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Edited by:

Prof. Muhammad Lawan Kaurangini, FICA

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Introduction

The Mathematical Association of Nigeria (MAN) is a non-profit and non-governmental academic organization committed to the promotion, advancement, and effective application of Mathematics in Nigeria. Since its establishment, MAN has consistently provided a strong platform for mathematical scientists, educators, researchers, and students to exchange ideas, foster collaborations, and contribute to national development through Mathematics.

One of the flagship activities of the Association is the **Annual National Conference**, which serves as a convergence point for members and stakeholders across the country. Beyond the presentation and discussion of scholarly ideas in Mathematics, the conference also emphasizes the training and development of the younger generation of mathematicians through workshops, quizzes, Olympiads, and mentorship programs. In this way, MAN continues to nurture mathematical talent and promote the relevance of Mathematics to national growth and global competitiveness.

The **2025 Annual National Conference**, hosted by **Usmanu Danfodiyo University, Sokoto**, was held under the theme: **"Rethinking Teaching and Learning of Mathematics as Tools for Creativity, Technological Innovation, Self-Reliance and Sustainable Development in Times of Economic Depression."** This theme reflects the urgent need to reposition Mathematics education as a catalyst for creativity, innovation, and resilience, particularly in the context of the socio-economic challenges facing Nigeria and the global community.

The conference brought together a wide range of participants including seasoned academics, teachers, students, policymakers, and young researchers. It provided a forum for presenting research papers, sharing innovative teaching methodologies, and exploring new applications of Mathematics in science, technology, and sustainable development.

This volume of **Proceedings** contains carefully reviewed papers and contributions presented during the conference. The diverse range of topics covered demonstrates not only the vitality of Mathematics as a discipline but also its indispensable role in driving technological progress, fostering self-reliance, and addressing pressing societal challenges.

The Association extends its sincere appreciation to the Local Organizing Committee, the host institution—Usmanu Danfodiyo University, Sokoto—authors, reviewers, and all participants for their invaluable contributions to the success of the 2025 Annual National Conference. It is our hope that the ideas, discussions, and innovations captured in these proceedings will inspire further research, strengthen Mathematics education in Nigeria, and support national development in meaningful ways.

Professor Muhammad Lawan Kaurangini
Editor-in-Chief

Table of Contents

Nirmala And Sombor Energy Of A Mixed Hourglass Graph
Kekere et al 1

A New Dimensional Type Hybrid Block Scheme Of Linear Multistep Method For Solving Stiff Systems
A.I. Bakari 6

A Survey On The Features Of Non-subgoup Cosets In The Factor Group Of Integers By Its Infinite Subgroups /*m¢*
Amina Muhammad Lawan 21

Solutions To Restricted Three-body Problem
Blessing et al 27

Computational Technique For Solving Fractional One-dimensional Hyperbolic Telegraph Equation
Falade 39

Prognostic Gamma Generalized Regression Modelling Of Determinants Influencing Variations In Under-5 Mortality
Joseph et al 52

Application Of Grey System Models For Energy Demand Forecasting And Load Balancing In Nigeria's Smart Grid Development
Adaji et al 80

Bayesian Spatial Analysis Of Diarrhoea Among Under-five Children In Nigeria: Unveiling Regional Disparities And Risk Factors
Aminu et al 91

Enhancing Data Security Through A Hybrid Approach: Integrating Transposition, Hill And Rsa Ciphers
Muhammad et al 102

Utilizing Sieve Of Eratosthenes For Generating Prime Number Patterns To Enhance Rsa Efficiency
Abubakar et al 113

Model Of A Relativistic R3bp Of The Collinear Equilibrium Points With A Bigger Oblate Primary
Tajudeen et al 124

Recommending Innovative Pedagogical Strategies For Transforming Learning Outcomes Through Technology And Active Learning
Danlami et al 142

Numerical Comparison Of Homotopy Perturbation And Modified New Iterative Method For Solving A 3-dimensional Steady-state Laplace Equation In Cartesian Coordinate
Muazu et al 153

Formulation Of A Predictor Corrector Linear Multistep Method In Solving First-order Ordinary Differential Equations (ode)
Yahaya et al 165

Nonlinear Contraction Mappings In Relation-theoretical Contractions With Application To Elastic Beams On Nonlinear Elastic Foundations
Muhammed et al 180

Optimizing Production Efficiency: A Linear Programming Approach At Nasco Foods Nigeria Limited
Koko et al 188

Arithmetic Return Model For Removing Autocorrelation From Statistical Process Control Data Exhibiting Geometric Brownian Motion
I. J. Dike et al 195

Prognostic Gamma Generalized Regression Modelling Of Determinants Influencing Variations In Under-5 Mortality
Joseph et al 202

The Dynamics Of A Probe Body In The Cr3bp With Triaxial Primaries
Joel JohnTaura 229

Integration Of Artificial Intelligence (ai) In Mathematics Education
Temitope et al 245

Application Of Mathematics In Financial Technologies, Financial Literacy, And Empowering Resilience
Abubakar et al 256

Effectiveness Of The Gamification Approach On Senior Secondary School Students' Achievement And Interest In Set Theory, Sokoto State, Nigeria
Aliyu et al 265

Model Of A Relativistic R3bp When The First Primary Is Oblate And The Second Primary Is Radiating
Tajudeen et al 275

Rethinking Mathematics Education As A Tool For Enhancing Creativity, Innovation, And Sustainable Development In Times Of Economic Hardship
Bello et al 288

Effect Of Zoom- Classroom Instructional Strategy On Students' Academic Achievement In Differential Equations In Federal University Of Education Zaria, Nigeria
Muawiya et al 298

Alleviating Students' Algebraic Misconceptions: An Application Of 7e's Model Among Senior Secondary School In Sokoto State, Nigeria
Moyi et al 311

Artificial Intelligence In Mathematics Education Research: Part I – From Topic Formulation To Literature Search
Yusuf Feyisara Zakariya 334

Artificial Intelligence In Mathematics Education Research: Part Ii – From Research Methods To Manuscript Proofreading
Yusuf Feyisara Zakariya 348

Impact Of Realistic Mathematics Education Stem-oriented Approach For Enhancing Senior Secondary School Students' Achievement In Geometry Representation In North Central, Nigeria
Abubakar et al 361

Adapting Van Hiele Theory In Geometric Thinking Among Senior Secondary Students Achievement In Sokoto State, Nigeria
Nura et al 371

Curriculum Innovations In Mathematics Education For Promoting Entrepreneurial Thinking And Self-employment Skills In Nigeria
Dr. Aliyu et al 390

Peer Influence And Mathematics Performance Of Senior Secondary School Students In Sokoto State, Nigeria
Malami 405

Rethinking Teaching And Learning Of Mathematics As Tools For Creativity, Technological Innovation, Self-reliance, And Sustainable Development In Times Of Economic Depression
Abubakar et al 417

Impact Of Language Of Immediate Environment (lie) On Performance In Mathematics Among Lower Basic Pupils In North-west Zone, Nigeria.
Ya'u Shehu (PhD) and Prof. M. Musa 427

Effects Of Computer-assisted Instruction On Geometry Performance And Retention Among Senior Secondary Students In Kano: Implications For Self-reliance In Times Of Economic Depression
Dr. Haruna et al 442

Impact Of Language Of Immediate Environment (lie) On Interest In Mathematics Among Lower Basic Pupils In North-west Zone, Nigeria.
Shehu Ya'u and Bala Sabiu Barwa 455

NIRMALA AND SOMBOR ENERGY OF A MIXED HOURGLASS GRAPH

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Abstract

The resulting matrix of WH factorization has been represented as a mixed hourglass graph, though not the topological index graph. The topological index graph is defined in terms of the degrees of its vertices, and the energy of the index graph can be obtained. There are over 50 types of graph energies obtained from degree-based topological indices; only the newest energies (Nirmala and Sombor) are considered for the mixed hourglass graph in this paper. The relationship between the Nirmala and Sombor energies is the similarity between their spectra, and that the energies are even

Keyword: *WH factorization, mixed hourglass graph, topological index, graph energy.*

1. INTRODUCTION

A mixed adjacency matrix $M(G) = [m_{ij}]$ *of a mixed graph* G *is an* $n \times n$ *matrix indexed by the vertices* $(v_1, \dots, v_n)$, *where* $m_{ij} = 1$ *if* $v_i v_j \in E(G), m_{ij} = -1$ *if* $v_i v_j \in A(G)$, *and* $m_{ij} = 0$ *otherwise, see (Babarinsa, Sofi, Ibrahim, Kamarulhaili, & Bashir, 2020; Beck, Blado, Crawford, Jean-Louis, & Young, 2015). The mixed adjacency matrix formed the mixed graph. A mixed graph is the orientation of a subset of the undirected graph, with its edge set being the combination of the arcs and undirected edges* $(e_1, \dots, e_n)$ *(Zhang & Kan, 2014). If a finite graph* G *possesses an even number of vertices and odd degree (number of edges* e *) then the degrees of vertices is twice the number of edges. The degrees of vertices form several vertex-degree-based graph invariants called topological indices.*

Index graph is a connectivity index with mathematical formula that can be applied to any graph which models some molecular structure (Randić, 2008). Nowadays we have 200 topological indices of mathematics used in chemistry and other fields. These indices can form adjacency matrices with unique eigenvalues and the energy (Gutman, 2021a). For instance, the Sombor index, $S^*(G)$, *in Equation (1) is constructed by using Euclidean metrics and the Nirmala index,* $N^*(G)$, *is a somewhat simplified variant of the Sombor index.*

$$S^*(G) = \sum_{v_i v_j \in E(G)} \sqrt{d(v_i)^2 + d(v_j)^2} \qquad (1)$$

$$N^*(G) = \sum_{v_i v_j \in E(G)} \sqrt{d(v_i) + d(v_j)} \qquad (2)$$

The spectrum and energy of Sombor and Nirmala can be found in (Gutman, 2021a, 2021b; Kulli, 2021). Ivan Gutman introduced, although originated from theoretical chemistry, the graph energy from an adjacency matrix (Gutman, 2011; Li, Shi, & Gutman, 2012). Therefore, mixed energy $E_M(G)$ *of a mixed graph* G *is the sum of absolute values of eigenvalues,* $\lambda_i(G)$, *of the adjacency matrix of the graph with order* n *(Alawiah, Rad, Jahanbani, & Kamarulhaili, 2018).*

$$E_M(G) = \sum_{i=1}^{n} |\lambda_i(G)|$$

2. BACKGROUND ON MIXED HOURGLASS GRAPH

The concept of mixed hourglass-adjacency matrix was coined from hourglass matrix which is obtained from quadratic interlocking factorization through modified Cramer's rule, see (Babarinsa & Kamarulhaili, 2017, 2019a). An hourglass matrix is a nonsingular matrix of order $n(n \geq 3)$ with nonzero entries from the t th to the $(n-i+1)$ th element of the i th and $(n-i+1)$ th row of the matrix and 0 's otherwise, for$i = 1,2,\ldots,\left\lfloor\frac{n+1}{2}\right\rfloor$ (Babarinsa & Kamarulhaili, 2018). A mixed hourglass-adjacency matrix $M(\mathcal{G})$ of a mixed hourglass graph $\mathcal{G}$ is a matrix $M(\mathcal{G})_{n\times n} = \left(h_{i,j}\right)_{n\times n}$ defined as

$$M(\mathcal{G}) = \begin{cases} 1 & \text{if } v_i, v_j \text{ is an edge} \\ -1 & \text{if } v_i, v_j \text{ is an arc} \\ 0 & \text{otherwise} \end{cases}$$

For $i = 1,2,\ldots,n,\ j = n-1+i$, and $n \geq 3$.

Thus, a mixed hourglass graph $\mathcal{G} = (V, E, A)$ is an ordered triple consisting of a set of vertices $V(\mathcal{G}) = \{v_1, v_2, \ldots, v_n\}$, a set of undirected edges $E(\mathcal{G}) = \{e_1, e_2, \ldots, e_n\}$, and a set of directed arcs $A(\mathcal{G})$ (Babarinsa & Kamarulhaili, 2019b). They gave the total number of undirected edges, $e(\mathcal{G})$, and the number of arcs $A(\mathcal{G})$ in a mixed hourglass graph is

$$e(\mathcal{G}) = \frac{n-\beta_n^*}{2} \quad \text{and} \quad A(\mathcal{G}) = \frac{n^2-2n+\beta_n^*}{2}$$

where $\beta_n^* = |(n+1)\text{mod}2 - 1|$.

Theorem 2.1. (Babarinsa & Kamarulhaili, 2019a) Let $\mathcal{E}_M(\mathcal{G})$ be the mixed energy of a mixed hourglass graph $\mathcal{G}$ of order n. Then

$$\mathcal{E}_M(\mathcal{G}) = n - \beta_n^*$$

where

$$n - \beta_n^* = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

Several results have been established on the mixed hourglass graph. Though the energy of a mixed hourglass graph was obtained; the Nirmala and Sombor energies of the mixed hourglass graph have not been considered.

3. NIRMALA AND SOMBOR ENERGY

To obtain the Nirmala and Sombor energy from the graph indices, we compute the degrees of the vertices, $D_{\mathcal{G}}$, of a mixed hourglass graph as follows.

order						Vertices			
3							1	2	1
4						1	3	3	1
5					1	3	4	3	1
6				1	3	5	5	.3	1
7			1	3	5	6	5	3	1
8		1	3	5	7	7	5	3	1
9	1	3	5	7	8	7	5	3	1

Since the indices can be established based on the degree of the vertices. Thus, the sum of the degrees of the vertices is important to establish the Nirmala and Sombor index and energy through the eigenvalues obtained from the adjacency matrices of their indices, see Table 1, Figure 1 and Figure 2.

Table 1. The values of Nirmala, and Sombor indices and energies

Order	Nirmala index	Sombor index	Nirmala energy	Sombor energy
3	4.87	5.89	2.82	2.82
4	11.86	18.29	7.72	11.30
5	21.64	32.14	7.72	11.30
6	36.14	67.08	14.04	25.44
7	54.08	110.35	14.04	25.44
8	77.07	172.30	21.52	45.24
9	97.87	245.69	21.52	45.24
10	140.36	344.48	30	66.86

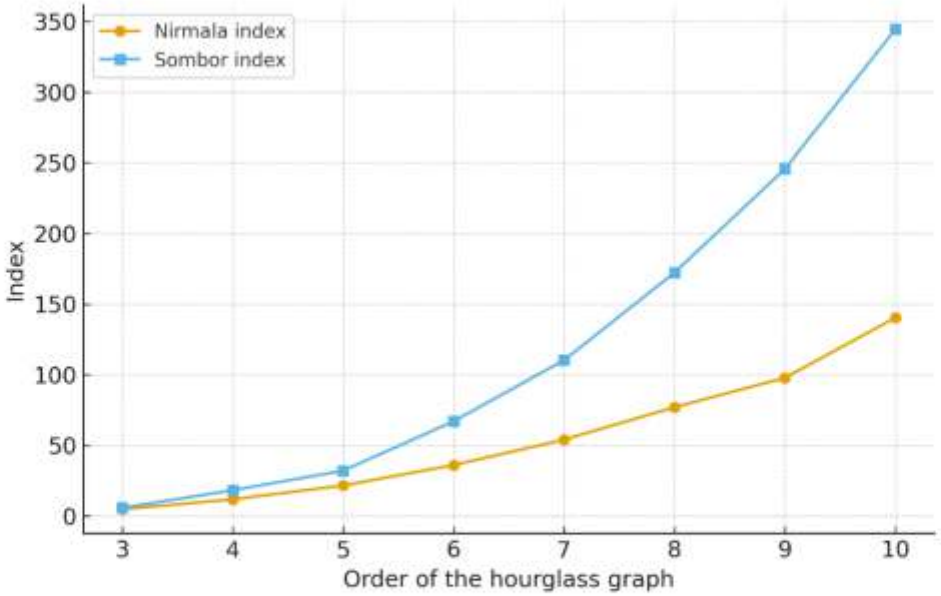


Figure 1. Nirmala and Sombor index of the mixed hourglass graph.

It can be observed, from Figure 1, that Sombor indices is about two-third of the Nirmala indices. The indices are predicator for the matrices as well as the energies of the mixed hourglass graph, see Figure 2.

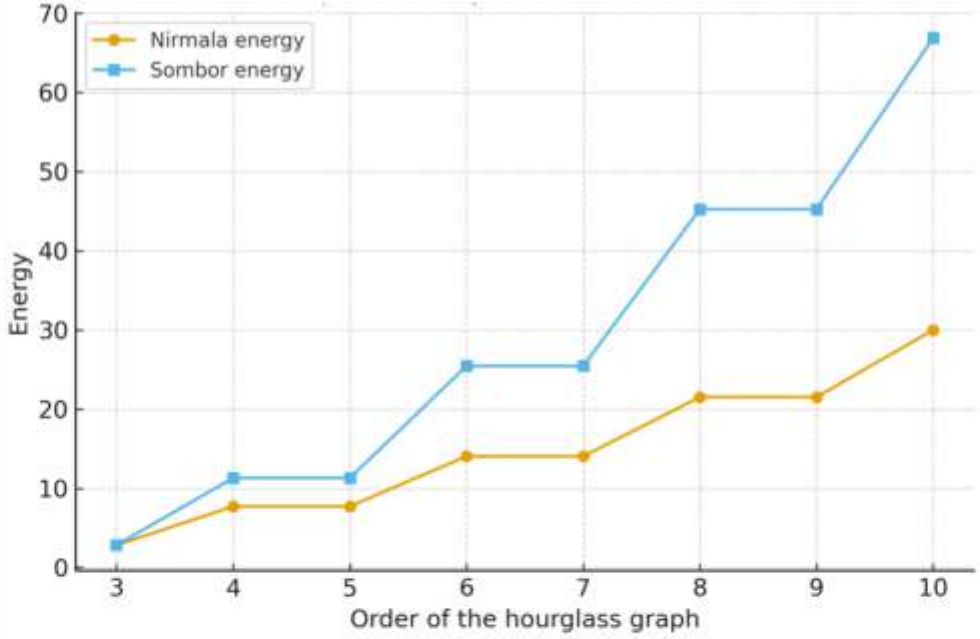


Figure 2. Nirmala and Sombor energy of the mixed hourglass graph.

Figure 1 illustrates the Nirmala energy and Sombor energy as stairs or zigzags due to the similarity in values between the odd and even orders of the graph. However, the mixed energy

from the graph is steep. The higher the order of the graph, the wider the differences between the values of the Nirmala energy and Sombor energy. In all, it can deduce that all values of graph energies are even.

4. CONCLUSION

The Nirmala energy and Sombor energy have decimal values, yet even. The occurrence of stairs between the consecutive even and the odd orders of the Nirmala energy and Sombor energy showed the similarities among spectrum of the graph.

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A NEW DIMENSIONAL TYPE HYBRID BLOCK SCHEME OF LINEAR MULTISTEP METHOD FOR SOLVING STIFF SYSTEMS

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Abstract

In this paper, A newly Constructed hybrid block method first derivative linear multistep method for solving systems of first order ordinary differential equation have been proposed. The development of the method employed the power series as the basis function for the approximate solution. The techniques of interpolation and collocation were adopted for the derivation of the method. The single continuous formulation derived is evaluated at off-grid points $x = x_{n+j} j = \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{3}{2}, \frac{13}{8}, \frac{7}{4}$, *and at grid-point* $x = x_{n+j} j = 1,2$ *. Yielded the multi discrete schemes that form a self-starting of uniform order eleven (11) block method with Error Constants*$[2.88205 \times 10^{-11}, 2.83019 \times 10^{-11}, 2.85444 \times 10^{-11}, 2.74030 \times 10^{-11}, 2.85106 \times 10^{-11}, 2.74030 \times 10^{-11}, 2.85444 \times 10^{-11}, 2.83019 \times 10^{-11}, 2.88205 \times 10^{-11}, 0]^T$. *The methods were applied to solve linear and nonlinear stiff first order initial value problems. Errors in the results obtained were compared with those of the existing methods. The comparison shows a better performance than the existing methods.*

KEYWORDS: *Hybrid, Block Method, linear multi-step method first Derivative, Stiff ODEs*

1. Introduction

The first order stiff initial value problems (IVPs) of ordinary differential equations (ODEs) of the form

$$y'(x) = f(x, y), (a \leq x \leq b), y(x_0) = y_0 \quad (1)$$

The design and analysis of numerical techniques that generate continuous approximation has been subject fact that continuous methods enjoy certain advantages, such as the potential to measure and manage error (Enright, 2000). (Owolabi, 2016) proposed an order seven implicit symmetric scheme employed to attain solution of second order initial value problems of ordinary differential equations. (James *et al*, 2013) construct a continuous block method for the solution of second order IVPs with constant step size the method was developed by interpolation and

collocation of power series approximate solution with minimal computational estimate error. (Mohammed, 2011) proposed newly constructed linear multistep method with continuous coefficients and used it to get multiple finite difference methods which were directly put into practice to solve first-order ODEs. (Ebadi *et al*. 2010) Developed the hybrid block BDF with one additional off-grid point introduced in the first derivative of the solution to enhance the absolute stability region of the method to perform efficiently. (Zurni *et al*., 2016) Proposed direct Solution of second-order ordinary differential equation using a single-step hybrid block method of order five with better accurate approximate solutions as the step-size becomes small.

2. Derivation of the Hybrid Block Method

In this section, the construction of linear multistep method of two-step hybrid block with eight off-grid points and the approximate solution of (1) in the form:

$$y(x) = \sum_{j=0}^{m+t-1} \alpha_j x^j \qquad (2)$$

With first derivative given as

$$y'(x) = \sum_{j=0}^{m+t-1} j\alpha_j x^{j-1} \qquad (3)$$

And $\alpha_j s'$ are the parameters to be estimated. By this approach involve interpolating (2) at point $x_{n+j} j = 0$ and collocating (3) at points $x_{n+j} j = 0, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{13}{8}, \frac{7}{4}, 2$ and the continuous hybrid linear multistep method (2SHBM) reduces to

$$y(x) = \alpha_0 y_n + h(\beta_0 f_n + \beta_{\frac{1}{4}} f_{n+\frac{1}{4}} + \beta_{\frac{3}{8}} f_{n+\frac{3}{8}} + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_1 f_{n+1} + \beta_{\frac{5}{4}} f_{n+\frac{5}{4}} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_{\frac{13}{8}} f_{n+\frac{13}{8}} + \beta_{\frac{7}{4}} f_{n+\frac{7}{4}} + \beta_2 f_{n+2} \qquad (4)$$

Using maple 18 software to perform some algebraic manipulations to determine the values of a's and get the continuous schemes as follows:

$$y(x) = y_n + \left(h\lambda - \frac{38639}{5460}\lambda^2 + \frac{689567}{24570h}\lambda^3 - \frac{342119}{4914h^2}\lambda^4 + \frac{107726}{945h^3}\lambda^5 - \frac{221792}{1755h^4}\lambda^6 + \frac{390464}{4095h^5}\lambda^7 - \frac{39488}{819h^6}\lambda^8 + \frac{12800}{819h^7}\lambda^9 - \frac{180224}{61425h^8}\lambda^{10} + \frac{32768}{135135h^9}\lambda^{11}\right) f_n + \left(\frac{624}{11}\lambda^2 - \frac{443504}{1155h}\lambda^3 + \frac{129796}{105h^2}\lambda^4 - \frac{40945792}{17325h^3}\lambda^5 + \frac{4306208}{1485h^4}\lambda^6 - \frac{1626880}{693h^5}\lambda^7 + \frac{618368}{495h^6}\lambda^8 - \frac{13119488}{31185h^7}\lambda^9 + \frac{1409024}{17325h^8}\lambda^{10} - \frac{262144}{38115h^9}\lambda^{11}\right) f_{n+\frac{1}{4}} + \left(-\frac{65536}{495}\lambda^2 + \frac{685047808}{675675h}\lambda^3 - \frac{1437515776}{405405h^2}\lambda^4 + \frac{14561312768}{2027025h^3}\lambda^5 - \frac{532283392}{57915h^4}\lambda^6 + \frac{5186256896}{675675h^5}\lambda^7 - \frac{7340032}{1755h^6}\lambda^8 + \frac{134217728}{93555h^7}\lambda^9 - \frac{570425344}{2027025h^8}\lambda^{10} + \frac{536870912}{22297275h^9}\lambda^{11}\right) f_{n+\frac{3}{8}} + \cdots + \left(-\frac{1}{4}\lambda^2 + \frac{18637}{8190h}\lambda^3 - \frac{79207}{8190h^2}\lambda^4 + \frac{1472974}{61425h^3}\lambda^5 - \frac{65824}{1755h^4}\lambda^6 + \frac{155968}{4095h^5}\lambda^7 - \frac{14656}{585h^6}\lambda^8 + \frac{9728}{945h^7}\lambda^9 - \frac{16384}{6825h^8}\lambda^{10} + \frac{32768}{135135h^9}\lambda^{11}\right) f_{n+2} \qquad (5)$$

Where $\lambda = \frac{x-x_n}{h}$ evaluating hybrid scheme at off-grid points $x = x_{n+j} j = \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{3}{2}, \frac{13}{8}, \frac{7}{4}$, and at grid-point $x = x_{n+j} j = 1,2$ gives

$$y_{n+\frac{1}{4}} = y_n + \frac{h}{68497228800}[4101525351f_n + 44427087354f_{n+\frac{1}{4}} - 75539955712f_{n+\frac{3}{8}} + 63006339682f_{n+\frac{1}{2}} - 33251030670f_{n+\frac{3}{4}} + 25263432480f_{n+1} - 18858288306f_{n+\frac{5}{4}} + 18092955166f_{n+\frac{3}{2}} - 13599490048f_{n+\frac{13}{8}} + 3591776838f_{n+\frac{7}{4}} - 110044935f_{n+2}] \quad (6)$$

$$y_{n+\frac{3}{8}} = y_n + \frac{h}{72161689600}[4307779597f_n + 50088631086f_{n+\frac{1}{4}} - 71866605568f_{n+\frac{3}{8}} + 63845168530f_{n+\frac{1}{2}} - 34095001226f_{n+\frac{3}{4}} + 25987871052f_{n+1} - 19428242262f_{n+\frac{5}{4}} + 18656068574f_{n+\frac{3}{2}} - 14027137024f_{n+\frac{13}{8}} + 3705681330f_{n+\frac{7}{4}} - 113580489f_{n+2}] \quad (7)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{2140538400}[127909287f_n + 1477168680f_{n+\frac{1}{4}} - 1997307904f_{n+\frac{3}{8}} + 2047380478f_{n+\frac{1}{2}} - 1029829944f_{n+\frac{3}{4}} + 781613976f_{n+1} - 583374792f_{n+\frac{5}{4}} + 559706290f_{n+\frac{3}{2}} - 420708352f_{n+\frac{13}{8}} + 111115992f_{n+\frac{7}{4}} - 3404511f_{n+2}] \quad (8)$$

$$y_{n+\frac{3}{4}} = y_n + \frac{h}{281881600}[16788893f_n + 197163486f_{n+\frac{1}{4}} - 277266432f_{n+\frac{3}{8}} + 316678934f_{n+\frac{1}{2}} - 93877498f_{n+\frac{3}{4}} + 92705184f_{n+1} - 71068998f_{n+\frac{5}{4}} + 68931434f_{n+\frac{3}{2}} - 51986432f_{n+\frac{13}{8}} + 13765986f_{n+\frac{7}{4}} - 423357] \quad (9)$$

$$y_{n+1} = y_n + \frac{h}{267567300}[15972561f_n + 185706768f_{n+\frac{1}{4}} - 256835584f_{n+\frac{3}{8}} + 290144140f_{n+\frac{1}{2}} - 50567088f_{n+\frac{3}{4}} + 127817976f_{n+1} - 76389456f_{n+\frac{5}{4}} + 71597812f_{n+\frac{3}{2}} - 53542912f_{n+\frac{13}{8}} + 14093040f_{n+\frac{7}{4}} - 429957f_{n+2}] \quad (10)$$

$$y_{n+\frac{5}{4}} = y_n + \frac{5h}{2739889152}[32654259f_n + 382428930f_{n+\frac{1}{4}} - 534593536f_{n+\frac{3}{8}} + 606844810f_{n+\frac{1}{2}} - 121848870f_{n+\frac{3}{4}} + 343323552f_{n+1} - 77509146f_{n+\frac{5}{4}} + 125223670f_{n+\frac{3}{2}} - 96649216f_{n+\frac{13}{8}} + 25904190f_{n+\frac{7}{4}} - 806355f_{n+2}] \quad (11)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{h}{8808800}[525701f_n + 6120504f_{n+\frac{1}{4}} - 8486912f_{n+\frac{3}{8}} + 9605882f_{n+\frac{1}{2}} - 1778920f_{n+\frac{3}{4}} + 5199480f_{n+1} - 58344f_{n+\frac{5}{4}} + 3483766f_{n+\frac{3}{2}} - 1998848f_{n+\frac{13}{8}} + 498888f_{n+\frac{7}{4}} - 14685f_{n+2}] \quad (12)$$

$$y_{n+\frac{13}{8}} = y_n + \frac{13h}{1348868505600}[6190533459f_n + 72151477242f_{n+\frac{1}{4}} - 100191543296f_{n+\frac{3}{8}} + 113453821910f_{n+\frac{1}{2}} - 21296811822f_{n+\frac{3}{4}} + 61765118844f_{n+1} - 207921714f_{n+\frac{5}{4}} + 48477778778f_{n+\frac{3}{2}} - 17025916928f_{n+\frac{13}{8}} + 5458833510f_{n+\frac{7}{4}} - 166806783f_{n+2}] \quad (13)$$

$$y_{n+\frac{7}{4}} =$$

$$y_n + \frac{7h}{9785318400}[83447991f_n + 970550490f_{n+\frac{1}{4}} - 1344028672f_{n+\frac{3}{8}} + 1520673154f_{n+\frac{1}{2}} - 278420142f_{n+\frac{3}{4}} + 819987168f_{n+1} - 15309294f_{n+\frac{5}{4}} + 604073470f_{n+\frac{3}{2}} - 79937536f_{n+\frac{13}{8}} + 137176806f_{n+\frac{7}{4}} - 2502423f_{n+2}) \quad (14)$$

$$y_{n+2} = y_n + \frac{h}{66891825}[3885651f_n + 49949952f_{n+\frac{1}{4}} - 77594624f_{n+\frac{3}{8}} + 90435488f_{n+\frac{1}{2}} - 31739136f_{n+\frac{3}{4}} + 63908988f_{n+1} - 31739136f_{n+\frac{5}{4}} + 90435488f_{n+\frac{3}{2}} - 77594624f_{n+\frac{13}{8}} + 49949952f_{n+\frac{7}{4}} - 3885651f_{n+2}) \quad (15)$$

Writing equations (6)-(15) in block form gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{3}{8}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \\ y_{n+\frac{5}{4}} \\ y_{n+\frac{3}{2}} \\ y_{n+\frac{13}{8}} \\ y_{n+\frac{7}{4}} \\ y_{n+2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-9} \\ y_{n-8} \\ y_{-7} \\ y_{-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{-1} \\ y_n \end{pmatrix}$$

$$+h\left[\begin{pmatrix} \frac{569578043}{878169600} & -\frac{5674576}{5145525} & \frac{220301887}{239500800} & -\frac{7750823}{15966720} & \frac{368057}{997920} & -\frac{21979357}{79833600} & \frac{63262081}{239500800} & -\\ \frac{1926485811}{2775449600} & -\frac{8772779}{8808800} & \frac{44646971}{50462720} & -\frac{119213291}{252313600} & \frac{45433341}{126156800} & -\frac{67930917}{252313600} & \frac{65231009}{252313600} & -\\ \frac{946903}{1372140} & -\frac{62415872}{66891825} & \frac{7158673}{7484400} & -\frac{300067}{623700} & \frac{227743}{623700} & -\frac{24283}{89100} & \frac{391403}{1496880} & -\\ \frac{7583211}{10841600} & -\frac{270768}{275275} & \frac{1107269}{985600} & -\frac{328243}{985600} & \frac{20259}{61600} & -\frac{35499}{140800} & \frac{241019}{985600} & -\\ \frac{1190428}{1715175} & -\frac{64208896}{66891825} & \frac{101449}{93555} & -\frac{29468}{155925} & \frac{74486}{155925} & \frac{44516}{155925} & \frac{125171}{467775} & -\\ \frac{24514675}{35126784} & -\frac{2610320}{2675673} & \frac{10609175}{9580032} & -\frac{710075}{3193344} & \frac{125045}{199584} & -\frac{451685}{3193344} & \frac{2189225}{9580032} & -\\ \frac{58851}{84700} & -\frac{37888}{39325} & \frac{33587}{30800} & -\frac{311}{1540} & \frac{909}{1540} & \frac{51}{7700} & \frac{12181}{30800} & -\\ \frac{156328200691}{224811417600} & -\frac{158995369}{164656800} & \frac{13408178953}{12262440960} & -\frac{4194826571}{20437401600} & \frac{6082928371}{10218700800} & -\frac{5850611}{2919628800} & \frac{28645960187}{61312204800} & -\\ \frac{17420137}{25090560} & -\frac{9187696}{9555975} & \frac{37219273}{34214400} & -\frac{2271493}{11404800} & \frac{418117}{712800} & \frac{124901}{11404800} & \frac{2957003}{6842880} & -\\ \frac{1280768}{1715175} & -\frac{77594624}{66891825} & \frac{632416}{467775} & -\frac{73984}{155925} & \frac{148972}{155925} & -\frac{73984}{155925} & \frac{632416}{467775} & - \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5918507}{98841600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{391616327}{6560153600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{143557}{2402400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1526263}{25625600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{161339}{2702700} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{183245}{3075072} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{47791}{800800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{90322037}{1513881600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5900363}{98841600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{623}{10725} \end{pmatrix}\begin{pmatrix} f_{n-9} \\ f_{n-8} \\ f_{n-7} \\ f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}\right]$$

(15)

The hybrid block method is of order $(11,11,11,11,11,11,11,11,11,11)^T$ and error constant of $(\frac{132349}{4592179032883200}, \frac{766753}{27091966508400640}, \frac{59}{2066953011200}, \frac{145}{5291399708672}, \frac{179}{6278369771520}, \frac{145}{5291399708672}, \frac{59}{2066953011200},$

3. Stability Analysis

From (15) the arrangement of the block method as a matrix finite difference equation of the form

$$A^{(1)}y_{u+1} = A^{(0)}y_u + hB^{(1)}F_u \quad (16)$$

Where

$$y_{u+1} = (y_{n+\frac{1}{4}}, y_{n+\frac{3}{8}}, y_{n+\frac{1}{2}}, y_{n+\frac{3}{4}}, y_{n+1}, y_{n+\frac{5}{4}}, y_{n+\frac{3}{2}}, y_{n+\frac{13}{8}}, y_{n+\frac{7}{4}}, y_{n+2})^T \quad (17)$$

$$y_u = (y_{n-9}, y_{n-8}, y_{n-7}, y_{n-6}, y_{n-5}, y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}, y_n)^T \quad (18)$$

.

$$F_u = (f_{n+\frac{1}{4}}, f_{n+\frac{3}{8}}, f, y_{n+\frac{3}{4}}, f_{n+1}, f_{n+\frac{5}{4}}, f_{n+\frac{3}{2}}, f_{n+\frac{13}{8}}, f_{n+\frac{7}{4}}, f_{n+2})^T \quad (19)$$

For $U = 0 \ldots \ldots$ $and\ n = 0 \ldots \ldots$ and matrices $A^{(1)}, A^{(0)}, B^{(1)}$ are 12 by 12 matrices whose entries are given by the coefficients of (15).

3.1 Zero Stability block method

According to Fatunla (1991), a block method is said to be zero stable if $\lambda_{i,j} = 1,2, \ldots k$ specified as $\rho(\lambda) = \left|\sum_{j=0}^{k} A^{(i)}\lambda^{(k-i)}\right| = 0$ (20)

Satisfied$\rho(\lambda) = |\lambda_j| \leq 1$. The multiplicity must exceed two.

The block method (6)-(15) is expressed in matrix equation form to obtain

Where $A^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$,

$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$$B^{(0)} = \begin{pmatrix} \frac{569578043}{878169600} & -\frac{5674576}{5145525} & \frac{220301887}{239500800} & -\frac{7750823}{15966720} & \frac{368057}{997920} & -\frac{2197935}{7983600} \\ \frac{1926485811}{2775449600} & -\frac{8772779}{8808800} & \frac{44646971}{50462720} & -\frac{119213291}{252313600} & \frac{45433341}{126156800} & -\frac{6793091}{25231360} \\ \frac{946903}{1372140} & -\frac{62415872}{66891825} & \frac{7158673}{7484400} & -\frac{300067}{623700} & \frac{227743}{623700} & -\frac{24283}{89100} \\ \frac{7583211}{10841600} & -\frac{270768}{275275} & \frac{1107269}{985600} & -\frac{328243}{985600} & \frac{20259}{61600} & -\frac{35499}{140800} \\ \frac{1190428}{1715175} & -\frac{64208896}{66891825} & \frac{101449}{93555} & -\frac{29468}{155925} & \frac{74486}{155925} & -\frac{44516}{155925} \\ \frac{24514675}{35126784} & -\frac{2610320}{2675673} & \frac{10609175}{9580032} & -\frac{710075}{3193344} & \frac{125045}{199584} & -\frac{451685}{3193344} \\ \frac{58851}{84700} & -\frac{37888}{39325} & \frac{33587}{30800} & -\frac{311}{1540} & \frac{909}{1540} & -\frac{51}{7700} \\ \frac{156328200691}{224811417600} & -\frac{158995369}{164656800} & \frac{13408178953}{12262440960} & -\frac{4194826571}{20437401600} & \frac{6082928371}{10218700800} & -\frac{585061}{29196288} \\ \frac{17420137}{25090560} & -\frac{9187696}{9555975} & \frac{37219273}{34214400} & -\frac{2271493}{11404800} & \frac{418117}{712800} & -\frac{124901}{11404800} \\ \frac{1280768}{1715175} & -\frac{77594624}{66891825} & \frac{632416}{467775} & -\frac{73984}{155925} & \frac{148972}{155925} & -\frac{73984}{155925} \end{pmatrix}$$

$$B^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5918507}{98841600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{391616327}{6560153600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{143557}{2402400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1526263}{25625600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{161339}{2702700} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{183245}{3075072} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{47791}{800800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{90322037}{1513881600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5900363}{98841600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{623}{10725} \end{pmatrix}$$

Substituting $A^{(0)}$ and $A^{(1)}$ into (15) gives the characteristic polynomial of the block method$\rho(\lambda) = \det(\lambda A^{(0)} - A^{(1)})$

$$= det\left[\lambda\begin{pmatrix}1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&1\end{pmatrix}\right]$$

$$= det\begin{bmatrix}\lambda&0&0&0&0&0&0&0&0&\lambda\\0&\lambda&0&0&0&0&0&0&0&\lambda\\0&0&\lambda&0&0&0&0&0&0&\lambda\\0&0&0&\lambda&0&0&0&0&0&\lambda\\0&0&0&0&\lambda&0&0&0&0&\lambda\\0&0&0&0&0&\lambda&0&0&0&\lambda\\0&0&0&0&0&0&\lambda&0&0&\lambda\\0&0&0&0&0&0&0&\lambda&0&\lambda\\0&0&0&0&0&0&0&0&\lambda&\lambda\\0&0&0&0&0&0&0&0&0&0\end{bmatrix}$$

$= \lambda^9(\lambda - 1) = 0$

Therefore$\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = 0$ following Henrici (1962), the new block method (15) is zero stable and consistent since its order $P = 11 > 1$. the method is therefore convergent.

3.2 Region of Absolute Stability of the Block Method

The absolute stability region of the block method is obtained using (Chollom *etal.*2007), and shown below.

The stability polynomial is given by

$$\frac{13}{1384120320}r^{10}z^{10}-\frac{38639}{145332633600}r^{10}z^{9}+\frac{689567}{145332633600}r^{10}z^{8}-\frac{10001}{72666316800}r^{9}z^{9}$$
$$-\frac{342119}{5449973760}r^{10}z^{7}-\frac{431801563}{198379044864000}r^{9}z^{8}+\frac{700219}{1089994752}r^{10}z^{6}$$
$$-\frac{269143}{6900940800}r^{9}z^{7}-\frac{6931}{1351680}r^{10}z^{5}-\frac{1365996637}{3381460992000}r^{9}z^{6}$$
$$+\frac{42707}{1351680}r^{10}z^{4}-\frac{129766121}{34871316480}r^{9}z^{5}-\frac{617}{4224}r^{10}z^{3}$$
$$-\frac{41801341343}{1743565824000}r^{9}z^{4}+\frac{675}{1408}r^{10}z^{2}-\frac{10027379}{82555200}r^{9}z^{3}-r^{10}z$$
$$-\frac{578381}{1372800}r^{9}z^{2}+r^{10}-\frac{10102}{10725}r^{9}z-r^{9}$$
$$\frac{13}{138412032}r^{10}z^{9}-\frac{38639}{16148070400}r^{10}z^{8}+\frac{689567}{18166579200}r^{10}z^{7}-\frac{10001}{8074035200}r^{9}z^{8}$$
$$-\frac{342119}{778567680}r^{10}z^{6}-\frac{431801563}{24797380608000}r^{9}z^{7}+\frac{700219}{181665792}r^{10}z^{5}$$
$$-\frac{1884001}{6900940800}r^{9}z^{6}-\frac{6931}{270336}r^{10}z^{4}-\frac{1365996637}{563576832000}r^{9}z^{5}+\frac{42707}{337920}r^{10}z^{3}$$
$$-\frac{129766121}{6974263296}r^{9}z^{4}-\frac{617}{1408}r^{10}z^{2}-\frac{41801341343}{435891456000}r^{9}z^{3}+\frac{675}{704}r^{10}z$$
$$-\frac{10027379}{27518400}r^{9}z^{2}-r^{10}-\frac{578381}{1372800}r^{9}z^{2}+r^{10}-\frac{10102}{10725}r^{9}z-r^{9}$$

Which is plotted using MATLAB R2008b software and the absolute stability region of the Two-Step hybrid Block Method (2SHBM) is shown in the Fig 1.

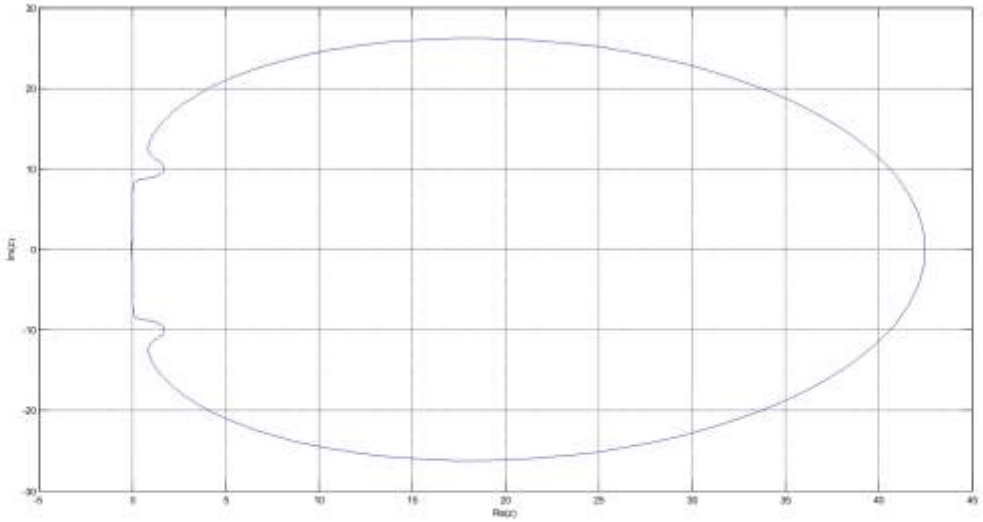


Figure 1: Region of Absolute Stability of 2SHBM

4. Numerical Experiments

The following numerical experiments are performed with the aid of MATLAB R2008b software package in order to ascertain the performance of the method

PROBLEM 1

Consider the mildly stiffly differential equation

Source: (Donald *etal*., 2022)

$y_1' = -8y_1 + 7y_2 \qquad y_1(0) = 1$

$y_2' = 42y_1 - 43y_2 \quad y_2(0) = 8, \quad h = 0.1$

With exact solution

$$y_{1(x)} = 2e^{-x} - e^{-50x}$$

$$y_{2(x)} = 2e^{-x} - 6e^{-50x}$$

PROBLEM 2

$y_1' = -1002y_1 + 1000y_2^2$

$y_2' = y_1 - y_2 - y_2^2$

$$\text{Exact} \quad y_{1(x)} = e^{-2x}y_1(0) = 1$$

$$y_2(x) = e^{-x}y_2(0) = 1$$

$$with\ h = 0.1$$

Table 1. Absolute Errors of numerical solutions of problem 1 within the interval $0 \leq x \leq 1$

(x)	Absolute errors Of Method		Absolute errors	(Yunusa *etal*.,2020)
	$y_1(x)$	$y_2(x)$	$y_1(x)$	$y_2(x)$
0.1	0	0	1.53587E −08	1.22622E −05
0.2	8.46212E-04	2.82071E-04	3.39481E − 08	1.35518E − 05
0.3	4.77367E-05	1.5912E-05	5.62776E −08	1.49771E −05
0.4	2.69458E-06	8.98438E-07	8.29285E −08	1.65523E −05
0.5	1.50494E-07	4.99257E-08	1.14562E −07	1.82931E −05
0.6	9.96641E-09	3.55459E-09	1.51934E −07	2.02170E −05
0.7	8.73494E-10	5.17202E-10	1.95898E −07	2.23432E −05
0.8	1.34661E-09	6.6863E-10	2.47430E −07	246931. E −05
0.9	1.28054E-09	6.40398E-10	3.07633E −07	2.72901E −05
1.0	1.24587E-09	6.22791E-10	3.77764E −07	3.01602E −05

Table 2. Absolute Errors of numerical solutions of problem 2 within the interval $0 \leq x \leq 1$

	Absolute errors Of Method		Absolute errors	(Donald *etal*.,2022)
	$y_1(x)$ $y_2(x)$		y_2(x)	y_2(x)
0.1	0	0	7.730000E −04	1.95000E −02
0.2	5.07727E-4	1.69243E-04	7.74000E − 03	5.72000E − 03
0.3	2.86420E-05	9.54718E-06	7.29000E −07	1.18000E −02
0.4	1.61675E-06	5.39063E-07	5.86000E −05	9.67000E −03
0.5	9.02965E-08	2.99554E-08	4.65000E −08	8.00000E −03

0.6	5.97985E-09	2.13275E-09	4.42000E −08	6.41000E −03
0.7	5.24097E-10	3.10321E-10	1.60000E −09	5.24000E −03
0.8	8.07968E-10	4.01178E-10	1.20000E −09	4.29000E −03
0.9	7.68325E-10	3.84239E-10	1.80000E −09	3.51000E −03
1.0	7.47521E-10	3.73675E-10	2.00000E −10	2.87000E −03

5. Conclusion

In this study, two-step first derivative hybrid block method with eight(8) off-grid point for integrating stiff differential equations have been developed. The error obtained showed that the method is more accurate and efficiency. The new method is highly recommended for general use due to its effectiveness.

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A SURVEY ON THE FEATURES OF NON-SUBGOUP COSETS IN THE FACTOR GROUP OF INTEGERS BY ITS INFINITE SUBGROUPS $\mathbb{Z}/\langle m\rangle$

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Abstract

This study is aimed at finding properties of the non-subgroup cosets in the factor group of integers $\mathbb{Z}$ by its infinite subgroups $\langle m\rangle$. we first display the $\langle m\rangle$-cosets for $m\in\mathbb{Z}$ $(m\geq 2)$ determined by $k\in\mathbb{Z}$ for the cases when $m=2,3,4,...,8,9$. In each case from the lists of these cosets, only $\langle m\rangle$ is a subgroup. We observed and investigate some properties of the non-subgroup $\langle m\rangle$-cosets. It was noticed that for even m, there exists unique non-subgroup $\langle m\rangle$-coset say $k+\langle m\rangle$ with the property that for all $x\in k+\langle m\rangle, x^{-1}\in k+\langle m\rangle$; we named this singular $\langle m\rangle$-coset . But for odd m such $\langle m\rangle$-coset does not exist. Investigating other cosets that do not satisfy this particular property, we have found that (i) for every x, y in each of the non-subgroup $h+\langle m\rangle\in\mathbb{Z}/\langle m\rangle$, there exists another non-subgroup $q+\langle m\rangle\in\mathbb{Z}/\langle m\rangle$ such that $x+y\in q+\langle m\rangle$. We call this closure $\langle m\rangle$-coset of $h+\langle m\rangle$. (ii) for every $y\in r+\langle m\rangle$, there exists some coset $s+\langle m\rangle$ such that $y^{-1}\in s+\langle m\rangle$. In this case $r+\langle m\rangle$ and $s+\langle m\rangle$ are said to be inverse pair $\langle m\rangle$-cosets. We proceed to determine what type(s) of binary relation do closure $\langle m\rangle$-coset and inverse pair $\langle m\rangle$-coset properties satisfy on $\mathbb{Z}/\langle m\rangle$ respectively? It was found that the relation "closure $\langle m\rangle$-coset" is irreflexive and asymmetric while "inverse pair" relation is irreflexive and symmetric.

Keywords: $\langle m\rangle$-coset, non-subgroup $\langle m\rangle$-coset, singular, closure and inverse pair $\langle m\rangle$-cosets

Introduction

Groups are the most rudimentary forms of algebraic structures. Cosets are basic tool in the study of groups and their most important role come in understanding a factor group. That is cosets of a subgroup enable us to construct a new group from one given group. Cosets are not subgroups.

The infinite additive abelian group $\mathbb{Z}$ has infinitely many proper normal subgroups $\langle m\rangle$ for $m\in\mathbb{Z}$ $(m\geq 2)$. In general, the $\langle m\rangle$-coset determined by $a\in\mathbb{Z}$ is of the form $a+\langle m\rangle$.

First, we will consider/study the features of the non-subgroup $\langle m\rangle$-cosets determined by $a\in\mathbb{Z}$ for the cases when $m=2,3,4,...,8,9$ and then generalize for any m.

In the factor group $\mathbb{Z}/\langle m\rangle$, all the $\langle m\rangle$-cosets are disjoint and it is only $\langle m\rangle$ that is a subgroup of $\mathbb{Z}$. In this study, we will investigate the properties of the other $\langle m\rangle$-cosets that are not subgroups.

Research questions

I. What are the properties of the non-subgroup $\langle m\rangle$-cosets?

II. How do we identify non-subgroup $\langle m\rangle$-cosets in $\mathbb{Z}/\langle m\rangle$ with special properties?

III. What type(s) of binary relation do "closure $\langle m\rangle$-coset" and "inverse pair $\langle m\rangle$-coset" satisfy on $\mathbb{Z}/\langle m\rangle$ respectively?

2. Preliminaries

In this section we give definitions of some basic terms, concepts and relevant properties/results on cosets, factor group and binary relation needed for the proper understanding of this paper. Refer to any introductory book on group theory. For instance, see

Definition 2.1 (Coset of a subgroup): Let G be a group and H be a subgroup of G. For each $g\in \mathrm{G}$, the sets $g\mathrm{H}=\{gh:h\in \mathrm{H}\}$, $\mathrm{H}g=\{hg:h\in \mathrm{H}\}$ are called respectively a left H-coset and a right H-coset determined by $g\in \mathrm{G}$.

Theorem 2.2: Every H-coset determined by $g\in \mathrm{G}$ has the same number of elements as H. i.e $|g\mathrm{H}|=|\mathrm{H}|$.

Theorem 2,3: Any two left (or right) H-cosets are identical or disjoint.

Theorem 2.4: The group G is equal to the union of all left (or right) H-cosets in G.

Definition 2.5 (Invariant/Normal subgroup): A subgroup H of a group G is said to be normal (also called invariant) in G if every left H-coset is the same as the right H-coset i.e $g\mathrm{H}=\mathrm{H}g$ for all $g\in \mathrm{G}$.

Definition 2.6 (Factor Group): Let G be a group and N be a normal subgroup of G. The collection of all the distinct N-cosets determined by the elements G is a group under the coset operation is called the factor (or quotient) group of G by N and is denoted by G/N. i.e $\mathrm{G}/\mathrm{N}=\{g\,\mathrm{N}:g\in \mathrm{G}\}$.

Theorem 2.7: The factor group G/N partitions the group G.

Definition 2.8 (Index of a normal subgroup): The number of elements in the factor group of G by N is called the index of a normal subgroup N of G.

Theorem 2.9: In the group $(\mathbb{Z},+)$ the subgroup generated by $n\in\mathbb{Z}$ is a normal subgroup.

Theorem 2.10: For all $n\in\mathbb{Z}$, there is one to one correspondence between $\mathbb{Z}/\langle n\rangle$ and $\mathbb{Z}_n$.

Definition 2.11 (Irreflexive relation): A relation R on a non-empty set A is irreflexive if a is not related to itself for every $a\in \mathrm{A}$.

Definition 2.12 (Symmetric relation): Let A be a non-empty set and $a,b\in \mathrm{A}$. A relation is said to be symmetric if whenever a is related b then b is related to a.

Definition 2.13 (Asymmetric relation): A relation R on a non-empty set A is called asymmetric if a is related b but b is not related a for all $a,b\in$ A .

3. The $\langle m\rangle$-cosets in the Factor group $\mathbb{Z}/\langle m\rangle$

In this section, the $\langle m\rangle$-cosets determined by $a\in\mathbb{Z}$ where $0\le a<m$ were displayed and examined to enable us establish our results. Let's consider the cases when m is even or odd for $m=2,3,4,...,8,9$. From **Theorem 2.10**, the index of $\langle m\rangle$ in $\mathbb{Z}$ is m and hence the number of non-subgroup $\langle m\rangle$-cosets will be $m-1$.

A. $\langle 2\rangle$-cosets are:

$\langle 2\rangle=\{\ldots,-6,-4,-2,0,2,4,6,\ldots\}$ and $1+\langle 2\rangle=\{\ldots,-7,-5,-3,-1,1,3,5,7,\ldots\}$.

B. $\langle 3\rangle$ -cosets are:

$\langle 3\rangle=\{\ldots,-9,-6,-3,0,3,6,9,\ldots\}$

$1+\langle 3\rangle=\{\ldots,-8,-5,-2,1,4,7,10,\ldots\}$

$2+\langle 3\rangle=\{\ldots,-7,-4,-1,2,5,8,11,\ldots\}$

C. $\langle 4\rangle$ -cosets are

$\langle 4\rangle=\{\ldots,-12,-8,-4,0,4,8,12,\ldots\}$

$1+\langle 4\rangle=\{\ldots,-11,-7,-3,1,5,9,13,\ldots\}$

$2+\langle 4\rangle=\{\ldots,-10,-6,-2,2,6,10,14,\ldots\}$

$3+\langle 4\rangle=\{\ldots,-9,-5,-1,3,7,11,15,\ldots\}$

D. $\langle 5\rangle$ -cosets are

$\langle 5\rangle=\{\ldots,-15,-10,-5,0,5,10,15,\ldots\}$

$1+\langle 5\rangle=\{\ldots,-14,-9,-4,1,6,11,16,\ldots\}$

$2+\langle 5\rangle=\{\ldots,-13,-8,-3,2,7,12,17,\ldots\}$

$3+\langle 5\rangle=\{\ldots,-12,-7,-2,3,8,13,18,\ldots\}$

$4+\langle 5\rangle=\{\ldots,-11,-6,-1,4,9,14,19,\ldots\}$

E. $\langle 6 \rangle$ **-cosets are**

$\langle 6 \rangle = \{\ldots, -18, -12, -6, 0, 6, 12, 18, \ldots\}$

$1 + \langle 6 \rangle = \{\ldots, -17, -11, -5, 1, 7, 13, 19, \ldots\}$

$2 + \langle 6 \rangle = \{\ldots, -16, -10, -4, 2, 8, 14, 20, \ldots\}$

$3 + \langle 6 \rangle = \{\ldots, -15, -9, -3, 3, 9, 15, 21, \ldots\}$

$4 + \langle 6 \rangle = \{\ldots, -14, -8, -2, 4, 10, 16, 22, \ldots\}$

$5 + \langle 6 \rangle = \{\ldots, -13, -7, -1, 5, 11, 17, 23, \ldots\}$

F. $\langle 7 \rangle$ **-cosets are**

$\langle 7 \rangle = \{\ldots, -21, -14, -7, 0, 7, 14, 21, \ldots\}$

$1 + \langle 7 \rangle = \{\ldots, -20, -13, -6, 1, 8, 15, 22, \ldots\}$

$2 + \langle 7 \rangle = \{\ldots, -19, -12, -5, 2, 9, 16, 23, \ldots\}$

$3 + \langle 7 \rangle = \{\ldots, -18, -11, -4, 3, 10, 17, 24, \ldots\}$

$4 + \langle 7 \rangle = \{\ldots, -17, -10, -3, 4, 11, 18, 25, \ldots\}$

$5 + \langle 7 \rangle = \{\ldots, -16, -9, -2, 5, 12, 19, 26, \ldots\}$

$6 + \langle 7 \rangle = \{\ldots, -15, -8, -1, 6, 13, 20, 27, \ldots\}$

G. $\langle 8 \rangle$ **-cosets are**

$\langle 8 \rangle = \{\ldots, -24, -16, -8, 0, 8, 16, 24, \ldots\}$

$1 + \langle 8 \rangle = \{\ldots, -23, -17, -7, 1, 9, 17, 25, \ldots\}$

$2 + \langle 8 \rangle = \{\ldots, -22, -16, -6, 2, 10, 18, 26, \ldots\}$

$3 + \langle 8 \rangle = \{\ldots, -21, -15, -5, 3, 11, 19, 27, \ldots\}$

$4 + \langle 8 \rangle = \{\ldots, -20, -14, -4, 4, 12, 20, 28, \ldots\}$

$5 + \langle 8 \rangle = \{\ldots, -19, -13, -3, 5, 13, 21, 29, \ldots\}$

$6 + \langle 8 \rangle = \{\ldots, -18, -12, -2, 6, 14, 22, 30, \ldots\}$

$7 + \langle 8 \rangle = \{\ldots, -17, -11, -1, 7, 15, 23, 31, \ldots\}$

H. $\langle 9\rangle$ -cosets are

$\langle 9\rangle = \{\ldots, -27, -18, -9, 0, 9, 18, 27, \ldots\}$

$1+\langle 9\rangle = \{\ldots, -26, -17, -8, 1, 10, 19, 28, \ldots\}$

$2+\langle 9\rangle = \{\ldots, -25, -16, -7, 2, 11, 20, 29, \ldots\}$

$3+\langle 9\rangle = \{\ldots, -24, -15, -6, 3, 12, 21, 30, \ldots\}$

$4+\langle 9\rangle = \{\ldots, -23, -14, -5, 4, 13, 22, 31, \ldots\}$

$5+\langle 9\rangle = \{\ldots, -22, -13, -4, 5, 14, 23, 32, \ldots\}$

$6+\langle 9\rangle = \{\ldots, -21, -12, -3, 6, 15, 24, 33, \ldots\}$

$7+\langle 9\rangle = \{\ldots, -20, -11, -2, 7, 16, 25, 34, \ldots\}$

$8+\langle 9\rangle = \{\ldots, -19, -10, -1, 8, 17, 26, 35, \ldots\}$

Observations

Investigating the properties of the non-subgroup $\langle m\rangle$-cosets above, we can observe that

a) For even m, there exists unique non-subgroup $\langle m\rangle$-coset say $k+\langle m\rangle$ with the property that for all $x\in k+\langle m\rangle, x^{-1}\in k+\langle m\rangle$. But for odd m such coset does not exists. For instance, when $m=4$, the unique $\langle 4\rangle$-coset **is** $2+\langle 4\rangle$ and when $m=7$ there is no such $\langle 7\rangle$-coset **.**

b) In each case, for every x, y in the non-subgroup $h+\langle m\rangle\in \square/\langle m\rangle$, there exists another $q+\langle m\rangle\in \square/\langle m\rangle$ such that $x+y\in q+\langle m\rangle$. For instance, when $m=9$, for every $x, y\in 4+\langle 9\rangle$; $x+y\in 5+\langle 9\rangle$.

c) For every $y\in r+\langle m\rangle$, there exists some $s+\langle m\rangle$ such that $y^{-1}\in s+\langle m\rangle$ For instance, when $m=9$, for every $y\in 4+\langle 9\rangle$; $-y\in 5+\langle 9\rangle$.

4. Findings/Results

In this section, we present the results of our findings.

Result 4.1: Let $m\in \square$, for even m, we call the unique non-subgroup $\langle m\rangle$-coset $k+\langle m\rangle$ with the property that for each $x\in k+\langle m\rangle$, there is $x^{-1}\in k+\langle m\rangle$ a singular $\langle m\rangle$-coset.

Result 4.2: Let $m\in \square$, for even m, there exists singular $\langle m\rangle$-coset of the form $m/2+\langle m\rangle$.

Result 4.3: Let $m\in \square (m\geq 3)$. The non-subgroup $\langle m\rangle$-coset $\mathrm{A}=p+\langle m\rangle$ is call a closure coset of another non-subgroup $\langle m\rangle$-coset $\mathrm{B}=q+\langle m\rangle$ if for all $x, y\in \mathrm{B}$; $x+y\in \mathrm{A}$.

Result 4.4: Let $m \in \square (m \geq 3)$. The closure $\langle m \rangle$-coset of each $k + \langle m \rangle$ is the $\langle m \rangle$-coset $2k (\text{mod}\, m) + \langle m \rangle$.

Result 4.5: Define a relation by "A $\square$ B" if $A = p + \langle m \rangle$ is a closure coset of $B = q + \langle m \rangle$. The relation closure $\langle m \rangle$-coset on $\square / \langle m \rangle$ is irreflexive and asymmetric.

Result 4.6: Let $m \in \square (m \geq 3)$. We call a non-subgroup $\langle m \rangle$-coset $A = s + \langle m \rangle$ an inverse pair of another
$\langle m \rangle$-coset say $B = r + \langle m \rangle$. i.e if $(r + s) \bmod m = 0$ then $A = r + \langle m \rangle$ and $B = s + \langle m \rangle$ are inverse pair $\langle m \rangle$-cosets.

Result 4.7: Define a relation by "A $\square$ B" if A is an inverse pair $\langle m \rangle$-coset of B. The relation inverse pair on $\square / \langle m \rangle$ is irreflexive and symmetric.

5. Conclusion

This study is an in-depth survey on the features of non-subgroup $\langle m \rangle$-cosets in $\square / \langle m \rangle$. We have found some properties that these $\langle m \rangle$-cosets have satisfied.

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SOLUTIONS TO RESTRICTED THREE-BODY PROBLEM

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Abstract

We examine the solution of the restricted three-body problem. Using the equation of motion, we solved for the equilibrium points, where five solutions were obtained. Three of which are collinear, and the remaining two are triangular. We have solved numerically, to compute values for the triangular and collinear points, using a mathematical software (MATHEMATICA 12.1) with mass ratio $0 < \mu < \frac{1}{2}$*, and the result after computing the values was graphs showing the actual picture of triangular and collinear points as stated in our problem (Figure 1).*

Keywords: Restricted three-body problem (R3BP), Equilibrium points, Triangular points, Collinear points

1. Introduction

The three-body problem (3BP) is one of the most important problems in celestial mechanics. In the 3BP, three bodies move in space under their mutual gravitational interactions as described by Newton's theory of gravity. It has many applications in scientific research, especially in the fields of astrophysics and astrodynamics. Typical examples of the 3BPs we may consider are; Sun-Planet-Moon or Sun-Planet-Planet or Sun-Planet-Asteroid systems. However, the restricted three body problem (R3BP) is a mathematical model that describes the motion of a small body that moves under the gravitational influence of two larger bodies. If one of these particles is so much smaller (called infinitesimal mass) than the other two particles (primaries) and has negligible effect on the motion, then their body problem reduces to a restricted three-body problem. The model is used to study the motion of objects, such as asteroids and space crafts in the presence of celestial bodies such as planets and the moon. The motions of the bodies take place in 3 dimensions (3D), and there are restrictions on their masses or the initial connections. In the past, many physicists, astronomers, and mathematicians attempted unsuccessfully to find closed-form solutions to the most challenging problems in the history of science. The 3BP has been studied for over three hundred years.

In recent years, there has been a growing interest in the R3BP due to its applications to space exploration and mission design. For example, the R3BP can be used to design lunar and interplanetary trajectories, as well as to study the dynamics of asteroids and comets. Related articles on restricted three-body problems are thus: Meltzer and Gurfil (2006) designed libration-point reference trajectories in the elliptic restricted three-body problem. Minglibayev and Zhumabek (2021) studied the restricted three-body problem. Minesaki (2023) studied the Quasi-conservative integration method for Restricted three-body problems. Peterson and Daniel (2023) observed the local orbital elements for the circular restricted three-body problem. Liu (2023) studied the orbital motion and calculation method in the restricted three-body problem. Sharan, et al.(2023) studied the accurate uncertainty characterization of impulsive thrust manoeuvres in the restricted three-body problem. Quartullo, et al. (2023) analyzed the periodic model predictive control for tracking Halo orbits in the elliptic restricted three-body problem. Charette, et al. (2022) observed the restricted three-body problem in celestial mechanics. Verma, et al. (2023) studied the perturbed restricted problem of three bodies with elongated smaller primary.

Simó (2016) studied the applications of the restricted three-body problem to space exploration. Celletti, et al. (2020) studied The Three-Body Problem. Simó (2016) studied the advances in the study of the restricted three-body problem. Hénon (2009) worked on the restricted three-body problem, from its early history to its most recent developments. Liu and Zhao (2017)studied on new method for constructing periodic orbits in the restricted three-body problem. Li and Wisdom (2013) studied the applications of the restricted three-body problem to asteroid mining. Giorgilli and Lega (2011) analyzed the applications of the restricted three-body problem to space colonization.

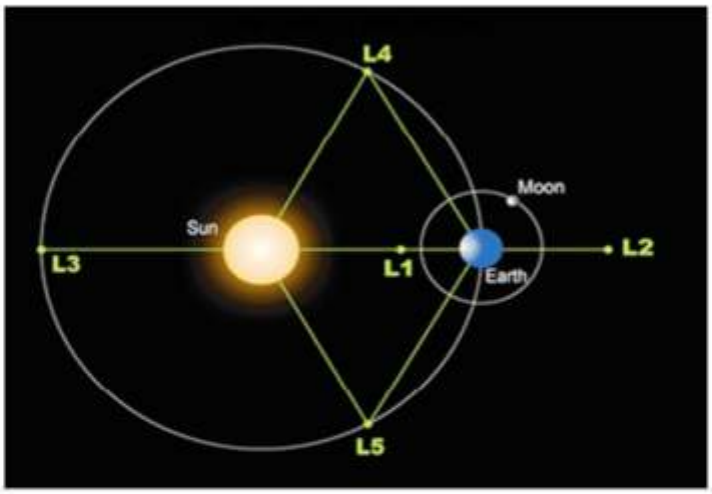


Figure 1

Consider two rigid bodies moving about their common centre of mass on account of their mutual gravitational attraction in the framework of the restricted three-body problem, and a third body of mass known as the infinitesimal body (smaller body) is attracted by the two massive bodies even though it exerts no influence on them. At the equilibrium point, five solutions (L1, L2, L3, L4 and L5) are obtained as seen in the diagram above.

This research aims at investigating the solution of the restricted three-body problem around the equilibrium points.

Hence, we set our objectives as follows;

- To write down the equations of motion of the infinitesimal mass.
- To determine the triangular and collinear equilibrium points
- To solve numerically, the solution of triangular and collinear points.

This work is divided into seven chapters; Chapter one is the introduction, Chapter two explains the equations of motion. Chapter three provides the location of triangular point. Chapter four solves the collinear points, while chapter five explore their numerical applications. The discussion and conclusion are shown in chapter six and chapter seven respectively.

2. Equations of motion

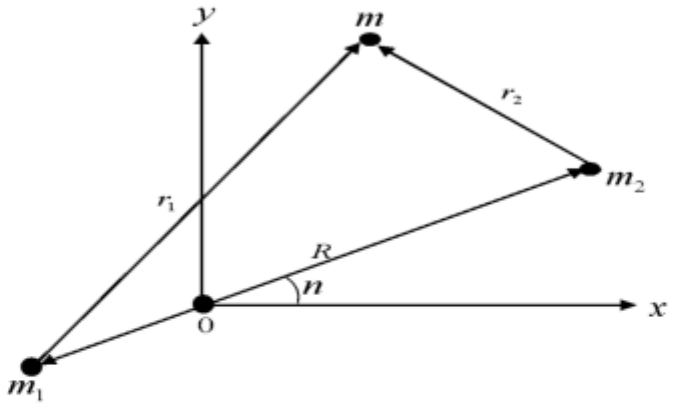


Figure 2

In figure 2 above, let m_1 and m_2 be the masses of bigger and smaller primaries, respectively, while we denote the mass of the test particles by (m). Supposed (x,y,z) are coordinates of m in a rotating barycentric co-ordinate system 0xyz relative to an inertia system with angular velocity (n). following the terminologies of Szebehely (1967), the equation of motion of the infinitesimal mass in the R3BP is given by

$$\ddot{x} - 2\dot{y} = \Omega_x, \quad \ddot{y} + 2\dot{x} = \Omega_y, \quad \ddot{z} = \Omega_y \tag{1}$$

with,

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{2}$$

$$r_i^2 = (x - x_i)^2 + y^2 + z^2, i = 1,2 \tag{3}$$

$$x_1 = -\mu, \qquad x_2 = 1 - \mu, \qquad 0 < \mu = \frac{m_2}{m_1 + m_2} < \frac{1}{2}$$

Where m_1 *and* m_2 are masses of the primaries, positioned at the points (x,0,0), i=1,2.

3. Location of triangular point

At the equilibrium point, Ω_x, Ω_y, *and* Ω_z =0

Hence,

$$\Omega_x = \frac{1}{2}(x^2 + y^2)\frac{\partial}{\partial x} + \frac{1-\mu}{[(x+\mu)^2 + y^2 + z^2]^{\frac{1}{2}}}\frac{\partial}{\partial x} + \frac{\mu}{[(x+\mu-1)^2 + y^2 + z^2]^{\frac{1}{2}}}\frac{\partial}{\partial x}$$

Hence,

$$\Omega_x = x - \frac{(1-\mu)(x+\mu)}{[(x+\mu)^2 + y^2 + z^2][(x+\mu)^2 + y^2 + z^2]^{\frac{1}{2}}} - \frac{\mu(x+\mu-1)}{[(x+\mu-1)^2 + y^2 + z^2]^{\frac{1}{2}}[(x+\mu-1)^2 + y^2 + z^2]}$$

$$\Omega_x = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} \tag{4}$$

$$\Omega_y = \frac{1}{2}(x^2+y^2)\frac{\partial}{\partial y} + \frac{1-\mu}{[(x+\mu)^2+y^2+z^2]^{\frac{1}{2}}}\frac{\partial}{\partial y} + \frac{\mu}{[(x+\mu-1)^2+y^2+z^2]^{\frac{1}{2}}}\frac{\partial}{\partial y}$$

Hence,

$$\Omega_y = y - \frac{y(1-\mu)}{[(x+\mu)^2+y^2+z^2][(x+\mu)^2+y^2+z^2]^{\frac{1}{2}}} - \frac{y\mu}{[(x+\mu-1)^2+y^2+z^2]^{\frac{1}{2}}[(x+\mu-1)^2+y^2+z^2]}$$

$$\Omega_y = y - \frac{y(1-\mu)}{r_1^3} - \frac{y\mu}{r_2^3} \tag{5}$$

$$\Omega_z = \frac{1}{2}(x^2+y^2)\frac{\partial}{\partial z} + \frac{1-\mu}{[(x+\mu)^2+y^2+z^2]^{\frac{1}{2}}}\frac{\partial}{\partial z} + \frac{\mu}{[(x+\mu-1)^2+y^2+z^2]^{\frac{1}{2}}}\frac{\partial}{\partial z}$$

$$\frac{1-\mu}{[(x+\mu)^2+y^2+z^2]^{\frac{1}{2}}}\frac{\partial}{\partial z} = -\frac{(1-\mu)2z\frac{1}{2}[(x+\mu)^2+y^2+z^2]^{\frac{-1}{2}}}{[(x+\mu)^2+y^2+z^2]} - \frac{z(1-\mu)}{[(x+\mu)^2+y^2+z^2][(x+\mu)^2+y^2+z^2]^{\frac{1}{2}}}$$

$$\Omega_z = -\frac{z(1-\mu)}{r_1^3} - \frac{z\mu}{r_2^3} \tag{6}$$

At the triangular point, $\Omega_x = \Omega_y = \Omega_z = 0$. $y \neq 0, z = 0$ From equations (4), (5) and (6),

$$\Omega_x = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3}$$

$$\Omega_y = y - \frac{y(1-\mu)}{r_1^3} - \frac{y\mu}{r_2^3}$$

$$\Omega_y = y\left(1 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3}\right) = 0$$

then,

$$\left(1 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3}\right) = 0 \tag{7}$$

we multiply (7) *by* $(x+\mu)$ to get;

$$\left((x+\mu) - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x+\mu)}{r_2^3}\right) = 0 \tag{8}$$

Subtracting (8) from (4), we get;

$$x - (x+\mu) - \frac{(1-\mu)(x+\mu)}{r_1^3} + \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} + \frac{\mu(x+\mu)}{r_2^3} = 0$$

Then, $-\mu + \frac{\mu}{r_2^3} = 0$

$$r_2^3 = 1; \qquad \text{Therefore} \qquad r_2 = 1$$

$$multiply\ (7)\ by\ (x+\mu-1)$$

$$(x+\mu-1) - \frac{(x+\mu-1)(1-\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} = 0 \tag{9}$$

Subtracting (9) from (4), We get;

$$x - (x+\mu-1) - \frac{(1-\mu)(x+\mu)}{r_1^3} + \frac{(x+\mu-1)(1-\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} + \frac{\mu(x+\mu-1)}{r_2^3} = 0$$

Then, $-\mu - 1 + \frac{\mu-1}{r_1^3} = 0$

$$r_1^3(1-\mu) + \mu - 1 = 0$$
$$r_1^3(1-\mu) - (1-\mu) = 0$$
$$r_1^3(1-\mu) = 1-\mu$$
$$r_1 = 1$$

$putting\ r_1\ and\ r_2$ in (3), with z=0, we get

$$1 = (x+\mu)^2 + y^2 \tag{10}$$

$$1 = (x+\mu-1)^2 + y^2$$

$$0 = (x+\mu)^2 - 2(x+\mu) + y^2 \tag{11}$$

$subtracting\ (10) from (11)$, We get,

$$1 = 2(x+\mu)$$

Therefore, $x = \frac{1}{2} - \mu$ (12)

Putting (12) into (10), we have

$$1 = \left(\frac{1}{2} - \mu + \mu\right)^2 + y^2$$
$$1 = \frac{1}{4} + y^2$$

$$y = \pm\sqrt{\frac{3}{4}} \tag{13}$$

Values for x and y gotten in equations (12) and (13) are the solution for the triangular point $L_4\ and\ L_5$ respectively.

4. Location of Collinear Points

At collinear point, $\Omega_x = \Omega_y = \Omega_z = 0, y = z = 0,$

$$then,$$

$$x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} = 0$$

Where,

$r_1^2 = (x+\mu)^2$; $\quad r_2^2 = (x+\mu-1)^2$

$$x - \frac{(1-\mu)(x+\mu)}{(x+\mu)^3} - \frac{\mu(x+\mu-1)}{(x+\mu-1)^3} = 0$$

$$x - \frac{(1-\mu)}{(x+\mu)^2} - \frac{\mu}{(x+\mu-1)^2} = 0 \quad (14)$$

The collinear points L_1, L_2, L_3 are gotten on the three intervals $for\ L_1, -\mu < x < 1-\mu$, (L_1 lies between the two masses). Let $x + \mu - 1 = -\rho$

$$x + \mu = -\rho + 1\ ,$$

Substituting to (14), we have

$$x - \frac{(1-\mu)}{(-\rho+1)^2} - \frac{\mu}{-\rho^2} = 0$$

$$x - \frac{(1-\mu)}{(-\rho+1)^2} + \frac{\mu}{\rho^2} = 0$$

$$x - \frac{(1-\mu)}{r_1^2} + \frac{\mu}{r_2^2} = 0 \quad (15)$$

For $L_2, x > 1-\mu$, (L_2 $lies\ \ on\ the\ right\ hand\ side\ of\ the\ two\ masses$)

$$let, x + \mu - 1 = \rho$$

$x + \mu = \rho + 1$

substituting into (14) we have,

$$x - \frac{(1-\mu)}{(\rho+1)^2} - \frac{\mu}{\rho^2} = 0$$

$$x - \frac{(1-\mu)}{r_1^2} - \frac{\mu}{r_2^2} = 0 \quad (16)$$

$for\ L_3, x < -\mu$, (L_3 $lies\ on\ the\ left\ hand\ side\ of\ the\ two\ masses$)

$let, x + \mu = -\rho$

substituting to (14) we have,

$$x - \frac{(1-\mu)}{-\rho^2} - \frac{\mu}{(-\rho-1)^2} = 0$$

$$x + \frac{(1-\mu)}{\rho^2} + \frac{\mu}{(\rho-1)^2} = 0$$

$$x + \frac{(1-\mu)}{r_1^2} + \frac{\mu}{r_2^2} = 0 \quad (17)$$

Equations (15), (16) and (17) are the solutions to the collinear points L_1, L_2, and L_3 respectively.

5. Numerical Application

Here, we solve numerically showing arbitrary values of mass ratio (μ) of triangular points L_4 and L_5

μ	x	$\pm$y
0	0.5	0.866025
0.02	0.48	0.866025
0.04	0.46	0.866025
0.1	0.4	0.866025
0.12	0.38	0.866025
0.14	0.36	0.866025
0.2	0.3	0.866025
0.22	0.28	0.866025
0.24	0.26	0.866025
0.3	0.2	0.866025
0.46	0.04	0.866025

Table 1: Showing effect of the mass ratio on the triangular point

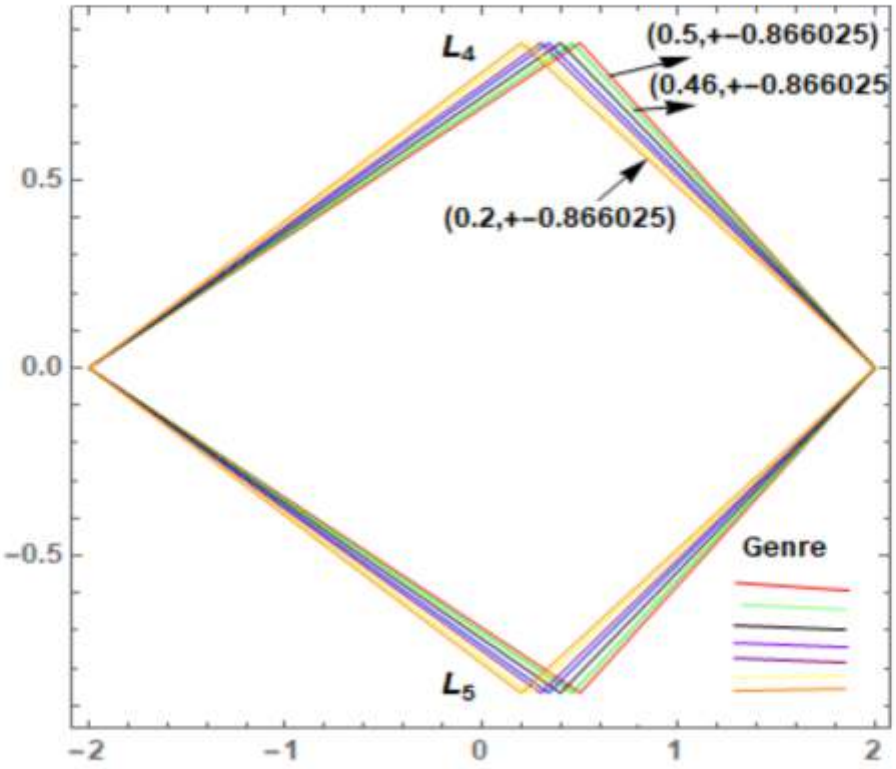


Figure 3: Showing arbitrary values of mass ratio on the triangular points

Here, we solve numerically showing arbitrary values of mass ratio (μ) of collinear points of the restricted three body problem. Using equation (15),(16), and (17) , We solve for L_1, L_2 and L_3 respectively.

μ	L_1	L_2	L_3
0	1	1	-1
0.02	0.975	0.985	-0.985
0.06	0.925	0.955	-0.955
0.1	0.875	0.925	-0.925
0.12	0.85	0.91	-0.91
0.26	0.675	0.805	-0.805
0.3	0.625	0.775	-0.775

0.36	0.55	0.73	-0.73
0.4	0.5	0.7	-0.7
0.44	0.45	0.67	-0.67

Table 2: Showing effect of the mass ratio on the collinear points

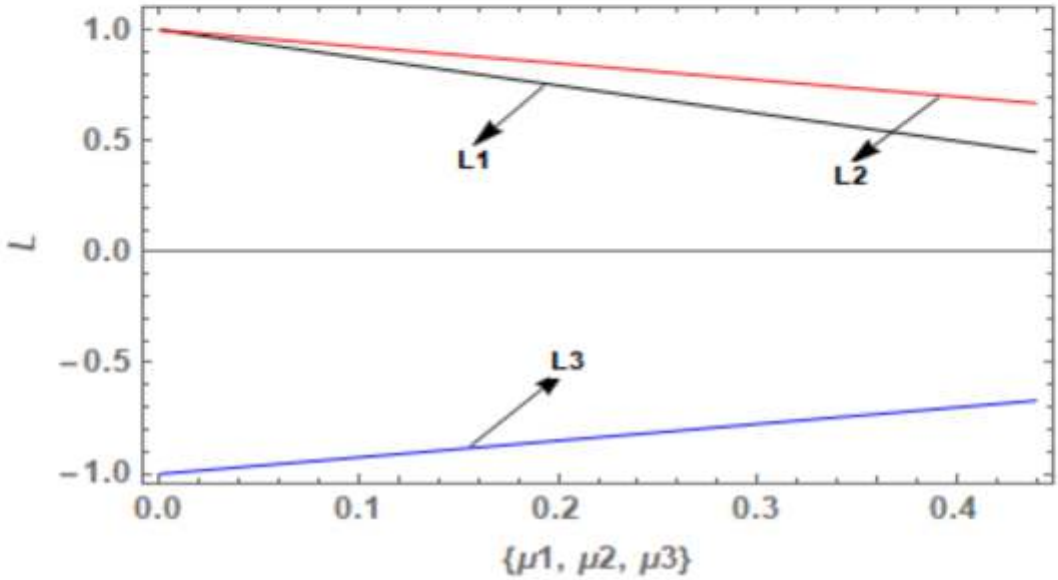


Figure 4: Showing combined effect of the mass ratio on the collinear points, L_1, L_2, and L_3

6. Discussions

Here, we studied the restricted three-body problem. The equation of motion of the system was used to solve the location of triangular and collinear points.

At the triangular point, values for L_4 and L_5 were computed as seen in Table 1 using equations (12) and (13). It was observed that for every increase in mass ratio (μ), there is a decrease in L_4, while L_5 remains constant throughout our calculations.

At the collinear point, values for L_1, L_2 *and* L_3 were computed as seen in Table 2 using equations (15), (16) and (17). It was observed that for every increase in the mass ratio, there is a decrease in L_1 *and* L_2 , and an increase in L_3 .

We compute numerically the solutions of the restricted three-body problem using the software MATHEMATICA 12.1. Plotting the graph for the triangular point (L_4, L_5), the result is a four-sided angle (scalene triangle) as seen in Figure 3, showing arbitrary values for mass ratio (μ) between the interval $\{0 < \mu < \frac{1}{2}\}$.

For the collinear point, it is noticed that L_1 *and* L_2 starts off at the same point, but L_1 reduces faster but stays between L_2 *and* L_3 in the interval $\{-\mu < x < 1-\mu\}$, and L_2 *and* L_3 mirrors themselves in positive and negative axis respectively, with intervals $\{x > 1-\mu\}$ and $\{x < -\mu\}$ respectively as seen in Figure 4.

7. Conclusion

Using the equation of motion of the infinitesimal mass we were able to determine the solution of triangular and collinear points. Also, computing numerically using MATHEMATICA 12.1, the solution of the equilibrium points are obtained as seen in Figure 1.

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COMPUTATIONAL TECHNIQUE FOR SOLVING FRACTIONAL ONE-DIMENSIONAL HYPERBOLIC TELEGRAPH EQUATION

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Abstract

This paper present computational scheme to implement the numerical investigating the effects of the fractional order parameter α and the electrical parameters resistance R, inductance L, capacitance C, and leakage conductance G. The fractional telegraph equation TFTE, a generalization of the classical telegraph model, incorporates fractional derivatives to better model memory and hereditary properties in transmission lines and electrical systems. We employ algorithmic approaches to solve the fractional hyperbolic telegraph equation. special attention is given to the variation of alpha $0 < \alpha \leq 2$, describing the fractional derivative order, which critically influences wave propagation characteristics such as dispersion and attenuation and adjusting α, the model can transition from classical integer-order behavior to fractional dynamic regimes that capture anomalous diffusion and wave propagation phenomena more accurately. Furthermore, the impacts of the electrical parameters on the solution profiles are systematically studied. Resistance R and leakage conductance G contribute to energy dissipation, affecting signal attenuation, while inductance L and capacitance C influence wave speed and oscillation patterns. Numerical simulations demonstrate how these parameters interact with fractional order effects to alter the characteristic waveforms and transmission properties. Overall, this work contributes to the understanding and numerical solution of fractional telegraph equations relevant to complex transmission line phenomena, providing flexible computational tools to model realistic electrical behaviors influenced by fractional dynamics and circuit parameters. The results have potential applications in electrical engineering, signal processing, and materials science where memory effects are significant.

Keywords: Computational scheme, one-dimensional telegraph equation, resistance R, inductance L, capacitance C, conductance of leakage current G.

1. Introduction

Circuit systems are essential in electronics engineering, particularly for transmitting electrical signals between locations. A key element in these systems is the transmission medium, which physically facilitates signal propagation. To precisely describe voltage behavior along a transmission wire, the telegraph equation was formulated. This mathematical model accounts for electromagnetic wave reflections and characterizes the wave propagation patterns occurring in transmission lines. The one-dimensional hyperbolic telegraph equation TFTE is a fundamental partial differential equation that models electrical signal transmission by capturing both wave-like and diffusive behaviors. The equation's dynamics are governed by critical electrical parameters: resistance (R), inductance (L), capacitance (C), and leakage conductance (G). These

parameters influence signal attenuation, velocity, dispersion, and distortion, all of which are vital considerations in telecommunications and electrical engineering. Given its wide applicability and inherent mathematical complexity, the telegraph equation remains a focal point of research, prompting the development of diverse analytical and numerical methods for its solution. In this paper, we consider the voltage form of time TFTE fractional one-dimensional telegraph equation of the form:

$$\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}} = LC\frac{\partial^{2} v(x,t)}{\partial x^{2}} + (RC + LC)\frac{\partial v(x,t)}{\partial t} + RGv(x,t) + f(x,t), \qquad 0 < \alpha \leq 2 \qquad (1)$$

with initial conditions:

$$\begin{cases} v(x,0) = g_1 \\ v_t(x,0) = g_2 \end{cases} \qquad (2)$$

and Dirichlet boundary conditions:

$$\begin{cases} v(L,t) = g_3 \\ v(x,L) = g_4 \end{cases} \qquad (3)$$

Where $v(x,t)$ the voltage at position x and time t, R the resistance per unit lenght (Ω/m), L the inductance per unit length $(\mathrm{H/m})$, C the capacitance per unit lenght $(\mathrm{F/m})$, G the leakage conductance per unit $(\mathrm{S/m})$, $f(x,t)$ the external force, g_1, g_2, g_3 and g_4 are smooth functions.

The telegraph equation is hyperbolic partial differential equation that models the propagation of the voltage $v(x,t)$ or current $I(x,t)$ along a tranmission line. It is derived from Kirchhoff's voltage and currevt law in conjuction with the distributed-parameter model of a tansmission line characterized .

Several numerical methods have been proposed over the years. For instance, [1] developed a method in the reproducing kernel space by piecewise technique to solve the TFTE, [2] applied the radial basis functions and finite difference method to solve the TFTE, [3] described a finite difference scheme for the generalized TFTE, [4] utilized a combination of method of line and group preserving scheme to solve the TFTE, [5] discussed a fully discrete local discontinuous Galerkin finite element method for the TFTE, [6] introduced the SincLegendre collocation method for solving the TFTE, Mollahasani et al. [7] presented an operational method based on hybrid functions of Legendre polynomials and Block-Pulse-Functions for solving the TFTE, [8] used a shifted Jacobi collocation scheme for a multidimensional TFTE, [9] proposed an accurate and efficient spectral algorithm for the numerical solution of the two-sided space-time TFTE with three types of non-homogeneous boundary conditions, [10] presented an efficient numerical method for a time-fractional telegraph equation, and [11] used Mamadu variational iteration method for the solution of time-fractional telegraph equation. This paper is aim to employ a suitable and efficient computational algorithm to solve fractional one-dimentional hyperbolic telegraph equation, while analyzing the effects of the fractional order parameter α and experimental parameters resistance R, inductance L, capacitance C, and leakage conductance G. This study focuses on understanding how variations in α and these system parameters influence the behavior of the telegraph equation, which models signal propagation with memory and damping effects in electrical transmission lines or related physical systems.

2. **Fractional Calculus** [12]

Definition 1.

Let $\alpha > 0$ we define the Riemann-Liouville fractional integral by

$$ {}_0^R D_t^{-\alpha} v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} v(\tau)\, d\tau, \qquad \alpha > 0 \tag{4} $$

Suppose $\alpha = 1$, then equation (4) reduces to the integral of the form

$$ {}_0^R D_x^{-1} v(t) = \int_0^t v(\tau)\, d\tau \qquad \alpha = 1 \tag{5} $$

Definition 2.

The fractional derivative in Caputo's sense is defined as

$$ \begin{cases} {}_0^R D_x^{-\alpha} v(x) = \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x (x-\tau)^{n-\alpha-1} v^{(n)}(\tau)\, d\tau \\ n-1 < \alpha \le n, \;\; n \in \mathbb{N}, t > 0, \end{cases} \tag{6} $$

3. **Description of the computational scheme**

The fractional time-stepping algorithm is a computational technique formulate to solve time-fractional partial differential equations (PDEs) that makes use of mathematical software packages like MATLAB, Mathematica, Python, and Maple. This technique substitutes fractional derivatives for the conventional time derivative, making it invaluable for simulating systems that display memory-dependent behavior via fractional derivatives, like Riemann-Liouville or Caputo. Some authors have proposed and applied some notable algorithms in recent years to handle both integer and non-integer differential and partial differential equations arises in applied mathematics [13, 14, 15, 16, 17].

3.1 A five step Algorithm

Step 1: Initialization:

Initialize key system parameters: resistance R, inductance L, capacitance C, and leakage conductance G. Define the spatial domain and set the unknown function with initial and boundary conditions. Specify computational settings with N by discretization points and numeric precision digits.

Step 2

Decompose the fractional telegraph equation (1) into the Riemann-Liouville fractional integral (4). We consider the Riemann-Liouville fractional derivative definition for order α and the operator can be represented as a fractional integral and integer derivative composition in looped maple algorithm as follows

$$ \begin{cases} for\ n = 0\ to\ N\ do \\ TFTE := LC \dfrac{\partial^2 v_n}{\partial x^2} + (RC + LC) \dfrac{\partial v_n}{\partial t} + RG v_n + f(x,t) \\ v_{n+1} := simply\left(\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} TFTE ds,\ \ assume = nonnegative\right) \\ end\ do \end{cases} \tag{7} $$

Step 3

Discretize the time domain into NN steps with a fixed step size, defining the numerical length of the finite computational process. Use the Riemann-Liouville fractional integral approximation to evaluate fractional integrals and sums at each discrete time. This approach enables calculating fourth-order mixed derivatives efficiently. The variable vv represents the sum of values from step 0 $to\ N+1$ as follows:

$$v := \sum_{n=0}^{N+1} v[k] \qquad (8)$$

Step 4

Factorize and collect like terms at each time step. Compute the fractional time derivative using previous values. Assemble the spatially discretized operator and form the right-hand side with prior step solutions and fractional derivatives. Simulate results for varying fractional orders $\alpha := [2.0, 1.6, 1.2, 0.8, 0.4]$

Step 5

Display the output of the simulations in visualized 3D surfaces, and appendices.

