rho\eta_\theta} & 0 \\ 0 & 0 & \epsilon \end{pmatrix}$$

(2.16)

$$C = F^TF\begin{pmatrix} (\frac{dr(\rho)}{d\rho})^2 & 0 & 0 \\ 0 & (\frac{r}{\rho}\eta_\theta)^2 & 0 \\ 0 & 0 & (\epsilon)^2 \end{pmatrix}$$

(2.17)

Solve to find the determinant which implies

$$I = \left[\frac{dr(\rho)}{d\rho}\frac{r}{\rho}\eta_\theta\epsilon\right]^2 = 1$$

(2.18)

Firstly, we have that

(a) $r = r(\rho)$, $(b)\ \theta = \eta_\theta(\rho)\varphi$, $(c)\ \varphi = K_\theta(R)\theta$, $(d)\ E_g = \frac{1}{2}(F_g^TF_e - 1)$, $(e)\ F = F_eF_g$, $(f)\ \frac{1}{K\theta} = \eta_\theta(\rho)$, $(g)\ \rho = R$

(2.19)

Eqn.(2.18) can be integrated to obtain an expression for the growth radius r as

$$\frac{dr(\rho)}{d\rho}\frac{r}{\rho}\eta_\theta\epsilon$$

(2.20)

From eqn. (2.19f) solving for η_θ we obtain

$K_\theta\eta_\theta(\rho) = 1$, $\eta_\theta(\rho) = \frac{1}{K_\theta}$, $\eta_\theta = \frac{1}{K_\theta\rho}$ And substituting for η_θ in eqn. (2.20) we have

$$\frac{dr(\rho)}{d\rho}\frac{r}{\rho}\frac{1}{K_\theta\rho}\epsilon = \frac{dr(\rho)}{d\rho}\frac{r}{\rho}\frac{K_\theta\rho}{\epsilon} = \frac{dr(\rho)}{d\rho}\frac{r}{\rho}\frac{K_\theta\rho}{\epsilon} = \frac{dr(\rho)}{d\rho}\frac{rK_\theta}{\epsilon} \quad (2.21)$$

From eqn.(2.19a) solving for r by integration we have

$$\frac{dr.r}{d\rho}\frac{K_\theta}{\epsilon} = \frac{dr^2}{d\rho}\frac{K_\theta}{\epsilon} = \left(\frac{r^2K_\theta}{\epsilon}\right)\frac{d}{d\rho}$$ after integration yield

$\frac{r^2K_\theta}{\epsilon}R + C_2 = 1$, $r^2 = \frac{r^2K_\theta R}{\epsilon} + C_2$, hence

$$r = \sqrt{\frac{R^2K\theta}{\epsilon} + C_2}$$

(2.22)

Where C_2 is integration constant. Then, Green deformation tensor components referred to by coordinated form

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2.23)

$$E = \frac{1}{2}\ [F_e^TF_e - I]\begin{pmatrix} \left(\frac{dr(e)}{de}\right)^2 - 1 & 0 & 0 \\ 0 & \left(\frac{r}{e}\eta_0\right)^2 - 1 & 0 \\ 0 & 0 & (\epsilon)^2 - 1 \end{pmatrix}$$

(2.24)

Then,

$$E_{\rho\rho} = \frac{1}{2}\left[\left(\frac{RK_\theta}{r\epsilon}\right)^2 - 1\right], \quad E_{\varphi\varphi} = \frac{1}{2}\left[\left(\frac{r}{RK_\theta}\right)^2 - 1\right], \quad E_{\theta\theta} = \frac{1}{2}\ [(\epsilon)^2 - 1] \quad (2.25)$$

It is assumed the stress-strain relationship using

$$T_{ij} = \frac{1}{2}\ F_{ij}Fjt\ \left(\frac{dW}{dt_{3i}} + \frac{dW}{d\epsilon_{ts}}\right) - p_{\partial ij}$$

(2.26)

Which takes into account the strain energy function proposed by

$$W = \frac{C}{2}\ (\epsilon^\theta - 1)Q = 2\theta(E_{pp} + E_{\gamma\gamma} + E_{\epsilon\epsilon})$$

(2.27)

Being C = Matrix Constant, Q is function of principal deformation components that define module symmetry (Isotropic case). b_1 = material constant ($b_1 = 4.24$ $and\ C = 0.765Kp_a$). P is hydrostatic pressure.

$$T_{rr} = \frac{(r^2-C_2)K_0}{2r^2}\left[Cb_ie^{bi\left[\frac{(r^2-c^2)K_0}{r^2} + \frac{r^2}{(r^2-c^2)K_0} - 2\right]}\right] - P(r) \quad (2.28)$$

From equation. Where we have

$$T_{ij} = \frac{1}{2}\ F_{ij}Fjt\ \left(\frac{dW}{dt_{3i}} + \frac{dW}{d\epsilon_{ts}}\right) - p_{\partial ij}$$

(2.29)

$$F_{is} = \frac{K_\theta R^2}{r^2},\ F_{jT} = \frac{-K_\theta C}{r^2}$$

$$T_{\theta\theta} = \frac{r^2}{2(r^2-C_2)K_0} - \left[Cb_ie^{bi\left[\frac{(r^2-c^2)K_0}{r^2} + \frac{r^2}{(r^2-c^2)K_0} - 2\right]}\right] - P(r)$$

(2.30)

Considering $f = K_\theta\left(1 - \frac{C_2}{r_2}\right)$

(2.31)

gives

$$T_{rr} = \frac{f}{2}\left[Cb_ie^{bi\left[\frac{(f-1)^2}{f}\right]}\right] - P(r);\ T_{\theta\theta} = \frac{1}{2f}\left[Cb_ie^{bi\left[\frac{(f-1)^2}{f}\right]}\right] - P(r) \quad (2.32)$$

From equilibrium equations

$$\frac{dt_{rr}}{dr} + \frac{T_rr - T_{\theta\theta}}{r} = 0, \frac{dT_{r\theta}}{r} + \frac{2T_{r\theta}}{r} = 0, \frac{dT_{r2}}{dr} + \frac{T_{r^2}}{r} = 0 \quad (2.33)$$

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Since radius-stress boundary condition are assumed on the outer and inner walls, just the first of the above equations has to be solved. Then by integrating and using equation (2.29) Recall from $T_{\theta\theta}$ and T_{rr}

$$= \int_{r2}^{r} \frac{1}{2}\left(\frac{1}{f} - f\right)\left[Cb_i e^{bi\left[\frac{(f-1)^2}{f}\right]}\right] dr$$

(2.34)

T_{rr} in $r = r_2$ is radial stress at outer grown wall specified as zero, internal pressure is $-T_{rr}$ in $r = r_1$

The Model was solved numerically using Matlab because not all has analytical solution, by specifying the growth outer radius r_1 and solving for final inner radius value that gives a zero transmural pressure.

Processing Scheme: an initial value of external radius r_2 is specified with c_2 and r_1 are obtained. Then roots of T_{rr} are calculated to obtain new values of C_2,r_2 and r_1. The Scheme loops until the difference $(r_{2i+1} - r_{2i})$ approaches 100. (tolerance 10^{-12}) results for different values of K_θ Following the processing scheme on Matlab in solving a problem given K_θ to have values from 0.9 ,0.85 and 0.80 and having internal and external radius of 2 and 3 an T_{rr} and $T_{\theta\theta}$

Residual stress shown zero values of radii stress T_{rr} at inner and out endocardium walls since the cylinder was unloaded. Circumferential $T_{\theta\theta}$ shows nonlinear behaviour from compression at the endocardium to tension at the epicardium.

Using eqn. (2.31) gives

$$T_{rr} = \frac{f}{2}\left[Cb_i e^{bi\left[\frac{(f-1)^2}{f}\right]}\right] - P(r)$$

(2.35)

$$T_{\theta\theta} = \frac{1}{2f}\left[Cb_i e^{bi\left[\frac{(f-1)^2}{f}\right]}\right] - P(r)$$

(2.36)

Where $r = r_1$ and $r = r_2$ using the solution to plot graph of the rate of reabsorbtion or deformation and growth. And finally this method is coined out from the first step for future works in the implementation of material models using Matlab with the capability to grow or reabsorb as a response of differennt stress levels in order to predict zones of bones formation associated to growth and bone mass loose zone associated to reabsorption.

3.0 RESULT AND DISCUSSION

As earlier said and solving to find the values of T_{rr} in r = r_2 is radial stress of outer grown wall specified as zero, internal pressure is $-T_{rr}$ in $r = r_1$

The model was solved numerically using Matlab package because it has an analytic solution by specifying growth in outer radius r_2 and solving for final inner radius value that gives a zero transmural pressure.