Obtain simulation solution of fractional one dimensional telegraph at each time step for visualization and Maple plots commands to generate 3D surface plots for displacement of voltage $v(\mathrm{x}, \mathrm{t})$ at the selected times.

Table 1. Maple pseudo-code table of the five outlined algorithmic steps for simulating the fractional telegraph equation:

Step	Description	Maple Pseudocode
Step 1	Initialize parameters, domain, unknown function, and computation settings	`restart; with(PDEtools): with(plots): Digits := 10; N := 8; R := 0.8; L := 0.2; C := 0.4; G := 0.6; v[0] := sin(Pi*x);`
Step 2	Decompose PDE; apply Riemann-Liouville fractional integral and derivative definitions	`for alpha in FractionalOrders do RLFracInt := RLFracInt(v, alpha): end do;`
Step 3	Discretize time domain, approximate fractional integrals via sums	`dt := T/NN; for n from 1 to NN do v[n] := v[n-1] + dt * RLFracInt: vv := sum(v[k], k = 0..N+1): end do;`
Step 4	Factorize, collect terms, update time derivatives, assemble spatial operators and simulate for varying α	`for alpha in FractionalOrders do v := collect(factor(v), v): fTimeDeriv := FractionalTimeDeriv(v, alpha): end do;`
Step 5	Visualize results in 3D plots, output solutions at selected times	`for k from 1 to NN do plot3d(v[k], x = a .. b, t = c .. d): end do;`

4. Global Error and Convergence

This analysis focuses on the mathematical framework for the Global error and convergence of the numerical scheme described in Steps 2-4, specifically targeting the approximation of the fractional time derivative $D_t^{\alpha} v(x,t)$ using the Riemann-Liouville fractional integral and a time discretization algorithm

The global error $E_{i,n}$ is the difference between the exact solution V and the numerical solution v at the grid point (x_i, t_n).
The global error define as

$$E_{i,n} = V(x_i, t_n). - v_{i,n} \tag{9}$$

The global error satisfies the following difference equation.

$$L_{k,h}(E_{i,n}) = \tau_{i,n} \tag{10}$$

Where $L_{k,h}$ is the linearized discrete operator. To ensure the solution v converges to V, we require the numerical sheme to be both consistent and stable.

4.1 Stability condition. For linear schemes, stability often depends on the relationship between k and h. For explicit schemes, a severe constraint exists; for an implicit scheme (which Step 4 factorization suggests), the scheme is typically unconditionally stable.

$$\max_n \|E_n\| \leq C \max_n \|\tau_n\| \tag{11}$$

4.2 Convergence theorem

If the scheme is stable and the local truncation error $\tau_{i,n}$ is bounded, the scheme is convergent. The rate of convergence p is determined by the order of the local truncation error. The convergence order is define as

$$\|E\| = \max_{i,n} \|E_{i,n}\| \leq C(k^{pt} + h^{px}) \tag{12}$$

The final Convergence Rate:

$$\|E\| = \mathrm{O}(k^{2-\alpha} + h^2) \tag{13}$$

For the numerical simulation to yield meaningful results, the time step k and space step h should be balanced to maintain the convergence order $k^{2-\alpha} \approx h^2$. This yields the following step size relation for optimal computational efficiency: The optimal step size ratio is define as

$$\mathrm{k} \backsim \mathrm{O}h^{\left(\frac{2}{2-\alpha}\right)} \tag{14}$$

5. Implementation of the algorithm

5.1 Computational simulations

Case 1. We consider equation (1) when external force is homogenous $f(x,t)=0$, with initial conditions $v(x,0)=\sin(\pi x)$, $v_t(x,0)=0$, homogeneous Dirichlet boundary conditions $v(0,t)=0, v(1,t)=0$ [11]. We applied the proposed algorithm and performed the simulation with the experimental parameters $R=0.8, L=0.6, C=0.4, G=0.2$ and we obtain the series voltage solutions up to the 4th terms as follows:

$$v(x,t)\approx \sin(\pi x)\left(\begin{array}{c} 1+\frac{(-2.2087)^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(-2.2087)(0.56)t^{2\alpha-1}}{\Gamma(2\alpha)}+ \\ \frac{(-2.2087)^3t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{2(-2.2087)^2(0.56)t^{3\alpha-1}}{\Gamma(3\alpha)}+ \\ \frac{(-2.2087)(0.56)^2t^3}{\Gamma(3\alpha-1)}+\frac{(-2.2087)^4t^{4\alpha}}{\Gamma(4\alpha+1)}+ \\ \frac{3(-2.2087)^3(0.56)t^{4\alpha-1}}{\Gamma(4\alpha)}+\frac{3(-2.2087)^2(0.56)^2}{\Gamma(4\alpha-1)} \end{array}\right) \quad (15)$$

Case 2. We consider equation (1) when external force is non-homogenous $f(x,t)=\sin(t)\cos(x)$, with initial conditions $v(x,0)=\sin(\pi x)$, $v_t(x,0)=0$, boundary conditions $v(0,t)=0, v(1,t)=0$ [11]. We applied the proposed algorithm and performed the simulation with the experimental parameters $R=0.8, L=0.6, C=0.4, G=0.2$ and we obtain the series voltage solutions up to 4th terms as follows:

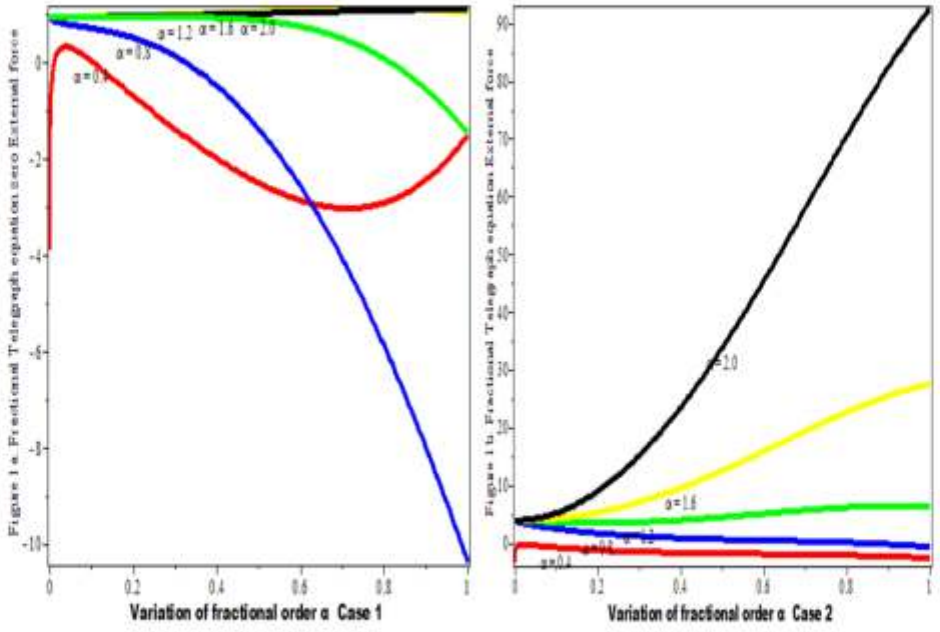
$$\left\{\begin{array}{c} v(x,t)^{(0)}=\sin(\pi x) \\ v(x,t)^{(1)}=\left\{\begin{array}{c} \sin(\pi x)\left(1+\frac{-2.20870t^{\alpha}}{\Gamma(1+\alpha)}\right)+ \\ \cos(x)\left(\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}-\frac{t^{\alpha+3}}{6\Gamma(4+\alpha)}+\frac{t^{\alpha+5}}{120\Gamma(5+\alpha)}\right) \end{array}\right. \\ v(x,t)^{(2)}=\left\{\begin{array}{c} \sin(\pi x)\left(1+\frac{-2.20870t^{\alpha}}{\Gamma(1+\alpha)}-1.23687\frac{t^{2\alpha-1}}{\Gamma(\alpha)\Gamma(2\alpha)}+..\right)+ \\ \cos(x)\left(\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}-\frac{t^{\alpha+3}}{6\Gamma(4+\alpha)}+0.56\frac{t^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}+..\right) \end{array}\right. \\ v(x,t)^{(3)}=\left\{\begin{array}{c} \sin(\pi x)\left(\begin{array}{c} 1+\frac{-2.208705t^{\alpha}}{\Gamma(1+\alpha)}-\frac{1.23687t^{2\alpha-1}}{\Gamma(\alpha)\Gamma(2\alpha)}+.. \\ \frac{4.878378t^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \end{array}\right)+ \\ \cos(x)\left(\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}+0.56\frac{t^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}+..\right) \end{array}\right. \\ v(x,t)\approx v(x,t)^{(0)}+v(x,t)^{(1)}+v(x,t)^{(2)}+v(x,t)^{(3)} \end{array}\right. \quad (16)$$

5.2 Voltage displacement output results

Table 2: Simulation voltage solutions for case 1and case 2

x,t		$\alpha = 0.4$	$\alpha = 0.8$	$\alpha = 1.2$	$\alpha = 1.6$	$\alpha = 2.0$
0,0	Case 1	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
	Case 2	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.2, 0.2	Case 1	-0.42252038	0.318888412	0.574109446	0.601488886	0.596107025
	Case 2	-0.70627796	1.051108644	2.169934384	3.406064793	6.001412512
0.4, 0.4	Case 1	-1.98841304	-0.47857718	0.923425026	1.025117484	0.997224723
	Case 2	-1.73988534	0.777316144	3.906344155	9.609384226	23.83533367
0.6, 0.6	Case 1	-2.86348297	-2.56771077	0.747251268	1.086563386	1.038662387
	Case 2	-2.00890875	0.353476909	4.985017646	15.81419906	45.35434867
0.8, 0.8	Case 1	-1.80091326	-3.63003819	0.054823878	0.689310639	0.666937935
	Case 2	-1.27490790	-0.05987688	3.598111312	13.94917377	44.59063988
1.0, 1.0	Case 1	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
	Case 2	0.000000000	0.000000000	-2.47105811	-3.84453977	-3.57680129

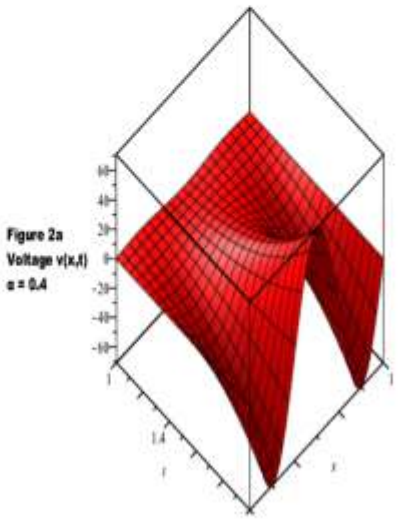
5.3 2D plots representation

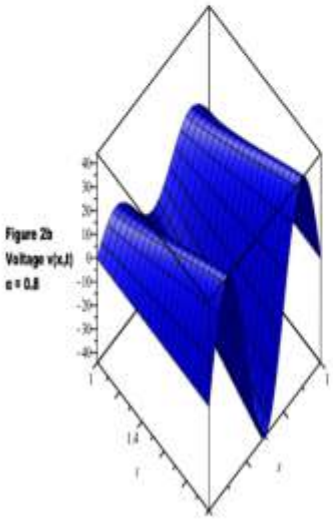


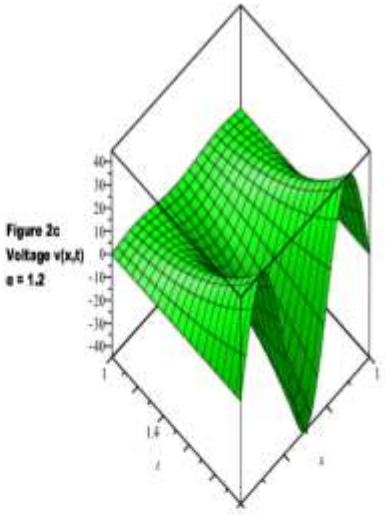
Figures 1a and 1b depict the trend of the displacement of the voltage v(t) obtained for the various fractional derivatives of $\alpha = [0.4, 0.8, 1.2, 1.6, 2.0]$ when external forces are $f(x,t) = 0$ and $f(x,t) = \sin(t)\cos(x)$.

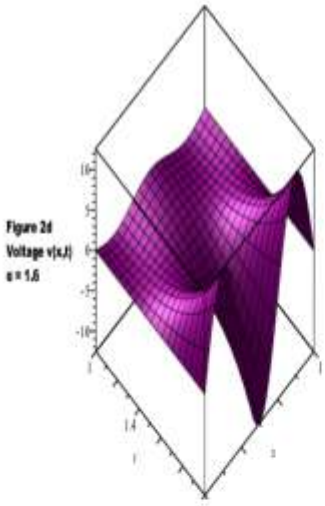
5.4 3D plots representation

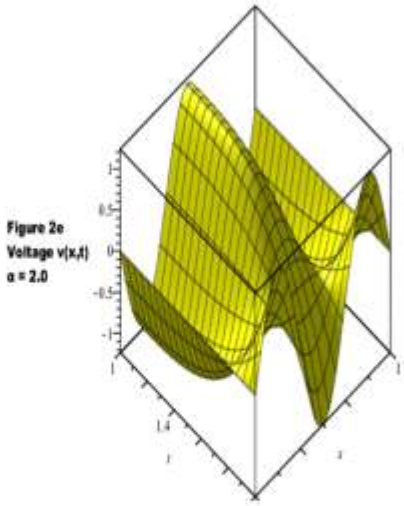
5.4.1 Case 1

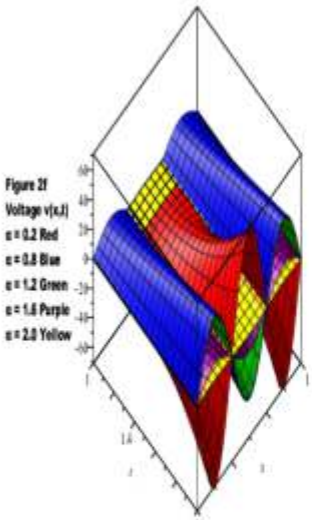






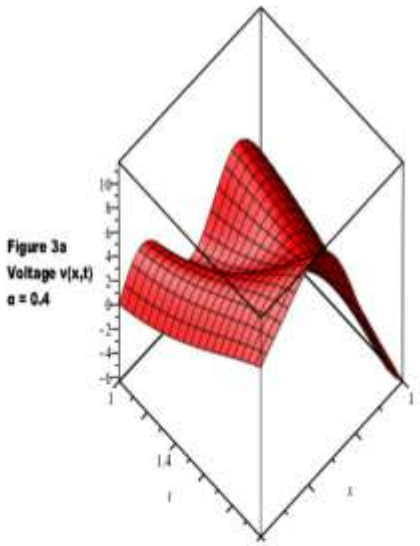


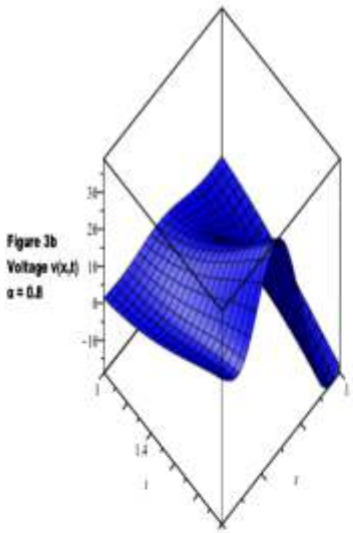


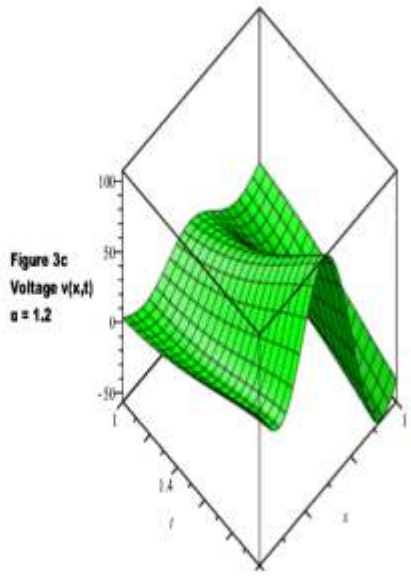


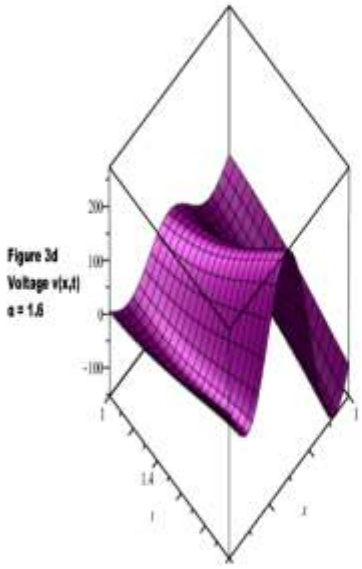
Figures 2a, 2b, 2c, 2d, and 2e depict the trend of the displacement of the voltage v(x, t) obtained for the various fractional derivatives of $\alpha = [0.4, 0.8, 1.2, 1.6, 2.0]$ when external force is $f(x,t) = 0$ and Figure 2f demonstrated the comparison of the displacement voltage for the case 1.

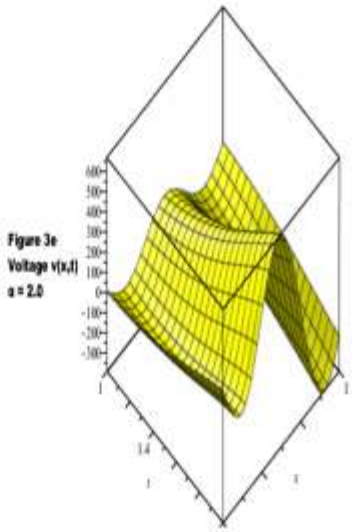
5.4.2 Case 2

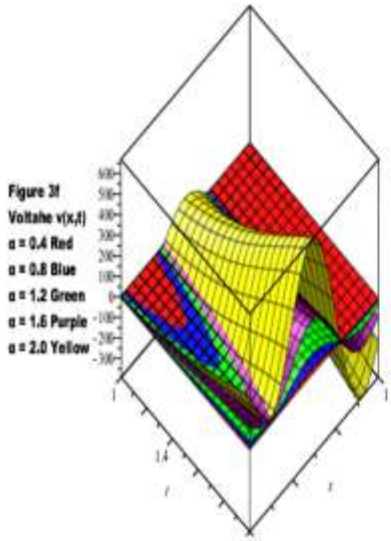












Figures 3a, 3b, 3c, 3d, and 3e depict the trend of the displacement of the voltage $v(x,t)$ obtained for the various fractional derivatives of $\alpha = [0.4, 0.8, 1.2, 1.6, 2.0]$ when external force is $f(x,t) = \sin(t)\cos(x)$ and Figure 3f demonstrated the comparison of the displacement voltages for the case 2.

6. Discussion and Conclusion

6.1 Discussion

Figure 1a: The plot illustrates the displacement of voltage $v(t)$ for the fractional telegraph equation under zero external force $f(x,t) = 0$. As the fractional order α increases from 0.4 to 2.0, the amplitude of $v(t)$ decreases, indicating faster decay and enhanced damping with higher fractional order. **Figure 1b:** The diagram shows the displacement of voltage $v(t)$ for the fractional telegraph equation with external force $f(x,t) = \sin(t)\cos(x)$. The voltage amplitude increases notably with higher fractional order α, demonstrating stronger oscillatory response and higher energy propagation for larger α values.

Figure 2a: The 3D surface illustrates the voltage displacement $v(x,t)$ for fractional order $\alpha = 0.4$ under zero external force $f(x,t) = 0$. The waveform shows strong oscillations and steep variations, indicating pronounced memory and diffusion effects at low fractional order. **Figure 2b:** For $\alpha = 0.8$, the surface of $v(x,t)$reveals a smoother oscillatory pattern with reduced amplitude. This demonstrates that increasing the fractional order leads to a more stabilized voltage distribution over time and space. **Figure 2c:** At $\alpha = 1.2$, the voltage displacement exhibits moderate attenuation and well-balanced oscillations. The behavior reflects improved damping effects and reduced wave propagation speed compared to lower fractional orders. **Figure 2d**: For $\alpha = 1.6$, the voltage $v(x,t)$ shows significant smoothing with diminished oscillatory magnitude. This implies a transition toward steady-state behavior as the fractional derivative approaches the classical integer order. **Figure 2e**: When $\alpha = 2.0$, the solution approximates the classical telegraph equation response, showing the most stable and least oscillatory voltage pattern. The system behaves almost linearly, confirming convergence to the standard integer-order model. **Figure 2f:** The comparison plot of case 1 demonstrates how the voltage displacement $v(x,t)$ varies with different fractional orders α. It clearly highlights that smaller α values yield stronger oscillations and memory effects, while larger α values enhance stability and damping**.**

Figure 3a: The 3D surface for $\alpha = 0.4$ shows highly oscillatory voltage $v(x,t)$ due to the strong influence of the external force $f(x,t) = \sin(t)\cos(x)$. at low fractional order. **Figure 3b:** For $\alpha = 0.8$, the voltage surface becomes smoother with reduced amplitude, indicating partial damping of oscillations from the external excitation. **Figure 3c:** At $\alpha = 1.2$, the voltage displacement exhibits balanced oscillatory behavior with moderate amplitude, reflecting a transition between diffusive and wave-like responses. **Figure 3d:** When $\alpha = 1.6$, the voltage $v(x,t)$ displays smoother and less fluctuating variations, suggesting stronger damping and faster stabilization under external forcing. **Figure 3e**: For $\alpha = 2.0$, the voltage pattern becomes nearly steady with minimal oscillation, closely resembling the classical telegraph equation behavior under sinusoidal excitation. **Figure 3f:** The comparison plot for case 2 illustrates that as α\alphaα increases, the amplitude of voltage displacement decreases, confirming that higher fractional orders enhance system stability and reduce oscillatory response.

6.2 Conclusion

The study on computational technique for solving the fractional one-dimensional hyperbolic telegraph equation FDFE has demonstrated the effectiveness of the proposed numerical framework in analyzing the dynamic behavior of voltage displacement under varying fractional orders. The results presented in Figures 1a and 1b clearly reveal that the fractional order α significantly influences the system's temporal evolution. For zero external force, lower values of α exhibit strong damping and rapid attenuation, while higher α values approach classical behavior with smoother, more stable responses. Conversely, in the presence of external forcing $f(x,t) = \sin(t)\cos(x)$, larger fractional orders amplify oscillatory voltage propagation, highlighting the sensitivity of fractional derivatives to external excitation. Overall, the computational technique efficiently captures the interplay between memory, damping, and wave speed in fractional telegraph equations. The findings confirm that fractional calculus provides a powerful modeling framework for real-world systems exhibiting nonlocal and history-dependent

behaviors, with potential applications in electrical circuit modeling, electromagnetic wave propagation, and signal transmission in complex media.

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PROGNOSTIC GAMMA GENERALIZED REGRESSION MODELLING OF DETERMINANTS INFLUENCING VARIATIONS IN UNDER-5 MORTALITY

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Abstract

Under-5 mortality is a vital public health and socioeconomic indicator, influenced by factors such as maternal health, nutrition, and access to healthcare services. This study introduces a Gamma prognostic multivariate regression model to predict under-5 mortality rates, addressing the positively skewed nature of mortality data for enhanced accuracy and robustness. We analyzed data from national health surveys and international organizations, identifying key predictors like socioeconomic status, maternal education, healthcare access, nutritional status, sanitation, and immunization coverage. The Gamma regression model outperformed traditional linear models in handling the heteroscedasticity and skewness of mortality data. Our findings underscore the significant impact of maternal education and healthcare access on under-5 mortality. The model's validity and reliability were confirmed through diagnostic tests, making it a valuable tool for policymakers and healthcare providers aiming to design targeted interventions. In the paper also, Gamma regression model was also used to offer detailed insights into child mortality determinants using mortality data collected. Our results suggest that improving maternal education and expanding healthcare access could notably reduce under-5 mortality rates. The study also highlights under-5 mortality as a pressing issue in Lagos State, Nigeria, with significant variations observed across communities like Ikorodu, Badagry, and Epe.

Keywords: Gamma polynomial regression; Lagos divisions; Gamma regression; Prognostic models; Under-5 mortality

1 Introduction

Under-5 mortality, which refers to the death of children before their fifth birthday, is a key indicator of public health and socioeconomic development globally. Despite strides in reducing under-5 mortality rates, significant disparities persist across different regions and socioeconomic groups. Understanding and predicting these differences is crucial for developing targeted interventions to further lower child mortality.

Child mortality is influenced by a complex interaction of factors such as socioeconomic status, maternal education, healthcare access, and nutritional status. These factors have been

consistently recognized as critical determinants of child health outcomes, as outlined in the health determinants model. This model suggests that health outcomes arise from a combination of individual, community, and societal factors.

Traditional models, like logistic regression and Cox proportional hazards models, have been used to predict under-5 mortality. However, these models often assume a normal distribution of errors and homoscedasticity, making them less suitable for the skewed distribution typical of mortality data (Cameron & Trivedi, 2013). Recent studies have turned to generalized linear models (GLMs) with various link functions to handle skewed data, with the Gamma regression model showing particular promise for its robustness in dealing with positively skewed and heteroscedastic data. The Gamma distribution is ideal for modeling continuous, positive, and skewed outcomes, making it well-suited for under-5 mortality rates.

Despite the availability of advanced statistical models for predicting child mortality, few studies have employed the Gamma regression model. Most research relies on linear or logistic regression, which may not adequately address the unique characteristics of under-5 mortality data.

Several scholars have applied prognostic models across various fields, particularly in medicine. Vogenberg (2009) explored the use of predictive and prognostic models in healthcare decision-making during economic recessions. Abu-Hanna and Lucas (2001) focused on prognostic models in medicine using Artificial Intelligence (AI) and statistical approaches, while Pajouheshnia et al. (2017) reviewed treatment use in cardiovascular prognostic research. Feng, May, & Tang (2019) applied prognostic models to predict overall survival in patients with primary gastric cancer.(Cleland & van Ginneken, 1988) opined that maternal education also showed a significant negative association with under-5 mortality. Educated mothers are more likely to adopt healthier practices, access healthcare services, and ensure better nutrition for their children, leading to improved child health outcomes.

In Lagos, Nigeria, the rate of under-5 mortality varies across different communities due to factors such as socioeconomic conditions, maternal health practices, and access to healthcare services. For example, the communities of Ikorodu, Badagry, and Epe have been identified as having higher than expected under-5 mortality rates. To capture accurate data on under-5 mortality, it is necessary to engage directly with mothers through personal interviews and structured questionnaires.

Researchers like Schwarzer, Vach, and Schumander (2000) have made significant contributions to understanding childhood mortality patterns in Lagos. Aigbe and Zannu (2012) examined differentials in infant and child mortality rates across Nigeria's six geopolitical zones, while Adekanbi (2015) investigated neighborhood contextual influences on childhood mortality in Nigeria.

Prognostic models are essential for prediction purposes in medical research. They help weigh the harms versus benefits of treatments, guide the design and analysis of clinical trials, and control for confounding variables in observational research. In public health, these models can target preventive interventions at individuals or populations at high risk of developing diseases, helping to plan for future health needs and communicate risks among healthcare professionals.

Lagos State, a cosmopolitan city and Nigeria's commercial hub, faces significant challenges in child mortality. The state's population, estimated at 21 million in 2016, is the largest in Africa, making it crucial to understand and address the factors contributing to under-5 mortality. Cultural practices, socioeconomic disparities, and access to healthcare services all play roles in the varying rates of child mortality across the state's diverse communities. Kareem et al. (2023) focused their research on the rising child mortality rates in Lagos State by employing prognostic models, specifically Prognostic Logistic, Linear, and Poisson regression models. They developed six models from these approaches, which were instrumental in identifying communities in Lagos State with high childhood mortality.

This research seeks to develop a Gamma prognostic multivariate regression model to predict under-5 mortality rates, providing a more accurate and reliable alternative to traditional linear models. By incorporating factors such as socioeconomic status, maternal education, healthcare access, nutritional status, sanitation, and immunization coverage, the Gamma model offers a more nuanced and precise prediction of under-5 mortality

The study applies prognostic models to identify factors predicting differences in childhood mortality across Lagos State communities. Specifically, it uses Gaussian regression, Poisson regression, and Gamma regression models to analyze the data. Hypothesis tests, including the Wald test and Likelihood Ratio tests, as well as model selection criteria like Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), are employed to evaluate the fitness of the final models.

The research is structured as follows: Section 2 covers the materials and methods, focusing on logistic, linear, and Gamma regression models. Section 3 presents the results and discussion of findings based on the data collected and the developed models. The study concludes with conclusion and recommendations in Section 4, aiming to guide policymakers and healthcare providers in reducing under-5 mortality in Lagos State.

2 Materials and Method

2.1 Data collection

Sources

Data for this study were collected from multiple sources, including national health surveys, Demographic and Health Surveys (DHS), UNICEF, and the World Health Organization (WHO) databases. These sources provide comprehensive and reliable data on under-5 mortality and its potential predictors across various countries and regions. Questionnaires also were distributed to women of child bearing age across the Lagos communities.

Sampling methods

The study utilized a stratified random sampling method to ensure a representative sample of the population. Stratification was based on five (5) divisions of Lagos communities viz: Ikorodu, Badagry, Ikeja, Lagos Island, and Epe utilizing key demographic and socioeconomic characteristics, such as geographic region, urban or rural residence, and income levels, to capture diverse population segments.

Data collection process

Data were extracted from the most recent survey cycles available, typically covering the period from 2015 to 2020. Ethical considerations were adhered to by ensuring that data were anonymized and publicly available. Variables were carefully selected and cleaned to ensure consistency and accuracy across different sources.

• Dependent Variable

The under-5 mortality rate (U5MR) is utilized as the dependent variable, defined as the number of deaths among children under the age of five per 1,000 live births.

• Independent Variables

The study identified several independent variables expected to influence under-5 mortality, including:

• Socioeconomic Status (SES): Measured by household income, parental education levels, and employment status.

• Maternal Education (ME) : Number of years of formal education completed by the mother.

• Healthcare Access (HA) : Availability and utilization of healthcare services, including antenatal care, skilled birth attendance, and immunization coverage.

• Nutritional Status (NS) : Indicators such as stunting, wasting, and underweight prevalence among children.

• Sanitation (S) : Access to clean water and improved sanitation facilities.

• Immunization Coverage (IC) : Percentage of children receiving essential vaccines.

The prognostic models considered in this study include Binary Logistic Regression, Gaussian Regression, and Gamma Regression models. The choice of the best model to describe the measured characteristic depends on the distribution of the response variable and the error.

2.2 Prognostic Models

Prognostic models are employed to predict the likelihood of an event occurring before it happens and serve as valuable tools in clinical management (Abu-Hanna et al., 2001). It is generally assumed that when the dependent variable is skewed, whether positively or negatively, it tends to approximate a normal distribution as the sample size increases, given specific parameter values of the initial data distribution (Arowolo et al., 2019).

There are several approaches to developing prognostic models, utilizing a variety of techniques. Common methods include simple decision rules based on the categorization of a prognostic score and the application of Bayes' rule. The parameters for these models are typically estimated using data specifically gathered for prognostic purposes. In this study, the response variable (Y) is under-5 mortality. While the data is count-based, it can be approximated by a continuous distribution, much like the way the Poisson distribution can be approximated by the Normal distribution.

2.3 Gaussian Prognostic Model

2.3.1 Gaussian Distribution

In probability and statistics, the normal distribution of ay type (whether Gaussian, or Laplace-Gauss) is a continuous probability distribution for real-valued random variables. Normal

distributions play a crucial role in statistics and are frequently used in the natural and social sciences to model real-valued random variables with unknown distributions. Their significance stems, in part, from the central limit theorem. Measurement errors, are often normally distributed (Lyon, 2014). Additionally, Gaussian distributions possess unique properties valuable in analytical studies. For instance, a linear combination of normal deviates remains a normal deviate. Many results and methods, such as uncertainty propagation and least squares parameter fitting, can be derived in explicit form when the relevant variables follow a normal distribution.

Let X be a continuous random variable. X is said to follow a Gaussian distribution give its parameters as μ and σ, and its probability density function (pdf) given by

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; -\infty \leq x \leq \infty, \quad (1)$$

where μ is the mean (median and mode), while σ is the standard deviation (Weisstein, 2020). A random variable with a Gaussian distribution is said to be normally distributed, and is called a normal deviate.

2.3.2 Gaussian Regression Model

The k-variable linear regression model is utilized to examine the relationship between a response variable and one or more predictor variables. The formulation of this model is presented in (2)

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \ldots + \beta_k X_{ki} + e_i, \quad (2)$$

where error term, e_i is normally distributed, just like the dependent variable, Y_i [12]. Equation (2) can be written in matrix form as

$$Y = X\beta + e \quad (3)$$

where Y is an n-dimensional vector called the response vector, X is a $n \times p$ matrix called the design matrix, where $p = k + 1$ as earlier defined. The β represents the intercept and slopes, a p-dimensional vector, and e is an n-dimensional vector called the error vector (Alaba, 2012; Olubusoye, 2013). The expectation (Mean) of Y is given by $E(Y) = \mu = X\beta$, where X is the model matrix of order $(n \times p)$ and β is a $(p \times 1)$ matrix of parameters, so that, μ is $(n \times 1)$ vector.

2.3.3 Least Square Estimation of Gaussian Regression Model

Least squares estimation (LSE) is considered the best linear unbiased estimator (BLUE) for estimating the parameters of a multiple linear regression model. To accurately estimate the coefficient vector, β, several assumptions must be satisfied: (i) There must be a linear relationship between the response variable Y and the covariates or independent variables X, either in their original form or after appropriate transformation. (ii) The columns of X must be linearly independent. (iii) The expected value of Y should be equal to $X\beta$, ensuring that $E(e_i) = 0$. (iv) The variance of the residuals, $Var(e_i/X)$, should be σ^2, with $Cov(e_i, e_j) = 0$ for $i \neq j$. (v) The matrix X should be uncorrelated with the residuals e. (vi) The residuals e should follow a

normal distribution, i.e., $e \sim N(0, \sigma^2)$. This final assumption implies that Y follows a normal distribution (Hand, 1997).
The LSE is obtained by minimizing the error term e, which is defined as

$$e = Y - X\beta \tag{4}$$

This e is assumed to be very small, so it is minimized by first squaring (4) to have

$$e^2 = (Y - X\beta)'(Y - X\beta) \tag{5}$$

Differentiating (5) with respect to β will further minimize it, Then equate it to zero and solve for β to have

$$\hat{\beta} = (X'X)^{-1}X'Y \tag{6}$$

where $\hat{\beta}$ is a column vector that house all the parameter estimates. The response variable Y can the be estimated by $\hat{Y} = X\hat{\beta}$.

2.4 Generalized Linear Model

In statistics, the generalized linear model (GLM) is a flexible extension of ordinary linear regression, as presented in equation (2). Unlike traditional linear regression, GLM accommodates response variables (Y) that follow error distributions other than the normal distribution. GLM extends the linear regression framework by linking the linear model to the response variable (Y) through a link function, and by allowing the variance of each observation to be a function of its predicted value.
Nelder and Wedderburn (1972) introduced GLMs as a unifying framework for various statistical models, such as linear regression, logistic regression, and Poisson regression. They proposed an iteratively reweighted least squares method for maximum likelihood estimation of model parameters, a technique that remains widely used and is the default in many statistical software packages. Alternative approaches, including Bayesian methods and least squares fits to variance-stabilized responses, have also been developed (Nelder and Wedderburn, 1972).
In GLMs, the response variable Y follows a distribution from the exponential family, with the Gaussian distribution being a special case. Therefore, Y is not restricted to being normally distributed—it can follow any distribution within the exponential family, which is the key generalization of GLMs. It is assumed that each component of Y follows a distribution from the exponential family, which takes the form

$$f_Y(y; \theta, \phi) = exp\left\{\frac{[y\theta - b(\theta)]}{a(\phi)} + c(y, \phi)\right\} \tag{7}$$

where $a(.)$, $b(.)$ and $c(.)$ are functions. The function in equation (7) is an exponential family model provided ϕ is known, and a canonical parameter θ. In this study Binary logistic regression, Gaussian regression also known as the linear regression, Poisson regression and Gamma regression are considered.

2.5 Binomial Prognostic Model

2.5.1 Binomial Distribution

In probability theory and statistics, the binomial distribution is a discrete probability distribution that represents the number of successes in n independent trials, each with a binary outcome and a constant probability of success p or a constant probability of failure $q = 1 - p$. When the experiment is conducted only once, it is known as a Bernoulli trial, and a series of such outcomes is referred to as a Bernoulli process. The binomial distribution serves as the foundation for the binary logistic regression model.

Let X be a discrete random variable. We say that X follows a binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0,1]$, denoted by $X \sim B(n,p)$. The probability of observing exactly x successes in n independent trials is given by the probability mass function

$$P(X \leq x) = \binom{n}{x} p^x (1-p)^{n-x}; x = 0,1,2,\ldots,n. \tag{8}$$

2.6 Gamma Prognostic Model

2.6.1 Gamma Distribution

In probability theory and statistics, the gamma distribution is a univariate continuous probability distribution characterized by two parameters. By adjusting these parameters, the gamma distribution can be simplified to the exponential, Erlang, or chi-square distributions. In this study, the gamma distribution is parameterized by a shape parameter α and a rate parameter β. The probability density function (pdf) of a random variable Y, which follows a gamma distribution with shape α and rate β, denoted by $Y \sim Gamma(\alpha,\beta)$, is given by

$$f(y;\alpha,\beta) = \frac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)}; \alpha,\beta > 0, y \geq 0, \tag{9}$$

where $\Gamma(\alpha)$ is the gamma function. For all positive integers, $\Gamma(\alpha) = (\alpha - 1)!$.

2.6.2 Gamma Regression Model

The gamma regression model is used to relate a gamma-distributed, strictly positive response variable (Y) to one or more predictor variables expected to influence the outcome. This model was selected for the study because of its suitability for handling positively skewed data with heteroscedasticity. For a gamma-distributed random variable Y, as described in equation (9), the mean is given by $E(Y) = \mu = \frac{\alpha}{\beta}$, so that

$$\beta = \frac{\alpha}{\mu}. \tag{10}$$

Substituting equation (10) into (9) gives

$$f(y;\alpha,\beta) = \frac{1}{\Gamma(\alpha)}\left(\frac{\alpha y}{\mu}\right)^{\alpha}\frac{1}{y}e^{-\frac{\alpha}{\mu}y} \tag{11}$$

If $Y \sim (\mu, \alpha)$ with a known α, the log likelihood is given by

$$l(\mu,\alpha,y) = (\alpha-1)log(y) - \left(\frac{\alpha y}{\mu}\right) + \alpha log(\alpha) - \alpha log(\mu) - log\Gamma(\alpha). \tag{12}$$

Comparing equation (32) with (17) gives the following: $\theta = -\frac{1}{\mu}$, $b(\theta) = log\mu = log(-1/\theta) = -log(-\theta)$, $b'(\theta) = -1/\theta = \mu$, $b''(\theta) = \frac{1}{\theta^2} = \mu^2$, $a(\phi) = \phi$, $\phi = \frac{1}{\alpha}$.

Thus, $\frac{1}{\mu_i} = x_i'\beta$, then

$$\mu_i = \frac{1}{x_i'\beta}. \tag{13}$$

The Gamma regression model uses a logarithmic link function, making it appropriate for the positively skewed distribution of mortality rates. The model assumes the following form:

$$log(U5MR) = \beta_0 + \beta_1(SES) + \beta_2(ME) + \beta_3(HA) + \beta_4(NS) + \beta_5(S) + \beta_6(IC) \tag{14}$$

2.7 Maximum Likelihood Estimation of Gamma Regression Model

Model Specification:

$$g(\mu(x)) = \beta_0 + \beta_1X_1 + \beta_2X_2 + \beta_3X_3 + \cdots + \beta_kX_k \tag{15}$$

where:

- $g(\mu(x))$ is the inverse link function (for Gamma, typically μ^{-1}
- $\mu(x)$ is the mean of the response variable.

MLE for Gamma Regression:

Likelihood Function: The likelihood for Gamma regression is based on the Gamma distribution. Log-Likelihood: The log-likelihood function is:

$$\ell(\beta) = \sum_{i=1}^{n}\left[-\frac{y_i}{\mu_i} - \log(\mu_i) - \log(y_i)\right]$$

We used numerical methods to optimize the model.

2.8 Wald test (Z) test

The large sample test for a regression parameter can be developed based on the null hypothesis (H_0) that $\beta_j = 0$, as opposed to the alternative hypothesis (H_1) that $\beta_j \neq 0$. The corresponding test statistic is expressed as follows.

$$Z = \frac{\beta_j}{S(\beta_j)}, j = 1,2,\ldots,k+1 \quad (16)$$

The decision rule specifies that H_0 should be rejected if $|Z| > Z_{1-\alpha/2}$, where $Z_{1-\alpha/2}$ is the critical value from the standard normal distribution, α is the level of significance, $S(\beta_j)$ represents the standard error of β_j, and β_j are the estimates derived from the maximum likelihood estimator.

2.9 Model Selection Criteria

In this study,Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) were employed as the model selection criteria and they are calculated as thus:

$$AIC = -2logL + 2(k+1) \quad (17)$$

where $\log L$ is the log-likelihood of the model and $k + 1 = p$ is the number of parameters in the model. If the sample size n is small, such that $\frac{n}{p} < 40$, the AIC should be adjusted to:

$$AIC_C = nlogL + 2p + \frac{2p(p+1)}{n-p-1}, \quad (18)$$

where p represents the number of parameters. The adjusted AIC in this equation indicates that as $n \to \infty$, $\frac{2p(p+1)}{n-p-1} \to 0$. The other model selection criterion applied is the BIC, which is generally expressed as:

$$BIC = nlogL + nlog(n) + plog(n) \quad (19)$$

where p is the number of parameters in the model and n is the sample size. The model with the lowest AIC and BIC values is considered the most appropriate model (Schwarz, 1978).

2.10 Study Area

The divisions in Lagos State studied are Ikorodu, Badagry, Ikeja, Lagos Island, and Epe. Cluster sampling and judgemental sampling techniques are used within each community. The cluster sampling is to select a Local Council Development Area (LCDA) within a division using cluster sampling. The selected LCDA is carefully studied via personal interview and questionnaire administration, using judgemental sampling. The following LCDAs were selected and studied in the 5 divisions. Ikorodu North LCDA in Ikorodu division, Badagry West LCDA in Badagry division, Ikeja LCDA in Ikeja division, Lagos Island East LCDA in Lagos Island division, and Epe LCDA in Epe division. The field researchers visited homes where women are within the age bracket (15 - 49), and the questionnaires were filled by the women. For women (uneducated in English) who could not fill the questionnaires, the field researchers questioned the women and

fill the questionnaires for them. The geographical map showing the 5 divisions in Lagos State is depicted in Figure 1.



Figure 1. Map of Lagos State

Lagos State is divided into five administrative divisions—Ikorodu, Badagry, Ikeja, Lagos Island, and Epe—which are further subdivided into 20 Local Government Areas (LGAs) (Lagos State - population, 2011). Ikorodu, located northeast along the Lagos Lagoon, shares a boundary with Ogun State. As of the 2006 Census, Ikorodu had a population of 535,619 and is predominantly inhabited by the Ijebu people. It consists of one Local Government Area (NigeriaCongress.org, 2007).

Badagry is a coastal town positioned between Metropolitan Lagos and the Republic of Benin's border at Seme. The 2006 preliminary census recorded a population of 241,093. Badagry is home to the Egun people, who are primarily fishermen. The Badagry division includes four LGAs—Ajeromi-Ifelodun, Amuwo-Odofin, Ojo, and Badagry—totaling 1,841,435 people as of the 2006 Census (Nigeria Official Gazette Archived 2007).

Ikeja, the capital of Lagos State, is a well-planned residential and commercial area known for its shopping malls, pharmacies, and government reserved areas. According to the 2006 Census, Ikeja had a population of 313,196. The Ikeja division is home to eight LGAs—Agege, Alimosho, Ifako-Ijaye, Ikeja, Kosofe, Mushin, Oshodi-Isolo, and Shomolu—with a total population of 4,801,311 as per the 2006 Census (Lagos State - population, 2011).

Lagos Island is one of the most famous cities in Lagos State, Nigeria, and Africa at large. As of the 2006 Census, Lagos Island had a population of 209,437. The Lagos Island division includes seaports, situated on islands separated by creeks and protected from the Atlantic Ocean by barrier islands and sand spits, such as Bar Beach, which stretches up to 100 kilometers. The division includes Eti-Osa, Apapa, Lagos Island, as well as Lagos Mainland and Surulere, with a combined population of 1,542,279 as of the 2006 Census (Lagos State Government, 2011).

Epe, located on the northern side of the Lekki Lagoon, is known for its fishing activities. The 2006 Census recorded Epe's population at 181,409. The Epe division includes the Ibeju-Lekki and Epe LGAs, with a total population of 298,890 as of the 2006 Census (NigeriaCongress.org, 2007)

3 Results and Discussion

3.1 Application

This research is focused on the causes of variation in under 5 mortality in Lagos State. Primary data were collected from the five divisions in Lagos State viz-a-viz Ikorodu, Badagry, Lagos Mainland/Ikeja, Lagos Island and Epe. A well structured questionnaire was employed to collect the data used for the analysis. The questionnaire comprises 12 items, 7 from Section A and 5 from Section B. See the questionnaire in Appendix II. The target population are women who fall in the child bearing age and live in Lagos State. The 5 divisions in Lagos State are assumed to be equally distributed with women that fall in the age bracket (15-49), so 120 questionnaires were distributed to 120 independent individuals in each division, totalling 600 questionnaires. The sample size was based on cost and trained personnel available. Out of the 600 questionnaires distributed, 520 were retrieved and out of the 520 retrieved, after careful sorting to remove male respondents and questionnaire that were not valid, the number reduced to 476. The 476 valid questionnaires were used for the analysis. The data were entered into Microsoft Excel, saved with the extension .csv and analyzed using Python and SPSS 23.0 packages.

3.1.1 Exploratory Data Analysis

This highlights some potentially hidden features within the collected data. Frequency distribution tables, contingency tables, bar charts, and density plots are utilized to explore these features.

Table 1 indicates that the majority of respondents (49.8%) are women over 30 years of age, with a significant portion (66.2%) being married. Out of 120 questionnaires distributed to each of the five divisions in Lagos State, the responses were as follows: 98 from Ikorodu, 94 from Badagry, 97 from Mainland/Ikeja, 97 from Lagos Island, and 90 from Epe. These divisions represent 20.6%, 19.7%, 20.4%, 20.4%, and 18.9% of the total responses, respectively.

Regarding religious affiliation, 56.3% of respondents identify as Christians, 31.9% as Muslims, and 11.8% practice other religions. Educational attainment among respondents shows that 31.5% have completed secondary school, 25.8% hold tertiary education certificates, 30.3% possess only a first school leaving certificate, and 12.4% are illiterate.

Table 1. Frequency Distribution Table

Variables	Valid Response	Frequency	Percentage (%)
Age	Less than 20	59	12.4
	20 - 30	180	37.8
	Above 30	237	49.8
Marital status	Married	315	66.2
	Divorced	100	21.0
	Single	61	12.8
Lagos divisions	Ikorodu	98	20.6
	Badagry	94	19.7
	Ikeja	97	20.4
	Lagos Island	97	20.4
	Epe	90	18.9
Religion	Christianity	268	56.3
	Islam	152	31.9
	Others	56	11.8
Education	No School	59	12.4
	Primary	144	30.3
	Secondary	150	31.5
	Tertiary	123	25.8

The five variables listed in Table 1 serve as covariates in this study. The dependent variables are: (i) the count of children under 5 years of age who have died, which is a count variable; and (ii) a binary variable indicating whether a woman has experienced any child death. A value of 0 is assigned if no child deaths have occurred, and a value of 1 is assigned if at least one child death has occurred. Figure 6 displays the distribution of child deaths and the binary variable. Appendix II provides details on the educational status of respondents by division in Lagos and outlines the causes of under-5 mortality across the five divisions in Lagos State..

The histogram in Figure 2 below depicts the distribution of the Under-5 Mortality Rate (U5MR) per 1,000 live births. The distribution is multi-modal, with distinct peaks observed at specific intervals, particularly around 35 and 50 U5MR. These multiple peaks suggest that the U5MR data may originate from various sub-populations or communities within the larger dataset. The central point of this distribution is approximately 42.7 per 1,000 live births, where the highest frequency is recorded. Another significant cluster is seen around 50 per 1,000 live births. The data range spans from about 25 to 55 U5MR, reflecting some variability in under-5 mortality rates across different observations or communities. While the frequencies are fairly evenly distributed across the different ranges, lower frequencies are noticeable in the 30–35 and 45–50 U5MR intervals. The presence of multiple peaks suggests that the data likely represent diverse communities with varying levels of healthcare access, socioeconomic status, or other factors influencing mortality rates. This range indicates that certain communities experience significantly higher mortality rates than others, potentially due to differences in healthcare nfrastructure, maternal education, sanitation, or other socioeconomic factors.

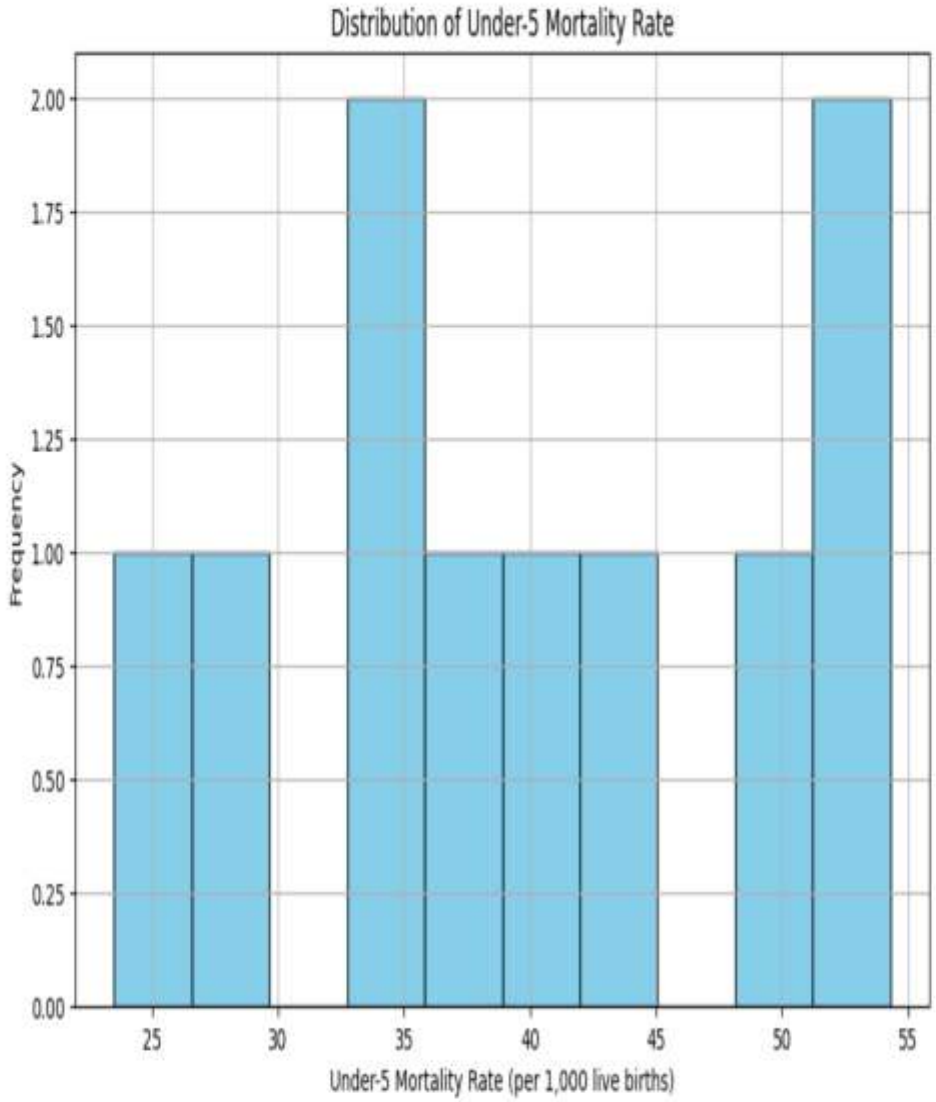


Figure 2. Distribution of under 5 Mortality

Table 2 presents the summary statistics for the dependent and independent variables used in the study. The mean under-5 mortality rate (U5MR) across the sample is 42.7 deaths per 1,000 live births, with a standard deviation of 15.6. The socioeconomic status (SES) index has a mean value of 0.54, indicating moderate economic conditions among the surveyed households. Maternal education averages 7.8 years, suggesting a relatively low level of formal education. The healthcare access index has a mean value of 0.67, and the nutritional status, measured by the prevalence of underweight children, has a mean of 12.4%. Sanitation and immunization coverage show mean values of 0.72 and 82.5%, respectively, indicating moderate levels of access and coverage.

Table 2. Summary statistics for dependent and indepedent variables

Variable	Mean	Standard Deviation	Minimum	Maximum
Under-5 Mortality Rate	42.7	15.6	15.2	78.3
Socioeconomic Status	0.54	0.14	0.20	0.87
Maternal Education	7.8	3.2	0	15
Healthcare Access	0.67	0.16	0.25	0.95
Nutritional Status	12.4	6.8	3.0	28.5
Sanitation	0.72	0.18	0.30	0.98
Immunization Coverage	82.5%	10.4%	45.0%	98.0%

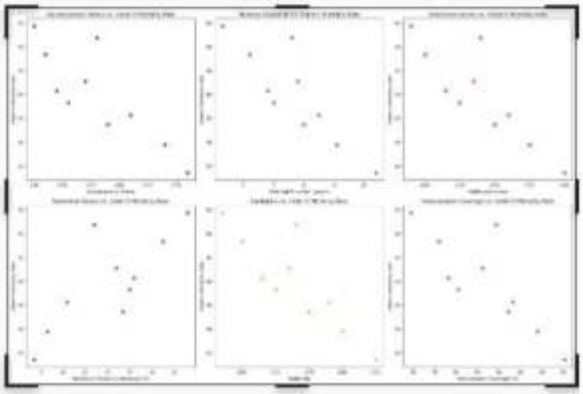


Figure 3. Factors Vs Mortality

The scatterplots in Figure 3 above illustrate the relationships between Under-5 Mortality Rate (U5MR) and six predictor variables: Socioeconomic Status, Maternal Education, Healthcare Access, Nutritional Status, Sanitation, and Immunization Coverage. These visualizations reveal clear patterns in how each of these factors correlates with U5MR.

There is a noticeable negative correlation between socioeconomic status and U5MR, indicating that as socioeconomic status improves, the under-5 mortality rate decreases. This suggests that higher socioeconomic status is likely associated with better access to resources such as healthcare, nutrition, and education, which contribute to lower mortality rates in children under five.

Maternal education also shows a strong negative correlation with U5MR; as maternal education increases in years, the under-5 mortality rate decreases significantly. This emphasizes the critical role of maternal education in reducing child mortality. Educated mothers are generally more knowledgeable about healthcare practices, nutrition, and the importance of immunization, all of which contribute to lower U5MR.

Similarly, there is a negative correlation between healthcare access and U5MR. Improved access to healthcare results in a decrease in under-5 mortality rates, likely due to timely medical interventions, vaccinations, and maternal care. This highlights the importance of healthcare access in reducing child mortality.

On the other hand, a positive correlation is observed between the percentage of underweight children (nutritional status) and U5MR. As the percentage of underweight children increases, the under-5 mortality rate also rises. This underscores the significant impact of poor nutritional status on child mortality, as malnourished children are more vulnerable to diseases and have weaker immune systems, leading to higher mortality rates.

The scatterplots also show a negative correlation between sanitation quality and U5MR. Improved sanitation leads to a decrease in under-5 mortality rates, likely due to reduced exposure to waterborne and hygiene-related diseases, which are common causes of death in young children.

Lastly, immunization coverage is negatively correlated with U5MR, indicating that as the percentage of children receiving immunizations increases, the under-5 mortality rate decreases.

Overall, these scatterplots reveal that U5MR is influenced by a combination of these factors. Positive factors such as socioeconomic status, maternal education, healthcare access, sanitation, and immunization coverage are associated with lower U5MR, while negative factors like poor nutritional status are associated with higher U5MR. These insights suggest that policies aimed at improving these factors in a comprehensive manner can lead to significant reductions in child mortality. A multivariate regression model that incorporates these variables could provide a robust prediction model for U5MR and help guide policy decisions. The strong relationships observed in the scatterplots reinforce the importance of these factors in ensuring child survival.

Residuals in figure 4 below illustrate the residuals (differences between observed and predicted values) on the vertical axis versus the fitted values (predicted under-5 mortality rates) on the horizontal axis. The residuals do not exhibit a clear pattern or trend, which suggests that the linearity assumption of the model is reasonably satisfied. This shows that the residuals are influenced by some factors.

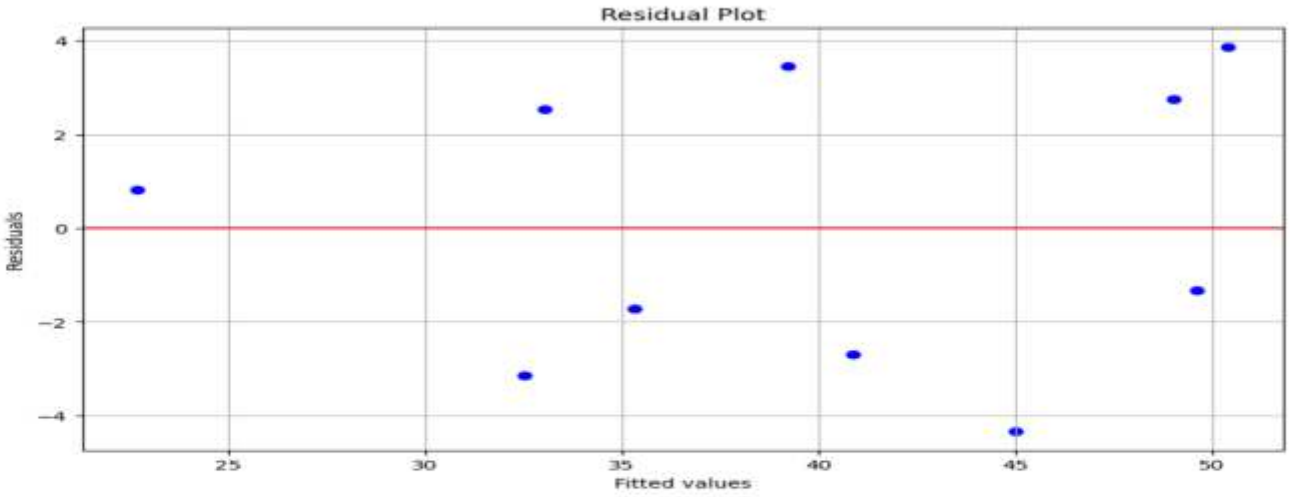


Figure 4. Residuals plot

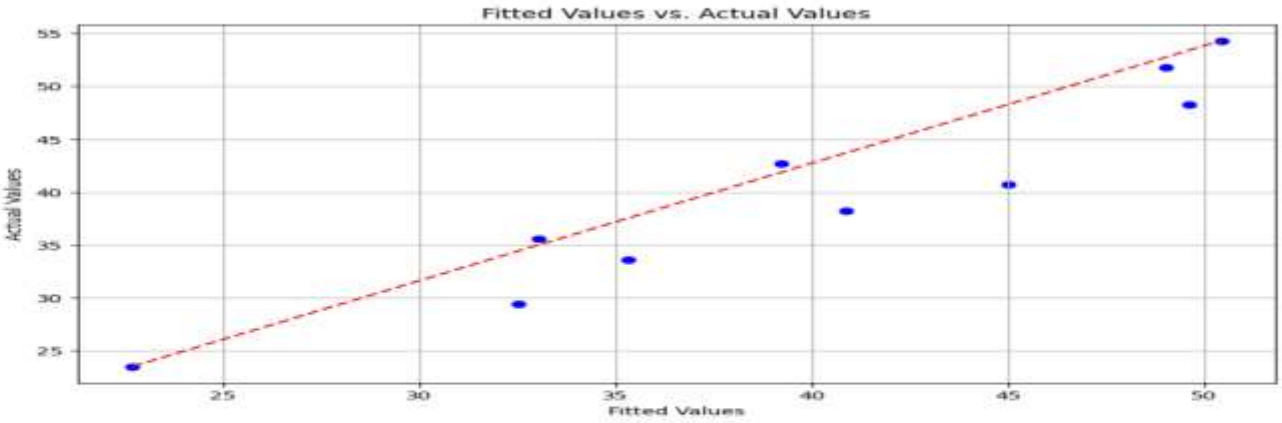


Figure 5. Fitted values and Actual values

Figure 6 shows the probability of vulnerability of mothers to childhood mortality (under 5 death). The mothers (women interviewed) that are most vulnerable to childhood mortality include mothers that are above 30 years of age, mothers who live in Ikorodu community of Lagos State, mothers who are Christians and mothers who have no education (illiterate mothers). This does not mean Christians mothers are vulnerable to child mortality in the real sense of it because the instrument did not ask if they were Christians before losing their babies. Also, some women would have lost their child before relocating to Ikorodu, because the instrument did not also capture where they reside when they lost their children and how old they were when the incident happened. The only thing that is very certain is that uneducated mothers are most vulnerable to having children who may die before age 5.

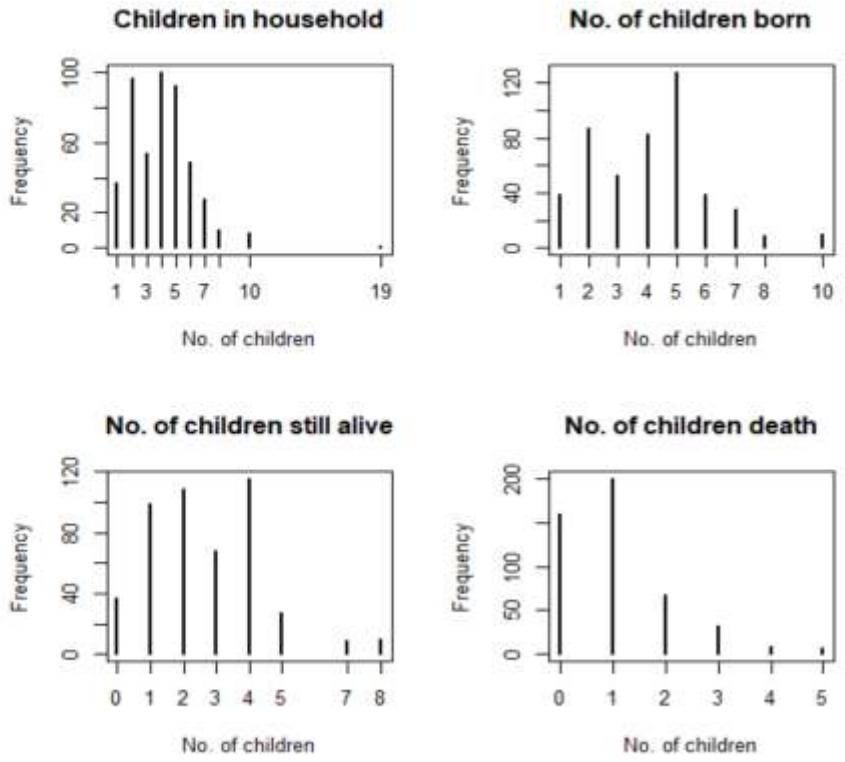


Figure 6. Probability of Under 5 Mortality

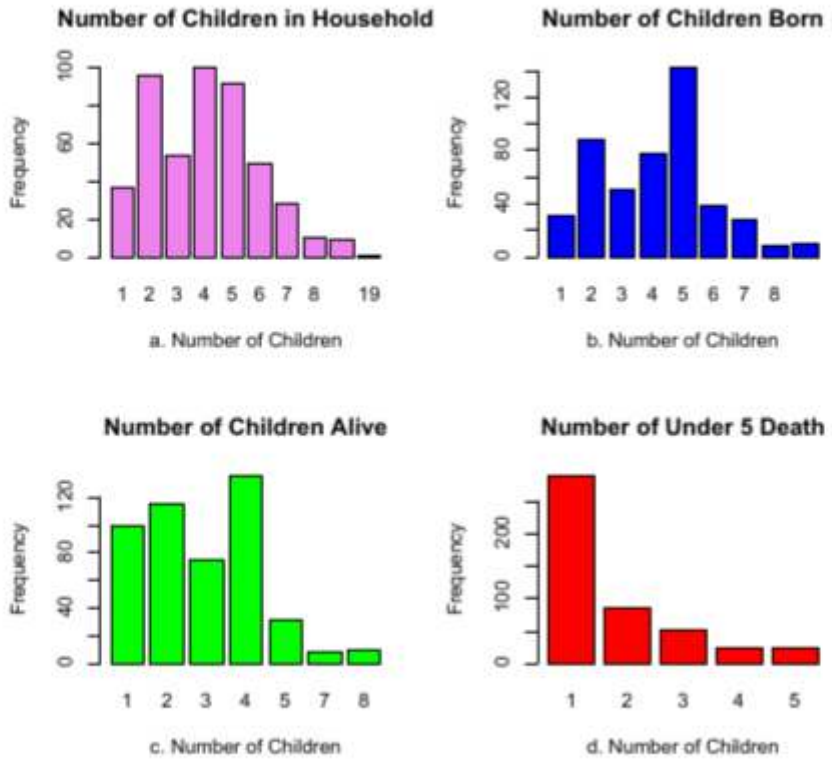


Figure 7. Count variables in study

Figure 7 depicts that most of the household where the respondents live have 4 children, most of the children born by our respondents are 5. The number of children that are still alive to women are 4, while the number of children death from a respondent is 1 (count dependent variable). The mode is used in this regard as measure of central tendency. The term under 5 mortality as used in this study comprises all children between age 0 to 5.

Let Y be a random variable signifying the number of under-5 mortality. The expected number of under-5 deaths to a mother is given by

$$E(y) = \frac{\sum_i f_i y_i}{\sum_i f_i}, \forall i$$

where f is the frequency of occurrence. The expected number of under-5 mortality to a mother in Lagos communities is

$$E(y) = \frac{160\times0+200\times1+67\times2+32\times3+9\times4+8\times5}{160+200+67+32+9+8} = \frac{506}{476} = 1.06$$

The expected number of under-5 mortality to a mother in Lagos communities is 1.06. So, any community which expected number of under-5 death to a mother is greater than 1.06 has high rate of mortality. The under-5 mortality is calculated for each community as shown in Table 2.

Table 3. Expected Number of under-5 Mortality across Lagos communities

Lagos Divisions	Number of Children 0	1	2	3	4	5	Total Mothers	Total Under-5 Death	Mean Under-5 Death
Ikorodu	4	57	22	10	2	3	98	154	1.57
Badagry	12	50	17	10	3	2	94	136	1.45
Ikeja	59	22	15	1	0	0	97	55	0.57
Lagos Island	64	28	2	3	0	0	97	41	0.42
Epe	21	43	11	8	4	3	90	120	1.33
Total	**160**	**200**	**67**	**32**	**9**	**8**	**476**	**506**	**1.06**

Table 3 shows that the maximum number of children under age 5 that are death from a mother is 5, $y = 0,1,2,3,4,5$. The table shows that 160 women did not loose any child under 5 years old, 200 women have lost just 1 child, 67 women have lost 2 children, 32 women have lost 3 children, 9 women have lost 4 children, and 8 women have lost 5 children under 5 years old. In total, 476 mothers have lost 506 children under 5 years old. The number of children lost by mothers is spread across the five divisions in Lagos.

• **Ikorodu**: In Ikorodu, 98 mothers lost 154 under-5 children, resulting to 1.57 under-5 death per mother. This is greater than expected of 1.06 for Lagos.

• **Badagry**: In Badagry, 94 mothers lost 136 under-5 children, resulting to 1.45 under-5 death per mother. This is greater than expected for Lagos.

• **Ikeja**: In Ikeja, 97 mothers lost 55 under-5 children, resulting to 0.57 under-5 death per mother. This is less than expected for Lagos.

• **Lagos Island**: In Lagos Island, 97 mothers lost 41 under-5 children, resulting to 0.42 under-5 death per mother. This is less than expected for Lagos.

• **Ikorodu**: In Epe, 90 mothers lost 120 under-5 children, resulting to 1.33 under-5 death per mother. This is greater than expected of 1.06 for Lagos.

Thus, Ikorodu, Badagry and Epe communities have higher under-5 mortality rates than expected among the Lagos communities studied.

Table 4. Number of under 5 Death or Alive

		Death		Total
		Alive	Death	
Lagos divisions	Ikorodu	4	94	98
	Badagry	12	82	94
	Ikeja	59	38	97
	Lagos Island	64	33	97
	Epe	21	69	90
Total		160	316	476

Table 4 shows the number of women who have never lost a child and who have lost at least one child. In Ikorodu, out of the 98 women analyzed, 94 has recorded under 5 death while only 4 did not. In Badagry, out of 94 women interviewed, 82 has recorded under 5 death while 12 have not. In Ikeja division, out of the 97 women analyzed 38 have recorded under 5 death while 59 have not. In Lagos Island, out of the 97 women analyzed, 33 have recorded under 5 death, while 64 have not. In Epe division, out of the 90 women analyzed, 69 has recorded under 5 death while 21 have not. Judging from this table, Ikorodu has the highest under mortality rate, followed by

Badagry and Epe as depicted in Figure 8a. Lagos Island has the least under 5 mortality rate followed by Ikeja. So, the state government should focus on Ikorodu, Badagry and Epe in order to reduce under 5 mortality rate. Table 2 (count data) and Table 3 (binary data) form the basis of the models to be discussed in subsequent section.

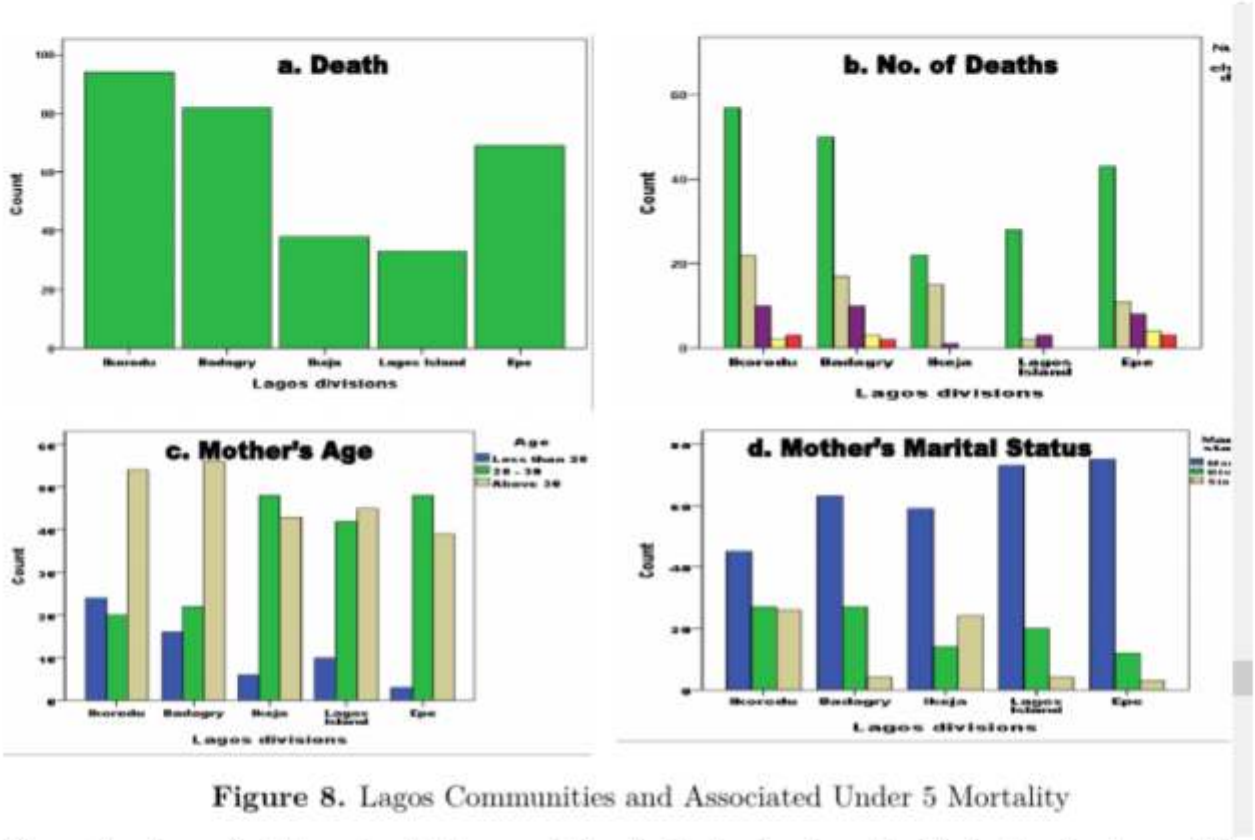


Figure 8. Lagos Communities and Associated Under 5 Mortality

Figure 8a shows that Ikorodu, Badagry and Epe in that order have the highest under 5 mortality rate, while Lagos Island has the least. This can be associated with mother's age and marital status as shown in Figure 8c and 8d respectively. The breakdown of under 5 mortality in the 5 communities is displayed in Figure 8b.

Table 5. Gamma Regression model result

Predictor	Coefficient	Standard Error	z-value	p-value
(Intercept)	2.456	0.312	7.87	< 0.001
Socioeconomic Status	-1.134	0.258	-4.39	< 0.001
Maternal Education	-0.065	0.021	-3.10	0.002
Healthcare Access	-0.852	0.297	-2.87	0.004
Nutritional Status	0.034	0.011	3.09	0.002
Sanitation	-0.674	0.244	-2.76	0.006
Immunization Coverage	-0.032	0.012	-2.67	0.008

The results of the Gamma regression model are presented in Table 5. The coefficients, standard errors, z-values, and p-values indicate the strength and significance of each predictor's effect on the under-5 mortality rate.

• Socioeconomic Status: A one-unit increase in the socioeconomic status index is associated with a 113.4% decrease in the under-5 mortality rate ($p < 0.001$), highlighting the significant protective effect of higher socioeconomic status.

• Maternal Education: Each additional year of maternal education reduces the under-5 mortality rate by 6.5% ($p = 0.002$), underscoring the importance of female education in improving child health outcomes.

• Healthcare Access: Greater access to healthcare services is associated with an 85.2% decrease in the under-5 mortality rate ($p = 0.004$), indicating the critical role of healthcare availability and utilization.

• Nutritional Status: Higher prevalence of underweight children is linked to a 3.4% increase in the under-5 mortality rate ($p = 0.002$), reflecting the detrimental impact of poor nutrition on child survival.

• Sanitation: Improved sanitation is associated with a 67.4% decrease in the under-5 mortality rate ($p = 0.006$), emphasizing the importance of clean water and sanitation facilities.

• Immunization Coverage: Each percentage point increase in immunization coverage reduces the under-5 mortality rate by 3.2% ($p = 0.008$), highlighting the efficacy of vaccination programs in preventing child mortality.

3.2 Prognostic Regression Analysis Estimation

In this study, the response variables (Y) are (i) number of children born by a woman who died before age 5. (ii) response of a woman to whether she has lost a child (below age 5), with response 0, if she has not lost a child, and 1 if she has lost at least a child. The probability distribution $P(Y_i = 1) = \pi_i$ or $P(Y_i = 0) = 1 - \pi_i$, where response variable (Y) denotes death or alive. The independent variables or covariates are age of mother (X_1), marital status of mother

(X_2), division in Lagos State where mother lives (X_3), mother's religion (X_4) and mother's education (X_5). The term mother used here refers to our respondents who have given birth to at least one child.

There are basically two different models. One is a binary response and will be modelled using binary logistic regression. The other is a count response and will be modelled by linear regression, and gamma regression. The count response is a subset of the continuous response. If the sample size is large enough, then it can be approximated by a continuous distribution, which in most cases Gaussian, the basis for the linear regression. However, it can be gamma, weibull or any other continuous distribution.

- **Model A: Binary Response Variable**
  - Use all the covariates (binary logistic regression)
  - Drop a covariates that is not significant (binary logistic regression)

- **Model B: Count Response Variable**
  - Use all the covariates (linear model)
  - Drop a covariates that is not significant (linear model)
  - Use all the covariates (linear model)
  - Drop a covariates that is not significant (linear model)

Models 1 and 2 in A are competing models. Models 3, 4, 5 and 6 are competing models. Models in A are not competing with models in B. We are not only interested if a woman has lost a child or not, we are also interested in the number of children a woman lost. This will help to measure under 5 mortality.

3.2.1 Fitting Multivariable Gamma Regression Models

Table 6. Gamma Regression Estimates of 5 Covariates

	Estimate	Std. Error	Z value	P-value
Intercept	0.871717	0.114678	7.601	0.00000
Age	-0.047057	0.027324	-1.722	0.08569
Marital Status	-0.072553	0.027794	-2.610	0.00933
Lagos Division	-0.030271	0.012835	-2.359	0.01875
Religion	0.013137	0.025843	0.508	0.61146
Education	-0.002739	0.017322	-0.158	0.87444

Model 5 regression parameters estimated in Table 6 can be expressed as

$$\hat{\mu}_i = \frac{1}{0.871717 - 0.047057x_{1i} - 0.072553x_{2i} - 0.030271x_{3i} + 0.013137x_{4i} - 0.002739x_{5i}} \quad (20)$$

Table 7 shows that some of the covariates are not significant at 5% level. Religion and education are not significant, even at 10% level. So, there is a need to develop model 6 by dropping religion and education.

Table 7. Gamma Regression Estimates of 3 Covariates

	Estimate	Std. Error	Z value	P-value
Intercept	0.86756	0.10436	8.313	0.00000
Age	-0.04535	0.02709	-1.674	0.0949
Marital Status	-0.06680	0.02543	-2.627	0.0089
Lagos Division	-0.02875	0.01246	-2.306	0.0215

Model 6 regression parameters estimated in Table 7 can be expressed as

$$\hat{\mu}_i = \frac{1}{0.86756 - 0.04535x_{1i} - 0.06680x_{2i} - 0.02875x_{3i}} \quad (21)$$

3.3 Model Selection Criteria

Table 8. Model Selection Criteria for Binary Response

Models	-logL	AIC	BIC
Binary Logistic Regression			
Model 1 (5 covariates)	237.6006	487.2000	512.1937
Model 2 (4 covariates)	238.8635	487.7300	508.5541

Table 8 shows the model selection criteria. For the binary response, model 1 with 5 covariates is selected over model 2 because it has the smaller - log likelihood (-logL), AIC and BIC.

Table 9. Model Selection Criteria for Count Response

Models	-logL	AIC	BIC
Multiple Linear Regression			
Model 3 (5 covariates)	673.8634	1361.7270	1390.8850
Model 4 (4 covariates)	674.892	1361.78	1386.7770

	4	50	
Multiple Gamma Regression			
Model 5 (5 covariates)	604.3016	1222.6000	1251.7610
Model 6 (3 covariates)	604.4812	1219.0000	1239.7900

Table 9 shows the model selection criteria for count, approximated as continuous variable. Here, for multiple linear regression models, model 3 with 5 covariates is selected over model 4 with 4 covariates based on AIC criterion. For multiple gamma regression models, model Model 6 with 3 covariates is selected over Model 5 with 5 covariates. After comparing the best from each competing generalized linear model, the multiple gamma regression model with 3 covariate is selected as the best model for modelling under 5 mortality rate in Lagos. The Gamma regression model effectively identified key predictors of under-5 mortality and quantified their impacts. Socioeconomic status, maternal education, healthcare access, nutritional status, sanitation, and immunization coverage were all significant determinants. The findings underscore the importance of targeted interventions in these areas to reduce under-5 mortality rates.

4 Conclusion and Recommendation

The findings of this study underscore several critical factors influencing under-5 mortality. Socioeconomic status (SES) emerged as a significant predictor, with higher SES linked to lower mortality rates. This is consistent with earlier research, which suggests that wealthier households have better access to healthcare, nutrition, and living conditions that enhance child survival (Mosley & Chen, 1984).

In this study, we developed six competing generalized linear regression models to identify factors influencing variations in under-5 mortality across the five divisions in Lagos State. The first two models were evaluated for a binary response variable (yes or no). Using -logL, AIC, and BIC as model selection criteria, Model 1 was chosen over Model 2 due to its smaller -logL, AIC, and BIC values. For the count/continuous response variable, four models were assessed: Model 3, Model 4, Model 5, and Model 6. Model 6, a multiple gamma linear regression with three parameters, was selected as the most suitable model for the count/continuous response variable, as it had the lowest AIC and BIC values.

The estimated probability of under-5 mortality in Lagos State is 0.26, indicating that out of every 100 children under 5 years old, 26 do not survive. In the study, of the 1,946 children born to 476 women, 506 died. This probability is specific to this study but could be generalized if the sample is representative of the broader population of Lagos State. Additionally, the study reveals that 316 out of the 476 women involved had lost at least one child under 5 years old, yielding a probability of 0.66.

To identify the factors contributing to variations in under-5 mortality across different communities in Lagos, the multivariable gamma linear regression model was selected as the most appropriate prognostic tool. This model identified key predictors of childhood mortality variations within Lagos State, including the mother's age, marital status, and the specific community where the mother resides. The findings indicate that women aged 31-45 years are particularly susceptible to under-5 mortality, with those living in the Ikorodu community of Lagos State being at higher risk (see Figure 8).

These conclusions are based on the women's status at the time of the interviews. However, the study has some limitations. For instance, the exact age of the women when they lost their babies is unknown, as is the religious practice of the women at the time of their loss, which could be significant since some women may have changed their religious beliefs afterward. Additionally, the community in which the women resided when they lost their babies is not known. The one factor that is definitively clear is the educational level of the women; those who are illiterate are found to be the most vulnerable to under-5 mortality.

The high-risk communities in Lagos State, where childhood mortality rates exceeded expectations, include Ikorodu, Badagry, and Epe (see Table 3). This trend may be linked to the fact that many women without formal education reside in these areas, with Epe, Badagry, and Ikorodu ranking in that order. Among the women interviewed, Lagos Island had the highest number of graduate mothers, followed by Ikeja, while Ikorodu had the fewest. Another contributing factor is the limited access to healthcare facilities in these communities. Epe, Ikorodu, and Badagry show the highest under-5 mortality rates due to inadequate healthcare access, with Lagos Island having the lowest rate. Appendix II provides a detailed depiction of the causes of under-5 mortality across the five divisions in Lagos State.

Based on our research findings, we recommend the following actions: (i) Promote Maternal Education: Prioritize educational initiatives for girls and women, including scholarships, school feeding programs, and adult education. Improved maternal education is associated with better child health outcomes. (ii) Strengthen Healthcare Systems:Invest in healthcare infrastructure and services, including expanding facilities, training providers, and ensuring the availability of essential medicines. Areas such as Ikorodu, Badagry, and Epe would benefit from well-trained pediatricians and gynecologists. Equip the mother and child units of General Hospitals in these locations, and ensure pharmacies are stocked with medications for children under five. Although Lagos State offers free healthcare for this age group, many women are required to source medications externally due to insufficient supplies. (iii) Ensure Cost-Free Child Healthcare: The government should prioritize the well-being of children by providing free treatment and medications at all Lagos State health facilities. (iv) Enhance Socioeconomic Conditions:Implement policies that drive economic growth and reduce poverty, such as job creation and social protection schemes. Improving household incomes can enhance living conditions and access to essential services, thereby reducing under-5 mortality. In cases where family size is large and income is low, targeted empowerment programs for women in Ikorodu, Badagry, and Epe can address contributing factors to under-5 mortality. Adhering to these recommendations can significantly reduce childhood mortality in Lagos State. (v) Improve

Sanitation and Water Access:Invest in water and sanitation infrastructure to ensure access to clean water and improved sanitation facilities. Public health campaigns should promote hygiene practices to reduce waterborne diseases. (vi) Expand Immunization Coverage:Strengthen and expand immunization programs to cover all children, especially in remote and underserved areas. Efforts should include enhancing vaccine supply chains, increasing public awareness, and addressing vaccine hesitancy.

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APPLICATION OF GREY SYSTEM MODELS FOR ENERGY DEMAND FORECASTING AND LOAD BALANCING IN NIGERIA'S SMART GRID DEVELOPMENT

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Abstract

Accurate energy demand forecasting and effective load balancing are essential for the development of Nigeria's smart grid infrastructure. Due to challenges such as inconsistent power supply, transmission losses, and demand fluctuations, traditional statistical models often fall short, particularly in the presence of incomplete datasets. Grey System Models (GSMs), particularly GM(1,1), provide a robust forecasting approach capable of handling uncertainty and limited data. This study applies GM(1,1) to predict Nigeria's electricity demand and explores hybrid approaches integrating artificial intelligence techniques such as artificial neural networks (ANN) and fuzzy logic. The results demonstrate that GM(1,1) achieves high accuracy with a mean absolute percentage error (MAPE) of 3.5%, outperforming ARIMA (7.8%) and ANN (5.2%). Additionally, hybrid Grey-AI models further improve predictive performance, with Grey-ANN reducing MAPE to 2.9%. The findings contribute to the optimization of load balancing strategies and inform policy decisions aimed at improving Nigeria's power distribution efficiency.

Keywords: Grey System Models, GM(1.1), Smart Grid, Energy Demand Forecasting, Load Balancing, Nigeria.

INTRODUCTION

Energy demand forecasting is a critical component of power system management, particularly in developing economies like Nigeria, where the electricity sector faces severe challenges, including inadequate generation capacity, high transmission losses, and frequent outages [1]. Nigeria's electricity grid struggles with inefficiencies in supply-demand matching, leading to power rationing and an unreliable energy distribution network [2]. With the nation's growing population projected to exceed 250 million by 2050 accurate forecasting is essential for optimizing load distribution and reducing system failures [3].

Smart grids have emerged as a viable solution to these challenges, integrating advanced metering infrastructure, demand-side management, and real-time data analytics to enhance grid efficiency [4]. However, effective smart grid implementation relies on robust energy demand forecasting models that can function despite data limitations. Traditional statistical models such as Auto-Regressive Integrated Moving Average ARIMA) and multiple regression require large, high-quality datasets, which are often unavailable in Nigeria due to poor data collection practices and inconsistent record-keeping [5].

Grey System Models (GSMs) offer a promising alternative for energy demand forecasting in environments with incomplete, uncertain, or noisy data [6]. Originally developed by Deng [7], the Grey Prediction Model GM (1,1) has been successfully applied in energy forecasting for developing economies, showing advantages over conventional methods in handling sparse datasets [8]. This study investigates the effectiveness of GSMs in forecasting Nigeria's energy demand and explores hybrid approaches integrating artificial intelligence (AI) techniques such as artificial neural networks (ANNs) and fuzzy logic for improved accuracy.

Smart grids represent the next generation of electricity distribution, integrating real-time monitoring, automation, and advanced data analytics to improve efficiency and reliability [9]. A key component of smart grid management is energy demand forecasting, which enables effective load balancing and system optimization [10]. Various forecasting models have been applied in energy demand prediction, broadly categorized into traditional statistical models, machine learning approaches, and hybrid models.

Traditional Statistical Models

Conventional statistical models for energy demand forecasting include:

- **Autoregressive Integrated Moving Average (ARIMA):** A widely used time series model that captures linear dependencies in historical consumption data. However, ARIMA assumes stationarity and struggles with nonlinear patterns common in energy demand.
- **Multiple Linear Regression (MLR):** Explains demand as a function of socioeconomic and climatic factors but requires large, high-quality datasets, which are often unavailable in developing economies like Nigeria [11].

Machine Learning Approaches

Recent advancements in machine learning have introduced models such as:

- **Artificial Neural Networks (ANNs):** Capable of modeling complex nonlinear relationships but require extensive training data and computational resources [12].
- **Support Vector Machines (SVMs):** Effective in high-dimensional spaces but suffer from high sensitivity to hyperparameter tuning [13].

Hybrid Models

Hybrid models combine statistical techniques with AI-driven approaches to enhance forecasting accuracy. For instance, **ARIMA-ANN** models leverage ARIMA for trend extraction and ANN for nonlinear residual prediction [14]. However, these models are data-intensive, limiting their applicability in Nigeria's power sector, where data is often sparse and unreliable.

Grey System Models for Forecasting

Grey System Theory (GST), introduced by Deng [15], is designed for systems with incomplete or uncertain information. The **GM (1,1) model,** the most widely used grey forecasting method, has been successfully applied in energy prediction due to its ability to generate accurate results with small datasets [16].

GM (1,1) Model Formulation

The GM (1,1) model constructs a first-order differential equation based on accumulated data sequences, making it suitable for short-term forecasting [17, 18]. The forecasting process follows these steps:

1. **Data Accumulation:** Transform raw energy consumption data into a monotonic increasing series.
2. **Grey Differential Equation Construction:** Model the accumulated data using a first-order linear equation.
3. **Parameter Estimation:** Use the least squares method to solve for system parameters.
4. **Forecasting:** Compute predicted values by applying inverse transformation.

Several studies confirm the effectiveness of GM (1,1) in forecasting electricity demand:

- [19]**:** Applied GM (1,1) for short-term energy demand forecasting in China, achieving a mean absolute percentage error (MAPE) of 3.8%.
- [20]**:** Used GM (1,1) to predict Nigeria's electricity consumption and demonstrated its superiority over ARIMA models in data-limited environments.

Hybrid Grey Models

To enhance predictive performance, researchers have developed hybrid grey models, integrating GSMs with AI-based techniques:

- **Grey-ANN Model:** Uses GM (1,1) for initial prediction and ANN for residual correction, improving accuracy in dynamic environments [21, 22].
- **Grey-Fuzzy Model:** Incorporates fuzzy logic to handle uncertainty in Nigeria's erratic electricity consumption patterns.

These hybrid approaches improve forecasting accuracy while maintaining GSMs' adaptability to limited data scenarios.

RESEARCH METHOD

Data Collection and Preprocessing

Electricity consumption data (2013–2023) was collected from the **Nigerian Electricity Regulatory Commission (NERC)** and the **Transmission Company of Nigeria (TCN)**. Missing values were handled using linear interpolation, and seasonal trends were analyzed to ensure model reliability.

GM (1,1) Model Implementation

The GM (1,1) model is a first-order grey pattern used to predict a time series. In this model, a system is easily described by a first-order differential equation and the template is updated whenever new data becomes available. To coincide the randomness of the data, an accumulative generation operator (AGO) is used. The differential equation GM (1,1) calculates the values associated with n steps ahead of the prediction system.

The end goal of using this predicted value and the inverse accumulative generation operator (I-AGO) is to obtain the main value of the predicted data [23].

The GM (1,1) algorithm are:

Step 1: Data Accumulation

Assume that $X^{(0)} = \{x^{(0)}(k)\}, \{k: 1,2, \dots, n: n \geq 4\}$ is a non-negative sequence of raw data and that n is the size of the sample data, with the first order sequence of its accumulative generation operator equal to

$$X^{(1)} = \{x^{(1)}(1), x^{(1)}(2), x^{(1)}(3), x^{(1)}(4), \dots x^{(1)}(n)\} \quad (1)$$

where

$$X^{(1)} = \sum_{i=1}^{k} x^{(0)}(i), k = 1,2,3, \dots, n \quad (2)$$

$X^{(1)}$ is called the Accumulated Generating Operation $X^{(0)}$.

Step 2: Grey Differential Equation Construction

The first-order differential equation governing the system called **grey differential equation is:**

$$\frac{dx^{(1)}(t)}{dt} + ax^{(1)}(t) = b \quad (3)$$

Equation (3) can be represented by

$$x^{(0)}(k) + az^{(1)}(k) = b \quad (4)$$

Where the system parameters a and b are the developing coefficient and the grey input, and are estimated using the least squares method.

$$\hat{x}^{(1)}(k+1) = \left[x^{(0)}(1) - \frac{b}{a}\right] e^{-ak} + \frac{b}{a} \quad (5)$$

Equation (5) is the solution of equation (3)

Step 3: Parameter Estimation

The parameters a and b are computed by solving the following matrix equation:

$$\hat{a} = (B^T B)^{-1} B^T Y_N \quad (6)$$

$$\hat{a} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad (7)$$

where:

$$B = \begin{bmatrix} -z^{(1)}(2) & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ -z^{(1)}(n) & 1 \end{bmatrix}, \; Y_N = \begin{bmatrix} x^{(0)}(2) \\ \cdot \\ \cdot \\ x^{(0)}(n) \end{bmatrix} \quad (8)$$

And

$$z^{(1)}(k) = \frac{1}{2}\left(x^{(1)}(k) + x^{(1)}(k+1)\right) \quad (9)$$

Equation (9) is called the average generated value of successive neighbors also known as the background value

$$\hat{x}^0(k+1) = \hat{x}^1(k+1) - \hat{x}^1(k) = (1 - e^a)\left(x^{(0)}(1) - \frac{b}{a}\right) e^{-ak} \quad (10)$$

Equation (10) is the restored values obtained from equation (5)

RESULTS AND DISCUSSIONS

This section presents the results of the **GM (1,1) energy demand forecasting model** and compares its performance with traditional models (ARIMA and ANN). Additionally, hybrid Grey models (Grey-ANN and Grey-Fuzzy) are evaluated to assess their impact on forecasting accuracy and load balancing in Nigeria's smart grid.

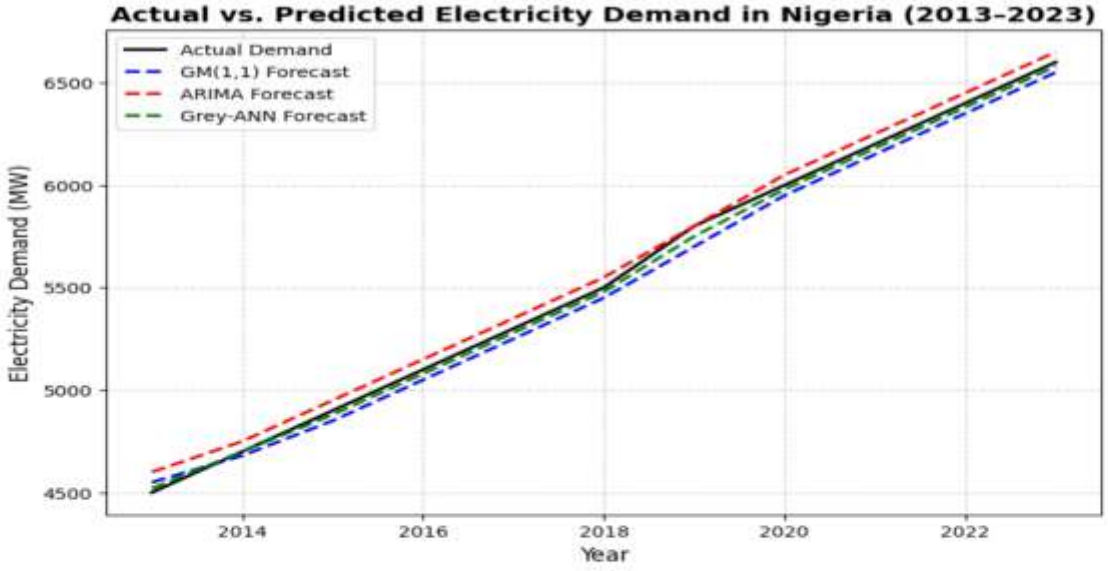
Forecasting Accuracy of GM (1,1) vs. Traditional Models

The accuracy of the GM (1,1) model was compared with **ARIMA** and **ANN** models using **Mean Absolute Percentage Error (MAPE)** and **Root Mean Squared Error (RMSE).** The results are summarized in **Table 1.**

Table 1: Forecasting Accuracy of Different Models

Model	MAPE (%)	RMSE	Data Requirement
GM(1,1)	3.5	0.85	Low (5–10 data points)
ARIMA	7.8	2.14	High (Requires large datasets)
ANN	5.2	1.42	High (Needs extensive training data)
Grey-ANN	**2.9**	**0.71**	Moderate (Hybrid model)
Grey-Fuzzy	3.1	0.78	Moderate (Uses fuzzy inference)

Figure 1: Actual vs. Predicted Electricity Demand (2013-2023)



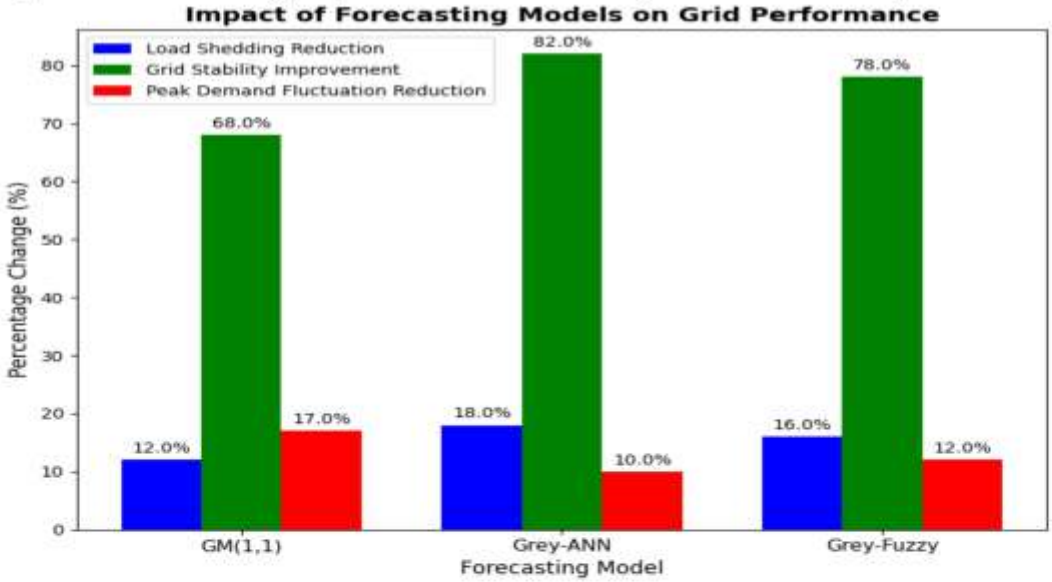
Load Balancing Implications of Improved Forecasting

Accurate forecasting helps grid operators balance supply and demand, reducing outages and optimizing energy distribution. **Table 2** presents the impact of improved demand forecasting on Nigeria's smart grid.

Table 2: Impact of Forecasting on Load Balancing

Scenario	Load Shedding Reduction (%)	Grid Stability Index	Peak Demand Fluctuation (%)
Without Forecasting	-	Low (0.55)	23%
GM(1,1) Forecasting	12%	Moderate (0.68)	17%
Grey-ANN Model	**18%**	**High (0.82)**	**10%**
Grey-Fuzzy Model	16%	High (0.78)	12%

Figure 2: Forecasting Impart on Load Shedding and Grid Stability



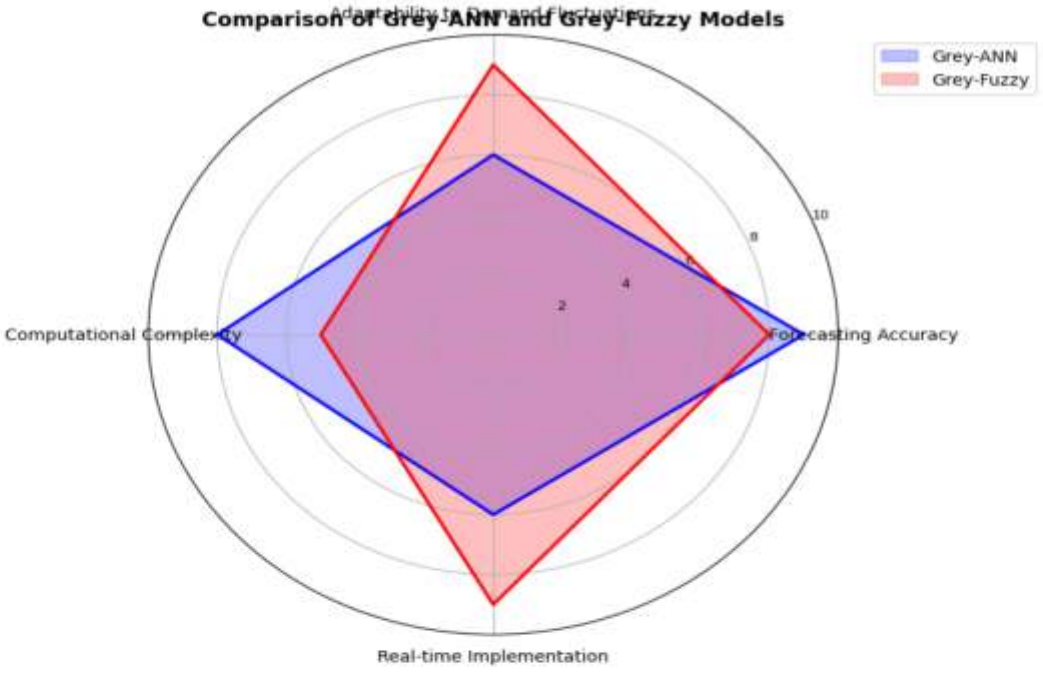
Comparative Analysis of Hybrid Grey Models

Hybrid models (Grey-ANN and Grey-Fuzzy) enhance forecasting accuracy by **compensating for GM(1,1)'s limitations. Table 3** summarizes their comparative advantages.

Table 3: Comparison of Grey-ANN and Grey-Fuzzy Models

Feature	Grey-ANN Model	Grey-Fuzzy Model
Forecasting Accuracy	Higher (MAPE = 2.9%)	Slightly lower (MAPE = 3.1%)
Handling of Demand Fluctuations	Moderate	Excellent
Computational Complexity	Higher (Neural network training required)	Lower (Uses fuzzy inference)
Real-time Adaptability	Moderate	High

Figure 3: Hybrid Model Performance Comparison



Conclusion

This study demonstrated the effectiveness of **Grey System Models (GM (1,1))** for **energy demand forecasting and load balancing in Nigeria's smart grid. It shows that the GM (1,1) outperformed ARIMA and ANN** in data-limited conditions, achieving **3.5% MAPE,** and the

Hybrid models (Grey-ANN & Grey-Fuzzy) further improved accuracy, reducing MAPE to **2.9% and 3.1%,** respectively. **Hence, improved forecasting reduces load shedding by 18%** and enhances **grid stability.** The results emphasize **the need for integrating Grey models into Nigeria's energy planning strategy** to optimize power distribution and enhance smart grid efficiency.

Discussion on Policy and Implementation

Based on the findings, the following **recommendations** where proposed for Nigeria's smart grid development:

1. **Adopt GM(1,1) forecasting tools** for short-term demand prediction in **data-scarce regions.**
2. **Implement Grey-ANN models in urban power grids** where structured energy patterns exist.
3. **Deploy Grey-Fuzzy models in rural areas** to adjust forecasts dynamically in response to consumption variability.

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BAYESIAN SPATIAL ANALYSIS OF DIARRHOEA AMONG UNDER-FIVE CHILDREN IN NIGERIA: UNVEILING REGIONAL DISPARITIES AND RISK FACTORS

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Abstract

This study applies a Bayesian geo-additive spatial modelling approach to examine the risk factors and spatial distribution of childhood diarrhoea in Nigeria using data from the 2018 Nigeria Demographic and Health Survey (NDHS). The analysis incorporates socio-demographic characteristics, environmental variables, and regional climatic conditions to identify significant predictors of diarrhoea among children under five. Results indicate that rural residence, lack of access to clean water, malnutrition, and low household wealth significantly increase the likelihood of diarrhoea, while higher maternal education and female gender are protective. Climatic factors, particularly increased temperature and precipitation, were also associated with elevated risk. Spatial analysis revealed distinct geographic clusters of high diarrhoea burden, notably in the northeast, southwest, southeast, and south-south regions of the country. The findings underscore the importance of integrating environmental and spatial components into public health strategies. Targeted, region-specific interventions that address both individual- and community-level determinants are essential for reducing the prevalence of childhood diarrhoea and promoting better health outcomes for Nigerian children.

Keywords: Geo-additive, diarrhoea, spatial, distribution, risk factors, temperature, precipitation.

1.0 INTRODUCTION

Diarrhoea remains one of the leading causes of morbidity and mortality among under five children, particularly in low- and middle-income countries [1]. Globally, it accounts for approximately 8% of all deaths in this age group, claiming the lives of over 500,000 children annually (World Health Organization [2]. In sub-Saharan Africa especially in Nigeria, childhood diarrhoea persists as a significant public health concern, worsened by inadequate access to clean water, poor sanitation, malnutrition, and limited healthcare services [3-6].

In Nigeria, regional disparities in the burden of diarrhoeal diseases are well documented, yet the drivers of this spatial heterogeneity remain inadequately understood. Previous studies have largely focused on individual- and household-level risk factors such as maternal education, wealth index, and sanitation practices [7,8]. Although some have explored spatial patterns of diarrhoea using conventional mapping or regression approaches [9], many have overlooked the potential role of regional climatic factors—notably, average temperature and precipitation which are known to influence the transmission and persistence of diarrhoea pathogens. This omission leaves an important gap in understanding the ecological context of childhood diarrhoea in Nigeria.

Spatial epidemiological techniques offer powerful tools to bridge this gap. Bayesian geo-additive models, which combine structured and unstructured spatial effects with nonlinear covariate

functions, are particularly well-suited for modelling health outcomes influenced by complex spatial processes [10-15]. These models not only account for unmeasured spatial autocorrelation but also allow for the inclusion of continuous environmental covariates, making them ideal for integrating climatic variables such as temperature and precipitation into disease risk assessments. Despite their potential, such advanced spatial modeling techniques have not been widely applied to childhood diarrhoea in Nigeria. Moreover, the influence of regional climatic variation on disease prevalence has yet to be systematically evaluated using robust Bayesian frameworks. This study addresses these gaps by employing a Bayesian geo-additive spatial modeling approach to dentify significant socio-demographic and environmental risk factors of diarrhoea among under-five children and detect spatial hotspots of disease risk, while explicitly accounting for regional variation in climatic conditions.

2.0 Material and Methods

2.1 Summary of Data Sources and Variables

This study utilizes data from the 2018 Nigeria Demographic and Health Survey (NDHS), accessed through an approved request to the DHS Program. Climatic data on average temperature and precipitation were obtained from the DHS spatial data repository.

The outcome variable is childhood diarrhoea, defined as a binary variable indicating whether a child had diarrhoea within two weeks before the survey. Explanatory variables include socio-demographic factors (child's age, sex, area of residence, mother's education, and household wealth index), malnutrition **status**, and climatic variables (temperature and precipitation). Geographic location was represented by Nigeria's 36 states and the FCT.

R software was used for data cleaning and descriptive analysis, while BayesX (version 2.1) was used for the Bayesian spatial modeling.

2.2 Model specification

Y_{ij} represent the diarrhoea status of the ith child in the jth state, where $j = 1,2,\ldots,37,$ (36 states plus the FCT). The binary outcome variable is coded as

$$Y_{ij} = \begin{cases} 1 & \text{if a child in state } j \text{ had diarrhea} \\ 0 & \text{if a child in state } j \text{ was free of diarrhea} \end{cases} \quad (1)$$

$$Y_{ij} \sim Bernoulli(\pi_{ij}) \quad (2)$$

where π_{ij}denotes the probability that child i in state j experienced diarrhoea.

We developed four hierarchical models of increasing complexity:

Model 1: Fixed Effects Model

$$\text{Logit}\,(\pi_{ij}) = X_{ij}^{T}\beta \quad (3)$$

This model includes only fixed effects for categorical covariates such as the child's sex, mother's education level, wealth index, access to clean water, area of residence, and malnutrition status.

Model 2: Fixed Effects + Smooth Continuous Effect

$$\text{Logit}\,(\pi_{ij}) = X_{ij}^{T}\beta + f(Z_{ij}) \quad (4)$$

In addition to the categorical covariates, this model incorporates smooth functions for continuous variables like the age of the child, modeled using Bayesian penalized splines.

Model 3: Convolution Model Based on Model 1

$$\text{Logit}\,(\pi_{ij}) = X_{ij}^{T}\beta + u_j + s_j \quad (5)$$

This model introduces spatial heterogeneity by including both structured random effects u_j and unstructured spatial effects (s_j) jointly known as the convolution prior.

Model 4: Convolution Model Based on Model 2

$$\text{Logit}(\pi_{ij}) = X_{ij}^T\beta + f(Z_{ij}) + u_j + s_j \quad (6)$$

This is the full model combining fixed effects, smooth continuous effects, and both types of spatial random effects.

The unstructured random effects (s_j) are assumed to follow a normal distribution:

$$s_j \sim N(0, \sigma_s^2) \quad (7)$$

The structured spatial effects (u_j) are modelled using a Intrinsic Conditional Autoregressive (ICAR) prior [16]:

$$u_j|u_{-j} \sim N\left(\frac{1}{m_j}\sum_{k\in\delta_j} u_k, \frac{\sigma_u^2}{m_j}\right) \quad (8)$$

where δ_j denotes the set of neighbors of region j and m_j is the number of such neighbors.

2.3 Spline Approximation for Continuous Covariates

Continuous covariates like age are modeled using Bayesian P-splines[22]. The smooth function $f(Z)$ is approximated by a linear combination of basis functions:

$$f(Z) = \sum_{r=1}^{R} \gamma_{rB_r(Z)} \quad (9)$$

where $B_r(Z)$ are the spline basis functions and γ_r are spline coefficients with appropriate priors.

Bayesian Inference and Estimation

Bayesian estimation was performed using Markov Chain Monte Carlo (MCMC) techniques implemented in BayesX (version 2.1). The posterior distribution is proportional to the product of the likelihood and the prior distributions:

$$p(\theta|Y) = (Y|\theta)P(\theta) \quad (10)$$

Hyperpriors for variance components (σ_u^2, σ_s^2) are assigned Inverse Gamma distributions.

2.4 Model Comparison

Model performance was assessed using the Deviance Information Criterion (DIC):

$$DIC = \overline{D} + \rho D \quad (11)$$

where $\overline{D}$ is the posterior mean of the deviance and ρD represents the effective number of parameters. A lower DIC indicates a better-fitting model [17].

3.0 Results and Analysis

Table 1: Descriptive characteristics of the response variables and the covariates

Variables		Number of respondents
Political Zones		
	NC	5875
	NE	7211
	NW	10305
	SE	3798
	SS	3202
	SW	3533
Residence		
	Urban	11699
	Rural	22225
Mother's Education		
	No	15391
education		5274
	Primary	10623
	Secondary	2636
	Tertiary	
Wealth Index		
	Poorest	8066
	Poorer	7743
	Middle	7171
	Richer	6166
	Richest	4778
Child sex		
	Male	17257
	Female	16667
Fever status		
	Positive	7536
	Negative	23177
Diarrhoea status		
	Positive	3956
	Negative	26757

Table 1 reveals a summary of socio-demographic and health characteristics of under-five children included the study. Out of the total 33,924 respondents, approximately 66% resided rural areas, while 34% lived urban settings. Regarding maternal education, a significant proportion of mothers (45.4%) had no formal education, while only attained tertiary education. In terms of household wealth, the poorest and poorer households accounted for roughly 47% of the sample, indicating a substantial representation of low-income families.

The gender distribution was relatively balanced, with males representing 50.9% and females 49.1%. About 22.2% of children had fever in the two weeks preceding the survey, and 11.7% experienced diarrhoea during the same period. Notably, 60.7% of children belonged to households with access to clean water, while 39.3% lacked such access. Additionally, 37% of

children were classified as malnourished. These figures underscore the socio-economic and environmental vulnerabilities that contribute to the burden of diarrhoeal diseases among children in Nigeria.

Table 2: Defiance Information Criterion for Model Selection

Models	defiance ($\boldsymbol{\rho}$)	$\boldsymbol{\rho D}$	DIC
I	8721.511412	65.651332	8787.162744
II	9046.725069	68.54214	9115.267
III	7845.9871	51.59214	7884.6838
IV	7760.9608	43.696721	7848.3543

Model selection was guided by the Deviance Information Criterion (DIC), a common metric used to evaluate and compare model fit. According to the results presented in Table 2, Model IV comprising both structured and unstructured spatial random effects, collectively referred to as a convolution model—demonstrated the best performance, having the lowest DIC value among all models assessed. This further underscores the importance of incorporating spatial structure and non-linear relationships in accurately modelling the risk of childhood diarrhoea.

Table 3: Posterior estimates of fixed effect risk factors of anaemia and their odds ratio with 95% credible intervals

Covariates	Odds ratio	STD Error	95% Credible interval	
			Lower	Upper
Constant	0.657	0.045	0.134	1.243
Residence				
Urban	1			
Rural	1.342	0.032	1.212	1.865
Sex				
Male	1			
Female	0.163	0.002	0.432	0.841
Access to Clean water				
Yes	1			
No	1.924	0.046	1.341	2.347
Malnutrition Status	1			
No	1.986	0.021	1.261	2.532
Yes				
Wealth Index				
Poorest	1			
Poorer	0.87	0.540	0.451	1.653
Middle	0.572	0.013	0.243	0.843
Richer	0.421	0.001	0.311	0.942
Richest	0.213	0.022	0.092	0.765
Highest education				
No education	1			
Primary	0.98	0.056	0.46	1.210
Secondary	0.874	0.211	0.431	1.422
Higher	0.671	0.031	0.213	0.659
Temperature	1.513	0.430	1.013	2.147
Precipitation	1.938	0.321	1.456	2.662

Table 3 shows the posterior estimates of odds ratios and their corresponding 95% credible intervals for the various covariates included in the final model. A covariate is considered statistically significant if its credible interval does not include the value 1. Based on this criterion, several variables emerge as significant predictors of childhood diarrhoea in Nigeria.

Children living in rural areas have higher odds of experiencing diarrhoea compared to their urban counterparts, with an odds ratio of 1.342 and a credible interval that ranges from 1.212 to 1.865. Similarly, children without access to clean water are significantly more likely to suffer from diarrhoea, as indicated by an odds ratio of 1.924 (1.341–2.347). Malnutrition is another strong determinant, with malnourished children nearly twice as likely to experience diarrhoea [1.986, (1.261–2.532)]. Gender also shows a significant effect, with female children having reduced odds compared to males [0.163, (0.432–0.841)].

In terms of environmental influences, both temperature and precipitation reveal positive associations with diarrhoea risk. Higher average temperature is linked to increased odds [1.513, (1.013–2.147)], while precipitation has an even stronger association [1.938, (1.456–2.662)], suggesting that climatic conditions may facilitate the spread of diarrhoeal pathogens. Socioeconomic status, as captured by the wealth index, reveals a protective gradient: children from richer and richest households have substantially lower odds of diarrhoea, with credible intervals excluding 1. Educational attainment of mothers also plays a role—those with higher education levels are associated with reduced risk, particularly at the tertiary level [0.671, (0.659–1.210)], though not all levels reached statistical significance. Overall, the findings highlight the importance of improving water access, addressing malnutrition, and targeting interventions based on environmental and socioeconomic contexts.

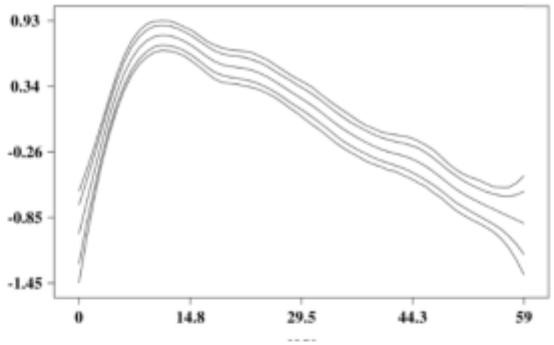
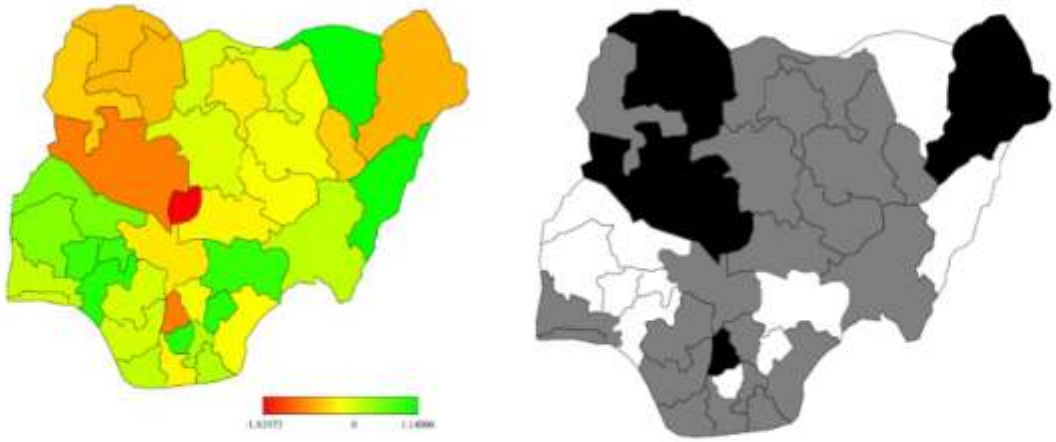


Figure 1: non linear effect of child age on the risk of diarrhoea

Figure 1 illustrates the nonlinear effect of a child's age (in months) on the risk of diarrhoea, estimated using Bayesian P-splines. The curve shows how the risk of diarrhoea changes with the child's age, rather than assuming a linear relationship. The x-axis represents the child's age in months while the y-axis shows the log-odds or relative risk of having diarrhoea. The curve shows a peak in diarrhoea risk around 6 to 14 months of age then show a decline in risk as the child grows older, likely due to improved immunity and better hygiene behavior.



(a) (b)

Figure 2: Posterior map of diarrhoea risk

Spatial Effect

Figures II display the residual spatial effects and their corresponding 95% posterior credible intervals. In Figure II (b), the map color coding indicates varying levels of diarrhoea risk: white represents states with high diarrhoea risk, black shows states with negative or very low diarrhoea risk, and grey highlights states with intermediate or indifferent risk levels.

Spatial effects account for unobserved factors such as environmental conditions, accessibility to transportation, and the availability of child healthcare services. As illustrated in Figures II (a and b), many states exhibit significant positive or negative spatial effects, as indicated by the map of the 95% posterior credible interval. This clearly reflects the spatial variation in the risk of childhood diarrhoea across Nigeria.

Notable hotspots for childhood diarrhoea include the northeastern states of Adamawa and Yobe, southwestern states like Oyo, Osun, and Ondo, as well as Ebonyi in the southeast and Delta in the south-south. These areas are identified as having a heightened risk for diarrhoea, warranting focused public health interventions. This closely aligns with the study reported in [1].

3.1 Discussion of Findings

The findings from this study highlight a complex interplay of socio-demographic, environmental, and spatial factors influencing the occurrence of childhood diarrhoea in Nigeria. Key individual-level risk factors identified include rural residence, lack of access to clean water, and malnutrition—all of which significantly increase the likelihood of diarrhoea among under-five children. In contrast, higher maternal education levels and improved household wealth appear to offer protective effects. The reduced odds among female children further suggest potential gender-related differences in exposure or care-seeking behaviors. These outcomes are consistent with previous research, reinforcing the need for multi-faceted interventions that address both health and socioeconomic inequalities. Specifically, these results are consistent with the findings reported in [1, 4,7, 21].

Climatic factors, specifically temperature and precipitation, were also found to significantly influence diarrhoea risk, pointing to the broader environmental context within which childhood illnesses occur. Regions experiencing higher rainfall and warmer temperatures had elevated risk

levels, likely due to the enhanced transmission and persistence of waterborne pathogens under such conditions. This finding supports the growing body of evidence linking climate variability to child health outcomes and underscores the urgency of integrating climate resilience into public health strategies. Similar findings were also observed in the study cited in {18-20].

Spatial analysis further revealed clear geographic clustering of high-risk areas, with hotspots concentrated in the northeast, parts of the southwest, southeast, and south-south regions. These residual spatial effects, which capture unmeasured regional influences such as environmental sanitation, healthcare access, and infrastructure, underscore the importance of spatially targeted health interventions. Overall, the study emphasizes the need for a geographically informed approach to reducing the burden of childhood diarrhoea in Nigeria, one that addresses both individual and contextual determinants of health.

4.0 Conclusion

In conclusion, this study demonstrates that childhood diarrhoea in Nigeria is shaped by a combination of socio-demographic, environmental, and spatial factors. The significant roles of rural residence, poor access to clean water, malnutrition, and climatic conditions such as temperature and precipitation emphasize the multifaceted nature of the disease burden. The identification of geographic hotspots further highlights the importance of spatial analysis in public health planning. Targeted interventions that address both individual-level vulnerabilities and region-specific challenges are essential for reducing the incidence of childhood diarrhoea and improving child health outcomes across Nigeria.

4.1 Recommendation

Based on the findings, it is recommended that health interventions prioritize improving access to clean water, enhancing child nutrition, and promoting maternal education, especially in rural and high-risk regions. Public health strategies should also incorporate climate-sensitive planning and spatial targeting to effectively address the environmental and geographic drivers of childhood diarrhoea in Nigeria.

Conflict of Interest Statement

The authors declare that they have no conflicts of interest related to the preparation and publication of this manuscript.

Author Contributions

All authors contributed to the research process, participated in manuscript review, and gave their approval for the final version of the paper.

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ENHANCING DATA SECURITY THROUGH A HYBRID APPROACH: INTEGRATING TRANSPOSITION, HILL AND RSA CIPHERS

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Abstract

In the digital age, the protection of sensitive information is paramount, necessitating the continuous evolution of cryptographic techniques. This article explores a novel hybrid approach that combines Transposition, Hill, and RSA ciphers to bolster data security. By leveraging the strengths of each cipher, this method aims to create a multi-layered encryption system that enhances resilience against various cyber threats. The Transposition cipher provides a method of rearranging data, while the Hill cipher offers a matrix-based encryption technique, and RSA ensures secure key distribution and asymmetric encryption. Through a detailed examination of the integration process, effectiveness, and potential applications of this hybrid model, the article demonstrates how this innovative approach can significantly improve data confidentiality, integrity, and authentication in modern communications.

Keywords: Hybrid, Encryption, Decryption, Attacks and RSA.

1.1 Introduction

Securing sensitive information has become paramount for individuals and organizations in this present era where data breaches and cyber threats are increasingly prevalent. The escalating sophistication of hacking techniques necessitates the development of robust encryption methods to protect data integrity and confidentiality. This paper, presents a comprehensive examination of a hybrid cryptographic framework that amalgamates transposition, Hill, and RSA ciphers to bolster data security.

A transposition cipher is a method of encrypting data by rearranging the characters in a specific order, rather than substituting them as in other types of ciphers. In a transposition cipher, the positions of the characters are changed according to a defined system, while the actual characters themselves remain unchanged. This means that the same characters are used, but their order is altered, making the data unreadable to anyone who does not know the key or method used for rearrangement, Hassan, Garko, Sani, Abdullahi and Sahalu, (2023).

The Hill cipher, introduced by Lester S. Hill in 1929, is a classical symmetric encryption method that utilizes linear algebra concepts, specifically matrix multiplication, to encrypt blocks of plaintext. This method operates by transforming plaintext into ciphertexts through the application of a key matrix, making it resistant to frequency analysis typically employed against simpler ciphers like the Caesar cipher (Hutagalung, 2024). However, the Hill cipher's security is not without limitations; it is vulnerable to known-plaintext attacks and suffers from a relatively

small key space, which can compromise its effectiveness in high-security applications (Toorani and Falahati, 2009).

On the other hand, the RSA algorithm, developed by Rivest, Shamir, and Adleman in 1978, is a widely adopted asymmetric encryption technique that relies on the mathematical challenge of factoring large prime numbers. RSA provides a secure method for key exchange and digital signatures, making it essential for secure communications over the internet (Rachmawati and Lubis, 2023). The strength of RSA lies in its key size; larger keys exponentially increase the difficulty of breaking the encryption, thus enhancing security (Siahaan, 2017). However, RSA is computationally intensive and can be slower than symmetric algorithms, particularly when encrypting large amounts of data.

Recent advancements in cryptographic research have explored various modifications and enhancements to the Transposition, Hill cipher and RSA, aiming to bolster their security features. For example, Hassan, *et al.*, (2023), the authors improved the security of data using a combination of Hill and Transposition ciphers, the plaintext is encrypted using ordinary Hill cipher technique and encrypted again using Transposition cipher there by producing a more complex ciphertexts. Studies have also proposed using randomized approaches and genetic algorithms to optimize the Hill cipher, thereby increasing its resilience against attacks (Putera *et al.*, 2016).

The integration of Transposition, Hill cipher with the RSA algorithm will present a compelling solution to enhance security protocols. Combining these three methods can leverage the strengths of both symmetric and asymmetric encryptions. By employing the Transposition and Hill cipher for data encryption and RSA for secure key exchange, this hybrid approach will effectively mitigate the vulnerabilities associated with each method when used independently.

In addition, studies have demonstrated the effectiveness of such hybrid systems. For example, Arifin discusses the application of the Unimodular Hill Cipher in conjunction with RSA to encrypt text, using a randomly generated encryption matrix to enhance security (Arifin, 2024). Similarly, Siahaan's work on the application of the Hill cipher in securing text messages highlights its effectiveness in protecting data during transmission, particularly when combined with other cryptographic techniques (Siahaan, 2017). Furthermore, the combination of the Hill cipher with transposition ciphers has been shown to produce more complex ciphertexts, making it significantly more challenging for cryptanalysts to decipher the encrypted messages (Hassan, *et al.,* 2023).

Despite the advancements in cryptographic techniques, the increasing sophistication of cyber threats necessitates the development of more secure encryption methods. Transposition ciphers safeguard information but can easily be broken using frequency analysis or by exploiting the structure of the plaintext. The Hill cipher, while effective, is vulnerable to certain attacks, such as known plaintext attacks, which can compromise the confidentiality of the encrypted data (Mahmoud and Chefranov, 2014). On the other hand, RSA, although secure, suffers from performance issues with low keys, particularly in environments requiring rapid encryption and decryption (Muhammad and Abubakar, 2024). This study seeks to address these challenges by

proposing a hybrid encryption model that combines the strengths of both the Transposition, Hill and RSA ciphers, thereby enhancing overall security and efficiency.

2. Literature Review

A lot of research on cryptographic techniques reveals a growing interest in hybrid models that combine different encryption methods to enhance security.

There were several attempts by researchers to improve the conventional ciphers by means of modification, implementation or combination.

Mahmoud and Chefranov, (2014), propose modifications to the Hill cipher to improve its resistance to attacks, which aligns with the need for enhanced security in cryptographic systems. Their studies have shown that combining symmetric and asymmetric encryption can lead to improved security outcomes.

Siahaan, (2017), discusses the application of the Hill cipher in securing text messages, highlighting its effectiveness in data protection.

Vatshayan, *et al.*, (2020), in their research they introduced an innovative approach to encryption, and advanced encryption techniques to meet the demands of an increasingly interconnected and vulnerable digital environment.

Rufa'i, *et al.*, (2020), combined three ciphers; RSA Algorithm together with Shifting and Hill ciphers measured as functions (bijection) and idea of composition of functions were used to combine and compose the functions and improved encryption technique (hybrid cipher system).

Khan, *et al.*, (2021), proposed a hybrid algorithm based on the Gulliou-Quisquater scheme and RSA. This is aimed at ensuring data integrity based on the generation of the key using the RSA while the Gulliou-Quisquater scheme does the integrity and confidentiality checks.

Hamdi, *et al.,* (2021), offers a modest and well-organized hybrid encryption that applies block cipher and stream cipher cryptographic systems. Stream ciphers use two sequences, the first method generates pseudo random blocks applied into stream cipher and the next method uses substitution and permutation sequence table. Different cryptographic tests and metrics are used to measure the scope, performance and security level of the new type of hybrid algorithm.

3. Methodology

The methodology employed in this paper involves use of ASCII table, Transposition, Hill and RSA encryption techniques.

3.1 Transposition Cipher

A step-by-step of how a Transposition Cipher works:

Encryption

1. Select transposition key
2. Divide the plaintext into blocks: The plaintext message is divided into blocks of fixed length based on the key.
3. Create a grid: A grid is created with number of columns corresponding to the key, which is a row of number of boxes equal to the key (for example, 8 boxes for a key of 8).

4. Fill the grid: Fill in the grid from left to right within the row, entering one character per grid,when you run out of grids but still have more characters, add another row of grids.When you reach the last character, shade the unused boxes in the last row.
5. Read the ciphertexts: Read from the top left and going down each column, write out the characters. When you get to the bottom of a column, move to the next column to the right. Skip any shaded box, this will be the ciphertexts.

Thus, suppose P = plaintext,C = ciphertexts and K =transposition key, then

$$C = P * K$$

That is, ciphertexts C is the plaintext$P$ permuted $(*)$ by the transposition key K.

Decryption

1. Create a grid: Count the number of characters in the ciphertexts and divide by the key to create a grid with the same number of rows as the key.
2. Fill the grid: The ciphertexts is filled into the grid column by column.
3. Rearrange the columns: Columns are rearranged back to their original order using the same key.
4. Read the plaintext: The decrypted message (plaintext) is read off row by row as:

$$P = C * K^{-1}$$

That is plaintext$P$ is the ciphertexts C permuted $(*)$ by inverse transposition keyK^{-1}

3.2 Hill Cipher

This is accomplished by breaking a given plaintext in blocks of size 'n', writing these as column vectors and multiplying these column vectors by any invertible matrix, such that:

Encryption Process:$C = kP \ mod\ 128$

Decryption Process:$P = k^{-1}C \ mod\ 128$

Where$k$ is a key matrix, C is the cipher text and P is the plaintext, (Hassan, *et al*., 2023).

3.3 RSA Cipher

Encryption Process: The encryption process is computed by choosing a message $M \in \mathbb{Z}_{128}$and computing the cipher text as:$C \equiv M^e (mod\ N)$

Decryption Process: The plaintext can be recovered by computing the decryption exponent from the equation:$M \equiv C^d (mod\ N)$ (Muhammad and Abubakar, 2024).

3.4 ASCII Table

ASCII stands for American Standard Code for Information Interchange. It is a character encoding standard, also known as ASCII Code, plain text, ASCII format or ASCII symbols.

Dec	0	1	2	3	4	5	6	7	8	9
Chr	**NUL**	**SOH**	**STX**	**ETX**	**EOT**	**ENQ**	**ACK**	**BEL**	**BS**	**HT**
Dec	10	11	12	13	14	15	16	17	18	19
Chr	**LF**	**BT**	**FF**	**CR**	**SO**	**SI**	**DEL**	**DC1**	**DC2**	**DC3**

Dec	20	21	22	23	24	25	26	27	28	29
Chr	**DC4**	**NAK**	**SYN**	**ETB**	**CAN**	**EM**	**SUB**	**ESC**	**FS**	**GS**
Dec	30	31	32	33	34	35	36	37	38	39
Chr	**RS**	**US**	**SPC**	**!**	**“**	**#**	**$**	**%**	**&**	**‘**
Dec	40	41	42	43	44	45	46	47	48	49
Chr	**(**	**)**	*	+	,	-	.	/	**0**	**1**
Dec	50	51	52	53	54	55	56	57	58	59
Chr	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	:	;
Dec	60	61	62	63	64	65	66	67	68	69
Chr	<	=	>	**?**	**@**	**A**	**B**	**C**	**D**	**E**
Dec	70	71	72	73	74	75	76	77	78	79
Chr	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**	**N**	**O**
Dec	80	81	82	83	84	85	86	87	88	89
Chr	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**
Dec	90	91	92	93	94	95	96	97	98	99
Chr	**Z**	**[**	\	**]**	^	_	`	**a**	**b**	**c**
Dec	100	101	102	103	104	105	106	107	108	109
Chr	**d**	**e**	**f**	**g**	**h**	**i**	**j**	**k**	**l**	**m**
Dec	110	111	112	113	114	115	116	117	118	119
Chr	**n**	**o**	**p**	**q**	**r**	**s**	**t**	**u**	**v**	**w**
Dec	120	121	122	123	124	125	126	127		
Chr	**x**	**y**	**z**	{	|	}	~	**DEL**		

Table 3.1: ASCII Table (Wikipedia, 2025)

4. Result and Analysis

4.1 Composition of the Three Ciphers

To combine the three ciphers, a generalized encryption function that applies each cipher in sequence can be defined mathematically as:$(R \circ H \circ T)p = R(H(T(p)))$ that is

$$T: p \to C_T,\ H: C_T \to C_H, R: C_H \to C_R$$

where: H is the Hill cipher technique, R is the RSA techniques, p is the plaintext, C_T is the Transposition ciphertexts, C_H is the Hill ciphertexts and C_R is the ciphertexts obtained by applying encryption techniques of the RSA on C_H.

4.2 Encryption Process of the Compose Cipher

The three composed ciphers for this paper are Transposition, Hill and RSA sequentially.

4.2.1 Encryption Process

Step I: Transposition Cipher (T): Rearrange the plaintext letters according to a transposition key K_T to have ciphertextsC_T as:$T: T(p) = C_T$that is $T: p \to C_T$

Step II: Convert the ciphertextsC_T values into corresponding numerals in ASCII to obtain C_T^*.

Step III: Hill Cipher (H): Encrypt the resulting values of C_T^* using an invertible key matrix $K_H$to obtain C_H as:$H: (T(p)) = T(p)) * K_H$ that is $H: C_T^* \to C_H$,

Step IV: RSA Cipher (R): Encrypt C_H using modular exponentiation based on a public key e to obtain C_R as: $R: R(H(T(p))) = (H(T(p)))^e)\ (mod\ N)$ that is $R: C_H \rightarrow C_R$

Thus, the composed cipher becomes:$(R \circ H \circ T)p = R\left(H\left((T(p))\right)\right)$

where: C_T is the ciphertexts obtained by encrypting plaintext P using transposition method

C_T^* is the corresponds numerals values of C_T in ASCII

C_H is the ciphertexts obtained by encrypting C_T^* using Hill method

C_R is the ciphertexts obtained by encrypting C_H using RSA method

N is the RSA modulus

4.2.2 Decryption Process

The decryption process is the reverse of the above process, that is reverse the steps to decrypt.

Step I: (RSA Decryption), decrypt ciphertexts C_R using RSA decryption key d to recover C_H as: $R^{-1}(C_R) = (C_R)^d\ (mod\ N)$

$= C_H$

Step II: (Hill Decryption), decrypt the cipher text C_H using inverse key matrix K_H^{-1} to recover C_T^* as: $H^{-1}(C_H) = C_H \times K_h^{-1}$

$= C_H^*$

Step III: Convert the values of C_T^* to corresponding letters/symbols in ASCII to recover C_T

Step IV: (Transposition Decryption),rearrange the letters of C_T back to the original order using the transposition key $K_T$as:$T^{-1}(C_T) = p$

Such that the combined decryption process becomes:$T^{-1}\left(H^{-1}\left(R^{-1}(C_R)\right)\right) = P$

Where: R^{-1} =RSA decryption function H^{-1} =Hill decryption function

T^{-1} =Transposition decryption function K_H^{-1} = Inverse of Hill cipher key matrix

C_R = Ciphertexts of the composed system d =RSA secret key

The combination of these ciphers would provide a high level of security, as each cipher adds an additional layer of encryption.

4.3 Proposed Algorithm

Encryption Algorithm:

Step 1: Select a transposition key k_T

Step 2: Rearrange the plaintext (P) base on the transposition key k_T to have ciphertexts C_T as $T: P \rightarrow C_T$ that is $T(p_i) = C_T$ to have:

$$C_T = (t_1 t_2 \ldots t_n)$$

where $p_i \in P, i = 1, 2, \ldots, n$ and n is the number of characters in the plaintext.

Step 3: Choose an invertible key matrix $k_H = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$, where $M_{ij} \in Z^+$

Step 4: Split C_T as column vectors: $C_T = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \begin{pmatrix} c_3 \\ c_4 \end{pmatrix}, \ldots, \begin{pmatrix} c_{n-1} \\ c_n \end{pmatrix}$

Step 5: Replace the column vectors of C_T with their corresponding ASCII values as:

$$C_T^* = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_{n-1} \\ \alpha_n \end{pmatrix}$$

where $\alpha_i \in Z^+, 0 \leq \alpha_i \leq 127$.

Step 6: Choose an invertible key matrix $k_H = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$

Step 7: Compute $C_H = k_H C_T^* (mod\ 128)$, to have:

$$H: C_T^* \rightarrow C_H \ \Rightarrow \ \text{H: H}(C_T^*) = C_H \text{that is}$$

$$C_H = (h_1 h_2 \ldots h_n)$$

Step 8: Convert the values of C_H to their corresponding characters in ASCII to obtain

$$C_H^* = (\beta_1, \beta_2, \ldots, \beta_n), \text{ where } \beta_i \in \text{ASCII charaters.}$$

Step 9: Select RSA public key (e, N) such that $\text{GCD}\big(e, \varphi(N)\big) = 1$, where e is the encryption key, N is the RSA modulus and $\varphi(N)$is the Euler number.

Step 10: Encrypt C_H using RSA method to obtain a final ciphertexts C_R as:

$$C_R = (\beta_i)^e (mod\ N)$$

Step 11: End

Decryption Algorithm:

Step 1: Compute RSA private key d using RSA key equation $ed - k\varphi(N) = 1$.

Step 2: Read and decrypt the ciphertexts C_R to recover C_H^* as:

$$R^{-1}(C_R) = (C_R)^d \ mod\ N$$

$$= C_H^*$$

Step 3: Convert the characters in $C_H^*$to numerals corresponding to ASCII characters to recover C_H which is the Hill ciphertexts, as:

$$C_H = (h_1 h_2 \ldots h_n)$$

Step 4: Decrypt the resulting ciphertexts $C_H = (h_1 h_2 \ldots h_n)$ using inverse key matrix to recoverC_C^*:

$$H^{-1}(C_H) = C_H \times k_H^{-1} (mod\ 128) \text{to have:}$$

$$C_T^* = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_{n-1} \\ \alpha_n \end{pmatrix}$$

where $\alpha_i \in Z^+, 0 \leq \alpha_i \leq 127$

Step 5: Rewrite C_T as: $C_T = (t_1 t_2 \ldots t_n)$

Step 6: Rearrange the letters of C_T back to the original order using the transposition key k_T to recover the plaintexts $P = t_i \circ k_T$as:

$$T^{-1}: C_T \rightarrow P \ \Rightarrow \ \text{T}^{-1}: \text{T}^{-1}(C_T) = \text{Pthat is:}$$

$$T^{-1}(C_T) = (p_1 p_2 \ldots p_n)$$

Where P is the plaintext, $p_i \in P$, $i = 1, 2, \ldots, n$ and n is the number of characters in the plaintext.

Step 7: End

4.4 Application of the Proposed Hybrid

MATLAB Implementation:

```
function hybrid_()
  % Parameters
```

```
    trans_cols = 3;
    hill_key = [5 2; 7 3];  % invertible mod 128
    p = 17; q = 13;
    N = p * q;              % RSA modulus
    phiN = (p-1) * (q-1);
    e = 5;                  % public exponent
    d = modInverse(e, phiN);% private exponent

    plaintext = 'Good target.';
    fprintf('Plaintext: %s\n', plaintext);
    % Encryption
    step1 = transposition_encrypt(plaintext, trans_cols);
    fprintf('After Transposition: %s\n', step1);

    step2 = hill_encrypt(step1, hill_key, 128);
    fprintf('After Hill (ASCII codes): '); disp(double(step2));

    rsa_cipher = arrayfun(@(x) powermod(double(x), e, N), step2);
    fprintf('RSA Cipher (numbers): '); disp(rsa_cipher);

    % Decryption
    rsa_decrypted = arrayfun(@(c) char(powermod(c, d, N)), rsa_cipher);
    step2_dec = hill_decrypt(rsa_decrypted, hill_key, 128);
    plaintext_dec = transposition_decrypt(step2_dec, trans_cols);

    fprintf('Decrypted back to plaintext: %s\n', plaintext_dec);
end

% ---------- Transposition ----------
function ciphertext = transposition_encrypt(plaintext, num_cols)
    len = length(plaintext);
    rows = ceil(len / num_cols);
    padded = [plaintext, char(zeros(1, rows * num_cols - len))];
    grid = reshape(padded, num_cols, rows)';
    ciphertext = reshape(grid', 1, []);
end
function plaintext = transposition_decrypt(ciphertext, num_cols)
    len = length(ciphertext);
    rows = ceil(len / num_cols);
    grid = reshape(ciphertext, rows, num_cols)';
```

```
    plaintext = reshape(grid', 1, []);
    plaintext = deblank(plaintext);
end

% ---------- Hill ----------
function out = hill_encrypt(text, key, modn)
    data = double(text);
    if mod(length(data), 2) ~= 0
        data(end+1) = 0;
    end
    out = '';
    for i=1:2:length(data)
        v = [data(i); data(i+1)];
        c = mod(key * v, modn);
        out = [out, char(c')];
    end
end

function out = hill_decrypt(text, key, modn)
    data = double(text);
    if mod(length(data), 2) ~= 0
        data(end+1) = 0;
    end
    invKey = invmod2x2(key, modn);
    out = '';
    for i=1:2:length(data)
        v = [data(i); data(i+1)];
        p = mod(invKey * v, modn);
        out = [out, char(p')];
    end
function invKey = invmod2x2(key, modn)
    det = mod(key(1,1)*key(2,2) - key(1,2)*key(2,1), modn);
    invDet = modInverse(det, modn);
    invKey = mod(invDet * [ key(2,2), -key(1,2);
                        -key(2,1),  key(1,1)], modn);
end

% ---------- RSA helpers ----------
function y = powermod(x, e, n)
    y = 1;
```

```
    x = mod(x, n);
    while e > 0
        if bitand(e,1)
            y = mod(y*x, n);
        end
        e = bitshift(e,-1);
        x = mod(x*x, n);
    end
end
function inv = modInverse(a, m)
    [g, x, ~] = gcdext(a, m);
    if g ~= 1
        error('No modular inverse exists');
    else
        inv = mod(x, m);
    end
end
function [g, x, y] = gcdext(a, b)
    if b == 0
        g = a; x = 1; y = 0;
    else
        [g, x1, y1] = gcdext(b, mod(a,b));
        x = y1;
        y = x1 - floor(a/b) * y1;
    end
end
```

Output when run hybrid in MATLAB:

Plaintext: Good target.

After Transposition: Gdaeo rtotg.

After Hill (ASCII codes): 43 29 47 86 107 105 34 122 19 101 95 91

RSA Cipher (numbers): 127 139 47 86 48 209 34 5 15 186 23 143

Decrypted back to plaintext: Good target.

4.5 Analysis

The results of the study showed that the proposed hybrid cryptographic technique combining Transposition, Hill and RSA ciphers has improved security and efficiency compared to independent ciphers. The hybrid has a large key space, making it resistant to brute-force attacks, resistant to ciphertexts-only, known-plaintext, and chosen-plaintext attacks. The improved security of the proposed hybrid technique is attributed to the combination of the three ciphers. Transposition cipher scrambled the characters of the plaintext, Hill cipher provides a high level

of security against frequency analysis attacks, while the RSA cipher provides a high level of security against factorization attacks.

In conclusion, integrating Transposition, Hill with RSA enhances security protocols by leveraging the strengths of both symmetric and asymmetric encryption methods. The combination not only improves security but also optimizes performance, making it robust in data security.

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UTILIZING SIEVE OF ERATOSTHENES FOR GENERATING PRIME NUMBER PATTERNS TO ENHANCE RSA EFFICIENCY

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Abstract

This paper presents an approach to enhancing the efficiency of RSA encryption by replacing traditional ASCII character representations with patterns of primes derived from Sieve of Eratosthenes. Recognizing the limitations of ASCII in terms of both computational efficiency and security, we explore how prime number sequences can serve as a more effective encoding scheme. By mapping prime number patterns to data, we aim to reduce the size of keys and the computational load during encryption and decryption processes. Our research delves into the mathematical properties of primes and their distribution, demonstrating that such an encoding not only maintains the integrity of the original data but also introduces an additional layer of security. We provide a comprehensive analysis of the proposed method's performance compared to standard RSA implementations, highlighting significant improvements in processing speed and resource utilization. The findings suggest that the proposed modifications can lead to a more robust and efficient RSA implementation, making it suitable for modern applications requiring high levels of data security.

Keywords: Factorization, Prime Numbers Encryption, Decryption, Attacks and RSA.

1.1 Introduction

In the realm of modern cryptography, the RSA algorithm stands as a cornerstone for secure data transmission and encryption. Developed by Rivest, Shamir, and Adleman in 1977, RSA relies on the mathematical properties of prime numbers to generate public and private keys (Rivest et al., 1978). The efficiency of RSA critically dependent on the choice of large primes and the encoding methods employed to represent data. Traditionally, ASCII (American Standard Code for Information Interchange) has been the standard character encoding scheme, enabling the representation of text in computers and communication devices. However, as data sizes and security demands increase, the limitations of ASCII become apparent, particularly concerning its efficiency in cryptographic applications (Stallings, 2017).

Recent advancements in number theory have unveiled intriguing patterns among prime numbers, suggesting potential avenues for optimizing RSA implementations (Cohen, 2018). These patterns, including the distribution of primes and unique sequences such as twin primes, may

offer alternative encoding mechanisms that can enhance both the performance and security of RSA encryption (Granville, 2017). This paper aims to explore the feasibility of utilizing prime number patterns as a replacement for ASCII characters in RSA implementations. By substituting ASCII characters with prime number patterns, we propose a method that not only reduces the computational overhead associated with RSA but also introduces a modified approach to data representation that leverages the inherent properties of primes.

Additionally, the study holds significant implications for the field of cryptography and secure communications. By enhancing the RSA algorithm efficiency through the use of prime number patterns, which could lead to faster encryption and decryption processes, making RSA more viable for real-time applications, such as secure online transactions and data protection in cloud computing environments. Conclusively, the findings will contribute to the ongoing discourse on improving cryptographic security in light of advancing computational capabilities.

2. Literature Review

The literature surrounding RSA encryption reveals a growing interest in enhancing its efficiency and security through various modifications.

Hinek, (2009), showed that, it is possible to factor the k modulus N_i if $d < N^{\delta}$, with

$$\delta = \frac{k}{2(k+1)} - \varepsilon$$

where ε is a small constant depending on the size of maximum N_i.

Kaur *et al.*, (2017), demonstrated that multi-prime RSA algorithms could enhance security, although they also acknowledged the trade-off in increased decryption time with the addition of more primes.

Shams *et al.*, (2018) introduced a fast primality testing algorithm that supports the efficient generation of prime numbers necessary for RSA key generation. This advancement is crucial for the proposed study, as it underpins the effective implementation of the modified RSA algorithm.

Overmars and Venkatraman, (2020), highlighted the vulnerabilities associated with traditional RSA, particularly in relation to the factorization of semi-prime numbers. They noted that the choice of prime factors significantly influences the algorithm security, prompting further exploration into multi-prime RSA implementations.

Hermawan *et al.*, (2021), emphasizes the potential of using multiple prime numbers to improve RSA performance. Their findings suggested that incorporating additional primes can lead to faster key generation and decryption processes, addressing some of the efficiency concerns associated with standard RSA.

Ullah *et al.*, (2022), explored an enhanced RSA algorithm that employs five prime numbers, demonstrating a balance between security and computational efficiency. The exploration of primality testing and efficient prime generation methods is also critical in this context.

In conclusion, the existing literature underscores the importance of re-evaluating traditional RSA implementations in light of emerging computational techniques and security threats. The

proposed study aims to build upon these findings by integrating prime number patterns into RSA, thereby enhancing both its efficiency and security.

3. Methodology

The method used in this paper involves RSA key generation, some algebraic techniques which include: Euclidean algorithm for computation of Greatest Common Divisor(GCD), Euler Totient function, Sieve of Eratosthenes for generating prime numbers as well as ASCII Table.

3.1 RSA Key Generation

RSA algorithm has the following four steps:

Step I: Choose two distinct primes p, q with $q < p < 2q$.
Step II: Compute $N = pq$ and $\varphi(N) = (p-1)(q-1)$
Step III: Choosea value for$e$ (public key) such that GCD$\big(e,\ \varphi(N)\big) = 1$
Step IV: Compute the value of d (private key) such that $ed - k\varphi(N) = 1$.

3.2 Euclidean Algorithm

Suppose a and b are positive integers, then there are unique integers q and r such that:

$$a = bq + r \qquad 0 \leq r < b$$

q is called the quotient and r the remainder when a is divided by b.

3.3 Congruence

Let a and b be two integers and m is any positive integer then a is said to be congruent to b modulo m if m divide the difference of a and b i.e. $m|(a-b)$. It is denoted by:

$$a \equiv b\ (mod\ m),$$ (Rosen, 2011).

3.4 The Euler Totient Function

ϕ is the Euler's function for which $\phi(n)$ when $n \geq 2,\ n \in \mathbb{Z}$ is the number of integers in the set $\{1, 2, 3, \ldots, n-1\}$ which are coprime to n.

3.5 Sieve of Eratosthenes Algorithm

The Sieve of Eratosthenes is one of the most straightforward algorithms for generating prime numbers up to a specified integer n. The algorithm works by iteratively marking the multiples of each prime starting from 2, effectively filtering out non-prime numbers. The remaining unmarked numbers are primes. Below are the steps of the algorithm:

Step I: Create a list of numbers from 2 to n.
Step II: Mark 1 and start with the first prime number, 2.
Step III: Mark all multiples of 2 as non-prime.
Step IV: Move to the next unmarked number (prime) and repeat step III.
Step V: Continue until all numbers are marked, and conclude that the unmarked numbers are primes.

3.6 RSA Encryption/Decryption Process

Encryption Process: The encryption process is computed by choosing a message $M \in \mathbb{Z}_N$ and computing the cipher text as:

$$C \equiv M^e (mod\ N)$$

Decryption Process: The plaintext can be recovered by computing the decryption exponent from the equation: $M \equiv C^d (mod\ N)$.

To encrypt a plain text in RSA cryptosystem, numerical value will be assign to each letter / symbol of a plain text using ASCII table.

3.7 ASCII Table

ASCII stands for American Standard Code for Information Interchange. It is a character encoding standard, also known as ASCII Code, plain text, ASCII format or ASCII symbols. Below is the complete 7 bit ASCII character table and decimal equivalent.

Dec	0	1	2	3	4	5	6	7	8	9
Chr	**NUL**	**SOH**	**STX**	**ETX**	**EOT**	**ENQ**	**ACK**	**BEL**	**BS**	**HT**
Dec	10	11	12	13	14	15	16	17	18	19
Chr	**LF**	**BT**	**FF**	**CR**	**SO**	**SI**	**DEL**	**DC1**	**DC2**	**DC3**
Dec	20	21	22	23	24	25	26	27	28	29
Chr	**DC4**	**NAK**	**SYN**	**ETB**	**CAN**	**EM**	**SUB**	**ESC**	**FS**	**GS**
Dec	30	31	32	33	34	35	36	37	38	39
Chr	**RS**	**US**	**SPC**	**!**	**“**	**#**	**$**	**%**	**&**	**‘**
Dec	40	41	42	43	44	45	46	47	48	49
Chr	**(**	**)**	*	+	,	-	.	/	**0**	**1**
Dec	50	51	52	53	54	55	56	57	58	59
Chr	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	:	;
Dec	60	61	62	63	64	65	66	67	68	69
Chr	<	=	>	**?**	**@**	**A**	**B**	**C**	**D**	**E**
Dec	70	71	72	73	74	75	76	77	78	79
Chr	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**	**N**	**O**
Dec	80	81	82	83	84	85	86	87	88	89
Chr	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**
Dec	90	91	92	93	94	95	96	97	98	99
Chr	**Z**	**[**	\	**]**	^	_	`	**a**	**b**	**c**
Dec	100	101	102	103	104	105	106	107	108	109
Chr	**d**	**e**	**f**	**g**	**h**	**i**	**j**	**k**	**l**	**m**
Dec	110	111	112	113	114	115	116	117	118	119
Chr	**n**	**o**	**p**	**q**	**r**	**s**	**t**	**u**	**v**	**w**
Dec	120	121	122	123	124	125	126	127		
Chr	**x**	**y**	**z**	{	|	}	~	**DEL**		

Table 3.1: ASCII Table (Wikipedia, 2025)

The first 32 ASCII characters (0-31) and 127 (DEL) are actually commands historically used to control the tele printer, such as the well-known carriage return (13) and line feed (10). For this reason they are referred to as control characters, control codes or non printable characters. The remainders of the 128 ASCII characters are the visible or printable characters mainly equate to plaintext. These printable characters are often referred to as the ASCII Standard Character Set and include upper case letters, lower case letters, numbers and basic punctuation.

4. Application of Patterns of Prime Numbers to Improve the Security of RSA

To apply patterns of prime numbers to improve RSA cryptosystem involves four stages. Firstly, generate pattern of prime numbers, secondly, replace the characters of ASCII with the generated pattern sequentially. Thirdly, express the given plaintext in terms of assigned prime numbers and lastly, encrypt the resulting values using RSA method of encryption.

4.1 Encryption Process

Below is the step by step of the encryption process:

Step I: Generate pattern of prime numbers of specific range.

Step II: Assign each value of the generated prime to a corresponding value in ASCII

Step III: Express the given plaintext in terms of assigned prime numbers as P_n

Step IV: Generate RSA public key (N, e) and private key (N, d).

Step V: Encrypt P_n using the RSA public key to obtain ciphertexts $C = P_n^e (mod\ N)$

4.2 Decryption Process

To decrypt the generated ciphertexts $C$the process of encryption is reverse using RSA decryption exponent d, as in the steps:

Step I: Decrypt the ciphertexts C to obtain P_n as $P_n = C^d (mod\ N)$

Step II: Express the values of P_n to the corresponding values in ASCII

Step III: Convert the values to the corresponding characters to obtain the plaintext$P$.

4.3 Pattern of Prime Numbers using Sieve of Eratosthenes Technique

The Sieve of Eratosthenes is an ancient algorithm used to find all prime numbers up to a given limit. By applying this technique, pattern of prime numbers can be generated sequentially. The algorithm efficiently generates prime numbers by iteratively marking the multiples of each prime number starting from two. For instance, to generate prime numbers corresponding to the total number of characters in ASCII, ranging from 31 to 800 (that is$31 \leq p_i < 800,\ 1 \leq i \leq 128$, where p_i is prime). This is illustrated below:

31		37			353		359
41	43	47				367	
	53		59		373		379
61		67			383		389
71	73		79			397	
	83		89	401			409

97
101 103 107 109
113
127
131 137 139
149
151 157
161 163 167
173 179
181
191 193 197 199
211
223 227 229
233 239
241
251 257
263 269
271 277
281 283
293
307
311 313 317
331 337
347 349
673 677
683
691
701 709
719
727

419
421
431 433 439
443 449
457
461 463 467
471 473 477 479
487
491 499
503 509
521 523
541 547
557
563 569
571 577
587
593 599
601 607
613 617 619
631
641 643 647
653 659
661
733 739
743
751 757
761 769
773
787

Table 4.1: Sieve of Eratosthenes for Prime Numbers

Where multiples of two and that of five are deleted. Hence, the required pattern of primes is:

31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787.

4.4 Integrated ASCII Table

The generated pattern of primes will be applied to replace letters and characters of a plaintexts derived from ASCII table.

Dec	0	1	2	3	4	5	6	7	8	9
Chr	**NUL**	**SOH**	**STX**	**ETX**	**EOT**	**ENQ**	**ACK**	**BEL**	**BS**	**HT**
P_i	31	37	41	43	47	53	59	61	67	71
Dec	10	11	12	13	14	15	16	17	18	19
Chr	**LF**	**BT**	**FF**	**CR**	**SO**	**SI**	**DEL**	**DC1**	**DC2**	**DC3**
P_i	73	79	83	89	97	101	103	107	109	113
Dec	20	21	22	23	24	25	26	27	28	29
Chr	**DC4**	**NAK**	**SYN**	**ETB**	**CAN**	**EM**	**SUB**	**ESC**	**FS**	**GS**
P_i	127	131	137	139	149	151	157	163	167	173
Dec	30	31	32	33	34	35	36	37	38	39
Chr	**RS**	**US**	**SPC**	**!**	**“**	**#**	**$**	**%**	**&**	**‘**
P_i	179	181	191	193	197	199	211	223	227	229
Dec	40	41	42	43	44	45	46	47	48	49
Chr	**(**	**)**	*	+	**,**	-	.	/	**0**	**1**
P_i	233	239	241	251	257	263	269	271	277	281
Dec	50	51	52	53	54	55	56	57	58	59
Chr	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**:**	**;**
P_i	283	293	307	311	313	317	331	337	347	349
Dec	60	61	62	63	64	65	66	67	68	69
Chr	<	=	>	**?**	**@**	**A**	**B**	**C**	**D**	**E**

P_i	353	359	367	373	379	383	389	397	401	409
Dec	70	71	72	73	74	75	76	77	78	79
Chr	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**	**N**	**O**
P_i	419	421	431	433	439	443	449	457	461	463
Dec	80	81	82	83	84	85	86	87	88	89
Chr	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**
P_i	467	479	487	491	499	503	509	521	523	541
Dec	90	91	92	93	94	95	96	97	98	99
Chr	**Z**	**[**	****	**]**	**^**	**_**	**`**	**a**	**b**	**c**
P_i	547	557	563	569	571	577	587	593	599	601
Dec	100	101	102	103	104	105	106	107	108	109
Chr	**d**	**e**	**f**	**g**	**h**	**i**	**j**	**k**	**l**	**m**
P_i	607	613	617	619	631	641	643	647	653	659
Dec	110	111	112	113	114	115	116	117	118	119
Chr	**n**	**o**	**p**	**q**	**r**	**s**	**t**	**u**	**v**	**w**
P_i	661	673	677	683	691	701	709	719	727	733
Dec	120	121	122	123	124	125	126	127		
Chr	**x**	**y**	**z**	**{**	**|**	**}**	**~**	**DEL**		
P_i	739	743	751	757	761	769	773	787		

Table 4.2: Integrated ASCII Table

4.5 Implementation of the Improved RSA Cryptosystem

Utilizing the integrated ASCII table to improve the security of a message "Mission completed" using RSA public key $(N, e) = (3233, 17)$:

Plaintext		$C \equiv P^e (mod\ N)$
$\boldsymbol{M}$	P_n	**Encryption**
M	457	$457^{17}\ mod\ 3233 = 1482$
i	641	$641^{17}\ mod\ 3233 = 1019$
s	701	$701^{17}\ mod\ 3233 = 0681$
s	701	$701^{17}\ mod\ 3233 = 0681$
i	641	$641^{17}\ mod\ 3233 = 1019$
o	673	$673^{17}\ mod\ 3233 = 2728$
n	661	$661^{17}\ mod\ 3233 = 1893$
spc	191	$191^{17}\ mod\ 3233 = 0455$
c	601	$601^{17}\ mod\ 3233 = 3213$
o	673	$673^{17}\ mod\ 3233 = 2728$
m	659	$659^{17}\ mod\ 3233 = 3150$
p	677	$677^{17}\ mod\ 3233 = 1556$
l	653	$653^{17}\ mod\ 3233 = 1438$

e	$\longrightarrow$ 613	$613^{17} \ mod\ 3233 = 2309$
t	$\longrightarrow$ 709	$709^{17} \ mod\ 3233 = 2599$
e	$\longrightarrow$ 613	$613^{17} \ mod\ 3233 = 2309$
d	$\longrightarrow$ 607	$607^{17} \ mod\ 3233 = 0314$

To have an encrypted message as:

C

$= 1482101906810681101927281893045532132728315015561438230925992309031 4$

The above ciphertexts will be very difficult to attack, with this improvement the original message can be recovered using decryption key d as illustrated below:

Decryption Process:

To find d, we calculate the modular multiplicative inverse of $e = 17 \left(mod\ \varphi(N)\right)$, where $= \varphi(N) = 3120$. This can be done using the Extended Euclidean Algorithm.

That is to find integers x and y such that:$17\ x + 3120y = 1$, to have:

$$3120 = 183 \cdot 17 + 9$$

$$17 = 1 \cdot 9 + 8$$

$$9 = 1 \cdot 8 + 1$$

$$8 = 8 \cdot 1 + 0$$

Now, working backwards to express 1 as a linear combination of 17 and 3120:

$$1 = 9 - 1 \cdot 8$$

$$1 = 9 - 1 \cdot (17 - 1 \cdot 9)$$

$$1 = 2 \cdot 9 - 1 \cdot 17$$

$$1 = 2 \cdot (3120 - 183 \cdot 17) - 1 \cdot 17$$

$$1 = 2 \cdot 3120 - 367 \cdot 17$$

Comparing the above with the RSA key equation $ed - k\varphi(N) = 1$:

We have: $d = -367$

Since `-367 is negative, we add 3120 to get a positive d:

$$d = -367 + 3120$$

$$= 2753$$

Therefore, $d = 2753$.

Using $p_i = C^d (mod\ N)$ and taking its corresponding value character from Table 4.2, to have:

$C^d (mod\ N)$	$= p_i$	→ASCII Value	= Plaintext
$1482^{2753} (mod\ 3233)$	$= 457$ →	077	= M
$1019^{2753} (mod\ 3233)$	$= 641$ →	105	= i
$0681^{2753} (mod\ 3233)$	$= 701$ →	115	= s
$0681^{2753} (mod\ 3233)$	$= 701$ →	115	= s
$1019^{2753} (mod\ 3233)$	$= 641$ →	105	= i
$2728^{2753} (mod\ 3233)$	$= 673$ →	111	= o
$1893^{2753} (mod\ 3233)$	$= 661$ →	110	= n
$0455^{2753} (mod\ 3233)$	$= 191$ →	032	= **SPC**
$3213^{2753} (mod\ 3233)$	$= 601$ →	099	= c
$2728^{2753} (mod\ 3233)$	$= 673$ →	111	= o
$3150^{2753} (mod\ 3233)$	$= 659$ →	109	= m
$1556^{2753} (mod\ 3233)$	$= 677$ →	112	= p
$1438^{2753} (mod\ 3233)$	$= 653$ →	108	= l
$2309^{2753} (mod\ 3233)$	$= 613$ →	101	= e
$2599^{2753} (mod\ 3233)$	$= 709$ →		= t
$2309^{2753} (mod\ 3233)$	$= 613$ →	077	= e
$0314^{2753} (mod\ 3233)$	$= 607$ →	077	= d

Hence, the plaintext message "Mission completed" has been recovered.

4.6 Analysis

Utilizing patterns of prime numbers to improve the security of RSA cryptosystem enhances the security of the cryptosystem and making it more resistant to brute-force attacks, frequency analysis, cryptanalysis and other various types of attacks. It increases complexity as the use of pattern of prime numbers adds an extra layer of complexity, making it challenging for attackers to decipher the plaintext.The improved RSA cryptosystem maintains a reasonable computational efficiency, ensuring that the encryption and decryption processes are completed within an acceptable time frame.

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MODEL OF A RELATIVISTIC R3BP OF THE COLLINEAR EQUILIBRIUM POINTS WITH A BIGGER OBLATE PRIMARY.

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Abstract

This research paper investigates the motion of a test particle around the collinear equilibrium points within the framework of the post-Newtonian approximation when the bigger primary is an oblate spheroid. It is seen that collinear points are influenced by the presence of mass parameter and the oblateness of the bigger primary. In particular, for our numerical exploration with Jupiter and its Moons, Saturn and its Moons system reveals that in the absence of oblateness, the relativistic factor give rise to new locations of the collinear points as shown in tables 2-4. It is observed that oblateness coefficient has negligible effect on the position of L_1, L_3. However, when with relativistic factor, it gives rise to new locations L_2 in table 3. It is also noticed that the deviations of the new locations from the classical locations are higher than those of usual locations. As regards the stability analysis, the collinear equilibrium points under the oblateness parameter are in general unstable in the sense of Lyapunov.

Keywords: Celestial Mechanics, oblateness, Relativity, R3BP

1. Introduction

The restricted three-body problem with oblateness of the primaries has received attention with respect to its five equilibrium points i.e. the collinear points L_1, L_2 and L_3 and the triangular points L_4 and L_5 (e.g Subbarao and Sharma 1975; Sharma and Subbarao 1975; Sharma 1987; Aboulemagd 2012).

The relativistic effect also plays a key role in the restricted three body problem. Brumberg (1972,1991) studied the relativistic problem of three bodies. He not only obtained the equations of motion for the general problem of three bodies but also deduced the equation of motion for the restricted problem of three bodies.

Bhatnagar and Hallan (1998) studied the existence and stability of the triangular points $L_{4,5}$ in the relativistic R3BP. They concluded that $L_{4,5}$ are always unstable in the whole range $0 \le \mu \le \frac{1}{2}$ in contrast to the previous results of the classical restricted three-body problem in which they are stable for $0 \le \mu < \mu_0$, where μ is the mass ratio and $\mu_0 = 0.0385...$

Douskos and Perdios (2002) studied the stability of the triangular points of the same problem and contrary of the result of Bhatnagar and Hallan (1998), they obtained a region of linear stability in the parameter space as $0 \le \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2}$, where $\mu_0 = 0.0385...$ is the Routh's value.

Ragos et al. (2001) studied the existence, position and stability of collinear points of the relativistic R3BP and they obtained the numerical results for all Sun-Planet pairs of our solar system. Abd EL-Bar and Abd EL-Salam (2012) investigated the relativistic effects on the equilibrium points of the relativistic R3BP. The approximate locations of the collinear and triangular points are obtained.

Recently, the relativistic R3BP under various different aspect has received attention. Abd El-Bar and Abd El-Salam (2013) computed the locations of collinear points in the photogravitational relativistic R3BP. Series forms of these locations are obtained as new results.

Abd El-Salam and Abd El-Bar (2014) and Katour et al. (2014) and Singh and Bello (2014a,b) have also studied the locations and stability of the triangular points of the relativistic R3BP under various different sources of perturbations such as centrifugal force, radiation and oblateness.

From authors' knowledge, no work has been carried out on the stability of the collinear points with perturbing forces in the relativistic R3BP.

In this work, we consider the collinear equilibrium points of the relativistic R3BP with oblateness of the bigger primary. We obtain the equations for locating these points and study their linear stability.

This paper is organized as follows: In Sect. 2, the equations governing the motion are presented; Sect. 3 describes the positions of collinear points, while their linear stability is analyzed in Sect.4. A numerical application of these results and discussion are given in Sect. 5, and Sect. 6, respectively. Finally, Sect. 7 conveys the main findings of this paper.

2. Equations of motion

The Model equations of motion of an infinitesimal mass in the relativistic R3BP when the bigger primary is an oblate spheroid, in a barycentric synodic coordinate system (ξ, η) and dimensionless variables, can be written as Singh and Bello (2018)Brumberg (1972) and Bhatnagar and Hallan (1998):

$$\ddot{\xi} - 2n\dot{\eta} = \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right),$$

(1)

$$\ddot{\eta} + 2n\dot{\xi} = \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right),$$

with

$$W = \frac{1}{2}n^2\left(\xi^2+\eta^2\right) + \frac{1-\mu}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right) + \frac{\mu}{\rho_2} + \frac{1}{c^2}\left[\frac{1}{8}\left\{\dot{\xi}^2+\dot{\eta}^2+2n(\xi\dot{\eta}-\eta\dot{\xi})+n^2(\xi^2+\eta^2)\right\}^2\right.$$

$$+\frac{3}{2}\left(\frac{1-\mu}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right)+\frac{\mu}{\rho_2}\right)\left\{\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+(\xi^2+\eta^2)\right\} - \frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}\left(1+\frac{A_1}{2\rho_1^2}\right)^2+\frac{\mu^2}{\rho_2^2}\right)$$

$$+\mu(1-\mu)\left\{n\left(4\dot{\eta}+\frac{7}{2}n\xi\right)\left(\frac{1}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right)-\frac{1}{\rho_2}\right)-\frac{n^2\eta^2}{2}\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+n^2\left(-\frac{1}{\rho_1\rho_2}+\frac{\mu}{2\rho_1}-\frac{(1-\mu)}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right)\right.\right.$$

$$\left.\left.\left.+\frac{1-3\mu}{2\rho_2}\right)\right\}\right],$$

(2)

$$\rho_1^2 = (\xi+\mu)^2+\eta^2$$

(3)

$$\rho_2^2 = (\xi+\mu-1)^2+\eta^2$$

$$n = 1+\frac{3}{4}A_1 - \frac{3}{2c^2}\left(1-\frac{\mu(1-\mu)}{3}\right) \qquad (4)$$

where $0<\mu\le\frac{1}{2}$ is the ratio of the mass of the smaller primary to the total mass of the primaries, ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and smaller primary, respectively; c is the velocity of light. A_1 characterizes the oblateness of the bigger primary and is given by $A_1 = \frac{AE^2-AP^2}{5R^2} << 1$ (McCuskey 1963) where AE and AP are the equatorial and polar radii of the bigger primary, and R is the distance between the primaries.

3. Locations of collinear points

Equilibrium points are those points at which no resultant force acts on the third infinitesimal body. Therefore, if it is placed at any of those points with zero velocity, it will stay there. In fact all derivatives of the coordinates with respect to the time are zero at these points.

$W_\xi \neq 0$ and $W_\eta = 0$ (5)

W_ξ and W_η may be written as

$$W_\xi = \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{\mu(\xi-1+\mu)}{\rho_2^3} + \frac{3}{2}A_1\left\{\xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^5}\right\} + \frac{1}{c^2}\left[-3\xi\left\{1 - \frac{\mu(1-\mu)}{3}\right\} + \frac{1}{2}\xi(\xi^2+\eta^2)\right.$$
$$-\frac{3}{2}(\xi^2+\eta^2)\left\{\frac{(1-\mu)(\xi+\mu)}{\rho_1^3} + \frac{\mu(\xi-1+\mu)}{\rho_2^3}\right\} + 3\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right)\xi + \frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \frac{\mu^2(\xi-1+\mu)}{\rho_2^4}$$
$$+\mu(1-\mu)\left\{\frac{7}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3}\right) + \frac{3}{2}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5} + \frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right) + \frac{(\xi+\mu)}{\rho_1^3\rho_2}\right.$$
$$+\frac{(\xi-1+\mu)}{\rho_1\rho_2^3} - \frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3} - \frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\Big\} + A_1\left\{-\frac{9}{4}\left(1 - \frac{\mu(1-\mu)}{3}\right)\xi + \frac{3}{2}(\xi^2+\eta^2)\xi\right.$$
$$+\left(-\frac{9(1-\mu)(\xi+\mu)}{4\rho_1^5} - \frac{9(1-\mu)(\xi+\mu)}{4\rho_1^3} - \frac{9\mu(\xi-1+\mu)}{4\rho_2^3}\right)(\xi^2+\eta^2) + \left(\frac{3(1-\mu)}{2\rho_1^3} + \frac{9(1-\mu)}{2\rho_1} + \frac{9\mu}{2\rho_2}\right)\xi$$
$$+\frac{2(1-\mu)^2(\xi+\mu)}{\rho_1^6}\Big\} + \mu(1-\mu)A_1\left\{\frac{7}{4\rho_1^3} + \frac{21}{4}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \frac{7}{2}\xi\left(-\frac{3(\xi+\mu)}{2\rho_1^5}\right) + \frac{21}{4}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3}\right)\right.$$
$$+\frac{9}{4}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5} + \frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right) + \frac{3(1-\mu)(\xi+\mu)}{2\rho_1^5} + \frac{3(\xi+\mu)}{2\rho_1^3\rho_2} + \frac{3(\xi-1+\mu)}{2\rho_1\rho_2^3} - \frac{3\mu(\xi+\mu)}{4\rho_1^3}$$
$$\left.\left.-\frac{3(1-3\mu)(\xi-1+\mu)}{4\rho_2^3} + \frac{3(1-\mu)(\xi+\mu)}{2\rho_1^3}\right\}\right]$$

and

$$W_\eta = \eta F,$$

with

$$F = 1 - \frac{1-\mu}{\rho_1^3} - \frac{\mu}{\rho_2^3} + \frac{3}{2}A_1\left(1 - \frac{1-\mu}{\rho_1^5}\right) + \frac{1}{c^2}\left[-3\left(1 - \frac{\mu(1-\mu)}{3}\right) + \frac{1}{2}(\xi^2+\eta^2) - \frac{3}{2}(\xi^2+\eta^2)\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right)\right.$$
$$+3\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right) + \left(\frac{(1-\mu)^2}{\rho_1^4} + \frac{\mu^2}{\rho_2^4}\right) + \mu(1-\mu)\left\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3} + \frac{1}{\rho_2^3}\right) - \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3}\right) + \frac{3}{2}\eta^2\left(\frac{\mu}{\rho_1^5} + \frac{(1-\mu)}{\rho_2^5}\right)\right.$$
$$+\frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3} - \frac{(3\mu-2)}{2\rho_1^3} - \frac{(1-3\mu)}{2\rho_2^3}\Big\} + A_1\left\{-\frac{9}{4}\left(1 - \frac{\mu(1-\mu)}{3}\right) + \frac{3}{2}(\xi^2+\eta^2) + \left(\frac{-9(1-\mu)}{4\rho_1^5} - \frac{9(1-\mu)}{4\rho_1^3} - \frac{9\mu}{4\rho_2^3}\right)(\xi^2+\eta^2)\right.$$
$$+\frac{3(1-\mu)}{2\rho_1^3} + \frac{9(1-\mu)}{2\rho_1} + \frac{9\mu}{2\rho_2} + \frac{2(1-\mu)^2}{\rho_1^6}\Big\} + \mu(1-\mu)A_1\left\{\frac{21}{4}\xi\left(-\frac{1}{\rho_1^5} - \frac{1}{\rho_1^3} + \frac{1}{\rho_2^3}\right) - \frac{3}{2}\left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3}\right) + \frac{9}{4}\eta^2\left(\frac{\mu}{\rho_1^5} + \frac{1-\mu}{\rho_2^5}\right)\right.$$
$$\left.\left.+\frac{3}{2}\left(\frac{1-\mu}{\rho_1^5} + \frac{1-\mu}{\rho_1^3} + \frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3} - \frac{\mu}{2\rho_1^3} - \frac{(1-3\mu)}{2\rho_2^3}\right)\right\}\right]$$

In order to find the collinear points, we put $\eta = 0$ in equation (5). The abscissa are the root of the equation.

$$f(\xi) = \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1} - \frac{\mu(\xi-1+\mu)}{\rho_2^3} + \frac{3}{2}A_1\left\{\xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^5}\right\} + \frac{1}{c^2}\left[-3\xi\left\{1 - \frac{\mu(1-\mu)}{3}\right\} + \frac{1}{2}\xi^3\right.$$
$$-\frac{3}{2}\xi^2\left\{\frac{(1-\mu)(\xi+\mu)}{\rho_1^3} + \frac{\mu(\xi-1+\mu)}{\rho_2^3}\right\} + 3\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right)\xi + \frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \frac{\mu^2(\xi-1+\mu)}{\rho_2^4}$$
$$+\mu(1-\mu)\left\{\frac{7}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3}\right) + \frac{(\xi+\mu)}{\rho_1^3\rho_2} + \frac{(\xi-1+\mu)}{\rho_1\rho_2^3} - \frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3}\right.$$
$$-\frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\Big\} + A_1\left\{-\frac{9}{4}\left(1 - \frac{\mu(1-\mu)}{3}\right)\xi + \frac{3}{2}\xi^3 + \left(-\frac{9(1-\mu)(\xi+\mu)}{4\rho_1^5} - \frac{9(1-\mu)(\xi+\mu)}{4\rho_1^3} - \frac{9\mu(\xi-1+\mu)}{4\rho_2^3}\right)\xi^2\right.$$
$$+\left(\frac{3(1-\mu)}{2\rho_1^3} + \frac{9(1-\mu)}{2\rho_1} + \frac{9\mu}{2\rho_2}\right)\xi + \frac{2(1-\mu)^2(\xi+\mu)}{\rho_1^6}\Big\} + \mu(1-\mu)A_1\left\{\frac{7}{4\rho_1^3} + \frac{21}{4}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \frac{7}{2}\xi\left(-\frac{3(\xi+\mu)}{2\rho_1^5}\right)\right.$$
$$+\frac{21}{4}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3}\right) + \frac{3(1-\mu)(\xi+\mu)}{2\rho_1^5} + \frac{3(\xi+\mu)}{2\rho_1^3\rho_2} + \frac{3(\xi-1+\mu)}{2\rho_1\rho_2^3} - \frac{3\mu(\xi+\mu)}{4\rho_1^3} - \frac{3(1-3\mu)(\xi-1+\mu)}{4\rho_2^3}$$
$$\left.\left.+\frac{3(1-\mu)(\xi+\mu)}{2\rho_1^3}\Big\}\right]\right\} = 0 \qquad (6)$$

with $\rho_1 = |\xi+\mu|$, $\rho_2 = |\xi-1+\mu|$

To locate the collinear points on the $\xi-$axis, we divide the orbital plane into three parts: $\xi < \xi_1$, $\xi_1 < \xi < \xi_2$ and $\xi_2 < \xi$ with respect to the primaries.

3.1 Position of $L_1(\xi > \xi_2)$

Let $\xi - \xi_2 = \lambda;\ \xi - \xi_1 = 1+\lambda \Rightarrow \xi = 1+\lambda+\xi_1;$ since the distance between the primaries is unity,

i.e $\xi_2 - \xi_1 = 1 \Rightarrow \xi_1 = -\mu$ and $\xi_2 = 1-\mu$ then $\xi = 1+\lambda-\mu; \rho_1 = 1+\lambda;\ \rho_2 = \lambda$ (7)

Now substituting equation (7) in (6), we obtain

$$(2+6A_1)\lambda^{11} + (-18A_1\mu - 6\mu + 16 + 48A_1)\lambda^{10} + (44 - 38\mu - 123A_1\mu + 159A_1 + 15A_1\mu^2 + 4c^2 + 6A_1c^2 + 2\mu^2)\lambda^9$$
$$+(46 - 351A_1\mu + 291A_1 - 4\mu c^2 + 24c^2 + 36A_1c^2 + 2\mu^3 + 8\mu^2 - 3A_1\mu^3 - 90\mu - 6A_1c^2\mu + 87A_1\mu^2)\lambda^8$$
$$+(90A_1c^2 - 10 - 90\mu + 60c^2 + 10\mu^2 - 540A_1\mu + 210A_1\mu^2 + 10\mu^3 - 30A_1\mu c^2 - 20\mu c^2 + 330A_1 - 15A_1\mu^3)\lambda^7$$
$$+(76c^2 - 40\mu c^2 + 255A_1\mu^2 + 120A_1c^2 - 30A_1\mu^3 - 68 - 462A_1\mu + 243A_1 + 20\mu^3 - 10\mu^2 - 60A_1c^2\mu)\lambda^6 +$$
$$+(24\mu^3 + 90A_1c^2 + 96\mu - 48\mu c^2 + 116A_1\mu^2 - 60 + 114A_1 - 56\mu^2 - 24A_1\mu^3 - 60A_1c^2\mu - 161A_1\mu + 48c^2)\lambda^5$$
$$+(126\mu - 24A_1c^2\mu - 92A_1\mu^2 + 98A_1\mu + 12c^2 - 48\mu c^2 + 30A_1c^2 + 30A_1 + 24\mu^3 - 94\mu^2 - 18)\lambda^4 +$$
$$(11A_1 + 102\mu + 135A_1\mu + 20\mu^3 - 76\mu^2 + 18A_1\mu^3 - 155A_1\mu^2 - 40\mu c^2)\lambda^3 + (75A_1\mu + 50\mu - 20\mu c^2 - 20\mu^2 + 15A_1\mu^3$$
$$+10\mu^3 - 90A_1\mu^2)\lambda^2 + (-4\mu c^2 + 8\mu^2 - 18A_1\mu^2 + 10\mu + 3A_1\mu^3 + 15A_1\mu + 2\mu^3)\lambda + 4\mu^2 = 0 \qquad (8)$$

3.2 Position of $L_2(\xi_1 < \xi < \xi_2)$

Let $\xi_2 - \xi = \lambda;\ \xi - \xi_1 = 1-\lambda \Rightarrow \xi = 1-\lambda-\mu;\ \rho_1 = 1-\lambda; \rho_2 = \lambda$ (9)

Substituting equation (9) in (6), we obtain

$$(2+6A_1)\lambda^{11} + (18A_1\mu + 6\mu - 48A_1 - 16)\lambda^{10} + (15A_1\mu^2 + 159A_1 + 2\mu^2 - 38\mu + 4c^2 + 44 - 123A_1\mu + 6A_1c^2)\lambda^9$$
$$+ (-8\mu^2 + 6A_1c^2\mu - 36A_1c^2 + 102\mu + 369A_1\mu - 87A_1\mu^2 - 46 - 24c^2 - 291A_1 + 3A_1\mu^3 - 2\mu^3 + 4\mu c^2)\lambda^8$$
$$+ (-630A_1\mu + 10\mu^3 + 210A_1\mu^2 - 30A_1\mu c^2 + 90A_1c^2 + 60c^2 + 330A_1 - 150\mu - 15A_1\mu^3 - 10 + 10\mu^2 - 20\mu c^2)\lambda^7$$
$$+ (132\mu - 16\mu^3 - 279A_1\mu^2 + 36A_1\mu^3 - 76c^2 - 6\mu^2 - 120A_1c^2 + 660A_1\mu + 68 - 243A_1 + 60A_1c^2\mu + 32\mu c^2)\lambda^6 +$$
$$+ (-100\mu - 455A_1\mu + 4\mu^3 + 40\mu^2 + 260A_1\mu^2 - 54A_1\mu^3 + 114A_1 - 60 - 60A_1c^2\mu + 90A_1c^2 + 48c^2 - 8\mu c^2)\lambda^5$$
$$+ (256A_1\mu + 16\mu^3 - 30A_1c^2 - 122\mu^2 + 60A_1\mu^3 - 12c^2 + 24A_1c^2\mu - 232A_1\mu^2 - 30A_1 + 18 + 110\mu - 32\mu c^2)\lambda^4 +$$
$$(-20\mu^3 - 171A_1\mu - 102\mu + 156\mu^2 - 42A_1\mu^3 + 193A_1\mu^2 40\mu c^2 + 11A_1)\lambda^3 + (-90A_1\mu^2 + 10\mu^3 + 50\mu + 75A_1\mu + 15A_1\mu^3$$
$$-100\mu^2 - 20\mu c^2)\lambda^2 + (18A_1\mu^2 - 15A_1\mu - 3A_1\mu^3 + 32\mu^2 - 10\mu - 2\mu^3 + 4\mu c^2)\lambda - 4\mu^2 = 0 \quad (10)$$

3.3 Position of $L_3(\xi < \xi_1)$

Let the distance of the point L_3 from the bigger primary be $1-\lambda$.