From the process of numerical computation, the constant, C_2 , b_1 are obtain and r_1, r_2, K_θ, T_{rr}, $T_{\theta\theta}$ are presented in a tabular form.

The variables r_1, r_2, K_θ, T_{rr} , $T_{\theta\theta}$ which determines the effect of growth rate of the bone tissue when it grows under stress and when it has deformation or reabsorption. When $K_\theta < 1$ denotes reabsorption and $K_\theta > 1$ denotes growth.

Table 3.1: $K_\theta < 1$ for the reabsorption using the formula derived.

K_θ	r_1	r_2
0.90	1.755	2.753
0.85	1.633	2.630
0.80	1.512	2.507
0.75	1.441	2.475

For K_θ = 0.90 and r_1 and r_2 we have the table below of more than 8 iteration

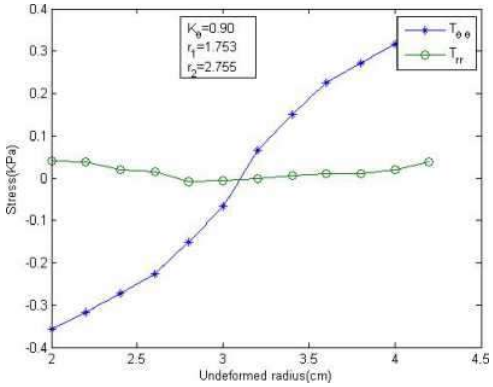

Table 3.2: $K_\theta > 1$ growth, and initial value of inner and outer radius r_1, r_2

$T_{\theta\theta}$	T_{rr}
-0.3573	0.0419
-0.3178	0.0392
-0.2731	0.0210
-0.2251	-0.0110
-0.2000	-0.0100
-0.1660	-0.0050

0.1660	-0.0004
0.2000	-0.0050
0.2251	0.0100
0.2731	0.0210
0.3178	0.0392
0.3573	0.0419

K_θ	r_1	r_2
1.10	2.247	3.248
1.15	2.370	3.372
1.20	2.490	3.496
1.25	2.552	3.548

4.0 CONCLUSION

The combined effect of volumetric growth of thermoelastic materials using hard tissue (bone) as a case study was presented. A mathematical model for studying the effect of volumetric growth of thermoelastic materials using bone as a case study of hard tissue biomechanics was presented giving specific attention to thermal term.. Using stress function under a specific temperature, a set of ordinary differential equation were derived to find the growth rate, deformation, tensile stress and strain and energy equation. An example of finite adaptive growth due to abnormal loads was analyzed, discussed and the result presented in tabular and graphic form. The implementation of material using finite element method (FEM) with the capability to grow or reabsorb as a response of different stress level in order to predict zones of bones formation associated to growth and reabsorbtion.

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On Some Special Cases of Opial's and Hardy's Inequality

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Abstract
In this work, we obtain a relationship between Opial and Hardy-types integral inequalities of Anthonio and Rauf Opial-type inequalities for convex function.

1.0 INTRODUCTION

Following integral inequalities are the statements of Hardy and Opial respectively:

Theorem 1.1. In 1920 G.H Hardy gives the famous inequality as

(1) $\int_0^\infty \left[\frac{1}{X}\left(\int_0^X f(t)dt\right)\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx$.

where $p > 1$,

Theorem 1.2. Let $x(t) \in c'[0, h]$ be such that $x(0) = x(h) = 0 \; and \; x(t) > 0 \; in \; (0, h)$ then the following inequality holds

(2) $\int_0^h |x(t)x'(t)|dt \leq \frac{h}{4}\int_0^h (x'(t))^2\, dt$

The constant $\frac{h}{4}$ is the best possible.

Many researchers have shown various methods to establish the necessary and sufficient condition on p, q, v, w for the Hardy-type inequality as applicable in analysis. The following inequality:

(3) $\left[\int_a^b |u(x)|^q w(x)dx\right]^{\frac{1}{q}} \leq C\left[\int_a^b |u'(x)|^p v(x)dx\right]^{\frac{1}{p}}$

holds, where C is a constant depending on p and q [8].

Theorem 1.3. Let g and h be continuous and non-decreasing function on $[a, b]$ $0 \leq a < b < \infty$, with $g(x), h(x) > 0, r = 1, for\; x > 0$. Let $q \geq p \geq 1 \; and \; f(x)$ be a non-negative and lebesgue-Stieltjes integrable with respect to $(x) and\; h(x) \;\; on\; [a, b]$. Suppose δ is a real number such that $-\frac{p}{q} < \delta < 0$ then,

(4)

$$\left[\int_a^b h(x)g(x)^{\frac{\delta q}{p}}\left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{r}{q}} \leq C(a,b,p,q,\delta)\left[\int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p dg(x)\right]^{\frac{r}{p}}$$

where

(5) $C(a,b,p,q,\delta) = (-\delta)^{\frac{(1-p)}{p}}\left(\frac{p}{q\delta+p}\right)^{\frac{1}{q}}\left(g(b)^{-\delta} - g(a)^{-\delta}\right)^{\frac{(p-1)}{p}}\left(g(b)^{\frac{q\delta+p}{p}} - g(a)^{\frac{q\delta+p}{p}}\right)^{\frac{1}{q}}$

Following the conditions in the theorem, (4) reduces to

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}}\left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{1}{q}} \leq C(a,b,p,q,\delta)\left[\int_a^b g(x)^{(p-1)(1+\delta)} f(x)^p dg(x)\right]^{\frac{1}{p}}$$

The constant C still retaining its valve.

2.0 RESULTS

For the purpose of this paper we shall define: $k(x,t) = r(x)^{\varphi} f(t)^p r(t)^{p(1+\varphi)}$, $h(x,t) = g(x)^{\varphi} f(t)^p g(t)^{p(1+\varphi)}$, $d\varepsilon(t) = g(t)^{-(1+\varphi)}dg(t)$ and put all to use where necessary throughout this paper.

Lemma 2.1. Let $k(x,t) and\; h(x,t)$ be non negative, $x \geq 0, t \geq 0$ and $\varepsilon \geq 0$ be non decreasing.

Let $-\infty \leq 0 \leq x < \infty$, where $r(t) \geq g(t)$ then the following holds:

(7) $\left[\int_0^x k(x,t)d\varepsilon(\lambda)\right]^{\frac{v}{p}} = r(x)^{\frac{\varphi v}{p}} \left[\int_a^x f(t)^p r(x)^{(p-1)(1+\varphi)} dg(t)\right]^{\frac{v}{p}}$

Proof:

Let f (t) and r(x) be absolutely continuous functions.

$$(8)\qquad \left[\int_0^x k(x,t)d\varepsilon(t)\right]^{\frac{v}{p}} = \left[\int_a^x r(x)^{\varphi} f(t)^p r(t)^{p(1+\varphi)} g(t)^{-(1+\varphi)} dg(t)\right]^{\frac{v}{p}}$$
$$= r(x)^{\frac{\varphi v}{p}}\left[\int_a^x f(t)^p\, r(t)^{p(1+\varphi)} g(t)^{-(1+\varphi)} dg(t)\right]^{\frac{v}{p}}$$
$$= r(x)^{\frac{\varphi v}{p}}\left[\int_a^x f(t)^p\, r(t)^{(p-1)} g(t)^{(1+\varphi)} dg(t)\right]^{\frac{v}{p}}$$
$$= r(x)^{\frac{\varphi v}{p}}\left[\int_a^x f(t)^p r(t)^{(p-1)(1+\varphi)} dg(t)\right]^{\frac{v}{p}}.$$

Lemma 2.2. Let all the conditions stated in lemma 2.1 still remain in contact then,

$$(9)\ \left[\int_a^x h(x,t)k(x,t)d\varepsilon(t)\right]^{\frac{v}{p}} = (g(x)r(x))^{\frac{\varphi v}{p}}\left[\, f(t)^p \int_a^x g(t)^{(p+p\varphi)(1+\varphi)(p-1)}\, dg(t)\right]^{\frac{v}{p}}$$