Since $\xi_2 - \xi_1 = 1 \Rightarrow \xi_1 - \xi = 1-\lambda; \xi_2 - \xi = 2-\lambda$ and $\xi = \lambda - \mu - 1;\ \rho_1 = 1-\lambda;\ \rho_2 = 2-\lambda$

(11)

substituting equation (11) in (6), we obtain

$$(2+6A_1)\lambda^{11} + (-18A_1\mu - 6\mu - 28 - 84A_1)\lambda^{10} + (164 + 4c^2 + 82\mu + 519A_1 + 2\mu^2 + 6A_1c^2 + 15A_1\mu^2 + 237A_1\mu)\lambda^9$$
$$+ (-72A_1c^2 - 4\mu c^2 - 6A_1c^2\mu - 486\mu - 518 - 28\mu^2 - 48c^2 - 1869A_1 - 3A_1\mu^3 + 2\mu^3 - 1377A_1\mu - 183A_1\mu^2)\lambda^8$$
$$+ (44\mu c^2 + 922 + 33A_1\mu^3 - 22\mu^3 + 1638\mu + 378A_1c^2 + 252c^2 + 170\mu^2 + 66A_1c^2\mu + 4644A_1\mu + 4368A_1 + 978A_1\mu^2)\lambda^7$$
$$+ (-578\mu^2 - 208\mu c^2 - 6987A_1 - 3464\mu - 1140A_1c^2 - 10074A_1\mu - 776 - 156A_1\mu^3 - 312A_1\mu c^2 - 756c^3 + 104\mu^3$$
$$- 2991\mu^2A_1)\lambda^6 + (1204\mu^2 + 5734A_1\mu^2 + 7860A_1 - 280\mu^3 + 4788\mu + 14747A_1\mu + 828A_1c^2\mu + 408A_1\mu^3 - 220 + 560\mu c^2$$
$$+ 1416c^2 + 2178A_1c^2)\lambda^5 + (-2730A_1c^2 - 1356A_1c^2\mu + 476\mu^3 - 14914A_1\mu - 6270A_1 - 952\mu c^2 - 1586\mu^2 - 630A_1\mu^3$$
$$- 4362\mu - 7082A_1\mu^2 - 1692c^2 + 1274)\lambda^4 + (-1372 + 2550\mu + 1428A_1c^2\mu + 1056\mu c^2 + 1256c^2 + 2220A_1c^2 + 5559A_1\mu^2$$
$$- 528\mu^3 + 10513A_1\mu + 552A_1\mu^3 + 3503A_1 + 1320\mu^2)\lambda^3 + (696 - 1080A_1c^2 - 5077A_1\mu - 862\mu - 676\mu^2 - 528c^2 - 2636A_1\mu^2$$
$$- 1338A_1 - 984A_1c^2\mu + 374\mu^3 - 748c^2\mu - 207A_1\mu^3)\lambda^2 + (1493A_1\mu - 154\mu^3 + 706A_1\mu^2 + 200\mu^2 - 144 + 240A_1c^2 + 372A_1$$
$$+ 308\mu c^2 - 39A_1\mu^3 + 110\mu + 432A_1c^2\mu + 96c^2)\lambda + (12\mu - 88A_1 + 28\mu^3 - 96A_1c^2\mu - 158A_1\mu + 42A_1\mu^3 - 108A_1\mu^2 - 28\mu^2$$
$$- 56\mu c^2) = 0 \quad (12)$$

4. Stability of collinear points

We examine the stability of an equilibrium configuration that is its ability to restrain the body motion in its vicinity. To do so we displace the infinitesimal body a little from an equilibrium point with small velocity. If its motion is rapid departure from vicinity of the point, we call such a position of equilibrium an unstable one. If the body oscillates about the point, it is said to be a stable position.

In order to study the stability of the collinear points, following Singh and Bello (2014), the characteristic equation is given by

$$(P_1q_2 - P_2q_1)\lambda^4 + (P_1q_6 + P_5q_2 + P_3q_4 - P_6q_1 - P_2q_5 - P_4q_3)\lambda^2 + P_5q_6 - P_6q_5 = 0 \qquad (13)$$

where,

$$P_1 = 1 + W^0_{\xi\xi}, P_2 = W^0_{\eta\xi}, P_3 = W^0_{\xi\dot{\xi}} - W^0_{\xi\dot{\xi}} = 0, P_4 = \left\{W^0_{\eta\dot{\xi}} - 2\left(1 + \frac{3}{4}A_1 - \frac{3}{2c^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)\right) - W^0_{\xi\dot{\eta}}\right\},$$

$$P_5 = -W^0_{\xi\dot{\xi}}, P_6 = -W^0_{\xi\eta}, q_1 = W^0_{\dot{\xi}\dot{\eta}}, q_2 = 1 + W^0_{\dot{\eta}\dot{\eta}}, q_3 = 2\left\{1 + \frac{3}{4}A_1 - \frac{3}{2c^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)\right\} + W^0_{\xi\dot{\eta}} - W^0_{\eta\dot{\xi}},$$

$$q_4 = W^0_{\eta\dot{\eta}} - W^0_{\eta\dot{\eta}} = 0, q_5 = -W^0_{\xi\dot{\xi}}, q_6 = -W^0_{\eta\eta}.$$

The second order partial derivative of W are denoted by subscripts. The superscripts 0 indicates that the derivative is to be evaluated at the equilibrium points (ξ_0, η_0).

The second order derivatives are

$$W_{\xi\xi} = 1+\frac{3(1-\mu)(\xi+\mu)^2}{\rho_1^5}-\frac{(1-\mu)}{\rho_1^3}+\frac{3\mu(\xi-1+\mu)^2}{\rho_2^5}-\frac{\mu}{\rho_2^3}+A_1\left\{\frac{3}{2}+\frac{15(1-\mu)(\xi+\mu)^2}{2\rho_1^7}-\frac{3(1-\mu)}{2\rho_1^5}\right\}+$$

$$\frac{1}{c^2}\left[(\mu(1-\mu)-3)+(\dot\eta+\xi)^2+\frac{1}{2}(\dot\xi^2+\dot\eta^2)+\xi\dot\eta-\eta\dot\xi+\frac{1}{2}(\xi^2+\eta^2)+\left(\frac{9(1-\mu)(\xi+\mu)^2}{2\rho_1^5}-\frac{3(1-\mu)}{2\rho_1^3}-\frac{3\mu}{2\rho_2^3}\right.\right.$$

$$\left.+\frac{9\mu(\xi-1+\mu)^2}{2\rho_2^5}\right)\left(\dot\xi^2+\dot\eta^2+2\xi\dot\eta-2\eta\dot\xi+\xi^2+\eta^2\right)+2\left(\frac{-3(1-\mu)(\xi+\mu)}{\rho_1^3}-\frac{3\mu(\xi-1+\mu)}{\rho_2^3}\right)(\dot\eta+\xi)$$

$$+\frac{3(1-\mu)}{\rho_1}+\frac{3\mu}{\rho_2}-\frac{4(1-\mu)^2(\xi+\mu)^2}{\rho_1^6}+\frac{(1-\mu)^2}{\rho_1^4}-\frac{4\mu^2(\xi-1+\mu)^2}{\rho_2^6}+\frac{\mu^2}{\rho_2^4}+\mu(1-\mu)\left\{\left(-\frac{7(\xi+\mu)}{\rho_1^3}+\frac{7(\xi-1+\mu)}{\rho_2^3}\right)\right.$$

$$+\left(\frac{3(\xi+\mu)^2}{\rho_1^5}-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}-\frac{3(\xi-1+\mu)^2}{\rho_2^5}\right)\left(4\dot\eta+\frac{7}{2}\xi\right)-\frac{\eta^2}{2}\left(\frac{15\mu(\xi+\mu)^2}{\rho_1^7}-\frac{3\mu}{\rho_1^5}+\frac{15(1-\mu)(\xi-1+\mu)^2}{\rho_2^7}-\frac{3(1-\mu)}{\rho_2^5}\right)$$

$$-\frac{3(\xi+\mu)^2}{\rho_1^5\rho_2}-\frac{2(\xi+\mu)(\xi-1+\mu)}{\rho_1^3\rho_2^3}+\frac{1}{\rho_1^3\rho_2}-\frac{3(\xi-1+\mu)^2}{\rho_1\rho_2^5}+\frac{1}{\rho_1\rho_2^3}+\frac{3(3\mu-2)(\xi+\mu)^2}{2\rho_1^5}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}$$

$$\left.+\frac{3(1-3\mu)(\xi-1+\mu)^2}{2\rho_2^5}\right\}+A_1\left\{\frac{3}{4}(\mu(1-\mu)-3)+\frac{3}{2}(\dot\eta+\xi)(\dot\eta+2\xi)+\frac{3}{4}(\dot\xi^2+\dot\eta^2)+\frac{9}{4}(\xi\dot\eta-\eta\dot\xi)+\frac{3}{2}(\xi^2+\eta^2)+\right.$$

$$\left(\frac{9(1-\mu)(\xi+\mu)^2}{2\rho_1^5}-\frac{3(1-\mu)}{2\rho_1^3}-\frac{3\mu}{2\rho_2^3}+\frac{9\mu(\xi-1+\mu)^2}{2\rho_2^5}\right)\left(\frac{3}{2}\xi\dot\eta-\frac{3}{2}\eta\dot\xi+\frac{3}{2}(\xi^2+\eta^2)\right)+\left(\frac{45(1-\mu)(\xi+\mu)^2}{4\rho_1^7}\right.$$

$$\left.-\frac{9(1-\mu)}{4\rho_1^5}\right)\left(\dot\xi^2+\dot\eta^2+2\xi\dot\eta-2\eta\dot\xi+\xi^2+\eta^2\right)+\left(\frac{-3(1-\mu)(\xi+\mu)}{\rho_1^3}-\frac{3\mu(\xi-1+\mu)}{\rho_2^3}\right)\left(\frac{3}{2}\dot\eta+3\xi\right)-\frac{9(1-\mu)(\xi+\mu)}{\rho_1^5}(\dot\eta+\xi)$$

$$+\left(\frac{9(1-\mu)}{2\rho_1}+\frac{9\mu)}{2\rho_2}+\frac{3(1-\mu)}{2\rho_1^3}\right)-\frac{16(1-\mu)^2(\xi+\mu)^2}{\rho_1^8}+\frac{2(1-\mu)^2}{\rho_1^6}\Bigg\}+\mu(1-\mu)A_1\left\{\frac{-21}{2}\left(\frac{\xi+\mu}{\rho_1^5}+\frac{\xi+\mu}{\rho_1^3}-\frac{(\xi-1+\mu)}{\rho_2^3}\right)\right.$$

$$+\left(\frac{15(1-\mu)^2}{2\rho_1^7}-\frac{3}{2\rho_1^5}\right)\left(4\dot\eta+\frac{7}{2}\xi\right)+\left(\frac{3(\xi+\mu)^2}{\rho_1^5}-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}-\frac{3(\xi-1+\mu)^2}{\rho_2^5}\right)\left(3\dot\eta+\frac{21}{4}\xi\right)-\frac{3}{4}\eta^2\left(\frac{15\mu(\xi+\mu)}{\rho_1^7}\right.$$

$$\left.-\frac{3\mu}{\rho_1^5}+\frac{15(1-\mu)(\xi-1+\mu)^2}{\rho_2^7}-\frac{3(1-\mu)}{\rho_2^5}\right)-\frac{15(1-\mu)(\xi+\mu)^2}{2\rho_1^7}+\frac{3(1-\mu)}{2\rho_1^5}-\frac{9(\xi+\mu)^2}{2\rho_1^5\rho_2}-\frac{3(\xi+\mu)(\xi-1+\mu)}{\rho_1^3\rho_2^3}$$

$$\left.\left.+\frac{3}{2\rho_1^3\rho_2}+\frac{3}{2\rho_1\rho_2^3}-\frac{9(\xi-1+\mu)^2}{2\rho_1\rho_2^5}+\frac{9(3\mu-2)(\xi+\mu)^2}{4\rho_1^5}-\frac{3(3\mu-2)}{4\rho_1^3}-\frac{3(1-3\mu)}{4\rho_2^3}+\frac{9(1-3\mu)(\xi-1+\mu)^2}{4\rho_2^5}\right\}\right] \qquad (14a)$$

$$W_{\eta\eta} = 1+\frac{3(1-\mu)\eta^2}{\rho_1^5}-\frac{(1-\mu)}{\rho_1^3}+\frac{3\mu\eta^2}{\rho_2^5}-\frac{\mu}{\rho_2^3}+A_1\left\{\frac{3}{2}+\frac{15(1-\mu)\eta^2}{2\rho_1^7}-\frac{3(1-\mu)}{2\rho_1^5}\right\}+\frac{1}{c^2}\left[\left(\mu(1-\mu)-3\right)+(\eta-\dot{\xi})^2\right.$$

$$\frac{3}{4}(\dot{\xi}^2+\dot{\eta}^2)+\frac{9}{4}(\xi\dot{\eta}-\eta\dot{\xi})+\frac{3}{2}(\xi^2+\eta^2)+\left(\frac{9(1-\mu)\eta^2}{2\rho_1^5}-\frac{3(1-\mu)}{2\rho_1^3}-\frac{3\mu}{2\rho_2^3}+\frac{9\mu\eta^2}{2\rho_2^5}\right)\left(\dot{\xi}^2+\dot{\eta}^2+2\xi\dot{\eta}-2\eta\dot{\xi}+\xi^2+\eta^2\right)$$

$$+2\left(-\frac{3(1-\mu)\eta}{\rho_1^3}-\frac{3\mu\eta}{\rho_2^3}\right)(\eta-\dot{\xi})+\frac{3(1-\mu)}{\rho_1}+\frac{3\mu}{\rho_2}-\frac{4(1-\mu)^2\eta^2}{\rho_1^6}+\frac{(1-\mu)^2}{\rho_1^4}-\frac{4\mu^2\eta^2}{\rho_2^6}+\frac{\mu^2}{\rho_2^4}$$

$$+\mu(1-\mu)\left\{\left(\frac{3\eta^2}{\rho_1^5}-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}-\frac{3\eta^2}{\rho_2^5}\right)\left(4\dot{\eta}+\frac{7}{2}\xi\right)-\frac{\mu}{\rho_1^3}-\frac{(1-\mu)}{\rho_2^3}+6\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{1-\mu}{\rho_2^5}\right)-\frac{\eta^2}{2}\left(\frac{15\mu\eta^2}{\rho_1^7}-\frac{3\mu}{\rho_1^5}\right.\right.$$

$$\left.+\frac{15(1-\mu)\eta^2}{\rho_2^7}-\frac{3(1-\mu)}{\rho_2^5}\right)-\frac{3\eta^2}{\rho_1^5\rho_2}-\frac{2\eta^2}{\rho_1^3\rho_2^3}+\frac{1}{\rho_1^3\rho_2}-\frac{3\eta^2}{\rho_1\rho_2^5}+\frac{1}{\rho_1\rho_2^3}+\frac{3(3\mu-2)\eta^2}{2\rho_1^5}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}$$

$$\left.+\frac{3(1-3\mu)\eta^2}{2\rho_2^5}\right\}+A_1\left\{\frac{3}{4}\left(\mu(1-\mu)-3\right)+(\eta-\dot{\xi})\left(3\eta-\frac{3}{2}\dot{\xi}\right)+\frac{3}{4}(\dot{\xi}^2+\dot{\eta}^2)+\frac{9}{4}\left(\xi\dot{\eta}-\eta\dot{\xi}\right)+\frac{3}{2}(\xi^2+\eta^2)\right.$$

$$+\left(\frac{9(1-\mu)\eta^2}{2\rho_1^5}-\frac{3(1-\mu)}{2\rho_1^3}-\frac{3\mu}{2\rho_2^3}+\frac{9\mu\eta^2}{2\rho_2^5}\right)\left(\frac{3}{2}\left(\xi\dot{\eta}-\eta\dot{\xi}+\xi^2+\eta^2\right)\right)+\left(\frac{45(1-\mu)\eta^2}{4\rho_1^7}-\frac{9(1-\mu)}{\rho_1^3}\right)\left(\dot{\xi}^2+\dot{\eta}^2+2\xi\dot{\eta}-2\eta\dot{\xi}+\xi^2+\eta^2\right)$$

$$\left.+\left(-\frac{3(1-\mu)\eta}{\rho_1^3}-\frac{3\mu\eta}{\rho_2^3}\right)\left(3\eta-\frac{3}{2}\dot{\xi}\right)-\frac{9(1-\mu)\eta}{\rho_1^5}(\eta-\dot{\xi})+\frac{9(1-\mu)}{2\rho_1}+\frac{9\mu}{2\rho_2}+\frac{3(1-\mu)}{2\rho_1^3}+\frac{2(1-\mu)^2}{\rho_1^6}-\frac{12(1-\mu)^2\eta^2}{\rho_1^8}\right\}$$

$$+\mu(1-\mu)A_1\left\{+\left(\frac{15\eta^2}{2\rho_1^7}-\frac{3}{2\rho_1^5}\right)\left(4\dot{\eta}+\frac{7}{2}\xi\right)+\left(\frac{3\eta^2}{\rho_1^5}-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}-\frac{3\eta^2}{\rho_2^5}\right)\left(3\dot{\eta}+\frac{21}{4}\xi\right)-\frac{3\mu}{2\rho_1^3}-\frac{3(1-\mu)}{2\rho_2^3}\right.$$

$$+9\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{1-\mu}{\rho_2^5}\right)-\frac{3\eta^2}{4}\left(\frac{15\mu\eta^2}{\rho_1^7}-\frac{3\mu}{\rho_1^5}+\frac{15(1-\mu)\eta^2}{\rho_2^7}-\frac{3(1-\mu)}{\rho_2^5}\right)-\frac{15(1-\mu)\eta^2}{2\rho_1^7}+\frac{3(1-\mu)}{2\rho_1^5}-\frac{9\eta^2}{2\rho_1^5\rho_2}-\frac{3\eta^2}{\rho_1^3\rho_3^3}$$

$$\left.\left.+\frac{3}{2\rho_1^3\rho_2}+\frac{3}{2\rho_1\rho_2^3}-\frac{9\eta^2}{2\rho_1\rho_2^5}+\frac{9(3\mu-2)\eta^2}{4\rho_1^5}-\frac{3(3\mu-2)}{4\rho_1^5}+\frac{9(1-3\mu)\eta^2}{4\rho_2^5}-\frac{3(1-3\mu)}{4\rho_2^3}\right\}\right] \qquad (15a)$$

$$W_{\xi\eta} = \frac{3(1-\mu)(\xi+\mu)\eta}{\rho_1^5} + \frac{3\mu(\xi-1+\mu)\eta}{\rho_2^5} + A_1\left\{\frac{15(1-\mu)(\xi+\mu)\eta}{2\rho_1^7}\right\} +$$
$$\frac{1}{c^2}\left[(\dot\eta+\xi)(\eta-\dot\xi) + \left(\frac{9(1-\mu)(\xi+\mu)\eta}{2\rho_1^5} + \frac{9\mu(\xi-1+\mu)\eta}{2\rho_2^5}\right)\left(\dot\xi^2+\dot\eta^2+2\xi\dot\eta-2\eta\dot\xi+\xi^2+\eta^2\right)\right.$$
$$+\left(-\frac{3(1-\mu)\eta}{\rho_1^3} - \frac{3\mu\eta}{\rho_2^3}\right)(\dot\eta+\xi) + \left(-\frac{3(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{3\mu(\xi-1+\mu)}{\rho_2^3}\right)(\eta-\dot\xi) - \frac{4(1-\mu)^2(\xi+\mu)\eta}{\rho_1^6}$$
$$-\frac{4\mu^2(\xi-1+\mu)\eta}{\rho_2^6} + \mu(1-\mu)\left\{\left(\frac{7\eta}{2\rho_2^3} - \frac{7\eta}{\rho_1^3} + \left(4\dot\eta+\frac{7}{2}\xi\right)\left(\frac{3(\xi+\mu)\eta}{\rho_1^5} - \frac{3(\xi-1+\mu)\eta}{\rho_2^5}\right) + \eta\left(\frac{3\mu(\xi+\mu)}{\rho_1^5}\right.\right.\right.$$
$$\left.+\frac{3(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right) - \frac{\eta^2}{2}\left(\frac{15\mu(\xi+\mu)\eta}{\rho_1^7} + \frac{15(1-\mu)(\xi-1+\mu)\eta}{\rho_2^7}\right) - \frac{3(\xi+\mu)\eta}{\rho_1^5\rho_2} - \frac{(\xi-1+\mu)\eta}{\rho_1^3\rho_2^3}$$
$$\left.-\frac{(\xi+\mu)\eta}{\rho_1^3\rho_2^3} - \frac{3\eta(\xi-1+\mu)}{\rho_1\rho_2^5} + \frac{3(3\mu-2)(\xi+\mu)\eta}{2\rho_1^5} + \frac{3(1-3\mu)(\xi-1+\mu)\eta}{2\rho_2^5}\right\} + A_1\left\{\frac{1}{2}(\dot\eta+\xi)\left(3\eta-\frac{3}{2}\dot\xi\right)\right.$$
$$+\frac{1}{2}\left(\frac{3}{2}\dot\eta+3\xi\right)(\eta-\dot\xi) + \frac{3}{4}\left(\frac{9(1-\mu)(\xi+\mu)\eta}{\rho_1^5} + \frac{9\mu(\xi-1+\mu)\eta}{\rho_2^5}\right)\left(\xi\dot\eta-\eta\dot\xi+\xi^2+\eta^2\right)$$
$$+\frac{45(1-\mu)(\xi+\mu)\eta}{4\rho_1^7}\left(\dot\xi^2+\dot\eta^2+2\xi\dot\eta-2\eta\dot\xi+\xi^2+\eta^2\right) + \left(-\frac{3(1-\mu)\eta}{2\rho_1^3} - \frac{3\mu\eta}{\rho_2^3}\right)\left(\frac{3}{2}\dot\eta+3\xi\right)$$
$$\left.+\left(-\frac{9(1-\mu)(\xi+\mu)}{2\rho_1^5}\right)(\eta-\dot\xi) - \frac{12(1-\mu)^2(\xi+\mu)\eta}{\rho_1^8}\right\} + \mu(1-\mu)A_1\left\{\left(\frac{-21\eta}{4}\left(\frac{1}{\rho_1^5}+\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right)\right.\right.$$
$$+\frac{15(\xi+\mu)\eta}{2\rho_1^7}\left(4\dot\eta+\frac{7}{2}\xi\right) + \left(3\dot\eta+\frac{21}{4}\xi\right)\left(\frac{3(\xi+\mu)\eta}{\rho_1^5} - \frac{3(\xi-1+\mu)\eta}{\rho_2^5}\right) + \frac{9}{2}\eta\left(\frac{\mu(\xi+\mu)}{\rho_1^5} + \frac{(1-\mu)(\xi+\mu)}{\rho_2^5}\right)$$
$$-\frac{3}{4}\eta^2\left(\frac{15\mu(\xi+\mu)\eta}{\rho_1^7} + \frac{15(1-\mu)(\xi-1+\mu)\eta}{\rho_2^7}\right) - \frac{15(\xi+\mu)\eta}{\rho_1^7} - \frac{9(\xi+\mu)\eta}{2\rho_1^5\rho_2} - \frac{3\eta(\xi-1+\mu)}{2\rho_1^3\rho_2^3}$$
$$\left.\left.-\frac{3\eta(\xi+\mu)}{2\rho_1^3\rho_2^3} - \frac{9\eta(\xi-1+\mu)}{2\rho_1\rho_2^5} + \frac{9(3\mu-2)(\xi+\mu)\eta}{4\rho_1^5} + \frac{9(1-3\mu)(\xi-1+\mu)\eta}{4\rho_2^5}\right\}\right] \qquad (16a)$$

$$W_{\dot\xi\dot\xi} = \frac{1}{c^2}\left[(\dot\xi-\eta)^2 + \frac{1}{2}\left(\dot\xi^2+\dot\eta^2\right) + \left(\xi\dot\eta-\eta\dot\xi\right) + \frac{1}{2}\left(\xi^2+\eta^2\right) + \frac{3(1-\mu)}{\rho_1} + \frac{3\mu}{\rho_2} + A_1\left\{\frac{3}{2}\eta\left(\eta-\dot\xi\right) + \frac{3}{4}\left(\xi\dot\eta-\eta\dot\xi\right)\right.\right.$$
$$\left.\left.\frac{3}{4}\left(\xi^2+\eta^2\right) + \frac{3(1-\mu)}{2\rho_1^3}\right\}\right] \qquad (17a)$$

$$W_{\dot\eta\dot\eta} = \frac{1}{c^2}\left[(\dot\eta+\xi)^2 + \frac{1}{2}\left(\dot\xi+\dot\eta\right)^2 + \left(\xi\dot\eta-\eta\dot\xi\right) + \frac{1}{2}\left(\xi^2+\eta^2\right) + \frac{3(1-\mu)}{\rho_1} + \frac{3\mu}{\rho_2} + A_1\left\{\frac{3}{2}\left(\dot\eta+\xi\right)\xi + \frac{3}{4}\left(\xi\dot\eta-\eta\dot\xi\right)\right.\right.$$
$$\left.\left.\frac{3}{4}\left(\xi^2+\eta^2\right) + \frac{3(1-\mu)}{2\rho_1^3}\right\}\right] \qquad (18a)$$

$$W_{\eta\dot{\xi}} = \frac{1}{c^2}\left[(\dot{\xi}-\eta)(\dot{\eta}+\xi)+\frac{3}{4}A_1\{(\dot{\xi}-\eta)\xi-\eta(\dot{\eta}+\xi)\}\right] \qquad (19a)$$

$$W_{\xi\dot{\xi}} = \frac{1}{c^2}\left[(\dot{\eta}+\xi)(\dot{\xi}-\eta)-3(\dot{\xi}-\eta)\left(\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}+\frac{\mu(\xi-1+\mu)}{\rho_2^3}\right)+A_1\left\{-\frac{3}{4}(\dot{\eta}+\xi)\eta+\frac{1}{2}\left(\frac{3}{2}\dot{\eta}+3\xi\right)(\dot{\xi}-\eta)\right.\right.$$
$$\left.\left.\frac{9}{4}\eta\left(\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}+\frac{\mu(\xi-1+\mu)}{\rho_2^3}\right)-\frac{9(1-\mu)(\xi+\mu)}{2\rho_1^5}(\dot{\xi}-\eta)\right\}\right] \qquad (20a)$$

$$W_{\eta\dot{\eta}} = \frac{1}{c^2}\left[(\eta-\dot{\xi})(\xi+\dot{\eta})-3(\dot{\eta}+\xi)\left(\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}\right)\eta+\mu(1-\mu)\left(-\frac{4\eta}{\rho_1^3}+\frac{4\eta}{\rho_2^3}\right)\right]+A_1\left\{\frac{3}{2}(\eta-\dot{\xi})\xi+\frac{1}{2}(\dot{\eta}+\xi)\left(-\frac{3}{4}\dot{\xi}+3\eta\right)\right.$$
$$\left.-\frac{9}{4}\eta\left(\frac{(1-\mu)}{\rho_1^3}+\frac{\mu}{\rho_2^3}\right)\xi-\frac{9\eta(1-\mu)}{2\rho_1^5}(\dot{\eta}+\xi)+\mu(1-\mu)\left(-\frac{6\eta}{\rho_1^5}-\frac{3\eta}{\rho_1^3}+\frac{3\eta}{\rho_2^3}\right)\right\}\Bigg] \qquad (21a)$$

In order to study the stability of the collinear points we have to study the motion in the proximity of these points, hence in this case (14a)-(21a) can be written as:

$$W^0_{\xi\xi} = 1+\frac{3(1-\mu)(\xi+\mu)^2}{\rho_1^5}-\frac{(1-\mu)}{\rho_1^3}+\frac{3\mu(\xi-1+\mu)^2}{\rho_2^5}-\frac{\mu}{\rho_2^3}+A_1\left\{\frac{3}{2}+\frac{15(1-\mu)(\xi+\mu)^2}{2\rho_1^7}-\frac{3(1-\mu)}{2\rho_1^5}\right\}+$$

$$\frac{1}{c^2}\left[(\mu(1-\mu)-3)+\frac{3}{2}\xi^2+\left(\frac{9(1-\mu)(\xi+\mu)^2}{2\rho_1^5}-\frac{3(1-\mu)}{2\rho_1^3}-\frac{3\mu}{2\rho_2^3}+\frac{9\mu(\xi-1+\mu)^2}{2\rho_2^5}\right)\xi^2+2\left(\frac{-3(1-\mu)(\xi+\mu)}{\rho_1^3}\right.\right.$$

$$\left.-\frac{3\mu(\xi-1+\mu)}{\rho_2^3}\right)\xi+\frac{3(1-\mu)}{\rho_1}+\frac{3\mu}{\rho_2}-\frac{4(1-\mu)^2(\xi+\mu)^2}{\rho_1^6}+\frac{(1-\mu)^2}{\rho_1^4}-\frac{4\mu^2(\xi-1+\mu)^2}{\rho_2^6}+\frac{\mu^2}{\rho_2^4}+\mu(1-\mu)\left\{\left(-\frac{7(\xi+\mu)}{\rho_1^3}\right.\right.$$

$$\left.+\frac{7(\xi-1+\mu)}{\rho_2^3}\right)+\frac{7}{2}\xi\left(\frac{3(\xi+\mu)^2}{\rho_1^5}-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}-\frac{3(\xi-1+\mu)^2}{\rho_2^5}\right)-\frac{3(\xi+\mu)^2}{\rho_1^5\rho_2}-\frac{2(\xi+\mu)(\xi-1+\mu)}{\rho_1^3\rho_2^3}+\frac{1}{\rho_1^3\rho_2}-\frac{3(\xi-1+\mu)^2}{\rho_1\rho_2^5}$$

$$+\frac{1}{\rho_1\rho_2^3}+\frac{3(3\mu-2)(\xi+\mu)^2}{2\rho_1^5}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}+\frac{3(1-3\mu)(\xi-1+\mu)^2}{2\rho_2^5}\Bigg\}$$

$$+A_1\left\{\frac{3}{4}(\mu(1-\mu)-3)+\frac{9}{2}\xi^2+\frac{3}{4}\xi^2\left(\frac{9(1-\mu)(\xi+\mu)^2}{\rho_1^5}-\frac{3(1-\mu)}{\rho_1^3}-\frac{3\mu}{\rho_2^3}+\frac{9\mu(\xi-1+\mu)^2}{\rho_2^5}\right)+\left(\frac{45(1-\mu)(\xi+\mu)^2}{4\rho_1^7}\right.\right.$$

$$\left.-\frac{9(1-\mu)}{\rho_1^5}\right)\xi^2+3\xi\left(-\frac{3(1-\mu)(\xi+\mu)}{\rho_1^3}-\frac{3\mu(\xi-1+\mu)}{\rho_2^3}\right)-\frac{9(1-\mu)(\xi+\mu)}{\rho_1^5}\xi+\left(\frac{9(1-\mu)}{2\rho_1}+\frac{9\mu)}{2\rho_2}+\frac{3(1-\mu)}{2\rho_1^3}\right)$$

$$-\frac{16(1-\mu)^2(\xi+\mu)^2}{\rho_1^8}+\frac{2(1-\mu)^2}{\rho_1^6}\Bigg\}+\mu(1-\mu)A_1\left\{\left\{\frac{-21}{2}\left(\frac{\xi+\mu}{\rho_1^5}+\frac{\xi+\mu}{\rho_1^3}-\frac{(\xi-1+\mu)}{\rho_2^3}\right)+\frac{7}{2}\xi\left(\frac{15(1-\mu)^2}{2\rho_1^7}-\frac{3}{2\rho_1^5}\right)\right.\right.$$

$$+\frac{21}{4}\xi\left(\frac{3(\xi+\mu)^2}{\rho_1^5}-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}-\frac{3(\xi-1+\mu)^2}{\rho_2^5}\right)-\frac{15(1-\mu)(\xi+\mu)^2}{2\rho_1^7}+\frac{3(1-\mu)}{2\rho_1^5}-\frac{9(\xi+\mu)^2}{2\rho_1^5\rho_2}-\frac{3(\xi+\mu)(\xi-1+\mu)^2}{\rho_1^3\rho_2^3}$$

$$\left.\left.+\frac{3}{2\rho_1^3\rho_2}+\frac{3}{2\rho_1\rho_2^3}-\frac{9(\xi-1+\mu)^2}{2\rho_1\rho_2^5}+\frac{9(3\mu-2)(\xi+\mu)^2}{4\rho_1^5}-\frac{3(3\mu-2)}{4\rho_1^3}-\frac{3(1-3\mu)}{4\rho_2^3}+\frac{9(1-3\mu)(\xi-1+\mu)^2}{4\rho_2^5}\right\}\right]\Bigg] \qquad (14b)$$

$$W^0_{\eta\eta} = 1-\frac{(1-\mu)}{\rho_1^3}--\frac{\mu}{\rho_2^3}+A_1\left\{\frac{3}{2}-\frac{3(1-\mu)}{2\rho_1^5}\right\}+\frac{1}{c^2}\left[(\mu(1-\mu)-3)+\frac{3}{2}\xi^2+\left(-\frac{3(1-\mu)}{2\rho_1^3}-\frac{3\mu}{2\rho_2^3}\right)\xi^2+\frac{3(1-\mu)}{\rho_1}+\frac{3\mu}{\rho_2}\right.$$

$$+\frac{(1-\mu)^2}{\rho_1^4}+\frac{\mu^2}{\rho_2^4}+\mu(1-\mu)\left\{\left\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\frac{\mu}{\rho_1^3}-\frac{(1-\mu)}{\rho_2^3}+\frac{1}{\rho_1^3\rho_2}+\frac{1}{\rho_1\rho_2^3}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}\right\}\right\}$$

$$+A_1\left\{\left\{\frac{3}{4}(\mu(1-\mu)-3)+\frac{3}{2}\xi^2+\frac{3}{2}\xi^2\left(-\frac{3(1-\mu)}{2\rho_1^3}-\frac{3\mu}{2\rho_2^3}\right)-\frac{9(1-\mu)}{4\rho_1^5}\xi^2+\frac{9(1-\mu)}{2\rho_1}+\frac{9\mu}{2\rho_2}+\frac{3(1-\mu)}{2\rho_1^3}+\frac{2(1-\mu)^2}{\rho_1^6}\right\}\right\}$$

$$\left.+\mu(1-\mu)A_1\left\{\left\{-\frac{21\xi}{4\rho_1^5}+\frac{21\xi}{4}\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\frac{3\mu}{2\rho_1^3}-\frac{3(1-\mu)}{2\rho_2^3}+\frac{3(1-\mu)}{2\rho_1^5}+\frac{3}{2\rho_1^3\rho_2}+\frac{3}{2\rho_1\rho_2^3}-\frac{3(3\mu-2)}{4\rho_1^3}-\frac{3(1-3\mu)}{4\rho_2^3}\right\}\right\}\right] \qquad (15b)$$

$$W^0_{\xi\eta} = 0 \qquad (16b)$$

$$W^0_{\dot{\xi}\dot{\xi}} = \frac{1}{c^2}\left[\frac{1}{2}\xi^2+\frac{3(1-\mu)}{\rho_1}+\frac{3\mu}{\rho_2}+A_1\left(\frac{3}{2}\xi^2+\frac{3(1-\mu)}{2\rho_1^3}\right)\right] \qquad (17b)$$

$$W^0_{\dot\eta\dot\eta} = \frac{1}{c^2}\left[\frac{3}{2}\xi^2 + \frac{3(1-\mu)}{\rho_1} + \frac{3\mu}{\rho_2} + A_1\left(\frac{9}{4}\xi^2 + \frac{3(1-\mu)}{2\rho_1^3}\right)\right] \qquad (18b)$$

$$W^0_{\dot\eta\dot\xi} = 0 \qquad (19b)$$

$$W^0_{\xi\dot\xi} = 0 \qquad (20b)$$

$$W^0_{\eta\dot\eta} = 0 \qquad (21b)$$

Now we will show that the discriminant Δ of (13) is positive at the collinear points $L_i(i=1,2,3)$.

To show Δ is positive it is sufficient to show that

$$L = -4(p_1q_2 - p_2q_1)(p_5q_6 - p_6q_5) > 0 \qquad (22)$$

L can also be written as

$$L = -4W^0_{\xi\xi}W^0_{\eta\eta}\left(1+W^0_{\dot\xi\dot\xi}\right)\left(1+W^0_{\dot\eta\dot\eta}\right) \qquad (23)$$

From (17b) and (18b) it clear that $\left(1+W^0_{\dot\xi\dot\xi}\right) > 0$ and $\left(1+W^0_{\dot\eta\dot\eta}\right) > 0$

Now we will study the signs of $W^0_{\xi\xi}$ and $W^0_{\eta\eta}$ at the collinear points $L_i(i=1,2,3)$

Firstly we will do this at L_1, since the coordinate of this point is $(1+\lambda-\mu,0)$, then $r_1 = 1+\lambda$ and $r_2 = \lambda$ where $0<\lambda<<1$, hence we can write $W^0_{\xi\xi}$ and $W^0_{\eta\eta}$ as a function in λ say $h(\lambda)$ and $g(\lambda)$, respectively. Therefore in this case from (14b), $h(\lambda) \cong h(0^+) = -\infty$ and from (15b), $g(\lambda) \cong g(0^+) = +\infty$, hence $W^0_{\xi\xi} > 0$ and $W^0_{\eta\eta} < 0$ in which $W^0_{\xi\xi}W^0_{\eta\eta} < 0$ and consequently $L > 0$. Hence the discriminant of the equation (13) is positive, and the characteristic roots can be written as $\lambda_{1,2} = \pm\sigma$, $\lambda_{3,4} = \pm i\tau$. where σ and τ are real.

Thus $\lambda_{1,2}$ are real and $\lambda_{3,4}$ are pure imaginary, hence the motion around the collinear point L_1 is unbounded and the solution is unstable.

Similarly, it can be shown that the points L_2, L_3 are also unstable.

5. Numerical results

We have used the above mentioned analysis to calculate the location of the collinear points of Jupiter and its Moons and Saturn and its Moons. The necessary data has been borrowed from Sharma and SubbaRao (1975) and Dermott and Murray (1999). The mass parameter(μ), the

oblateness(A_1) and the speed of light the systems consisting of a major planet and its satellite have been presented in table 1. It should be noted here that, the oblateness effect of the bigger primary only is considered.

Table 1: Parameters of the system

S/No	System	μ	c	A_1
1	Jupiter-Io	0.0000415622	17294.81937	0.0006885582
2	Jupiter-Europa	0.00002526534711	21817.21721	0.0002717680
3	Saturn-Mimas	0.0000000668	20967.08625	0.0038673922
4	Saturn-Thethy	0.0000011421	26424.27518	0.0015345696

Table 2: Location of first Lagrangian point(L_1)

S/No	Classical	Without oblateness $(A_1 = 0)$	With oblateness of the primary
1	1.024166773125	1.024166771	1.024153112
		0.9999584378	0.9999584378
2	1.020425563162	1.020457368	1.020452793
		0.9999747347	0.9999747347
3	1.002815442311	1.002815283	1.002806883
		0.9999999332	0.9999999332
4	1.007263912709	1.007263928	1.007254705
		0.9999988579	0.9999988579

Table 3: Location of second Lagrangian point(L_2)

S/No	Classical	Without oblateness $(A_1 = 0)$	With oblateness of the primary
1	0.976134628032	0.9761346306	0.9761485083
		0.9999584378	0.9999584378
			−0.0000352314
2	0.979799229733	0.9797680445	0.9797726819
		0.9999747347	0.9999747347
			−0.00001903184711
3	0.997189698986	0.9971892586	0.9971982757

		0.9999999332	0.9999999332
			0.0000017641
4	0.992768821174	0.9927688062	0.9927780731
		0.9999988579	0.9999988579
			−0.00000008227

Table 4: Location of Third Lagrangian point (L_3)

S/No	Classical	Without oblateness $(A_1 = 0)$ of the primary	With oblateness of the primary
1	−1.000017317589	−1.000041562	−1.000041562
		−0.9764824012	−0.9764956895
2	−1.000010478409	−1.000025265	−1.000025265
		−0.9799801469	−0.9799846277
3	−1.000000027280	−1.000000067	−1.000000067
		−0.9971898154	−0.9971988268
4	−1.000000475872	−1.000001142	−1.000001142
		−0.9927783769	−0.9927876044

6. Discussion

Equations (1)-(4) describe the motion of a third body under the influence of the oblateness of the bigger primary. Equations (8), (10), (12) give respective locations of the collinear equilibrium points L_1, L_2, L_3. In section 4, we have discovered that the oblateness coefficient and relativistic terms are unable to change the instability characteristic behavior of the collinear points. It can be seen from table 2 that the relativistic factor has a noticeable effect on the position of the collinear point L_1, of Jupiter and its Moons and Saturn and its Moons. This can be explained by the appearance of the new location on each pair as indicated in column 3 (second entries) in the absence of oblateness $(i.e.\ A_1 = 0)$. From column 4 (first entries) in table 2, it is also clear that the relativistic factor in the absence of oblateness $(i.e.\ A_1 = 0)$. gives rise to a new position of the

collinear point L_2 of each pair of the systems in consideration as in indicated in column 3 (second entries) while the combined effect of relativistic and oblateness give rise to a new location as in indicated in column 4 (third entries). Similarly from table 4, it is observed that the relativistic factor has a remarkable effect on the position of L_3 in the sense that, in the absence of oblateness $(i.e.\ A_1 = 0)$ it gives rise to a new location (Second entries) in Column 3. From column 4 of table 4 (second entries), it can be said that the oblateness has a little effect.

It is also important to notice that the shifts (i.e deviations) of a new positions (second and third entries) from the classical positions are higher than those of usual positions (first entries) from the classical positions.

6. Conclusion

By considering the bigger primary as an oblate spheroid, we have obtained the location of collinear points and have studied their linear stability.

It is noticed that their locations are prominently affcetd by the relativistic terms as it gives rise to new locations as indicated in table 2-4 in column 3 (second entries).

It is also observed that the oblateness factor has a negligible effect on the positions of collinear points L_1 and L_3 as shown in column 4 (first entries) of table 2 and column 4 (second entries) of table 4. However, the combined effect of relativistic and oblateness has a remarkable effect on the position of L_2 as it gives rise to new location (third entries) in Column 4 of table3. It is further observed that in spite of the inclusion of relativistic terms and oblateness coefficient, the instability characteristic behavior of the collinear points remains unchanged.

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RECOMMENDING INNOVATIVE PEDAGOGICAL STRATEGIES FOR TRANSFORMING LEARNING OUTCOMES THROUGH TECHNOLOGY AND ACTIVE LEARNING

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Abstract

This paper recommends a set of innovative pedagogical strategies, technology integration and active learning as effective approaches to improving student learning outcomes. These strategies foster engagement, motivation, and critical thinking while addressing systemic challenges such as inequality, limited resources, and resistance to change. It argues that combining active learning with digital tools enhances comprehension, retention, and problem-solving, while technology expands collaboration and access to resources. To support implementation, the paper emphasises the importance of continuous professional development for educators, institutional support, and curriculum alignment with modern methodologies. By adopting these pedagogical innovations, educators and policymakers can create inclusive, student-centred learning environments that equip learners with essential 21st-century skills.

Keywords: educational innovation, learning outcomes, technology integration, active learning, instructional strategies

Introduction

Improved learning outcomes refer to measurable enhancements in students' knowledge, skills, attitudes, and competencies resulting from effective educational practices. These outcomes are pivotal in modern education as they prepare students to navigate complex societal challenges and contribute meaningfully to the workforce (Wakhidah & Bandono, 2022). Clearly defined learning outcomes can increase student motivation and engagement, leading to more active participation in educational activities (Lo et al., 2022). However, achieving these improved outcomes is fraught with challenges. Educational systems in developing nations, particularly in Nigeria, frequently encounter significant challenges, including low-quality instruction and a shortage of qualified teachers (Angwaomaodoko, 2023). These issues can hinder educational

advancement and the overall development of the country. Additionally, the digital divide excludes many students from the benefits of technology-enhanced learning, exacerbating educational inequalities. Furthermore, traditional curricula that emphasise rote memorisation over critical thinking impede the development of essential problem-solving skills. Measuring learning outcomes also presents challenges, impacting the validity and reliability of assessments (Ukobizaba et al., 2021).

To effectively address these gaps, this paper argues that educational systems must adopt innovative pedagogical strategies. Specifically, it recommends integrating technology and active learning approaches as vital pathways to enhance the personalisation of learning experiences and respond to diverse student needs (Singh & Awasthi, 2024). Active learning strategies, including problem-solving, blended learning, flipped learning, inquiry-based learning, and project-based learning, have been shown to enhance student engagement and retention. Additionally, providing continuous professional development for educators enables the adoption of modern pedagogical approaches, thereby improving instructional quality (Ambon et al., 2024). Implementing outcome-based education further aligns course content and teaching methodologies with desired learning outcomes, ensuring that students acquire relevant skills and knowledge (Juma, 2024). By embracing these innovations, educational institutions can overcome existing challenges and significantly improve learning outcomes, equipping students with the competencies required for success in the 21st century.

Educational Innovations

The integration of educational innovations has transformed traditional teaching and learning approaches, offering promising pathways to improve learning outcomes significantly. To meet diverse educational needs, this paper recommends the adoption of technology-driven strategies as a central component of contemporary pedagogy.

Technology Integration

Technology has emerged as a vital driver of innovation in education, reshaping how knowledge is delivered and experienced. According to Ghory and Ghafory (2021), technological advancements have profoundly influenced all aspects of life, including education, by enhancing access to resources and improving learning experiences. We recommend that educators and institutions embrace these tools as essential for preparing students to thrive in a digitally connected world. Emerging technologies such as Augmented Reality (AR) and Virtual Reality

(VR) are increasingly being integrated into classrooms to create immersive, interactive environments that engage students in practical learning (Hajirasouli et al., 2024; Sami Ur Rehman et al., 2023). These innovations support students' acquisition of knowledge, skills, and competencies through digitally mediated experiences (Abidoye et al., 2022; Ghanbaripour et al., 2024). Velayutham et al. (2022) further note that technology has redefined human activity across various domains, and its impact on education is particularly transformative, enhancing teaching efficiency, enriching classroom engagement, and making learning more enjoyable. We recommend that institutions leverage technology not simply as a supplement but as a core component of instruction. For example, collaborative platforms such as WhatsApp, Zoom, and Google Meet enable real-time communication and peer-to-peer engagement that were previously unimaginable in traditional classrooms. Tools such as wikis, Google Docs, and cloud-based applications allow students to co-create knowledge, collaborate on assignments, and extend learning beyond the classroom. By adopting such approaches, educators can foster inclusivity, engagement, and higher-order learning outcomes.

Active Learning

Alongside technology integration, this paper recommends active learning approaches as a powerful strategy for transforming educational outcomes. Traditional teacher-centred methods often limit learner participation and emphasise rote memorisation, which constrains creativity, motivation, and problem-solving abilities. In contrast, active learning fosters student engagement, autonomy, and critical thinking, making it an indispensable approach for modern education. Active learning strategies, including problem-solving, project-based learning, flipped classrooms, collaborative inquiry, and debates, promote learner agency and deepen understanding (Cheng & Ding, 2021; Evangelou, 2023). When learners actively construct knowledge through discussion, exploration, and reflection, they are more likely to retain information and apply concepts to real-world contexts. Reeve and Cheon (2021) emphasise that autonomy-supportive teaching fosters intrinsic motivation and initiative, both of which are crucial for success in mathematics and other disciplines.

We recommend that educators adopt active learning approaches in combination with technology to strengthen classroom practice. For example, flipped classrooms enable students to review lecture content digitally before class, thereby freeing up time for collaborative problem-solving and critical engagement during lessons (Balarabe et al., 2024). Project-based and inquiry-based

learning enable learners to work on authentic tasks, developing both subject knowledge and transferable skills such as teamwork, communication, and creativity. Moreover, cooperative learning fosters socio-emotional skills, building stronger peer relationships and a positive classroom climate (Isohätälä et al., 2020). For these strategies to succeed, educators require ongoing professional development to transition from teacher-centred to learner-centred instruction. Training, mentorship, and institutional support are crucial in equipping teachers to manage active learning environments effectively (Ambon et al., 2024). Therefore, we recommend that schools and policymakers prioritise teacher training and curriculum alignment to embed active learning practices into everyday instruction.

One of the most effective ways to operationalise active learning is through blended learning models, which combine traditional face-to-face teaching with digital instruction. This hybrid approach addresses the limitations of conventional teaching, particularly the lack of interaction between educators and students, by creating a flexible and adaptive learning environment (Tong et al., 2023). We recommend that higher education institutions adopt blended learning as a core pedagogical model to enhance both engagement and accessibility. Blended learning allows educators to combine the strengths of in-person and digital approaches, tailoring instruction to diverse learning needs (Zakariya et al., 2024). By integrating classroom activities with online resources, students can take an active role in their education, accessing materials at their own pace while still benefiting from personal interaction with instructors. This dual structure not only promotes autonomy but also fosters collaboration and critical engagement among peers (Liu et al., 2024).

A particularly effective form of blended learning is the flipped classroom model, where students review lecture content online before attending class. This preparation frees up classroom time for interactive activities such as group discussions, problem-solving, and collaborative projects, which reinforce a deeper understanding of subject matter (Balarabe et al., 2024). Such approaches also empower learners to manage their study schedules more flexibly while ensuring that face-to-face sessions are dedicated to higher-order learning. Research further highlights that blended learning promotes inclusivity by accommodating different learning preferences and enabling greater flexibility (Tabassum et al., 2024). We therefore recommend that policymakers and institutions institutionalise blended learning as part of active learning reform. By adopting blended learning, schools and universities can create student-centred educational experiences

that strengthen comprehension, motivation, and retention while equipping learners with 21st-century competencies.

Effective Strategies

Effective strategies, such as clear communication, engaging curriculum design, individualised learning plans, and recognition of achievement, are essential for enhancing student motivation and academic success (Kunicina et al., 2023). We recommend that schools and universities prioritise the systematic adoption of these strategies to strengthen engagement, improve educational outcomes, and prepare learners for real-world challenges. Among these, learner-centred approaches, collaborative learning, inquiry-based and project-based learning, differentiated instruction, and technology integration stand out as effective means of fostering engagement and personalisation (Al Hussaini et al., 2024).

Learner-Centered Approaches

This paper specifically recommends learner-centred approaches as a core strategy within active learning. Learner-centred instruction shifts the classroom focus from teacher-dominated delivery to practices that prioritise student autonomy, active participation, and critical engagement. Although such approaches demand changes in teacher behaviour, they significantly enhance student learning behaviours by promoting self-direction and responsibility (Cheng & Ding, 2021). A range of learner-centred techniques, including brainstorming, concept maps, working groups, role-playing, problem-solving, debates, and De Bono's Six Thinking Hats, have proven effective in making lessons more engaging while strengthening collaboration and self-confidence (Evangelou, 2023). For these strategies to succeed, teachers must adopt new professional roles: managers, supervisors, resource persons, and counsellors who facilitate learning rather than dictate it (Meenambal & Meenakshi, 2022). We therefore recommend that teacher training programmes emphasise this professional transformation, equipping educators to foster autonomy and deeper student–teacher collaboration.

Mathematics, long perceived as an abstract and rigid discipline, particularly benefits from learner-centred methodologies. Research demonstrates that alternative approaches can make mathematics more relevant and engaging (Danlami et al., 2024; Meeran & Van Wyk, 2022). Learner-centred teaching helps students construct meaning from prior knowledge and experiences, emphasising active knowledge construction rather than passive reception (Shah Ph & Kumar, 2020). Collaborative learning, moreover, integrates cognitive and socio-emotional

engagement, building critical and metacognitive capacities (Isohätälä et al., 2020). Reeve and Cheon (2021) further argue that autonomy-supportive teaching improves learner initiative and intrinsic motivation in mathematics classrooms. We therefore recommend that mathematics teachers, in particular, be recognised as pivotal agents in advancing learner-centred strategies. By facilitating opportunities for collaboration, critical thinking, and autonomy, they can significantly enhance students' mathematical competencies and foster transferable 21st-century skills.

Overcoming Barriers

Achieving improved learning outcomes requires navigating significant challenges that hinder innovation in education. These obstacles include systemic inequality, inadequate resources, and resistance to change among educators and institutions. This paper argues that these barriers can be overcome through deliberate strategies, scalable solutions, and sustained institutional commitment.

Inequality in Education

Educational inequality in Nigeria is shaped by social, geopolitical, economic, and religious factors, demanding reforms that guarantee equal opportunities for all learners. We recommend the adoption of mobile learning initiatives as a practical strategy to bridge these gaps. Evidence shows that project-based, collaborative, and game-based approaches delivered through mobile devices enhance learning outcomes across diverse contexts (Tlili et al., 2024). For instance, Eneza Education in Kenya has successfully used SMS technology to deliver lessons to rural learners, improving literacy and numeracy for students previously excluded from quality education (Unicef, 2020). By prioritising low-tech, scalable solutions, policymakers can extend educational access to underserved populations and reduce inequality.

Lack of Resources

Resource scarcity in low-income communities severely constrains the adoption of modern teaching methodologies. Schools often struggle with outdated materials, inadequate infrastructure, and limited funding, challenges that undermine student engagement and performance (Mncube et al., 2023; Tibane et al., 2024). Research further indicates that simply increasing funding is insufficient without comprehensive systemic reforms that address governance, curriculum development, and teacher preparation (Syafaruddin et al., 2024).

We therefore recommend a holistic strategy for overcoming resource challenges, which includes:

1. Strengthening teacher training to maximise existing resources.

2. Aligning curricula with digital and active learning practices.
3. Promoting community and stakeholder involvement in resource mobilisation.

This integrated approach will ensure that additional resources translate into real improvements in learning outcomes.

Resistance to Change

Resistance to pedagogical innovation remains a widespread challenge, rooted in traditional teaching cultures and educators' reluctance to abandon familiar methods. Psychological barriers such as fear of the unknown and lack of self-confidence contribute to resistance (Kucukkaragoz & Meylani, 2025), while institutional norms often reinforce traditional practices (González et al., 2024).

To mitigate these challenges, we recommend the following strategies:

- Ongoing professional development: Training should provide hands-on experiences with new methods, including flipped classroom models, to build teacher confidence (Kucukkaragoz & Meylani, 2025).
- Mentorship and peer collaboration: Establishing support networks for teachers can sustain innovation and create a culture of continuous improvement (Li, 2024).
- Inclusive decision-making: Involving educators in the design and implementation of new initiatives fosters ownership and reduces resistance (González et al., 2024).

While some educators may retain preferences for traditional approaches, gradual adaptation supported by professional development and institutional encouragement can bridge the gap between old and new practices.

Conclusion and Recommendations

This paper has argued that transforming learning outcomes in contemporary education requires deliberate adoption of innovative pedagogical strategies, particularly technology integration and active learning. These approaches are essential for fostering engagement, motivation, and critical thinking while equipping learners with the competencies necessary for the 21st century. Evidence drawn from diverse studies demonstrates that digital tools and learner-centred methods not only enhance comprehension and retention but also promote inclusivity and collaboration. However, the effectiveness of these strategies depends on overcoming systemic barriers such as inequality, inadequate resources, and resistance to change. By prioritising scalable technological

solutions, strengthening teacher training, and embedding active learning practices into curricula, education systems can create sustainable improvements in learning outcomes.

Based on the arguments advanced in this paper, the following recommendations are made:

1. For Educators: Adopt active learning techniques such as flipped classrooms, problem-solving, and collaborative inquiry supported by technology to foster autonomy, motivation, and critical thinking.
2. For Institutions: Provide continuous professional development, mentorship, and supportive infrastructure to enable teachers to effectively implement technology-enhanced active learning.
3. For Policymakers: Develop policies that reduce digital inequality, promote low-cost and mobile-based solutions, and ensure equitable access to quality education for all learners.
4. For Researchers: Expand context-specific studies on innovative pedagogical strategies to refine best practices and guide evidence-based reforms.

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NUMERICAL COMPARISON OF HOMOTOPY PERTURBATION AND MODIFIED NEW ITERATIVE METHOD FOR SOLVING A 3-DIMENSIONAL STEADY-STATE LAPLACE EQUATION IN CARTESIAN COORDINATE

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Abstract

In this paper, we analyzes and compared the Homotopy Perturbation method (HPM) and Modified New Iterative method (MNIM) in solving three-dimensional steady-state Laplace equation in rectangular coordinates. The Laplace equation was derived from the general Heat Conduction Equation and was subjected to an initial trigonometric; algebraic; and exponential function, since there was no heat generation, the Energy Constant ($\dot{e}_{gen}$), thermal diffusivity(α) and thermal conductivity (k) ware all subjected to zero. The usage of MAPLE 18 software was employed to generate the numerical solutions; the solutions obtained were illustrated in 3D and 2D plots profiles. From the numerical point of view and results obtained, the Modified new iterative algorithm (MNIA) performed better than Homotopy perturbation algorithm (HPA), and it demonstrated a very good agreement with the exact solutions.

Keyword: Laplace Equation, Modified New Iterative Method, Homotopy Perturbation Method.

INTRODUCTION

One of the most fundamental partial differential Equations in Mathematics, Physics and Engineering is the Laplace equation, this equation arises naturally in situations where a system has reached a steady state with no internal sources or sinks. In physical terms, it describes equilibrium phenomena such as the distribution of electric potential in electrostatics, steady-state heat conduction, incompressible fluid flow, and gravitational potentials. The equation, which is a second order linear partial differential equation of elliptic type was derived from the general heat equation of the form;

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (1)$$

Where;

Where T is the temperature of the medium, α is the thermal diffusivity of the medium given by $\alpha=\frac{k}{\rho c}$, $\dot{e}_{gen} is$ the Heat (energy) generation constant, K is the thermal conductivity along the x, y, z planes and t is the time.

The Laplace equation was derived from (1) under the following assumptions;

i. The Heat distribution is in Steady-state. This implies that $\frac{\partial T}{\partial t}=0$
ii. There is no Heat generation; this implies that there is no medium through which heat is conducted, whether it involves the conversion of mechanical, electrical, nuclear, or chemical energy into heat or thermal energy.

Applying these assumptions to (1) will yield;

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0 \quad (2)$$

With initial trigonometric condition

$$T(x,y,z,0)=\cos(y)\sin(z)+x \quad (3)$$

2.0 HOMOTOPY PERTURBATION METHOD

The Homotopy perturbation approach was first used in 1998 by Dr. Ji Huan HE, a Chinese scholar. Because of its simplicity and ability to produce incredibly effective answers for complex issues in a wide range of scientific and technological fields, this method has recently gained popularity and acceptance as an elegant tool in the hands of researchers. Numerous physical issues can be described by mathematical models that use partial differential equations. Stated otherwise, a mathematical model is a condensed representation of the physical world. The input data for the specific problem—the forcing function, the partial differential equation's coefficient functions, and the boundary or initial conditions—determine how each model behaves.

We provide a concise overview of HPM to elucidate the fundamental concepts of the Homotopy Perturbation Method, considering the subsequent differential equation [7].

$$A(u)-f(\beta)=0 \qquad \beta\epsilon\Omega \quad (4)$$

With boundary conditions

$$B\left(u,\frac{\partial u}{\partial \beta}\right)=0 \qquad \beta\epsilon\partial\Omega \quad (5)$$

Where A is the general operator, B is the Boundary operator, $f(\beta)$ is a known analytic function and $\partial\Omega$ is the boundary of the domainΩ. The operator A can be generally divided into two parts of L and N where L is linear part, while N is the nonlinear part in the DE, Therefore Equation (4) can be rewritten as follows

$$L(u)+N(u)-f(\beta)=0 \quad (6)$$

Applying Homotopy technique, one can construct a Homotopy

$$G(\beta,k):\Omega\times[0,1]\mapsto\mathbb{R} \quad (7)$$

Which satisfies

$$H(g,k)=(1-k)[L(g)-L(u_o)]+k[L(g)+N(g)-f(\beta)]=0 \quad (8)$$

Or $$H(g,k)=L(g)-L(u_o)+kL(u_o)+k[N(g)-f(\beta)]=0 \quad (9)$$

Where $k\in[0,1]$, $\beta\epsilon\Omega$ *and* k is called homotopy parameter and u_o is an initial approximation for the solution of equation (3.1.1) which satisfies the boundary conditions obviously, using equation (3.1.5) or (3.1.6), we have the following equation:

$$H(g,0) = L(g) - L(u_o) = 0 \quad (10)$$

$$H(g,1) = L(g) + N(g) - f(\beta) = 0 \quad (11)$$

Assume that the solution of (3.1.5) or (3.1.6) can be expressed as a series in m as follows:

$$G = g_0 + kg_1 + k^2 g_2 + k^3 g_3 + \cdots = \sum_{i=0}^{\infty} k^i g_i \quad (12)$$

Set $k \rightarrow 1$ results in the approximate solution of (4).

Consequently,

$$u(\beta) = \lim_{k\to 1} G = g_0 + g_1 + g_2 + g_3 + \cdots = \sum_{i=0}^{\infty} g_i \quad (13)$$

The primary advantage of He's Homotopy Perturbation approach is the flexibility in constructing the perturbation equation and selecting the approximation approach. The convergence of series of (13) is widely recognized in most instances, with the rate of convergence being contingent upon L(z). The comparisons of equivalent powers of m yield solutions of differing orders. In summary, as per H.K. Liu (2011), He's HPM regards the solution $u(x)$ of the homotopy equation as a series of m;

$$g(x) = \sum_{i=0}^{\infty} k^i g_i = g_0 + kg_1 + k^2 g_2 + k^3 g_3 + \cdots \quad (14)$$

and the method considers the nonlinear $N(u)$ as

$$N(u) = \sum_{i=0}^{\infty} k^i H_i = H_0 + kH_1 + k^2 H_2 + k^3 H_3 + \cdots \quad (15)$$

where Hn are the so-called He's polynomials (A. Ghorbani and J. S. Nadjf 2007) which can be calculated by using the formula

$$Hn(g_0, g_1, \ldots, g_n) = \frac{1}{n!}\frac{\partial^n}{\partial k^n}\left(N\left(\sum_{i=0}^{\infty} k^i g_i\right)\right)_{k=0} \quad (16)$$

3.0 Modified New Iterative Method (MNIM)

The New Iterative Method was described using a partial differential equation of the form;

$$u(\bar{x}) = f(\bar{x}) + N(u(\bar{x})). \quad (17)$$

An extension of the New Iterative Method is the modified new iterative Method (MNIM), which was introduced by [11]. The Method is based on adding specific terms from the source term of the non-homogeneous Klein-Gordon equation to the integral that represents N(u) in NIM., this is done under the following conditions;

i. If the source term is function of the independent variable, x only, we include it in $N(u)$.

ii. If the source term is function of both independent variables x and t, we include it in$N(u)$.

iii. If the source term contains the terms which are functions of x, t and both x *and* t, then we include in $N(u)$ the terms involving t and both x and t.

iv. If the source term is $sin(x)sin(t)$, then NIM can be applied to obtain the exact solution.

3.0 FORMULATION OF ALGORITHMS

We have used the above submission on HPM and MNIM to formulate the following algorithms that will solve equation (2) subject to (3).

3.1 SIX-STEP HOMOTOPY PERTURBATION ALGORITHM

Step 1

```
restart : with(plots) : Digits := 20 : N := 14 :
> T[0] := cos(y)·sin(z) + x :
>
```

Step 2

```
>
  for i from 0 to 0 do
  A[0] := ( Diff(T0, [y, y]) + Diff(T0, [z, z])) :
  A[1] := -Int(A[0], x, x) :
  Ti+1 := value(A[1]) :
  end do:
```

> Step 3

```
>
  for i from 1 to N do
   B[0] := (Diff(Ti, [z, z])) :
  B[1] := -Int(B[0], x, x) :
  Ti+1 := value(B[1]) :
  end do:
```

> Step 4

```
> T := sum(Tj, j = 0 ..N + 1) :
> M := eval(T, [x = 1]) :
> plot3d(M, y = − 2 ..2, z = -2 ..2, grid = [50, 50], color = "blue")
> sol1 := eval(T, [x = 0, y = 0, z = 0]) :
> sol1 := eval(T, [x = 0.1, y = 0.1, z = 0.1]) :
> sol1 := eval(T, [x = 0.2, y = 0.2, z = 0.2]) :
> sol1 := eval(T, [x = 0.3, y = 0.3, z = 0.3]) :
> sol1 := eval(T, [x = 0.4, y = 0.4, z = 0.4]) :
> sol1 := eval(T, [x = 0.5, y = 0.5, z = 0.5]) :
> sol1 := eval(T, [x = 0.6, y = 0.6, z = 0.6]) :
> sol1 := eval(T, [x = 0.7, y = 0.7, z = 0.7]) :
> sol1 := eval(T, [x = 0.8, y = 0.8, z = 0.8]) :
> sol1 := eval(T, [x = 0.9, y = 0.9, z = 0.9]) :
```

> $sol1 := eval(T, [x = 1.0, y = 1.0, z = 1.0])$:

> Step 5

$C[0] := eval(M, z = 0)$:
$C[1] := eval(M, z = 2)$:
$C[2] := eval(M, z = 4)$:
$C[3] := eval(M, z = 6)$:
$C[4] := eval(M, z = 8)$:
$C[5] := eval(M, z = 10)$:

> Step 6

$plot([C[1], C[2], C[3], C[4], C[5]], y = 0..10, color = [red, blue, yellow, purple, black, green], axes = BOXED, title = "LAPLACE");$
$logplot([C[1], C[2], C[3], C[4], C[5]], y = 0..10, color = [red, blue, yellow, purple, black, green], axes = BOXED, title = "LAPLACE")$

Output: see tables2 to 8
where N is the computational length.

3.2 Seven steps modified new iterative algorithm (SSMNIA)

Step 1

$restart : with(plots) : Digits := 20 : N := 20 :$

> $T[0] := \cos(y) \cdot \sin(z) + x :$

> $A[0] := value(-Diff(T[0], y\$2) - Diff(T[0], z\$2)) :$

> $T[1] := value(Int(A[0], x = 0..x, x = 0..x)) :$

Step 2

for m **from** 1 **to** N **do**
$T[m+1] := value(Int(-Diff(sum(T[n], n = 0..m), y\$2) - Diff(sum(T[n], n = 0..m), z\$2), x = 0..x, x = 0..x) - Int(-Diff(sum(T[n], n = 0..m-1), y\$2) - Diff(sum(T[n], n = 0..m-1), z\$2), x = 0..x, x = 0..x)) :$
end do:

Step 3

> $T := sum(T_j, j = 0..N+1) :$

> $sol1 := eval(T, [x = 0, y = 0, z = 0]) :$

> $sol1 := eval(T, [x = 0.1, y = 0.1, z = 0.1]) :$

> $sol1 := eval(T, [x = 0.2, y = 0.2, z = 0.2]) :$

> $sol1 := eval(T, [x = 0.3, y = 0.3, z = 0.3]) :$

> $sol1 := eval(T, [x = 0.4, y = 0.4, z = 0.4]) :$

> $sol1 := eval(T, [x = 0.5, y = 0.5, z = 0.5]) :$

> $sol1 := eval(T, [x = 0.6, y = 0.6, z = 0.6]) :$

```
> sol1 := eval(T, [x = 0.7, y = 0.7, z = 0.7]) :
> sol1 := eval(T, [x = 0.8, y = 0.8, z = 0.8]) :
> sol1 := eval(T, [x = 0.9, y = 0.9, z = 0.9]) :
> sol1 := eval(T, [x = 1.0, y = 1.0, z = 1.0]) :
```

Step 4

```
> M := eval(T, [x = 1.0]) : P := eval(T, [x = 2.0]) :
```

Step 5

```
>
    plot3d(M, y = -2..2, z = -2..2, grid = [100, 100], color = "red") : plot3d(P, y = -2..2, z = -2..2,
      grid = [100, 100], color = "blue")
```

Step 6

```
> C[0] := eval(M, z = 0) :
> C[1] := eval(M, z = 2) :
> C[2] := eval(M, z = 4) :
> C[3] := eval(M, z = 6) :
> C[4] := eval(M, z = 8) :
> C[5] := eval(M, z = 10) :
```

Step 7

```
> plot([C[1], C[2], C[3], C[4], C[5]], y = 0..10, color = [red, blue, yellow, purple, black, ], axes
    = BOXED, title = "LAPLACE");
  logplot([C[1], C[2], C[3], C[4], C[5]], y = 0..100, color = [red, blue, yellow, purple, black, ],
    axes = BOXED, title = "LAPLACE")
```

Output: *see Tabe 1*
where N is the computational length.

4.0 NUMERICAL EXPERIMENTS

The table below shows the numerical solution of Laplace equation at $(k = \dot{e}_{gen} = \alpha = \frac{k}{\rho c} = 0)$

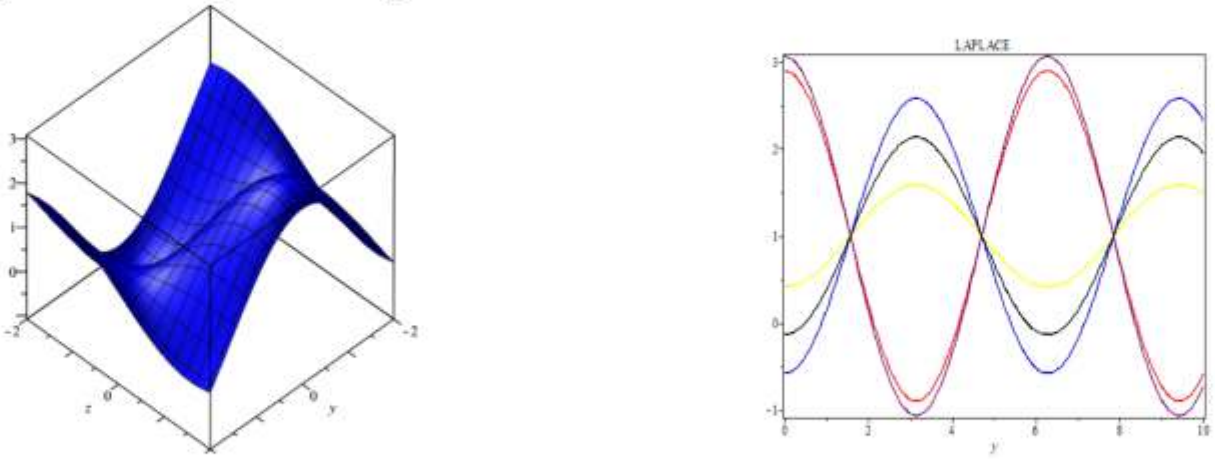
TABLE 1

(x,y,z)	Method	Solutions
(0,0,0)	Exact	0.0000000000000000000
	HPA	0.0000000000000000000
	MNIA	0.0000000000000000000
(0.1,0.1,0.1)	Exact	0.20032966873337531653
	HPA	0.20032884011636313129
	MNIA	0.20032966873337531653
(0.2,0.2,0.2)	Exact	0.40254959910397017371

	HPA	0.40252353386301855945
	MNIA	0.4025495991039701737 1
(0.3,0.3,0.3)	Exact	0.60811357583700916047
	HPA	0.60792128745529321965
	MNIA	0.60811357583700916047
(0.4,0.4,0.4)	Exact	0.81761330983021779171
	HPA	0.81683580519174787825
	MNIA	0.81761330983021779171
(0.5,0.5,0.5)	Exact	1.0303757270592102060
	HPA	1.0281290390332198273
	MNIA	1.0303757270592102060
(0.6,0.6,0.6)	Exact	1.2440973161888416415
	HPA	1.2388803754728485198
	MNIA	1.2440973161888416415
(0.7,0.7,0.7)	Exact	1.4545327462624241295
	HPA	1.4441810926946667631
	MNIA	1.4545327462624241295
(0.8,0.8,0.8)	Exact	1.6552586198314611303
	HPA	1.6370778665908029262
	MNIA	1.6552586198314611303
(0.9,0.9,0.9)	Exact	1.8375371537992745265
	HPA	1.8086839658739873962
	MNIA	1.8375371537992745265
(1.0,1.0,1.0)	Exact	1.9903083515890925341
	HPA	1.9484705372091997404
	MNIA	1.9903083515890925341

5.0 GRAPH PRESENTATION

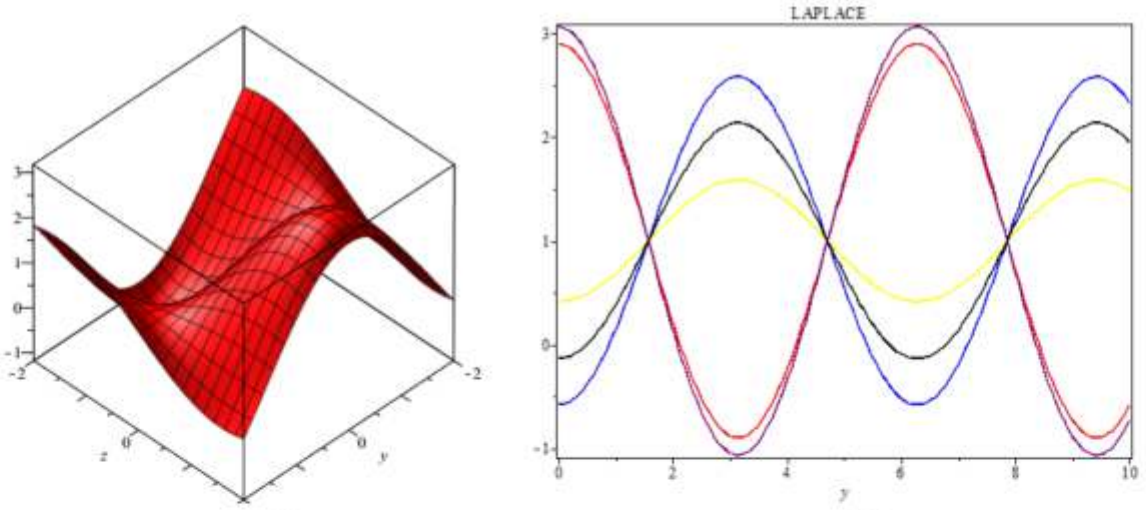
we presented the graph we plotted using HPA and MNIA to show the nature of heat Laplace equation when subjected to a trigonometric conditions



(a) (b)

FIGURE 1: (a) 3D plot representation of distribution and flow pattern in two coordinate's domain using HPA.

(b) 2D plot representation of distribution and flow pattern in two coordinate's domain using HPA.



(a) (b)

FIGURE 2: (a) 3D plot representation of distribution and flow pertain in two coordinates domain .

(b) 2D plot representation of distribution and flow pertain in two coordinate's domain.

6.0 DISCUSSION

Having carefully examined the above Numerical Experiments and the corresponding graphs, we can clearly make the following submissions;

- 1. **Influence of Co-ordinate Directions:**

the results reveal that as the coordinate variables increases in magnitude, the numerical solutions also exhibit a corresponding increase, this shows a strong dependency of the solution behavior on the spatial directions further reflecting the sensitivity of the equation to boundary condition

- 2. **Comparison of Approximation methods:** both The Homotopy pertubation (HPM) and modified new iterative methods (MNIM) demonstrated close aggreements in their solution profiles, their consistency confirms the robustness of these semi-analytical techniques In handling laplacian-type problems with each other.
- 3. **Superiority of MNIM over HPM:** while HPM provides an accurate approximation, MNIM exhibit superior performance, yielding results that perfectly aligned with the exact analytical solution, this indicates that MNIM has a stronger convergence property and greater numerical stability than the HPM.

7.0 CONCLUSION

- The present study is devoted to explore the feasibility and applicability of two proposed algorithm (Homotopy perturbation algorithm (HPA) and Modified new iterative algorithm (MNIA)) for obtaining numerical solutions for Laplace heat equation in rectangular coordinate, the primary objective is to evaluate the efficiency, accuracy and convergence properties of these methods in solving classical boundary value problems.
- A comparative analysis of the results demonstrates that while both algorithms are effective in approximating the solution, the MNIA consistently exhibits superior performance in terms of accuracy, stability and computational efficiency. In particular, the MNIA-generated solutions shows a remarkable agreement with the exact analytical solution, therbey confirming its robustness and reliability.

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APPENDICES

APPENDIX 1: HPA CODES

```
> restart : with(plots) : Digits := 20 : N := 14 :
> T[0] := cos(y)·sin(z) + x :
>

>
   for i from 0 to 0 do
   A[0] := ( Diff(T_0, [y, y]) + Diff(T_0, [z, z]) ) :
   A[1] := -Int(A[0], x, x) :
   T_{i+1} := value(A[1]) :
   end do:

>
>
   for i from 1 to N do
   B[0] := ( Diff(T_i, [z, z]) ) :
   B[1] := -Int(B[0], x, x) :
   T_{i+1} := value(B[1]) :
   end do:

>
> T := sum(T_j, j = 0 ..N + 1) :
> M := eval(T, [x = 1]) :
> plot3d(M, y = -2 ..2, z = -2 ..2, grid = [50, 50], color = "blue")
> sol1 := eval(T, [x = 0, y = 0, z = 0]) :
> sol1 := eval(T, [x = 0.1, y = 0.1, z = 0.1]) :
> sol1 := eval(T, [x = 0.2, y = 0.2, z = 0.2]) :
> sol1 := eval(T, [x = 0.3, y = 0.3, z = 0.3]) :
> sol1 := eval(T, [x = 0.4, y = 0.4, z = 0.4]) :
> sol1 := eval(T, [x = 0.5, y = 0.5, z = 0.5]) :
> sol1 := eval(T, [x = 0.6, y = 0.6, z = 0.6]) :
> sol1 := eval(T, [x = 0.7, y = 0.7, z = 0.7]) :
> sol1 := eval(T, [x = 0.8, y = 0.8, z = 0.8]) :
> sol1 := eval(T, [x = 0.9, y = 0.9, z = 0.9]) :
> sol1 := eval(T, [x = 1.0, y = 1.0, z = 1.0]) :
>
C[0] := eval(M, z = 0) :
C[1] := eval(M, z = 2) :
C[2] := eval(M, z = 4) :
C[3] := eval(M, z = 6) :
C[4] := eval(M, z = 8) :
C[5] := eval(M, z = 10) :
```

```
> plot([C[1], C[2], C[3], C[4], C[5]], y = 0..10, color = [red, blue, yellow, purple, black,
      green], axes = BOXED, title = "LAPLACE");
  logplot([C[1], C[2], C[3], C[4], C[5]], y = 0..10, color = [red, blue, yellow, purple, black,
      green], axes = BOXED, title = "LAPLACE")
```

APPENDIX II: MNIA CODE

```
> restart : with(plots) : Digits := 20 : N := 20 :
> T[0] := cos(y)·sin(z) + x :
> A[0] := value(-Diff(T[0], y$2) - Diff(T[0], z$2)) :
> T[1] := value(Int(A[0], x = 0..x, x = 0..x)) :
>
  for m from 1 to N do
   T[m + 1] := value(Int(-Diff(sum(T[n], n = 0..m), y$2) - Diff(sum(T[n], n = 0..m), z$2), x
       = 0..x, x = 0..x) - Int(-Diff(sum(T[n], n = 0..m - 1), y$2) - Diff(sum(T[n], n = 0..m
       - 1), z$2), x = 0..x, x = 0..x)) :
   end do:
```

```
> T := sum(T_j, j = 0..N + 1) :
> sol1 := eval(T, [x = 0, y = 0, z = 0]) :
> sol1 := eval(T, [x = 0.1, y = 0.1, z = 0.1]) :
> sol1 := eval(T, [x = 0.2, y = 0.2, z = 0.2]) :
> sol1 := eval(T, [x = 0.3, y = 0.3, z = 0.3]) :
> sol1 := eval(T, [x = 0.4, y = 0.4, z = 0.4]) :
> sol1 := eval(T, [x = 0.5, y = 0.5, z = 0.5]) :
> sol1 := eval(T, [x = 0.6, y = 0.6, z = 0.6]) :
> sol1 := eval(T, [x = 0.7, y = 0.7, z = 0.7]) :
> sol1 := eval(T, [x = 0.8, y = 0.8, z = 0.8]) :
> sol1 := eval(T, [x = 0.9, y = 0.9, z = 0.9]) :
> sol1 := eval(T, [x = 1.0, y = 1.0, z = 1.0]) :
> M := eval(T, [x = 1.0]) : P := eval(T, [x = 2.0]) :
> plot3d(M, y = -2..2, z = -2..2, grid = [100, 100], color = "red") : plot3d(P, y = -2..2, z = -2..2,
      grid = [100, 100], color = "blue")
> C[0] := eval(M, z = 0) :
> C[1] := eval(M, z = 2) :
> C[2] := eval(M, z = 4) :
> C[3] := eval(M, z = 6) :
> C[4] := eval(M, z = 8) :
> C[5] := eval(M, z = 10) :
>
  plot([C[1], C[2], C[3], C[4], C[5]], y = 0..10, color = [red, blue, yellow, purple, black, ], axes
      = BOXED, title = "LAPLACE");
  logplot([C[1], C[2], C[3], C[4], C[5]], y = 0..100, color = [red, blue, yellow, purple, black, ],
      axes = BOXED, title = "LAPLACE")
```

FORMULATION OF A PREDICTOR CORRECTOR LINEAR MULTISTEP METHOD IN SOLVING FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS (ODE)

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Abstract

In this work, a 4 - step predictor - corrector Linear Multistep methods of numerical schemes that provides very accurate approximations to the solution of first-order initial value problem is presented. The methods were derived through Taylor series expansion of the general solution of linear multistep methods, using explicit and implicit methods to obtain the predictor and corrector respectively. The scheme proved to be accurate, zero-stabled, convergent and consistent.

Keywords: Linear multistep, predictor-corrector, explicit and implicit methods, zero stabled, consistency.

1.0 Introduction

Many fields of human endeavour, notably Engineering, Sciences, Astronomy, Economics and so on are basically focused on proffering solution to critical problems that guarantee development for the betterment of mankind. Some of these problems yields to first order ordinary differential equations of the form:

$$\frac{dy}{dx} = f(x, y) \qquad y(0) = y_0 \qquad (1)$$

Analytical solution to these differential equations might be cumbersome and, in some cases, very difficult or impossible, hence the need to device a numerical solution or an accurate approximation of the solution. Here, we shall be looking at Linear Multistep method of Solution to equation (1). According to Okunuga et al [4], existing methods of derivation or formulation o of Linear Multistep Method in discrete form includes interpolation approach, numerical integration, Taylor series expansion and through determination of the order of Linear Multistep Method. According to [1], [5] and [6], the linear multistep method has the form;

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \qquad \alpha_k = 1, \ n \geq 0 \qquad (2)$$

For the approximate solution of the initial value problem of Equation (1), we approximate the solution $y(x)$ on a $a \leq x \leq b$ at $x_i = a + ih$, $0 \leq \frac{b-a}{h}$, $y_i = y(x_i)$ and $f_{n+j} = f(x_{n+j}, y_{n+j})$.

The Linear Multistep method is called a K−step method if at least one of the coefficients α_0 and β_0 is non zero.

If $\beta_k = 0$, the Linear multistep method is explicit, once we know the preceding values of the results, that is $y_n, y_{n+1}, y_{n+2}, \ldots, y_{n+k-1}$, we then promptly proceed to calculate the succeeding values given as; $y_{n+k} = \sum_{j=0}^{k}\left(h\beta_j f_{n+j} - \alpha_j y_{n+j}\right)$. Otherwise if $\beta_k \neq 0$, the Linear Multistep method is called Implicit.

Because of the importance and potency of Linear Multistep Method in solving the first order Ordinary Differential Equations described in Equation (1), attention of researchers around the world had been sustained on the subject through contributions that the method had recorded in the past and recent times while some attentions are now being focused on Predictor – Corrector method. According to [5], Predictor- Corrector Formulae: In numerical analysis, the predictor–corrector belongs to a class of algorithms that are prepared to integrate ordinary differential equations, that is, to find a solution to ordinary differential equations, done basically in two steps. In the first step, we have a scheme called "Predictor" that takes values at a preceding set of points to anticipate the function values at a subsequent new point, before the second step which present another scheme called the " Corrector", that will refines the initial approximation by using the predicted value of the function to interpolate the values at the same subsequent point. When considering the numerical solution of ordinary differential equations, a Predictor – Corrector method typically uses an Explicit method for the Predictor step and an Implicit method for the Corrector step

In the works of Dhaigude et al [7], they developed a new sixth-order predictor-corrector method by using Newton's forward interpolation formula to obtain the solution of first-order initial value problems. Another notable work on the subject worth mentioning is the work of Shior et al [8] where they presented various description of numerical methods such as Adams method, Predictor – Corrector Method, Milne's method and Runge – Kutta methods, they used Adam – moulton Predictor – Corrector to solve ordinary differential equation with Initial Value

Problems. We shall like to reference the contributions of Kedar et al [9] for their work in which they presented fourth-order Adams Predictor-Corrector numerical schemes for solving Initial Value Problems, using finite difference approximations, leading to the development of a numerical scheme that provides algebraic systems of difference equation.

Drawing on insights from previous works, we are motivated to contribute to the knowledge on this subject matter. In this work, we shall present numerical schemes derived through Taylor series expansion, leading to Explicit and Implicit numerical methods called the Predictor and corrector methods, respectively. The schemes are not self–starting, so we use the Runge–Kutta fourth-order method as a starter method.

2.0 Derivation of the Method

We seek a 4 −step Linear Multistep method of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \quad \alpha_k = 1, \qquad n \geq 0$$

$$= \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} + \cdots + \alpha_k y_{n+k}$$

$$= h[\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4 f_{n+4} + \cdots + \beta_k f_{n+k}] \qquad (3)$$

From Taylor series;

$$y(x_n + h) = y(x_n) + hy'(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + \cdots + + \frac{h^r}{r!} y^{(r)}(x_n) + \cdots$$

$$x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$

$$x_3 = x_0 + 3h$$

That is $h = x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots$

Then we have;

$$y(x_n + h) = y_{n+1}$$

$$y_{n+1} = y_n + hy_n{}' + \frac{h^2}{2!} y_n{}'' + \frac{h^3}{3!} y_n{}''' + \frac{h^4}{4!} y_n{}^{iv} + \cdots$$

$$y_{n+2} = y_n + 2hy_n{}' + \frac{4h^2}{2!} y_n{}'' + \frac{8h^3}{3!} y_n{}''' + \frac{16h^4}{4!} y_n{}^{iv} + \cdots$$

$$y_{n+3} = y_n + 3hy_n{}' + \frac{9h^2}{2!} y_n{}'' + \frac{27h^3}{3!} y_n{}''' + \frac{81h^4}{4!} y_n{}^{iv} + \cdots$$

$$y_{n+4} = y_n + 4hy_n{}' + \frac{16h^2}{2!} y_n{}'' + \frac{64h^3}{3!} y_n{}''' + \frac{256h^4}{4!} y_n{}^{iv} + \cdots$$

From equation (1);

$$\frac{dy}{dx} = f(x_n, y_n);\ y(x_0) = y_0$$

$$f_n = y'(x_n)$$

$$f_{n+1} = y'(x_n + h) = y_n{}' + hy_n'' + \frac{h^2}{2!}y_n{}''' + \frac{h^3}{3!}y_n{}^{iv} + \cdots$$

$$f_{n+2} = y'(x_n + 2h) = y_n{}' + 2hy_n'' + \frac{4h^2}{2!}y_n{}''' + \frac{8h^3}{3!}y_n{}^{iv} + \cdots$$

$$f_{n+3} = y'(x_n + h) = y_n{}' + 3hy_n'' + \frac{9h^2}{2!}y_n{}''' + \frac{27h^3}{3!}y_n{}^{iv} + \cdots$$

$$f_{n+4} = y'(x_n + h) = y_n{}' + 4hy_n'' + \frac{16h^2}{2!}y_n{}''' + \frac{64h^3}{3!}y_n{}^{iv} + \cdots$$

Now equation (3) becomes;

$$\begin{aligned}
&\alpha_0 y_0 + \alpha_1\left(y_n + hy_n{}' + \frac{h^2}{2!}y_n{}'' + \frac{h^3}{3!}y_n{}''' + \frac{h^4}{4!}y_n{}^{iv} + \cdots\right)\\
&\quad + \alpha_2\left(y_n + 2hy_n{}' + \frac{4h^2}{2!}y_n{}'' + \frac{8h^3}{3!}y_n{}''' + \frac{16h^4}{4!}y_n{}^{iv} + \cdots\right)\\
&\quad + \alpha_3\left(y_n + 3hy_n{}' + \frac{9h^2}{2!}y_n{}'' + \frac{27h^3}{3!}y_n{}''' + \frac{81h^4}{4!}y_n{}^{iv} + \cdots\right)\\
&\quad + \alpha_4\left(y_n + 4hy_n{}' + \frac{16h^2}{2!}y_n{}'' + \frac{64h^3}{3!}y_n{}''' + \frac{256h^4}{4!}y_n{}^{iv} + \cdots\right)\\
&\quad - h\left\{\beta_0 f_n{}' + \beta_1\left(y_n{}' + hy_n'' + \frac{h^2}{2!}y_n{}''' + \frac{h^3}{3!}y_n{}^{iv} + \cdots\right)\right.\\
&\quad + \beta_2\left(y_n{}' + 2hy_n'' + \frac{4h^2}{2!}y_n{}''' + \frac{8h^3}{3!}y_n{}^{iv} + \cdots\right)\\
&\quad + \beta_3\left(y_n{}' + 3hy_n'' + \frac{9h^2}{2!}y_n{}''' + \frac{27h^3}{3!}y_n{}^{iv} + \cdots\right)\\
&\quad \left. + \beta_4\left(y_n{}' + 4hy_n'' + \frac{16h^2}{2!}y_n{}''' + \frac{64h^3}{3!}y_n{}^{iv} + \cdots\right) + \cdots\right\} = 0
\end{aligned}$$

$= (\alpha_0 + \alpha_1 + \alpha_3 + \alpha_4 + \cdots)y_n + h\{(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + \cdots) - (\beta_0 + \beta_1 + \beta_3 + \beta_4 + \cdots)\}y_n' +$

$\left\{\frac{h^2}{2!}(\alpha_1 + 4\alpha_2 + 9\alpha_3 + 16\alpha_4 + \cdots) - h^2(\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + \cdots)\right\}y_n'' + \left\{\frac{h^3}{3!}(\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4 + \cdots) - \frac{h^2}{2!}(\beta_1 + 4\beta_2 + 9\beta_3 + 16\beta_4 + \cdots)\right\}y_n''' + \left\{\frac{h^4}{4!}(\alpha_1 + 16\alpha_2 + 81\alpha_3 + 256\alpha_4 + \cdots) - \frac{h^4}{3!}(\beta_1 + 8\beta_2 + 27\beta_3 + 64\beta_4 + \cdots)\right\} + \cdots = 0$

This can further be expressed as;

$$
\begin{aligned}
&(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \cdots)y_n \\
&\quad + h\{(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + \cdots) - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \cdots)\}y_n' \\
&\quad + \left\{\frac{h^2}{2!}(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 + 4^2\alpha_4 + \cdots) - h^2(\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + \cdots)\right\}y_n'' \\
&\quad + \left\{\frac{h^3}{3!}(\alpha_1 + 2^3\alpha_2 + 3^3\alpha_3 + 4^3\alpha_4 + \cdots)\right. \\
&\quad \left. - \frac{h^3}{2!}(\beta_1 + 2^2\beta_2 + 3^2\beta_3 + 4^2\beta_4 + \cdots)\right\}y_n''' \\
&\quad + \left\{\frac{h^4}{4!}(\alpha_1 + 2^4\alpha_2 + 3^4\alpha_3 + 4^4\alpha_4 + \cdots)\right. \\
&\quad \left. - \frac{h^4}{3!}(\beta_1 + 2^3\beta_2 + 3^23 + 4^3\beta_4 + \cdots)\right\}y_n{}^{iv} + \cdots = 0
\end{aligned}
$$

In general, we have,

$$c_0 y_n + c_1 y_n' + c_2 y_n'' + c_3 y_n''' + + c_4 y^{iv} + \cdots + c_q y^{(q)} = 0$$

Where;

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \cdots \alpha_k$$

$$c_1 = (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + \cdots + k\alpha_k) - (\beta_0 + \beta_1 + \beta_3 + \beta_4 + \cdots + \beta_k)$$

$$c_2 = \left\{\frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 + 4^2\alpha_4 + \cdots + k^2\alpha_k) - (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + \cdots + \beta_k)\right\}$$

$$c_3 = \left\{\frac{1}{3!}(\alpha_1 + 2^3\alpha_2 + 3^3\alpha_3 + 4^3\alpha_4 + \cdots + k^3\alpha_k) - \frac{1}{2!}(\beta_1 + 2^2\beta_2 + 3^2\beta_3 + 4^2\beta_4 + \cdots + k^2\beta_k)\right\}$$

$$c_4 = \left\{\frac{1}{4!}(\alpha_1 + 2^4\alpha_2 + 3^4\alpha_3 + 4^4\alpha_4 + \cdots k^4\alpha_4) - \frac{1}{3!}(\beta_1 + 2^3\beta_2 + 3^2\beta_3 + 4^3\beta_4 + \cdots k^3\beta_k)\right\}$$

$$c_5 = \left\{\frac{1}{5!}(\alpha_1 + 2^5\alpha_2 + 3^5\alpha_3 + 4^5\alpha_4 + \cdots k^5\alpha_4) - \frac{1}{4!}(\beta_1 + 2^4\beta_2 + 3^4\beta_3 + 4^4\beta_4 + \cdots k^4\beta_k)\right\}$$

$$c_q = \left\{\frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 + 4^q\alpha_4 + \cdots k^q\alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + 3^{q-1}\beta_3 + 4^{q-1}\beta_4 + \cdots k^{q-1}\beta_k)\right\}$$

Since $y_n, y_n', y_n'', y_n''', y_n{}^{iv} \ldots, y_n{}^{(q)}$ cannot be zero then;

$$c_0 = c_1 = c_2 = c_3 = c_4 = \cdots = c_q = 0$$

4 −step method implies that $k = 4$.

Implicit Method. $(\alpha_4 = 1, \beta_4 \neq 0)$

We have to solve the following equations simultaneously;

$$c_0: \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = -1 \quad (4)$$

$$c_1: \alpha_1 + 2\alpha_2 + 3\alpha_3 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4 = -4 \quad (5)$$

$$c_2: \ \alpha_1 + 4\alpha_2 + 9\alpha_3 - 2\beta_1 - 4\beta_2 - 6\beta_3 - 8\beta_4 = -16 \quad (6)$$

$$c_3: \ \alpha_1 + 8\alpha_2 + 27\alpha_3 - 3\beta_1 - 12\beta_2 - 27\beta_3 - 48\beta_4 = -64 \quad (7)$$

$$c_4: \ \alpha_1 + 16 + 81\alpha_3 - 4\beta_1 - 32\beta_2 - 108\beta_3 - 256\beta_4 = -256 \quad (8)$$

$$c_5: \ \alpha_1 + 32\alpha_2 + 243\alpha_3 - 5\beta_1 - 80\beta_2 - 405\beta_3 - 1280\beta_4 = -1024 \quad (9)$$

Now taking $\alpha_0 = c, \alpha_1 = -b - c, \ \alpha_2 = a + b, \ \alpha_3 = -1 - a$ and $\alpha_4 = 1$ (10)

we have;

$$\beta_0 = \frac{1}{720}(-19 - 11a - 19b - 251c),$$

$$\beta_1 = \frac{1}{360}(53 + 37a + 173b - 323c),$$

$$\beta_2 = \frac{1}{30}(-11 - 19a + 19b + 11c),$$

$$\beta_3 = \frac{1}{360}(323 - 173a - 37b - 53c),$$

$$\beta_4 = \frac{1}{720}(251 + 19a + 11b + 19c) \quad (11)$$

For step $k = 4$, Equation (3) becomes;

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + y_{n+4} = h[\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4 f_{n+4}] \quad (12)$$

Putting (10) and (11) in (12), we have;

$$cy_n + (-b-c)y_{n1} + (a+b)y_{n+2} + (-1-a)y_{n+3} + y_{n+4}$$
$$= \frac{h}{720}[(-19-11a-19b-251c)f_n + (106 = 74a + 346b - 646c)f_{n+1}$$
$$+ (-264 - 456a + 456b + 264c)f_{n+2} + (646 - 346a - 74b - 106c)f_{n+3}$$
$$+ (251 + 19a + 11b + 19c)] \quad (13)$$

Explicit Method:

Let us re-write Equation (12) as;

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + y_{n+4}$$
$$= h[\lambda_0 f_n + \lambda_1 f_{n+1} + \lambda_2 f_{n+2} + \lambda_3 f_{n+3}] \quad (14)$$

Because $\lambda_4 = 0$;

The Resulting equations are;

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1 - a + b - c \quad (15)$$
$$2\lambda_1 + 4\lambda_2 + 6\lambda_3 = 7 - 5a + 3b - c \quad (16)$$
$$3\lambda_1 + 12\lambda_2 + 27\lambda_3 = 37 - 19a + 7b - c \quad (17)$$
$$4\lambda_1 + 32\lambda_2 + 108\lambda_3 = 175 - 65a + 15b - c \quad (18)$$

Solving the equations, we obtain;

$$\lambda_0 = \frac{1}{24}(-9 - a - b - 6c),$$
$$\lambda_1 = \frac{1}{24}(37 + 5a + 13b - 19c),$$
$$\lambda_2 = \frac{1}{24}(-59 - 19a + 13b + 5c),$$
$$\lambda_3 = \frac{1}{24}(55 = 9a - b - c) \quad (19)$$

By putting (19) into (14), we obtain;

$$cy_n + (-b-c)y_{n+1} + (a+b)y_{n+2} + (-1-a)y_{n+3} + y_{n+4}$$
$$= \frac{h}{24}[(-9 - a - b - 6c)f_n + (37 + 5a + 13b - 19c)f_{n+1}$$
$$+ (-59 - 19a + 13b + 5c)f_{n+3}] \quad (20)$$

Taking $a = 0, b = 1, c = 1$, into (20) and (13), yields;

$$\bar{y}_{n+4} = -y_n + 2y_{n+1} - y_{n+2} + y_{n+3}$$
$$+ \frac{h}{24}(-16f_n + 31f_{n+1} - 41f_{n+2} + 53f_{n+3}) \quad (21)$$

$$y_{n+4} = -y_n + 2y_{n+1} - y_{n+2} + y_{n+3}`$$
$$+ \frac{h}{720}(-289f_n - 194f_{n+1} + 256f_{n+2} + 466f_{n+3} + 281\bar{f}_{n+4}) \quad (22)$$

(21) and (22) above are the Explicit and Implicit schemes respectively, which are the Predictor−Corrector Schemes.

3.0 Analysis of the Basic Properties of the Method.

3.1 Order of Accuracy and Error Constant.

According to [1 - 3] , a k −step, linear multistep method Equation (2) has order p if $c_0 = c_1 = c_2 = \ldots, = c_{p-1} = c_p$ and $c_{p+1} \neq 0$. Thus c_{p+1} is the error constant of the method.

Taking $p = 5$, implies that the methods are of order 5, then $c_{p+1} = c_6 = \frac{1}{720}(\alpha_1 + 64\alpha_2 + 729\alpha_3 + 4096) - \frac{1}{120}(\beta_1 + 32\beta_2 + 243\beta_3 + 1024\beta_4)$.

$c_6 = -\frac{1}{1440}(27 + 11a + 11b + 27c)$, with $a = 0, b = 1$ *and* $c = 1, c_6 = -0.04518888889$, which represents a high order of accuracy and relatively small error constant.

3.2 Consistency

The conditions for consistency according to [2] and [3] are as follows;

(i) The order $p \geq 1$

$(ii).\ \sum_{j=0}^{k} \alpha_j = 0,$

$(iii).\ \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j,$

$(iv).\ \rho(1) = 0, \rho'(1) = \sigma(1),$

Where $\rho(\xi) = \sum_{r=0}^{k} \alpha_r \xi^r$, $\sigma(\gamma) = \sum_{r=0}^{k} \beta_r \gamma^r$, which are the first and second characteristic polynomials of equation (2). Applying these conditions to our main scheme, we have;

$(i).\ p = 5 \geq 1.$

$(ii).\sum_{j=0}^{4} \alpha_j = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = c - b - c + a + b - 1 - a + 1 = 0.$

$(iii).\ \sum_{j=}^{4} j\alpha_j = -b - c + 2a + 2b - 3 - 3a + 4 = 1 - a + b - c = 1,$ for $a = 0, b = 1$ and $c = 1$

$\sum_{j=0}^{4}\beta_j = \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4$ for $a = 0, b = 1$ and $c = 1$, $\sum_{j=0}^{4}\beta_j = \frac{1}{720}(-289) + \frac{1}{360}(-97) + \frac{1}{30}(19) + \frac{1}{360}(233) + \frac{1}{720}(281) = 1$. So, $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k}\beta_j$

(iv). $\rho(\xi) = \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 + \alpha_4\xi^4 = 1 - 2\xi + \xi^2 - \xi^3 + \xi^4,\ \rho(1) = 0.$

$\rho'(\xi) = -2 + 2\xi - 3\xi^2 + 4\xi^3,\ \sigma(\gamma) = \beta_0 + \beta_1\gamma + \beta_2\gamma^2 + \beta_3\gamma^3 + \beta_4\gamma^4.$

$\sigma(\gamma) = \frac{-289}{720} - \frac{97}{360}\gamma + \frac{19}{30}\gamma^2 + \frac{233}{360}\gamma^3 + \frac{281}{720}\gamma^4.$

$$\rho'(1) = 1, \sigma(1) = -\frac{289}{720} - \frac{97}{360} + + \frac{19}{30} + \frac{233}{360} + \frac{281}{720} = 1.$$

Since the scheme satisfies the conditions $(i) - (iv)$. The scheme is consistent.

3.3 Zero Stability of the Method.

According to [1 - 3], a linear Multistep method is called Zero Stable if it satisfies the dahlquist root conditions, that is the zeros of the first characteristic polynomial of the method provided that all zeros ξ satisfy $|\xi| \leq 1$ and Multiple zeros ξ satisfy $|\xi| < 1$.

The first characteristic polynomial of the method is given as $\rho(\xi) = 1 - 2\xi + \xi^2 - \xi^3 + \xi^4$. Solving $\rho(\xi) = 0$, proves that the method is zero-stable.

3.4 Implementation of the Method

During the implementation of this method, we combined the two schemes, tagged the predictor−corrector method. We used the initial value as provided by the problems to compute the next three initial values, $y_n, n = 1, 2, 3$. This was implemented by Runge-Kutta, fourth order method.

3.5 Numerical Results.

The Numerical schemes (21) and (22) are programmed with the aid of C++ standard compiler and implemented on test problems to ascertain the good performance of the schemes in terms of consistency and accuracy.

Test problem 1

We consider the initial value problem;

$\frac{dy}{dx} = 2x, \qquad y(0) = 1$ with $h = 0.1$.

Its exact solution is given as $y(x) = x^2 + 1$.

The Results are presented in Table 1 below.

x	Numerical Solution	Exact Solution	Error
0.0	1.00000000	1.00000000	0.00000000
0.1	1.01000000	1.01000000	0.00000000
0.2	1.04000000	1.04000000	0.00000000
0.3	1.09000000	1.09000000	0.00000000
0.4	1.16000000	1.16000000	0.00000000
0.5	1.25000000	1.25000000	0.00000000
0.6	1.36000000	1.36000000	0.00000000
0.7	1.49000000	1.49000000	0.00000000
0.8	1.64000000	1.64000000	0.00000000
0.9	1.81000000	1.81000000	0.00000000

Table 1: Numerical and exact values of the test problem 1

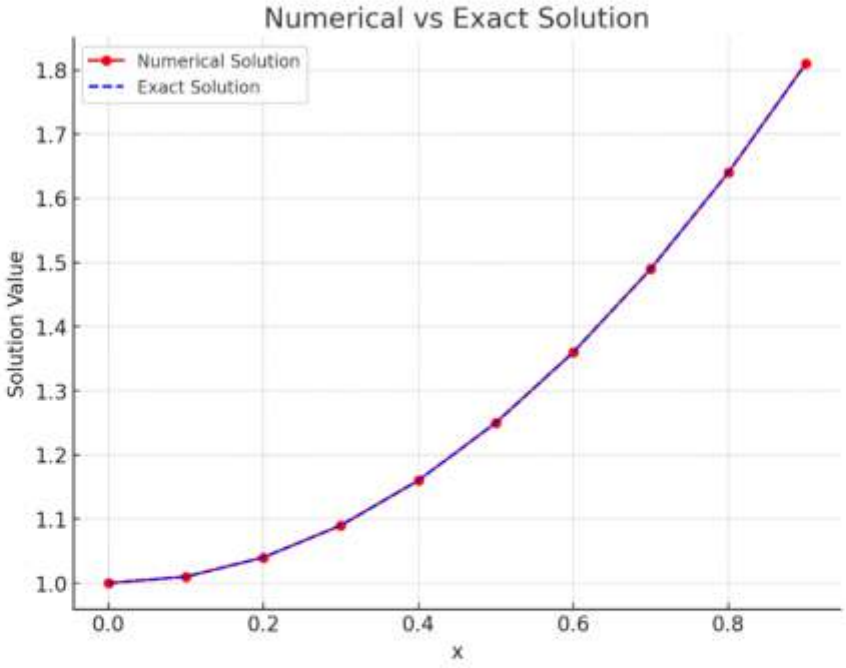


Figure 1: Numerical vs exact values for test problem 1

Test problem 2

We looked at the initial value problem given as;

$\frac{dy}{dx} = 3x^2 - 6x + 5, \quad y(0) = 1 \quad h = 0.1$ whose exact solution is: $y(x) = x^3 - 3x^2 + 5x + 1$. The results are presented in Table 2 below.

x	Numerical Solution	Exact Solution	Error
0.0	1.000000000	1.000000000	0.000000000
0.1	1.471000000	1.471000000	0.000000000
0.2	1.88800000	1.88800000	0.000000000
0.3	2.25700000	2.25700000	0.000000000
0.4	2.58400000	2.58400000	0.000000000
0.5	2.87500000	2.87500000	0.000000000
0.6	3.13600000	3.13600000	0.000000000
0.7	3.37300000	3.37300000	0.000000000
0.8	3.59200000	3.59200000	0.000000000
0.9	3.79900000	3.79900000	0.000000000

Table 2: Numerical and exact values of the test problem 2

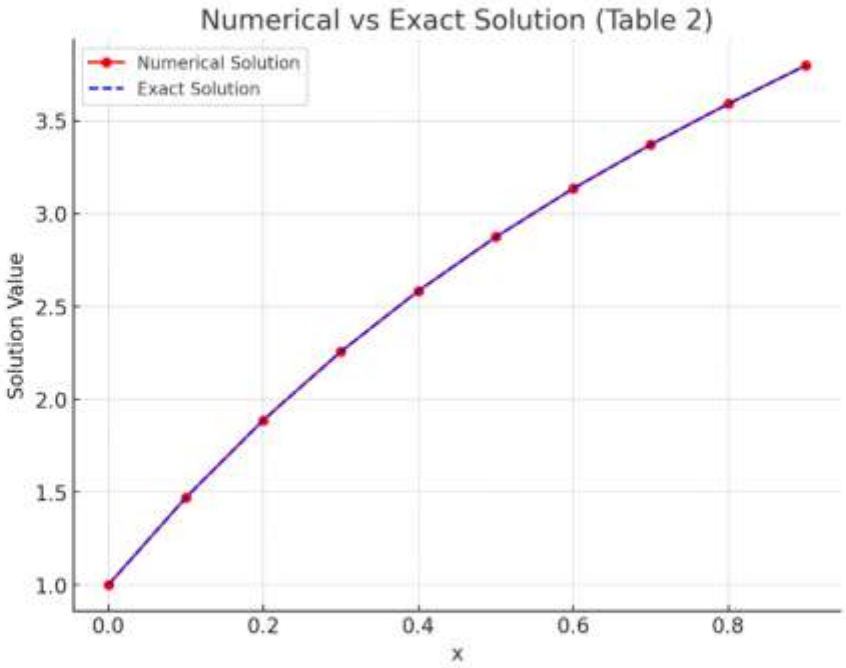


Figure 2: Numerical vs exact values for test problem 2

Test problem 3

We consider the initial value problem:

$\frac{dy}{dx} = x^2 - y, \;\; y(0) = 1 \qquad h = 0.01$, whose exact solution is given as: $y(x) = x^2 - 2x + 2 - e^{-x}$. The results are presented in Table 3 below.

x	Numerical Solution	Exact Solution	Error
0.0	1.000000	1.0000000000	0.0000000000
0.1	0.990050	0.9900501663	1.663×10^{-7}
0.2	0.980201	0.9802013269	3.267×10^{-7}
0.3	0.970454	0.9704544665	4.665×10^{-7}
0.4	0.960815	0.9608105608	-4.4395×10^{-7}
0.5	0.951280	0.9512705775	-9.4225×10^{-6}
0.6	0.941845	0.941835664	-9.5336×10^{-6}
0.7	0.932520	0.9325061801	-1.38199×10^{-5}

0.8	0.923307	0.9232836536	-2.33464× 10^{-5}
0.9	0.914192	0.9141688147	-2.31853× 10^{-5}

Table 3: Numerical and exact values of the test problem 3

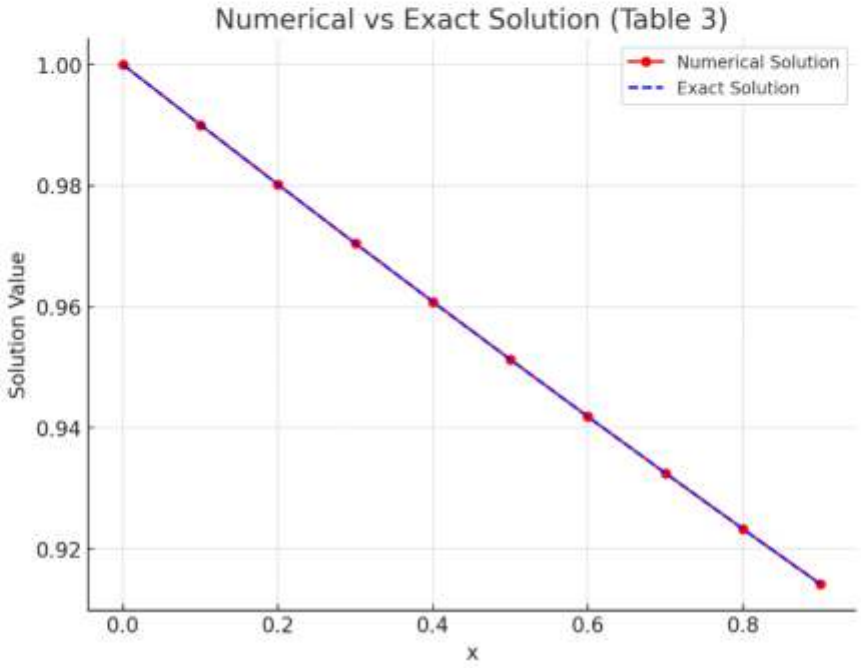


Figure 3: Numerical vs exact values for test problem 3

4.0 Results and Discussion

The performance of the proposed predictor–corrector linear multistep method was assessed using three representative test problems, with the corresponding outcomes summarized in Tables 1–3 and illustrated graphically in Figures 1–3.

For Test Problem 1 (Table 1 and Figure 1), the numerical solution was found to perfectly coincide with the exact analytical solution across the entire computational domain. The error remained identically zero, confirming that the proposed scheme is capable of reproducing the exact trajectory for problems of this form. This result is particularly significant as it demonstrates both the consistency and robustness of the derived method in handling relatively simple initial value problems with no accumulation of round-off or truncation error.

In Test Problem 2 (Table 2 and Figure 2), a similar trend was observed. The numerical values matched the exact solution to machine precision, again yielding zero error across all computed

steps. Such behaviour not only reinforces the accuracy of the scheme but also validates its zero-stability property, as established in the theoretical analysis. The perfect agreement between the computed and exact values for the first two test problems highlights the strength of the Taylor-series-based formulation of the predictor–corrector framework.

For Test Problem 3 (Table 3 and Figure 3), small but detectable discrepancies between the numerical and exact solutions emerged as the integration progressed. The maximum error recorded was on the order of 10^{-5}, which is negligible in practical computations. The graphical comparison shows that both curves remain virtually indistinguishable, indicating that the deviation is extremely minor and primarily attributable to cumulative truncation errors inherent in numerical approximations. Importantly, the observed bounded errors provide empirical confirmation of the convergence and stability properties established theoretically for the scheme.

Taken together, these results underscore the effectiveness of the proposed method. The exact reproduction of solutions in Problems 1 and 2, coupled with the negligible deviations in Problem 3, establishes the method as both highly accurate and reliable for solving first-order ordinary differential equations. The numerical experiments therefore demonstrate that the scheme not only competes favourably with existing predictor–corrector formulations but also offers a promising alternative for real-world applications where accuracy and stability are paramount

5.0 Conclusion

In this work, we proposed a four-step predictor–corrector linear multistep method derived via Taylor series expansion for solving first-order ordinary differential equations. The scheme was shown analytically to be consistent, convergent, and zero-stable. Numerical experiments confirmed these properties: for two benchmark problems, the method reproduced exact solutions with zero error, while the third test case showed only negligible deviations on the order of 10^{-5}. These results highlight the accuracy and reliability of the approach and suggest its potential as a practical tool for scientific and engineering computations.

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NONLINEAR CONTRACTION MAPPINGS IN RELATION-THEORETICAL (ϕ, φ)-CONTRACTIONS WITH APPLICATION TO ELASTIC BEAMS ON NONLINEAR ELASTIC FOUNDATIONS

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Abstract

The paper aims to extend the classic Banach contraction principle to a complete metric space endowed with a binary relation. We accomplish this by generalizing several key notions from metric space equipped with an amorphous relation to prove results on the existence and uniqueness of fixed points for (ϕ, φ)-contraction mapping to the relation-theoretic setting. We also apply our results to determine a unique solution of a boundary value problem associated with nonlinear elastic beam equations. We further provide illustrative example that demonstrate the main results.

Keywords Fixed point, (ϕ, φ)-contraction mapping, boundary value problems; fixed points; binary relations

Mathematics Subject Classification 47H10, 54H25

1 Introduction

The Banach contraction principle [2] is widely used in the existence and uniqueness problems. Because of its applicability, many authors extended Banach's principle in various directions [5, 6, 9, 10, 11].

In 2001, Turinici [12] introduced the concept of order-theoretic fixed point results, which provided a novel approach to proving fixed point theorems. Later, Alam and Imdad [3] introduced a new variant of Banach contraction principle on a complete metric space endowed with a binary relation which reduces to Banach contraction principle under universal relation. This variant is a weaker version of Banach's theorem in the sense of contractive condition used by them, and generalizes the fixed point results of Ran and Reurings [17] and Nieto and Rodríguez-López [18]. After the introduction of Alam–Imdad's relation-theoretic contraction principle, the field of metric fixed point theory has attracted much attention. A number of fixed point theorems in the context of relational metric space employing various types of contractions has been appeared over the years. Recently, Algehyne et. al [7] proved some results on existence and uniqueness of fixed points for an almost ϕ-contraction mapping defined on a metric space endowed with an amorphous relation.

Considering the insights by Alam and Imdad [3] and Algehyne et. al [7], we introduce a new class of mappings which is called relation theoretic (ϕ, φ)-contraction mapping and prove results on the existence and uniqueness of fixed points for (ϕ, φ)-contraction mapping to the relation-theoretic setting. As an application of this result, we prove existence and uniqueness

solution of a boundary value problem associated with nonlinear elastic beam equations. Results are supported by suitable example.

2 Preliminaries

We start this section by presenting some basic relevant definitions, proposition and theorems.

Definition 2.1 [14] A mapping $\phi: [0,\infty) \to [0,\infty)$ is said to be comparison function if it satisfies the following

(i) ϕ is monotonic increasing,

(ii) $\lim_{n\to\infty} \phi^n(t) = 0$, for each $t > 0$.

Definition 2.2 [16] A mapping $\phi: [0,\infty) \to [0,\infty)$ is said to be (c)-comparison function if it satisfies the following

(i) ϕ is monotonic increasing,

(ii) $\sum_{n=1}^{+\infty} \phi^n(t) < \infty$ for each $t > 0$.

Clearly, every (c)-comparison function is a comparison function.

Remark 2.3 Let ϕ be a (c)-comparison function. Then

(i) $\phi(0) = 0$,

(ii) $\phi(t) < t$, for each $t > 0$,

(iii) ϕ is right continuous at 0.

Definition 2.4 Let Φ be the set of all function $\varphi: [0,\infty)^5 \to [0,\infty)$ satisfying the following

(i) φ is continuous,

(ii) $\varphi(t_1,t_2,t_3,t_4,t_5) = 0$ if and only if $t_1t_2t_3t_4t_5 = 0$.

Example 2.5 The following function $\varphi: [0,\infty)^5 \to [0,\infty)$ belong to Φ:

(i) $\varphi(t_1,t_2,t_3,t_4,t_5) = t_1t_2t_3t_4t_5$,

(ii) $\varphi(t_1,t_2,t_3,t_4,t_5) = e^{t_1t_2t_3t_4t_5} - 1$,

(iii) $\varphi(t_1,t_2,t_3,t_4,t_5) = ln(1 + t_1t_2t_3t_4t_5)$.

Definition 2.6 [13] A function T from a metric space (X,d) into itself is said to be ϕ-contraction if there exist a function $\phi: [0,\infty) \to [0,\infty)$ satisfying

$$d(Tx,Ty) \leq \phi\big(d(x,y)\big), \forall x,y \in X. \tag{2.1}$$

Definition 2.7 A function T from a metric space (X,d) into itself is said to be (ϕ,φ)-contraction if there exist a function $\phi: [0,\infty) \to [0,\infty)$ and $\varphi: [0,\infty)^5 \to [0,\infty)$ satisfying

$$d(Tx,Ty) \leq \phi\big(d(x,y)\big) - \varphi\big(d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\big), \forall x,y \in X. \tag{2.2}$$

Proposition 2.8 [4] A function T from a metric space (X,d) into itself equipped with a relation R. If ϕ is a (c)-comparison function, then the following are equivalent:

(i) $d(Tx,Ty) \leq \phi\big(d(x,y)\big), \forall x,y \in X$, with $(x,y) \in R$,

(ii)$d(Tx,Ty) \leq \phi\big(d(x,y)\big), \forall x,y \in X$, with $[x,y] \in R$.

The result below follows from proposition 2.8.

Proposition 2.9 A function T from a metric space (X, d) into itself equipped with a relation R. If ϕ is a (c)-comparison function and a function $\varphi: [0,\infty)^5 \to [0,\infty)$, then the following are equivalent:

(i) $d(Tx,Ty) \leq \phi\big(d(x,y)\big) - \varphi\big(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big), \forall x, y \in X$, with $(x,y) \in R$,

(ii)$d(Tx,Ty) \leq \phi\big(d(x,y)\big) - \varphi\big(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big), \forall x, y \in X$, with $[x,y] \in R$.

3 Main Results

We begin this section with the existence of fixed point under (ϕ, φ)-contraction.

Theorem 3.1 Let X be a nonempty set equipped with a binary relation R and a metric d such that (X, d) is a metric space. Let $T: X \to X$ be a self-mapping. Suppose that the following conditions hold:

(a) (X, d) is R-complete metric space,

(b) $X(T, R)$ is nonempty,

(c) R is T-closed,

(d) either T is R-continuous or R is d-self-closed,

(e) there exists a (c)-comparison function and a function $\varphi: [0,\infty)^5 \to [0,\infty)$ such that

$$d(Tx,Ty) \leq \phi\big(d(x,y)\big) - \varphi\big(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big), \forall x, y \in X, \text{ with } (x,y) \in R \tag{3.1}$$

Then, T has a fixed point.

Proof Using the hypothesis (b), there exist $x_0 \in X(T, R)$ and construct sequence $\{x_n\} \subset X$ satisfying

$$x_n = T^n(x_0) = T(x_{n-1}), \forall n \in \mathbb{N} \tag{3.2}$$

Since $(x_0, Tx_0) \in R$, by T-closedness of R and proposition 2.2, we have

$$(T^n x_0, T^{n+1} x_0) \in R \tag{3.3}$$

Using (3.2) and (3.3), we have

$$(x_n, x_{n+1}) \in R, \forall n \in \mathbb{N} \tag{3.4}$$

Hence, $\{x_n\}$ is R-preserving sequence.

Set $d_n := d(x_n, x_{n+1})$. Using the (3.1), (3.2) and (3.4), we have

$$d_n = d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq \phi\big(d(x_{n-1}, x_n)\big) - \varphi\big(d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\big)$$
$$= \phi\big(d(x_{n-1}, x_n)\big) - \varphi\big(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\big)$$

By definition 2.4 (ii), we have

$$d_n \leq \phi(d_{n-1}). \tag{3.5}$$

By induction, (3.5) reduces to

$$d_n \leq \phi(d_{n-1}) \leq \phi^2(d_{n-2}) \leq \cdots \leq \phi^n(d_0)$$

which implies

$$d_n \leq \phi^n(d_0), \forall n \in \mathbb{N} \tag{3.6}$$

Using (3.6), $\forall m, n \in \mathbb{N}$ and $m < n$, we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq \phi^m(d_0) \leq \phi^{m+1}(d_0) \leq \cdots \leq \phi^{n-1}(d_0) \\ &= \sum_{i=m}^{n-1} \phi^i(d_0) \leq \sum_{i \geq m} \phi^i(d_0) \to 0 \end{aligned}$$

as $m, n \to \infty$.

Thus, $\{x_n\}$ is a Cauchy sequence. Therefore, $\{x_n\}$ is R-preserving Cauchy sequence and hence the R-completeness of X, there exists $x \in X$ such that $x_n \xrightarrow{d} x$.

We now use hypothesis (d) to show that $x \in F(T)$. If T is R-continuous, then we have

$$x_{n+1} = T(x_n) \xrightarrow{d} T(x)$$

Using the uniqueness of the limit, we have $T(x) = x$. Or if R is d-self-closed, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in R, \forall k \in \mathbb{N}$. Now, using Proposition 2.9, $[x_{n_k}, x] \in R$ and (3.1), we have

$$\begin{aligned} d(x_{n_k+1}, Tx) &= d(Tx_{n_k}, Tx) \\ &\leq \phi\left(d(x_{n_k}, x)\right) - \varphi\left(d(x_{n_k}, x), d(x_{n_k}, Tx_{n_k}), d(x, Tx), d(x_{n_k}, Tx), d(x, Tx_{n_k})\right) \\ &= \phi\left(d(x_{n_k}, x)\right) - \varphi\left(d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, Tx), d(x_{n_k}, Tx), d(x, x_{n_k+1})\right) \quad (3.7) \end{aligned}$$

Using (i), (ii) of Remark 2.3 and $x_{n_k} \xrightarrow{d} x$ in (3.7), we get

$$\begin{aligned} d(x_{n_k+1}, Tx) \leq \phi\left(d(x_{n_k}, x)\right) \\ - \varphi\left(d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, Tx), d(x_{n_k}, Tx), d(x, x_{n_k+1})\right) \to 0 \end{aligned}$$

as $k \to \infty$, which implies that $x_{n_k+1} \xrightarrow{d} Tx$. Again, using the uniqueness of limit, we have $Tx = x$. Thus, T has a fixed point.

In the following Theorem we show the existence and uniqueness of a fixed point.

Theorem 3.2 In addition to the hypotheses of Theorem 3.1, suppose that the following condition holds:

(f) $R|_{T(X)}$-complete.

Then T has a unique fixed point.

Proof Following the arguments of the proof Theorem 3.1, $F(T) \neq \emptyset$. Suppose $x, y \in F(T)$, then, there exist $x, y \in X$ such that

$$T(x) = x \quad \text{and } T(y) = y \qquad (3.8)$$

On contrary, assume that $x \neq y$. Clearly $x, y \in T(X)$. Then, by the hypothesis (f), we get

$$[x, y] \in R. \qquad (3.9)$$

Using (3.8), (3.9) and hypothesis (e), we have

$$\begin{aligned} &d(x, y) = d(Tx, Ty) \\ &\leq \phi\big(d(x, y)\big) - \varphi\big(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) \end{aligned}$$

By definition 2.5, we have

$$d(x, y) \leq \phi\big(d(x, y)\big)$$

Consider item (ii) of Remark 2.3, we get

$$d(x,y) \leq \phi\big(d(x,y)\big) < d(x,y)$$

a contradiction. Hence, $x = y$.

Remark 3.3 The generalization and the improvement of Theorem 3.1 and Theorem 3.2 can be found in the results of literatures as following:

i. Setting $\phi(a) = ka, 0 \leq k < 1$ and $\varphi(a) = 0$, then our results reduce to relational-theoretical contraction principle by Alam and Imdad [3].

ii. Putting $\varphi(a) = 0$ in our results, then the results of Algehyne et. al. [7].

iii. On setting $\phi(a) = ka, 0 \leq k < 1$ and $\varphi(a) = Ld(x,Ty)$ where $L \geq 0$ is a constant, then the results of Khan [8].

We now present example that demonstrate the practical implications of our main theorems.

Example 3.4 Consider $X = [0,1]$ endowed with usual metric $(X,d) = |x-y|$ and define a binary relation $R = \geq$. Then (X,d) is an R-complete metric space. Consider the mappings $T: X \to X$ defined by

$$T(x) = \begin{cases} 0, & if\ x = 1 \\ 1/3, & otherwise. \end{cases}$$

Then, T is R-continuous and R is T-closed. Also define a (c)-comparison function $\phi(a) = \frac{a}{3}$ and a function $\varphi(a) = \frac{1}{3}$ and for all $x, y \in X$ satisfying $(x,y) \in R$ such that

$$d(Tx,Ty) \leq \phi\big(d(x,y)\big) - \varphi\big(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big)$$

So that contraction condition (e) of Theorem 3.1 is satisfied. Thus, all the hypothesis of Theorem 3.1 are satisfied. Hence T has a fixed point. Moreover, all the hypothesis, all the hypothesis of Theorem 3.2 also holds. Then, T has a unique fixed point $x = \frac{1}{3}$.

4 Application

In this section, we consider positive solutions for a fourth-order equation with nonlinear boundary conditions of the type

$$\begin{aligned} \vartheta^{iv}(s) &= h\big(s, \vartheta(s)\big), \qquad 0 \leq s \leq 1, \\ \vartheta(0) &= \vartheta'(0) = 0, \\ \vartheta''(1) &= 0, \qquad \vartheta'''(1) = 0 \end{aligned} \tag{4.1}$$

where $h: [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function. This kind of problem arises in the study of deflections of elastic beams on nonlinear elastic foundations which is usually characterized with bending equilibria of an elastic beam model of length 1, whose both ends are rigidly fixed usually referred to as cantilever beam in material mechanics. For more details on this, we refer to [1,15].

Theorem 4.1 Consider the following boundary value problem BVP in (3.19). Suppose there exists a (c)-comparison function and a function $\varphi: [0,\infty)^5 \to [0,\infty)$ satisfying

$$0 \leq h(s,x) - h(s,y) \leq \phi(x-y) - \varphi(x-y), \tag{4.2}$$

for all $s \in [0,1], x, y \in \mathbb{R}$ with $x \geq y$. Assume there exist $\lambda \in [0,1]$ satisfying

$$\lambda(s) \geq \int_0^1 \Omega(s,t) h\big(t, \lambda(t)\big) dt, \ \forall s \in [0,1], \tag{4.3}$$

where Green function $\Omega(s,t)$ is defined as

$$\Omega(s,t) = \frac{1}{6}\begin{cases} t^2(3s-t), 0 \le t \le s \le 1, \\ s^2(3t-t), 0 \le s \le t \le 1. \end{cases} \quad (4.4)$$

Then, the boundary value problem BVP in (4.1) has a unique solution.

Proof The boundary value problem BVP in (4.1) can be transformed to integral equation as

$$\rho(s) \ge \int_0^1 \Omega(s,t)h\big(t,\rho(t)\big)dt, \ \forall s \in [0,1], \quad (4.5)$$

Clearly, we get

$$0 \le \Omega(s,x) \le \frac{1}{2}s^2 t, \forall s,t \in [0,1]. \quad (4.6)$$

Consider metric d on $X \coloneqq C[0,1]$ defined as

$$d(u,v) = \max_{s\in[0,1]}|u(s)-v(s)|, \ \forall u,v \in X, \quad (4.7)$$

Also, consider a relation R on $X \coloneqq C[0,1]$ defined as

$$(u,v) \in R \Leftrightarrow u(s) \ge v(s), \forall u,v \in X, \forall s \in [0,1].$$

To prove that (X,d) is R-complete and R is d-self-closed.

Defined a function $T: X \to X$ as

$$T(u)(s) = \int_0^1 \Omega(s,t)h\big(t,\rho(t)\big)dt, \ \forall s \in [0,1], \forall u \in X,$$

With (4.2), we can show that T is R-closed and $R|_{T(X)}$ is complete.

With (4.2), we find $\lambda(s) \ge T(\lambda)(s)$ so that $\lambda \in X(T,R)$. Using (4.2), $\forall s \in [0,1]$ and $\forall u,v \in X$ satisfying $(u,v) \in R$ as

$$|T(u)(s) - T(v)(s)| = \int_0^1 \Omega(s,t)\Big(h\big(t,u(t)\big) - h\big(t,v(t)\big)\Big)dt$$

$$\le \int_0^1 \Omega(s,t)\Big(\phi\big(u(t)-v(t)\big) - \varphi\big(u(t)-v(t)\big)\Big)dt$$

$$\le \left(\int_0^1 \Omega(s,t)dt\right)\Big(\phi\big(u(t)-v(t)\big) - \varphi\big(u(t)-v(t)\big)\Big)$$

$$\le \frac{\phi\big(d(u,v)\big)}{4} - \frac{\varphi\big(d(u,v)\big)}{4}$$

Using (4.6), we have

$$\le \phi\big(d(u,v)\big) - \varphi\big(d(u,v)\big)$$

$$\le \phi\big(d(x,y)\big) - \varphi\big(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big)$$

So that

$$d(Tx,Ty)$$

$$\le \phi\big(d(x,y)\big) - \varphi\big(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big)$$

Hence, all the hypothesis of Theorem 3.2 are satisfied. Consequently, there exists a unique $\vartheta^* \in C([0,1])$ satisfying $T\vartheta^* = \vartheta^*$, which is the unique solution of (4.5). Thus, the solution of (4.1).

Conclusion

The main findings of this article is to introduce a new class of mappings which is called relation theoretic (ϕ, φ)-contraction mapping and prove results on the existence and uniqueness of fixed points for (ϕ, φ)-contraction mapping to the relation-theoretic setting. As an application of this result, we prove existence and uniqueness solution of a boundary value problem associated with nonlinear elastic beam equations. Results are supported by suitable example. This study provides significant advancements in the understanding of relation-theoretic setting in metric spaces, with potential applications in fractional differential equations and nonlinear elliptic problems.

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Author's contribution
The authors have contributed equally to the writing and development of this paper.

Compliance with ethical standards

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OPTIMIZING PRODUCTION EFFICIENCY: A LINEAR PROGRAMMING APPROACH AT NASCO FOODS NIGERIA LIMITED

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Abstract

This study applies linear programming (LP) to optimize production planning at NASCO Foods Nigeria Limited, addressing the challenge of inefficient resource allocation in Nigeria's manufacturing sector. The model maximizes weekly profit (₦872,045.46) under constraints of raw material availability (wheat flour, maize grits, sugar, etc.) and processing time capacities (mixing, wrapping, baking, etc.). Using data from the company's production line, we formulate a BILP problem with 11 products, incorporating selection variables to enforce minimum production thresholds (500–10,000 cartons) and product diversity (≥5 distinct items).

Results reveal that time constraints, particularly in the wrapping process (100% utilization), are the primary bottleneck, while raw materials remain significantly underutilized (e.g., 12.5% wheat flour usage). The optimal production mix prioritizes high-margin products like Fancy Biscuit and Vanilla Cornflakes, excluding less profitable items. These findings underscore the need for capacity expansion in time-bound processes to unlock higher profitability. The study demonstrates LP's efficacy in bridging operational gaps and informs strategic decision-making for Nigerian food manufacturers.

Keywords: Linear Programming, Production Optimization, Resource Allocation, NASCO Foods.

1. Introduction

The primary objective of any company is to generate revenue and ensure long-term viability. To achieve this, businesses must make strategic decisions about operational processes, maximizing efficiency and productivity to sustain profitability. However, many Nigerian manufacturing firms still rely on trial-and-error methods for production scheduling, often leading to profit losses and, in extreme cases, business failure (Anthony & Koko, 2022).

Linear programming (LP) is a mathematical optimization technique that deals with linear objective functions subject to linear constraints (Hillier, 2018). It plays a crucial role in managerial decision-making, particularly in allocating scarce resources, a challenge emphasized by classical economists. Efficient resource allocation is vital for production planning, as poor decisions can lead to operational inefficiencies, financial instability and even organizational collapse. Many firms operate below installed capacity, highlighting the need for effective production planning to maximize resource utilization (Solaja et al., 2019).

Production planning involves strategically allocating raw materials, labor, and workspace to meet demand efficiently. In make-to-order systems, production begins only after customer orders are placed, while make-to-stock models rely on pre-planned work orders. The production

department oversees shop-floor operations, ensuring schedules align with delivery timelines. A well-structured production plan optimizes available resources, minimizing waste while meeting targets.

This study applies linear programming to determine an optimal production schedule that maximizes profit for NASCO Foods Nigerian Limited, Jos, Plateau State, Nigeria. A mathematical model is formulated, incorporating production materials and time constraints as a linear programming problem.

The remainder of this paper is organized as follows: In section 2, we discussed the Historical Context of Linear Programming and Empirical Evidence from Previous Studies, materials and method in section 3, problem formulation in section 4, result and discussion in section 5 and conclusion in section 6.

2. Historical Context of Linear Programming and Empirical Evidence from Previous Studies

LP was first developed by Soviet mathematician Leonid Kantorovich in the late 1930s. Later, in 1947, George B. Dantzig introduced the simplex method, revolutionizing LP applications in logistics, including aircraft development and personnel optimization (Dantzig, 1947). Today, LP is a cornerstone of operations research, widely used in production scheduling, resource allocation, and transportation planning. Several studies demonstrate LP's effectiveness in production optimization.

Linear programming (LP) has proven particularly effective in Nigeria's food and beverage sector. Balogun et al. (2012) demonstrated this at Coca-Cola bottling company, Ilorin, where LP identified two high-margin soft drinks for production prioritization. Similar success was observed by Adamu (2013) at Coca-Cola Kaduna through parametric LP, which dynamically adjusted to profit fluctuations. The approach was further validated by Erinle et al. (2020) at Rite Foods Limited in Ogun State using the simplex method, while Akanbi (2018) extended its utility at Nigerian Bottling Company by combining LP with Integer Linear Programming for enhanced decision-making.

Beyond consumables, LP optimizes production in other manufacturing domains. Anthony and Koko (2022) applied it to nylon production at Albarka Plastics in Kaduna, with four of six product lines driving maximum profitability under constraints. Andrew et al. (2014) achieved similar results in textile manufacturing at Niger Mills, Calabar. Notably, Oluwafemi et al. (2013) documented dual successes - optimizing soap production at Sokat Soap Industry (where 1kg Salem soap yielded highest returns) and animal feed formulation at Feed Masters Limited, Ilorin, where strategic focus on two key products maximized profits despite broader market demands. Building on these demonstrated successes, this study applies LP to NASCO Foods' production planning. The following section details the materials and methodological framework adopted to optimize their production schedule.

3. Materials and Method

The data used for this study were obtained through a structured questionnaire administered to the production manager of NASCO Foods Nigeria. The questionnaire gathered detailed information on the company's product types, raw material requirements per unit of production, available weekly quantities of materials, processing time requirements, and production time capacities across various stages. The company produces eleven distinct products, each requiring a unique combination of materials and processing times. The key raw materials include wheat flour, raw maize grits, sugar, salt, and an additive labeled ABC. Time requirements were also captured for six production stages: mixing, forming, baking, wrapping, cooking, and flaking. All available weekly capacities were based on current operational constraints provided by the company.

To determine the optimal weekly production plan that maximizes total profit without exceeding resource or time limits, a linear programming model was developed. The model includes constraints for material availability, process capacity, and minimum production levels for each product. The formulation was implemented and solved using the PuLP optimization library in Python, which provided a practical and efficient approach to modeling and solving the problem. This method enabled the identification of an optimal production strategy under realistic and data-driven operational constraints.

4. Problem Formulation

The company manufactures eleven distinct products, each with unique packaging configurations and production profiles. These products include NASCO Shortcake (120g×36), NASCO Rodeo Biscuit (75g×60), NASCO Fancy Biscuit (50g×48), NASCO Savoury Crackers (100g×48), NASCO Cream Crackers (100g×24), NASCO Cornflakes (350g×10), NASCO Cornflakes (35g×120), NASCO Banana Cornflakes (350g×10), NASCO Strawberry Cornflakes (350g×10), NASCO Vanilla Cornflakes (350g×10) and NASCO Frostyz Cornflakes (350g×10). Each of these products consumes specific quantities of raw materials per carton. For the biscuit products: NASCO Shortcake requires 5□kg of wheat flour, 0.035□kg of sugar and 0.01□kg of salt; NASCO Rodeo Biscuit uses 4□kg of wheat flour, 0.035□kg of sugar and 0.01□kg of salt; NASCO Fancy Biscuit needs 6□kg of wheat flour, 0.035□kg of sugar, 0.01□kg of salt and 0.05□kg of ABC; NASCO Savoury Crackers consume 4□kg of wheat flour, 0.035□kg of sugar, and 0.01□kg of salt; while NASCO Cream Crackers require 5□kg of wheat flour, 0.035□kg of sugar and 0.01□kg of salt. For the cornflakes products: NASCO Cornflakes (350g×10) consumes 0.3□kg of maize grits, 0.035□kg of sugar and 0.01□kg of salt; NASCO Cornflakes (35g×120) uses 0.1□kg of maize grits, 0.035□kg of sugar and 0.01□kg of salt; NASCO Banana, Strawberry, Vanilla, and Frostyz Cornflakes each require 0.2□kg of maize grits, 0.035□kg of sugar, and 0.01□kg of salt per carton. The weekly resource limits are 40,000□kg of wheat flour, 30,000□kg of maize grits, 20,000□kg of sugar, 10,000□kg of salt and 6,000□kg of ABC.

Each product also consumes production time across different processes. All products undergo mixing and wrapping. Specifically, each carton of every product requires 0.35 minutes of mixing and 0.55 minutes of wrapping. Biscuit products also undergo forming and baking: each carton of NASCO Shortcake, Rodeo Biscuit, and Fancy Biscuit requires 0.2 minutes of forming and 0.5 minutes of baking. Meanwhile, flavored cornflakes, Banana, Strawberry, Vanilla and Frostyz

require cooking and flaking: each of these products uses 0.35 minutes of cooking and 0.55 minutes of flaking per carton. The total time available weekly for each of these six processes: mixing, forming, baking, wrapping, cooking and flaking is capped at 1,500 minutes. These time constraints necessitate careful scheduling and selection of product combinations to avoid exceeding the processing capacity.

Operational constraints further guide production decisions. A product can only be produced in quantities between 500 and 10,000 cartons and this only applies if the product is selected for production in that week. This is enforced through selection variables for each product. Additionally, to ensure product diversity, at least three different products must be produced each week. There are also category-specific requirements: a minimum of 1,000 cartons must be produced from the biscuit group, which includes NASCO Shortcake, Rodeo Biscuit, Fancy Biscuit, Savoury Crackers, and Cream Crackers; likewise, at least 1,000 cartons must be produced from the cornflakes group, comprising all six cornflakes variants. These conditions ensure a balanced product lineup and help the company meet its market obligations across different consumer preferences.

The ultimate goal of the model is to maximize total profit from the weekly production. Each product contributes a different profit margin per carton. Specifically, the profits are ₦200 for NASCO Shortcake, ₦250 for NASCO Rodeo Biscuit, ₦300 for NASCO Fancy Biscuit, ₦180 for NASCO Savoury Crackers, ₦220 for NASCO Cream Crackers, ₦350 for NASCO Cornflakes (350g×10), ₦100 for NASCO Cornflakes (35g×120), ₦330 for Banana Cornflakes, ₦340 for Strawberry Cornflakes, ₦345 for Vanilla Cornflakes and ₦320 for Frostyz Cornflakes. These values, combined with the material and time constraints as well as business rules, form a linear programming model. This model determines the optimal mix and quantity of products to manufacture in a given week to achieve the highest possible profit. To accomplish this, we define decision variables representing the weekly production quantities of each product, state the objective function and the constraints to which the objective function should be maximize as follows:

Decision variables:

Let x_1 be the number of cartons of NASCO Shortcake 120g×36 to be produced , x_2 be the number of cartons of NASCO Rodeo Biscuit 75g×60 to be produced , x_3 be the number of cartons of NASCO Fancy biscuit 50g×48 to be produced ,x_4 be the number of cartons of NASCO Savoury Crackers 100g×48 to be produced, x_5 be the number of cartons of NASCO Cream Crackers 100g×24 to be produced, x_6 be the number of cartons of NASCO Cornflakes 350g×10 to be produced, x_7 be the number of cartons of NASCO Cornflakes 350g×120 to be produced, x_8 be the number of cartons of NASCO Banana Cornflakes 350g×10 to be produced, x_9 be the number of cartons of NASCO Strawberry Cornflakes 350g×10 to be produced, x_{10} be the number of cartons of NASCO Vanilla Cornflakes 350g×10 to be produced and x_{11} be the number of cartons of NASCO Frostyz Cornflakes 350g×10 to be produced.

Decision variable:

$y_i = 1$ if product x_i is produced, zero otherwise.

Objective function: Since the goal of the production manager is to know the production mix that will maximize the profit of the company, the objective function is therefore:

Maximize: $Z = 200x_1 + 250x_2 + 300x_3 + 180x_4 + 220x_5 + 350x_6 + 100x_7 + 330x_8 + +340x_9 + 345x_{10} + 320x_{11},$

Subject to:

$$5x_1 + 4x_2 + 6x_3 + 4x_4 + 5x_5 \leq 40000,$$
$$0.3x_6 + 0.1x_7 + 0.2x_8 + 0.2x_9 + 0.2x_{10} + 0.2x_{11} \leq 30000,$$
$$0.035x_1 + 0.035x_2 + 0.035x_3 + 0.035x_4 + 0.035x_5 + 0.035x_6 + + 0.035x_7 + 0.035x_8 + 0.035x_9 + 0.035x_{10} + 0.035x_{11} \leq 20000,$$
$$0.01x_1 + 0.01x_2 + 0.01x_3 + 0.01x_4 + 0.01x_5 + 0.015x_6 + 0.01x_7 + +0.01x_8 + 0.01x_9 + 0.01x_{10} + 0.01x_{11} \leq 10000,$$
$$0.005x_{11} \leq 6000,$$
$$0.35x_1 + 0.35x_2 + 0.35x_3 + 0.35x_4 + 0.35x_5 + 0.35x_6 + 0.35x_7 + +0.35x_8 + 0.35x_9 + 0.35x_{10} + 0.35x_{11} \leq 1500,$$
$$0.2x_1 + 0.2x_2 + 0.2x_3 \leq 1500,$$
$$0.5x_1 + 0.5x_2 + 0.5x_3 \leq 1500,$$
$$0.55x_1 + 0.55x_2 + 0.55x_3 + 0.55x_4 + 0.55x_5 + 0.55x_6 + 0.55x_7 + +0.55x_8 + 0.55x_9 + 0.55x_{10} + 0.55x_{11} \leq 1500,$$
$$0.35x_8 + 0.35x_9 + 0.35x_{10} + 0.35x_{11} \leq 1500,$$
$$0.55x_8 + 0.55x_9 + 0.55x_{10} + 0.55x_{11} \leq 1500,$$
$$x_i \geq 500y_i, \text{ for all } i = 1,2,3,..,11,$$
$$x_i \leq 10000y_i, \text{ for all } i = 1,2,3,..,11.$$
$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} \geq 5,$$
$$x_1 + x_2 + x_3 + x_4 + x_5 \geq 1000,$$
$$x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} \geq 1000.$$

Having established the mathematical formulation of the problem, the following section presents the computational results obtained from the model and provides a detailed analysis of the production mix, resource utilization and the implications of the findings for operational decision-making.

5. Result and Discussion

The production optimization model was successfully solved using the PuLP library in Python. The solver returned an optimal solution, indicating that the maximum profit achievable under the given constraints is ₦872,045.46 per week. This value represents the highest possible profit while ensuring that all material and operational limits are respected.

The optimal production plan recommends focusing on a selective set of products. Specifically, Rodeo Biscuit (500 cartons), Fancy Biscuit (500 cartons), Cornflakes 350g (727 cartons), Strawberry Cornflakes (500 cartons) and Vanilla Cornflakes (500 cartons) are included in the production mix. No production is recommended for the other six products, Shortcakes, Savoury Crackers, Cream Crackers and Banana or Frostyz Cornflakes. This implies that these excluded products either consume resources less efficiently or contribute relatively less to the overall profit when resource constraints are binding.

An analysis of resource utilization reveals that most raw materials are significantly underutilized. For example, only 12.5% of the wheat flour and 1.4% of the raw maize grits were used. Similarly, the sugar and salt usage stood at only 0.5% and 0.3% of their respective limits. This suggests that material constraints did not serve as the primary limiting factors in the optimization process.

In contrast, operational (time-related) constraints were more binding. The wrapping process reached full capacity (100%), making it the most critical bottleneck in the production system. Mixing also utilized 63.6% of its available time, while forming and baking stood at 13.3% and 33.3%, respectively. Cooking and flaking, processes more specific to cornflake production, consumed 23.3% and 36.7% of their capacities.

The implication of these findings is that future strategies for improving production efficiency and profitability should focus on alleviating time-related bottlenecks, particularly in the wrapping stage. Additionally, the underutilization of raw materials presents an opportunity to revise process flows or consider alternative product mixes that can better leverage available resources without exceeding time limits. Overall, the model provides actionable insights into production planning and highlights key areas for operational improvement.

6. CONCLUSION

This study formulated and solved a linear programming model to determine the optimal product mix for maximizing weekly profit for NASCO foods Nigeria Limited. By integrating both production quantities and decisions on whether to produce specific products, the model effectively balanced profit generation with practical constraints on raw materials and processing times.

The results demonstrated that a focused production plan, emphasizing a subset of high-profit products yields an optimal profit of ₦872,045.46 per week. Notably, the solution highlighted the wrapping process as the most critical constraint, reaching full capacity, while many raw materials remained significantly underutilized. This suggests that operational time constraints, rather than material shortages, currently limit production scalability.

The findings offer valuable insights for managerial decision-making. To enhance profitability further, efforts should be directed toward expanding capacity in key bottleneck operations, particularly wrapping. Additionally, the substantial underutilization of raw materials suggests the potential for introducing new products or restructuring current production processes to better exploit available inputs.

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ARITHMETIC RETURN MODEL FOR REMOVING AUTOCORRELATION FROM STATISTICAL PROCESS CONTROL DATA EXHIBITING GEOMETRIC BROWNIAN MOTION

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Abstract

The presence of positive autocorrelation in a controlled process is a major problem especially when traditional quality control charts are to be used in monitoring the process. This is because the two major assumptions in using the traditional control charts are that the process data are independently and normally distributed. In this work, a novel method of removing autocorrelation from data exhibiting Geometric Brownian Motion (GBM) is proposed. This GBM is autoregressive of order AR(1). A chemical process dataset and furnace temperature dataset were transformed to Arithmetic Return model (ARM). The fitted ARM for both datasets were fitted and residuals obtained from both datasets were subjected to DW test for the presence of positive autocorrelation. Initial Durbin Watson's (DW) test result for both processes before the transformation were 0.0538 and 1.5045 respectively which indicated the presence of positive autocorrelation. Final DW test results from the ARM transformation were 2.0047 and 1.7848 respectively indicating that positive autocorrelation was removed from both datasets. The proposed method is simple to understand and easy to use provided that the process data is GBM and autocorrelation is the major concern.

Keywords: Autocorrelation, Geometric Brownian Motion (GBM), Autoregressive of order 1 AR(1), Arithmetic Return Model (ARM), Logarithmic Return Model (LRM), Durbin Watson (DW)

Introduction

Statistical quality control charts have been designed to work with the two basic assumptions that the process data are normally distributed and that they should be independent. In essence, a process outcome data should not determine the outcome of the next successive process data and so on. This is established in the literature but in real life, some process that are being controlled are autocorrelated in nature. Most of such processes are time dependent. Autocorrelation is a

representation of the degree of similarity mathematically, between a given time series data and a lagged version of itself over successive time intervals (Tim, 2023). It is conceptually similar to the correlation between two different time series, but autocorrelation uses the same time series twice. Once in its original form and lagged in one or more time periods. A process is only said to be in a state of statistical control if it is operating in the presence of natural causes of variation and is free from assignable causes (Dike *et al.,* 2018). When autocorrelation is present in the process, unnecessary out of control signals may be triggered which makes a problem that has to be treated with urgency before monitoring of the process.

In the continuous industries such as the chemical industry, most process data are autocorrelated. In other circumstances, such as when a process follows an adaptive model or when the process mean is a deterministic function, for example, a harmonic function or a linear or nonlinear trend, the data will also be autocorrelated (Nien, 1999).

Douglas (2013) stated that the most important of the assumptions made concerning control charts is that of independence of the observations, for conventional control charts do not work well if the quality characteristic exhibits even low levels of correlation over time. Specifically, these control charts will give misleading results in the form of too many false alarms if the data are positively correlated. To avoid such unnecessary disturbances, the process data needs to be cleared of autocorrelation before monitoring.

In discrete as well as in continuous production processes, data often show some autocorrelation, or serial dependence. Several monitoring tools are found in the literature that deals with the case of multiple process variables, but a few of them deal with the case of autocorrelated data (Hussam *et al.,* 2021).

From literatures, the two major ways to remove autocorrelation from the process is by fitting a time series model to the process data, thereby obtaining independent residuals, and the residuals are now used as the process control data. Classical control charts for independent variables can be applied to residual processes, provided that the residuals are independent Erna *et al.* (2018). Or, the traditional control charts limits are simply adjusted to account for autocorrelation in the process. Thirdly, a less popular approach is the model-free charts as stated by (Aytaç, 2020). These are control charts for autocorrelation that do not apply any modeling techniques to the data. They can be considered as a separate category of charts where no residuals are used and no modifications are done to existing charts. However, the first method involved the use of some

sophisticated statistical skills to build a satisfactory model which led to the work of Siaw *et al.* (2013) where a new method of fitting a time series model to the process with ease was proposed by using Logarithmic Return transformation. Djauhari *et al.* (2014) established that the positive time series data are governed by Geometric Brownian Motion law, and this GBM is an autoregressive process of order 1 AR(1). The derivations are found in Siaw *et al.* (2022).

This study is an exploration of the work done by siaw *et al.* (2022), where Logarithmic return transformation was done on the AR(1) process to obtain i.i.n.d residuals. According to literatures, when the rate of return is not large, logarithmic return is approximately equal to the arithmetic return. This work is intended by using arithmetic return transformation instead of the logarithmic return on a GBM process to see the response.

The Proposed Arithmetic Return Model (ARM)

The proposed model is intended by converting the AR (1) process to Arithmetic Return Model, this because the Geometric Brownian Motion (GBM) is an AR(1) process, proof of this is found in Siaw *et al.* (2022). To consider the Arithmetic Return Model for dependent data, transformation of Arithmetic Return (AR) is needed and shown as

$$AR = \frac{X_t - X_{t-1}}{X_{t-1}} \tag{1}$$

Consider the AR (1) process which can be written as;

$$AR_t = C + \Theta AR_{t-1} + \mathcal{E}_t \tag{2}$$

where C is the intercept, Θ the slope in AR(1) model and the error terms $\mathcal{E}_t$ are i.i.n.d. with zero mean and constant variance.

Then the Arithmetic Return transformation of Equation (2) is given as

$$\frac{X_t - X_{t-1}}{X_{t-1}} = C + \Theta\left(\frac{X_{t-1} - X_{t-2}}{X_{t-2}}\right) + \mathcal{E}_t \tag{3}$$

thus,

$$\left.\begin{aligned} \frac{X_t}{X_{t-1}} - 1 &= C + \Theta\left(\frac{X_{t-1}}{X_{t-2}} - 1\right) + \mathcal{E}_t \\ \frac{X_t}{X_{t-1}} &= C + 1 + \Theta\left(\frac{X_{t-1}}{X_{t-2}}\right) - \Theta + \mathcal{E}_t \\ \frac{X_t}{X_{t-1}} &= [C + 1 - \Theta] + \Theta\left(\frac{X_{t-1}}{X_{t-2}}\right) + \mathcal{E}_t \\ \frac{X_t}{X_{t-1}} &= \mathcal{B} + \Theta\left(\frac{X_{t-1}}{X_{t-2}}\right) + \mathcal{E}_t \end{aligned}\right\} \tag{4}$$

where $\mathcal{B} = [C + 1 - \Theta]$, and Θ are the regression parameters, $\mathcal{B}$ is the intercept and Θ is the slope in the model.

Equation (4) is the Arithmetic Return model (ARM).

The residuals are obtained using Equation (5)

$$\widehat{\mathcal{E}_t} = X_t - \hat{X}_t \quad (5)$$

The Durbin Watsons test for autocorrelation

The Durbin-Watson (DW) test is a statistical test used to detect autocorrelation in the residuals of a linear regression model.

The null and alternative hypotheses are;

H_0 : There is no positive autocorrelation among the residuals.

H_1 : The residuals are autocorrelated

Test statistic:

$$d = \frac{\sum_{t=2}^{T}(e_t - e_{t-1})^2}{\sum_{t=1}^{T} e_t{}^2} \quad (6)$$

where **T:** The total number of observations

e_t: The t^{th} residual from the regression model

Decision rule for the Durbin Watsons (DW) test for the presence of positive autocorrelation

At significance level α (alpha), the test statistic DW is compared at lower and upper critical values.

If DW > dU, there is no statistical evidence that positive autocorrelation exists in the data

If DW < dL, there is statistical evidence that positive autocorrelation exists in the data.

If DW is between the two bounds, the test is inconclusive (CFI, 2024).

RESULTS FROM THE ANALYSIS

The ARM was fitted to two sets of data, a chemical process simulations data and furnace temperature data used by (Aytaç, 2020) in Microsoft excel.

Results of analysis from the chemical process data

Table 1: Summary of results from the chemical process data

Statistic	Chemical process data	ARM	LRM
Slope	0.025249611	0.072793689	-0.00495605

Intercept	0.983811403	-0.00382924	-0.16939806
R squared	0.99977547	1.46633E-05	0.028703173
DW	0.053810486	2.003625015	2.338180758
Correlation	0.994906871	-0.00382927	-0.16942011

$$\left(\frac{X_t}{X_{t-1}}\right) = -0.00382924 + 0.072793689\left(\frac{X_{t-1}}{X_{t-2}}\right) \quad (7)$$

Equation (7) is the fitted ARM for the chemical process data. Estimated values for the regression parameters $\mathcal{B}$ and θ are 0.072793689 and -0.00382924 respectively. From the Durbin Watson's table at 5% significance level, n = 200, k = 1, dL = 1.758 and dU = 1.779. Initial DW test was carried on the residuals of the process giving the value 0.053810486 which is less than dL indicating that positive autocorrelation exists in the process. ARM was fitted to the process dataset, and DW test was carried on the residuals obtained from the ARM which gave the value 2.003625015. The DW 2.003625015 > 1.779 which indicates that positive autocorrelation was removed from the chemical process data.

Results of analysis from the furnace temperature data

Table 2: Summary of results from the furnace temperature data

Statistic	Furnace data	ARM	LRM
Slope	0.723881222	0.145460909	0.14545076
Intercept	435.9849614	-4.8315×10^{-6}	-4.8926×10^{-6}
R squared	0.500212167	0.021732047	0.021729192
Correlation	0.70725679	0.147417933	0.14740825
DW	1.504463924	1.784796405	2.004645495

$$\left(\frac{X_t}{X_{t-1}}\right) = -4.8315 \times 10^{-6} + 0.145460909\left(\frac{X_{t-1}}{X_{t-2}}\right) \quad (8)$$

Equation (8) is the fitted ARM for the furnace temperature data. Regression estimated values for $\mathcal{B}$ and θ are -4.8315×10^{-6} and 0.145460909 respectively. From the Durbin Watson's table at

5% significance level, n = 78, k = 1 the lower and upper values are dL = 1.611 and dU = 1.662. Initial DW test value obtained from the residuals of the process data is 1.504463924 which is less than the dL indication that positive autocorrelation exists in the process data. ARM was fitted to the process dataset, and DW test was carried on the residuals obtained from the ARM which gave the value 1.784796405. The DW 1.784796405 > 1.662 which indicates that positive autocorrelation was removed from the furnace temperature data.

Discussion

From results of the analysis, ARM successfully removed the effect of autocorrelation from the chemical process dataset and furnace temperature datasets used with initial DW test results of 0.053810486 and 1.504463924 to an autocorrelation-free results of 2.003625015 and 1.784796405 respectively with ease and no log transformation of the data was needed for this purpose. The proposed ARM is a simple method with ease of understanding and implementation which makes it a better option over the LRM transformation of the process data if the process is GBM and autocorrelation is the cause for concern.

Conclusion

In conclusion, the ARM has proven effectiveness in removing autocorrelation from both process data which are Geometric Brownian Motion with much simplicity and ease of understanding and implementation than the LRM which has to go through the log return transformation of the process data. DW test does not provide a judgement on how better the autocorrelation is removed, once DW > dU there is no statistical evidence that autocorrelation exists in the data, which makes the ARM a better choice to use if the process data is GBM and the cause for concern is the presence of positive autocorrelation. Further research is currently ongoing to find out other strengths of the ARM in statistical process control for autocorrelated data which is GBM driven.

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PROGNOSTIC GAMMA GENERALIZED REGRESSION MODELLING OF DETERMINANTS INFLUENCING VARIATIONS IN UNDER-5 MORTALITY

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Abstract

Under-5 mortality is a vital public health and socioeconomic indicator, influenced by factors such as maternal health, nutrition, and access to healthcare services. This study introduces a Gamma prognostic multivariate regression model to predict under-5 mortality rates, addressing the positively skewed nature of mortality data for enhanced accuracy and robustness. We analyzed data from national health surveys and international organizations, identifying key predictors like socioeconomic status, maternal education, healthcare access, nutritional status, sanitation, and immunization coverage. The Gamma regression model outperformed traditional linear models in handling the heteroscedasticity and skewness of mortality data. Our findings underscore the significant impact of maternal education and healthcare access on under-5 mortality. The model's validity and reliability were confirmed through diagnostic tests, making it a valuable tool for policymakers and healthcare providers aiming to design targeted interventions. In the paper also, Gamma regression model was also used to offer detailed insights into child mortality determinants using mortality data collected. Our results suggest that improving maternal education and expanding healthcare access could notably reduce under-5 mortality rates. The study also highlights under-5 mortality as a pressing issue in Lagos State, Nigeria, with significant variations observed across communities like Ikorodu, Badagry, and Epe.

Keywords: Gamma polynomial regression; Lagos divisions; Gamma regression; Prognostic models; Under-5 mortality

1 Introduction

Under-5 mortality, which refers to the death of children before their fifth birthday, is a key indicator of public health and socioeconomic development globally. Despite strides in reducing under-5 mortality rates, significant disparities persist across different regions and socioeconomic groups. Understanding and predicting these differences is crucial for developing targeted interventions to further lower child mortality.

Child mortality is influenced by a complex interaction of factors such as socioeconomic status, maternal education, healthcare access, and nutritional status. These factors have been consistently recognized as critical determinants of child health outcomes, as outlined in the

health determinants model. This model suggests that health outcomes arise from a combination of individual, community, and societal factors.

Traditional models, like logistic regression and Cox proportional hazards models, have been used to predict under-5 mortality. However, these models often assume a normal distribution of errors and homoscedasticity, making them less suitable for the skewed distribution typical of mortality data (Cameron & Trivedi, 2013). Recent studies have turned to generalized linear models (GLMs) with various link functions to handle skewed data, with the Gamma regression model showing particular promise for its robustness in dealing with positively skewed and heteroscedastic data. The Gamma distribution is ideal for modeling continuous, positive, and skewed outcomes, making it well-suited for under-5 mortality rates.

Despite the availability of advanced statistical models for predicting child mortality, few studies have employed the Gamma regression model. Most research relies on linear or logistic regression, which may not adequately address the unique characteristics of under-5 mortality data.

Several scholars have applied prognostic models across various fields, particularly in medicine. Vogenberg (2009) explored the use of predictive and prognostic models in healthcare decision-making during economic recessions. Abu-Hanna and Lucas (2001) focused on prognostic models in medicine using Artificial Intelligence (AI) and statistical approaches, while Pajouheshnia et al. (2017) reviewed treatment use in cardiovascular prognostic research. Feng, May, & Tang (2019) applied prognostic models to predict overall survival in patients with primary gastric cancer.(Cleland & van Ginneken, 1988) opined that maternal education also showed a significant negative association with under-5 mortality. Educated mothers are more likely to adopt healthier practices, access healthcare services, and ensure better nutrition for their children, leading to improved child health outcomes.

In Lagos, Nigeria, the rate of under-5 mortality varies across different communities due to factors such as socioeconomic conditions, maternal health practices, and access to healthcare services. For example, the communities of Ikorodu, Badagry, and Epe have been identified as having higher than expected under-5 mortality rates. To capture accurate data on under-5 mortality, it is necessary to engage directly with mothers through personal interviews and structured questionnaires.

Researchers like Schwarzer, Vach, and Schumander (2000) have made significant contributions to understanding childhood mortality patterns in Lagos. Aigbe and Zannu (2012) examined differentials in infant and child mortality rates across Nigeria's six geopolitical zones, while Adekanbi (2015) investigated neighborhood contextual influences on childhood mortality in Nigeria.

Prognostic models are essential for prediction purposes in medical research. They help weigh the harms versus benefits of treatments, guide the design and analysis of clinical trials, and control for confounding variables in observational research. In public health, these models can target preventive interventions at individuals or populations at high risk of developing diseases, helping to plan for future health needs and communicate risks among healthcare professionals.

Lagos State, a cosmopolitan city and Nigeria's commercial hub, faces significant challenges in child mortality. The state's population, estimated at 21 million in 2016, is the largest in Africa, making it crucial to understand and address the factors contributing to under-5 mortality.

Cultural practices, socioeconomic disparities, and access to healthcare services all play roles in the varying rates of child mortality across the state's diverse communities. Kareem et al. (2023) focused their research on the rising child mortality rates in Lagos State by employing prognostic models, specifically Prognostic Logistic, Linear, and Poisson regression models. They developed six models from these approaches, which were instrumental in identifying communities in Lagos State with high childhood mortality.

This research seeks to develop a Gamma prognostic multivariate regression model to predict under-5 mortality rates, providing a more accurate and reliable alternative to traditional linear models. By incorporating factors such as socioeconomic status, maternal education, healthcare access, nutritional status, sanitation, and immunization coverage, the Gamma model offers a more nuanced and precise prediction of under-5 mortality

The study applies prognostic models to identify factors predicting differences in childhood mortality across Lagos State communities. Specifically, it uses Gaussian regression, Poisson regression, and Gamma regression models to analyze the data. Hypothesis tests, including the Wald test and Likelihood Ratio tests, as well as model selection criteria like Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), are employed to evaluate the fitness of the final models.

The research is structured as follows: Section 2 covers the materials and methods, focusing on logistic, linear, and Gamma regression models. Section 3 presents the results and discussion of findings based on the data collected and the developed models. The study concludes with conclusion and recommendations in Section 4, aiming to guide policymakers and healthcare providers in reducing under-5 mortality in Lagos State.

2 Materials and Method

2.1 Data collection

Sources

Data for this study were collected from multiple sources, including national health surveys, Demographic and Health Surveys (DHS), UNICEF, and the World Health Organization (WHO) databases. These sources provide comprehensive and reliable data on under-5 mortality and its potential predictors across various countries and regions. Questionnaires also were distributed to women of child bearing age across the Lagos communities.

Sampling methods

The study utilized a stratified random sampling method to ensure a representative sample of the population. Stratification was based on five (5) divisions of Lagos communities viz: Ikorodu, Badagry, Ikeja, Lagos Island, and Epe utilizing key demographic and socioeconomic characteristics, such as geographic region, urban or rural residence, and income levels, to capture diverse population segments.

Data collection process

Data were extracted from the most recent survey cycles available, typically covering the period from 2015 to 2020. Ethical considerations were adhered to by ensuring that data were anonymized and publicly available. Variables were carefully selected and cleaned to ensure consistency and accuracy across different sources.

• Dependent Variable

The under-5 mortality rate (U5MR) is utilized as the dependent variable, defined as the number of deaths among children under the age of five per 1,000 live births.

• Independent Variables

The study identified several independent variables expected to influence under-5 mortality, including:

• Socioeconomic Status (SES): Measured by household income, parental education levels, and employment status.

• Maternal Education (ME) : Number of years of formal education completed by the mother.

• Healthcare Access (HA) : Availability and utilization of healthcare services, including antenatal care, skilled birth attendance, and immunization coverage.

• Nutritional Status (NS) : Indicators such as stunting, wasting, and underweight prevalence among children.

• Sanitation (S) : Access to clean water and improved sanitation facilities.

• Immunization Coverage (IC) : Percentage of children receiving essential vaccines.

The prognostic models considered in this study include Binary Logistic Regression, Gaussian Regression, and Gamma Regression models. The choice of the best model to describe the measured characteristic depends on the distribution of the response variable and the error.

2.2 Prognostic Models

Prognostic models are employed to predict the likelihood of an event occurring before it happens and serve as valuable tools in clinical management (Abu-Hanna et al., 2001). It is generally assumed that when the dependent variable is skewed, whether positively or negatively, it tends to approximate a normal distribution as the sample size increases, given specific parameter values of the initial data distribution (Arowolo et al., 2019).

There are several approaches to developing prognostic models, utilizing a variety of techniques. Common methods include simple decision rules based on the categorization of a prognostic score and the application of Bayes' rule. The parameters for these models are typically estimated using data specifically gathered for prognostic purposes. In this study, the response variable (Y) is under-5 mortality. While the data is count-based, it can be approximated by a continuous distribution, much like the way the Poisson distribution can be approximated by the Normal distribution.

2.3 Gaussian Prognostic Model

2.3.1 Gaussian Distribution

In probability and statistics, the normal distribution of ay type (whether Gaussian, or Laplace-Gauss) is a continuous probability distribution for real-valued random variables. Normal distributions play a crucial role in statistics and are frequently used in the natural and social sciences to model real-valued random variables with unknown distributions. Their significance stems, in part, from the central limit theorem. Measurement errors, are often normally distributed (Lyon, 2014). Additionally, Gaussian distributions possess unique properties valuable in

analytical studies. For instance, a linear combination of normal deviates remains a normal deviate. Many results and methods, such as uncertainty propagation and least squares parameter fitting, can be derived in explicit form when the relevant variables follow a normal distribution.

Let X be a continuous random variable. X is said to follow a Gaussian distribution give its parameters as μ and σ, and its probability density function (pdf) given by

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; -\infty \leq x \leq \infty, \quad (1)$$

where μ is the mean (median and mode), while σ is the standard deviation (Weisstein, 2020). A random variable with a Gaussian distribution is said to be normally distributed, and is called a normal deviate.

2.3.2 Gaussian Regression Model

The k-variable linear regression model is utilized to examine the relationship between a response variable and one or more predictor variables. The formulation of this model is presented in (2)

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \ldots + \beta_k X_{ki} + e_i, \quad (2)$$

where error term, e_i is normally distributed, just like the dependent variable, Y_i [12]. Equation (2) can be written in matrix form as

$$Y = X\beta + e \quad (3)$$

where Y is an n-dimensional vector called the response vector, X is a $n \times p$ matrix called the design matrix, where $p = k + 1$ as earlier defined. The β represents the intercept and slopes, a p-dimensional vector, and e is an n-dimensional vector called the error vector (Alaba, 2012; Olubusoye, 2013). The expectation (Mean) of Y is given by $E(Y) = \mu = X\beta$, where X is the model matrix of order $(n \times p)$ and β is a $(p \times 1)$ matrix of parameters, so that, μ is $(n \times 1)$ vector.

2.3.3 Least Square Estimation of Gaussian Regression Model

Least squares estimation (LSE) is considered the best linear unbiased estimator (BLUE) for estimating the parameters of a multiple linear regression model. To accurately estimate the coefficient vector, β, several assumptions must be satisfied: (i) There must be a linear relationship between the response variable Y and the covariates or independent variables X, either in their original form or after appropriate transformation. (ii) The columns of X must be linearly independent. (iii) The expected value of Y should be equal to $X\beta$, ensuring that $E(e_i) = 0$. (iv) The variance of the residuals, $Var(e_i/X)$, should be σ^2, with $Cov(e_i, e_j) = 0$ for $i \neq j$. (v) The matrix X should be uncorrelated with the residuals e. (vi) The residuals e should follow a normal distribution, i.e., $e \sim N(0, \sigma^2)$. This final assumption implies that Y follows a normal distribution (Hand, 1997).

The LSE is obtained by minimizing the error term e, which is defined as

$$e = Y - X\beta \tag{4}$$

This e is assumed to be very small, so it is minimized by first squaring (4) to have

$$e^2 = (Y - X\beta)'(Y - X\beta) \tag{5}$$

Differentiating (5) with respect to β will further minimize it, Then equate it to zero and solve for β to have

$$\hat{\beta} = (X'X)^{-1}X'Y \tag{6}$$

where $\hat{\beta}$ is a column vector that house all the parameter estimates. The response variable Y can the be estimated by $\hat{Y} = X\hat{\beta}$.

2.4 Generalized Linear Model

In statistics, the generalized linear model (GLM) is a flexible extension of ordinary linear regression, as presented in equation (2). Unlike traditional linear regression, GLM accommodates response variables (Y) that follow error distributions other than the normal distribution. GLM extends the linear regression framework by linking the linear model to the response variable (Y) through a link function, and by allowing the variance of each observation to be a function of its predicted value.

Nelder and Wedderburn (1972) introduced GLMs as a unifying framework for various statistical models, such as linear regression, logistic regression, and Poisson regression. They proposed an iteratively reweighted least squares method for maximum likelihood estimation of model parameters, a technique that remains widely used and is the default in many statistical software packages. Alternative approaches, including Bayesian methods and least squares fits to variance-stabilized responses, have also been developed (Nelder and Wedderburn, 1972).

In GLMs, the response variable Y follows a distribution from the exponential family, with the Gaussian distribution being a special case. Therefore, Y is not restricted to being normally distributed—it can follow any distribution within the exponential family, which is the key generalization of GLMs. It is assumed that each component of Y follows a distribution from the exponential family, which takes the form

$$f_Y(y;\theta,\phi) = exp\left\{\frac{[y\theta - b(\theta)]}{a(\phi)} + c(y,\phi)\right\} \tag{7}$$

where $a(.)$, $b(.)$ and $c(.)$ are functions. The function in equation (7) is an exponential family model provided ϕ is known, and a canonical parameter θ. In this study Binary logistic regression, Gaussian regression also known as the linear regression, Poisson regression and Gamma regression are considered.

2.5 Binomial Prognostic Model

2.5.1 Binomial Distribution

In probability theory and statistics, the binomial distribution is a discrete probability distribution that represents the number of successes in n independent trials, each with a binary outcome and a constant probability of success p or a constant probability of failure $q = 1 - p$. When the experiment is conducted only once, it is known as a Bernoulli trial, and a series of such outcomes is referred to as a Bernoulli process. The binomial distribution serves as the foundation for the binary logistic regression model.

Let X be a discrete random variable. We say that X follows a binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0,1]$, denoted by $X \sim B(n,p)$. The probability of observing exactly x successes in n independent trials is given by the probability mass function

$$P(X \leq x) = \binom{n}{x} p^x (1-p)^{n-x}; x = 0,1,2,\ldots,n. \tag{8}$$

2.6 Gamma Prognostic Model

2.6.1 Gamma Distribution

In probability theory and statistics, the gamma distribution is a univariate continuous probability distribution characterized by two parameters. By adjusting these parameters, the gamma distribution can be simplified to the exponential, Erlang, or chi-square distributions. In this study, the gamma distribution is parameterized by a shape parameter α and a rate parameter β. The probability density function (pdf) of a random variable Y, which follows a gamma distribution with shape α and rate β, denoted by $Y \sim Gamma(\alpha,\beta)$, is given by

$$f(y;\alpha,\beta) = \frac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)}; \alpha,\beta > 0, y \geq 0, \tag{9}$$

where $\Gamma(\alpha)$ is the gamma function. For all positive integers, $\Gamma(\alpha) = (\alpha - 1)!$.

2.6.2 Gamma Regression Model

The gamma regression model is used to relate a gamma-distributed, strictly positive response variable (Y) to one or more predictor variables expected to influence the outcome. This model was selected for the study because of its suitability for handling positively skewed data with heteroscedasticity. For a gamma-distributed random variable Y, as described in equation (9), the mean is given by $E(Y) = \mu = \frac{\alpha}{\beta}$, so that

$$\beta = \frac{\alpha}{\mu}. \tag{10}$$

Substituting equation (10) into (9) gives

$$f(y;\alpha,\beta) = \frac{1}{\Gamma(\alpha)} \left(\frac{\alpha y}{\mu}\right)^\alpha \frac{1}{y} e^{-\frac{\alpha}{\mu} y} \tag{11}$$

If $Y \sim (\mu,\alpha)$ with a known α, the log likelihood is given by

$$l(\mu, \alpha, y) = (\alpha - 1)log(y) - \left(\frac{\alpha y}{\mu}\right) + \alpha log(\alpha) - \alpha log(\mu) - log\Gamma(\alpha). \quad (12)$$

Comparing equation (32) with (17) gives the following: $\theta = -\frac{1}{\mu}$, $b(\theta) = log\mu = log(-1/\theta) = -log(-\theta)$, $b'(\theta) = -1/\theta = \mu$, $b''(\theta) = \frac{1}{\theta^2} = \mu^2$, $a(\phi) = \phi$, $\phi = \frac{1}{\alpha}$.

Thus, $\frac{1}{\mu_i} = x_i'\beta$, then

$$\mu_i = \frac{1}{x_i'\beta}. \quad (13)$$

The Gamma regression model uses a logarithmic link function, making it appropriate for the positively skewed distribution of mortality rates. The model assumes the following form:

$$log(U5MR) = \beta_0 + \beta_1(SES) + \beta_2(ME) + \beta_3(HA) + \beta_4(NS) + \beta_5(S) + \beta_6(IC) \quad (14)$$

2.7 Maximum Likelihood Estimation of Gamma Regression Model

Model Specification:

$$g(\mu(x)) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + . \cdots + \beta_k X_k \quad (15)$$

where:

- $g(\mu(x))$ is the inverse link function (for Gamma, typically μ^{-1}
- $\mu(x)$ is the mean of the response variable.

MLE for Gamma Regression:

Likelihood Function: The likelihood for Gamma regression is based on the Gamma distribution. Log-Likelihood: The log-likelihood function is:

$$\ell(\beta) = \sum_{i=1}^{n} \left[-\frac{y_i}{\mu_i} - \log(\mu_i) - \log(y_i)\right]$$

We used numerical methods to optimize the model.

2.8 Wald test (Z) test

The large sample test for a regression parameter can be developed based on the null hypothesis (H_0) that $\beta_j = 0$, as opposed to the alternative hypothesis (H_1) that $\beta_j \neq 0$. The corresponding test statistic is expressed as follows.

$$Z = \frac{\beta_j}{S(\beta_j)}, j = 1,2,\ldots,k+1 \quad (16)$$

The decision rule specifies that H_0 should be rejected if $|Z| > Z_{1-\alpha/2}$, where $Z_{1-\alpha/2}$ is the critical value from the standard normal distribution, α is the level of significance, $S(\beta_j)$ represents the standard error of β_j, and β_j are the estimates derived from the maximum likelihood estimator.

2.9 Model Selection Criteria

In this study,Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) were employed as the model selection criteria and they are calculated as thus:

$$AIC = -2logL + 2(k+1) \tag{17}$$

where $\log L$ is the log-likelihood of the model and $k+1 = p$ is the number of parameters in the model. If the sample size n is small, such that $\frac{n}{p} < 40$, the AIC should be adjusted to:

$$AIC_C = nlogL + 2p + \frac{2p(p+1)}{n-p-1}, \tag{18}$$

where p represents the number of parameters. The adjusted AIC in this equation indicates that as $n \to \infty$, $\frac{2p(p+1)}{n-p-1} \to 0$. The other model selection criterion applied is the BIC, which is generally expressed as:

$$BIC = nlogL + nlog(n) + plog(n) \tag{19}$$

where p is the number of parameters in the model and n is the sample size. The model with the lowest AIC and BIC values is considered the most appropriate model (Schwarz, 1978).

2.10 Study Area

The divisions in Lagos State studied are Ikorodu, Badagry, Ikeja, Lagos Island, and Epe. Cluster sampling and judgemental sampling techniques are used within each community. The cluster sampling is to select a Local Council Development Area (LCDA) within a division using cluster sampling. The selected LCDA is carefully studied via personal interview and questionnaire administration, using judgemental sampling. The following LCDAs were selected and studied in the 5 divisions. Ikorodu North LCDA in Ikorodu division, Badagry West LCDA in Badagry division, Ikeja LCDA in Ikeja division, Lagos Island East LCDA in Lagos Island division, and Epe LCDA in Epe division. The field researchers visited homes where women are within the age bracket (15 - 49), and the questionnaires were filled by the women. For women (uneducated in English) who could not fill the questionnaires, the field researchers questioned the women and fill the questionnaires for them. The geographical map showing the 5 divisions in Lagos State is depicted in Figure 1.



Figure 1. Map of Lagos State

Lagos State is divided into five administrative divisions—Ikorodu, Badagry, Ikeja, Lagos Island, and Epe—which are further subdivided into 20 Local Government Areas (LGAs) (Lagos State - population, 2011). Ikorodu, located northeast along the Lagos Lagoon, shares a boundary with Ogun State. As of the 2006 Census, Ikorodu had a population of 535,619 and is predominantly inhabited by the Ijebu people. It consists of one Local Government Area (NigeriaCongress.org, 2007).

Badagry is a coastal town positioned between Metropolitan Lagos and the Republic of Benin's border at Seme. The 2006 preliminary census recorded a population of 241,093. Badagry is home to the Egun people, who are primarily fishermen. The Badagry division includes four LGAs—Ajeromi-Ifelodun, Amuwo-Odofin, Ojo, and Badagry—totaling 1,841,435 people as of the 2006 Census (Nigeria Official Gazette Archived 2007).

Ikeja, the capital of Lagos State, is a well-planned residential and commercial area known for its shopping malls, pharmacies, and government reserved areas. According to the 2006 Census, Ikeja had a population of 313,196. The Ikeja division is home to eight LGAs—Agege, Alimosho, Ifako-Ijaye, Ikeja, Kosofe, Mushin, Oshodi-Isolo, and Shomolu—with a total population of 4,801,311 as per the 2006 Census (Lagos State - population, 2011).

Lagos Island is one of the most famous cities in Lagos State, Nigeria, and Africa at large. As of the 2006 Census, Lagos Island had a population of 209,437. The Lagos Island division includes seaports, situated on islands separated by creeks and protected from the Atlantic Ocean by barrier islands and sand spits, such as Bar Beach, which stretches up to 100 kilometers. The division includes Eti-Osa, Apapa, Lagos Island, as well as Lagos Mainland and Surulere, with a combined population of 1,542,279 as of the 2006 Census (Lagos State Government, 2011).

Epe, located on the northern side of the Lekki Lagoon, is known for its fishing activities. The 2006 Census recorded Epe's population at 181,409. The Epe division includes the Ibeju-Lekki and Epe LGAs, with a total population of 298,890 as of the 2006 Census (NigeriaCongress.org, 2007)

3 Results and Discussion

3.1 Application

This research is focused on the causes of variation in under 5 mortality in Lagos State. Primary data were collected from the five divisions in Lagos State viz-a-viz Ikorodu, Badagry, Lagos Mainland/Ikeja, Lagos Island and Epe. A well structured questionnaire was employed to collect the data used for the analysis. The questionnaire comprises 12 items, 7 from Section A and 5 from Section B. See the questionnaire in Appendix II. The target population are women who fall in the child bearing age and live in Lagos State. The 5 divisions in Lagos State are assumed to be equally distributed with women that fall in the age bracket (15-49), so 120 questionnaires were distributed to 120 independent individuals in each division, totalling 600 questionnaires. The sample size was based on cost and trained personnel available. Out of the 600 questionnaires distributed, 520 were retrieved and out of the 520 retrieved, after careful sorting to remove male respondents and questionnaire that were not valid, the number reduced to 476. The 476 valid questionnaires were used for the analysis. The data were entered into Microsoft Excel, saved with the extension .csv and analyzed using Python and SPSS 23.0 packages.

3.1.1 Exploratory Data Analysis

This highlights some potentially hidden features within the collected data. Frequency distribution tables, contingency tables, bar charts, and density plots are utilized to explore these features.

Table 1 indicates that the majority of respondents (49.8%) are women over 30 years of age, with a significant portion (66.2%) being married. Out of 120 questionnaires distributed to each of the five divisions in Lagos State, the responses were as follows: 98 from Ikorodu, 94 from Badagry, 97 from Mainland/Ikeja, 97 from Lagos Island, and 90 from Epe. These divisions represent 20.6%, 19.7%, 20.4%, 20.4%, and 18.9% of the total responses, respectively.

Regarding religious affiliation, 56.3% of respondents identify as Christians, 31.9% as Muslims, and 11.8% practice other religions. Educational attainment among respondents shows that 31.5% have completed secondary school, 25.8% hold tertiary education certificates, 30.3% possess only a first school leaving certificate, and 12.4% are illiterate.

Table 1. Frequency Distribution Table

Variables	Valid Response	Frequency	Percentage (%)
Age	Less than 20	59	12.4
	20 - 30	180	37.8
	Above 30	237	49.8
Marital status	Married	315	66.2
	Divorced	100	21.0
	Single	61	12.8
Lagos divisions	Ikorodu	98	20.6
	Badagry	94	19.7
	Ikeja	97	20.4
	Lagos Island	97	20.4
	Epe	90	18.9
Religion	Christianity	268	56.3
	Islam	152	31.9
	Others	56	11.8
Education	No School	59	12.4
	Primary	144	30.3
	Secondary	150	31.5
	Tertiary	123	25.8

The five variables listed in Table 1 serve as covariates in this study. The dependent variables are: (i) the count of children under 5 years of age who have died, which is a count variable; and (ii) a binary variable indicating whether a woman has experienced any child death. A value of 0 is assigned if no child deaths have occurred, and a value of 1 is assigned if at least one child death has occurred. Figure 6 displays the distribution of child deaths and the binary variable. Appendix II provides details on the educational status of respondents by division in Lagos and outlines the causes of under-5 mortality across the five divisions in Lagos State..

The histogram in Figure 2 below depicts the distribution of the Under-5 Mortality Rate (U5MR) per 1,000 live births. The distribution is multi-modal, with distinct peaks observed at specific intervals, particularly around 35 and 50 U5MR. These multiple peaks suggest that the U5MR data may originate from various sub-populations or communities within the larger dataset. The central point of this distribution is approximately 42.7 per 1,000 live births, where the highest frequency is recorded. Another significant cluster is seen around 50 per 1,000 live births. The data range spans from about 25 to 55 U5MR, reflecting some variability in under-5 mortality rates across different observations or communities. While the frequencies are fairly evenly distributed across the different ranges, lower frequencies are noticeable in the 30–35 and 45–50 U5MR intervals. The presence of multiple peaks suggests that the data likely represent diverse communities with varying levels of healthcare access, socioeconomic status, or other factors influencing mortality rates. This range indicates that certain communities experience significantly higher mortality rates than others, potentially due to differences in healthcare nfrastructure, maternal education, sanitation, or other socioeconomic factors.

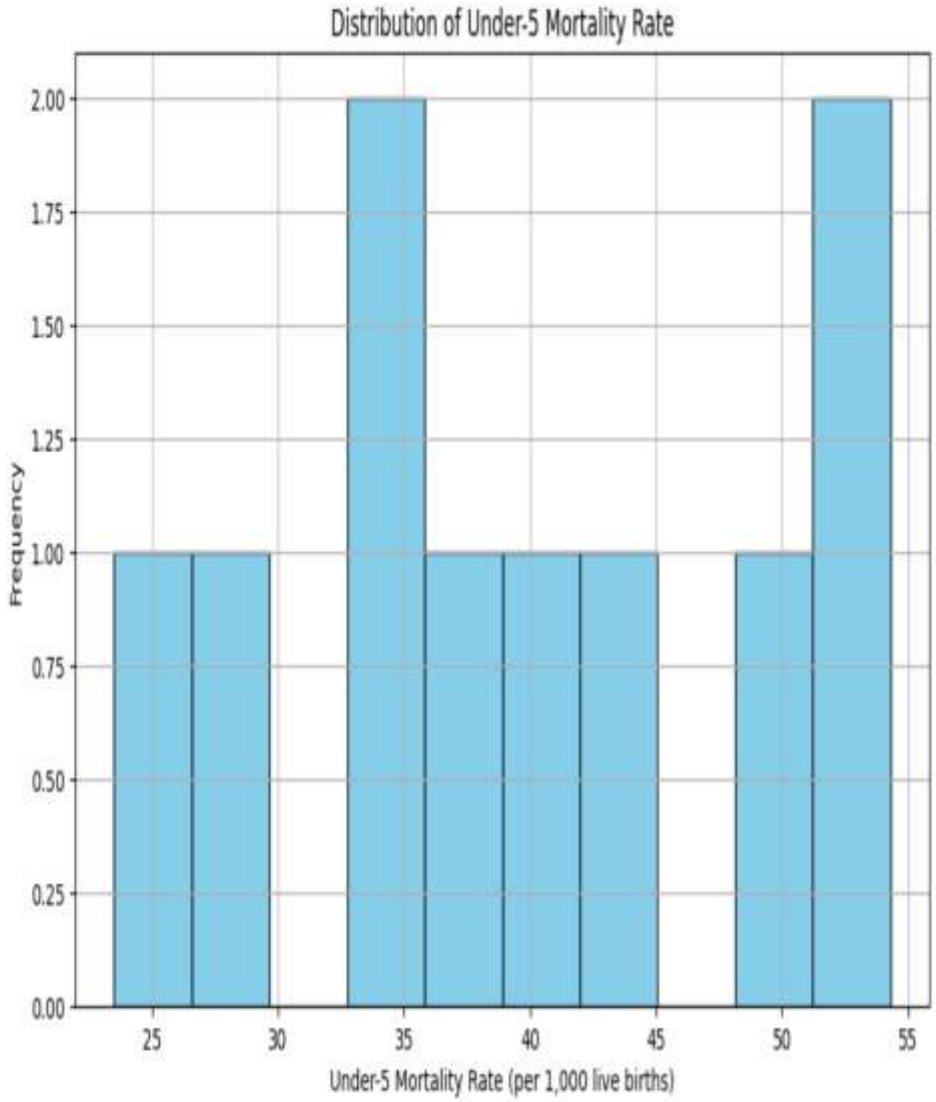


Figure 2. Distribution of under 5 Mortality

Table 2 presents the summary statistics for the dependent and independent variables used in the study. The mean under-5 mortality rate (U5MR) across the sample is 42.7 deaths per 1,000 live births, with a standard deviation of 15.6. The socioeconomic status (SES) index has a mean value of 0.54, indicating moderate economic conditions among the surveyed households. Maternal education averages 7.8 years, suggesting a relatively low level of formal education. The healthcare access index has a mean value of 0.67, and the nutritional status, measured by the prevalence of underweight children, has a mean of 12.4%. Sanitation and immunization coverage show mean values of 0.72 and 82.5%, respectively, indicating moderate levels of access and coverage.