Proof:

$$\left[\int_a^x h(x,t)k(x,t)d\varepsilon(t)\right]^{\frac{v}{p}}$$
$$= \left[\int_a^x g(x)^{\varphi} f(t)^p g(t)^{p(1+\varphi)} r(x)^{\varphi} f(t)^p\, r(t)^{p(1+\varphi)} g(t)^{-(1+\varphi)} dg(t)\right]^{\frac{v}{p}}$$
$$= (g(x)r(x))^{\frac{\varphi v}{p}}\left[\int_a^x f(t)^p g(t)^{p(1+\varphi)} f(t)^p\, r(t)^{p(1+\varphi)} g(t)^{-(1+\varphi)} dg(t)\right]^{\frac{v}{p}}$$
$$= (g(x)r(x))^{\frac{\varphi v}{p}}\left[\, f(t)^p \int_a^x g(t)^{p(1+\varphi)} r(t)^{p(1+\varphi)}\, g(t)^{-(1+\varphi)} dg(t)\right]^{\frac{v}{p}}$$
$$= (g(x)r(x))^{\frac{\varphi v}{p}}\left[\, f(t)^p \int_a^x g(t)^{(p-1)(1+\varphi)} r(t)^{p(1+\varphi)}\, dg(t)\right]^{\frac{v}{p}}$$

hence

$$\left[\int_a^x h(x,t)k(x,t)d\varepsilon(t)\right]^{\frac{v}{p}} = (g(x)r(x))^{\frac{\varphi v}{p}}\left[\, f(t)^p \int_a^x g(t)^{(p+p\varphi)(1+\varphi)(p-1)}\, dg(t)\right]^{\frac{v}{p}}$$

Lemma 2.3. Let $k(x,t)$ be non negative, $t \geq 0, x \geq 0, p \geq 1$ *and* $\varepsilon \geq 0$ be non decreasing. Let $-\infty \leq 0 \leq x < \infty$, *then the following holds*:

$$(10)\ \left[\int_a^x d\varepsilon(t)\right]^{\frac{(1-p)v}{p}} = \left[\frac{g(t)^{-\varphi}}{-\varphi}\Big|_a^x\right]^{\frac{(1-p)v}{p}} = [g(x) - g(a)]^{\frac{(p-1)\varphi v}{p}}\,(-\varphi)^{\frac{-(p-1)v}{p}}$$

Proof:

$$\left[\int_a^x d\varepsilon(t)\right]^{\frac{(1-p)v}{p}} = \left[\int_a^x g(t)^{-(1+\varphi)} dg(t)\right]^{\frac{(1-p)v}{p}}$$
$$\left[\int_a^x d\varepsilon(t)\right]^{\frac{(1-p)v}{p}} = \left[\frac{g(t)^{(-1-\varphi+1)}}{-1-\varphi+1}\Big|_a^x\right]^{\frac{(1-p)v}{p}}$$
$$\left[\int_a^x d\varepsilon(t)\right]^{\frac{(1-p)v}{p}} = \left[\frac{g(t)^{-\varphi}}{-\varphi}\Big|_a^x\right]^{\frac{(1-p)v}{p}}$$
$$\left[\int_a^x d\varepsilon(t)\right]^{\frac{(1-p)v}{p}} = [g(x)^{\varphi} - g(a)^{\varphi}]^{\frac{(p-1)v}{p}}\,(-\varphi)^{\frac{-(p-1)v}{p}}$$
$$\left[\int_a^x d\varepsilon(t)\right]^{\frac{(1-p)v}{p}} = [g(x)^{\varphi} - g(a)^{\varphi}]^{\frac{(p-1)v}{p}}\,(-\varphi)^{\frac{-(p-1)v}{p}}$$

Lemma 2.4. Assume all the conditions for lemma 2.3 hold, considering the case where $k(x,t) \neq 0$ and choosing it to be 1 then we have:

$$(11)\ \left[\int_a^x h(x,t)^{\frac{1}{p}} k(x,t)d\varepsilon(t)\right]^{v} \geq g(x)^{\frac{\varphi v}{p}}\left[\int_a^x f(t)dg(t)\right]^{v}$$

Proof:

$$\left[\int_a^x h(x,t)^{\frac{1}{p}} k(x,t)d\varepsilon(t)\right]^{v} =$$

$$\left[\int_a^x \left(g(x)^{\varphi} f(t)^p g(t)^{p(1+\varphi)}\right)^{\frac{1}{p}} \left(r(x)^{\varphi} f(t)^p r(t)^{p(1+\varphi)}\right) g(t)^{-(1+\varphi)} dg(x)\right]^v$$

$$= \left[\int_a^x \left(g(x)^{\varphi} f(t)^p g(t)^{p(1+\varphi)}\right)^{\frac{1}{p}} g(t)^{-(1+\varphi)}\, dg(x)\right]^v$$

$$= \left[\int_a^x g(x)^{\frac{\varphi}{p}} f(t) g(t)^{(1+\varphi)} g(x)^{-\varphi+1}\, dg(x)\right]^v$$

$$= \left[\int_a^x g(x)^{\frac{\varphi}{p}} f(t) dg(x)\right]^v$$

$$\geq\ g(x)^{\frac{\varphi v}{p}} \left[\int_a^x f(t) dg(t)\right]^v$$

The proof is complete.

Putting the well-known Jensen's inequality into use as stated in [3] and [4] yields

(14) $$\int_a^x h(x,t) d\varepsilon(s)\ \geq\ \left[\int_a^x d\varepsilon(s)\right]^{1-p} \left[\int_a^x h(x,t)^{\frac{1}{p}} d\varepsilon(s)\right]^p$$

Raising both sides of inequality (14) to the power $\frac{vr}{p}$ with $r \geq 1$ gives:

(15) $$\left[\int_a^x h(x,t) d\varepsilon(s)\right]^{\frac{vr}{p}} \geq \left[\int_a^x d\varepsilon(s)\right]^{\frac{(1-p)vr}{p}} \left[\int_a^x h(x,t)^{\frac{1}{p}} d\varepsilon(s)\right]^{\frac{pvr}{p}}$$

(16) $$\int_a^x h(x,t) d\varepsilon(s)^{\frac{v}{p}} \geq \left[\int_a^x d\varepsilon(s)\right]^{\frac{(1-p)v}{p}} \left[\int_a^x h(x,t)^{\frac{1}{p}} d\varepsilon(s)\right]^v$$

Theorem 2.4. Let $f(t), k(x,t) and\ g(t)$ be an absolutely continuous function which is non-decreasing on $[a,b], 0 \leq a \leq b < \infty$. suppose that $p \geq v \geq 1, \varphi > 0\ and\ f(x)$ is lebesque-stieltges integrable with respect to $g(x)$ on $[a,b]$ and taking into consideration the given condition in lemma 2.4 then,

(17) $$\left[\int_a^x \prod_{t=1}^2 f(t)\, dg(t)\right] \leq\ (-\varphi)^{\frac{(1-p)v}{p}} [g(x) - g(a)]^{\frac{(1-p)\varphi v}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\varphi)} dg(t)\right]^{\frac{v}{p}}.$$

Proof:

Multiplying both sides of (17) with $g(x)^{\frac{\varphi v}{p}}$, taking the second term of $f(t)$ to be $k(x,t)$ gives

(18) $$g(x)^{\frac{\varphi v}{p}} \left[\int_a^x f(t) dg(t)\right] \leq g(x)^{\frac{\varphi v}{p}} \left[(-\varphi)^{\frac{(1-p)v}{p}} [g(x) - g(a)]^{\frac{(1-p)\varphi v}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\varphi)} dg(t)\right]^{\frac{v}{p}}.\right]$$

Combining the outcomes of lemma 2.1, 2.3 and 2.4 in the inequality (18), we have

(19) $$g(x)^{\frac{\varphi v}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\varphi)} dg(t)\right]^{\frac{v}{p}} \geq (-\varphi)^{\frac{(p-1)v}{p}} [g(x) - g(a)]^{\frac{(p-1)\varphi v}{p}} g(x)^{\frac{\varphi v}{p}} \left[\int_a^x f(t) dg(t)\right]^v$$

that is,

(20) $$(-\varphi)^{\frac{(p-1)v}{p}} [g(x) - g(a)]^{\frac{(p-1)\varphi v}{p}} g(x)^{\frac{\varphi v}{p}} \left[\int_a^x f(t) dg(t)\right]^v \leq g(x)^{\frac{\varphi v}{p}} \left[\int_a^x f(t)^p g(t)^{(p-1)(1+\varphi)} dg(t)\right]^{\frac{v}{p}}$$

which implies

(21) $$g(x)^{\frac{\varphi v}{p}} \left[\int_a^x f(t) dg(t)\right]^v \leq (-\varphi)^{\frac{(1-p)v}{p}} [g(x) - g(a)]^{\frac{(1-p)\varphi v}{p}} g(x)^{\frac{\varphi v}{p}} \left[\int_a^x f(t)^p\, g(t)^{(p-1)(1+\varphi)} dg(t)\right]^{\frac{v}{p}}$$

Integrating both sides of (20) with respect to g(x) on $[a,b]$ and then raising both sides to power $\frac{p}{v}$ to obtain the following inequality:

(22) $$\left[\int_a^b g(x)^{\frac{\varphi v}{p}} \left[\int_a^x f(t) dg(t)\right]^v dg(x)\right]^{\frac{p}{v}} \leq (-\varphi)^{\frac{(1-p)v}{p}} \left[\int_a^b [g(x) -$$

$$g(a)]^{\frac{(1-p)\varphi v}{p}} g(x)^{\frac{\varphi v}{p}} \left[\int_a^x f(t)^p g(x)^{(p-1)(1+\varphi)}\, dg(t)\right]^{\frac{v}{p}} dg(x)\Bigg]^{\frac{p}{v}}$$

Which gives the generalization in [2] and [9].