Table 2. Summary statistics for dependent and indepedent variables

Variable	Mean	Standard Deviation	Minimum	Maximum
Under-5 Mortality Rate	42.7	15.6	15.2	78.3
Socioeconomic Status	0.54	0.14	0.20	0.87
Maternal Education	7.8	3.2	0	15
Healthcare Access	0.67	0.16	0.25	0.95
Nutritional Status	12.4	6.8	3.0	28.5
Sanitation	0.72	0.18	0.30	0.98
Immunization Coverage	82.5%	10.4%	45.0%	98.0%

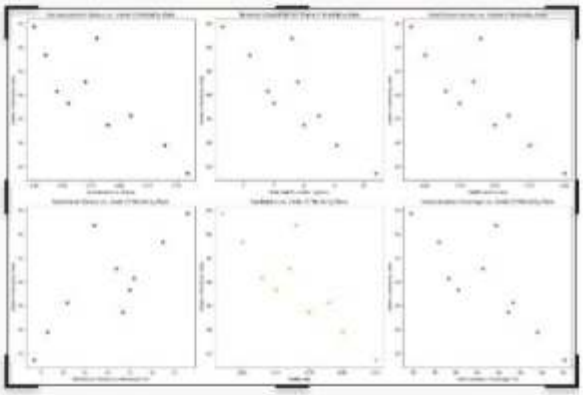


Figure 3. Factors Vs Mortality

The scatterplots in Figure 3 above illustrate the relationships between Under-5 Mortality Rate (U5MR) and six predictor variables: Socioeconomic Status, Maternal Education, Healthcare Access, Nutritional Status, Sanitation, and Immunization Coverage. These visualizations reveal clear patterns in how each of these factors correlates with U5MR.

There is a noticeable negative correlation between socioeconomic status and U5MR, indicating that as socioeconomic status improves, the under-5 mortality rate decreases. This suggests that higher socioeconomic status is likely associated with better access to resources such as healthcare, nutrition, and education, which contribute to lower mortality rates in children under five.

Maternal education also shows a strong negative correlation with U5MR; as maternal education increases in years, the under-5 mortality rate decreases significantly. This emphasizes the critical role of maternal education in reducing child mortality. Educated mothers are generally more knowledgeable about healthcare practices, nutrition, and the importance of immunization, all of which contribute to lower U5MR.

Similarly, there is a negative correlation between healthcare access and U5MR. Improved access to healthcare results in a decrease in under-5 mortality rates, likely due to timely medical

interventions, vaccinations, and maternal care. This highlights the importance of healthcare access in reducing child mortality.

On the other hand, a positive correlation is observed between the percentage of underweight children (nutritional status) and U5MR. As the percentage of underweight children increases, the under-5 mortality rate also rises. This underscores the significant impact of poor nutritional status on child mortality, as malnourished children are more vulnerable to diseases and have weaker immune systems, leading to higher mortality rates.

The scatterplots also show a negative correlation between sanitation quality and U5MR. Improved sanitation leads to a decrease in under-5 mortality rates, likely due to reduced exposure to waterborne and hygiene-related diseases, which are common causes of death in young children.

Lastly, immunization coverage is negatively correlated with U5MR, indicating that as the percentage of children receiving immunizations increases, the under-5 mortality rate decreases.

Overall, these scatterplots reveal that U5MR is influenced by a combination of these factors. Positive factors such as socioeconomic status, maternal education, healthcare access, sanitation, and immunization coverage are associated with lower U5MR, while negative factors like poor nutritional status are associated with higher U5MR. These insights suggest that policies aimed at improving these factors in a comprehensive manner can lead to significant reductions in child mortality. A multivariate regression model that incorporates these variables could provide a robust prediction model for U5MR and help guide policy decisions. The strong relationships observed in the scatterplots reinforce the importance of these factors in ensuring child survival.

Residuals in figure 4 below illustrate the residuals (differences between observed and predicted values) on the vertical axis versus the fitted values (predicted under-5 mortality rates) on the horizontal axis. The residuals do not exhibit a clear pattern or trend, which suggests that the linearity assumption of the model is reasonably satisfied. This shows that the residuals are influenced by some factors.

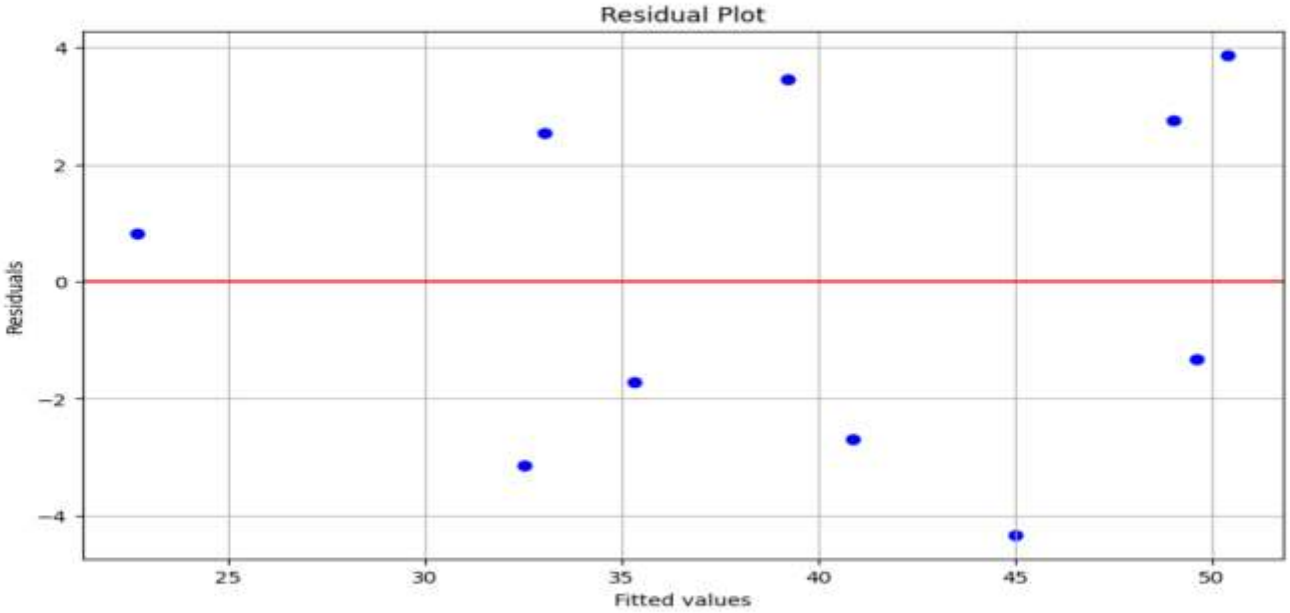


Figure 4. Residuals plot

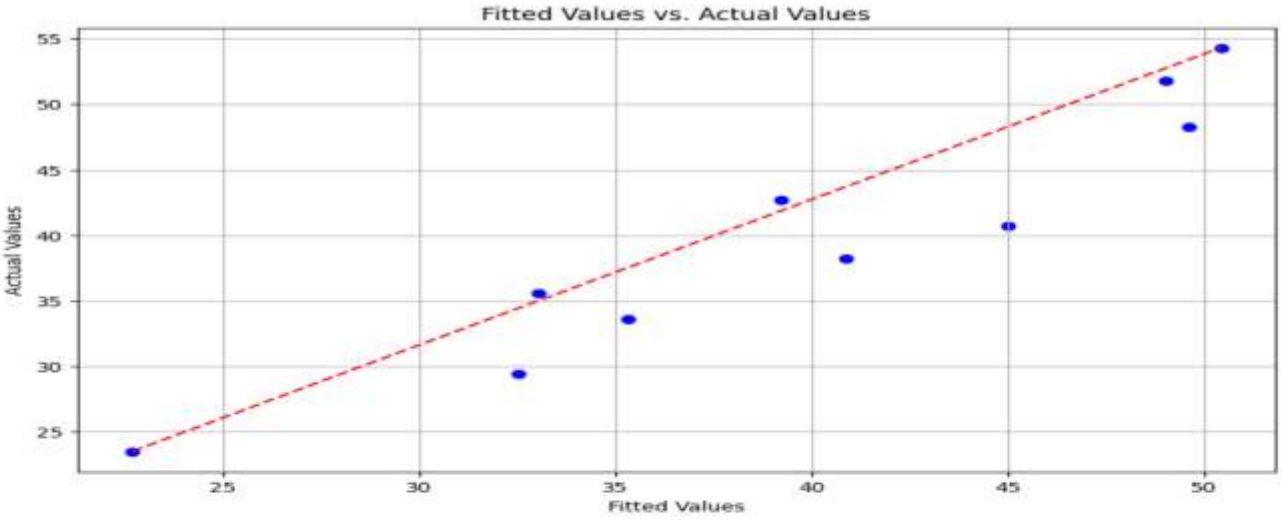


Figure 5. Fitted values and Actual values

Figure 6 shows the probability of vulnerability of mothers to childhood mortality (under 5 death). The mothers (women interviewed) that are most vulnerable to childhood mortality include mothers that are above 30 years of age, mothers who live in Ikorodu community of Lagos State, mothers who are Christians and mothers who have no education (illiterate mothers). This does not mean Christians mothers are vulnerable to child mortality in the real sense of it

because the instrument did not ask if they were Christians before losing their babies. Also, some women would have lost their child before relocating to Ikorodu, because the instrument did not also capture where they reside when they lost their children and how old they were when the incident happened. The only thing that is very certain is that uneducated mothers are most vulnerable to having children who may die before age 5.

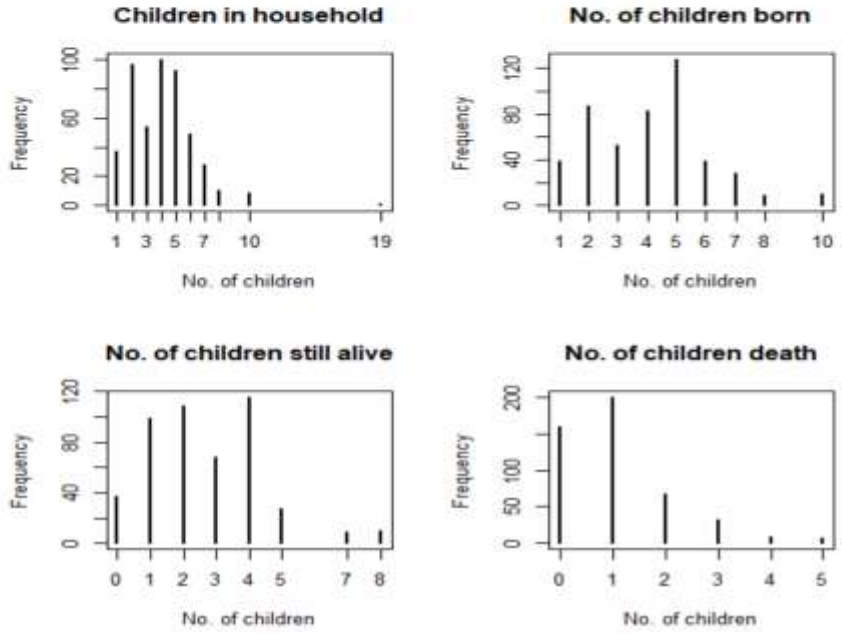


Figure 6. Probability of Under 5 Mortality

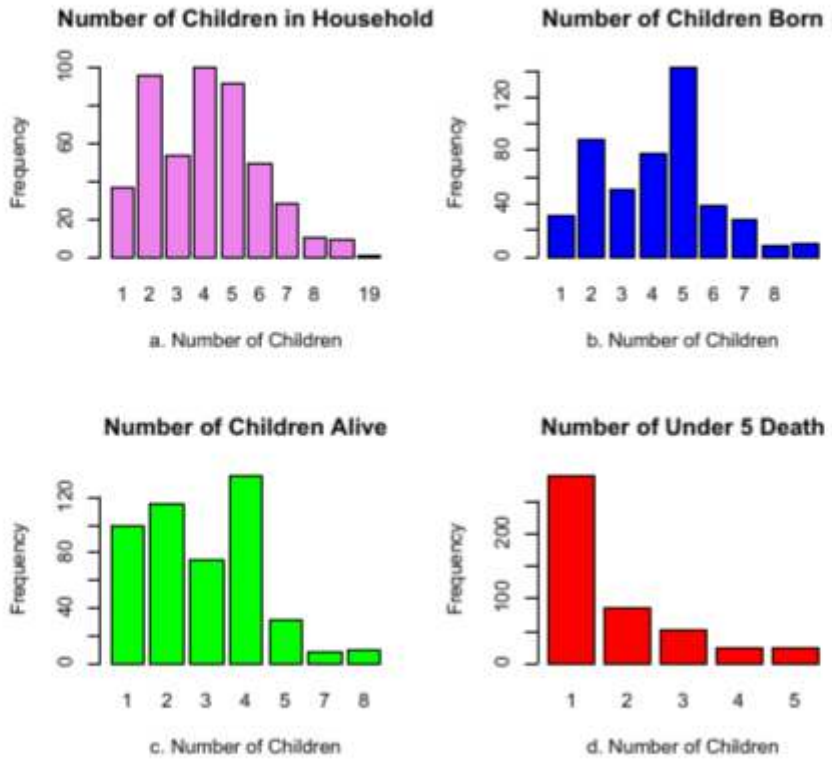


Figure 7. Count variables in study

Figure 7 depicts that most of the household where the respondents live have 4 children, most of the children born by our respondents are 5. The number of children that are still alive to women are 4, while the number of children death from a respondent is 1 (count dependent variable). The mode is used in this regard as measure of central tendency. The term under 5 mortality as used in this study comprises all children between age 0 to 5.

Let Y be a random variable signifying the number of under-5 mortality. The expected number of under-5 deaths to a mother is given by

$$E(y) = \frac{\sum_i f_i y_i}{\sum_i f_i}, \forall i$$

where f is the frequency of occurrence. The expected number of under-5 mortality to a mother in Lagos communities is

$$E(y) = \frac{160\times0+200\times1+67\times2+32\times3+9\times4+8\times5}{160+200+67+32+9+8} = \frac{506}{476} = 1.06$$

The expected number of under-5 mortality to a mother in Lagos communities is 1.06. So, any community which expected number of under-5 death to a mother is greater than 1.06 has high rate of mortality. The under-5 mortality is calculated for each community as shown in Table 2.

Table 3. Expected Number of under-5 Mortality across Lagos communities

Lagos Divisions	Number of Children 0	1	2	3	4	5	Total Mothers	Total Under-5 Death	Mean Under-5 Death
Ikorodu	4	57	22	10	2	3	98	154	1.57
Badagry	12	50	17	10	3	2	94	136	1.45
Ikeja	59	22	15	1	0	0	97	55	0.57
Lagos Island	64	28	2	3	0	0	97	41	0.42
Epe	21	43	11	8	4	3	90	120	1.33
Total	**160**	**200**	**67**	**32**	**9**	**8**	**476**	**506**	**1.06**

Table 3 shows that the maximum number of children under age 5 that are death from a mother is 5, $y = 0,1,2,3,4,5$. The table shows that 160 women did not loose any child under 5 years old, 200 women have lost just 1 child, 67 women have lost 2 children, 32 women have lost 3 children, 9 women have lost 4 children, and 8 women have lost 5 children under 5 years old. In total, 476 mothers have lost 506 children under 5 years old. The number of children lost by mothers is spread across the five divisions in Lagos.

• **Ikorodu**: In Ikorodu, 98 mothers lost 154 under-5 children, resulting to 1.57 under-5 death per mother. This is greater than expected of 1.06 for Lagos.

• **Badagry**: In Badagry, 94 mothers lost 136 under-5 children, resulting to 1.45 under-5 death per mother. This is greater than expected for Lagos.

• **Ikeja**: In Ikeja, 97 mothers lost 55 under-5 children, resulting to 0.57 under-5 death per mother. This is less than expected for Lagos.

• **Lagos Island**: In Lagos Island, 97 mothers lost 41 under-5 children, resulting to 0.42 under-5 death per mother. This is less than expected for Lagos.

• **Ikorodu**: In Epe, 90 mothers lost 120 under-5 children, resulting to 1.33 under-5 death per mother. This is greater than expected of 1.06 for Lagos.

Thus, Ikorodu, Badagry and Epe communities have higher under-5 mortality rates than expected among the Lagos communities studied.

Table 4. Number of under 5 Death or Alive

		Death		Total
		Alive	Death	
Lagos divisions	Ikorodu	4	94	98
	Badagry	12	82	94
	Ikeja	59	38	97
	Lagos Island	64	33	97
	Epe	21	69	90
Total		160	316	476

Table 4 shows the number of women who have never lost a child and who have lost at least one child. In Ikorodu, out of the 98 women analyzed, 94 has recorded under 5 death while only 4 did not. In Badagry, out of 94 women interviewed, 82 has recorded under 5 death while 12 have not. In Ikeja division, out of the 97 women analyzed 38 have recorded under 5 death while 59 have not. In Lagos Island, out of the 97 women analyzed, 33 have recorded under 5 death, while 64

have not. In Epe division, out of the 90 women analyzed, 69 has recorded under 5 death while 21 have not. Judging from this table, Ikorodu has the highest under mortality rate, followed by Badagry and Epe as depicted in Figure 8a. Lagos Island has the least under 5 mortality rate followed by Ikeja. So, the state government should focus on Ikorodu, Badagry and Epe in order to reduce under 5 mortality rate. Table 2 (count data) and Table 3 (binary data) form the basis of the models to be discussed in subsequent section.

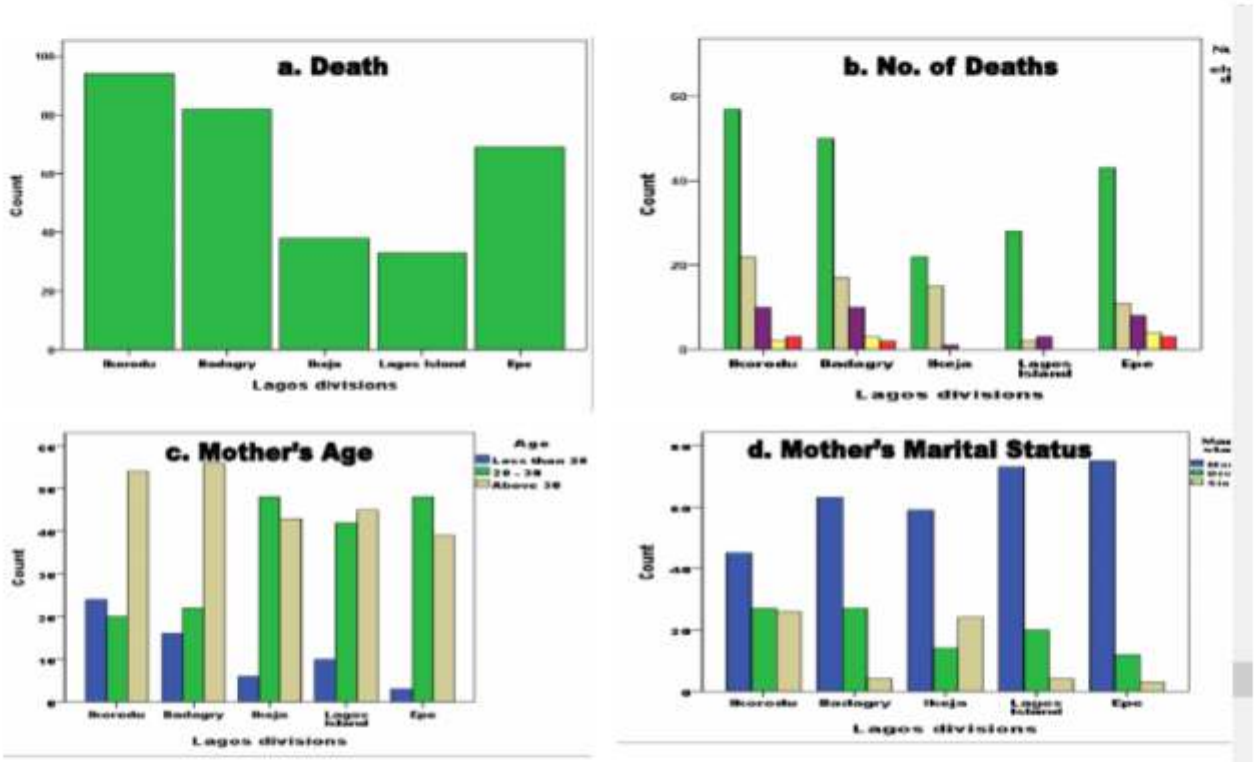


Figure 8. Lagos Communities and Associated Under 5 Mortality

Figure 8a shows that Ikorodu, Badagry and Epe in that order have the highest under 5 mortality rate, while Lagos Island has the least. This can be associated with mother's age and marital status as shown in Figure 8c and 8d respectively. The breakdown of under 5 mortality in the 5 communities is displayed in Figure 8b.

Table 5. Gamma Regression model result

Predictor	Coefficient	Standard Error	z-value	p-value
(Intercept)	2.456	0.312	7.87	< 0.001
Socioeconomic Status	-1.134	0.258	-4.39	< 0.001
Maternal Education	-0.065	0.021	-3.10	0.002
Healthcare Access	-0.852	0.297	-2.87	0.004
Nutritional Status	0.034	0.011	3.09	0.002
Sanitation	-0.674	0.244	-2.76	0.006
Immunization Coverage	-0.032	0.012	-2.67	0.008

The results of the Gamma regression model are presented in Table 5. The coefficients, standard errors, z-values, and p-values indicate the strength and significance of each predictor's effect on the under-5 mortality rate.

• Socioeconomic Status: A one-unit increase in the socioeconomic status index is associated with a 113.4% decrease in the under-5 mortality rate ($p < 0.001$), highlighting the significant protective effect of higher socioeconomic status.

• Maternal Education: Each additional year of maternal education reduces the under-5 mortality rate by 6.5% ($p = 0.002$), underscoring the importance of female education in improving child health outcomes.

• Healthcare Access: Greater access to healthcare services is associated with an 85.2% decrease in the under-5 mortality rate ($p = 0.004$), indicating the critical role of healthcare availability and utilization.

• Nutritional Status: Higher prevalence of underweight children is linked to a 3.4% increase in the under-5 mortality rate ($p = 0.002$), reflecting the detrimental impact of poor nutrition on child survival.

• Sanitation: Improved sanitation is associated with a 67.4% decrease in the under-5 mortality rate ($p = 0.006$), emphasizing the importance of clean water and sanitation facilities.

• Immunization Coverage: Each percentage point increase in immunization coverage reduces the under-5 mortality rate by 3.2% ($p = 0.008$), highlighting the efficacy of vaccination programs in preventing child mortality.

3.2 Prognostic Regression Analysis Estimation

In this study, the response variables (Y) are (i) number of children born by a woman who died before age 5. (ii) response of a woman to whether she has lost a child (below age 5), with response 0, if she has not lost a child, and 1 if she has lost at least a child. The probability distribution $P(Y_i = 1) = \pi_i$ or $P(Y_i = 0) = 1 - \pi_i$, where response variable (Y) denotes death

or alive. The independent variables or covariates are age of mother (X_1), marital status of mother (X_2), division in Lagos State where mother lives (X_3), mother's religion (X_4) and mother's education (X_5). The term mother used here refers to our respondents who have given birth to at least one child.

There are basically two different models. One is a binary response and will be modelled using binary logistic regression. The other is a count response and will be modelled by linear regression, and gamma regression. The count response is a subset of the continuous response. If the sample size is large enough, then it can be approximated by a continuous distribution, which in most cases Gaussian, the basis for the linear regression. However, it can be gamma, weibull or any other continuous distribution.

- **Model A: Binary Response Variable**
  - Use all the covariates (binary logistic regression)
  - Drop a covariates that is not significant (binary logistic regression)

- **Model B: Count Response Variable**
  - Use all the covariates (linear model)
  - Drop a covariates that is not significant (linear model)
  - Use all the covariates (linear model)
  - Drop a covariates that is not significant (linear model)

Models 1 and 2 in A are competing models. Models 3, 4, 5 and 6 are competing models. Models in A are not competing with models in B. We are not only interested if a woman has lost a child or not, we are also interested in the number of children a woman lost. This will help to measure under 5 mortality.

3.2.1 Fitting Multivariable Gamma Regression Models

Table 6. Gamma Regression Estimates of 5 Covariates

	Estimate	Std. Error	Z value	P-value
Intercept	0.871717	0.114678	7.601	0.00000
Age	-0.047057	0.027324	-1.722	0.08569
Marital Status	-0.072553	0.027794	-2.610	0.00933
Lagos Division	-0.030271	0.012835	-2.359	0.01875
Religion	0.013137	0.025843	0.508	0.61146
Education	-0.002739	0.017322	-0.158	0.87444

Model 5 regression parameters estimated in Table 6 can be expressed as

$$\hat{\mu}_i = \frac{1}{0.871717 - 0.047057x_{1i} - 0.072553x_{2i} - 0.030271x_{3i} + 0.013137x_{4i} - 0.002739x_{5i}} \quad (20)$$

Table 7 shows that some of the covariates are not significant at 5% level. Religion and education are not significant, even at 10% level. So, there is a need to develop model 6 by dropping religion and education.

Table 7. Gamma Regression Estimates of 3 Covariates

	Estimate	Std. Error	Z value	P-value
Intercept	0.86756	0.10436	8.313	0.00000
Age	-0.04535	0.02709	-1.674	0.0949
Marital Status	-0.06680	0.02543	-2.627	0.0089
Lagos Division	-0.02875	0.01246	-2.306	0.0215

Model 6 regression parameters estimated in Table 7 can be expressed as

$$\hat{\mu}_i = \frac{1}{0.86756 - 0.04535x_{1i} - 0.06680x_{2i} - 0.02875x_{3i}} \quad (21)$$

3.3 Model Selection Criteria

Table 8. Model Selection Criteria for Binary Response

Models	-logL	AIC	BIC
Binary Logistic Regression			
Model 1 (5 covariates)	237.6006	487.2000	512.1937
Model 2 (4 covariates)	238.8635	487.7300	508.5541

Table 8 shows the model selection criteria. For the binary response, model 1 with 5 covariates is selected over model 2 because it has the smaller - log likelihood (-logL), AIC and BIC.

Table 9. Model Selection Criteria for Count Response

Models	-logL	AIC	BIC
Multiple Linear Regression			
Model 3 (5 covariates)	673.8634	1361.7270	1390.8850

Model 4 (4 covariates)	674.8924	1361.7850	1386.7770
Multiple Gamma Regression			
Model 5 (5 covariates)	604.3016	1222.6000	1251.7610
Model 6 (3 covariates)	604.4812	1219.0000	1239.7900

Table 9 shows the model selection criteria for count, approximated as continuous variable. Here, for multiple linear regression models, model 3 with 5 covariates is selected over model 4 with 4 covariates based on AIC criterion. For multiple gamma regression models, model Model 6 with 3 covariates is selected over Model 5 with 5 covariates. After comparing the best from each competing generalized linear model, the multiple gamma regression model with 3 covariate is selected as the best model for modelling under 5 mortality rate in Lagos. The Gamma regression model effectively identified key predictors of under-5 mortality and quantified their impacts. Socioeconomic status, maternal education, healthcare access, nutritional status, sanitation, and immunization coverage were all significant determinants. The findings underscore the importance of targeted interventions in these areas to reduce under-5 mortality rates.

4 Conclusion and Recommendation

The findings of this study underscore several critical factors influencing under-5 mortality. Socioeconomic status (SES) emerged as a significant predictor, with higher SES linked to lower mortality rates. This is consistent with earlier research, which suggests that wealthier households have better access to healthcare, nutrition, and living conditions that enhance child survival (Mosley & Chen, 1984).

In this study, we developed six competing generalized linear regression models to identify factors influencing variations in under-5 mortality across the five divisions in Lagos State. The first two models were evaluated for a binary response variable (yes or no). Using -logL, AIC, and BIC as model selection criteria, Model 1 was chosen over Model 2 due to its smaller -logL, AIC, and BIC values. For the count/continuous response variable, four models were assessed: Model 3, Model 4, Model 5, and Model 6. Model 6, a multiple gamma linear regression with three parameters, was selected as the most suitable model for the count/continuous response variable, as it had the lowest AIC and BIC values.

The estimated probability of under-5 mortality in Lagos State is 0.26, indicating that out of every 100 children under 5 years old, 26 do not survive. In the study, of the 1,946 children born to 476 women, 506 died. This probability is specific to this study but could be generalized if the sample is representative of the broader population of Lagos State. Additionally, the study reveals that

316 out of the 476 women involved had lost at least one child under 5 years old, yielding a probability of 0.66.

To identify the factors contributing to variations in under-5 mortality across different communities in Lagos, the multivariable gamma linear regression model was selected as the most appropriate prognostic tool. This model identified key predictors of childhood mortality variations within Lagos State, including the mother's age, marital status, and the specific community where the mother resides. The findings indicate that women aged 31-45 years are particularly susceptible to under-5 mortality, with those living in the Ikorodu community of Lagos State being at higher risk (see Figure 8).

These conclusions are based on the women's status at the time of the interviews. However, the study has some limitations. For instance, the exact age of the women when they lost their babies is unknown, as is the religious practice of the women at the time of their loss, which could be significant since some women may have changed their religious beliefs afterward. Additionally, the community in which the women resided when they lost their babies is not known. The one factor that is definitively clear is the educational level of the women; those who are illiterate are found to be the most vulnerable to under-5 mortality.

The high-risk communities in Lagos State, where childhood mortality rates exceeded expectations, include Ikorodu, Badagry, and Epe (see Table 3). This trend may be linked to the fact that many women without formal education reside in these areas, with Epe, Badagry, and Ikorodu ranking in that order. Among the women interviewed, Lagos Island had the highest number of graduate mothers, followed by Ikeja, while Ikorodu had the fewest. Another contributing factor is the limited access to healthcare facilities in these communities. Epe, Ikorodu, and Badagry show the highest under-5 mortality rates due to inadequate healthcare access, with Lagos Island having the lowest rate. Appendix II provides a detailed depiction of the causes of under-5 mortality across the five divisions in Lagos State.

Based on our research findings, we recommend the following actions: (i) Promote Maternal Education: Prioritize educational initiatives for girls and women, including scholarships, school feeding programs, and adult education. Improved maternal education is associated with better child health outcomes. (ii) Strengthen Healthcare Systems:Invest in healthcare infrastructure and services, including expanding facilities, training providers, and ensuring the availability of essential medicines. Areas such as Ikorodu, Badagry, and Epe would benefit from well-trained pediatricians and gynecologists. Equip the mother and child units of General Hospitals in these locations, and ensure pharmacies are stocked with medications for children under five. Although Lagos State offers free healthcare for this age group, many women are required to source medications externally due to insufficient supplies. (iii) Ensure Cost-Free Child Healthcare: The government should prioritize the well-being of children by providing free treatment and medications at all Lagos State health facilities. (iv) Enhance Socioeconomic Conditions:Implement policies that drive economic growth and reduce poverty, such as job creation and social protection schemes. Improving household incomes can enhance living conditions and access to essential services, thereby reducing under-5 mortality. In cases where

family size is large and income is low, targeted empowerment programs for women in Ikorodu, Badagry, and Epe can address contributing factors to under-5 mortality. Adhering to these recommendations can significantly reduce childhood mortality in Lagos State. (v) Improve Sanitation and Water Access:Invest in water and sanitation infrastructure to ensure access to clean water and improved sanitation facilities. Public health campaigns should promote hygiene practices to reduce waterborne diseases. (vi) Expand Immunization Coverage:Strengthen and expand immunization programs to cover all children, especially in remote and underserved areas. Efforts should include enhancing vaccine supply chains, increasing public awareness, and addressing vaccine hesitancy.

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THE DYNAMICS OF A PROBE BODY IN THE CR3BP WITH TRIAXIAL PRIMARIES

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Abstract

By utilizing the Lagrangian-Hamiltonian formulations, this work presents detailed derivation of equations of motion of a probe body under the impact of triaxiality of the primary bodies in the structure of the circular restricted three-body problem (CR3BP). It is found that the equations of motion of the probe body are affected by the triaxiality factor parameters of the primaries. Furthermore, the dynamical equations differ from those of the classical restricted three-body problem owing to the presents of the triaxiality parameters.

Keywords: CR3BP; Probe body; Dynamics; Triaxiality.

INTRODUCTION

The Restricted Three-Body Problem (R3BP) is a fundamental problem in astrodynamics and celestial mechanics, and its understanding has far- reaching implications for space exploration. It characterizes the motion of three objects in space, where one of the objects is much smaller than the other two. The infinitesimal mass object moves under the gravitational effects of the two larger objects, called primaries, which orbit each other in circular orbits around their center of

mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The objective of the CR3BP is to determine the trajectory of the infinitesimal body (probe body) under the gravitational influence of the primary bodies.

In the CR3BP, the primaries are fixed in a coordinate system rotating at an orbital angular velocity, with the origin at the centre of their masses. Lagrange showed that in this rotating frame there are five libration points at which the infinitesimal mass would remain fixed if placed there with zero velocity. There are three such points lying on the line joining the primaries, while the other two, form equilateral triangles with the primaries (Szebehely, 1967).

The classical CR3BP considers only the effects of gravitational attraction on the infinitesimal body, but there are other perturbing forces that affect the motion of the probe body. In the presence of a luminous body, it is altogether inadequate to consider merely the gravitational interaction force. Poynting (1903) pointed out that particles, such as small meteors or cosmic dust are comparably affected by the gravitation and light radiation force as they approach luminous celestial objects. Radzievskii (1950) found that an allowance for direct solar radiation impact results in a change in the positions of the libration points.

In the classical CR3BP the sphericity of the bodies is presumed, whereas in reality numerous celestial bodies are non-spherical in shape. For instance, Jupiter and Saturn are oblate while Pluto and its moon Charon are triaxial in shape. Singh and Ishwar (1999) revealed that lack of sphericity of the primaries affects the motion of the probe body.

Thus, the equations of classical CR3BP do not adequately represent the motion of the probe body in the presence of any perturbing agent such as non-sphericity of a body. The canonical equation of a non-spherical shape in the Cartesian coordinate system is of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad (1)$$

where a and b are the equatorial radii and c is the polar radius. If

(i) $a>b>c$ the shape is triaxial

(ii) $a=b>c$ the shape is oblate

(iii) $a=b<c$ the shape is a prolate

(iv) $a=b=c$ the shape is a sphere.

The interest is in the first case, when the primary bodies are triaxial in shape. Hence, in the present work, the aim is to explicitly derive the equations of motion of the CR3BP to include the

effects of triaxiality of the primary bodies. Section one contains the introduction, while section two deals with the dynamical exploration of the problem. The discussion and conclusions are drawn in sections three and four, respectively.

2. DERIVATION OF EQUATIONS OF MOTION

2.1 Mathematical formulations of the problem

Denote m_1 and m_2 with $m_2 < m_1$ as the masses of the primaries moving in the circular orbits around their center of mass *O*. A probe body of infinitesimal mass m is moving in the plane of motion defined by m_1 and m_2. Let us consider a baryentric coordinate system *oxyz* where the *x*-axis is the line joining the primaries; *y*-axis is perpendicular to it; the *z*-axis is perpendicular to the orbital plane of the primaries. The distances of m from m_1, m_2 are r_1, r_2 respectively, and the distance between the primaries is *R*. The coordinates of m_1, m_2 and m are $(x_1, 0)$, $(x_2, 0)$ and (x, y), respectively. We are to derive the equations of motion of m considering the influence of triaxiality of the primaries in the coordinate system *oxyz* Fig. 1.

2.2 Energy of the infinitesimal mass

The velocity of the probe body in the barycentric coordinate system oxyz rotating about *z*-axis with uniform angular velocity ω (see Fig1) *is* $v = \dot{x}\hat{i} + \dot{y}\hat{j} + \omega(x\hat{j} - y\hat{i})$.

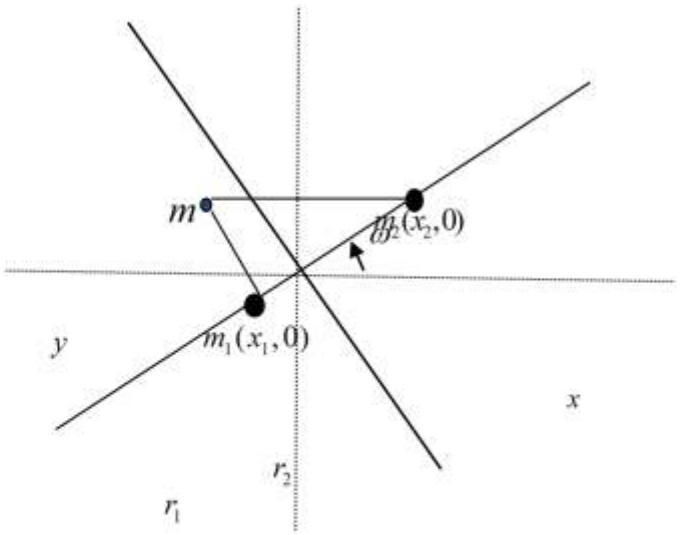


Fig.1: The planar view of the rotating system.

Thus, its kinetic energy (T) is given as

$$T = \frac{m\left(\dot{x}\hat{i} + \dot{y}\hat{j} + \omega(x\hat{j} - y\hat{i})\right)^2}{2}$$
$$= \frac{m\left[(\dot{x}\hat{i} + \dot{y}\hat{j})^2 + 2\omega(\dot{x}\hat{i} + \dot{y}\hat{j})(x\hat{j} - y\hat{i}) + \omega^2(x\hat{j} - y\hat{i})^2\right]}{2}$$
$$= \frac{m\left[\dot{x}^2 + \dot{y}^2 + 2\omega(-\dot{x}y + x\dot{y}) + \omega^2(x^2 + y^2)\right]}{2} \qquad (2)$$
$$= \frac{m\left(\dot{x}^2 + \dot{y}^2\right)}{2} + m\omega\left(x\dot{y} - \dot{x}y\right) + \frac{m\omega^2\left(x^2 + y^2\right)}{2}.$$

The potential energy

The gravitational potential due to a triaxial body Eq.(1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$ with $a > b > c,$ of mass m_t to an external particle is given by (McCuskii, 1963)

$$V_t = -\frac{Gm_t}{r_t}\left[1 + \frac{1}{2m_t r_t^2}\left(I_1 + I_2 + I_3 - 3I\right)\right], \qquad (3)$$

where $I_1 = \frac{\left(b^2 + c^2\right)m_t}{5}$, $I_2 = \frac{\left(c^2 + a^2\right)m_t}{5}$, $I_3 = \frac{\left(b^2 + a^2\right)m_t}{5}$, are the principal moments of inertia of m_t about its centre of mass, $I = I_1 l'^2 + I_2 m'^2 + I_3 n'^2$ is the moment of inertia about the line joining the centre of the mass of the body m_t and the particle. l', m', n' are the direction cosines of the vector from the particle relative to the principal axes of m_t.

Now, the total potential energy of the infinitesimal mass under the influence of the triaxiality of the primaries is given in view of Eq. (3) as

$$V = -Gm\left\langle\left(\frac{m_1}{r_1} + \frac{1}{2r_1^3}\left(I_1 + I_2 + I_3 - 3I\right)\right) + \left(\frac{m_2}{r_2} + \frac{1}{2r_2^3}\left(I_1' + I_2' + I_3' - 3I'\right)\right)\right\rangle \qquad (4)$$

where I_1, I_2, I_3 and I_1', I_2', I_3' are the principal moments of inertia of m_1 and m_2 about their centre of mass respectively, I is the moment of inertia of the vector from m_1 to m and I' is the moment of inertia of the vector from m_2 to m.

with

$$r_1^2 = (x - x_1)^2 + y^2, \quad r_2^2 = (x - x_2)^2 + y^2, \tag{5}$$

G is the gravitational constant.

2.3 Equations of motion in dimensional variables

Starting from the Lagrangian (L) of the system, which is the kinetic energy Eq. (2) minus the potential energy Eq. (4) of the infinitesimal body. That is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + m\omega(x\dot{y} - \dot{x}y) + \frac{1}{2}m\omega^2(x^2 + y^2) - V. \tag{6}$$

Lagrangian equations of motion are of

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0,$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0.$$

Or

$$\ddot{x} - 2\omega\dot{y} = \omega^2 x - \frac{\partial V}{m\partial x},$$
$$\ddot{y} + 2\omega\dot{x} = \omega^2 y - \frac{\partial V}{m\partial y}.$$

The Lagrangian equations are in terms of the generalized co-ordinates and generalized velocities with time t as a parameter. Hamilton's formulation is an alternative to the Lagrangian formulation in which the independent variables are generalized coordinates and momenta P_x and P_y defined as

$$P_x = \frac{\partial L}{\partial \dot{x}} = (\dot{x} - \omega y)m,$$
$$P_y = \frac{\partial L}{\partial \dot{y}} = (\dot{y} + \omega x)m. \tag{7}$$

Hence, we can write

$$\dot{x} = \frac{P_x}{m} + \omega y,$$
$$\dot{y} = \frac{P_y}{m} - \omega x. \tag{8}$$

The Hamiltonian function H equivalent to the motion of the probe body m is specified as

$$H = \dot{x}P_x + \dot{y}P_y - L$$
$$= \left(2\dot{x}P_x + 2\dot{y}P_y - m\left(\dot{x}^2 + \dot{y}^2\right) - 2m\omega\left(x\dot{y} - \dot{x}y\right) - m\omega^2\left(x^2 + y^2\right)\right) + V.$$

Or

$$H = \dot{x}P_x + \dot{y}P_y - \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - m\omega\left(x\dot{y} - \dot{x}y\right) + \Gamma, \tag{9}$$

where $\Gamma = V - \frac{1}{2}m\omega^2\left(x^2 + y^2\right).$

Employing Eq. (8) in Eq. (9) produces

$$H = \left(\frac{P_x}{m} + \omega y\right)P_x + \left(\frac{P_y}{m} - \omega x\right)P_y - \frac{1}{2}m\left(\left(\frac{P_x}{m} + \omega y\right)^2 + \left(\frac{P_y}{m} - \omega x\right)^2\right) - m\omega\left(x\left(\frac{P_y}{m} - \omega x\right) - \left(\frac{P_x}{m} + \omega y\right)y\right) + \Gamma.$$
$$= \frac{P_x^2}{m} + \omega y P_x + \frac{P_y^2}{m} - \omega x P_y - \frac{P_x^2}{2m} - \frac{P_y^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 y^2 + \Gamma.$$

$$= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \omega\left(yp_x - xp_y\right) + \frac{1}{2}m\omega^2\left(x^2 + y^2\right) + \Gamma. \tag{10}$$

Hamilton's standard equations of motion generated by the Legendre transformation are

$$\dot{x} = \frac{\partial H}{\partial p_x}, \qquad \dot{y} = \frac{\partial H}{\partial p_y},$$
$$\dot{p}_x = -\frac{\partial H}{\partial x}, \qquad \dot{p}_y = -\frac{\partial H}{\partial y}.$$

These yield four first order ordinary differential equations which control the movement of the probe body in expressions of the Hamiltonian variables:

$$\left.\begin{aligned}
\dot{x} &= \frac{1}{m}\left(P_x + m\omega y\right), \\
\dot{y} &= \frac{1}{m}\left(P_y - m\omega x\right), \\
\dot{p}_x &= \omega\left(p_y - m\omega x\right) - \frac{\partial \Gamma}{\partial x}, \\
\dot{p}_y &= -\omega\left(p_x + m\omega y\right) - \frac{\partial \Gamma}{\partial y}.
\end{aligned}\right\} \tag{11}$$

Now, the first two equations of Eq. (11) yield

$$\begin{aligned}
P_x &= \left(\dot{x} - \omega y\right)m, \\
P_y &= \left(\dot{y} + \omega x\right)m.
\end{aligned} \tag{12}$$

On, differentiating Eq. (12) with respect to time *t*, gives

$$\begin{aligned}
\dot{P}_x &= (\ddot{x} - \omega \dot{y})m, \\
\dot{P}_y &= (\ddot{y} + \omega \dot{x})m.
\end{aligned} \tag{13}$$

Utilizing Eq. (12) and Eq. (13) in the last two equations of Eq. (11), produces

$$\begin{aligned}
m(\ddot{x} - \omega \dot{y}) &= \omega m(\dot{y} + \omega x) - m\omega^2 x - \frac{\partial \Gamma}{\partial x}, \\
m(\ddot{y} + \omega \dot{x}) &= -\omega m(\dot{x} - \omega y) - m\omega^2 y - \frac{\partial \Gamma}{\partial y}.
\end{aligned} \tag{14}$$

Equations of Eq. (14) by simplifies to

$$\begin{aligned}
\ddot{x} - \omega \dot{y} &= \omega(\dot{y} + \omega x) - \omega^2 x - \frac{\partial \Gamma}{m\partial x}, \\
\ddot{y} + \omega \dot{x} &= -\omega(\dot{x} - \omega y) - \omega^2 y - \frac{\partial \Gamma}{m\partial y}.
\end{aligned}$$

Or

$$\ddot{x} - 2\omega \dot{y} = -\frac{\partial \Gamma}{m\partial x}, \tag{15}$$

$$\ddot{y}+2\omega\dot{x}=-\frac{\partial\Gamma}{m\partial y}. \qquad (16)$$

Eq. (15) and Eq. (16) provide the equations that regulate the motion of the probe body in dimensional variables.

Next is to evaluate the partial derivatives $\frac{\partial\Gamma}{\partial x}$ *and* $\frac{\partial\Gamma}{\partial y}$ of Eq. (15) and Eq. (16) respectively.

Considering Eq. (9) and Eq. (4) results to

$$\frac{\partial\Gamma}{\partial x}=-Gm\frac{\partial}{\partial x}\left\langle\left(\frac{m_1}{r_1}+\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)\right)+\left(\frac{m_2}{r_2}+\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)\right)\right\rangle-\frac{1}{2}m\omega^2\frac{\partial}{\partial x}\left(x^2+y^2\right),$$

$$=-Gm\left\langle\left(m_1\frac{\partial}{\partial x}(\frac{1}{r_1})+\frac{\partial}{\partial x}\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)\right)+\left(m_2\frac{\partial}{\partial x}(\frac{1}{r_1})+\frac{\partial}{\partial x}\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)\right)\right\rangle-m\omega^2x.$$

and

$$\frac{\partial\Gamma}{\partial y}=-Gm\frac{\partial}{\partial y}\left\langle\left(\frac{m_1}{r_1}+\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)\right)+\left(\frac{m_2}{r_2}+\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)\right)\right\rangle-\frac{1}{2}m\omega^2\frac{\partial}{\partial y}\left(x^2+y^2\right).$$

$$=-Gm\left\langle\left(\frac{\partial}{\partial y}\frac{m_1}{r_1}+\frac{\partial}{\partial y}\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)\right)+\left(\frac{\partial}{\partial y}\frac{m_2}{r_2}+\frac{\partial}{\partial y}\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)\right)\right\rangle-m\omega^2y.$$

In view of Eq. (5) we find by implicit differentiation:

$$2r_1\frac{\partial r_1}{\partial x}=2\left(x-x_1\right)$$

$$\Rightarrow\frac{\partial r_1}{\partial x}=\frac{\left(x-x_1\right)}{r_1},$$

$$2r_2\frac{\partial r_2}{\partial x}=2\left(x-x_2\right)$$

$$\Rightarrow\frac{\partial r_2}{\partial x}=\frac{\left(x-x_2\right)}{r_2},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_1}\right)=-\frac{1}{r_1^2}\frac{\partial r_1}{\partial x}$$

$$=-\frac{\left(x-x_1\right)}{r_1^3},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r_2}\right)=-\frac{1}{r_2^2}\frac{\partial r_2}{\partial x}$$

$$=-\frac{\left(x-x_2\right)}{r_2^3},$$

Making use of the above relations in Eq. (15) and Eq. (16), produce

$$\ddot{x}-2n\dot{y}=G\left\langle\left(-\frac{m_1(x-x_1)}{r_1^3}+\frac{\partial}{\partial x}\frac{1}{2r_1^3}(I_1+I_2+I_3-3I)\right)+\left(-\frac{m_2(x-x_2)}{r_2^3}+\frac{\partial}{\partial x}\frac{1}{2r_2^3}(I_1'+I_2'+I_3'-3I')\right)\right\rangle+\omega^2 x,$$

$$\ddot{y}+2n\dot{x}=G\left\langle\left(-\frac{m_1 y}{r_1^3}+\frac{\partial}{\partial y}\frac{1}{2r_1^3}(I_1+I_2+I_3-3I)\right)+\left(-\frac{m_2 y}{r_2^3}+\frac{\partial}{\partial y}\frac{1}{2r_2^3}(I_1'+I_2'+I_3'-3I')\right)\right\rangle+\omega^2 y.$$

(17) ***2.4***

Equations of motion in dimensionless variables

To rationalize and universalize the equations of motion defined by Eq. (17), it is appropriate to make the variables dimensionless such that the properties of the system rely on a particular parameter.

(i) *Unit of mass*: The sum of the masses of the primaries is taken as the unit of mass. For this, we let $m_2=\mu$ *and* $m_1=1-\mu$, where $0<\mu=\dfrac{m_2}{m_1+m_2}\le\dfrac{1}{2}$, is the mass ratio.

(ii) *Unit of length*: The unit of length is defined by the distance between m_1 and m_2.

(iii)*Unit of time:* The unit of time is taken as 2π the periodic time of rotation of m_1 and m_2, so that G is unity i.e., G=1.

Now since the origin is at the barycentre of m_1 and m_2, we have

$$m_1x_1+m_2x_2=0.$$

That is

$$\begin{aligned}
&x_1(1-\mu)+x_2\mu=0\\
&x_1-x_1\mu+x_2\mu=0\\
&x_1+(-x_1+x_2)\mu=0\\
&x_1+(|x_1|+x_2)\mu=0\\
&x_1+(1)\mu=0\\
&x_1=-\mu.
\end{aligned}$$

And we obtain $-\mu(1-\mu)+x_2\mu=0$

$$x_2\mu=\mu(1-\mu)$$

$$x_2=(1-\mu)$$

Next, we detremine the partial derivatives in Eq. 17. We start by evaluating $(I_1+I_2+I_3-3I)$ *and* $(I_1'+I_2'+I_3'-3I')$.

With respect to Eq. (3), we obtain

$$I_1=\frac{1}{5}(b^2+c^2)m_1, \qquad I_1'=\frac{1}{5}(b'^2+c'^2)m_2,$$

$$I_2=\frac{1}{5}(c^2+a^2)m_1, \qquad I_2'=\frac{1}{5}(c'^2+a'^2)m_2, \qquad (18)$$

$$I_3=\frac{1}{5}(b^2+a^2)m_1, \qquad I_3'=\frac{1}{5}(b'^2+a'^2)m_2,$$

with $(a,b,c)\,a(\,a',b',c')$ are the semi- axes of m_1 *and* m_2 correspondingly.

Now

$$I_1+I_2+I_3=\frac{2m_1}{5}(a^2+b^2+c^2) \qquad (19)$$

and

$$I_1'+I_2'+I_3'=\frac{2m_2}{5}(a'^2+b'^2+c'^2). \qquad (20)$$

Utilizing Section 2.4 and in view of Fig. (2), Eq. (19) and Eq. (20) yeild

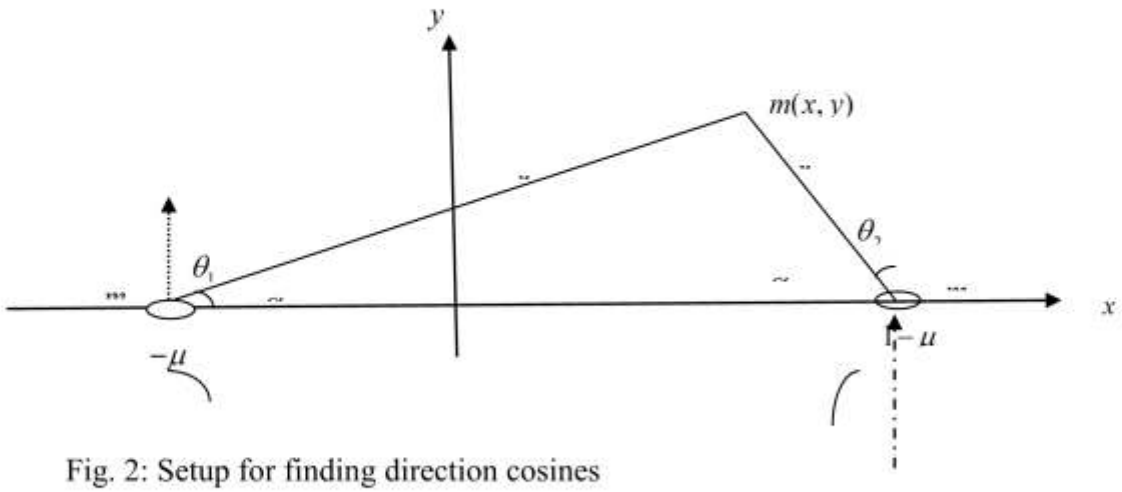


Fig. 2: Setup for finding direction cosines

$$I = I_1 l'^2 + I_2 m'^2 + I_3(0)$$

and

$$I' = I_1' l''^2 + I_2' m''^2 + I_3'(0)$$

with

$\left\{ l' = \frac{\mu + x}{r_1},\ m' = \frac{y}{r_1}, 0 \right\}$ the direction cosines of $\overline{m_1 m}$ and $\left\{ l'' = \frac{1-(\mu + x)}{r_2}, m'' = \frac{y}{r_2}, 0 \right\}$ is the direction cosines of $\overline{m_2 m}$. Furthermore, Eq.(5) gives

$$r_1^2 = (x+\mu)^2 + y^2\ , \qquad r_2^2 = \left(x - (\mu + 1)\right)^2 + y^2. \tag{21}$$

Hence, the moment of inertia I becomes

$$I = \frac{m_1}{5}\left(b^2 + c^2\right)\left(\frac{x+\mu}{r_1}\right)^2 + \frac{m_1}{5}\left(c^2 + a^2\right)\left(\frac{y}{r_1}\right)^2.$$

$$= \frac{1}{5r_1^2} m_1 \left\{ \left(b^2 + c^2\right)(x+\mu)^2 + \left(c^2 + a^2\right) y^2 \right\}.$$

$$= \frac{1}{5r_1^2} m_1 \left\{ \left(a^2 + b^2 + c^2\right)\left\{(x+\mu)^2 + y^2\right\} - a^2(x+\mu)^2 - b^2 y^2 \right\}.$$

$$= \frac{1}{5r_1^2} m_1 \left\{ r_1^2\left(b^2 + c^2\right) + y^2\left(a^2 - b^2\right) \right\}.$$

Thus,

$$\left(I_1 + I_2 + I_3 - 3I\right) = \left\{ \frac{\left[(a^2 - b^2) + (a^2 - c^2)\right]}{5} - \frac{3\left(a^2 - b^2\right) y^2}{5r_1^2} \right\} m_1.$$

From the first equation of Eq. (17), the partial derivatives therein become

$$\frac{\partial}{\partial x}\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)=\frac{\partial}{\partial x}\frac{1}{2r_1^3}\left\{\left[\frac{\left[(a^2-b^2)+(a^2-c^2)\right]}{5}-\frac{3\left(a^2-b^2\right)y^2}{5r_1^2}\right]\right\}m_1.$$
$$=-\frac{3(x+\mu)m_1}{2r_1^5}\left[\frac{(a^2-b^2)}{5}+\frac{(a^2-c^2)}{5}\right]+\frac{15(x+\mu)y^2}{2r_1^7}\frac{\left(a^2-b^2\right)m_1}{5} \tag{22}$$

and

$$\frac{\partial}{\partial y}\frac{1}{2r_1^3}\left(I_1+I_2+I_3-3I\right)=\frac{\partial}{\partial y}\frac{1}{2r_1^3}\left\{\left[\frac{\left[(a^2-b^2)+(a^2-c^2)\right]}{5}-\frac{3\left(a^2-b^2\right)y^2}{5r_1^2}\right]\right\}m_1.$$
$$=-\frac{3y}{2r_1^5}\left[\frac{(a^2-b^2)}{5}+\frac{(a^2-c^2)}{5}\right]m_1-\frac{3y}{r_1^5}\frac{\left(a^2-b^2\right)m_1}{5}+\frac{15y^3}{2r_1^7}\frac{\left(a^2-b^2\right)m_1}{5}. \tag{23}$$

Similarly,

$$I'=\frac{\left[\left(b'^2+c'^2\right)r_2^2+\left(a'^2-b'^2\right)y^2\right]}{5r_2^2}m_2,$$

and hence

$$\left(I_1'+I_2'+I_3'-3I'\right)=\frac{\left[(a'^2-b'^2)+(a'^2-c'^2)\right]}{5}m_2-\frac{3\left(a'^2-b'^2\right)y^2}{5r_2^2}m_2.$$

Therefore, the two partial derivatives in the second equation of Eq. (17) turn to

$$\frac{\partial}{\partial x}\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)=\frac{\partial}{\partial x}\frac{1}{2r_2^3}\left\{\left[\frac{\left[(a'^2-b'^2)+(a'^2-c'^2)\right]}{5}m_2-\frac{3\left(a'^2-b'^2\right)y^2}{5r_2^2}m_2\right]\right\}.$$
$$=-\frac{3(x+\mu-1)}{2r_2^5}\left[\frac{(a'^2-b'^2)}{5}+\frac{(a'^2-c'^2)}{5}\right]m_2+\frac{15(x+\mu-1)y^2}{2r_2^7}\frac{\left(a'^2-b'^2\right)m_2}{5} \tag{24}$$

and

$$\frac{\partial}{\partial y}\frac{1}{2r_2^3}\left(I_1'+I_2'+I_3'-3I'\right)=\frac{\partial}{\partial y}\frac{1}{2r_2^3}\left\{\frac{\left[(a'^2-b'^2)+(a'^2-c'^2)\right]}{5}m_2-\frac{3\left(a'^2-b'^2\right)y^2}{5r_2^2}m_2\right\}.$$

$$=-\left\{\frac{3y}{2r_2^5}\left[\frac{(a'^2-b'^2)}{5}+\frac{(a'^2-c'^2)}{5}\right]m_2+\frac{3y}{r_2^5}\frac{\left(a'^2-b'^2\right)m_2}{5}-\frac{15y^3}{2r_2^7}\frac{\left(a'^2-b'^2\right)m_2}{5}\right\}. \qquad (25)$$

Employing Eq. (22) to Eq. (25), normalized variables: $m_1=1-\mu$, $m_2=\mu$, G=1 in Equation (17), produce the equations of motion of m in non-dimensional form as

$$\ddot{x}-2\omega\dot{y}=\left\{\left(-\frac{(1-\mu)(x+\mu)}{r_1^3}-\frac{3(1-\mu)(x+\mu)}{2r_1^5}\left[\frac{(a^2-b^2)}{5R^2}+\frac{(a^2-c^2)}{5R^2}\right]+\frac{15(1-\mu)(x+\mu)y^2}{2r_1^7}\frac{\left(a^2-b^2\right)}{5R^2}\right)\right.$$
$$\left.+\left(-\frac{\mu(x+\mu-1)}{r_2^3}-\frac{3\mu(x+\mu-1)}{2r_2^5}\left[\frac{(a'^2-b'^2)}{5R^2}+\frac{(a'^2-c'^2)}{5R^2}\right]+\frac{15\mu(x+\mu-1)y^2}{2r_2^7}\frac{\left(a'^2-b'^2\right)}{5R^2}\right)\right\}+\omega^2x,$$

$$\ddot{y}+2\omega\dot{x}=\left\{\left(-\frac{(1-\mu)y}{r_1^3}-\frac{3(1-\mu)y}{2r_1^5}\left[\frac{(a^2-b^2)}{5R^2}+\frac{(a^2-c^2)}{5R^2}\right]-\frac{3(1-\mu)y}{r_1^5}\frac{\left(a^2-b^2\right)}{5R^2}+\frac{15(1-\mu)y^3}{2r_1^7}\frac{\left(a^2-b^2\right)}{5R^2}\right)\right.$$
$$\left.+\left(-\frac{\mu y}{r_2^3}-\frac{3\mu y}{2r_2^5}\left[\frac{(a'^2-b'^2)}{5R^2}+\frac{(a'^2-c'^2)}{5R^2}\right]-\frac{3\mu y}{r_2^5}\frac{\left(a'^2-b'^2\right)}{5R^2}+\frac{15\mu y^3}{2r_2^7}\frac{\left(a'^2-b'^2\right)}{5R^2}\right)\right\}+\omega^2y.$$

Or

$$\ddot{x}-2\omega\dot{y}=\omega^2x-\left\{\frac{1}{r_1^3}+\frac{3(2\sigma_1-\sigma_2)}{2r_1^5}-\frac{15(\sigma_1-\sigma_2)y^2}{2r_1^7}\right\}(1-\mu)(x+\mu)$$
$$-\left\{\frac{1}{r_2^3}+\frac{3(2\sigma_1'-\sigma_2')}{2r_2^5}-\frac{15(\sigma_1'-\sigma_2')y^2}{2r_2^7}\right\}\mu(x+\mu-1),$$

(26)

$$\ddot{y}+2\omega\dot{x}=\omega^2y-\left\{\frac{1}{r_1^3}+\frac{3(2\sigma_1-\sigma_2)}{2r_1^5}+\frac{3(\sigma_1-\sigma_2)}{r_1^5}-\frac{15(\sigma_1-\sigma_2)y^2}{2r_1^7}\right\}(1-\mu)y$$
$$-\left\{\frac{1}{r_2^3}+\frac{3(2\sigma_1'-\sigma_2')}{2r_2^5}+\frac{3(\sigma_1'-\sigma_2')}{r_2^5}-\frac{15(\sigma_1'-\sigma_2')y^2}{2r_2^7}\right\}\mu y,$$

with,

$$\sigma_1=\frac{1}{5R^2}\left(a^2-c^2\right),\quad \sigma_2=\frac{1}{5R^2}\left(b^2-c^2\right),\ \sigma_1'=\frac{1}{5R^2}\left(a'^2-c'^2\right),\quad \sigma_2'=\frac{1}{5R^2}\left(b'^2-c'^2\right).$$

$(\sigma_1,\sigma_2,\sigma_1',\sigma_2'\ are\ strictly\ less\ \ than\ one)$

Determination of Mean Motion ω

In view of Fig. (3), the distances of m_1 and m_2 from the origin O are $|x_1|$ *and* x_2, correspondingly. Hence, the distance between the primaries $|x_1| + x_2 = R$. Now, m_1 and m_2 are moving in circular parths round O, therefore, the mean motion satisfies

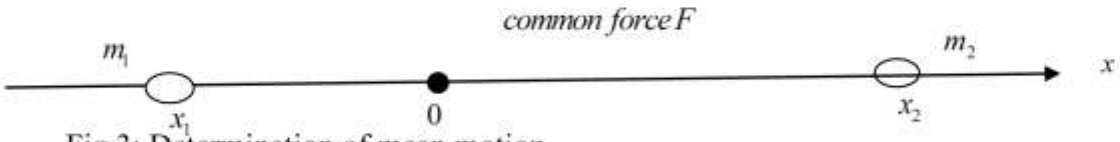


Fig.3: Determination of mean motion

$$F = |x_1| m_1 \omega^2 = x_2 m_2 \omega^2,$$

where F is the common force of gravity between the primaries which is given with respect to Equation (4) as

$$F = -Gm_1 m_2 \frac{\partial}{\partial R}\left[\frac{1}{R} + \frac{1}{2R^3 m_1}(I_1 + I_2 + I_3 - 3I) + \frac{1}{2R^3 m_2}(I_1' + I_2' + I_3' - 3I')\right]$$

Thus,

$$m_1\omega^2 |x_1| = -Gm_1 m_2 \frac{\partial}{\partial R}\left[\frac{1}{R} + \frac{1}{2R^3 m_1}(-2I_1 + I_2 + I_3) + \frac{1}{2R^3 m_2}(-2I_1' + I_2' + I_3')\right]$$

$$= \frac{Gm_1 m_2}{R^2}\left\{1 + \frac{3}{2R^2 m_1}(-2I_1 + I_2 + I_3) + \frac{3}{2R^2 m_2}(-2I_1' + I_2' + I_3')\right\}.$$

Therefore,

$$\omega^2 |x_1| = \frac{Gm_2}{R^2}\left\{1 + \frac{3}{2R^2 m_1}(-2I_1 + I_2 + I_3) + \frac{3}{2R^2 m_2}(-2I_1' + I_2' + I_3')\right\}.$$

Similarly, we get

$$\omega^2 x_2 = \frac{Gm_1}{R^2}\left\{1 + \frac{3}{2R^2 m_1}(-2I_1 + I_2 + I_3) + \frac{3}{2R^2 m_2}(-2I_1' + I_2' + I_3')\right\}.$$

Adding ,

$$\omega^2\left(|x_1|+x_2\right)=\frac{G\left(m_1+m_2\right)}{R^2}\left\{1+\frac{3}{2R^2m_1}\left(-2I_1+I_2+I_3\right)+\frac{3}{2R^2m_2}\left(-2I_1'+I_2'+I_3'\right)\right\}.$$

Or

$$\omega^2=\frac{G\left(m_1+m_2\right)}{R^3}\left\{1+\frac{3}{2R^2m_1}\left(-2I_1+I_2+I_3\right)+\frac{3}{2R^2m_2}\left(-2I_1'+I_2'+I_3'\right)\right\}. \tag{27}$$

Making use of Eq. (18) in Eq. (27), yields

$$\omega^2=\frac{G\left(m_1+m_2\right)}{R^3}\left\{1+\frac{3}{2R^2}\left(\frac{(a^2-b^2)+(a^2-c'^2)}{5}\right)+\frac{3}{2R^2}\left(\frac{(a'^2-b'^2)+(a'^2-c'^2)}{5}\right)\right\}. \tag{28}$$

Employing the non-dimentional variables, Eq. (28) transforms to

$$\omega^2=1+\frac{3}{2}\left(\frac{(a^2-b^2)+(a^2-c'^2)}{5R^2}\right)+\frac{3}{2}\left(\frac{(a'^2-b'^2)+(a'^2-c'^2)}{5R^2}\right).$$

Or

$$\omega^2=1+\frac{3}{2}\left(2\sigma_1-\sigma_2\right)+\frac{3}{2}\left(2\sigma_1'-\sigma_2'\right). \tag{29}$$

3 DISCUSSION

The equations of motion (26) of the CR3BP under the influence of triaxiality of the primaries has been derived using the Lagrangean – Hamiltonian method.

These equations are different from those of the classical case of Szebehely (1967) due to the inclusion of influences of the triaxiality of the primaries in the set up. however, if the effect of the triaxility of the primaries are ignored, that is the primaries are spherical instead triaxial

$$a=b=c\Rightarrow\sigma_1=\frac{a^2-c^2}{5R^2}=0,\quad \sigma_2=\frac{b^2-c^2}{5R^2}=0\ and\ \ a'=b'=c'implying\,\sigma_1'=0,\quad \sigma_2'=0$$

equations (26) will fully coincide with those of the classical case of Szebehely (1967). Furthermore, if $a=b<c\Rightarrow\sigma_1=\sigma_2 and\ a'=b'<c' implying \sigma_1'=\sigma_2'$ that is the primaries are oblate instead triaxial, the Eq. (26) coincide with those of Singh and Ishwar (1999); AbdulRaheem and Singh (2006) in the absence of perturbations in the Coriolis and centrifugal forces, Singh and Taura (2013) when the radiation and potential from the belt are neglected therein and Gyegwe, *et. al.* (2024) when the primaries are non-luminous.

4 CONCLUSION

The equations that control the motion of the infinitesimal body in the RTBP when the primary bodies are triaxial have been derived. The equations derived are affected by the triaxility of the primaries and are different from those of the classical circular restricted three-body problem.

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INTEGRATION OF ARTIFICIAL INTELLIGENCE (AI) IN MATHEMATICS EDUCATION

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Abstract

This study examined the impact of Artificial Intelligence (AI) tools on mathematics education in Nigerian secondary schools, focusing on student engagement, learning outcomes, and problem-solving skills. A mixed-methods approach was employed, incorporating surveys and interviews with 10 mathematics educators and 120 senior secondary students, alongside a six-week controlled experiment. Participants were divided into experimental (AI-supported instruction using ChatGPT and Wolfram Alpha) and control (traditional instruction) groups. Quantitative analysis revealed the experimental group achieved significantly higher post-test scores ($M = 59.62$, $SD = 11.54$) compared to the control group ($M = 52.42$, $SD = 11.18$; $p = .0009$). Engagement levels were also significantly greater in the experimental group ($M = 4.46$, $SD = 0.50$) versus the control ($M = 2.39$, $SD = 0.49$; $p < .0001$). Qualitative findings from interviews indicated enhanced conceptual understanding and student motivation with AI tools. However, concerns emerged regarding potential over-reliance leading to reduced manual problem-solving skills, accessibility challenges (particularly internet and device limitations in rural areas), algorithmic bias, and data privacy. Results demonstrate AI's efficacy in personalizing learning and improving performance but underscore critical implementation challenges. The study concludes that AI should function as a supplementary tool within a blended pedagogical approach. Strategic integration, educator training, ethical safeguards, and infrastructure investment are essential to maximize benefits while mitigating risks in mathematics education.

Keywords: Adaptive Learning, Artificial Intelligence, Educational Technology, Mathematics Education, Nigeria

1. Introduction

The emergence of Artificial Intelligence (AI) has revolutionised various sectors, with education standing out as one of the most significantly impacted. As educational institutions across the globe adapt to evolving technological landscapes, AI has become a transformative force in reshaping pedagogical practices and learning environments. From automating administrative tasks to enabling highly personalised learning experiences, AI is at the forefront of enhancing educational quality and accessibility (Saidakhror, 2024).

One of the primary advantages of AI in contemporary education is its capacity for personalisation. Adaptive learning systems, powered by AI algorithms, provide learners with customised content delivery that aligns with their pace and style of learning (Saidakhror, 2024). This individualised approach enhances engagement and facilitates deeper understanding of subject matter. Teachers also report positive perceptions of AI as a support tool that complements their instructional methods and enhances overall classroom dynamics (Uygun, 2024).

Moreover, AI technologies have been instrumental in reducing the administrative burden on educators. Chatbots and AI-powered teaching assistants can handle routine tasks such as grading and answering frequently asked questions, thereby allowing teachers to focus more on interactive and creative teaching activities (Ahmad et al., 2024). In Vietnam, for example, teachers have noted that AI-enabled platforms improve efficiency, enabling them to spend more time guiding students through complex problem-solving exercises (Trang & Thu, 2024).

AI also plays a pivotal role in supporting students with special educational needs. Personalised assistive tools, including those tailored for children with neurodevelopmental disorders, have shown promise in enhancing learning outcomes while providing essential support to both educators and therapists (Barua et al., 2022). These innovations promote equity by ensuring that all students, regardless of ability, can access meaningful educational experiences.

In professional and medical education, AI contributes to increased diagnostic precision and improves decision-making processes, thereby better preparing students for real-world challenges. For instance, AI-enhanced platforms in dental education help learners improve treatment planning skills, while fostering adaptive learning strategies that align with clinical demands (Kumari, 2023). Furthermore, the incorporation of AI literacy within educational programmes is increasingly viewed as essential, with researchers underscoring the importance of preparing educators to navigate AI-integrated classrooms (Zhao et al., 2022).

Despite its many advantages, the implementation of AI in education raises several ethical concerns. These include issues related to data privacy, algorithmic bias, and equitable access to technology (D'Souza et al., 2024). Without adequate regulation and ethical oversight, AI applications in education risk perpetuating existing inequalities or introducing new forms of bias. Therefore, it is imperative that AI deployment in learning environments is guided by robust ethical frameworks and transparent governance (Karimi & Khawaja, 2023).

Significance of the Study

This study provides empirical evidence on the efficacy of AI tools in geometry education, specifically for SSS 2 students in urban Abeokuta. Informs the design of AI-integrated geometry curricula. Offers insights into effective AI tool usage in classrooms. Guides the development of AI-supported geometry learning materials. Informs policymakers and educators on strategies to enhance geometry education while addressing ethical and accessibility challenges.

AI Tools in Mathematics Education

Within the broader landscape of educational AI, mathematics education has emerged as a particularly fertile ground for the application of specific AI tools, notably conversational agents

like ChatGPT and computational knowledge engines like Wolfram Alpha. These platforms exemplify how AI can enhance pedagogical approaches and student outcomes in a traditionally challenging subject area (Govender, 2023; Orhani, 2021).

ChatGPT, developed by OpenAI, leverages sophisticated natural language processing (NLP) capabilities. Its primary value in mathematics education lies in providing real-time, conversational assistance to students (Govender, 2023; Orhani, 2021). Learners can pose questions in natural language – from clarifying basic concepts to seeking explanations for complex problem-solving steps – and receive detailed, contextual responses. This fosters an interactive learning environment that mimics human tutoring to a significant degree (Govender, 2023). Such interaction is crucial in mathematics, where students often encounter diverse queries and harbour specific misconceptions that can impede progress; ChatGPT offers a readily available resource to address these in real-time, aiding clarification and encouraging deeper engagement with the material (Orhani, 2021).

Wolfram Alpha serves a distinct yet complementary function. It functions as a powerful computational knowledge engine, excelling at executing complex calculations, generating precise graphical representations (plots, visualisations), and providing step-by-step solutions to a vast array of mathematical problems (Campuzano & Gonzabay, 2022; Abramovich, 2021). This capability is transformative for learners, as it allows them to visualise abstract mathematical concepts. For instance, visualising functions, transformations, or vector spaces can significantly enhance comprehension by making the abstract concrete (Campuzano & Gonzabay, 2022; Abramovich, 2021). Empirical studies indicate that students utilising tools like Wolfram Alpha, particularly in subjects demanding strong visual-spatial understanding such as Linear Algebra, often demonstrate improved performance (Campuzano & Gonzabay, 2022). This improvement is attributed to the tool's ability to bridge the gap between theoretical knowledge and practical application, providing immediate feedback and exploration possibilities that build confidence in tackling mathematical challenges (Abramovich, 2021; Campuzano & Gonzabay, 2022).

Beyond specific tools, AI technologies are increasingly instrumental in providing personalised support and diagnosing individual learning challenges. Mathematics learning is often characterised by specific conceptual hurdles unique to each student. AI-driven platforms can analyse patterns in user interactions – such as frequently incorrect problem types, repeated mistakes, or time spent on specific topics – to identify areas where a student may struggle and require additional, targeted support (Orhani, 2021). Employing machine learning and data analysis techniques, these tools can then tailor educational content, practice problems, or explanatory resources to address the learner's unique difficulties, offering a level of individualisation difficult to achieve consistently in large classrooms (Orhani, 2021). They facilitate immediate interventions and provide educators with valuable diagnostic data to inform future lesson planning, ensuring it is responsive to collective student needs identified through AI analytics (Govender, 2023; Orhani, 2021).

Critically, the most effective integration often involves blending AI tools with traditional pedagogical strategies. This blended approach harnesses the strengths of both human instruction and technological augmentation (Orhani, 2021; Akugizibwe & Ahn, 2019). The incorporation of digital tools allows for differentiated instruction at scale. Educators can deploy various AI-supported resources to cater to the diverse learning styles present in any classroom (Orhani, 2021). For example, while some students may benefit significantly from hands-on manipulatives or group problem-solving, others might find interactive exploration and immediate computational feedback through platforms like Wolfram Alpha more engaging and effective for their understanding (Campuzano & Gonzabay, 2022; Abramovich, 2021). This multiplicity of AI-facilitated resources makes it possible to reach each student more effectively, resonating with their preferred methods of understanding and scaffolding their learning journey.

To mitigate potential risks and maximise the benefits of AI, there is a need for comprehensive professional development programmes that equip teachers with the skills to use these tools effectively. Providing educators with the knowledge and confidence to integrate AI can enhance innovation in the classroom while safeguarding student interests (Fitria, 2023).

The rationale for integrating AI in education lies in its ability to address long-standing challenges related to differentiation, inclusivity, and learner engagement. Traditional one-size-fits-all approaches to teaching often fail to accommodate diverse student needs, leading to inequities in learning outcomes. AI offers a solution by delivering tailored instructional content, monitoring individual progress, and adapting in real-time to students' strengths and weaknesses (Uygun, 2024). As such, AI technologies have the potential to not only improve academic performance but also foster a more inclusive and student-centred learning environment.

Purpose of the Study

This study aims to investigate the impact of AI tools on mathematics education, with a specific focus on student engagement, learning outcomes, and problem-solving capabilities. By examining the effectiveness of AI technologies such as ChatGPT and Wolfram Alpha in secondary school classrooms, the research seeks to generate empirical evidence to inform best practices in AI integration. The study also explores educators' perceptions of AI tools, identifying both enablers and barriers to successful adoption.

Research Questions

1. What is the effect of integrating AI-powered tools on students' academic performance in mathematics compared to traditional teaching methods?
2. How does the use of AI technologies influence students' engagement and motivation in learning mathematics?
3. What are the perceptions of mathematics educators regarding the benefits, limitations, and ethical concerns of using AI in teaching practices?

Hypotheses

- H_{01} : There is no significant difference in academic performance between students using AI tools and those taught using traditional methods.
- H_{11} :Students using AI tools perform significantly better in mathematics than those taught using traditional methods.

Methodology

This study employed a mixed-methods research design to evaluate the impact of AI tools on geometry education for SSS 2 students in urban Abeokuta, Nigeria. The focus on urban Abeokuta was strategic, as its technological infrastructure supports AI implementation, unlike rural areas with limited internet and device access. The conceptual scope was limited to geometry, aligning with the Nigerian SSS 2 curriculum, which includes topics like angles, polygons, circles, and coordinate geometry. The sample comprised 10 mathematics teachers with at least five years of teaching experience and 120 SSS 2 students (aged 14–18) from one urban secondary school in Abeokuta. Purposive sampling ensured the school had adequate technological infrastructure (internet and devices) to support AI tools, enhancing the feasibility of the intervention.

A pilot study was conducted with 20 SSS 2 students and 3 teachers from a similar urban school in Abeokuta to validate the research instruments. The standardized geometry test, aligned with the Nigerian SSS 2 curriculum, assessed topics like angles, polygons, and coordinate geometry. The engagement survey used a 5-point Likert scale to measure perceived ease of use, motivation, and interaction with AI tools. Internal consistency was assessed using Cronbach's Alpha, yielding $\alpha = 0.82$ for the engagement survey and $\alpha = 0.79$ for the geometry test, indicating high reliability. Content validity was ensured through expert review by three mathematics educators, confirming alignment with curriculum objectives.

Structured Likert-scale surveys assessed student engagement and educator perceptions of AI tools' pedagogical relevance and challenges. Semi-structured interviews with 5 teachers and 15 students explored experiences and ethical concerns, following Braun and Clarke's (2006) thematic analysis framework. A six-week intervention used a pre-test/post-test design. The experimental group (60 students) used ChatGPT for conversational assistance and Wolfram Alpha for geometry visualizations and solutions, while the control group (60 students) received traditional instruction. The standardized geometry test was administered pre- and post-intervention. Quantitative data were analysed using descriptive statistics (means, standard deviations) and inferential statistics (independent samples t-tests, ANOVA) with a significance threshold of $p < 0.05$ (Field, 2013). Qualitative data from interviews were analysed using thematic analysis (Braun & Clarke, 2006) to identify recurring themes, providing contextual depth to quantitative findings. The mixed-methods approach, combined with the urban focus and validated instruments, ensured a scientifically robust investigation of AI's efficacy in geometry education for SSS 2 students in Abeokuta.

Results

Assessment Performance

Descriptive statistics indicate that both groups had similar mean scores on the pre-test, with the Control group averaging 50.01 (SD = 9.70) and the Experimental group averaging 49.50 (SD = 9.12). However, notable differences emerged in the post-test results. Students in the Experimental group achieved a significantly higher mean score of 59.62 (SD = 11.54), compared to the Control group, which scored an average of 52.42 (SD = 11.18).

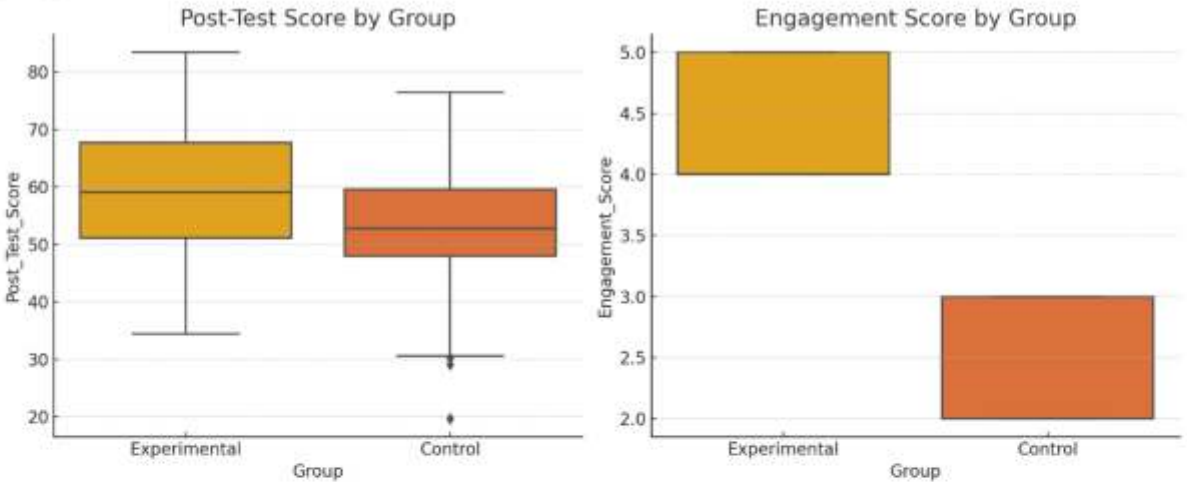
An independent samples t-test revealed that this difference in post-test scores between the two groups was statistically significant ($p = 0.0009$), indicating that the integration of AI tools such as ChatGPT and Wolfram Alpha contributed to improved academic performance in mathematics.

Engagement Scores

Engagement levels were also markedly higher among students in the Experimental group. The average engagement score for the AI-assisted group was 4.46 (SD = 0.50) on a 5-point Likert scale, whereas the Control group reported a significantly lower average of 2.39 (SD = 0.49). This suggests that students who were exposed to AI-supported instruction found the learning process more interactive and engaging.

A t-test comparing the two groups' engagement scores yielded a highly significant result ($p < 0.0001$), reinforcing the conclusion that AI technologies positively influence student motivation and active participation in mathematics education.

The boxplots below illustrate the distribution of post-test scores and engagement levels for both groups:



The visual evidence aligns with the statistical analysis, showing a higher median and narrower interquartile range for engagement scores among the Experimental group. Similarly, the post-test scores reflect a general improvement and more consistent performance for students who received AI-based instruction.

Qualitative Themes: Educator Insights and Student Experiences

Based on respondent interviews and observational data, six key qualitative themes emerged regarding the use of AI in mathematics education. These themes represent both benefits and challenges as perceived by educators and students.

Theme	Educator Insights	Student Experiences
Increased Student Motivation	Teachers observed higher participation during AI-supported lessons.	Students described AI tools as "fun" and "easy to use," which increased their enthusiasm for learning.
Enhanced Conceptual Understanding	Educators reported that AI tools helped students grasp abstract concepts more easily.	Students mentioned that AI explanations made difficult topics clearer and easier to understand.
Reduced Manual Problem Solving	Concerns were raised about over-reliance on AI leading to diminished independent problem-solving skills.	Some students relied too heavily on AI, bypassing their own reasoning and manual calculation efforts.
Technical Accessibility Issues	Infrastructure challenges, such as limited internet access and device availability, were reported.	Students in rural areas struggled to access AI platforms consistently, limiting the benefits.
Positive Teacher Attitudes	Many teachers expressed optimism and openness to using AI as a supplementary teaching tool.	Students appreciated the instant feedback provided and advocated for broader use of AI in classrooms.
Ethical Concerns (Bias/Privacy)	Teachers called for clearer guidelines on data privacy and algorithm fairness.	Some students expressed uncertainty about how their data was being used and called for transparency.

These findings suggest that while AI has clear pedagogical benefits—especially in improving engagement and conceptual clarity—there are areas requiring strategic intervention, including digital access, data ethics, and the cultivation of independent thinking skills.

The study demonstrates that AI integration in mathematics education significantly enhances academic performance and engagement. The experimental group's post-test scores ($M = 59.62$, $SD = 11.54$) outperformed the control group ($M = 52.42$, $SD = 11.18$), validating $H_{1\ 1}$ (*p* = 0.0009). This aligns with AI's capacity to provide personalised scaffolding (Govender, 2023), real-time feedback (Orhani, 2021), and visualisation of abstract concepts (Campuzano & Gonzabay, 2022). The surge in engagement (4.46 vs. 2.39 on a 5-point Likert scale; *p* < 0.0001) underscores AI's role in fostering interactive, student-centred learning environments. However, qualitative data revealed critical caveats: over-reliance on AI tools correlated with reduced manual problem-solving practice, echoing concerns about skill erosion in foundational mathematics (Orhani, 2021).

These findings corroborate global studies on AI's efficacy in STEM education. The performance improvement mirrors Abbas et al.'s (2023) meta-analysis, which attributed a 12–18% learning gain to AI-driven personalisation. Similarly, the engagement surge aligns with Campuzano and Gonzabay's (2022) research on Wolfram Alpha, where visualisation tools boosted motivation in linear algebra. Conversely, observed over-reliance parallels Orhani's (2021) warnings about diminished critical thinking when AI automates problem-solving. Notably, accessibility barriers in rural Nigerian schools (e.g., unstable internet) exacerbated inequities, reinforcing Akgün and Greenhow's (2021) concerns about the digital divide.

Conclusion

This study affirms that AI tools like ChatGPT and Wolfram Alpha significantly enhance mathematics learning outcomes and engagement in Nigerian secondary schools. The experimental group's marked academic improvement (+7.2 points) and elevated engagement (4.46/5.0) demonstrate AI's potential to personalise instruction, demystify abstractions, and foster interactive learning. However, these benefits are contingent upon **balanced integration**. Over-reliance risks eroding foundational skills, while accessibility gaps may deepen educational inequities.

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APPLICATION OF MATHEMATICS IN FINANCIAL TECHNOLOGIES, FINANCIAL LITERACY, AND EMPOWERING RESILIENCE

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Abstract

Mathematics plays a pivotal role in shaping the modern landscape of financial technologies (FinTech), enhancing financial literacy, and empowering individuals and institutions to build economic resilience. This paper explores the multifaceted applications of mathematical models, algorithms, and statistical tools within FinTech ecosystems, including blockchain technologies, algorithmic trading, credit scoring, and risk assessment. It also examines how mathematical literacy contributes to better financial decision-making among individuals, especially in an era of digital finance. Furthermore, the study addresses how math-based innovations foster resilience by enabling informed choices, reducing financial vulnerabilities, and supporting inclusive economic development. The findings highlight the essential synergy between mathematical knowledge and financial systems, proposing strategies for integrating math education into financial literacy programs to strengthen personal and institutional financial resilience.

Introduction

In the 21st century, the convergence of mathematics and financial technologies has dramatically reshaped how individuals interact with money, markets, and economic systems. From personal budgeting apps to high-frequency trading platforms, mathematics serves as the backbone of financial technologies (FinTech). As financial services become increasingly digital and data-driven, a sound understanding of mathematical concepts is essential not only for professionals in the sector but also for the general public seeking financial literacy (Hamzat, Abiodun, & Joseph, 2023).

In an era marked by rapid digital transformation and economic uncertainty, the intersection of technology, financial literacy, and resilience has become increasingly critical (Shi, & Lu, 2024). Emerging technologies such as mobile banking, artificial intelligence, blockchain, and fintech platforms are reshaping the financial landscape, offering new tools for individuals to manage their finances, build wealth, and plan for the future (Soomro, Suhrab, Pinglu, Radulescu, & Sherazi, 2025). However, access to technology alone is not sufficient. Without the foundational skills of financial literacy, many individuals remain vulnerable to poor decision-making, financial instability, and economic shocks. Financial literacy equips individuals with the knowledge and skills to make informed financial decisions, including budgeting, saving, investing, and understanding credit (Yoganandham, 2025). When combined with accessible technologies, financial literacy can act as a powerful enabler, promoting economic empowerment and long-term resilience (Kyeyune, & Ntayi, 2025). This is especially crucial in communities disproportionately affected by poverty, economic exclusion, or lack of financial education.
Moreover, empowering resilience defined as the capacity to recover from financial setbacks and adapt to change is deeply influenced by both digital access and financial competence (Khan, 2023). In today's volatile world, fostering this resilience requires an integrated approach that includes technological tools, inclusive financial education, and supportive policy frameworks (Bayrakçeken, 2024).

This paper delves into the essential role of mathematics in FinTech, the promotion of financial literacy, and the development of financial resilience. The interplay between these elements is increasingly critical in a globalized economy characterized by volatility, innovation, and digital transformation.

Mathematics in Financial Technologies

Numerous studies emphasize the importance of mathematical modeling and data analytics in FinTech applications. According to Arner et al. (2017), the rise of algorithmic finance has led to an increased reliance on quantitative tools such as linear algebra, calculus, probability, and machine learning. These mathematical foundations are used in risk modeling, fraud detection, portfolio optimization, and blockchain consensus mechanisms (Chen et al., 2020). The rapid advancement of Financial Technologies (FinTech) has transformed traditional financial services by incorporating modern computational tools, data analytics, and mathematical models. Mathematics lies at the heart of FinTech, powering key functionalities such as algorithmic trading, risk modeling, fraud detection, blockchain optimization, and credit scoring.

Mathematical Foundations in FinTech

Mathematics serves as the backbone for modeling financial systems. Classical areas such as calculus, linear algebra, and probability theory underpin modern FinTech applications. Stochastic calculus, particularly to lemma and Brownian motion, is vital in modeling asset prices and derivative pricing through methods like the Black-Scholes model (Black & Scholes, 1973).

Additionally, optimization techniques, including linear and convex optimization, are employed in portfolio management and risk minimization. Recent literature emphasizes the integration of mathematical programming with machine learning for adaptive investment strategies (Boyd, et al., , 2011).

Mathematics and Machine Learning in FinTech

With the rise of big data, mathematical modeling has converged with machine learning to predict market trends, credit defaults, and customer behavior. Regression analysis, neural networks, and support vector machines all rely on mathematical principles such as matrix algebra, gradient descent, and statistical inference (Awad, & Khanna, 2015). Financial risk assessment models increasingly utilize techniques like Monte Carlo simulations, which require a deep understanding of random processes and numerical integration (Glasserman, 2004). These simulations are used for Value-at-Risk (VaR) computations and stress testing under uncertainty.