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Monte Carlo Studies On Conway-Maxwell Poisson Generalized Linear Mixed Effects Model For *Under-Dispersed* Count Data.

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ABSTRACT
Poisson regression is the traditional technique for handling count data. The assumption of equality of mean and variance which is an important property of the Poisson distribution makes the application of the distribution on count data highly restrictive since in reality count data do not always satisfy this assumption. The Generalized Poisson distribution and the Conway-Maxwell Poisson regression are some of the proposed remedies for handling under dispersed data. Our recent work on theoretical exposition of the re-parameterization and extension of the Conway-Maxwell-Poisson regression models to accommodate random effects appeared in the literature. This paper presents a simulation study to evaluate the performance of the re-parameterized Conway-Maxwell-Poisson Generalized Linear Mixed Effects Model (CMPGLMM) for handling the problem of under-dispersion in clustered data. The re-parameterization allows the response to be directly related to the regression coefficients via an approximation of the mean, thereby, leading to straightforward interpretation of the coefficients. The simulation result showed that the implementation is reliable and the CMPGLMM produced results that are better than the traditional Poisson and Negative-Binomial models which imply that the CMPGLMM is a better alternative for under-dispersed clustered count data.

Keywords: Under-dispersion, Clustered data, Poisson regression, Mixed-effect, Monte Carlo Studies.

1.0 INTRODUCTION.

Generalized linear mixed models (GLMMs) also known as multilevel generalized linear models (GLMs) are popular for multilevel data with units nested in clusters. GLMMs combine the properties of GLMs and linear mixed effects models (LMMs). As GLMs they have the ability to fit non-linear and non-Normal response data by using link functions and responses drawn from distributions in the exponential family. As mixed models they have the ability to include both fixed and random effects. Mixed-effects models represent the covariance structure related to the clustering of data by associating the common random effects to observations that have the same level of a clustering variable.

Poisson distribution using the mixed effect framework is a traditional method for handling count data. One of such technique is the Poisson-Gamma distribution, though unsuitable for under-dispersed data McCullagh and Nelder (1997) also noted that the procedure is an unpopular option with problematic link. The Com-Poisson regression proposed by Sellers and Shmueli (2010) was recently extended by Dikko et al (2017) to accommodate random effects for handling clustered count data which are frequently encountered in observational or experimental studies.

Statistical methods for the analysis of cross-sectional count data where only one measurement is made for a variable of interest for each individual/observational unit in the study are well developed in literature. An important assumption for modelling cross-sectional data is that observations are independent of each other. Therefore, statistical methods for analyzing cross-sectional data cannot directly be used for analyzing longitudinal or clustered data. Clustered data can be defined as data in which the observations are grouped into disjoint classes called clusters according to some classification criterion (Pinheiro, 1994). These includes longitudinal data where individuals in a longitudinal setting are followed over a period of time and data collected at multiple time point for each individual (Wu, 2010). Here observations from

each individual constitute a cluster. Mixed models were developed to handle clustered data and have attracted lots of interest in Statistics for decades. Observations in the same cluster usually cannot be considered independent therefore mixed effects models constitute a convenient tool for modelling cluster dependence.

Dikko et al (2017) combined the ability of the GLMMs to account for correlation within clustered data and the flexibility of the COM-Poisson distribution in handling any dispersion level in count data to propose a COM-Poisson GLMM (CMPGLMM). In this paper, we present a simulation study to evaluate the performance of the CMPGLMM alongside the Poisson and Negative-Binomial GLMM in the presence of under-dispersion.

The rest of the paper is organized as follows. Section 2 provides an overview of count distributions and regression models; section 3 gives some details on the re-parameterization used by Dikko et al (2017) as well as some information on implementation; section 4 consists of the simulation setting, results and discussion while section 5 provides concluding remarks.

2.0 COUNT MODELS

Poisson Regression

The Poisson distribution characterizes the probability of observing any discrete number of events given an underlying mean count of events, assuming that the timing of the events is random and independent (Le, 2003).

The Poisson regression model is a model where the mean of the distribution is a function of the explanatory variables, with the defining characteristic that the conditional mean of the outcome is equal to the conditional variance (Algamal, 2012; Algamal and Lee, 2015).

In Poisson regression model, the number of events y_i has a Poisson distribution with a conditional mean that depends on individual characteristics according to the structural model;

$$P(Y_i) = \frac{e^{-\theta_i}{\theta_i}^{y_i}}{y_i!};\ y_i = 0,1 \dots; \qquad (1)$$

and the conditional mean parameter $\theta_i = exp(X_i^T\beta)$, where, $\beta^T = [\beta_1,\beta_2 \dots \beta_p]$denotes a $1 \times p$ vector of regression parameters and $X_iap \times 1$ vector, $p = k + 1$.

The interpretation of each coefficient depends on whether the corresponding covariate is categorical or continuous. If the covariates are continuous then $exp(\beta_j)$ represents a multiplicative effect of the X_j on the expected mean (Liao, 1994).

There are two main approaches for interpreting coefficients in regression models (Long, 1997). The first approach examines the changes in the conditional mean for a unit change in a single predictor via the additive or the multiplicative model. The second approach used in non-linear regression models is to examine directly the relationship between the fitted values and the changes in a predictor through graphical plots.

The Com-Poisson Distribution

The Conway- Maxwell-Poisson distribution was developed in 1962 by Conway and Maxwell primarily for studying the processes involved in queues. The basic properties of the Conway-Maxwell –Poisson distribution were derived by Shmueli *et al* in 2005. The distribution is a two way parameter extension of the Poisson distribution; it possesses some unique properties such that it can effectively fit count data with varying dispersion levels.

The probability mass function (PMF) of the Com-Poisson for the discrete count Y_i, according to Shmueli *et al.*(2005), is given as:

$$P(y_i\ ;\ \lambda_i, \nu) = \frac{{\lambda_i}^{y_i}}{(y_i!)^{\nu} z(\lambda_i, \nu)} \qquad (2)$$

where

$$Z(\lambda_i, \nu) = \sum_{h=0}^{\infty} \frac{{\lambda_i}^{h}}{(h!)^{\nu}}$$

$\lambda_i > 0, \nu \geq 0, y_i = 0,1,2, \dots$ and $i = 1, 2, \dots, n$

From the Probability Mass Function above λ_i is the parameter representing the mean of the observation, ν is the shape parameter while Z is the normalizing constant. The dispersion parameter in the Com-Poisson formulation is such that $\nu = 1$ represents data that is equally dispersed. $\nu > 1$ represents under-dispersed data while $\nu < 1$ refers to over-dispersed data.

Shmueli *et al.*(2005)used an asymptotic expression to derive the approximation for Z with the mean and variance given as:

$$E(Y) \approx \lambda^{\frac{1}{\nu}} + {}^{1}/_{2\nu} - {}^{1}/_{2} \qquad (3)$$

$$Var(Y) \approx \frac{1}{\nu}\lambda^{\frac{1}{\nu}} \qquad (4)$$

2.0 MATERIALS AND METHODS

A New Parameterization Of The Com-Poisson Distribution

Let $\omega = \lambda^{\frac{1}{\nu}} + \frac{1}{2\nu} - \frac{1}{2}$ which is the approximated mean of the distribution. Guikema and Coffelt

(2008) expressed λ in terms of α, that is, $\lambda = \alpha^{\nu}$ and then model the response via

$$\alpha_i = \exp(X_i^T \beta)$$

Here, we express λ in terms of ω:

$$\lambda = \left(\omega - \frac{1}{2\nu} + \frac{1}{2}\right)^{\nu}.$$

The relationship between Y_i and X_i is modelled via

$$E(Y_i) \approx \omega_i = \exp(X_i^T \beta)$$

The Com-Poisson PMF under our re-parameterization is given as

$$P(y_i |\ \omega_i, \nu) = \frac{\left(\left(\omega_i - \frac{1}{2\nu} + \frac{1}{2}\right)^{y_i}\right)^{\nu}}{(y_i!)^{\nu} z(\omega_i, \nu)} \quad (5)$$

where

$$Z(\omega_i, \nu) = \sum_{h=0}^{\infty} \left(\frac{\left(\omega_i - \frac{1}{2\nu} + \frac{1}{2}\right)^{h}}{h!} \right)^{\nu}. \quad (6)$$

Based on the re-parameterized Com-Poisson distribution given above, we present the formulation of our mixed effect model in the next subsection.

Let y_{ij}denote the jth response for the ith cluster, $i = 1, \ldots, N$and $j = 1, \ldots, n_i$. For each i, conditional on random effect b_i, the y_{ij}, $j = 1, \ldots, n_i$ are assumed to be independent and follow a Com-Poisson (CMP) distribution where the probability mass function of the CMP distribution using our proposed re-parameterization is

$$P(y_{ij} |\ b_i, \omega_{ij}, \nu) = \frac{\left(\left(\omega_{ij} - \frac{1}{2\nu} + \frac{1}{2}\right)^{y_{ij}}\right)^{\nu}}{(y_{ij}!)^{\nu} z\left(\left(\omega_{ij} - \frac{1}{2\nu} + \frac{1}{2}\right), \nu\right)} \quad (6)$$

where,

$$Z(\omega_{ij}, \nu) = \sum_{h=0}^{\infty} \left(\frac{\left(\omega_{ij} - \frac{1}{2\nu} + \frac{1}{2}\right)^{h}}{h!} \right)^{\nu}$$

and

$E[Y_{ij}] \approx \omega_{ij}$ and $Var[Y_{ij}] = \frac{\omega_{ij} - \frac{1}{2\nu} + \frac{1}{2}}{\nu}$ where $\omega_{ij} > 0, \nu \geq 0, i = 1, 2, \ldots, N; y_{ij} = 0, 1, 2, \ldots$.

The heirechical representation of our CMPMM formulation is,

$$y_{ij} |\ b_i \sim independent\ CMP(\omega_{ij}, \nu) \quad (7)$$

$$b_i \sim i.i.d\ N(0, \sigma^2)$$

and

$$g\left(E(Y_{ij} |\ b_i)\right) = \log \omega_{ij} = X_{ij}^T \beta + b_i \quad (8)$$

and

$$\omega_{ij} = \exp(X_{ij}^T \beta + b_i). \quad (9)$$

Details of the procedure for obtaining estimates of the parameters in the CMPGLMM can be found in Dikko *et al* (2017).

2.1 Implementation

The Conway-Maxwell Poisson Generalized Linear Mixed effect Model (CMPGLMM) has been implemented using R (R Core Team, 2017). To maximize the likelihood functions, we employ the Bound Optimization BY Quadratic Approximation (BOBYQA) Powell, (2009) algorithm which performs derivative-free bound-constrained optimization using an iteratively constructed quadratic approximation for the likelihood function. The algorithm is very robust for optimizing functions with many parameters. It uses a trust region method that forms quadratic models by interpolation. The algorithm optionally allows constraints to be placed on the parameters. For more details on the algorithm see (Powell, 2009). The BOBYQA algorithm adopted for this work is the one implanted by the nloptr R package version 1.0.4 (Johnson, 2017) which implements an R interface to NLopt. NLopt is a free/open-source library for nonlinear optimization which provides a common platform for a number of different free optimization routines as well as original implementations of various other algorithms.

Various R functions were written to carry out specific tasks. For example, the function COMP_Z, Q and likfncompute $z(\theta_{ij}, \nu)$, $Q(b_i)$ and $\ell_p(\beta, \hat{\nu}, \hat{\sigma})$ (the profiled loglikelihood for β respectively). The main R function that is called to fit a CMPGLMM given a dataset is cmpfitme. Calling the function will make calls to various necessary functions and return the estimated coefficients, random effects variance as well as standard errors. The function allows the response variable, predictors as well as the clustering variables to be specified. An example of the function usage is

```
Cmpfitme(No_casualties~month+Age+Gender+
Cause+Type_Accident+Nature_road+(1|location
)+(1|Type_Vehicle),data=cdat)
```

In the above example, No_casualties is the count response variable, month, Age, Gender, Cause, Type_Accident and Nature_road are the predictors while location and Type_Vehicle are the clustering variables which will constitute random effects terms.

SIMULATION STUDIES

The performance of the CMPGLMM for estimation at various sample sizes, dispersion level is examined through empirical simulations

vis-à-vis other clustered count modelling methods such as the Poisson GLMM (PGLMM) and the negative binomial GLMM (NBGLMM). All simulations and computations were carried out using R(R Core Team, 2017).

Simulation Setting

The true underlying model from which we simulate data is a model with one clustering variable and is given by

$$E(y_{ij}|X_{1ij}, X_{2i}, b_i) = \theta_{ij} = exp(\beta_0 + X_{1ij}\beta_1 + X_{2ij}\beta_2 + b_i), \quad b_i \sim N(0, \sigma) \quad (10)$$

, $i = 1, \ldots, m$, $j = 1, \ldots, n_i$.The parameters of the model were set as follows: $\beta_0 = 0.2$, $\beta_1 = -2$, $\beta_2 = 0.3$ and $\sigma = 1$. The number of clusters was varied as $m \in \{5, 10\}$ and the number of observations per cluster was set as $n_i \in \{5, 10\}$. Hence, the sample size setting considered are: $m = 5$, $n_i = 5$ (total number of observations $n = 25$); $m = 5$, $n_i = 10$ (total number of observations $n = 50$); $m = 10$, $n_i = 10$ (total number of observations $n = 100$); Furthermore, the predictors were generated as follows: $X_1 \sim N(0,1), X_2 \sim Unif(0,2)$.

The under-dispersed distribution considered is the double Poisson (DPOIS) distribution (Efron, 1986; Ridout and Besbeas, 2004). The under-dispersed responses were simulated such that $y_{ij} \sim DPOIS(\theta_{ij}, 0.3)$.

This model has three fixed effects parameters (β_0, β_1 and β_2), adding the random effects variance parameter σ^2 makes the total number of parameters to be four. It is important to note that there is only one random effects term in the model under consideration, therefore m random effects will be estimated for each case.

Estimates of the Poisson GLMM (PGLMM) and the negative binomial GLMM (NBGLMM) were obtained using the algorithms implemented in the lme4 R package while the CMPGLMM estimates were obtained using our own R implementation. The performances of the methods are evaluated over 100 replications of each setting discussed above. The evaluation criteria are: average estimation error (AE_j) defined as $E(|\hat{\beta}_j - \beta_j|) = \frac{\Sigma|\hat{\beta}_j - \beta_j|}{100}$; mean-squared errors of estimates (MSE_{β_j}) defined as $E\left([\hat{\beta}_j - \beta_j]^2\right) = \frac{\Sigma(\hat{\beta}_j - \beta_j)^2}{100}$, $j = 0,1,2$. Similarly, estimation of σ is also evaluated.

3.0 RESULT AND DISCUSSION

The results of the application of the techniques and simulation are presented in Table 1. Only the estimates of the major parameters (fixed effects and random effects standard deviation) are reported here.

The simulation results show that the CMPGLMM performed better and yielded better results than the PGLMM and NBGLMM when the correlated count data are under-dispersed. The simulation results also show that the estimate of the dispersion parameter $\hat{\boldsymbol{\nu}}$ of the CMPGLMM varies according to the nature of dispersion exhibited by the count data. For example, the average estimates $\hat{\nu}$ for $m = 10$ (10 clusters) and $n_i = 10$ (10 observations per cluster) under-dispersion is 3.807. This implies that during modelling of clustered count data, using the CMPGLMM the method detects the type of dispersion and fits the corresponding model. Furthermore, the average estimated dispersion parameter $\hat{\boldsymbol{\nu}}$ of the CMPGLMM is 4.009 for small sample size setting and 4.14 at the other sample size settings showing that the response data are highly under-dispersed. The CMPGLMM produced the lowest estimation and mean square errors for all parameters at all the different sample size settings. Also, the CMPGLMM produced the lowest errors for σ at all the sample size settings ($m = 5$, $n_i = 5$), ($m = 5, n_i = 10$) and ($m = 10, n_i = 10$).