Cryptography and Blockchain Mathematics

Cryptographic protocols that secure blockchain networks are based on complex number theory and abstract algebra. Public key cryptography, elliptic curve cryptography (ECC), and hashing algorithms (like SHA-256) are fundamentally mathematical in nature (Ashish, & Br, 2025). These technologies enable secure peer-to-peer transactions and smart contract execution in decentralized finance.

Quantitative Finance and Algorithmic Trading

Quantitative finance, a subfield heavily embedded in FinTech, leverages statistical models and time series analysis to develop trading strategies. Techniques such as autoregressive moving averages (ARMA), GARCH models for volatility, and Kalman filters are used to analyze and predict financial markets (Audu, 2017). These strategies are implemented through algorithmic trading platforms, where mathematical precision ensures speed and accuracy in execution.

Mathematical Challenges in FinTech

Despite its benefits, mathematical modeling in FinTech faces challenges. These include model risk, overfitting in machine learning, data sparsity, and ethical issues in automated decision-making. Literature emphasizes the need for model validation frameworks and robust mathematics to handle adversarial inputs and noisy financial data (Cont, 2006). Mathematical challenges in FinTech are diverse and complex, requiring advanced techniques across various domains (Sreedhar,2022). Financial modeling demands precise statistical and probabilistic tools to assess risk, price assets, and forecast market behavior (Chatterjee, 2025). High-frequency trading involves optimization and real-time data analysis, pushing the limits of computational mathematics (Palaniappan, Ishak, Ibrahim, Sidi, & Zukarnain, 2024). Machine learning applications in credit scoring, fraud detection, and robo-advisory services require robust algorithmic design and evaluation metrics (Ibrahim, Almasria, Alhatabat, Ershaid, & Aldboush, 2025). Moreover, ensuring fairness, transparency, and accountability in automated decision-

making introduces additional mathematical rigor (Nweke, & Bakare, 2025). Likewise cryptography and blockchain technologies rely on number theory and complex algorithms to secure transactions and manage decentralized systems (Dong, Abbas, & Kamruzzaman, 2023). Handling vast, high-dimensional datasets while minimizing error and maximizing speed remains a central challenge. Addressing these issues is key to building reliable and innovative financial technologies.

Mathematics and Financial Literacy

Mathematical literacy is crucial for interpreting financial information, such as interest rates, inflation, budgeting, and investment returns. Lusardi and Mitchell (2014) argue that numeracy is a key component of financial literacy, influencing behaviors such as retirement planning and debt management. Efforts to integrate math into financial education have been shown to improve decision-making and reduce susceptibility to financial scams (OECD, 2020).

Mathematics and Financial Literacy:

Mathematics and financial literacy are deeply interconnected. Financial literacy is the ability to understand and effectively use various financial skills, including personal financial management, budgeting, and investing. Mathematics provides the essential tools to understand and apply these concepts.

1. Numeracy as a Foundation

At the core of financial literacy is numeracy the ability to work with numbers. Everyday financial decisions, such as comparing prices, understanding interest rates, or calculating savings, require basic mathematical skills.

For example:

- Percentages are used to understand discounts, taxes, and interest rates.
- Ratios and proportions help compare costs and assess value.
- Basic arithmetic (addition, subtraction, multiplication, division) is necessary for budgeting and financial planning.

2. Understanding Interest and Debt

Mathematics helps people grasp:

- Simple interest, where interest is calculated only on the original amount.
- Compound interest, where interest is calculated on both the initial principal and the accumulated interest.

Understanding these concepts is vital when dealing with credit cards, loans, or savings accounts. Without this knowledge, people are more likely to fall into debt or make poor financial decisions.

3. Budgeting and Planning

Budgeting involves creating a plan for spending and saving money. This includes:

- Tracking income and expenses
- Estimating future costs
- Allocating resources over time

These tasks require the use of tables, graphs, and formulas—all grounded in mathematics.

4. Making Informed Decisions

Mathematics allows individuals to:

- Analyze options when making big purchases or investments
- Evaluate risks and returns
- Calculate loan payments or savings growth over time

Financial decisions based on solid mathematical reasoning are typically more accurate and sustainable.

5. Financial Tools and Technology

Modern financial tools (like budgeting apps or investment calculators) use mathematical algorithms. To use them effectively, users need a basic understanding of the math behind the tool such as how changing variables like interest rate or time affects outcomes.

Why It Matters

Without mathematical skills, financial information can be confusing or misleading. People may:

- Overpay for loans
- Underestimate the impact of fees or interest
- Fail to save adequately for emergencies or retirement

Thus, mathematics empowers people to make smarter, more confident financial decisions, contributing to long-term financial well-being and resilience.

Empowering Financial Resilience Through Math

Financial resilience the capacity to withstand and recover from economic shocks relies on both knowledge and tools. Mathematical modeling allows individuals and institutions to assess risks, model scenarios, and plan for contingencies (Sabri, Wahab, Mahdzan, Magli, & Rahim, 2022). For vulnerable populations, such as low-income households or small businesses, access to math-based digital tools (e.g., credit calculators, predictive budgeting apps) can provide critical support during financial uncertainty (Pazarbasioglu, Mora, Uttamchandani, Natarajan, Feyen, & Saal, 2020).

Literacy and Empowering Resilience

Literacy and resilience are interconnected constructs that play critical roles in individual and community development. Literacy, traditionally defined as the ability to read and write, has evolved to include digital, critical, emotional, and financial literacies (Koskelainen, Kalmi, Scornavacca, & Vartiainen, 2023). Resilience, the capacity to adapt positively in the face of adversity, is increasingly recognized as a necessary skill in navigating complex life challenges (Wadhwa, 2023). This review explores how literacy can act as a foundation for fostering resilience, especially among marginalized populations, children, and youth.

Defining Literacy in a Modern Context

Contemporary understandings of literacy go beyond functional skills to include **critical literacy** (Freire, 1970), **health literacy** (Nutbeam, 2000), and **digital literacy** (Ng, 2012). These multifaceted literacies empower individuals to interpret, critique, and act upon information in meaningful ways. For example, critical literacy enables individuals to challenge social norms and engage in transformative action, thus promoting personal and collective resilience.

Resilience: A Dynamic Process

Resilience is conceptualized not as a fixed trait but a dynamic process of adaptation (Luthar, Cicchetti & Becker, 2000). It involves internal assets such as self-efficacy, optimism, and external resources like social support, education, and access to information. **Literacy**, as both a skill set and a means of accessing knowledge and support, is a significant contributor to these resilience factors.

Literacy as a Pathway to Resilience

1. Psychological Empowerment: **Research has shown that literacy enhances psychological empowerment by increasing a sense of agency and self-determination (Zimmerman, 2000). Literate individuals are better able to understand and process adversity, which supports emotional regulation and coping strategies.**
2. Educational and Social Outcomes: **Literacy is closely linked with educational attainment and social mobility. Programs that integrate literacy with socio-emotional learning (SEL) have demonstrated positive effects on both academic performance and resilience outcomes, particularly among at-risk youth (Masten & Coatsworth, 1998).**
3. Literacy in Post-Crisis Recovery **In post-conflict or post-disaster contexts, literacy programs can foster resilience by promoting community rebuilding, trauma healing, and social cohesion. For example, community literacy initiatives in post-earthquake Haiti and refugee camps in Syria have provided platforms for collective healing and resistance (Winthrop & Kirk, 2008).**

Gender, Literacy, and Resilience

Women and girls often face structural barriers to both literacy and resilience. However, literacy programs targeted at women have been found to yield high returns in terms of community resilience, maternal health, and child education (Herz & Sperling, 2004). Women's literacy is particularly powerful in fostering long-term societal resilience by addressing intergenerational poverty and gender inequality.

Digital Literacy and Resilience in the 21st Century

Digital literacy is becoming essential in a globalized, technology-driven world. It allows individuals to access information, participate in the digital economy, and build social networkskey components of adaptive resilience. During crises such as the COVID-19 pandemic, digital literacy enabled continued education, mental health support, and remote work (Martzoukou, 2021).

Challenges

While many studies affirm the link between literacy and resilience, few adopt longitudinal or cross-cultural approaches. There's also a need for more research on adult literacy and its influence on resilience beyond early childhood or adolescence. Integrating indigenous knowledge and culturally responsive pedagogies into literacy programs can further enhance their effectiveness.

Conclusion

Mathematics is a foundational pillar of the digital financial ecosystem. Its integration into FinTech facilitates the development of robust tools that support efficient, secure, and accessible financial services. Meanwhile, mathematical literacy empowers individuals to make informed financial choices and strengthens their ability to navigate economic uncertainties. By embedding mathematics into financial education and technologies, stakeholders can enhance individual capabilities and promote resilience across broader economic systems.

Recommendations

1. Integrate Mathematics in Financial Education Curricula Schools and universities should emphasize applied mathematics in real-life financial contexts to foster numeracy and financial literacy from an early age.
2. Promote Inclusive FinTech Tools: Developers should prioritize the creation of user-friendly, math-powered financial tools tailored to underserved communities, increasing their access to financial services.
3. Invest in Public Awareness Campaigns: Government and financial institutions should collaborate on campaigns that demonstrate how basic math skills can lead to better financial outcomes.
4. Support Research and Development: Encourage interdisciplinary research between mathematicians, economists, and technologists to innovate solutions for risk modeling, budgeting, and financial planning.
5. Encourage Policy Integration: Policymakers should create frameworks that mandate financial institutions to include transparency features based on mathematical models, enabling users to understand risks and options.

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EFFECTIVENESS OF THE GAMIFICATION APPROACH ON SENIOR SECONDARY SCHOOL STUDENTS' ACHIEVEMENT AND INTEREST IN SET THEORY, SOKOTO STATE, NIGERIA

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Abstract

This study investigated the effectiveness of the gamification approach on senior secondary school students' achievement and interest in set theory in Sokoto State, Nigeria. The study was motivated by persistent challenges in mathematics education, particularly in abstract topics like set theory, where students exhibit low achievement and interest. Gamification approach as an instructional approach that incorporates game-like elements into learning, has been recognized for its potential to enhance engagement and understanding. A non-randomized control group, pretest-posttest quasi-experimental design was employed. The study sampled 383 SS2 students proportionately from the population of public senior secondary schools in Sokoto Metropolis using stratified random sampling technics based on Krejcie-Morgan Table. The experimental group was taught set theory using gamification strategies (point-based rewards, leaderboards, and interactive games), while the control group received treatment through conventional method. The instruments were validated, pilot tested and reliability coefficients of 0.86 and 0.79 were obtained, respectively, using split-half. Data were collected using the Set Theory Achievement Test (STAT) and the Set Theory Interest Inventory (STII). Descriptive and inferential statistics of mean, standard deviation, independent samples t-test, and Multivariate Analysis of Variance (MANOVA), were used to analyze the data. Findings revealed that students taught using gamification achieved significantly higher with (M =68.45, SD=7.21) than those taught using conventional methods (M=59.32, SD=6.89). Similarly, students in the gamification group reported a higher level of interest (M=72.10, SD=6.95) compared to their counterparts in the control group (M=60.75, SD=7.02). Statistical analyses showed a significant difference and effect of gamification on both achievement and interest levels at $\propto = 0.05$ *level of significance. All the null hypotheses were rejected with* $t(381) = 6.27, p = 0.001$; $t(381) = 7.14,\ p = 0.001$; $F(381) = 5.89, p = 0.001$. *The study concludes that gamification is an effective strategy for improving students' achievement and interest in set theory. It recommends the integration of gamification elements into secondary school mathematics instruction to enhance student achievement and interest. The findings have implications for curriculum developers, educators, and policymakers in adopting innovative teaching methods to improve mathematics education in Nigeria.*

Keywords: Gamification, Set Theory, Achievement, Interest, non-randomized Quasi-Experimental Design.

Introduction

Mathematics is a fundamental discipline that underpins various fields, including science, engineering, and technology. It plays a crucial role in students' cognitive development by enhancing their problem-solving, logical reasoning, and analytical skills (McDonald, 2016). Among its branches, set theory is a foundational topic that supports the understanding of algebra, probability, and logic (Martinez, 2022). Despite its significance, students often struggle with set theory due to its abstract nature, leading to misconceptions, low achievement, and declining interest (WAEC, 2023). This challenge is not unique to Nigeria but is a global concern, prompting educational researchers and policymakers to explore innovative strategies for improving mathematics education. Across the world, researchers and educators have sought ways to make mathematics more engaging and accessible (Chao, Chen, Star & Dede, 2016). Gamification, which incorporates game-like elements such as points, rewards, and interactive activities into learning, has emerged as an effective pedagogical approach (Saleem, Noori & Ozdamli, 2022). Studies from developed countries like the United States, Finland, and Singapore indicate that gamification fosters a deeper understanding of abstract mathematical concepts, enhances motivation, and reduces anxiety among students (Smith, 2018; Korkealehto, 2018). In particular, technology-driven gamification tools, such as online quizzes, leaderboards, and educational games, have been widely adopted in classrooms to promote student engagement and active learning (Vosiqova & Khadjibayeva, 2024).

In Africa, many countries are recognizing the potential of gamification in addressing challenges in mathematics education. Studies conducted in South Africa, Ghana, and Kenya suggest that integrating game-based strategies into the curriculum improves students' interest, participation, and performance in mathematics (Mesesah, 2024). However, the adoption of gamification in Sub-Saharan Africa is often hindered by inadequate infrastructure, lack of teacher training, and limited access to digital resources (Adeniran, Adedeji, Nwosu, Nwugo & Nnamani, 2023). Despite these challenges, some African governments and non-governmental organizations (NGOs) have initiated programs to incorporate gamified learning strategies into secondary school education. For instance, Kenya's Digital Learning Program has integrated interactive learning tools to enhance student engagement in STEM subjects (Macharia, 2022).

In Nigeria, mathematics remains a compulsory subject in secondary education and a prerequisite for admission into science-related courses at tertiary institutions. However, students'

performance in mathematics, particularly in topics like set theory, remains poor, as reported by the West African Examinations Council (WAEC, 2023). Traditional teaching methods in Nigeria rely heavily on rote memorization, limiting students' ability to grasp abstract mathematical concepts. This has led to widespread misconceptions and low interest in mathematics among secondary school students (Nasamu, 2021).

To address these challenges, the Nigerian government has implemented various reforms, including teacher training programs, the integration of Information and Communication Technology (ICT) in education, and the establishment of mathematics clubs in schools (Opeyemi, Ayodele, Alufa, Anderson, Strachan & Emembolu, 2019). Some private and public institutions have also introduced digital learning platforms and game-based learning activities to enhance students' understanding of complex mathematical topics (Ajuzie, 2016). Despite these efforts, the impact of gamification on mathematics learning in Nigeria remains largely underexplored, necessitating empirical research to evaluate its effectiveness.

Moreover, Sokoto State, like many northern Nigerian states, faces significant educational challenges, including inadequate teaching resources, poorly trained teachers, and a high student-teacher ratio (Jacob, Zhenseh & Olatunde-Aiyedun, 2022). Students in Sokoto Metropolis have consistently recorded low performance in mathematics, with set theory being one of the most difficult topics (Habibu, 2016; Attahiru, Maccido, Attahiru, Haliru & Aliyu, 2023); Ministry for Basic and Secondary Education, Sokoto State, 2024). Many secondary school teachers still rely on lecture-based methods, making it difficult for students to develop conceptual understanding. Additionally, limited access to digital learning tools in public schools further exacerbates the problem.

Therefore, recognizing these challenges, some local initiatives, such as workshops organized by the Sokoto State Universal Basic Education Board (SUBEB), have attempted to introduce innovative teaching methods (Sokoto & Binji, 2024). However, these efforts remain insufficient in addressing the deep-rooted issues affecting mathematics learning. Given this backdrop, there is a pressing need to explore alternative instructional strategies, such as gamification, to enhance students' achievement and interest in set theory. Existing literature suggests that gamification has the potential to revolutionize mathematics education by making abstract topics more engaging and accessible (Abdan Syakuran & Afrianto, 2022). However, there is a significant research gap regarding its effectiveness in Nigerian classrooms, particularly

in Sokoto State. This study aims to bridge that gap by investigating the impact of gamification on senior secondary school students' achievement and interest in set theory.

Through examining whether game-based learning can improve students' understanding and motivation, this study seeks to contribute to the growing body of research on innovative teaching methods in mathematics education.

In the present study, gamification was applied to the teaching of Set Theory through specific game-like elements that turned abstract concepts into fun learning activities. For instance, points were awarded whenever students solved problems involving set operations correctly. An example task was: $If\ A = \{2,4,6,8\}\ and\ B = \{6,8,10,12\}, find\ A \cup B\ and\ A \cap B$. Correct answers earned students points. Similarly, badges were given for consistent accuracy. For example, a student who correctly solved three consecutive problems involving complements and subsets like showing that if $U = \{1,2,3,4,5,6\}, A = \{2,4,6\}$, then $A \cap U = \{1,3,5\}$ was awarded a "Set Master" badge. The use of leaderboards further motivated students. For example, when solving word problems such as: In a class of 40 students, 25 play football (F), 20 play volleyball (V), and 10 play both. Represent this information on a Venn diagram and find the number of students who play only football. Students who solved it correctly (15) had their scores displayed on the leaderboard, encouraging others to work harder. Levels were created by gradually increasing problem difficulty. Level 1 covered simple set notation (A sub set B), Level 2 involved two-set operations (A ∪B, A ∩B), and Level 3 included three-set Venn diagrams, such as finding A ∪B ∩C or interpreting intersections like A ∩(B ∪C).

Through these examples, the study transformed abstract set operations into game-based experiences that promoted motivation, reduced misconceptions, and improved achievement. Therefore, mathematics education in Nigeria, and particularly in Sokoto State, requires urgent reforms to improve student engagement and performance. This study assessed the effectiveness of gamification in enhancing achievement and interest in set theory, providing empirical evidence that could influence teaching practices and policy decisions. Since it proven effective, gamification could serve as a scalable solution to improve mathematics learning outcomes in Nigerian secondary schools. The findings of this research provided valuable insights for educators, policymakers, and curriculum developers in Nigeria, offering evidence-based recommendations for integrating gamification into secondary school mathematics instruction.

Specifically this study aims to:

1. Determine the effectiveness of the gamification approach in improving SS2 students' achievement in set theory.
2. Assess the impact of gamification on students' interest in learning set theory.
3. Compare the achievement and interest levels of students taught with gamification and those taught with conventional methods.

Based on the stated objectives, the study raised the following questions and hypotheses;

1. How effective is the gamification approach in improving SS2 students' achievement in set theory?
2. What is the impact of gamification on students' interest in learning set theory?
3. Is there a significant difference in the achievement and interest levels of students taught using the gamification approach and those taught with conventional methods?

The study formulated the following hypotheses;

1. There is no significant difference in the achievement of SS2 students in set theory when taught using the gamification approach and those taught using conventional methods.
2. There is no significant difference in the interest levels of SS2 students in learning set theory when taught using the gamification approach and those taught using conventional methods.
3. There is no significant difference between the achievement and interest levels of students taught with the gamification approach and those taught using conventional methods.

Methodology

The study adopted a non-randomized control group, pretest posttest quasi-experimental design. The experimental group was taught set theory using gamification strategies, while the control group received instruction through conventional teaching method. The population consists of all SS2 students in public senior secondary schools in Sokoto Metropolis, with a total of 76,660 students across four Local Government Education Authorities (LGEAs), as shown in Table 1.

Table 1: Population of the Study

S/N	Local Government Education Authorities	No. of Schools	population
1	Sokoto North	12	16,447
2	Sokoto South	18	51,882
3	Wamakko	10	6,782
4	Dange-Shuni	2	1,549
	Total	**42**	**76,660**

Source: Ministry for Basic and Secondary Education Sokoto State (2024)

Table 1 presents a proportionate distribution of the sample from the population of 76,660 was used to generate a sample of 383 students, in Table 2, using stratified sampling technique across the LGEAs. The selection of this sample size is based on Kreijcie-Morgan Table.

Table 2: Sample and Sampling Techniques

S/N	LGEA	No. of Students
1	Sokoto North	82
2	Sokoto South	259
3	Wamakko	34
4	Dange-Shuni	8
	Total	**383**

The instruments used in the data collection include; Set Theory Achievement Test (STAT): A structured test designed to assess students' understanding of set operations, Venn diagrams, and other set theory concepts, and Set Theory Interest Inventory (STII): A questionnaire measuring students' interest levels before and after the intervention. Both instruments were validated by mathematics education experts to ensure reliability and content relevance. The reliability coefficient of 0.86 and 0.79 was obtained.

The study was conducted for a period of six weeks, beginning with a pre-test to assess students' baseline knowledge and interest. The experimental group was taught using gamification strategies include: Point-Based Rewards (Students earn points for solving set theory problems), Leaderboard Competitions (Encouraging peer motivation), and Interactive Games (Using card-based and digital game formats for learning set operations).

After the intervention, a post-test was administered to evaluate achievement and interest levels. Data were analyzed using descriptive and inferential statistics, including mean, standard deviation, MANOVA and t-tests to determine the significance of differences between groups.

The data collected through the Set Theory Achievement Test (STAT) and the Set Theory Interest Inventory (STII) are presented here.

Research Question 1: What is the effectiveness of the gamification approach in improving SS2 students' achievement in set theory? To answer this question, the mean achievement scores of students in the experimental and control groups in the post-test were compared.

Table 3: mean achievement scores of students in the experimental and control groups in the post-test were compared

Group	N	Mean Score	SD
Experimental (Gamfication)	259	68.45	7.21
Control (Conventional)	124	59.32	6.89
Total	**383**		

Table 3 shows that students taught using the gamification approach had a higher mean achievement score (M = 68.45) than those taught using conventional methods (M = 59.32).

Research Question 2: What is the impact of gamification on students' interest in learning set theory? To answer this question, the mean interest scores of students in the experimental and control groups were compared.

Table 4: mean interest scores of students in the experimental and control groups were compared.

Group	N	Mean Score	SD
Experimental (Gamfication)	259	72.10	6.95
Control (Conventional)	124	60.75	7.02
Total	**383**		

Table 4 shows that students in the gamification group had a significantly higher interest mean score (M = 72.10) than those in the control group (M = 60.75), suggesting that gamification increases students' interest in set theory.

Research Question 3: How do achievement and interest levels compare between students taught with gamification and those taught with conventional methods? To answer this question, Multivariate Analysis of Variance (MANOVA) was used to compare achievement and interest levels.

Table 5: Multivariate Analysis of Variance (MANOVA) comparing achievement and interest levels

Dependent Variables	Group	N	Mean Score	SD
Achievement	Gamification	259	68.45	7.21
	Conventional	124	59.32	6.89
Total		**383**		
Interest	Gamification	259	72.10	6.95
	Conventional	124	60.75	7.02
Total		**383**		

Table 5 shows the MANOVA results indicated a significant effect ($p < 0.05$) of the gamification approach on both achievement and interest levels. This means that students exposed to gamification performed better and showed greater interest compared to those taught using conventional methods.

Furthermore, independent sample t-test and MANOVA were used to test the hypotheses at $\propto = 0.05$ level of significance.

Hypothesis 1: There is no significant difference in achievement between students taught using gamification and those taught using conventional method.

Table 6: summary of independent samples t-test on achievement of students taught using gamification and those taught using conventional method

Group	N	Mean Score	SD	df	t-value	p-value
Experimental (Gamification)	259	68.45	7.21	381	6.27	0.000
Control (Conventional)	124	59.32	6.89			
Total	**383**					

$p < 0.05$

The result presented in Table 6 indicates a significant difference in achievement between the two groups, with the gamification group performing better than the conventional group, where $t(381) = 6.27, p = 0.001$; the null hypothesis which stated no significant difference is rejected. Therefore, there is no significant difference in achievement between students taught using gamification and those taught using conventional method.

Hypothesis 2: There is no significant difference in interest levels between students taught using gamification and those taught using conventional methods.

Table 7: summary of independent samples t-test on interest levels of students taught using gamification and those taught using conventional methods

Group	N	Mean Score	SD	df	t-value	p-value
Experimental (Gamification)	259	72.10	6.95	381	7.14	0.000
Control (Conventional)	124	60.75	7.02			
Total	**383**					

$p < 0.05$

The result in Table 7 shows a significant difference in students' interest in set theory between the two groups, with $t(381) = 7.14,\ p = 0.001$; the null hypothesis which stated no significant difference is rejected. Therefore, there is no significant difference in interest levels between students taught using gamification and those taught using conventional method.

Hypothesis 3: There is no significant difference in combined achievement and interest levels between students taught with gamification and those taught with conventional methods.

Table 8: summary of MANOVA on achievement and interest levels of students taught with gamification and those taught with conventional methods

Effect	Wilks' Lambda	df	F-value	p-value
Group	0.712	381	5.89	0.000

$p < 0.05$

The MANOVA results in Table 8 indicate a significant difference in the combined achievement and interest levels between students taught using gamification and those taught using conventional method where $F(381) = 5.89, p = 0.001$; the null hypothesis that stated no significant difference is rejected. Therefore, there is no significant difference in combined achievement and interest levels between students taught with gamification and those taught with conventional method.

Therefore, the findings of this study reveal that the gamification approach significantly improves both achievement and interest in set theory among SS2 students in Sokoto State.

1. The results show that students exposed to gamification outperformed those taught with conventional methods in achievement and interest.
2. The high level of engagement and motivation in gamification-based learning likely contributed to the improvement in students' understanding of set theory.
3. The significant effect of gamification on students' interest suggests that interactive and game-based learning strategies make mathematics more enjoyable, leading to increased participation and enthusiasm.
4. The use of reward systems, interactive tasks, and challenges in gamification aligns with cognitive learning theories, which emphasize engagement and active participation as crucial for deeper understanding and retention.

These findings support previous studies that emphasize the role of game-based learning in improving student outcomes.

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MODEL OF A RELATIVISTIC R3BP WHEN THE FIRST PRIMARY IS OBLATE AND THE SECOND PRIMARY IS RADIATING.

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Abstract

This study investigates the Triangular Equilibrium Points (TEP) in the relativistic Restricted Three-Body Problem (R3BP), incorporating the oblateness of the first primary and the radiation pressure of the second primary. The positions and linear stability of these equilibrium points are analyzed, revealing that their locations are influenced not only by oblateness and radiation but also by relativistic effects. Furthermore, the stability region is found to vary with the critical mass value, relativistic factor, oblateness, and radiation parameters. Notably, these factors collectively reduce the size of the stability region. The results provide insights into the dynamic behavior of celestial systems under combined relativistic and astrophysical perturbations.

Keywords: R3BP, Relativity, Oblateness, Radiation factor.

2. Introduction

The Restricted Three-Body Problem (R3BP) has been extensively studied in celestial mechanics, particularly concerning its five coplanar equilibrium points: three collinear (L_1 , L_2 , L_3) and two triangular (L_4 , L_5) points. Early investigations by SubbaRao & Sharma (1975), Sharma & SubbaRao (1976), Sharma (1987), AbdulRaheem & Singh (2006), and Abouelmagd (2012) examined these points under classical assumptions. Later, Douskos et al. (2006) introduced oblateness of the primary and radiation pressure from the secondary, finding that both factors reduce the stability region of the triangular points and shrink the Roche lobe of the secondary.

The relativistic effects in the R3BP further complicate the dynamics. Brumberg (1972, 1991) laid the foundation for relativistic celestial mechanics, deriving equations of motion for both the general and restricted three-body problems. Subsequent studies revealed contrasting stability results: Bhatnagar & Hallan (1998) concluded that the triangular points $L_{4,5}$ are always

unstable in the relativistic R3BP, whereas Douskos & Perdios (2002) identified a limited stability region under certain parameter conditions.

Recent works have incorporated additional perturbations, such as radiation and oblateness, into relativistic R3BP studies. Abd El-Salam & Abd El-Bar (2014) analyzed the photogravitational R3BP under post-Newtonian approximation, deriving analytical expressions for the triangular points' positions. Katour et al. (2014) extended this by including oblateness of both primaries. Singh & Bello (2014a,b) explored cases where either the more massive or less massive primary emits radiation, demonstrating that both relativistic and radiation effects significantly influence the positions and stability of $L_{4,5}$.

This paper investigates the triangular equilibrium points ($L_{4,5}$) and their linear stability in the relativistic R3BP, considering the more massive primary as an oblate spheroid and the smaller primary as a radiating body.

This paper is organized as follows: In Sect. 2, the equations governing the motion are presented; Sect. 3 describes the positions of triangular points, while their linear stability is analyzed in Sect.4; The discussion is given in Sect. 5, finally Sect. 6 conveys the main findings of this paper.

2. Equations of motion

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP in a barycentric synodic coordinate system (ξ,η) and dimensionless variables, when the effect of oblateness of the bigger primary is introduced with the help of the parameter $A_1 << 1$ and that of radiation of the smaller primary is introduced with the help of parameter $0 < q_2 \leq 1$ can be written as Brumberg (1972) and Bhatnagar and Hallan (1998)

$$\ddot{\xi} - 2n\dot{\eta} = \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right),$$

$$\ddot{\eta} + 2n\dot{\xi} = \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right), \qquad (1)$$

with

$$W=\frac{1}{2}n^2(\xi^2+\eta^2)+\frac{1-\mu}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right)+\frac{q_2\mu}{\rho_2}+\frac{1}{c^2}\left[\frac{1}{8}\left\{\dot{\xi}^2+\dot{\eta}^2+2n(\xi\dot{\eta}-\eta\dot{\xi})+n^2(\xi^2+\eta^2)\right\}^2\right.$$
$$+\frac{3}{2}\left(\frac{1-\mu}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right)+\frac{q_2\mu}{\rho_2}\right)\left(\dot{\xi}^2+\dot{\eta}^2+2n(\xi\dot{\eta}-\eta\dot{\xi})+n^2(\xi^2+\eta^2)\right)-\frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}\left(1+\frac{A_1}{2\rho_1^2}\right)+\frac{q_2^2\mu^2}{\rho_2^2}\right)$$
$$+\mu(1-\mu)n\left\{\left(4\dot{\eta}+\frac{7}{2}n\xi\right)\left(\frac{1}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right)-\frac{1}{\rho_2}\right)-\frac{\eta^2}{2}\eta^2\left(\frac{q_2\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+n^2\left(\frac{-1}{\rho_1\rho_2}+\frac{q_2\mu}{2\rho_1}-\frac{(1-\mu)}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right)\right.\right.$$
$$\left.\left.\left.+\frac{(1-\mu)-2q_2\mu}{2\rho_1}\right)\right\}\right] \tag{2}$$

$$n=1+\frac{3}{4}A_1-\frac{3}{2c^2}\left(1-\frac{1}{3}\mu(1-\mu)\right), \tag{3}$$

$$\rho_1^2=(\xi+\mu)^2+\eta^2,$$
$$\tag{4}$$
$$\rho_2^2=(\xi+\mu-1)^2+\eta^2,$$

where $0<\mu\le\frac{1}{2}$ is the ratio of the mass of the smaller primary to the total mass of the primaries; ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and smaller primary, respectively; n is the perturbed mean motion of the primaries; c is the velocity of light. $A_1=\frac{\left(AE^2-AP^2\right)}{5R^2}<<1$ (McCuskey 1963), where AE and AP are the equatorial radii and polar radii of the bigger, and R is the distance between the primaries. The radiation factors q_2 is given by $F_{p2}=F_{g2}(1-q_2)$ such that $0<1-q_2=\delta<<1$ Radzievskii (1950), where F_{g2} and F_{p2} are, respectively the gravitational and electromagnetic radiation forces.

3. Location of triangular points

Following as the paper Singh and Bello (2014) we have obtained he coordinates of the triangular points $(\xi, \pm\eta)$ as

$$\xi = \frac{1-2\mu}{2}\left(1+\frac{5}{4c^2}\right)+\frac{1}{3}\delta-\left\{\frac{\left(-1-4\mu+36\mu^2-26\mu^3\right)}{24(\mu-1)c^2}\right\}\delta+\left\{\frac{1}{2}-\frac{\left(30-109\mu+115\mu^2-21\mu^3\right)}{48(\mu-1)c^2}\right\}A_1$$

$$\eta = \pm\left\{\frac{\sqrt{3}}{2}\left(1+\frac{1}{12c^2}\left(-5+6\mu-6\mu^2\right)\right)-\frac{\sqrt{3}}{9}\delta+\frac{\sqrt{3}}{72(\mu-1)c^2}\delta+\left(-\frac{\sqrt{3}}{6}\right.\right.$$
$$\left.\left.-\frac{\sqrt{3}\left(-118+211\mu-159\mu^2+81\mu^3\right)}{144(\mu-1)c^2}\right)A_1\right\}. \qquad (5)$$

4. Stability of $L_{4,5}$

Since the nature of the linear stability about point L_5 will be similar to that of L_4, it will be sufficient to consider here the stability only near L_4.

Let (a,b) be the coordinates of the triangular points L_4.

We set $\xi = a+\alpha, \eta = b+\beta, (\alpha, \beta << 1)$, in the equations (1) of motion.

First, we compute the terms of their R.H.S, neglecting second and higher order terms, we get

$$\left(\frac{\partial W}{\partial \xi}\right)_{\xi=a+\alpha,\eta=b+\beta} = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta}$$

where,

$$A = \frac{3}{4}\left\{1+\frac{1}{2c^2}\left(2-19\mu+19\mu^2\right)\right\}-\left\{\frac{1}{48(\mu-1)c^2}\left(-28+82\mu+3\mu^2-321\mu^3+234\mu^4\right)+\left(\frac{3\mu}{2}-1\right)\right\}\delta$$
$$+\left\{-\frac{3}{8}(8\mu-9)-\frac{\left(226-1036\mu+1758\mu^2-1065\mu^3+87\mu^4\right)}{32(\mu-1)c^2}\right\}A_1,$$

$$B=\frac{3\sqrt{3}}{4}(1-2\mu)\left(1-\frac{2}{3c^2}\right)+\left\{-(\mu-2)+\frac{1}{48(\mu-1)c^2}\left(96-309\mu+314\mu^2-151\mu^3+40\mu^4\right)\right\}\delta$$
$$+\left\{-\frac{\sqrt{3}}{8}(26\mu-19)-\frac{\sqrt{3}\left(-46+393\mu-599\mu^2-135\mu^3+417\mu^4\right)}{96(\mu-1)\,c^2}\right\}A_1,$$

$$C=\frac{\sqrt{3}}{2c^2}(1-2\mu)+\left\{-\frac{2\sqrt{3}}{9c^2}(\mu+1)\right\}\delta+\left\{-\frac{\sqrt{3}}{24c^2}(46\mu-35)\right\}A_1,$$

$$D=\frac{6-5\mu+5\mu^2}{2c^2}+\left\{-\frac{1}{3c^2}\left(-2+3\mu-6\mu^2\right)\right\}\delta+\left\{\frac{22-33\mu+45\mu^2}{8c^2}\right\}A_1.$$

Similarly, we obtain

$$\left(\frac{\partial W}{\partial \eta}\right)_{\xi=a+\alpha,\eta=b+\beta}=E\alpha+B_1\beta+C_1\dot{\alpha}+D_1\dot{\beta},$$

where,

$$E=\frac{3\sqrt{3}}{4}(1-2\mu)\left(1-\frac{2}{3c^2}\right)+\left\{-(\mu-2)+\frac{1}{48(\mu-1)c^2}\left(96-309\mu+314\mu^2-151\mu^3+40\mu^4\right)\right\}\delta$$
$$+\left\{-\frac{\sqrt{3}}{8}(26\mu-19)+\frac{\sqrt{3}\left(46+393\mu+599\mu^2-135\mu^3+417\mu^4\right)}{96(\mu-1)\,c^2}\right\}A_1,$$

$$B_1=\frac{9}{4}\left\{1+\frac{7\left(-2+3\mu-3\mu^2\right)}{6c^2}\right\}+\left\{\left(\frac{3\mu}{2}-1\right)+\frac{1}{16(\mu-1)c^2}\left(4-26\mu+173\mu^2-279\mu^3+118\mu^4\right)\right\}\delta$$
$$+\left\{\frac{33}{8}+\frac{\left(290-1292\mu+2082\mu^2-1221\mu^3+1118\mu^4\right)}{32(\mu-1)c^2}\right\}A_1,$$

$$C_1=\frac{1}{2c^2}\left(-4+\mu-\mu^2\right)+\left\{-\frac{1}{3c^2}(7\mu-2)\right\}\delta+\left\{-\frac{\left(20-13\mu+9\mu^2\right)}{8c^2}\right\}A_1,$$

$$D_1=-\frac{\sqrt{3}}{2c^2}(1-2\mu)-\left\{\frac{2\sqrt{3}}{9c^2}\left(1-8\mu+9\mu^2\right)\right\}\delta+\left\{\frac{-\sqrt{3}\,(35+46\mu)}{24\,c^2}\right\}A_1.$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)_{\xi=a+\alpha,\eta=b+\beta} = A_2\dot{\alpha} + B_2\dot{\beta} + C_2\ddot{\alpha} + D_2\ddot{\beta}$$

where,

$$A_2 = \frac{\sqrt{3}}{2c^2}(1-2\mu) + \left\{\frac{2\sqrt{3}}{9c^2}(\mu+1)\right\}\delta + \left\{-\frac{\sqrt{3}(46\mu-35)}{24c^2}\right\}A_1,$$

$$B_2 = -\frac{4-\mu+\mu^2}{2c^2} + \left\{\frac{1}{3c^2}(7\mu-2)\right\}\delta + \left\{-\frac{(20-13\mu+9\mu^2)}{8c^2}\right\}A_1,$$

$$C_2 = \frac{17-2\mu+2\mu^2}{4c^2} + \left\{\frac{1}{3c^2}(7\mu+1)\right\}\delta + \left\{\frac{(23-10\mu+6\mu^2)}{8c^2}\right\}A_1,$$

$$D_2 = -\frac{\sqrt{3}(1-2\mu)}{4c^2} - \left\{\frac{\sqrt{3}}{9c^2}(\mu+1)\right\}\delta + \left\{\frac{\sqrt{3}(13+14\mu)}{24c^2}\right\}A_1.$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right)_{\xi=a+\alpha,\eta=b+\beta} = A_3\dot{\alpha} + B_3\dot{\beta} + C_3\ddot{\alpha} + D_3\ddot{\beta}$$

where,

$$A_3 = \frac{6-5\mu+5\mu^2}{2c^2} + \left\{\frac{1}{3c^2}(-2+3\mu-6\mu^2)\right\}\delta + \left\{\frac{22-33\mu+45\mu^2}{8c^2}\right\}A_1,$$

$$B_3 = -\frac{\sqrt{3}(1-2\mu)}{2c^2} + \left\{\frac{2\sqrt{3}}{9c^2}(1-8\mu+9\mu^2)\right\}\delta + \left\{\frac{\sqrt{3}(-35+46\mu)}{24\,c^2}\right\}A_1,$$

$$C_3 = -\frac{\sqrt{3}(1-2\mu)}{4c^2} + \left\{\frac{\sqrt{3}}{9c^2}(\mu+1)\right\}\delta + \left\{\frac{\sqrt{3}(-13+14\mu)}{24c^2}\right\}A_1,$$

$$D_3 = \frac{3(5-2\mu+2\mu^2)}{4c^2} + \left\{\frac{1}{3c^2}(9\mu-1)\right\}\delta + \left\{\frac{25-30\mu+18\mu^2}{8c^2}\right\}A_1$$

Thus, the variational equations of motion corresponding to equations (1), on making use of equation (3), can be obtained as

$$P_1\ddot{\alpha} + P_2\ddot{\beta} + P_3\dot{\alpha} + P_4\dot{\beta} + P_5\alpha + P_6\beta = 0,$$

$$q_1\ddot{\alpha} + q_2\ddot{\beta} + q_3\dot{\alpha} + q_4\dot{\beta} + q_5\alpha + q_6\beta = 0.$$

where,

$$P_1 = 1 + C_2, P_2 = D_2, P_3 = A_2 - C,\ P_4 = \left\{B_2 - 2\left(1 + \frac{3}{4}A_1 - \frac{3}{2c^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)\right) - D\right\}, P_5 = -A, P_6 = -B$$

$$q_1 = C_3, q_2 = 1 + D_3, q_3 = 2\left(1 + \frac{3}{4}A_1 - \frac{3}{2c^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)\right) - C_1 + A_3, q_4 = B_3 - D_1, q_5 = -E, q_6 = -B_1$$

Then, the characteristic equation is

$$(P_1q_2 - P_2q_1)\lambda^4 + (P_1q_6 + P_5q_2 + P_3q_4 - P_6q_1 - P_2q_5 - P_4q_3)\lambda^2 + P_5q_6 - P_6q_5 = 0 \qquad (6)$$

Substituting the values of $P_i, q_i, i = 1,2,...,6$ in (6), the characteristic equation (6) after normalizing becomes

$$\lambda^4 + b\lambda^2 + d = 0, \qquad (7)$$

$$b = \left(1 - \frac{9}{c^2}\right) + \left\{-\frac{3}{2} + 3\mu + \frac{-80 - 108\mu + 105\mu^2 - 18\mu^3}{8c^2}\right\}A_1 + \left(\frac{10 + 84\mu - 21\mu^2 - 12\mu^3}{12c^2}\right)\delta,$$

$$d = \left\{ \frac{27\mu(1-\mu)}{4} + \frac{9\mu\left(-65+77\mu-24\mu^2+12\mu^3\right)}{8c^2} \right\} + \left\{ \frac{117}{4}\mu(1-\mu) \right.$$
$$\left. + \frac{3\left(80-7245\mu+9624\mu^2-3366\mu^3+846\mu^4\right)}{64\,c^2} \right\} A_1 + \left(\frac{960-2253\mu-222\mu^2+9243\mu^3-6390\mu^4+816\mu^5}{12c^2} \right.$$
$$\left. + \frac{3\mu(1-\mu)}{2} \right) \delta$$

When $\frac{1}{c^2} \to 0$ and in the absence of radiation and oblateness $(i.e\ \delta = A_1 = 0)$, (7) reduces to its well-known classical restricted problem form (see e.g. Szebehely 1967):

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$$

The discriminant of (7) is

$$\Delta = 1 - \frac{9\mu\left(-65+77\mu-24\mu^2+12\mu^3\right)}{2c^2} - 27\mu(1-\mu) + \left(-117\mu(1-\mu) + 6\mu - 3\right.$$
$$\left. \frac{128+20439\mu-28452\mu^2+10026\mu^3-2538\mu^4}{16c^2} \right) A_1 + \left(\frac{10+84\mu-21\mu^2-12\mu^3}{16c^2} - 6\mu(1-\mu) \right. \quad (8)$$
$$\left. - \frac{960-2253\mu-2226\mu^2+9243\mu^3-6390\mu^4+816\mu^5}{16(\mu-1)c^2} \right) \delta$$

Its roots are

$$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2} \qquad (9)$$

where,

$$b = \left(1 - \frac{9}{c^2}\right) + \left\{ -\frac{3}{2} + 3\mu + \frac{-80-108\mu+105\mu^2-18\mu^3}{8c^2} \right\} A_1 + \left(\frac{10+84\mu-21\mu^2-12\mu^3}{12c^2} \right) \delta.$$

From (8), we have

$$\frac{d\Delta}{d\mu}=\frac{585-693\mu+206\mu^2108\mu^3-9\mu\left(77-48\mu+36\mu^3\right)}{2c^2}-27+54\mu$$
$$\left(\frac{-111+234\mu-20439-56904\mu+30078\mu^2-10152\mu^3}{16c^2}\right)A_1+\left(\frac{14-7\mu-6\mu^2-12\mu^3}{c^2}+12\mu-6\right.$$
$$\left.-\frac{-2253-4452\mu+27729\mu^2-25560\mu^3+4080\mu^4+816\mu^5}{48(\mu-1)c^2}\right.$$
$$\left.+\frac{960-2253\mu-2226\mu^2+9243\mu^3-6390\mu^4+816\mu^5}{48(\mu-1)^2c^2}\right)\delta<0\forall\,\mu\in\left(0,\frac{1}{2}\right) \qquad (10)$$

From (9), it follows that Δ is decreasing in $\left(0,\frac{1}{2}\right]$.

But

$$(\Delta)_{\mu=0}=1-3A_1-\frac{18}{c^2}-\frac{8A_1}{c^2}+\frac{65}{3}\delta>0$$

(11)

$$(\Delta)_{\mu=\frac{1}{2}}=-\frac{23}{4}-\frac{117A_1}{4}-\frac{3}{2}\delta+\frac{207}{4c^2}+\frac{32585A_1}{128c^2}+\frac{479}{48c^2}\delta<0$$

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and Δ is monotone and continuous, there is one value of μ, e.g. μ_c in the interval $\left(0,\frac{1}{2}\right)$ for which Δ vanishes. Solving the equation $\Delta=0$, using (8), we obtain critical value of the mass parameter as

$$\mu_c=\frac{1}{2}-\frac{1}{18}\sqrt{69}-\frac{1}{9}\left(1+\frac{13}{9\sqrt{69}}\right)A_1-\frac{2}{27\sqrt{69}}\delta-\frac{17\sqrt{69}}{486c^2}+\left(\frac{-197133+15493\sqrt{69}}{536544c^2}\right)A_1$$
$$+\left(\frac{-3690189+682606\sqrt{69}}{2414448c^2}\right)\delta \qquad (12)$$

$$\mu_c = \mu_0 - \frac{17\sqrt{69}}{486c^2} - \frac{1}{9}\left(1+\frac{13}{9\sqrt{69}}\right)A_1 + \left(\frac{-197133+15493\sqrt{69}}{536544c^2}\right)A_1 - \frac{2}{27\sqrt{69}}\delta + \left(\frac{-3690189+682606\sqrt{69}}{2414448c^2}\right)\delta$$

where $\mu_0 = 0.03852...$ is the Routh's value.

We consider the following three regions of the values of μ separately.

i. When $0 \le \mu < \mu_c$, the values of λ^2 given by (9) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.

ii. When $\mu_c < \mu \le \frac{1}{2}, \Delta < 0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.

iii. When $\mu = \mu_c, \Delta = 0$, the values of λ^2 given by (9) are the same. This induces instability of the triangular points.

Hence, the stability region is

$$0 < \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2} - \frac{1}{9}\left(1+\frac{13}{9\sqrt{69}}\right)A_1 + \left(\frac{-197133+15493\sqrt{69}}{536544c^2}\right)A_1 - \frac{2}{27\sqrt{69}}\delta + \left(\frac{-3690189+682606\sqrt{69}}{2414448c^2}\right)\delta \qquad (13)$$

5. Conclusion

We discuss the triangular equilibrium points in the relativistic R3BP under the assumption that the bigger primary is an oblate spheroid and the smaller one as a source of radiation.

Equation (5) gives the positions of triangular equilibrium points, which are affected by relativistic factor, radiation pressure and oblateness coefficient. It may also be noted that

triangular points no longer form equilateral triangles with the primaries as they do in classical case.

Also, the second equation of (5) indicates that with an increase in the values of , the triangular points move towards the line joining the primaries. Equation (12) gives the critical value of the mass parameter of the system which depends upon relativistic factors, oblateness coefficient and radiation factor. This critical value is used to determine the size of the region of stability of the triangular points and also helps in analyzing the behavior of the parameter involved therein. It is observed from (13) that relativistic, radiation and oblateness all have destabilizing effect, and therefore the size of the region of stability decreases with the increases of the values of these parameters. In the absence of radiation and oblateness, the stability results obtained in the study are in agreement with those of Douskos and Perdios (2002) and disagree with those of Bhatnagar and Hallan (1998). In the absence of relativistic terms, our results are accordant with those of Douskos et al. (2006)

7. Conclusion

By considering the bigger primary as an oblate spheroid and the smaller one as a source of radiation in the relativistic R3BP, we have determined the positions and stability are affected by the relativistic terms apart the oblateness and radiation factors. It is also noticed that these factors reduces the size of the stability region. We have also noticed that the expression A, D, A_2, C_2 in Bhatnagar and Hallan (1998) differ from the present study in the absence of oblateness and radiation$(i.e.\ A_1 = \delta = 0)$. Consequently, the expression P_1, P_3, P_4, P_5 and the characteristic equation are also different. This led them (Bhatnagar and Hallan 1998) to infer that triangular points are unstable, contrary to Douskos and Perdios and our results.

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RETHINKING MATHEMATICS EDUCATION AS A TOOL FOR ENHANCING CREATIVITY, INNOVATION, AND SUSTAINABLE DEVELOPMENT IN TIMES OF ECONOMIC HARDSHIP

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Abstract

In the face of 21st-century global disruptions including climate change, economic crises, pandemics, and technological transformation education must evolve to equip learners with the critical competencies for solving complex societal problems. This paper examines the pivotal role of mathematics education in fostering creativity, innovation, and sustainable development, particularly during times of economic hardship. Mathematics education has now moved beyond traditional rote-learning methods, it advocates for a transformative approach that integrates mathematical creativity, real-world modeling, technology, and sustainability. Mathematics has long been perceived as a rigid discipline governed by rules, formulas, and procedures. However, modern perspectives emphasize that creativity is central to mathematical thinking, creativity in mathematics is not limited to artistic expression only, it involves problem-solving, pattern recognition, generalization, logical reasoning, and the ability to view mathematical problems from multiple perspectives Furthermore, mathematics provides foundational tools for achieving the United Nations Sustainable Development Goals (SDGs), including data analysis, optimization, and systems thinking. In conclusion, the paper suggested ways in which creativity, innovation and sustainable development can be enhanced by shifting from rote learning to the use of digital tools in mathematics education.

Keywords: Creativity, Innovation, Sustainable development, mathematics education

Introduction

The 21st century has witnessed significant global economic disruptions from pandemics and climate change to inflation, debt crises, and resource scarcity. These challenges have amplified the need to reshape education systems to equip learners with the critical skills required for problem-solving, innovation, and sustainable living. In this context, rethinking mathematics education is essential not only for academic performance but as a driver of creativity, innovation, and sustainable development (UNESCO, 2023). Traditional approaches to mathematics that emphasize rote learning are insufficient to prepare learners for real-world complexity. Instead,

there is a need for transformative pedagogies that integrate creativity, technology, and sustainability within mathematical thinking. Mathematics has long been perceived as a rigid discipline governed by rules, formulas, and exact procedures. However, modern perspectives emphasize that creativity is central to mathematical thinking. Creativity in mathematics is not limited to artistic expression it involves problem-solving, pattern recognition, generalization, logical reasoning, and the ability to view mathematical problems from multiple perspectives (Leikin, 2023). As global education shifts towards equipping learners with 21st-century skills, fostering creativity in mathematics education becomes essential for innovation, critical thinking, and lifelong learning.

In today's fast-changing world, innovation has become a key requirement for solving real-world challenges in science, business, engineering, and society. Mathematics especially mathematical modeling provides a foundational tool for such innovation. When combined with digital technology, mathematical modeling enables learners to simulate complex systems, visualize abstract relationships, and develop creative solutions. Thus, integrating mathematical modeling and technology in education is vital to foster critical thinking, creativity, and 21st-century problem-solving skills (OECD, 2023). In the face of growing global challenges such as climate change, poverty, inequality, food insecurity, and energy crises the pursuit of sustainable development has become urgent. The United Nations Sustainable Development Goals (SDGs), adopted in 2015, provide a global framework for addressing these challenges in a comprehensive and integrated manner (UNESCO, 2023). Mathematics plays a critical yet often underappreciated role in achieving sustainable development. It provides the tools for measuring, modeling, analyzing, and optimizing systems skills essential for informed decision-making and innovation.

Research Objectives

1. **To examine the role of creativity in mathematics education** as a driver of problem-solving, critical thinking, and student engagement in the context of 21st-century learning.
2. **To investigate the impact of mathematical modeling and digital technologies** on fostering innovation and equipping learners with practical skills for addressing real-world challenges.

3. **To explore how mathematics education can be integrated with sustainability principles** to support the achievement of the United Nations Sustainable Development Goals (SDGs).
4. **To identify the challenges hindering creativity, innovation, and sustainable development** in mathematics education during periods of economic hardship.

Creativity in Mathematics Education

Creativity in mathematics involves more than producing unique answers; it is about flexible thinking, pattern recognition, and inventiveness in approach. Embedding open-ended problems, project-based learning, and interdisciplinary tasks in the mathematics curriculum can stimulate divergent thinking and creativity. For example, students could explore how to design energy-efficient buildings using geometry or optimize food distribution using linear programming (Leikin, 2023). Creativity in mathematics refers to the ability to produce original, valuable, and appropriate mathematical ideas. It is characterized by flexibility in thought, the construction of novel solutions, the exploration of patterns and structures, and the formulation of new conjectures or generalizations (Sriraman & English, 2024).

Mathematical creativity can occur at various levels such as;

i. Everyday creativity: Generating personal strategies or insights in solving problems.
ii. School-level creativity: Producing novel approaches within the curriculum.
iii. Professional creativity: Making original contributions to mathematical theory.

Importance of Creativity in Mathematics Education

1. Enhancing Problem-Solving Skills: Creative thinking enhances students' ability to approach non-routine and complex problems, enabling flexible thinking and multiple solution paths (Livne & Mevarech, 2023). This leads to deeper conceptual understanding rather than memorization.
2. Fostering Innovation and Critical Thinking: In a world dominated by uncertainty, the ability to innovate through mathematical modeling, abstraction, and algorithmic thinking is crucial. Creativity prepares learners for careers in science, technology, engineering, and innovation (STEMI) sectors (OECD, 2023).
3. Increasing Student Motivation and Engagement: Creative tasks often involve open-ended questions, real-world contexts, and exploratory learning, making mathematics more

enjoyable and meaningful to students. This boosts participation and reduces math anxiety (Mann & Lo, 2023).

Strategies for Promoting Creativity in Mathematics Classrooms

1. Open-Ended and Rich Tasks: Tasks that allow for multiple methods, representations, or solutions promote divergent thinking. For example, asking students to design a bridge with optimal cost using geometry and algebra fosters innovation and critical decision-making.
2. Project-Based and Inquiry-Based Learning: Students explore real-life problems (e.g., climate change, traffic optimization) using mathematical tools. These projects connect mathematics to everyday experiences, encouraging exploration and ownership of ideas.
3. Innovation through Mathematical Modeling and Technology: Innovation thrives where learners are encouraged to apply mathematical concepts to real-life challenges using digital tools. The integration of technology, coding, and AI in mathematics education enables students to simulate systems, test hypotheses, and iterate models. This transforms students from passive recipients of formulas to innovators capable of designing smart agriculture systems, analyzing climate data, or developing financial models for startups (Odili, 2024).

Innovation through Mathematical Modeling

Mathematical modeling serves as a Launchpad for innovation in multiple ways, such as in;

1. Problem-Solving: Students tackle authentic, messy problems where no single "right" answer exists similar to problems faced by engineers and scientists.
2. Decision-Making: Modeling supports decisions under uncertainty. For instance, students can model resource allocation in disaster relief or optimize logistics for supply chains.
3. Design Thinking: Modeling encourages iteration. Students revise assumptions, test variables, and improve models paralleling the innovation cycle in real-world industries (English, & Sriraman, 2024)
4.

Examples of Innovative Modeling in Schools

Context	Mathematical Concepts	Innovative Application
Climate	Exponential functions, data	Modeling CO_2 emissions and global

Change	analysis	temperature trends
Finance	Linear equations, interest rates	Designing and comparing savings and loan models
Health	Probability, statistics	Modeling virus transmission in populations
Agriculture	Optimization, geometry	Designing irrigation systems for water efficiency

Mathematics for Sustainable Development

Mathematics is pivotal to achieving the United Nations Sustainable Development Goals (SDGs). Skills in measurement, statistics, optimization, and logical reasoning underpin decision-making in health, environment, and economics. For instance, students can engage in sustainability projects such as calculating carbon footprints, analyzing water usage data, or budgeting for community development initiatives (UNESCO, 2023).

Understanding Sustainable Development in Education

Sustainable development is defined as "meeting the work toward sustainable lifestyles and equitable development that meets the needs of the present without compromising the ability of future generations to meet their own needs." (UNESCO, 2023). In education, this translates to equipping learners with the knowledge, skills, values, and attitudes needed to contribute to a more just, equitable, and environmentally sound society.

According to Boaler (2023), mathematics education for sustainable development focuses on:

i. Applying mathematics to real-world sustainability issues
ii. Promoting critical thinking and ethical reasoning
iii. Developing systems thinking and interdisciplinary problem-solving

Role of Mathematics in Achieving the SDGs

Mathematics contributes to nearly every one of the 17 SDGs. Here are key examples:

Table 1: Summary of United Nations Sustainable Development Goals

Sustainable Development Goal	Mathematical Applications
SDG 2 (Zero Hunger)	Optimization of agricultural yields through linear programming; statistical modeling of food production and consumption
SDG 3 (Good Health and Well-	Epidemiological modeling of disease spread (e.g.,

being)	COVID-19); risk analysis in public health
SDG 6 (Clean Water and Sanitation)	Measuring water quality and consumption rates; modeling distribution networks
SDG 7 (Affordable and Clean Energy)	Energy efficiency modeling; cost-benefit analysis of renewable energy systems
SDG 13 (Climate Action)	Climate data analysis; carbon footprint calculations; modeling greenhouse gas emissions

Source: UNDP (2023)

Teaching Mathematics for Sustainability: Approaches and Strategies

1. Contextual and Problem-Based Learning: Presenting students with real-life sustainability problems (e.g., calculating household energy use or planning sustainable urban transport) makes math relevant and socially meaningful.
2. Mathematical Modeling of Systems: Students can use mathematical models to represent and analyze environmental systems, like population dynamics or deforestation rates. Tools like GeoGebra, spreadsheets, and Python enhance these experiences.
3. Data Literacy and Statistical Analysis: Learners interpret graphs, analyze global datasets, and evaluate trends using mean, median, standard deviation, and correlation. Projects might involve comparing countries' progress toward SDG targets (Mavrikis & Holmes 2023)
4. Interdisciplinary Integration: Math lessons can integrate topics from geography, economics, and environmental science to deepen understanding of sustainability challenges.

Benefits of Integrating Sustainability into Mathematics Education

i. Relevance: Makes abstract math more meaningful by connecting it to students' lives and global issues.
ii. Engagement: Motivates students through real-world applications.
iii. Empowerment: Equips learners with tools to make informed and ethical decisions.
iv. Skill Development: Promotes critical thinking, systems thinking, and data analysis. (Adebayo, Yusuf & Okoro 2024).

Challenges hindering creativity, innovation and sustainable development in times economic hardship

1. Reduced Funding for Education and Innovation: One of the most immediate effects of economic downturns is budget cuts in education, research, and development (R&D). Without adequate funding, schools cannot implement programs that promote creative and critical thinking. Likewise, universities and research centers scale down innovation projects, stalling progress in sustainability.

2. Brain Drain and Talent Migration: In challenging economic times, highly skilled individuals often immigrate to countries with better job opportunities. This leads to loss of intellectual capital necessary for driving innovation and sustainable development at home (Agyemang & Boateng 2024)

3. Shift to Short-Term Solutions: Economic crises often shift focus from long-term sustainability goals to immediate economic recovery, neglecting creative or eco-friendly solutions that might initially appear more costly. For example, governments may invest in fossil fuels to revive industries instead of transitioning to clean energy (Sachs 2023).

4. Technological Inequality: Innovation often relies on access to digital infrastructure and technology. In low-resource environments, access to such tools is limited, creating a digital divide that prevents equitable participation in innovation and sustainable development (OECD 2023)

5. Weak Institutional Support: During recessions, institutional frameworks become overstretched or inefficient, reducing their capacity to support creative initiatives, entrepreneurship, or sustainable urban planning. Bureaucracy and policy inconsistency further delay developmental progress (UNDP 2023)

6. Psychological and Social Constraints: Economic hardship often causes stress, anxiety, and reduced motivation, particularly among young people and educators. This negatively impacts creativity, collaboration, and innovation in learning environments and workplaces (MGI 2024)

7. Inaccessibility of Sustainable Resources: Sustainable development practices such as renewable energy, green technologies, and eco-friendly materials—often require significant upfront investment. During economic hardships, these solutions become unaffordable or less prioritized, especially in low-income communities (IEA 2023)

8. Lack of Resources and Infrastructure: In many developing regions, especially Sub-Saharan Africa, schools face poor funding, large class sizes, and lack of access to learning materials and digital tools. These factors hinder the practical application of creativity-oriented and tech-enhanced math instruction (Adebayo, Yusuf & Okoro, 2024).
9. Rigid Curriculum and Assessment Systems: National education policies often prioritize examination performance over innovation or sustainability, leading to limited flexibility in curriculum design.
10. Teacher Capacity and Professional Development: Many educators lack the training to integrate innovative pedagogies and sustainability into their teaching. Ongoing professional development is crucial.

Conclusion

In these challenging economic times, mathematics education must be re-imagined as a catalyst for creativity, innovation, and sustainable development. Creativity is not an optional enrichment in mathematics, it is a foundational skill that supports meaningful learning, innovation, and real-world problem-solving. Innovation through mathematical modeling and technology is critical in transforming mathematics education from abstract theory to real-world application. By integrating modeling and digital tools into classrooms, educators can equip students with the skills to innovate, solve complex problems, and navigate uncertainty. For this to succeed, it is essential to invest in teacher training, rethink curricula, and ensure equitable access to technology. As mathematics becomes a tool for innovation, it empowers students to become future-ready thinkers, designers, and leaders. By moving beyond rigid instruction and embracing real-world relevance, educators can empower learners to become change-makers and innovators who contribute meaningfully to global progress. Sustainable development demands not just knowledge but the ability to use mathematics as a tool to improve lives, optimize resources, and design better futures. As we confront global challenges, integrating sustainability into mathematics education is not optional, it is essential. Economic hardship poses real threats to creativity, innovation, and sustainable development. However, these are precisely the tools needed to escape cycles of poverty and environmental degradation. Strategic investment, policy coherence, and inclusive digital access are key to unlocking innovation even in the toughest

economic times. The future depends not just on recovery, but on rethinking development through resilience and creativity

Suggestions

To strengthen the impact of the ideas presented and support effective implementation, the following suggestions are recommended:

1. Curriculum Transformation: There should be a shift from rote memorization to competency-based learning that integrates creativity, technology, and sustainability into mathematical concepts and applications.
2. Teacher Professional Development: Provide continuous training for educators on innovative pedagogies, digital tools, and sustainability-focused mathematics instruction.
3. Equitable Access to Technology: Bridge the digital divide by ensuring all learners, regardless of socioeconomic background, have access to the tools and infrastructure necessary for innovation.
4. Integration of Real-World Problems: Embed local and global sustainability challenges into mathematics curricula to enhance relevance, motivation, and problem-solving skills.
5. Policy Support and Investment: Advocate for education policies that prioritize creativity, STEM innovation, and sustainability, even during economic downturns.
6. Collaborative and Interdisciplinary Approach: Encourage partnerships between mathematics educators, environmental scientists, economists, and technologists to develop cross-disciplinary projects.
7. Student-Centered Learning: Incorporate project-based and inquiry-driven approaches that empower learners to explore multiple solution pathways and innovate independently.
8. Assessment Reform: Redesign assessment systems to value creative problem-solving, critical thinking, and real world applications

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EFFECT OF ZOOM- CLASSROOM INSTRUCTIONAL STRATEGY ON STUDENTS' ACADEMIC ACHIEVEMENT IN DIFFERENTIAL EQUATIONS IN FEDERAL UNIVERSITY OF EDUCATION ZARIA, NIGERIA

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Abstract

This study investigated the "Effects of Zoom Classroom Instructional Strategy on Students' Achievement in Differential Equations in Federal University of Education Zaria". The study adopted a quasi-experimental research of pre-test, post-test, non-equivalent, non-randomized, control groups design. The population for the study consists of (120) 300 level students. Simple random sampling techniques was used to sample (62) 300 level students, (38 males and 24 females) students were selected for the study. Two research questions were asked and two null hypotheses were formulated for the study. Differential Equations Achievement Test (DEAT)items was used as the instrument for data collection after it was validated by three experts and found good for data collection. Reliability index was also obtained to be 0.79. Descriptive statistics of mean and standard deviation was used to answer research questions, while inferential statistics of ANCOVA analysis was used to test the null hypotheses at 0.05 level of significant. The findings revealed that there is significant difference between the achievement of students exposed to Zoom classroom Instructional Strategy and their counterpart exposed to Conventional Method in favor of experimental group. Also, the findings revealed that there is no significant difference between male and female student's achievement on differential equations using Zoom Classroom Instructional Strategy. Hence the researcher recommended that the use of Zoom Classroom Instructional Strategy should be adopted for teaching/learning of differential equations in higher institution, In-service training programs for Mathematics lecturers in form of workshops, seminars and conferences should be organized to focus more on how to use Zoom Classroom Instructional Strategy in Teaching differential equations, among others.

Introduction

The need to acquire knowledge in mathematics in the world over has become very obvious, this is because it is relevant to everyday living and in various disciplines. As a follow up, it has been a compulsory subject for both primary and secondary school levels in Nigeria because of its usefulness in every field of study in the higher institutions (Umar, Tudunkaya, & Muawiya, 2019). E- learning is an educational activity that allows students to study via the interest, attend classes online, have digital interactions, share options with colleagues and be independent learners with more flexibility in the time and place where learning occurs (Laksana, 2021). The e-learning method of teaching, transforming learning materials and educational cooperation. Computers, tablets, mobile phones or any other electronic device that enables online connectivity between the participants in the learning process (Wang & Pi, 2019). For several decades, there has been continuous development in the processing capabilities of computers, increasing their power and the variety of uses they can be used for. At the same time, in many nations, there is also an accelerated development of communication infrastructures, such as optical cables and cellular networks that enable broadband data transmission at high speed, in a stable manner and with good quality. These technologies currently provide users with endless availability of information available on the internet, improved speed of data transfer and high level of interactive media that enables real-time video communication. On this basis, over the years, a solid technological and communication foundation was created for the existence of online environments for teaching and learning purposes in educational institutions in general and academic institution in particular (Bao, 2020).

According to Bio (2020) states that, as the internet began offering more advanced communication, varied and richer content, and network-based systems and applications were developed for study purposes, online learning becomes increasingly accepted and widespread during COVID-19 pandemic in 2020, with all the constraints that followed, gave a significant boost to e-learning throughout the world. academics institutions, which until that time had delayed and were not in a rush to integrate e-learning courses into their curricula, underwent a rapid adaption to web-based learning due to emerging situation, and quickly adopted different types of electronic teaching and learning using zoom classroom, synchronous, asynchronous and integrated software. In parallel, with technological and pedagogical factors contribute to

effective online learning and how to improve the effectiveness of e-learning in academic courses have become very common research topics.

Recent research on teaching and learning indicates that, technology can play a crucial role in changing classroom environment and restructuring schools to promote more meaningful and result oriented teaching and learning situation, some of the widely used technologies in the classroom are computers and their associated software, videos, apps and projectors. However, one of the recent additions to vital and online platforms is the Zoom classroom instructional strategy. Zoom classroom instructional strategy refers to the use of zoom video conferencing platform to deliver online instruction, facilitate students engagement, and promote learning through line virtual classes, that is by conducting real-time classes, lectures, or discussion via zoom; interactive sessions, that is by encouraging students participation through polls, questions and answers, and breakout rooms; screen shearing, that is by sharing presentations, videos or documents or lecture materials to students and also recorded sessions: by recording classes for latter review or for absent students may equally be updated. Zoom classroom instructional strategy can enhance flexibility, engagement in online learning environments if properly utilized by the teacher.

Differential equations are mathematical equations that describe how quantities change over time or space. They involve; rates of change that is derivatives of functions, representing how quantities change and relationships, that is equations that relate the rates of change to the quantities themselves. Differential equations are used to model various phenomena in physics (motion, energy and waves); Biology (population growth, disease spread); Engineering (control systems, electrical circuits); and Economics (growth models, financial markets). The differential equations are categorised into; Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs). Notes also that, solving differential equations helps to develop, understand and predict the behaviour of complex system in our society today. Ordinary differential equations are mathematical equations that, describe how a function changes with respect to one independent variable which involve derivatives that is rates of change of function and relationships of the equations that relate the function and its derivatives. ODEs are essential in many fields including physics, engineering and biology to model and analyse dynamic system while partial differential equations are also mathematical equations that describe how a function changes with respect to multiple independent variables which involves partial derivatives: rates

of change of the function with respect to each independent variable and relationships; equations that relate the function and its partial derivatives. PDEs are practically used to model various phenomena in; Physics (Heat transfer, fluid dynamics, wave propagation); Engineering (structural mechanics, electromagnetism); and Finance (Option pricing, risk management) etc.

In view of the forgoing, there is need for this concept be properly taught through involvement and engagement of students during the learning process by adopting technology app such as zoom classroom instructional strategy.

Academic Achievement can be referred to as the students overall learning experience resulting in an observable and measurable behaviour change, which includes the requisition of specific ideology, skills and techniques at the end of a teaching and learning situation (Umar & Musa, 2023). Quite a number of definitions of achievement exist in literature and most of these make attempt at comparing the essences of the concept. Musa (2021) define achievement as the event to which a person has achieved something acquired certain information or mastered certain skills, usually as a result of planned instruction or training. Akwagha, Agashi, & Obi, (2019) sees academic achieved as an assessment of how people have learnt or attained after a learning period. Thus, a degree of a student's learning programme is to be determined in the learner's achievement that is being sort for performance are usually determined through test. These tests provide means of evaluating students' performance in absolute turns such as percentage instead of comparing a students' result with those of his counterparts. Achievement in this study, the researcher wishes to investigate the academic achievement of students using Zoom classroom instructional strategy. Students' academic achievement can easily be enhanced when Mathematics teachers and educators use student-centred and activity-based strategies, this would go a long way in improving students' achievement in Mathematics and in other subject areas.

There is general belief that male perform better in Differential equations concepts and mathematical-related subjects than the female ones. A number of studies have looked into the influence of gender and environment on differential equation achievement of students. Musa (2021) emphasized such imbalance in male-female achievement in science-related descriptions and stressed its significance as a new country rolled in. Adams (2020) reported that male students performed better at higher level of Mathematics than female students. Other findings which support male superiority were that of Isah (2016), Aliyu (2023) and Muhammed (2023) opined that the achievement gap of male and female in Mathematics is interesting in the fact that

girls typically have better grades in differential equation classes, but tend to scores lower on standardized Mathematics test. On this premises therefore, the researcher deemed it necessary to determine the gender achievement among students because gender is part of the variable to be investigated under the study. A very current and innovative strategy can equally bring out best among male and female students in classroom situation if properly utilized. Therefore, the researcher looks into the gender and achievement in differential equations using Zoom classroom instructional strategy.

Statement of the problem

Students, parents, educators, government and the populace are worried because of the persistent poor academic achievement of students in differential equation at higher institutions. This poor academic achievement is directly connected with the methods, strategies or techniques employed by the lecturer in teaching the concepts (Muhammad, 2023). Lack of adopting technological tools is also one of the problems. In view of the empirical studies computer assisted instruction and online apps has been found to enhance students' academic achievement, especially mathematics than the conventional instructional methods, computer assisted instruction can be considered as effective than the traditional instruction. Furthermore, how e-learning delivered can affect its effectiveness, and that new studies are needed to clarify the effect of online strategy in contemporary students' environment. Therefore, the study investigated the effect of Zoom classroom instructional strategy on academic achievement in differential equations among mathematics Students in federal University of Education Zaria, Nigeria.

Objectives of the Study

This study main purpose is to investigating the effect of Zoom classroom instructional strategy on academic achievement in differential equations among mathematics students in federal University of Education, Zaria. While the specific objectives are:

1. Determine the Zoom classroom instructional strategy has effect on academic achievement of mathematics students taught differential equation concepts.
2. Verify the academic achievement of male and female mathematics students taught differential equations with Zoom classroom instructional strategy.

Research Questions

To achieve the stated objectives of the study, the following research questions were asked to guide the study:

1. What are the mean achievement scores of students taught differential equations using Zoom classroom instructional strategy and those taught using Conventional Method?
2. What are the mean achievement scores of male and female students taught differential equations using Zoom classroom instructional strategy?

Statement of the Hypotheses

Based on the specific objectives of the study, the following null hypotheses were formulated and tested at 0.05 level of significance

Ho_1: There is no significant difference in the mean achievement scores of students taught differential equations using Zoom classroom instructional strategy and those exposed to Conventional Method.

Ho_2: There is no significant difference in the mean achievement scores of male and female students taught differential equations using Zoom classroom instructional strategy.

Significance of the Study

This study is significant as it is likely to aid learning of differential equations. The result of this study might be of benefits to the following stakeholders in education; such as Mathematics teachers, policy maker, Mathematics curriculum planners, curriculum developers, researchers, teachers training institutions, Non-Governmental Organization and students.

Methodology

A quasi-experimental research design of pre-test, post-test nonrandomized control type was adopted for the study. The population of this study comprised of (120) 300 level students in department of mathematics Federal University of Education, Zaria. The choice of this population is because the concept of differential equation is taught in 300 levels. And the sample for this research consist of (62) 300 level students. Simple random sampling technique was used to select one intact class (combination) The two groups were randomized and assigned experimental and control groups respectively. The experimental group consist 35 students (22 males & 13 females) which were exposed to Zoom classroom instructional strategy while the control group 27 students (16 males & 11 females) were exposed to conventional method.

The instruments for data collection were Differential Equations Achievement Test (DEAT). The DEAT items consist of 20 multiple choice questions. The DEAT (CGAT) items was validated by experts from Department of Mathematics Federal University of Education Zaria. they did the required face validity by scrutinizing the entire work, by giving advice in terms of scope and

content coverage, content relevance, ambiguity and vagueness of expression. Content validity is also ensured in DEAT by constructing a test blueprint or table of specification to ensure adequate coverage of the content area covered in the study as well as maintain spread across the different lower and upper levels of cognitive domain such as remembering, understanding, application, analysis, syntheses and evaluation, according to new Benjamin Bloom's Taxonomy of education domain (2021).

The rating of each of the experts was 0.72, 0.84 and 0.80 indices of rationality of the instrument respectively. The average of the three validators gave 0.79 as the logical index validity and therefore, the instrument was considered suitable for data collection.

In a bid to ascertain the reliability of the Differential Equations Achievement Test (DEAT) instrument was trail tested to (20) 300 level Mathematics students. The DEAT was administered to a portion of the population of which was not part of the sample. The reliability of the DEAT instrument is determined using Kuder-Ricardson formula 21 ($K\text{-}R_{21}$) and the coefficient of internal consistency for DEAT is 0.78. The purpose of the trail tested was to confirm the reliability of the test instrument since the study adopted quasi-experimental research design.

Pre-test was given to both Experimental and Control groups to determine the level of their prior academic achievement in differential equation before treatment. The differential equations concepts were taught using Zoom classroom instructional strategy to experimental group while conventional method was used to teach control group. Lesson plans were designed for both experimental group and control group. The two groups had treatment for 4 weeks in first semester 2024/2025 academic session respectively to determine the significant difference in academic achievement between the two groups as well as gender difference.

Steps in Creating Zoom Classroom

Here are the steps to create Zoom Classroom:

Step 1: Create Zoom Account or sign up for Zoom account if you haven't already

Step 2: Scheduled a meeting or Lectures; by going to the Zoom website or app, and scheduled a new meeting or lecture. Choose the date, time, and duration.

Step 3: Set up meeting details: by adding a meeting title, description, and password (if desired).

Step 4: Invite Students; by sending the meeting/ lecture inviting to students via email, messaging app, or learning management system.

Step 5: Configure Audio/ Video settings: by choosing your audio and video settings such as enabling /disabling video and audio for students' participants.

Step 6: Prepare Materials: by sharing screens presentations, or files with students during the lecture.

Step 7: Start the Meeting; by beginning the virtual class on time, and engage with students, the session can be recorded for later viewing.

Below is the diagram illustrating zoom classroom procedure:

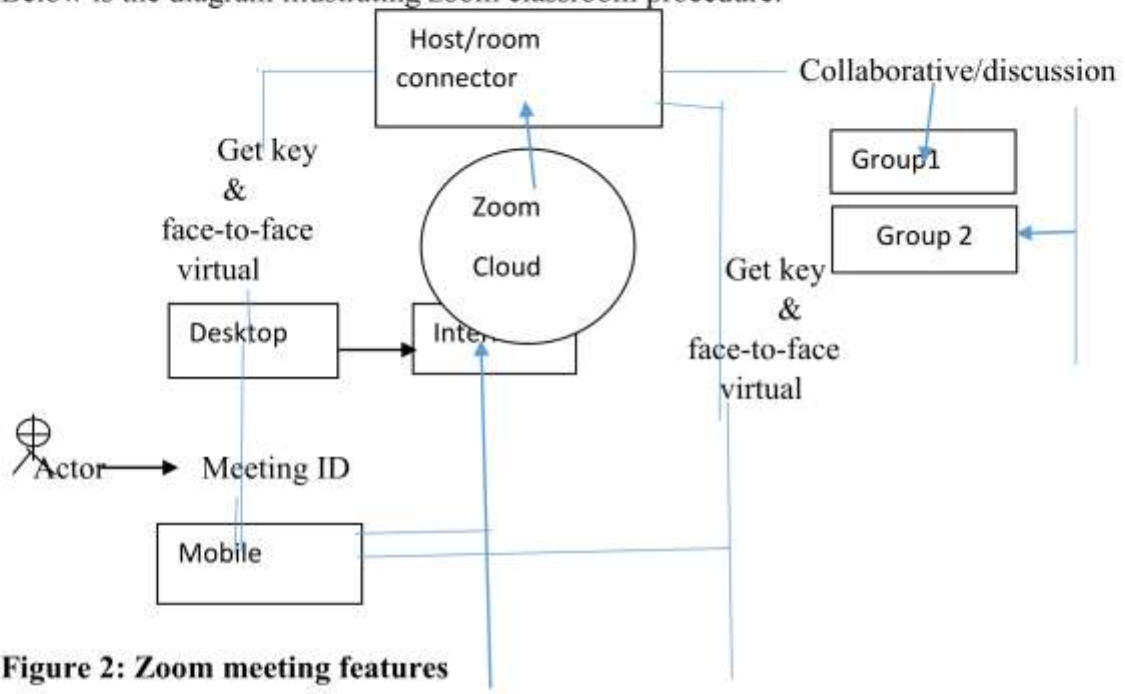


Figure 2: Zoom meeting features

Results and Discussion

The data collected was used to answer research questions and test the null hypotheses as follows:

Research Question One: What are the mean achievement scores of students taught differential equations using Zoom classroom instructional strategy and those taught using Conventional Method?

Table 1: Mean and Standard Deviation on Students Achievement Scores of Experimental and Control group Taught Differential Equations Using Zoom Classroom Instructional Strategy and Conventional Method

Groups	Test Types	N	Mean	St. Div.
EXPERIMENTAL	Pre-test	35	14.20	2.26
	Post-test	35	34.11	3.02
CONTROL	Pre-test	27	14.93	3.26
	Post-test	27	23.26	1.38

Table 1 shows the mean and standard deviation for pre-test and post-test on students' achievement scores of experimental and control groups exposed to differential equation has (N = 35, M = 14.20, SD = 2.26 & M = 34.11, SD = 3.02) while control group has (N = 27, M = 14.93, SD = 3.26, & M = 23.26, SD = 1.38). This indicates that the experimental group perform higher than the control group exposed to conventional method.

Hypothesis one: There is no significant difference in the mean achievement scores of students taught differential equations using Zoom classroom instructional strategy and those exposed to Conventional Method. The test of null hypothesis is presented in Table 2.

Table 2: Results of ANCOVA on Achievement Scores of Students Taught Differential Equation using Zoom Classroom and Conventional Method

Source	**Type III Sum of Squares**	**df**	**Mean Square**	**F**	**Sig.**	**Partial Eta Squared**
Corrected Model	1797.130[a]	2	898.565	148.262	0.000	0.834
Intercept	1600.029	1	1600.029	264.002	0.000	0.817
Pretest	1.148	1	1.148	0.189	0.665	0.003
Groups	**1776.399**	**1**	**1776.399**	**293.103**	**0.000**	**0.832**
Error	357.580	59	6.061			
Total	55698.000	62				
Corrected Total	2154.710	61				

a. R Squared = 0.834 (Adjusted R Squared =0.828)

Table 1 shows a significant difference in the mean achievement scores of students exposed to differential equations and Conventional Groups. The value of $F_{(2,59)} = 293.103$ was obtained with associated exact probability value of 0.000. Since the associated probability 0.000 is less

than 0.05 level of significance, the null hypothesis one is hereby rejected. The results implied that, the zoom classroom teaching strategy produced a significant effect on the post-test achievement scores of students' when covariate effect (pre-test) is controlled. Hence, there is a significant difference among the two groups.

Research Question Two: What are the mean achievement scores of male and female students taught differential equations using Zoom classroom instructional strategy?

Table 3: Mean and Standard Deviation for Achievement of Male and Female Students Exposed to GeoGebra Software

Groups	Test Types	N	Mean	St. Div.
Male	Pre-test	22	14.36	2.57
	Post-test	22	34.18	2.32
Female	Pre-test	13	13.92	1.66
	Post-test	13	34.00	4.04

Table 3 showed the pre test and post-test mean and standard deviation of male students exposed to Zoom classroom instructional strategy has (N = 22, M = 14.36, SD = 2.57 & M = 13.92, SD = 2.32) while female has (N = 13, M = 13.92, SD = 1.66, & M = 34.00, SD = 4.04). This indicates that the male and female students exposed to Zoom classroom instructional strategy perform equally well.

Hypothesis Two: There is no significant difference in the mean achievement scores of male and female students taught differential equations using Zoom classroom instructional strategy. The test of hypothesis two is presented in Table 4.

Table 4: The Result of ANCOVA of Mean Achievement Scores of Male and Female Students Exposed to Zoom Classroom Instructional Strategy

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	6.055[a]	2	3.028	.319	0.729	0.020
Intercept	831.235	1	831.235	87.646	0.000	0.733
Pretest	5.785	1	5.785	0.610	0.441	0.019
Gender	**0.083**	**1**	**00.083**	**0.009**	**0.926**	**0.000**

Error	303.488	32	9.484
Total	41042.000	35	
Corrected Total	309.543	34	

a. R Squared = 0.020 (Adjusted R Squared = -0.042)

Table 4 shows a significant difference in the mean achievement scores of male and female students taught differential equations using to Zoom classroom instructional strategy. The value of $F_{(2, 32)} = 0.009$ was obtained with associated exact probability value of 0.926 Since the associated probability 0.926 is greater than 0.05 level of significance, the null hypothesis two is hereby retained. The results implied that, the zoom classroom teaching strategy produced a significant effect on the post-test achievement scores of male and female students' when covariate effect (pre-test) is controlled. Hence, there is no significant difference among the two groups. The strategy is gender friendly.

Discussions of Findings

Based on the results in Table 2 revealed clearly that students who were taught using Zoom classroom strategy achieved higher academic achievement than their counterpart students taught with conventional method. This result is in agreement with the finding of Ogbu, et al (2017), Aliyu (2017) and Muawiya, (2019) who jointly discover that students tend to achieve positively when they are taught new mathematics concepts using student-centered method oriented.

Table 4 results indicates clearly that there is no significant difference between male and female in their academic achievement when taught differential equations using Zoom classroom instructional strategy. The result shows that, both male and female students performed almost equally same, therefore, this strategy is gender friendly. This finding is in contradiction with Muhammed & Bashir (2017), Adams (2020) and Sani (2019) who states that male tend to perform higher than female counterpart when exposed to activity-based method of teaching mathematics concepts. The result also in agreement with Muawiya, (2023), Simon (2024) and Muhammed, (2023) which states differently that when students are exposed to practical approach, male and female students tend to performed equally in mathematics proficiency.

Conclusions

Based on the findings of this research study, the researcher concludes that the Zoom classroom Instructional strategy can enhance student's academic achievement thereby bridging the gap between male and female students if properly utilized. Therefore, use of Zoom classroom

Instructional strategy is an important strategy for better academic achievement in learning and mastering of differential equations.

Recommendations

Based on the findings of the study the following recommendations was made such include:

1. Federal Ministry of Education should organize workshops for the lecturers of Mathematics on Zoom classroom Instructional strategy for effective teaching and learning of differential equations.
2. Professional bodies such as Science Teachers Association of Nigeria (STAN), Mathematical Association of Nigeria (MAN) among others, should organize workshops and seminars during their annual conferences to popularize and sensitize mathematics lecturers on the use of Zoom classroom Instructional strategy in teaching students the concepts of differential equations.
3. Teacher training institutions of learning should include the use of Zoom classroom Instructional strategy as a method of teaching differential equations. This will guarantee that, after the teacher training, they will be equipped on how to teach the concepts of differential equations effectively.
4. Further research is needed on this strategy to cover larger population in the state and Nigeria at large.

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ALLEVIATING STUDENTS' ALGEBRAIC MISCONCEPTIONS: AN APPLICATION OF 7E'S MODEL AMONG SENIOR SECONDARY SCHOOL IN SOKOTO STATE, NIGERIA

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Abstract

This study investigated the effectiveness of the 7E's Instructional Model, Peer Tutoring, and the traditional Demonstration Method in addressing students' misconceptions in algebraic problems among Senior Secondary School Two (SSII) students in Sokoto State, Nigeria. The study employed a non-randomized control group, pretest-posttest quasi-experimental design involving 378 students drawn from four educational zones (Sokoto North, Sokoto South, Bodinga, and Yabo) through stratified and simple random sampling. Experimental Group 1 received instruction using the 7E's Model, Experimental Group 2 used Peer Tutoring, and the Control Group was taught via the Demonstration Method. Instructional modules were developed based on the ADDIE model and implemented over six weeks. Data were collected using 5-Tier Students' Algebraic Processes Misconception Test (SAPMT; reliability = 0.92). Descriptive statistics of (percentages and frequencies) to answer research questions. Findings revealed that students in all groups initially exhibited low levels of conceptual understanding. The study also found that the 7E's Instructional Model is the most effective method for reducing misconceptions and enhancing students' understanding and problem-solving in algebra. While Peer Tutoring showed moderate success, the Demonstration Method was the least effective. The study recommends adopting the 7E's Model in senior secondary schools and reducing dependence on traditional, teacher-centered methods to improve students' algebraic understanding.

Key Words: Misconception, ADDIE Model, Instructional Modules, 7E's Model, Peer Tutoring.

Introduction

Algebra, as a major component of mathematics, encompasses various elements such as symbols, numbers, terms, polynomials, statements, and equations, and so on, making it a foundational aspect of mathematical study (Mangorsi & Solaiman, 2013). In the context of school mathematics, algebra is defined as the use of symbols to express and manipulate general numerical relationships, emphasizing its role in promoting logical reasoning and critical thinking towards problem solving (Ayhan & Nifuler, 2017). Furthermore, algebra plays a central role in developing problem-solving abilities and supports progress in Science, Technology, Engineering, and Mathematics (STEM) fields to suites the present technological development world over

(Schanzer, 2015). Alajmi (2016) notes that algebra enhances analytical skills transferable to areas such as economics, accounting, management, and the sciences. As such, algebra serves as a universal language for communicating mathematical ideas, and its mastery enables students to participate meaningfully in mathematical discussions and collaborative problem-solving (Arcavi, Drijvers & Stacey, 2016). At the senior secondary school level in Nigeria, proficiency in algebra is a vital part of the mathematics curriculum and is instrumental in shaping students' ability to succeed in both academic and professional domains (Son & Fatimah, 2021). According to Redick (2023), students with a strong foundation in algebra are more likely to pursue further studies in STEM-related fields.

However, despite its importance, many students in Nigerian secondary schools struggle with learning algebra effectively. A major challenge in algebra learning is the persistence of misconceptions deep-rooted, incorrect understandings of algebraic concepts which leads to persistent poor performance in mathematics and related areas. These misconceptions include misinterpreting algebraic symbols, misunderstanding basic operations, equal sign, confusing variables with constants, and misapplying rules for simplifying expressions and solving equations. Such misconceptions hinder students' ability to solve problems accurately by progression from solving factual problems, conceptual, procedural to metacognitive, and often persist despite the used of several teaching methods (Julius, 2022).

Addressing these challenges requires a shift from passive learning approaches to active and student-centered instructional methods. One such approach is the 7E's instructional model, which is rooted in constructivist learning theory and emphasizes student engagement and inquiry (Sewanu, 2022). The 7E's model comprises seven stages; Elicit, Engage, Explore, Explain, Elaborate, Evaluate, and Extend, each contributing to a cycle of deep learning and concept reinforcement. The Elicit phase helps teachers identify students' prior knowledge and misconceptions about the basic foundational skills required in solving algebraic problems (Bybee, et al., 2006). In the Engage phase, students' curiosity is stimulated when they are introduced to a new topic and learning resources, linking their existing ideas and skills to handle algebraic process problems (Järvelä, et al., 2016). The Explore phase involves hands-on minds-on activities where learners are engaged to investigate and explore relationships between algebraic process concepts. This phase is followed by the Explain phase, which allows them to articulate and refine their understanding of the novel algebraic process concepts introduced to

them (Lim, Ang, & Koh, 2016). In the Elaborate phase, students apply their previously acquired knowledge to the new situation in order to deepen their understanding of the concepts (Ruiz-Martín & Bybee, 2022). The Evaluate phase allows both teachers and students to assess their success and learning progress via solving algebraic process problems (Kibble, 2017), while the Extend phase, which is the final phase, enables learners to apply concepts in broader contexts which will give them room to clear any misconception in solving algebraic process problems (Türkmen, 2022).

Therefore, the present study applied the 7E's Instructional Model alongside Peer Tutoring to investigate their effectiveness in alleviating students' algebraic misconceptions.

Statement of the Problem

Algebra remains one of the most difficult topics for students in Nigerian secondary schools, particularly in Sokoto State. Many students struggle with basic algebraic tasks such as simplifying expressions, solving equations, and graphing linear or quadratic functions. These challenges have been consistently highlighted in the West African Examinations Council (WAEC) Chief Examiners' Reports of 2014 to 2023, which show recurring patterns of fluctuating poor performance and widespread misconception/errors in solving algebraic processes problems. A key factor behind these difficulties is the presence of deep-rooted misconceptions of algebraic concepts at the basic and middle basic levels of education. Students often misinterpret algebraic symbols, misuse rules, or apply procedures incorrectly. These misconceptions are not just errors but reflect a lack of understanding that traditional classroom teaching fails to address. Traditional instructional methods, such as the demonstration or lecture method, are mainly teacher-centered and focus on mere memorization rather than understanding. As a result, they do not encourage students to think critically or reflect on their mistakes, which limits their ability to correct misconceptions (Sadiq, 2019; Ginga & Zakariya, 2020; Faruk, Ayuba & Abubakar, 2023).

To address this gap, the present study developed and tested three instructional modules based on the 7E's instructional model, Peer Tutoring, and the Demonstration Method, using the ADDIE instructional design framework (Analysis, Design, Development, Implementation, and Evaluation). The goal of the study is to identify which of these methods is most effective in reducing students' algebraic misconceptions and improving their overall understanding of algebraic concepts.

Research Objectives

The specific objectives are to:

1. Find out the misconceptions of students in solving algebraic problems before using the three independent instructional strategies (7E's module, Peer tutoring, and Demonstration method) among senior secondary school students in Sokoto State.
2. Find the misconceptions of students in solving algebraic problems after using the three independent instructional strategies (7E's module, Peer tutoring, and Demonstration method) among senior secondary school students in Sokoto State.
3. Identify the misconceptions held by male and female students in solving algebraic problems after using the three independent instructional strategies (7E's module, Peer tutoring, and Demonstration method) among senior secondary school students in Sokoto State.

Research Questions

To achieve the stated objectives, the following research questions were formulated to guide the research:

1. What are the misconceptions of students in solving algebraic problems before using the three independent instructional strategies (7E's module, Peer tutoring, and Demonstration method) among senior secondary school students in Sokoto State?
2. What are the misconceptions of students in solving algebraic problems after using the three independent instructional strategies (7E's module, Peer tutoring, and Demonstration method) among senior secondary school students in Sokoto State?
3. What are the misconceptions held by male and female students in solving algebraic problems after using the three independent instructional strategies (7E's module, Peer tutoring, and Demonstration method) among senior secondary school students in Sokoto State?