Table 1:*Average estimation errors (**AE**) and mean squared errors of estimation (**MSE**) for underdispersion based on 100 replications over three different sample size settings.*

Sample Size setting	***Method***	**Avergae Estimation Error**				**MSE**			
		$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\sigma}$	$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\sigma}$
$m = 5, n_i = 5$	***PGLMM***	0.858	0.061	0.083	0.394	50.114	0.374	0.169	0.377
	NBGLMM	0.858	0.061	0.083	0.394	50.113	0.374	0.169	0.377
	CMPGLMM ($\bar{\hat{\boldsymbol{\nu}}} = 4.009$)	0.768	0.056	0.078	0.214	50.011	0.371	0.163	0.309

$m = 5$, $n_i = 10$	***PGLMM***	2.732	0.185	0.152	0.741	12.400	0.052	0.037	0.732
	NBGLMM	2.732	0.185	0.152	0.741	12.400	0.052	0.037	0.732
	CMPGLMM ($\bar{\hat{\nu}} = 4.14$)	2.639	0.136	0.146	0.207	11.495	0.028	0.035	0.066
$m = 10$, $n_i = 10$	***PGLMM***	1.862	0.087	0.105	0.499	6.812	0.012	0.021	0.313
	NBGLMM	1.861	0.087	0.105	0.499	6.812	0.012	0.021	0.313
	CMPGLMM ($\bar{\hat{\nu}} = 3.807$)	1.855	0.074	0.105	0.261	6.718	0.009	0.021	0.107

4.0 CONCLUSION

The implementation of the CMPGMM has been discussed. The performance of the Com-Poisson Generalized Linear Mixed Effects Model (CMPGLMM) has been evaluated compared to the Poisson and Negative Binomial linear mixed effects models (PGLMM and NBGLMM respectively) via Monte Carlo studies. The simulation result shows that our implementation is reliable. The implementation allows both the fixed effects and random effects parameters to be estimated at a relatively good computational cost. Also, the implementation allows the dispersion parameter $\boldsymbol{\nu}$ to be estimated from which one can deduce the type of dispersion.

The results from the simulation show that CMPGLMM produced the best results among the three methods used at different sample size settings, i.e, the model outperform the PGLMM and the NBGLMM this is obviously due to presence of under-dispersion in the response. The result here shows the versatility of CMPGLMM in handling under-dispersion in clustered count data.

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Investigation of Some Features Inherent in Analysis of Variance and one of its Variants in the Multivariate Setting

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ABSTRACT
It is common to have experiments conducted with the quest to seek an option in the use of analysis of variance (ANOVA) as incontrovertible choice among experimenters. The validity of this quest has been responsible for the considerable attention that has been given to the use of ANOVA. Different Types of experiments may throw up one or more response variable. With one response variable, the ANOVA option is indeed appropriate. However, for the simplicity of ANOVA the experimenter may have to contend with the somewhat a variant of it, even when the use of more than one response variable becomes expedient. The foregoing situation will be in violation of some well establishment ground norm of the need to overcome burdensome computations. This paper therefore presents a number of situations for the purpose of unveiling the inherent forms of statistics that are required to be computed. Although these statistics may appear to be burdensome or complicated in forms as they may present themselves in the setting of Multivariate ANOVA (MANOVA), which in reality has the other side of considerable reduction in the number of various statistics that are required to be computed. Such statistics are the sums of squares and sum of products for the two distinct independent components, in both the treatment and error, that the total sums of squares and products (SSPM) in the MANOVA has to be split into. Ditto, for the ANOVA that has its total sums of squares (SS), to be split into two orthogonal parts of treatment and error respectively. Furthermore, the appropriate number of independent components in which the treatment component will be partitioned is indeed determined by the number of factors say k, with the corresponding labels A_1, A_2, … , A_k, that the experimenter may have to adopt. A number of examples were presented for the illustration of distinct features inherent in both the ANOVA and MANOVA.

Keywords: ANOVA; MANOVA; Correction Factor; Independent Components; Structured and Unstructured Treatments; Sum of squares and Products.

1. INTRODUCTION
When three or more treatments are to be compared in a study, the student's 't' test becomes inefficient. The test procedure that is suitable for comparing three or more treatments simultaneously is the analysis of variance test in preference to the Student's 't' test. In designed experiments, the concern of the experimental is on the treatment effect as to whether it is significant or not, as this will make it possible as to whether all the treatments are of approximately the same effect. The implication of this is whether one or more of the treatments will be preferred to the others in terms of their yield potentials.

Further, treatments that are either structured or unstructured can conveniently be analysed using the analysis of variance test. Structured treatments will arise, when two or more factors are being studied simultaneously and a combination of levels of the two or more factors defines a treatment.

MANOVA as a variant of ANOVA satisfy all the features inherent in the ANOVA as described in the preceding part of this section with the replacement of a single response variable say Y with two or more response variables, say p response variables

$Y_1, Y_2, \ldots, Y_p$. For the purpose of simplicity and adequate attention to the main concern of this paper, equal replications for the treatments is assumed, with its attendant expectation of equal variances, which is inversely proportional to precision for the treatment estimates. This is with the usual stake of equal preference for all the treatments by the experimenter. Indeed ANOVA evaluates the mean difference in respect of a single dependent variable, while MANOVA deals with differences in mean vectors of two or more dependent variables, see for example, Korstarije (2019). Furthermore, Bobbitt (2021) emphasised that MANOVA is suitable in the testing of multiple dependent variables at a time, and thereby provides a viable option that adequately addresses the limitation of ANOVA.

The availability of homogeneous experimental material leads to the appropriateness of splitting of the total variation in a given experimental data into two orthogonal components of treatments and error, in contrast to when the experimental material is heterogeneous. For heterogeneous experimental material splitting of total variation in experimental data has three components, and these are ascribable to treatment, block, and error respectively, see for example Adeleke (2002). Indeed, heterogeneous experimental material is with attendant dissimilarity among its possible sub-divisions. Although heterogeneous experimental material may not be avoidable, when large number of treatments is to be compared. The appropriate experimental design for homogeneous experimental design is the completely randomised design (CRD). In tandem with the concern in this paper, the CRD can be used with respect to the following settings: one or more factors; and one or more response variables, leading to the following four options. Option (i) – one factor with one response variable; option (ii) – one factor with more than one factor; option (iii) – two or more factors with a single response variable; and option (iv) – two or more factors with more than one response variable.

The models that are suitable for homogeneous experimental material, which is of concern in this paper are given as Equations (1) and (3). However, equation (1) is referred to as ANOVA model, while Equation (3) is the MANOVA model, see for example Timm (1975).

$$Y_{ij} = \mu + \alpha i + \varepsilon_{ij} \quad i = 1, \ldots, t; j = 1, \ldots, r \qquad (1)$$

$$\varepsilon_{ij} \sim NIID(0, \sigma^2) \qquad (2)$$

The $Y'_{ij}s$ and $\varepsilon'_{ij}s$ have the same variance because the two differ by a constant and, since $E(Y_{ij}) = E(\varepsilon_{ij}) = 0$. The random variable ε_{ij} denotes the random error component of Y_{ij}; by Equation (2), the error are uncorrelated and normally distributed with mean zero and common variance σ^2 ; α_i denotes the effect of treatment 'i'; and μ denotes the overall mean, with all the terms being scalar quantities.

It is of note that the MANOVA model (3) is a replica of equation (1), except for the fact that the four distinct terms in the model are no longer scalars, but rather vectors $\mathbf{Y_{ij}}$, $\mathbf{1}$, $\boldsymbol{\alpha}_i$, $\boldsymbol{\varepsilon}_{ij}$, all being of dimension px1 respectively.

$$\boldsymbol{Y_{ij}} = \mathbf{1}\mu + \boldsymbol{\alpha i} + \boldsymbol{\varepsilon_{ij}} \quad i = 1, \ldots, t; j = 1, \ldots, r \qquad (3)$$

Section 2 of this paper presents the treatment structure, while sections 3 and 4 are concerned with data format and the procedure for partitioning of total variation into two distinct components namely, the treatments and error respectively. Section 5 deals with discussion of results and the concluding section to the study in focus in this paper is presented in section 6.

2. TREATMENT STRUCTURE

Of the two components into which the total variation be split into, the treatment are of immense priority to the experimenter. However, the use of treatment structure is a concept that originated from studies that are concerned with one or more factors. When a single factor is of choice, the resulting treatments are considered to be unstructured. This is in in concurrence with the fact that

treatment specification is without any difficulty, in the sense that each of the factor levels defines a treatment.

2.1 Single Factor Setting

For a single factor the treatments are considered to be unstructured and a typical format is presented in Table 2.1 below.

Table 2.1: Specification of treatments in a single factor study

Factor	Levels	Treatments
A	1 2 . . . m	1 2 . . . m

2.2 k Factors Setting

When k factors are of concern in an experiment each of possible treatments is specified not as a distinct levels of each of the factors, but rather as a combination of a level each of the k factors. This is demonstrated in Table 2.2 below.

Table 2.2: Specification of treatments for k factors

Factor	**Number of levels**	**Levels**
Factor A_1	**m_1**	**1 2 … m_1**
Factor A_2	**m_2**	**1 2 … m_2**
.	.	.
.	.	.
.	.	.
Factor A_k	**M_k**	**1 2 … m_k**

It is of note that the number of treatments or treatment combinations that can be formed with k factors is m_1 x m_2 x . . . x m_k, where m_i denotes the number of levels associated with factor A_i. Consequently, for a case of two factors, such that k = 2, and m_1= 3, m_2 = 4, and therefore m_1 x m_2 =12, a full specification of the twelve treatments or treatment combinations is presented in Table 2.3. Some further examples of full specification of structured treatments are also presented below in Table 2.3, in addition to the case with k = 2, and m_1= 3, m_2 = 4.

Table 2.3: Treatment Combinations for k =2, 3, 4

2x3	2x2	3X4	2x2x3	2x2x2x2
11	11	11	111	1111
12	12	12	112	1112
13	13	13	113	1121
21	22	14	121	1122
22	22	21	122	1211
…	…	22	123	1212
…	…	23	211	1221
…	…	24	212	1222
…	…	31	213	1111
…	…	32	221	1112
…	…	33	222	1121
…	…	34	223	1122
…	…	…	…	1211
…	…	…	…	1212
…	…	…	…	1221
…	…	…	…	1222

3. DATA FORMAT

Specification of treatments and the corresponding effects are of primary interest to the experimenter. However, the treatment effects can only be subjected to the test of significance with availability of data. Data format is specified without any ambiguity, when the number of factors and the number of response variables are specified. In this paper, cases of one or more factors in conjunction with the options of one and more than one response variables are considered in examining the inherent features of both ANOVA and MANOVA. In what follows the data format for the treatment structure given in Table 2.3 for m_1= 3, m_2 = 4 , is presented in Table 3.1 below. Treatment structures for other cases in Table 2.3 can be obtained in a similar manner.

Table 3.1: Data Format for Two Factors, with $m_1 = 3$, $m_2 = 4$

Treatments or treatment combinations	1	2	3	4	5	6
l^{th} Observations	Y_{111}	Y_{121}	Y_{131}	Y_{141}	Y_{211}	Y_{221}
	Y_{112}	Y_{122}	Y_{132}	Y_{142}	Y_{212}	Y_{222}
	.	.	.	.	.	.
	.	.	.	.	.	.
	.	.	.	.	.	.
	Y_{11r}	Y_{12r}	Y_{13r}	Y_{14r}	Y_{21r}	Y_{22r}

Treatments or treatment combinations	7	8	9	10	11	12
l^{th} Observations	Y_{321}	Y_{241}	Y_{311}	Y_{321}	Y_{331}	Y_{341}
	Y_{232}	Y_{242}	Y_{312}	Y_{322}	Y_{332}	Y_{342}
	.	.	.	.	.	.
	.	.	.	.	.	.
	.	.	.	.	.	.
	Y_{23r}	Y_{24r}	Y_{13r}	Y_{14r}	Y_{21r}	Y_{22r}

4. PARTITIONING OF TOTAL VARIATION INTO TWO DISTINCT COMPONENTS OF TREATMENTS AND ERROR

In designed experiments, there are two possibilities that available experimental materials may display in terms of their nature of homogeneity and heterogeneity respectively. One of the two options that will be considered in this paper is the type in which the experimental material is homogeneous. Consequently, the total variation in the resulting ANOVA and MANOVA, as the case may, can be split into only two independent components with respect to the treatment and error, as in the abridged ANOVA and MANOVA tables which are presented below as Tables 4.1 and 4.2.

4.1. Enumeration of Summary Statistics for the Treatments Effects

Data from designed experiments are usually analysed for the purpose of obtaining relevant and/ or required summary statistics that will facilitate the evaluation of the treatment effects. In what follows, consideration will be given to the summary statistics that are of main concern in both ANOVA and MANOVA, and these are, the treatments sums of squares and products, as well as the associated correction factors.

Meanwhile, the usual format for the abridged ANOVA and MANOVA tables are presented below as Tables 4.1 and 4.2 respectively.

Table 4.1: Abridged ANOVA Table

S.V	D.F	SS
Treatments	t – 1	Scalar
Error	n – t	Scalar
Total	n – 1	Scalar

Table 4.2: Abridged MANOVA Table

S.V	D.F	SSPM
Treatments	t – 1	pxp Square Matrix
Error	n – t	pxp Square Matrix

Total	n – 1	pxp Square Matrix

Summary statistics for the treatment effects that are of main importance in this paper are: for MANOVA, the sums of squares and products and their corresponding correction factors; and for ANOVA, the sums of squares and the corresponding correction factors. In Tables 4.3 and 4.4, the number of summary statistics for the following combinations of number of factors (k), and number of response variables (p), that is (k = 1, p = 1); (k = 2, p = 1); (k = 1, p = 3); (k = 2, p = 1); k = 1, p = 3; k = 1, p = 2; (k =3, p = 1);; (k = 2, p = 2); and (k = 2, p) are presented.

Table 4.3: Keeping Track or Counting the Number of Summary Statistics for ANOVA Table (p = 1)

Summary Statistics	Single factor A_1	Two Factor A_1, A_2	k factors A_1, A_2,.., A_k
Sums of squares	1	3	$\binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{k} = 2^n - 1$
Correctional factors	1	1	1
Sums of products	N/A	N/A	N/A

Table 4.4: Keeping Track or Counting the Number of Summary Statistics for MANOVA (k = 1, 2)

Summary Statistics	**Single factor**		**Two Factor A_1, A_2**		**k factors A_1, A_2,…, A_k**
	2 response	p response variable	2 responses variable	p response variable	p response variable
Sums of squares	2	p	6	3p	$\binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{k}$ - 1
Sums of products	1	$\binom{p}{2}$	3	$3\binom{p}{2}$	$\binom{p}{2}$
Correctional factors	2	$p + \binom{p}{2}$	3	p	$\binom{p}{2}$

4.2. Data Format for Specific Values of p and k

In general, the following combinations of number of factors and number or response variable are considered in this paper, and the number of appropriate summary statistics that is required for each of the data structure are presented in Tables 4.5 to 4.12. The resulting values are in agreement with the general rule presented in Tables 4.3 and 4.4.

Table 4.5: Data set for k =1, p = 1

Factor A	Level	Observation			
	1	Y_{11}	Y_{12}	Y_{13}	Y_{14}
	2	Y_{21}	Y_{22}	Y_{23}	Y_{24}
	4	Y_{31}	Y_{32}	Y_{33}	Y_{34}
		Y_{41}	Y_{42}	Y_{43}	Y_{44}

Here, there is only one (1) sum of squares, and only one (1) correction factor that are possible, while sums of product is not applicable.

Table 4.6a: k = 2, p = 1

		Factor A			
		Levels			
Levels		1	2	3	4
Factor B	1	Y_{111} Y_{112}	Y_{121} Y_{122}	Y_{131} Y_{132}	Y_{141} Y_{142}
	2	Y_{211} Y_{212}	Y_{221} Y_{222}	Y_{231} Y_{232}	Y_{241} Y_{242}
	3	Y_{311} Y_{312}	Y_{321} Y_{322}	Y_{331} Y_{332}	Y_{341} Y_{342}

Here, there are three (3) sums of squares, and only one (1) correction factor that are possible, while sums of product is not applicable.

Table 4.6b: k = 2, p = 1

		Factor A			
		1	2	3	4
Factor B	1	Y_{11}	Y_{11}	Y_{11}	Y_{11}
	2	Y_{11}	Y_{11}	Y_{11}	Y_{11}
	3	Y_{11}	Y_{11}	Y_{11}	Y_{11}

Here, there are two (2) sums of squares, and only one (1) correction factor that are possible, while sums of product is not applicable.

Table 4.7: k = 1, p = 3

Response value	Factor A 1	2	3
Y_1 Y_2 Y_3	$\begin{pmatrix} Y111 \\ Y112 \\ Y113 \end{pmatrix}$	$\begin{pmatrix} Y211 \\ Y212 \\ Y213 \end{pmatrix}$	$\begin{pmatrix} Y311 \\ Y312 \\ Y313 \end{pmatrix}$
Y_1 Y_2 Y_3	$\begin{pmatrix} Y121 \\ Y122 \\ Y123 \end{pmatrix}$	$\begin{pmatrix} Y221 \\ Y222 \\ Y223 \end{pmatrix}$	$\begin{pmatrix} Y321 \\ Y322 \\ Y323 \end{pmatrix}$
Y_1 Y_2 Y_3	$\begin{pmatrix} Y131 \\ Y132 \\ Y133 \end{pmatrix}$	$\begin{pmatrix} Y231 \\ Y232 \\ Y233 \end{pmatrix}$	$\begin{pmatrix} Y331 \\ Y332 \\ Y333 \end{pmatrix}$

Here, there are three (3) sums of squares, six (6) correction factors, and three (3) sums of products are possible.

Table 4.8: k = 1, p = 2

Response value	Factor A 1	2	3
Y_1 Y_2	$\begin{pmatrix} Y111 \\ Y112 \end{pmatrix}$	$\begin{pmatrix} Y211 \\ Y212 \end{pmatrix}$	$\begin{pmatrix} Y311 \\ Y312 \end{pmatrix}$
Y_1 Y_2	$\begin{pmatrix} Y121 \\ Y122 \end{pmatrix}$	$\begin{pmatrix} Y221 \\ Y222 \end{pmatrix}$	$\begin{pmatrix} Y321 \\ Y322 \end{pmatrix}$
Y_1 Y_2	$\begin{pmatrix} Y131 \\ Y132 \end{pmatrix}$	$\begin{pmatrix} Y231 \\ Y232 \end{pmatrix}$	$\begin{pmatrix} Y331 \\ Y332 \end{pmatrix}$

Here, there are two (2) sums of squares, three (3) correction factors, and only one (1) sum of products are possible.

Table 4.9: k = 2, p = 1

	Factor A (j) 1	2
1 Factor B 2 (i) 3	Y_{111}, Y_{112}	Y_{121}, Y_{122}
	Y_{211}, Y_{212}	Y_{221}, Y_{222}
	Y_{311}, Y_{312}	Y_{321}, Y_{322}

Here, there are three (3) sums of squares, and only one (1) correction factor that are possible, while sums of product is not applicable.

Table 4.10: k = 2, p = 2

Factor B	Response	Factor A 1	2
1	Y_1 Y_2 Y_3	$\begin{pmatrix} Y1111 \\ Y1112 \\ Y1113 \end{pmatrix}$	$\begin{pmatrix} Y1211 \\ Y1212 \\ Y1213 \end{pmatrix}$
	Y_1 Y_2 Y_3	$\begin{pmatrix} Y1121 \\ Y1122 \\ Y1123 \end{pmatrix}$	$\begin{pmatrix} Y1221 \\ Y1222 \\ Y1223 \end{pmatrix}$
2	Y_1 Y_2 Y_3	$\begin{pmatrix} Y1111 \\ Y1112 \\ Y1113 \end{pmatrix}$	$\begin{pmatrix} Y2211 \\ Y2212 \\ Y2213 \end{pmatrix}$
	Y_1 Y_2 Y_3	$\begin{pmatrix} Y1121 \\ Y1122 \\ Y1123 \end{pmatrix}$	$\begin{pmatrix} Y2221 \\ Y2222 \\ Y1123 \end{pmatrix}$

Here, there are nine (9) sums of squares, twelve (12) correction factors, and three (3) sum of products are possible.

Table 4.11: k = 3, p = 1

	Factor A (i)					
	1			2		
	Factor B (j)			Factor B (j)		
	1	2	3	1	2	3
1 Factor C 2 3 (l)	Y_{III}, Y_{III2}	Y_{I2II}, Y_{I2I2}	Y_{I3II}, Y_{13I2}	Y_{2III}, Y_{2II2}	Y_{22II}, Y_{22I2}	Y_{23II}, Y_{23I2}
	Y_{II2I}, Y_{II22}	Y_{I22I}, Y_{I222}	Y_{I32I}, Y_{I322}	Y_{2I2I}, Y_{2I22}	Y_{222I}, Y_{2222}	Y_{232I}, Y_{2322}

	Y_{II3I}, Y_{II32}	Y_{I23I}, Y_{I232}	Y_{I33I}, Y_{I332}	Y_{2I3I}, Y_{2I32}	Y_{223I}, Y_{2232}	Y_{233I}, Y_{I332}

Here, there are seven (7) sums of squares, and only one (1) correction factor that are possible, while sums of product is not applicable.

Table 4.12: k = 2, p

Factor B (j)	Response	Factor A (j) 1	2
1	Y_1 Y_2 . . . Y_p	$\begin{pmatrix} Y1111 \\ Y1112 \\ . \\ . \\ . \\ Y111p \end{pmatrix} \cdots \begin{pmatrix} Y11r1 \\ Y11r2 \\ . \\ . \\ . \\ Y11rp \end{pmatrix}$	$\begin{pmatrix} Y1211 \\ Y1212 \\ . \\ . \\ . \\ Y121p \end{pmatrix} \cdots \begin{pmatrix} Y12r1 \\ Y12r2 \\ . \\ . \\ . \\ Y12rp \end{pmatrix}$
2	Y_1 Y_2 . . . Y_p	$\begin{pmatrix} Y2111 \\ Y2112 \\ . \\ . \\ . \\ Y211p \end{pmatrix} \cdots \begin{pmatrix} Y21r1 \\ Y21r1 \\ . \\ . \\ . \\ Y21r1 \end{pmatrix}$	$\begin{pmatrix} Y2211 \\ Y2212 \\ . \\ . \\ . \\ Y221p \end{pmatrix} \cdots \begin{pmatrix} Y22r1 \\ Y22r2 \\ . \\ . \\ . \\ Y22rp \end{pmatrix}$

Here, there are '4p' sums of squares, 'p + $3\binom{p}{2}$ correction factors, and $\binom{p}{2}$ sum of products are possible.

5. DISCUSSION OF RESULTS

Treatment components with both structured and unstructured forms have been examined in terms of the various sums of squares that are expected to be calculated and the number of distinct correction factors that are utilised for cases with one or more factors.

For MANOVA, the corresponding treatment components with one or more factors in conjunction with two or more response variables have been adequately examined. Suitable examples were provided for the following settings: (k = 1, p = 2), (k = 1, p = 3), (k = 2, p = 2), and (k = 2, p). The number of possible sums of squares; sums of product, that are required to be calculated is specified, and the number of distinct correction factors that are utilized or used is provided. For instance, when the option of one factor and two response variables are considered, the following numbers of summary statistics that are possible: two (2) sums of squares, one (1) sum of products, and three (3) distinct number of correction factors.

In the case of ANOVA, treatment components were based on one response variable. This, by convention was implemented in conjunction with the adoption of one or more factors. The following combinations of the values of k and p were examined: (k = 1, p = 1); (k = 2, p = 1); (k = 2, p = 1); and (k =3, p = 1). For (k =3, p = 1), the possible summary statistics are: seven (7) sums of squares; one (1) correction factor; and sums of product is not applicable.

Overall, eight data structures with varied values of k and p are presented in Tables 4.5 to 4.12 for the purpose of illustration of the main concern of this paper.

6. CONCLUSION

With the aid of suitable examples, this paper has presented the required summary statistics and it is evident that the computational efforts that are required, with respect to the treatment effect, in both ANOVA and MANOVA increases rapidly as the number of response variables and the

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number of factors increases. For instance, in ANOVA with one factor and one response variable, one (1) sum of squares and one correction factor are to be calculated. The foregoing is a wide departure from the possible seven sums of squares and one correction factor in a situation with three factors and one response variable. Sums of products are not applicable in ANOVA. For MANOVA, the computation efforts to obtain two (2) sums of squares, three (3) correction factors, and one (1) sum of products in the case with $k = 1$, $p = 2$, is clearly lower than, with $k = 2$, $p = 3$, in that, twelve (12) sum of squares, six (6) correction factor, and three (3) sums of products are required to be calculated.

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