Methodology

The study adopted a non-randomized control group, pretest-posttest quasi-experimental design. Experimental Group 1 was taught algebraic processes using the 7E's instructional model, Experimental Group 2 received instruction through Peer Tutoring, while the Control Group was taught using the Demonstration Method. The population for the study comprised all Senior

Secondary School Two (SSII) students in public schools across the educational zones of Sokoto State, totaling 32,278 students with 19,594 males and 12,684 females. Due to security concerns, Goronyo and Gwadabawa Educational Zones were excluded, resulting in a revised population of 22,173 students across 92 schools in four educational zones: Sokoto North, Sokoto South, Bodinga, and Yabo. A stratified random sampling technique was employed to ensure fair representation from each zone and to maintain the proportional distribution of male and female students. Accordingly, the sample was proportionally distributed across the zones: Sokoto North (5,260 students) contributed 90 students; Sokoto South (9,904 students) contributed 169; Bodinga (3,292 students) contributed 56; and Yabo (4,025 students) contributed 63. Simple random sampling was then used to select one school from each zone, namely: Government Day Secondary School Kofar Marke, Government Day Secondary School Alu Quarters Bado, Government Day Secondary School Bodinga, and Government Day Secondary School Yabo. After school selection, each school was randomly assigned to one of the three groups, two experimental and one control ensuring equal chances of placement and minimizing bias. To preserve the natural school environment, one intact class was randomly selected from the available SSII arms in each school.

Instrumentation

The study utilized 5-Tier Students Algebraic Processes Misconception Test (SAPMT) for data collection: The SAPMT, a five-tier diagnostic test, was used to explore students' misconceptions and deeper understanding. This test assessed correctness, confidence, and conceptual understanding across multiple levels. To ensure the instrument validity and reliability, it was reviewed by experts in mathematics education and tested in a pilot test involving 40 students. The instrument demonstrated high reliability, with coefficients of 0.92. During the main study, students were divided into three groups: Experimental Group 1 (7E's model), Experimental Group 2 (Peer Tutoring), and a Control Group (Demonstration method). Each group received instruction for six weeks, followed by a post-test to evaluate learning outcomes. Data collected was securely stored for analysis. Table 3 presents marking, grading and decision guide for SAPMT

Table 3: marking, grading and decision for SAPMT

Sn	Decision	Description	Marking	Grading
1.	AC **(Algebraic Conception)**	Students provide correct answers at the macroscopic level (the main algebraic question) and sub-microscopic levels (Tiers 1-5), are confident in their answers, and their drawings are aligned with algebraic concepts.	Full marks are awarded. This is the highest level of understanding, reflecting complete mastery of the algebraic concepts, confidence in their answers, and accurate representation through drawings.	**Connected**
2.	**AAC (Almost Algebraic Conception)**	Students give correct answers at both the macroscopic and sub-microscopic levels, with high confidence. However, their drawings are either not fully accurate or do not entirely match the algebraic concept or explanation	High marks are given but slightly reduced compared to AC due to the issue with the drawings. The student understands the concepts but lacks full alignment between explanation and representation.	**Unconnected**
3.	**LC (Lack of Confidence)**	Students provide correct answers at both levels (macroscopic and sub-microscopic), and their drawings are accurate, but they express uncertainty about their answers.	Marks are awarded for correct answers and proper drawings, but there is a deduction due to the lack of confidence. This suggests understanding but hesitancy in applying the concepts.	**Connected**
4.	**LK (Lack of Knowledge)**	Students provide answers that are partly correct at either the macroscopic or sub-microscopic levels, with varying confidence. Their drawings only partially align with algebraic concepts.	Partial marks are awarded. The answers show some knowledge, but the incomplete or inaccurate drawings and answers suggest gaps in understanding	**Unconnected**
5.	**MC (Misconception)**	Students may provide either correct or incorrect answers at both levels, but they lack full confidence, and their drawings are not in line with algebraic concepts.	Marks are deducted due to misconceptions and incorrect or inappropriate drawings. This reflects confusion or misunderstanding of the	**Connected**

			algebraic processes	
6.	**HNC (Have No Conception)**	Students give incorrect answers at all levels, have no confidence in their responses, and their drawings are completely unrelated to algebraic concepts	Minimal or no marks are awarded. This decision reflects a lack of understanding and severe misconceptions	**Unconnected**

Source: (Widodo, Sopandi & Wu, 2020)

After marking session, the scores obtained from the study were subjected to analysis using SPSS Package. The researcher used descriptive statistics of (frequencies and percentages) to answer the research questions.

Research Question One

What are the misconceptions of students in solving algebraic problems before using the three independent instructional strategies (7E's module, Peer tutoring, and Demonstration method) among senior secondary school students in Sokoto State? The result of the analysis is presented in table 1, 2, 3 and 4 respectively.

Table 1: Levels of Misconception among Students for 7E's Instructional Model group (pretest)

S/N	Decision	Frequency	Percentages
1	AC	7	4%
2	AAC	12	7%
3	LC	15	9%
4	LK	30	17%
5	MC	50	30%
6	HNC	55	33%
Total		**169**	**100%**

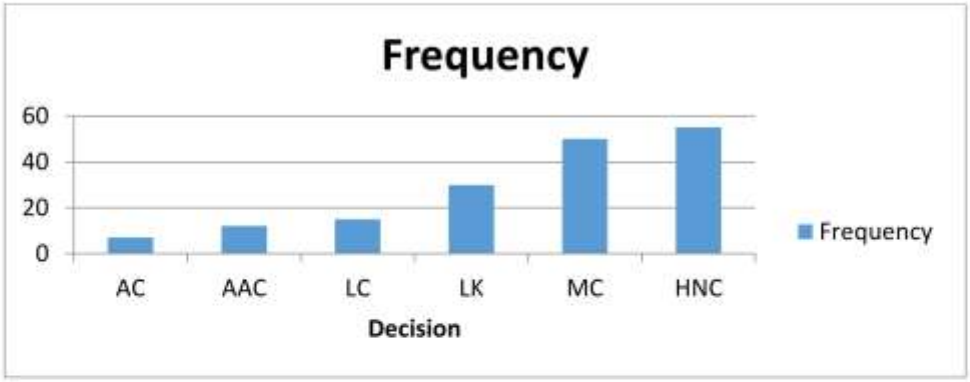


Figure 1: Levels of Misconception among Students for 7E's Instructional Model group (pretest)

Table 1 and Figure 1 presents the levels of misconceptions among students in the 7E's instructional model group before the treatment (pretest). Out of the total 169 students, Only 7 students (4%) demonstrated full Algebraic Conceptual (AC) understanding, while 12 students (7%) were categorized under Almost Algebraic Conceptual (AAC), showing they were close to understanding but still made minor errors. 15 students (9%) showed a lack of confidence (LC) in

applying algebraic concepts, and 30 students (17%) demonstrated a lack of knowledge (LK), indicating insufficient foundational understanding. Most notably, 50 students (30%) fell under the misconception (MC) category, showing significant misunderstandings of algebraic processes. Furthermore, 55 students (33%) were placed in the have no conception (HNC) category, reflecting a complete lack of understanding or deep confusion.

Table 2: Levels of Misconception among Students for Peer Tutoring Instructional Strategy group (pretest)

S/N	Decision	Frequency	Percentages
1	AC	4	4%
2	AAC	7	8%
3	LC	8	9%
4	LK	15	17%
5	MC	25	28%
6	HNC	31	34%
Total		**90**	**100%**

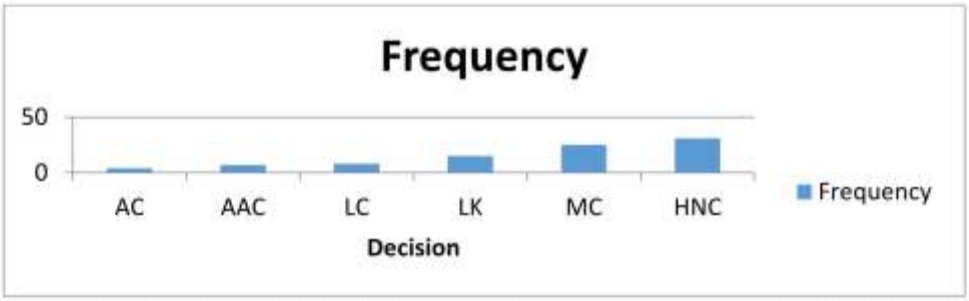


Figure 2: Levels of Misconception among Students for Peer Tutoring group (pretest)

Table 2 and Figure 2 shows the levels of misconceptions among students in the Peer Tutoring Instructional Strategy group before the treatment. Out of a total of 90 students, only 4 students (4%) demonstrated full Algebraic Conceptual (AC) understanding, while 7 students (8%) were in the Almost Algebraic Conceptual (AAC) category, indicating they were close to full understanding but still made some errors. However, a larger number of students showed signs of misconception and confusion. For instance, 8 students (9%) lacked confidence (LC) in solving algebraic problems, and 15 students (17%) showed a lack of knowledge (LK), meaning they lacked the foundational understanding required to engage with algebraic tasks effectively. Moreover, 25 students (28%) demonstrated misconceptions (MC) about algebraic concepts, and 31 students (34%) fell into the Have No Conception (HNC) category, indicating a complete misunderstanding or inability to approach algebraic problems.

Table 3: Levels of Misconception among Students for Demonstration Method group (pretest)

S/N	Decision	Frequency	Percentages
1	AC	5	4%
2	AAC	8	7%
3	LC	10	8%
4	LK	20	17%

5	MC	35	29%
6	HNC	41	35%
Total		**119**	**100%**

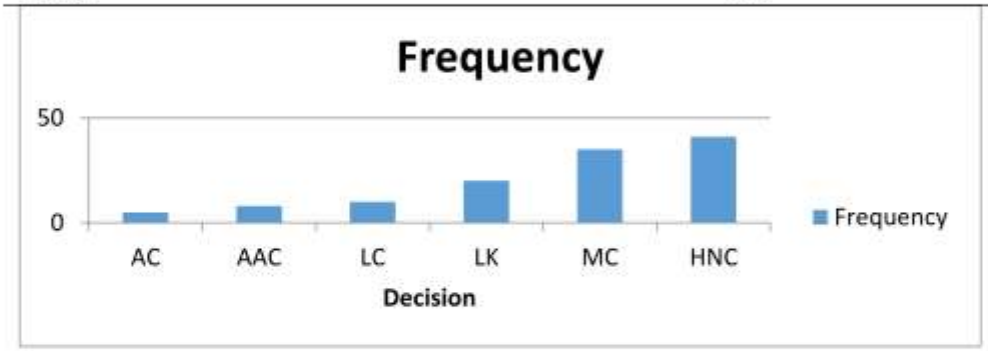


Figure 3: Levels of Misconception among Students for demonstration method group (pretest)

Table 3 and Figure 3 presents the levels of misconceptions among students in the Demonstration Method group before the treatment. Out of 119 students, only 5 students (4%) demonstrated full understanding of algebraic concepts (AC), and 8 students (7%) were classified under Almost Algebraic Conceptual (AAC), showing near-complete but not perfect understanding. On the other hand, a significant portion of the students exhibited various levels of misconceptions and confusion. Specifically, 10 students (8%) were categorized as having Lack of Confidence (LC), while 20 students (17%) showed Lack of Knowledge (LK) indicating weak foundational understanding of algebra. Notably, 35 students (29%) displayed misconceptions (MC), and 41 students (35%) were found to have No Conceptual Understanding (HNC), meaning they had serious difficulties grasping the core ideas of algebra.

Table 4: Summary of Students' Misconceptions Across the Three Instructional Strategies (pretest)

		7E's		Peer Tutoring		Demonstration	
S/N	**Decision**	**Frequency**	**%**	**Frequency**	**%**	**Frequency**	**%**
1	**AC**	7	4%	**Frequency**	**Percentages**	5	4%
2	**AAC**	12	7%	4	4%	8	7%
3	**LC**	15	9%	7	8%	10	8%
4	**LK**	30	17%	8	9%	20	17%
5	**MC**	50	30%	15	17%	35	29%
6	**HNC**	55	33%	25	28%	41	35%
	Total	**169**	**100%**	31	34%	**119**	**100%**

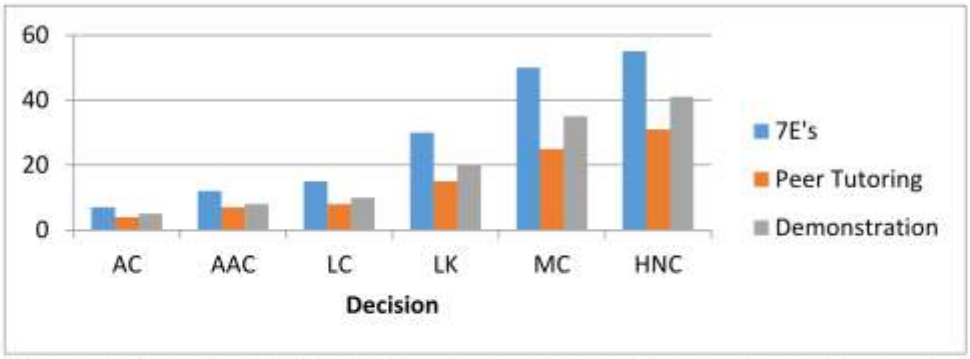


Figure 4: Levels of Misconception among Students across all the three Instructional Strategies (pretest)

Table 4 and Figure 4 indicated that, in the 7E's group, out of 169 students, only a small proportion7 students (4%) demonstrated full algebraic conception (AC), while 12 students (7%) were almost algebraic conception (AAC). However, a significant number exhibited misunderstanding: 15 students (9%) showed Lack of Confidence (LC), 30 students (17%) lacked basic knowledge (LK), and a combined 105 students (62%) fell into the Misconception (MC) and Have No Conceptual Understanding (HNC) categories. This indicates that prior to instruction; many students in the 7E's group already struggled with foundational algebraic concepts. Likewise, the Peer Tutoring group showed a similar trend. Out of 90 students, only 4 students (4%) demonstrated full algebraic conception, while 4 students (4%) were categorized as AAC. Misconceptions were also widespread in this group: 7 students (8%) had low confidence, 8 (9%) lacked knowledge, and 40 students (45%) held misconceptions (MC and HNC combined). This reinforces the observation that students entered the study with limited algebraic clarity. In the Demonstration Method group, the misconception levels were the highest. Out of 119 students, only 5 (4%) were in the AC category and 8 (7%) were close to it (AAC). Meanwhile, 10 students (8%) lacked confidence, 20 (17%) lacked knowledge, and the majority 76 students (64%) fell into the MC and HNC categories, indicating a serious gap in their conceptual grasp of algebra.

Research Question Two

What are the misconceptions of students in solving algebraic problems after using the three independent instructional strategies (7E's module, Peer tutoring, and Demonstration method) among senior secondary school students in Sokoto State? The result of the analysis is presented in table 5, 6, 7 and 8 respectively.

Table 5: Levels of Misconception among Students Taught Using the 7E's Instructional Model

S/N	Decision	Frequency	Percentages

1	AC	124	73%
2	AAC	19	11%
3	LC	3	2%
4	LK	5	3%
5	MC	6	4%
6	HNC	12	7%
Total		**169**	**100%**

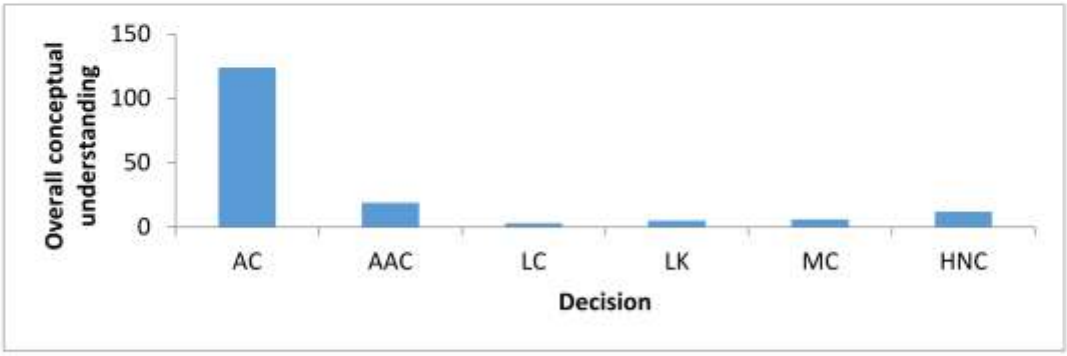


Figure 5: Students taught using the 7E's Model Levels of Misconception in Algebraic processes

Table 5 and Figure 5 summarizes the levels of misconceptions among students who were taught using the 7E's instructional model. Out of 169 students, only a small number showed signs of misunderstanding: 6 students (4%) demonstrated misconceptions (MC) about algebraic concepts, while 5 students (3%) showed lack of knowledge (LK), and 3 students (2%) exhibited lack of confidence (LC) in applying the concepts. Additionally, 12 students (7%) fell into the have no conception (HNC) category, indicating complete misunderstanding or confusion. In contrast, majority of the students 124 (73%) displayed no significant misconceptions, while 19 students (11%) showed only minor errors (Almost Algebraic Conceptual), suggesting partial misconceptions. These findings indicate that the 7E's instructional model was highly effective in minimizing algebra-related misconceptions.

Table 6: Levels of Misconception among Students Taught Using Peer Tutoring Instructional Strategy

S/N	Decision	Frequency	Percentages
1	AC	20	22%
2	AAC	16	18%
3	LC	6	7%
4	LK	7	8%
5	MC	19	21%
6	HNC	22	24%
Total		**90**	**100%**

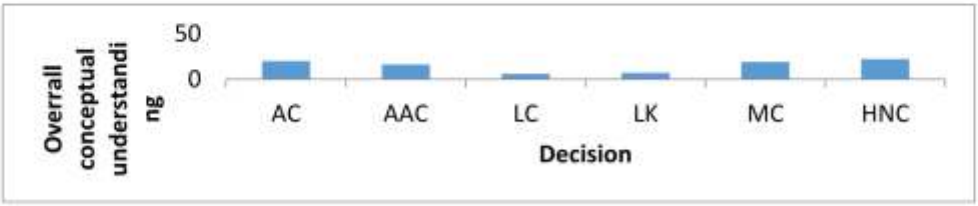


Figure 6: Misconception Levels among Students Taught with Peer Tutoring

Table 6 and Figure 6 shows that only a small portion of students (22%) taught with the Peer Tutoring Instructional Strategy had no algebra-related misconceptions, meaning they clearly understood the concepts. Another 18% had minor misconceptions, showing partial understanding. However, a large number of students (21%) demonstrated clear misconceptions in their reasoning, while 24% had no meaningful understanding of algebra at all. Others lacked confidence (7%) or basic knowledge (8%) to solve problems accurately. These results indicate that while Peer Tutoring helped some students reduce their misconceptions, it was not equally effective for all.

Table 7: Levels of Misconception among Students Taught Using the Demonstration Method

S/N	Decision	Frequency	Percentages
1	AC	6	5%
2	AAC	10	8%
3	LC	10	8%
4	LK	21	18%
5	MC	32	27%
6	HNC	40	34%
Total		**119**	**100%**

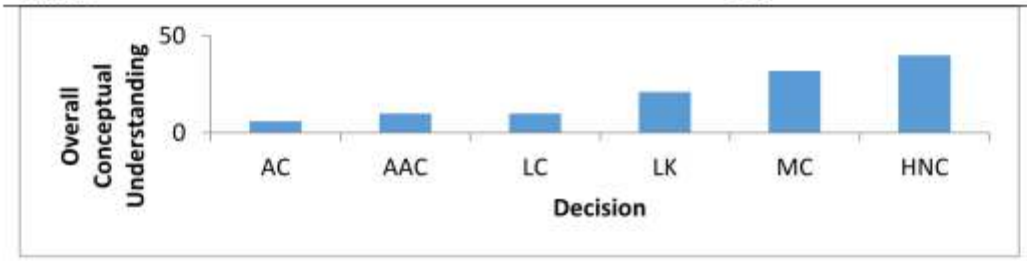


Figure 7: Misconception Levels among Students Taught with the Demonstration Method

Table 7 and Figure 7 shows the levels of algebra-related misconceptions among students taught using the Demonstration Method (Control Group). Only 6 students (5%) had no misconception, meaning they fully understood the algebraic concepts. A small portion (8%) showed minor misconceptions. However, a large number of students struggled significantly: 32 students (27%) demonstrated clear misconceptions in their reasoning, and 40 students (34%) had no conception at all indicating a serious lack of understanding. Additionally, 21 students (18%) lacked the necessary foundational knowledge, and 10 students (8%) showed a lack of confidence in applying what they learned. These results reveal that the Demonstration Method was the least effective in minimizing algebra misconceptions.

Table 8: Summary of Students' Misconceptions Across the Three Instructional Strategies

		7E's		Peer Tutoring		Demonstration	
S/N	Decision	Frequency	%	Frequency	%	Frequency	%
1	AC	124	73%	20	22%	6	5%
2	AAC	19	11%	16	18%	10	8%
3	LC	3	2%	6	7%	10	8%
4	LK	5	3%	7	8%	21	18%
5	MC	6	4%	19	21%	32	27%

6	**HNC**	12	7%	22	24%	40	34%
	Total	**169**	**100%**	**90**	**100%**	**119**	**100%**

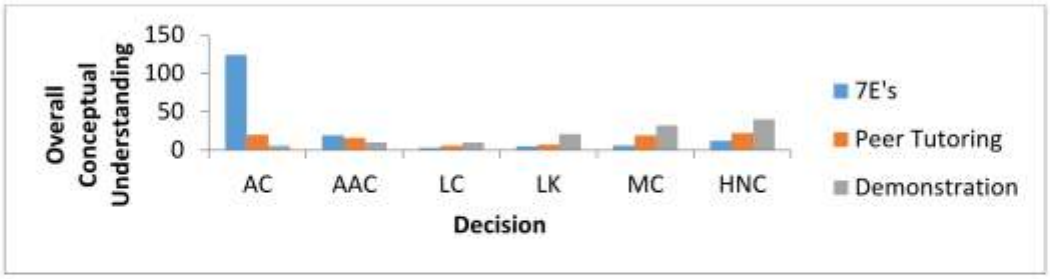


Figure 8: Comparison of Students' Misconception Levels by Instructional Strategy

Table 8 and Figure 8 shows the levels of misconceptions among students exposed to three instructional strategies: the 7E's Model, Peer Tutoring, and the Demonstration Method. The 7E's Model resulted in the lowest overall level of misconceptions, with only 4% of students showing clear misconceptions and 7% having no understanding (HNC). This indicates a strong ability of the 7E's Model to reduce confusion and foster correct algebraic reasoning. In contrast, students taught through Peer Tutoring had 21% showing clear misconceptions and 24% with no understanding, suggesting that while some students benefited, many still struggled with misunderstandings. The Demonstration Method showed the highest level of student misconceptions, with 27% of students demonstrating incorrect reasoning (MC) and 34% having no meaningful grasp of the content (HNC). This reflects the limitations of teacher-centered, lecture-based instruction in helping students overcome algebra-related misconceptions. Figure 4 visually confirms the clear advantage of the 7E's Model over the other teaching methods.

Research Question Three

What are the misconceptions held by male and female students in solving algebraic problems after using the three independent instructional strategies (7E's module, Peer tutoring, and Demonstration method) among senior secondary school students in Sokoto State? The result of the analysis is presented in table 9, 10 and 11 respectively.

Table 9: Levels of Misconception among Students for 7E's Instructional Model group based on gender

S/N	Decision	Gender				
		Male		Female		
		Freq	%	Freq	%	Total
1	AC	67	76	57	70	124
2	AAC	10	12	9	11	19
3	LC	1	1	2	2	3
4	LK	2	2	3	4	5
5	MC	3	3	3	4	6
6	HNC	5	6	7	9	12
Total		**88**	**100**	**81**	**100**	**169**

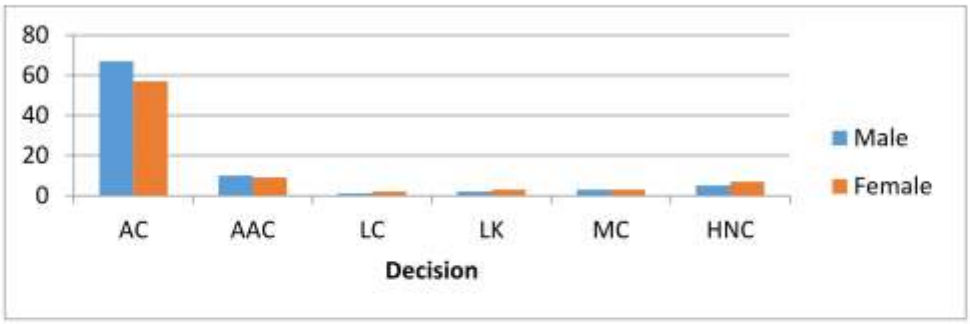


Table 9 and Figure 9 presents the levels of misconception among male and female students in the group taught using the 7E's Instructional Model, with a total of 169 students 88 males and 81 females. The data shows that a majority of both male and female students achieved Algebraic Conceptual Understanding (AC) after the treatment. Specifically, 67 male students (76%) and 57 female students (70%) were classified under AC, indicating a strong grasp of algebraic concepts in both groups. Additionally, 10 males (12%) and 9 females (11%) achieved Almost Algebraic Conceptual Understanding (AAC), reflecting partial comprehension with minor misunderstanding. While, fewer students fell into the misconception categories. Only 3 male students (3%) and 3 female students (4%) were found in the Misconception (MC) category. They Have No Conceptual Understanding (HNC) category included 5 males (6%) and 7 females (9%), while Lack of Knowledge (LK) was observed in 2 males (2%) and 3 females (4%). Lastly, the Lack of Confidence (LC) category had 1 male (1%) and 2 females (2%).

Table 10: Levels of Misconception among Students for Peer tutoring group based on gender

S/N	Decision	Gender				Total
		Male		Female		
		Freq	%	Freq	%	
1	AC	15	30	5	13	20
2	AAC	10	20	6	15	16
3	LC	2	4	4	10	6
4	LK	3	6	4	10	7
5	MC	9	18	10	25	19
6	HNC	11	22	11	27	22
Total		**50**	**100**	**40**	**100**	**90**

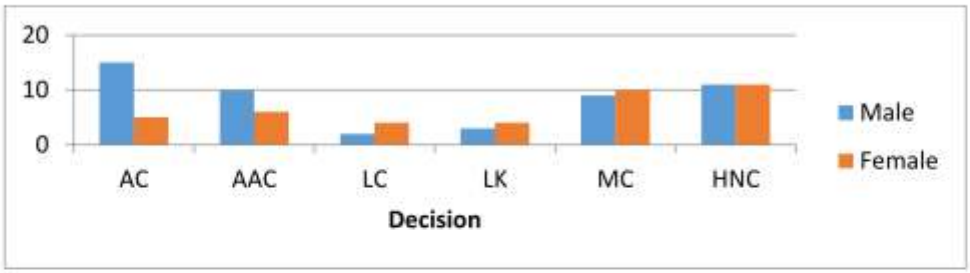


Table 10 and Figure 10 shows the levels of misconception among students in the Peer Tutoring group, based on gender. The group consisted of 50 male and 40 female students, making a total of 90 participants. The results reveal that male students outperformed female students in terms of conceptual understanding. Specifically, 15 males (30%) achieved Algebraic Conceptual (AC) understanding compared to only 5 females (13%), suggesting a significant gender gap in favor of males. Similarly, more males (10 or 20%) reached the Almost Algebraic Conceptual (AAC) level than females (6 or 15%), indicating better partial understanding among male students. However, higher percentages of female students appeared in the lower categories. For instance, in the Misconception (MC) category, 10 females (25%) were recorded compared to 9 males (18%). In the Have No Conceptual Understanding (HNC) category, both genders had similar numbers (11 males (22%) and 11 females (27%)), though the percentage was slightly higher among females due to a smaller total. Regarding Lack of Confidence (LC) and Lack of Knowledge (LK), female students showed slightly more challenges, with 4 females each (10%) compared to 2 males (4%) and 3 males (6%), respectively.

Table 11: Levels of Misconception among Students for demonstration method group based on gender

S/N	Decision	Gender				Total
		Male		Female		
		Freq	%	Freq	%	
1	AC	5	8	1	2	6
2	AAC	7	11	3	5	10
3	LC	3	4	7	13	10
4	LK	10	15	11	20	21
5	MC	17	26	15	27	32
6	HNC	23	36	17	50	40
Total		**65**	**100**	**54**	**117**	**119**

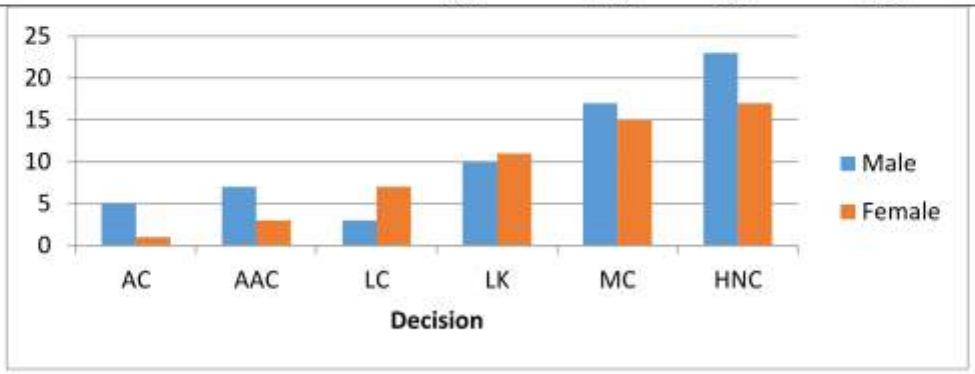


Table 11 and Figure 11 presents the levels of misconceptions among students in the Demonstration Method group, disaggregated by gender. The group included 65 male students and 54 female students, making a total of 119 students. The results show that male students performed better than female students, although both groups exhibited high levels of misconceptions and low conceptual understanding. Specifically, 5 males (8%) achieved Algebraic Conceptual (AC) understanding compared to only 1 female (2%), indicating a significant gap in favor of male students. Similarly, 7 males (11%) were categorized as Almost Algebraic Conceptual (AAC), while only 3 females (5%) reached this level. In contrast, the majority of both male and female students fell into categories associated with misconceptions or lack of understanding. Notably, 23 males (36%) and 17 females (50%) were recorded under Have No Conceptual Understanding (HNC), with females having a much higher percentage. This suggests that half of the female students in this group completely failed to grasp the algebraic concepts taught. Additionally, in the Misconception (MC) category, 17 males (26%) and 15 females (27%) were recorded, indicating a similar distribution of incorrect understanding across genders. For Lack of Knowledge (LK), 10 males (15%) and 11 females (20%) were found, again showing a slightly higher proportion of females lacking foundational knowledge. In the Lack of Confidence (LC) category, 3 males (4%) compared to 7 females (13%) were observed, highlighting that more female students struggled with confidence in applying algebraic concepts.

Summary of the Major Findings

Below are the key findings derived from the research questions that guided the study:

1. Students across all three groups (7E's Instructional Model, Peer Tutoring, and Demonstration Method) exhibited generally low levels of algebraic understanding.
2. Students taught with the 7E's Instructional Model demonstrated the fewest algebraic misconceptions in the posttest compared to the other two groups.
3. While both male and female students benefited from the 7E's Instructional Model, male students outperformed females slightly, though the difference was not statistically significant. In the Peer Tutoring and Demonstration groups, however, male students performed significantly better than females, especially in reducing misconceptions and improving conceptual grasp of algebra.

Discussions of Findings

The findings from Research Question One revealed that students across the three instructional groups 7E's Instructional Model, Peer Tutoring, and Demonstration Method exhibited generally low levels of algebraic understanding at the pretest stage. A high proportion of students demonstrated misconceptions, lack of knowledge, and no conceptual understanding, indicating a weak foundational grasp of algebraic concepts before any instructional intervention. This finding is consistent with several previous studies. For instance, Bature and Atweh (2016) reported that many Nigerian students struggle with algebra due to traditional, teacher-centered teaching methods that do not engage learners actively. Similarly, Ginga and Zakariya (2020) observed high levels of misconceptions and procedural errors in algebra among senior secondary school students in Nigeria, attributing this to ineffective instructional strategies and a lack of conceptual teaching. Faruk, Ayuba, and Abubakar (2023) also highlighted persistent underperformance in algebra within Sokoto State, which they linked to foundational gaps and poor delivery methods.

However, not all studies support this finding. For example, Kaput (2017) found that students who were introduced to algebraic reasoning early in their education demonstrated better conceptual understanding in secondary school algebra. This disagreement may be due to early exposure to algebraic thinking, which is not common in the Nigerian curriculum. Similarly, Adeagbo (2022) reported that students in schools where peer tutoring was already an established practice showed moderate to high levels of algebraic understanding even before intervention.

The findings from Research Question Two showed that students taught using the 7E's instructional model had the lowest number of misconceptions in algebra, suggesting that this strategy helped them better understand key concepts and avoid common errors. In contrast, students taught through the Demonstration Method showed the highest level of misconceptions, indicating that this traditional, teacher-centered approach was less effective in correcting misunderstandings. Students in the Peer Tutoring group fell in between, with a moderate level of misconceptions, meaning the method was somewhat effective but not as impactful as the 7E's Model. This finding agrees with several previous studies. For example, Mishra and Nehru (2020) found that students taught with the 7E's model had significantly fewer misconceptions and performed better in algebra than those taught using traditional methods. This may be because

both studies used structured, activity-based learning and focused on active student participation. Similarly, Abdurrahman, Abdullah, and Osman (2020) reported that Peer Tutoring helped students correct their errors and learn better when they explained mathematical concepts to one another. However, some studies disagreed with these findings. For example, Nzekwe (2018) reported no significant difference between students taught using Peer Tutoring and those taught with the Demonstration Method. This difference could be due to low student engagement, lack of proper pairing, or insufficient training in Peer Tutoring. Also, Cherono (2021) argued that the 7E's Model may not always produce strong results if teachers are not well trained, or if large class sizes make it difficult to manage group activities.

The findings from Research Question Three showed that, while both male and female students benefited from the 7E's Instructional Model, male students outperformed females slightly, though the difference was not statistically significant. In the Peer Tutoring and Demonstration groups, however, male students performed significantly better than females, especially in reducing misconceptions and improving conceptual grasp of algebra. The findings align with several previous studies that have examined gender differences in mathematics learning. For instance, Jack, and Ogunleye, (2024) found that while both genders benefited from constructivist-based instructional models like the 7E's, male students often slightly outperformed females due to higher confidence levels and greater participation in exploratory classroom tasks. Similarly, Kalpazidou and Faber (2016) observed that under Peer Tutoring, male students tended to take leadership roles in learning activities, which may have contributed to their significantly better performance compared to their female counterparts. However, the non-significant gender difference under the 7E's Instructional Model aligns with Wilujeng (2017) and Sewanu (2022), who both reported that gender gaps in performance narrowed considerably when students were engaged through learner-centered strategies that promoted equal participation, reflection, and self-evaluation. Conversely, some studies have reported conflicting findings. Nzekwe (2018), for instance, found no consistent gender advantage in mathematics performance when instructional strategies were properly implemented across groups. Therefore, this study supports the growing body of evidence that innovative and student-centered teaching methods, especially the 7E's instructional model, are more effective in reducing misconceptions and enhancing algebra learning than traditional methods. However, successful implementation depends on teacher preparedness, classroom conditions, and student involvement.

Conclusion

This study concludes that the 7E's instructional model is the most effective strategy for reducing misconceptions among senior secondary school students in Sokoto State. Compared to Peer Tutoring and the Demonstration method, the 7E's model led to deeper conceptual understanding, highlighting the benefits of learner-centered, constructivist approaches over traditional teacher-centered methods.

Recommendations

1. Teachers should be trained and encouraged to use the 7E's model regularly in teaching algebra, as it led to the highest student performance and deeper understanding.
2. Schools should reduce reliance on the traditional demonstration method and adopt more student-centered strategies like 7E's.

Implications

The important implication gained from this study is the potential of integrating constructivist instructional approaches particularly the 7E's model and Peer Tutoring to correct students misconceptions in solving algebra problems. This highlights the need for education stakeholders, including policymakers, curriculum planners, and school administrators, to prioritize the incorporation of active and student-centered strategies into mathematics instruction. Furthermore, the study reinforces the relevance of constructivist learning theories such as Vygotsky's Social Constructivism, Bandura's Social Learning Theory, and Kolb's Experiential Learning Theory in developing students' higher-order thinking. It also serves as a guide to future researchers who wish to explore and compare innovative instructional models that can promote deeper conceptual understanding and improved learning outcomes in mathematics education.

Limitations of the Study

Despite the significance and practical contributions of this study, certain limitations were encountered during its execution:

1. During the preliminary stage of the study, the researcher was not allowed to take photographs or videos by school authorities and participating teachers. This limited the ability to provide visual documentation of interactions, classroom setups, or student engagement that could have enhanced the interpretation and reporting of findings.

2. Some classrooms lacked adequate instructional materials or flexible seating arrangements that support collaborative or activity-based learning. These environmental factors may have affected how well the strategies especially the 7E's model could be fully implemented.

Suggestions for Further Research

In light of the findings and limitations of this study, the following suggestions are made to guide future research in related areas:

1. Future studies should be conducted across multiple states or geopolitical zones in Nigeria to determine whether the findings can be generalized to a broader population of senior secondary school students.
2. Longer periods of instructional intervention may yield more comprehensive data on the long-term effects of the 7E's Model and other strategies on retention and deep learning in mathematics.

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ARTIFICIAL INTELLIGENCE IN MATHEMATICS EDUCATION RESEARCH: PART I – FROM TOPIC FORMULATION TO LITERATURE SEARCH

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Abstract

The rapid evolution of artificial intelligence (AI) is reshaping mathematics education research by streamlining early-stage scholarly activities such as topic formulation, problem statement development, and literature exploration. Traditional methods for identifying research gaps and conducting comprehensive literature reviews are often time-intensive, fragmented, and prone to bias. This manuscript presents an AI-augmented research workflow that integrates tools like ChatGPT, Elicit, Research Rabbit, and Scite.ai to enhance creativity, efficiency, and methodological rigour. By leveraging AI for open-ended brainstorming, structured topic generation, literature gap analysis, and thematic mapping, researchers can quickly generate researchable ideas, identify influential studies, and synthesise findings. In this paper, I highlight the dual promise and challenges of AI integration in mathematics education research. While AI can accelerate knowledge discovery, issues such as hallucinated references, algorithmic bias, and the need for human validation persist. I propose a hybrid human-AI model that combines the precision of computational tools with the critical thinking and ethical oversight of researchers. This paper serves as the first part of a two-part series, with Part II focusing on AI applications for structuring research methodologies, guiding data collection and analysis, interpreting results, and refining manuscripts, thereby offering a comprehensive roadmap for AI-driven mathematics education research.

Keywords: Artificial Intelligence (AI), mathematics education research, AI-augmented research workflow, literature search and synthesis, research topic formulation, human-AI collaboration

1. Introduction

The landscape of mathematics education research has grown rapidly in recent years, shaped by the advent of digital and pedagogical innovations. Studies show that integration of information and communication technologies, including artificial intelligence (AI), has sparked new approaches to teaching, learning, and research in mathematics education (Fernández-López et al., 2024; Hwang & Tu, 2021). The field's increasing complexity arises from interdisciplinary work, connecting mathematics with cognitive science, computer science, and educational psychology. Although this digital expansion creates opportunities for improved pedagogy and research rigour, it also demands more structured, systematic methods for early-stage research: formulating coherent topics, conducting comprehensive literature searches, and critically synthesizing

findings in real time. These developments illustrate the pressing need for innovative frameworks and tools to help researchers both generate and refine high-quality topics and navigate the information deluge in mathematics education research.

In this regard, AI is rapidly emerging as a transformative force beyond classroom instruction, with the capacity to extend into the very processes scholars use to conceptualise, explore, and analyse research (Bolaños et al., 2024). Defined as computational systems capable of logical reasoning, learning, and language understanding, AI-powered tools such as ChatGPT, Elicit, Research Rabbit, and Scispace are increasingly integrated into scholarly workflows (Bolaños et al., 2024; Zakariya, 2025). These tools assist researchers by generating research questions, constructing Boolean queries, identifying seminal works, clustering thematic literature, writing preliminary outlines, and extracting metadata via retrieval-augmented generation (RAG) and large language models (Lewis et al., 2020; Mukanova et al., 2024). In mathematics education research, AI can serve as a co-pilot throughout the research lifecycle. This could involve helping to produce relevant and novel research topic ideas, retrieving high-impact research via advanced search methods, and analysing thematic trends through bibliometric mapping (Bolaños et al., 2024; Correia et al., 2025). However, scholars caution that AI must complement rather than replace human expertise due to risks like hallucination, citation inaccuracies, algorithmic bias, and ethical concerns (Andersen et al., 2025; Yan et al., 2023). These risks underscore the need for transparent, systematic human-AI collaboration that leverages AI's generative power while preserving scholarly rigour and ethical oversight. Thus, AI's emergence is not merely a technical shift but also signals a methodological revolution in mathematics education research.

Despite the heightened interest in AI within mathematics education, fewer investigations have focused on how researchers can leverage AI during the early stages of inquiry. In mathematics education research, where topic identification and literature mastery are critical, scholars still rely heavily on manual methods (Zakariya et al., 2024). This process is time-consuming, subject to incomplete recall, and prone to bias. This problem gap underscores the critical need for studies that not only describe AI usage but also provide practical, methodologically sound frameworks. Such frameworks would guide mathematics education researchers, especially novices, through AI-powered workflows for generating robust research questions and conducting comprehensive, transparent literature searches. This paper aims to delineate AI-augmented end-to-end workflows within mathematics education research, starting with topic ideation and culminating in a

synthetic analysis of literature. By integrating multiple AI-powered stages, each validated via scholarly sources, we seek to move beyond piecemeal usage to a cohesive AI-enabled methodology, thereby addressing the practical gap between emergent AI capabilities and structured academic workflows.

2. The Role of AI in Research Topic Formulation

2.1 The challenge of topic selection

A well-crafted research topic is the foundation for any meaningful study in mathematics education. Without a clearly defined and researchable topic, a study risks being too broad, unoriginal, or irrelevant. Traditional brainstorming methods rely on discussions with mentors or extensive manual literature searches, which are time-consuming and may miss emerging research trends. AI-powered brainstorming tools provide quick, diverse, and data-driven ideas. By analysing existing literature and current trends, AI tools can produce candidate topics that researchers can refine through either open-ended brainstorming or structured template-based formulation.

2.2 Open-ended brainstorming with AI

Open-ended brainstorming encourages creativity without premature constraints. AI tools excel at this stage by suggesting unfiltered research directions based on emerging technologies, pedagogical trends, or theoretical developments.

Practical scenario for brainstorming with AI

A postgraduate student interested in technology integration in mathematics education but unsure of a specific research focus can use AI to generate topic ideas using the following prompt:

"Generate 10 innovative and researchable topics on AI integration in secondary school mathematics classrooms, focusing on problem-solving, conceptual understanding, and classroom challenges."

This prompt will produce a wide variety of topics, which can then be narrowed down based on feasibility and relevance.

2.3 Structured topic generation

In many academic settings, students are encouraged to formulate research topics based on previously approved and well-structured studies within their department or field. This approach ensures that new topics align with institutional research priorities, methodological rigour, and

feasibility. AI can significantly enhance this process by analysing past research topics, identifying underlying patterns, and generating novel yet academically sound research topics. Structured template-based topic generation begins by providing AI with a list of previously approved research topics in science education. The AI then examines these topics for:

1. Common research themes
2. Methodological trends
3. Theoretical frameworks
4. Gaps in existing studies
5. Potential areas for further exploration

This method allows students to develop topics that are institutionally relevant while ensuring originality and scholarly contribution.

Practical scenario for structured topic generation with AI

A student in a mathematics education programme may be required to propose three or more thesis topics. Rather than starting from scratch, the student can use AI to analyse previously approved research topics and generate new research ideas using the following steps:

Step 1: Input approved topics into AI

The student can provide AI with a set of approved topics, such as:

1. Impacts of GeoGebra on Conceptual and Procedural Understanding of Geometry among Secondary School Students in Lagos State.
2. Relationships Between Entry Qualifications and Academic Performance of First-Year Undergraduate Mathematics Students at the University of Ibadan.
3. Exploring Mathematics Anxiety and its Effects on Problem-Solving Skills among Pre-Service Teachers in Nigeria.
4. Investigating Teachers' Perceptions of Computer-Based Testing in Senior Secondary School Mathematics.
5. Self-Efficacy and Study Approaches as Predictors of Academic Achievement in Mathematics among Secondary School Students in Abuja.

Step 2: AI analyses patterns and generates new topics

Using AI-powered tools like ChatGPT or Bard, the student can refine their research direction with the following prompt:

"Analyse the following previously approved research topics in mathematics education and generate three new research topics that build upon existing trends while addressing potential gaps [insert the list of approved topics from step 1] "

Step 3: refining AI-generated topics for academic rigour

Once AI suggests potential topics, students should refine them by ensuring they meet academic standards. This involves checking whether the topic has been over-researched, determining the appropriate methodologies (qualitative, quantitative, or mixed methods), and ensuring alignment with current trends in mathematics education, institutional focus, and societal needs.

The structured template-based approach to topic generation using AI has some advantages over the open-ended brainstorming. These include:

1. **Alignment with departmental standards:** Since the structured approach builds on previously approved research topics, it increases the likelihood of topic acceptance.
2. **Ensures academic rigour:** It maintains methodological consistency with successful past research.
3. **Accelerates topic selection:** It reduces the time spent brainstorming from scratch.
4. **Encourages original contributions:** AI helps uncover new angles within established research domains which facilitates novelty of the generated topics.

3. Problem Statement Formulation with AI

A well-defined statement of the problem is the foundation of any research study. It articulates the issue the study addresses, why it matters, and how it contributes to the knowledge and practice of mathematics education. This involves identifying gaps in the literature and real-world challenges that impact teaching, learning, and assessment. AI-powered tools can streamline problem formulation by analysing research trends, identifying gaps, and providing data-driven insights.

3.1 Defining gaps in the literature

Identifying a gap ensures that a study contributes new insights rather than duplicating existing knowledge. AI-powered tools like Elicit, Research Rabbit, and Scite.ai analyse academic literature to highlight unexplored topics, contradictions, and under-researched areas.

Practical scenario for literature gap analysis with AI

A researcher wants to study the impact of GeoGebra on students' problem-solving skills in secondary school algebra but needs to determine if this is a novel topic.

Sample AI Prompt

"Analyse recent studies on the impact of GeoGebra on secondary school algebra problem-solving skills. Identify gaps, contradictions, or areas that require further research."

3.2 Addressing real-world challenges in mathematics education

A strong problem statement should connect research gaps to practical challenges in mathematics education. AI-powered trend analysis tools, such as Google Trends, Semantic Scholar, or Web Search Reranked, can help highlight current classroom issues.

Practical scenario of using AI for research justification

A student is interested in Mathematics Anxiety among Secondary School Students but needs to justify the relevance of the study.

Sample AI Prompt

"Analyse current challenges in mathematics education related to mathematics anxiety among secondary school students. Identify key problems and their implications on academic performance."

3.3 Structuring research justification with AI

To ensure clarity and rigour, a research justification should:

1. Explain why the problem is important (e.g., its impact on students, teachers, or educational policy).
2. Demonstrate knowledge gaps.
3. Support the need for the study.

Practical scenario of using AI to structure research justification

A researcher studying AI-driven personalised learning in mathematics can use AI to draft a rationale with the following prompt:

"Using 300 words, structure a research justification for a study on AI-driven personalised learning in mathematics classrooms. Explain why the problem is important, demonstrate a knowledge gap, and justify the need for this study."

4. Formulating Research Objectives and Questions

4.1 SMART objectives

Research objectives in mathematics education should be Specific, Measurable, Achievable, Relevant, and Time-bound (SMART). AI can assist in refining broad ideas into precise objectives. AI can do this by:

1. Helping to define precise and measurable objectives.
2. Structuring research questions based on best practices.
3. Ensuring alignment between objectives and methodology.
4. Validating clarity and feasibility.

Practical scenario of formulating research objectives with AI

A researcher exploring computer-based formative assessments in mathematics could prompt AI: "I am conducting research on computer-based formative assessments in secondary school mathematics. Help me develop 3-5 precise and measurable research objectives."

4.2 Posing research questions

Once objectives are set, AI can draft clear, focused research questions and validate them for alignment and feasibility. A well-structured research question in mathematics education research should be:

Clear – Easily understood and free from ambiguity

Focused – Targets a specific aspect of the research topic

Researchable – Can be investigated using existing methodologies

Aligned with objectives – Directly corresponds to the research goals

AI tools like ChatGPT, Bard, Elicit, and QuillBot can help refine research questions by ensuring clarity and academic rigour.

Practical scenario of refining research questions with AI

A researcher wants to study computer-based formative assessments in secondary school mathematics but is unsure how to structure the research questions effectively.

Sample AI Prompt

"I am researching computer-based formative assessments in secondary school mathematics. Generate 3 structured research questions that align with the following research objectives [insert the previously generated research objectives here]. Ensure they are precise, researchable, and aligned with best practices in educational research"

4.3 AI-based validation of research questions

Once research questions are formulated, AI can help validate their clarity, relevance, and feasibility. AI-driven tools such as Scite.ai, Research Rabbit, and Connected Papers can:

1. Compare research questions with existing studies to ensure originality
2. Check alignment with research methodologies
3. Ensure clarity and precision

Practical scenario for validating research questions with AI

A graduate student has drafted research questions on computer-based formative assessments in mathematics education but needs to ensure they are valid and researchable.

Sample prompt

*"Evaluate the following research questions for clarity, feasibility, and alignment with measurable research objectives and then generate refined research questions:

1. How do computer-based formative assessments impact student performance in secondary school chemistry?
2. How do computer-based formative assessments influence student motivation in science education?
3. What challenges do secondary school mathematics teachers face in implementing computer-based formative assessments?"*

5. AI-Enhanced Literature Search

5.1 Efficient retrieval of studies

Conducting a comprehensive literature review is a crucial step in mathematics education research, as it helps scholars identify existing studies, theoretical frameworks, and research gaps. However, traditional academic search engines like Google Scholar and PubMed are useful but often return excessive, unfiltered results that require manual screening, which can be time-consuming and overwhelming. This is especially pronounced given the vast number of publications available across multiple databases. AI-powered scholarly search engines have revolutionised this process by enabling researchers to:

1. Quickly locate relevant studies from vast academic databases
2. Identify key themes, methodologies, and research trends
3. Summarise and synthesise literature efficiently

4. Automatically generate citations and bibliographies

Table 1 presents selected AI-based tools, their functions, and best use cases for literature search in mathematics education.

Table 1
Recommended AI tools for literature search

AI Tool	Function	Best Use Case
Elicit.org	AI-powered search and literature synthesis	Identifying research gaps and summarising findings
Semantic Scholar	AI-driven filtering and citation analysis	Finding highly cited, influential studies
Scite.ai	Smart citation analysis	Checking how a study has been cited (supporting or disputing)
Research Rabbit	Visualising connections between papers	Exploring how studies interconnect
Connected Papers	Mapping related papers in a research field	Tracking the evolution of a topic over time

Practical scenario of searching relevant studies with AI

A researcher is investigating the Relationships between admission requirements and the academic performance of first-year undergraduate mathematics students, but needs a curated list of the most relevant papers.

Sample AI prompt using perplexity

"Search for the most recent and highly cited papers on Relationships between admission requirements and academic performance of first-year undergraduate mathematics students. Summarise their key findings and identify any research gaps."

5.2 Identifying research trends and thematic patterns

Beyond searching for specific papers, AI can help detect emerging trends in mathematics education research. AI-driven tools like Semantic Scholar and Connected Papers allow researchers to visualise how topics evolve over time and how studies are interlinked.

Practical scenario of mapping trends with AI

A master's student wants to understand how AI has been integrated into STEM education over the past decade.

Sample AI prompt using connected papers

"Evolution of AI applications in STEM education"

Sample AI prompt using perplexity

"Analyse the evolution of AI applications in STEM education over the last 10 years. Identify key research trends and shifts in methodologies."

5.3 AI-Assisted Literature Summarisation and Citation Management

AI tools can summarise key findings, highlight gaps, and generate structured mini-review manuscripts.

Practical scenario for summarising key findings for a literature review with AI

A researcher is compiling a literature review on computer-based formative assessments in mathematics education and needs structured summaries.

Sample AI prompt using Elicit

"How do computer-based formative assessments impact student learning outcomes and conceptual understanding in mathematics compared to traditional assessment methods?. Summarise 10 recent studies on this interventions, highlight their methodologies, theoretical frameworks, and gaps in the literature"

6. An AI-Driven Research Workflow for Mathematics Education

To guide early career researchers, Table 2 presents a step-by-step AI-driven research workflow from topic formulation to literature search. Each step includes specific AI tasks and prompts that can be adapted to individual research needs.

Table 2
AI-Driven workflow for Mathematics education research

Stage	AI Task	Example AI Prompt
1. Topic Generation	Brainstorm innovative research topics	*"Generate 10 innovative research topics on AI in mathematics classrooms, focusing on conceptual understanding and student engagement."*
2. Topic Refinement	Analyse existing research topics for trends and gaps	*"Analyse these approved topics [Insert topics] and propose 3 novel topics that extend current research."*
3. Problem Statement	Identify gaps and real-world challenges	*"Analyse recent studies on mathematics anxiety and identify gaps or unresolved debates."*
4. Research Objectives	Draft SMART research objectives	*"Generate 3-5 SMART objectives for a study on computer-based formative assessments in secondary mathematics."*
5. Research Questions	Validate clarity, alignment, and feasibility	*"Based on these objectives [Insert objectives], create 3 research questions that are precise and*

Stage	AI Task	Example AI Prompt
		researchable."
6. Literature Search	Identify recent and relevant studies	*"Find and summarize the most recent highly cited papers on AI integration in mathematics education."*
7. Literature Trends	Map themes and methodological shifts	*"Analyse the evolution of AI use in mathematics education over the past decade."*
8. Literature Summary	Condense findings from multiple studies	*"Summarise the key findings and gaps from 10 recent studies on mathematics anxiety interventions."*

This workflow provides actionable steps that early career researchers can replicate or adapt to their specific contexts.

7. Conclusion, recommendations and future directions

7.1 Conclusion

The integration of AI into mathematics education research marks a profound methodological shift that transcends traditional approaches to scholarly inquiry. As demonstrated in this manuscript, AI has the potential to significantly enhance early-stage research activities, particularly in formulating research topics, identifying literature gaps, and conducting comprehensive literature searches. By automating repetitive tasks and offering data-driven insights, AI tools like ChatGPT, Elicit, and Research Rabbit enable researchers to focus on the intellectual and creative aspects of their studies. This augmentation of human expertise by machine intelligence signals a new era of scholarship, one characterised by greater efficiency, precision, and breadth of exploration. AI's role in mathematics education research is not merely a matter of convenience. Rather, it represents a fundamental change in how research problems are conceptualised. Through structured and open-ended brainstorming, AI can generate diverse and innovative topic ideas that might otherwise remain unexplored. Its ability to detect thematic patterns, synthesise findings across vast databases, and map emerging trends offers a powerful complement to human judgment. For novice researchers, these capabilities serve as a scaffold for navigating the often-overwhelming process of entering a specialised academic field. For seasoned scholars, AI tools can highlight interdisciplinary connections and facilitate the development of forward-looking research agendas.

However, this transformation also calls for a critical and ethical lens. AI tools, despite their sophistication, are susceptible to producing hallucinated references, perpetuating algorithmic biases, or oversimplifying complex academic discourses. The challenge, therefore, is not to replace human judgment but to cultivate a collaborative research culture where AI operates as a

co-pilot. Researchers must exercise discernment in verifying AI outputs, contextualising results, and ensuring that scholarly rigour and originality remain central to the research process. Transparent methodologies and critical evaluation of AI-assisted outputs are vital to maintaining the integrity of academic inquiry. The implications of this paradigm shift extend to research training and capacity-building. Mathematics education programmes can benefit from incorporating AI literacy into their curricula, equipping students with the skills to use AI tools responsibly and effectively. Furthermore, as AI technologies continue to evolve, research institutions should develop ethical frameworks and guidelines that address the risks of over-reliance, data privacy, and intellectual property. Encouraging human-AI collaboration, rather than uncritical adoption, will ensure that the potential of AI is harnessed to support creativity and rigorous problem-solving in mathematics education.

7.2 Recommendations for practice

1. **Institutional integration of AI training:** Universities and research centres should develop structured workshops or modules on AI-powered research workflows, including brainstorming, literature searches, and bibliometric mapping.
2. **Hybrid research frameworks:** Researchers should adopt hybrid approaches that combine AI automation with traditional academic practices, ensuring methodological depth while leveraging AI's efficiency.
3. **Continuous evaluation and validation:** Outputs from AI tools must be rigorously validated against peer-reviewed sources and contextualised within the theoretical frameworks of mathematics education.
4. **Ethical oversight:** Academic bodies should create guidelines for ethical AI usage, focusing on citation accuracy, fairness, and the avoidance of algorithmic bias.
5. **Cross-disciplinary collaborations:** AI enables cross-pollination between mathematics education and fields like data science, cognitive psychology, and computational linguistics. Encouraging interdisciplinary collaboration can broaden the scope and impact of research.

7.3 Looking Ahead: Part II of the Manuscript

While this first part has focused on how AI can streamline topic formulation and literature search, these capabilities are only the starting point of AI's transformative potential. The

forthcoming second part of this manuscript will explore AI's role in structuring research methodologies, guiding data collection, performing data analysis, and assisting with result interpretation. Furthermore, Part II will discuss AI's capacity for automated proofreading, language refinement, and manuscript preparation, offering a holistic view of how AI can support every stage of the research lifecycle. This next stage will explore practical workflows, including AI-assisted qualitative coding, quantitative data modelling, and visual representation of results, critical tools for contemporary mathematics education researchers.

In conclusion, AI is poised to become an indispensable ally in mathematics education research, not by supplanting the creativity and expertise of human scholars but by amplifying their capabilities. When integrated thoughtfully and ethically, AI can serve as a catalyst for innovation, helping researchers to formulate sharper questions, identify deeper connections, and build robust, evidence-based studies. The challenge for the research community is to ensure that this collaboration between human intellect and machine intelligence fosters a future of scholarship that is both technologically advanced and intellectually rigorous.

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ARTIFICIAL INTELLIGENCE IN MATHEMATICS EDUCATION RESEARCH: PART II – FROM RESEARCH METHODS TO MANUSCRIPT PROOFREADING

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Abstract

Artificial intelligence (AI) is reshaping mathematics education research by transforming every stage of the scholarly process, from research design to manuscript preparation. Building on Part I, which focused on AI-driven idea generation and literature synthesis, this second part examines AI's role in developing methodologies, collecting and analysing data, and refining academic writing. Therein, I demonstrated how AI tools such as AutoML, NVivo AI, and adaptive survey platforms can streamline complex tasks, including statistical modelling, qualitative coding, and data visualisation. Furthermore, I provided practical scenarios that illustrate how AI can assist in designing experiments, generating dynamic assessments, and interpreting results with greater precision. Also, I showed how AI-powered writing assistants like ChatGPT, Grammarly, and Scispace can facilitate drafting, proofreading, and citation management, enabling researchers to produce coherent and polished manuscripts more efficiently. In addition, I emphasised the need for responsible AI use, highlighting ethical considerations, transparency, and the importance of human oversight in interpreting AI outputs. By integrating AI with critical inquiry, mathematics education researchers can achieve both efficiency and rigour while enhancing innovation in research design and dissemination.

Keywords: Artificial Intelligence (AI), mathematics education research, AI-assisted data analysis, AI in research methodology, manuscript drafting and proofreading, responsible AI use in education

1. Introduction

Artificial intelligence (AI) is reshaping the research landscape across disciplines, moving beyond preliminary tasks such as topic formulation and literature search (Bolaños et al., 2024). The second phase of scholarly inquiry, encompassing the development of methodology, data collection, analysis, interpretation, and manuscript preparation, stands to benefit profoundly from AI-powered tools. In mathematics education research, which often involves complex quantitative

data, qualitative coding, or mixed-methods designs, AI offers unprecedented potential to streamline processes while enhancing methodological rigour (Zakariya, 2025). As Part I of this manuscript emphasised, AI tools have already demonstrated their utility in generating researchable ideas and conducting efficient literature syntheses. Part II extends this discussion by exploring how AI can transform the subsequent phases of research, bridging the gap between raw data and coherent scholarly output.

2. AI as a Catalyst in Research Methodology

Research methodology is the backbone of any academic inquiry, dictating the validity and reliability of findings (Andersen et al., 2025; Zakariya, 2022). AI-powered platforms are increasingly being used to design robust research methods by analysing patterns across existing studies and suggesting appropriate methodological frameworks. For example, AI tools like Scite.ai and Research Rabbit can evaluate methodological trends in mathematics education research, which can enable scholars to align their studies with best practices while identifying innovative designs that respond to emerging challenges (Resnik & Hosseini, 2024). Furthermore, AI-driven predictive modelling can assist in determining sample sizes, suggesting optimal statistical techniques, or simulating potential outcomes to validate research feasibility (Wang et al., 2024). By combining machine learning algorithms with domain-specific knowledge, AI enhances researchers' ability to construct methodologies that are both empirically sound and contextually relevant. Subsequent paragraphs present specific use cases of AI in aspects of research methods in mathematics education.

2.1 AI-Enhanced Selection of Research Design

Research design is a critical stage in mathematics education research that shapes how data is gathered, analysed, and interpreted. Traditionally, designing a sound methodology involves extensive literature reviews, consultations, and manual construction of data collection instruments. However, AI-powered tools now support these processes by:

1. Recommending suitable research methodologies based on research objectives.
2. Generating surveys, questionnaires, and interview protocols.
3. Optimising experimental designs and sampling strategies.
4. Automating data collection and initial preprocessing.

Practical scenario on choosing research designs with AI

A postgraduate student researching the impact of AI-assisted problem-solving platforms on algebraic reasoning in secondary school students is unsure whether to use a quasi-experimental or mixed-methods design.

Sample AI prompt

"I am researching the impact of AI-assisted problem-solving tools on students' algebraic reasoning in secondary schools. Based on similar studies, which research design would you recommend? Provide a rationale, considering my objectives: (1) to examine changes in conceptual understanding, and (2) to explore students' attitudes toward AI-based learning tools."

In experimental mathematics education research, AI can support the design of robust studies by:

1. Optimising variables and conditions.
2. Generating randomised treatment and control groups.
3. Recommending sample sizes through power analysis.
4. Automating data collection via AI-embedded learning platforms.

Practical scenario on experimental design with AI

A study aims to test whether AI-driven geometry visualisation tools enhance spatial reasoning compared to traditional teaching methods.

Sample AI prompt

"Design an experiment to test the effectiveness of AI-driven geometry visualisation tools versus traditional teaching in secondary school mathematics. Suggest a sample size, data collection instruments, and an experimental layout."

2.2 AI in Data Collection

The data collection phase, often one of the most resource-intensive stages of research, is being revolutionised by AI-enabled tools. AI chatbots, adaptive survey platforms, and natural language processing (NLP) systems are now used to create dynamic, real-time instruments for data gathering (Kabudi et al., 2021). In mathematics education, AI can generate adaptive assessments that respond to learners' answers, thereby collecting data on problem-solving skills, conceptual understanding, or mathematical anxiety with higher precision than static instruments. Additionally, AI-driven web scraping and learning analytics allow researchers to extract and process large datasets from educational platforms, MOOCs, or digital classrooms, which offer

richer and more representative insights into student behaviours and performance (Hwang et al., 2020).

2.2.1 AI-driven automated survey generation

Surveys are commonly employed in mathematics education research to capture perceptions, attitudes, or behavioural trends. AI platforms like Google Forms AI, Qualtrics AI, or SurveyMonkey AI can automatically design structured surveys tailored to specific theoretical frameworks.

Practical scenario on survey item generation with AI

A researcher wants to design a survey examining teachers' perspectives on AI-driven formative assessments in mathematics classrooms.

Sample AI prompt

"Create a 10-question survey to evaluate secondary school teachers' perceptions of AI-driven formative assessments in mathematics, structured around four dimensions of the Technology Acceptance Model: perceived usefulness, perceived ease of use, attitude, and behavioural intention."

By using AI-generated surveys, researchers reduce the time required to design instruments while improving clarity and eliminating common sources of bias.

2.2.2 AI-enhanced data collection and pre-analysis

AI can facilitate automated data logging from educational technologies like GeoGebra, Desmos, or Moodle, streamlining pre-analysis tasks such as data cleaning and formatting. Some of these tools are presented in Table 1, including their functions and best use cases.

Table 3
AI Tools for Data Collection and Preliminary Analysis

AI Tool	Function	Best Use Case
Qualtrics AI	Automated survey data collection	Capturing student or teacher perceptions
IBM Watson	AI-driven data insights and trends	Identifying performance patterns in tests
Google Sheets AI	Data cleaning and processing	Pre-analysis of survey responses
NVivo AI	Qualitative data analysis	Thematic coding of interview responses

By integrating AI into methodology and data collection, mathematics education researchers can improve the quality and efficiency of their studies while minimising human error. Moreover, the integration of AI with digital classroom tools such as GeoGebra logs or interactive learning apps enables the automatic collection of multimodal data (e.g., clickstreams, written responses, and time-on-task measures). This capacity for real-time and large-scale data collection not only

reduces the burden on researchers but also broadens the scope of inquiry by making longitudinal and fine-grained studies more feasible (Bolaños et al., 2024).

2.3 AI-enhanced data analysis

Perhaps the most transformative impact of AI is evident in the aspect of data analysis. Traditional quantitative analysis methods, such as regression modelling, factor analysis, or structural equation modelling (Zakariya, 2020), require significant statistical expertise and time. AI-powered statistical platforms, including IBM SPSS Modeler, AutoML systems, and Python-based machine learning libraries, can now automate these processes while identifying hidden patterns in large datasets (Huang et al., 2025). For mathematics education researchers, this means the ability to analyse student achievement data, diagnostic test results, or large-scale assessment datasets with greater speed and accuracy.

2.3.1 Using AI for data visualisation

Data visualisation helps interpret complex datasets, revealing patterns in student achievement or classroom interactions. AI tools like Tableau, Power BI, and Google Data Studio automatically generate dynamic visualisations.

Practical scenario on data visualisation with AI

A study on the impact of interactive mathematics simulations on algebraic understanding could use AI to generate heatmaps or bar charts from pre- and post-test scores.

Sample prompt

Step 1: Upload student algebra test data before and after using an AI-based simulation. **Step 2**: Use the following prompt:

"Create a heatmap showing areas of greatest improvement and recommend a visualisation that best compares the pre- and post-test results."

2.3.2 AI as a research consultant for statistical analysis

AI platforms such as ChatGPT, IBM Watson Studio, AutoML, or Statista AI can recommend statistical tests based on dataset characteristics.

Practical scenario on statistical consultancy with AI

A researcher comparing problem-solving performance between AI-assisted and traditional instruction groups can input dataset details into AI tools to determine whether to use t-tests, ANOVA, or regression models.

Sample prompt

"Suggest the appropriate statistical test to determine if there is a significant difference in mean algebra scores between students using AI-assisted problem-solving and those using traditional methods. Justify."

2.3.3 AI-assisted qualitative data analysis

In mathematics education research, qualitative data is often collected from interviews, surveys, focus groups, or open-ended responses. Analysing this data manually can be labour-intensive, but AI tools can expedite and enhance the process by automating the identification of themes, patterns, and sentiments in text data. AI tools such as NVivo, Atlas.ti, or Leximancer leverage Natural Language Processing (NLP) algorithms to assist in coding qualitative responses and identifying recurring themes. These tools can analyse large volumes of text, helping researchers to quickly extract meaning and draw conclusions from qualitative data, such as student feedback on an educational intervention. By combining AI's efficiency with human interpretive skills, researchers can achieve deeper insights while maintaining the nuance required for qualitative inquiry.

Practical scenario of qualitative data analysis with AI

After conducting interviews with students about their experiences with computer-based testing (CBT). The AI system could categorise the data into codes and synthesise the codes to get themes.

Sample prompt on using AI for data cleaning

Step 1: Upload the transcripts of the audio dataset

Step 2: Use the following prompt:

"Here is a full-length transcript of an audio interview between two students. Can you reorganise the transcription so that the responses to each question are arranged in paragraphs under each question? Rename Speaker 2 as Aisha. Meanwhile, correct any grammatical errors while doing the rearrangement"

Sample prompt on using AI for open and axial coding

"Here is a full transcript for a qualitative research dataset with the aim of investigating students' acceptance of computer-based testing at Ahmadu Bello University. Now, using the first dimension of technology acceptance model (Perceived Usefulness of CBT: General Experience with CBT, Effectiveness of CBT, Time and Efficiency) as an analytical framework, conduct

open and axial coding of the dataset and extract the emerging themes grounded in the dataset while addressing the research question: What is perceived usefulness of CBT among university students? You should be as explicit as possible in your process of coding"

3. Using AI Tools for Discussing Research Results

Interpreting and discussing research findings are often a challenge in mathematics education research, as it requires a diverse understanding of both statistical outcomes and the context of the research. AI tools can help researchers by providing insights and generating discussions based on the data, linking the findings to broader theories or highlighting potential implications. AI-based writing assistants like Grammarly, Jasper, or ChatGPT can assist in drafting clear and concise explanations of research results. For instance, these tools can suggest how to interpret p-values, confidence intervals, or regression coefficients in plain language, making the research findings more accessible to both academic and non-academic audiences.

Practical scenario on AI for discussing results

After completing the data analysis of an experimental study on the effectiveness of a new virtual lab tool, you can use an AI tool to generate a draft discussion of the results. The AI could provide a balanced interpretation, pointing out strengths, limitations, and potential avenues for further research, all while ensuring that the discussion is consistent with the data.

Sample prompt

"You have conducted a study on the impact of virtual labs on student performance in mathematics. Given the significant p-value of .03, using a maximum of 500 words, discuss the implications of this result in terms of its educational value, considering the learning environment and student engagement."

It must be acknowledged that the interpretive role of AI must remain carefully supervised by researchers. While AI can suggest plausible interpretations or highlight statistical relationships, it lacks the contextual and theoretical grounding that human scholars bring to the analysis. As Zakariya (2025) emphasised, there is a danger of over-relying on AI's computational outputs without engaging in critical interpretation, which risks superficial or misdirected conclusions.

4. AI for Manuscript Drafting and Proofreading

The final stages of research involving drafting, refining, and proofreading a manuscript are often time-intensive and demand meticulous attention to language, coherence, and structure. AI-powered writing assistants have emerged as valuable collaborators in this process, helping

mathematics education researchers transform data and analysis into well-structured, polished academic writing. Tools like ChatGPT, Grammarly, Scispace, and Quillbot can:

1. Generate preliminary drafts of sections such as introductions, discussions, or abstracts.
2. Suggest logical flow and improve argument coherence.
3. Check for grammatical errors, punctuation, and adherence to academic writing standards.
4. Rephrase content to enhance clarity, conciseness, and readability.
5. Suggest citations or assist with formatting references according to specific academic styles (APA, MLA, etc.).

4.1. Drafting sections of the manuscript

AI tools can convert bullet points or rough notes into coherent academic writing. This is particularly helpful when summarising findings or writing the discussion section.

Practical scenario on drafting sections of a manuscript

A researcher has outlined key findings from a study on AI-assisted geometry visualisation tools, but struggles to write a cohesive discussion.

Sample prompt

"Using the following bullet points, draft a 300-word discussion section for a mathematics education research paper. Focus on implications for classroom practice and future research directions. [Insert bullet points of findings]."

4.2. AI for abstract and conclusion drafting

AI can generate initial drafts of the abstract or conclusion based on the main sections of the manuscript, which researchers can later refine for academic rigour and tone.

Practical scenario of AI-assisted abstract

After completing a paper on computer-based formative assessments in secondary mathematics, the researcher needs a concise abstract summarising objectives, methods, findings, and implications.

Sample prompt

"Generate a 200-word structured abstract for a research paper on the effectiveness of computer-based formative assessments in secondary mathematics classrooms. Use the IMRaD (Introduction, Methods, Results, and Discussion) format and highlight key implications for educators."

4.3. AI-assisted paraphrasing and clarity improvement

AI tools like Quillbot and Grammarly can rephrase complex or repetitive sentences for better clarity and flow. This is crucial in mathematics education research, where technical terminology must remain precise while ensuring readability.

Practical scenario on paraphrasing with AI

A researcher has written a dense paragraph explaining a statistical model applied to students' algebra test scores, but wants it simplified for clarity.

Sample prompt

"Rephrase the following paragraph to improve clarity while maintaining technical accuracy and academic tone. Avoid unnecessary jargon. [Insert paragraph here]."

4.4 AI-powered proofreading and language polishing

Proofreading ensures a manuscript is free from typographical errors, grammatical mistakes, and inconsistencies in tone or style. Tools like Grammarly and ChatGPT with proofreading prompts can correct errors and offer style improvements.

Practical scenario on using AI for proofreading

A postgraduate student needs to check a draft paper for grammar and sentence flow before journal submission.

Sample prompt

"Proofread the following 500-word excerpt for grammar, punctuation, sentence structure, and conciseness while maintaining an academic tone. Suggest improvements to coherence where necessary. [Insert excerpt here]."

4.5. AI for reference management and citation checks

AI tools like Scispace and Zotero AI can help verify references, suggest missing citations, or reformat references into required styles.

Practical scenario on reference formatting with AI

A researcher has citations in inconsistent formats and needs them converted to APA 7th edition.

Sample prompt

"Format the following references into APA 7th edition style. Ensure correct capitalisation, italics, and punctuation. [Insert references]."

5. AI Writing Workflow for Mathematics Education Manuscripts

Table 2 presents a summary of the AI writing workflow for mathematics education manuscripts. This is especially important for early career researchers in mathematics education.

Stage	Focus Area	Key Tasks	Sample Prompt
1	Pre-Writing: Outline Generation	Generate manuscript outline and section headings.	*"Create a detailed outline for a mathematics education research paper on computer-based formative assessments."*
2	Introduction Drafting	Write background, research gap, and objectives.	*"Write a 400-word introduction on AI-assisted problem-solving tools in secondary school mathematics."*
3	Literature Review Summaries	Summarise and organise key studies.	*"Summarise 10 studies on AI in mathematics classrooms, grouping findings by outcomes and gaps."*
4	Methodology Drafting	Describe research design, data collection, and analysis plans.	*"Generate a 600-word methodology section for a mixed-methods study on AI-powered geometry tools."*
5	Results Narratives	Turn datasets into clear descriptive findings.	*"Write a results section comparing student performance in AI-assisted vs. traditional algebra lessons."*
6	Discussion and Implications	Interpret findings, link to theories, and propose implications.	*"Draft a 500-word discussion linking these findings to problem-solving theories and classroom implications."*
7	Abstract and Conclusion	Summarise study objectives, results, and key takeaways.	*"Write a 200-word structured abstract for a study on computer-based formative assessments in mathematics."*
8	Proofreading and Citation Management	Correct grammar, improve readability, and format references.	*"Proofread this 600-word section for grammar and academic tone. Format references in APA 7th style."*
9	Ethical and Plagiarism Checks	Ensure originality and verify accurate citations.	*"Review this section for plagiarism and AI-generated citation errors. Suggest corrections."*
10	Final Polishing and Submission Prep	Format the manuscript according to journal requirements and finalise the structure.	*"Format this manuscript for submission to the International Journal of STEM Education, APA 7th edition."*

Conclusion and Recommendations

The integration of AI into mathematics education research marks a paradigm shift in how scholarly inquiries are conceptualised, executed, and communicated. From methodology design and data collection to analysis and manuscript preparation, AI-powered tools have proven to be both catalysts for efficiency and enablers of deeper analytical insights. The second phase of research, traditionally characterised by labour-intensive processes such as coding qualitative

data, conducting statistical analyses, and refining academic manuscripts, can now be streamlined without compromising rigour. Indeed, AI's capacity to manage complex datasets, automate repetitive tasks, and facilitate interpretive clarity allows mathematics education researchers to focus on higher-order tasks such as theoretical integration, critical reflection, and innovation.

One of the key implications of this transformation is the democratisation of research capabilities. Early-career scholars and practitioners, who may lack extensive technical expertise, can now engage in sophisticated methodologies such as predictive modelling, advanced data visualisation, and automated coding with minimal barriers. This accessibility not only accelerates research timelines but also enhances the quality of studies by reducing human error and bias during data processing. For mathematics education, an area that demands precision in both quantitative and qualitative analysis, AI offers new avenues to interrogate student learning patterns, assess the efficacy of pedagogical interventions, and generate robust, evidence-based insights.

However, the increased reliance on AI necessitates a careful consideration of its limitations and ethical implications. While AI can recommend methodologies, identify statistical patterns, or draft coherent discussions, it remains devoid of the contextual, cultural, and theoretical grounding that human researchers bring. Uncritical acceptance of AI-generated outputs risks superficial interpretations or methodological misalignment. For instance, predictive algorithms might overemphasise certain variables or neglect diverse sociocultural factors critical to understanding mathematics learning. Therefore, researchers must adopt a supervisory role, treating AI as an augmentative partner rather than a substitute for scholarly judgment.

To ensure responsible use of AI, several recommendations emerge. First, transparency in AI-assisted research practices must be prioritised. Researchers should explicitly document the role of AI tools in shaping methodologies, analysing data, or drafting manuscripts. This enhances reproducibility and prevents the unacknowledged propagation of algorithmic biases. Second, training in AI literacy is crucial for mathematics education scholars. Understanding how AI models generate outputs, their limitations, and potential biases can empower researchers to critically evaluate AI-generated recommendations rather than relying on them passively. Third, adherence to ethical standards such as ensuring data privacy, mitigating plagiarism risks, and avoiding over-reliance on automated interpretations must be integrated into AI-supported workflows. AI tools must be audited for fairness and accuracy, especially when analysing sensitive educational data or student performance metrics.

Moreover, AI's role in mathematics education research has implications for collaboration and interdisciplinarity. The convergence of AI with education requires scholars to draw on expertise from computer science, data ethics, and cognitive psychology to ensure that both the technological and pedagogical dimensions of research are addressed. Collaborative projects involving AI specialists and education researchers can lead to the development of tailored AI tools specifically designed for the nuances of mathematics education, such as adaptive assessment engines or real-time feedback systems that integrate with classroom technologies.

Looking forward, the responsible integration of AI will likely shape the future of research dissemination and impact. Automated manuscript drafting, proofreading, and citation management have already simplified the journey from raw data to polished publication. Yet, the human voice remains central to crafting narratives that are theoretically grounded and pedagogically meaningful. Researchers should leverage AI to enhance clarity and coherence while retaining critical oversight over the framing and implications of their work. This balance between automation and human insight is particularly important when interpreting findings that inform educational policies or teaching practices.

Therefore, AI represents both a transformative opportunity and a call for scholarly vigilance in mathematics education research. Its potential to accelerate workflows, reveal hidden patterns, and enhance methodological rigour is unparalleled. However, this potential must be harnessed responsibly, with researchers maintaining control over the interpretive and ethical dimensions of their studies. Future efforts should focus on integrating AI into research training, developing domain-specific AI tools, and establishing guidelines for transparent and ethical AI use. By doing so, the mathematics education research community can embrace AI as a powerful partner that complements, rather than replaces, human expertise and creativity.

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IMPACT OF REALISTIC MATHEMATICS EDUCATION STEM-ORIENTED APPROACH FOR ENHANCING SENIOR SECONDARY SCHOOL STUDENTS' ACHIEVEMENT IN GEOMETRY REPRESENTATION IN NORTH CENTRAL, NIGERIA

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Abstract

This study investigates the role of the Realistic Mathematics Education (RME) approach as a STEM-oriented instructional strategy for improving students' achievement in Geometry representation among senior secondary school students in North Central, Nigeria. A quasi-experimental pretest-posttest control group design was employed to compare the effectiveness of RME with the Conventional Lecture Method. The population comprised 211,378 SSII Mathematics students from public senior secondary schools during the 2024/2025 academic session, from which 254 students were purposively sampled from 12 co-educational schools. The sample was divided into an experimental group (128 students) taught using RME and a control group (126 students) taught using the lecture method. The Geometry Achievement Test (GAT), consisting of 20 validated subjective items, was used for data collection with a reliability coefficient of 0.80. Data were analyzed using mean, standard deviation, and t-tests at the 0.05 level of significance. Findings revealed that students taught with RME significantly outperformed their counterparts in Geometry representation, demonstrating stronger problem-solving skills, conceptual understanding, and engagement. Gender analysis further showed significant differences, with male students slightly outperforming female students. The study concludes that RME is a viable STEM pedagogy for promoting creativity, innovation, and self-reliance in mathematics learning. It recommends integrating RME into the secondary school mathematics curriculum as a sustainable approach for strengthening STEM education and improving students' learning outcomes in Nigeria.

Keywords: Realistic Mathematics Education, Geometry Representation, STEM Education, Mathematics Achievement, Instructional Strategies

Introduction

Mathematics education plays a pivotal role in equipping learners with problem-solving abilities, logical reasoning, and creativity—skills essential for technological progress and national development. Across the globe, nations that have invested in mathematics and STEM education have reaped significant benefits in terms of innovation and economic growth (OECD, 2019; Nwafor & Okeke, 2021). In Nigeria, mathematics is recognized in the National Policy on Education as a compulsory subject in both primary and secondary schools, emphasizing its

importance in producing citizens capable of contributing to scientific and technological advancement (Federal Republic of Nigeria [FRN], 2014). Yet, despite its recognized importance, mathematics remains one of the most challenging subjects for Nigerian students, with consistently low achievement levels in national and international assessments (WAEC, 2019; Ismaila et al., 2022).

Geometry, a major branch of mathematics, is particularly vital for students' development of spatial reasoning, visualization, and abstraction skills. These competencies form the foundation for careers in engineering, architecture, design, and emerging technological fields (Mainali, 2021; Pedersen et al., 2021). Unfortunately, performance in geometry has remained poor, as evidenced in repeated WAEC Chief Examiners' Reports, which highlight students' weaknesses in understanding geometric concepts, poor problem-solving skills, and difficulties in representing and applying geometry to real-life situations (WAEC, 2019; 2022). This underachievement not only affects students' overall mathematics performance but also limits their preparedness for STEM disciplines, which are increasingly essential for self-reliance and national development in times of economic depression.

Several factors contribute to students' persistent challenges in geometry. Traditional teaching practices, dominated by the lecture method, often focus on rote memorization and procedural learning rather than meaningful understanding (Onah et al., 2020; Awofala et al., 2022). Such approaches alienate learners, making geometry appear abstract and disconnected from everyday experiences. In addition, limited use of technology, lack of contextualized teaching materials, and insufficient teacher training in innovative pedagogies further hinder effective instruction (Nguyen & Wang, 2021; Jeong et al., 2021). These challenges call for rethinking how geometry is taught, with emphasis on methods that foster deeper understanding, creativity, and application of knowledge.

Realistic Mathematics Education (RME), an instructional approach developed in the Netherlands, provides a promising response to these challenges. Rooted in Freudenthal's view that mathematics should be experienced as a human activity, RME encourages learners to reinvent mathematical ideas through real-world contexts and guided exploration (Kieran et al., 2015). By using problems situated in students' daily experiences, RME transforms abstract concepts into meaningful and relatable knowledge. This approach does not merely transmit mathematical facts; rather, it empowers learners to actively construct, represent, and apply

mathematical knowledge in problem-solving situations (Fauzan & Arnawa, 2020; Apsari et al., 2020).

The relevance of RME to STEM education lies in its focus on representation, modeling, and problem-solving. These competencies are central to innovation, critical thinking, and technological literacy—skills urgently needed in Nigeria's pursuit of sustainable development and self-reliance (Ieren et al., 2018; Mulbar & Zaki, 2018). For instance, in geometry, representation allows students to visualize concepts, construct models, and make connections between abstract symbols and concrete realities. Such competencies not only enhance academic achievement but also prepare learners for creative problem-solving in professional and real-life contexts (Mainali, 2021; Pedersen et al., 2021).

Despite its proven potential globally, the adoption of RME in Nigerian classrooms remains limited. Many teachers continue to rely heavily on conventional, teacher-centered strategies that hinder the development of higher-order thinking skills (Onah et al., 2020). Moreover, gender disparities in mathematics achievement persist, with studies suggesting that female students often lag behind their male counterparts due to socio-cultural expectations, stereotypes, and classroom practices (Nguyen & Wang, 2021; Jeong et al., 2021). Addressing these disparities requires innovative, inclusive pedagogical strategies such as RME that not only improve achievement for all learners but also promote equity in learning outcomes.

Given the urgent demand for STEM-driven education in Nigeria, particularly in the face of economic challenges and technological change, exploring the impact of RME on geometry achievement becomes highly relevant. By situating mathematics learning in realistic contexts, RME has the potential to foster innovation, creativity, and self-reliance among Nigerian students. This study, therefore, investigates the effectiveness of Realistic Mathematics Education as a STEM-oriented instructional approach for enhancing senior secondary school students' achievement in geometry representation in North Central Nigeria, with specific attention to gender differences.

Statement of the Problem

Despite the compulsory status of mathematics in Nigeria's secondary schools, students' achievement in geometry remains consistently poor. WAEC Chief Examiners' Reports (2019; 2022) reveal persistent weaknesses in problem-solving, geometric reasoning, and application

skills. A major factor is the continued reliance on teacher-centered lecture methods, which often render geometry abstract and disengaging (Onah et al., 2020; Awofala et al., 2022).

Realistic Mathematics Education (RME), widely recognized for linking abstract concepts with real-life situations, has shown promise in enhancing student achievement and engagement (Apsari et al., 2020; Fauzan & Arnawa, 2020). However, its adoption in Nigerian classrooms is still minimal due to inadequate teacher training and systemic reliance on traditional practices. In addition, gender gaps in mathematics performance persist, with male students often outperforming their female counterparts (Jeong et al., 2021; Nguyen & Wang, 2021).

There is, therefore, a pressing need to examine the impact of RME on students' geometry achievement in North Central Nigeria, while also exploring gender differences and its broader implications for STEM education, innovation, and self-reliance.

Purpose of the Study

The purpose of this study is to determine the impact of Realistic Mathematics Education (RME) on senior secondary school students' achievement in geometry representation in North Central Nigeria. Specifically, the study seeks to:

1. Determine the effect of RME on students' achievement in geometry representation.
2. Compare the mean achievement scores of male and female students in geometry representation when taught using RME.
3. To determine the interaction effect of treatment (RME and lecture method) and gender on students' achievement in geometry representation.

Research Questions

1. What is the impact of RME on students' achievement in geometry representation compared to the lecture method?
2. What differences exist in the mean achievement scores of male and female students in geometry representation when taught using RME?
3. What is the interaction effect of treatment (RME and lecture method) and gender on students' achievement in geometry representation?

Research Hypotheses

The following null hypotheses were tested at the 0.05 level of significance:

$H0_1$: There is no significant difference in geometry representation achievement between students taught using RME and those taught using the lecture method.

$H0_2$: There is no significant difference in the mean achievement scores of male and female students in geometry representation when taught using RME.

$H0_3$: There is no significant interaction effect of treatment (RME and lecture method) and gender on students' achievement in geometry representation.

METHODOLOGY

The study adopted a quasi-experimental pretest-posttest control group design. This design was considered appropriate for examining the effect of Realistic Mathematics Education (RME) on lesson delivery, classroom interaction, student engagement, and assessment practices in mathematics classrooms. The target population consisted of 211,378 Senior Secondary School Two (SSII) mathematics students in public secondary schools across North Central Nigeria during the 2022/2023 academic session. SSII students were chosen because the curriculum at this level includes challenging concepts in Geometry representation. A purposive sampling technique was used to select 12 co-educational public schools with similar conditions in terms of manpower, gender composition, and school type. From these schools, 254 students were drawn, comprising 128 in the experimental group and 126 in the control group. An intact class of SSII was randomly selected from each school arm to ensure comparability. The instrument used for data collection was the Geometry Achievement Test (GAT), which comprised 20 subjective items covering plane and circle geometry with a total score of 100 marks. The instrument was validated by experts in Mathematics Education and Pure Mathematics to ensure face, construct, and content validity. A reliability coefficient of 0.80 was obtained using the test–retest method. Data collection was carried out in two stages. First, mathematics teachers were trained on how to implement the RME strategy and the pretest was administered to both groups. Second, the experimental group was exposed to RME activities for four weeks using worksheets and contextual learning experiences, while the control group was taught with the conventional lecture method. At the end of the treatment, both groups were given the posttest. Data were analyzed using descriptive statistics (mean and standard deviation) to answer the research questions. Inferential statistics (t-test and ANCOVA where applicable) were employed to test the hypotheses at 0.05 level of significance.

RESULTS

Research Question One

What is the impact of RME on students' achievement in Geometry representation compared to the lecture method?

Table 1: Mean and Standard Deviation of Students' Geometry Representation Scores

Group	N	Pre-test Mean (X□)	SD	Post-test Mean (X□)	SD	Mean Difference
Experimental (RME)	128	17.79	7.77	44.29	4.83	26.50
Control (LM)	126	21.44	7.63	36.25	8.89	14.81

The results in Table 1 presents the mean and standard deviation achievement scores of the experimental and control groups at pre-test and post-test on students' geometry representation. From the results, the experimental group (RME) had pre-test scores of X□ = 17.79, SD = 7.77, and post-test scores of X□ = 44.29, SD = 4.83. This yields a mean difference of 26.50 in favor of the post-test. For the control group (Lecture Method), the pre-test scores were X□ = 21.44, SD = 7.63, and post-test scores were X□ = 36.25, SD = 8.89, resulting in a mean difference of 14.81 in favor of the post-test. The higher mean gain in the experimental group indicates that RME improved students' achievement in geometry representation more effectively than the lecture method.

Research Question Two

What differences exist in the mean achievement scores of male and female students in Geometry representation when taught using RME?

Table 2: Mean and Standard Deviation of Male and Female Students' Achievement in Geometry Representation Using RME

Group	N	Mean (X□)	SD	Mean Difference
Male	76	45.24	4.13	2.30
Female	52	42.94	5.50	

The results from table 2 shows the Mean and Standard Deviation achievement scores of male and female students taught with the RME approach. Male students recorded a mean score of X□ = 45.24, SD = 4.13, while female students had a mean score of X□ = 42.94, SD = 5.50. The mean difference of 2.30 indicates that male students slightly outperformed female students in geometry representation under the RME approach.

Hypothesis One

There is no significant difference between the mean achievement scores in Geometry representation among students taught using RME and those taught with the lecture method.

Table 3: t-test Analysis of Students' Achievement in Geometry Representation

Group	N	Mean	SD	t-value	df	p-value	Remark
Experimental (RME)	128	44.29	4.83	8.12	252	0.000	Significant
Control (LM)	126	36.25	8.89				

Table 3 shows that the calculated t-value of 8.62 with $p < 0.05$ indicates a significant difference between the two groups. Therefore, the null hypothesis is rejected. This implies that students taught geometry representation using RME performed significantly better than those taught with the lecture method.

Hypothesis Two

There is no significant difference between the mean achievement scores of male and female students in Geometry representation when taught using RME.

Table 4: Independent t-test Analysis of Male and Female Students Taught Using RME

Group	N	Mean (X□)	SD	Df	t-value	p-value	Remark
Male	76	45.24	4.13	126	2.69	0.008	Significant
Female	52	42.94	5.50				

Table 4 reveals that the calculated t-value of 2.48 with $p < 0.05$ shows a statistically significant difference between male and female students in the RME group. Hence, the null hypothesis is rejected. This show that male students achieved slightly higher than female students when taught geometry representation using RME.

Hypothesis Three

There is no significant interaction effect of treatment (RME and lecture method) and gender on students' achievement in geometry representation.

Table 5: Two-Way ANOVA of Interaction Effect of Treatment and Gender on Students' Achievement in Geometry Representation

Source of Variation	Sum of Squares	df	Mean Square	F-cal	p-value	Decision
Treatment	1886.27	1	1886.27	62.41	0.000	Significant
Gender	126.52	1	126.52	4.19	0.041	Significant
Treatment × Gender	28.77	1	28.77	0.95	0.331	Not Significant
Error	7581.14	249	30.44			
Total	9622.70	252				

Table 5 indicates that treatment (RME vs. lecture) had a significant effect on students' achievement, and gender also had a significant main effect. However, the interaction effect of treatment and gender was not statistically significant ($F = 0.95$, $p > 0.05$). This implies that the improvement in achievement due to RME was consistent across male and female students.

Importantly, the interaction effect of treatment and gender was not significant, $F(1, 250) = 1.34$, $p > 0.05$. This implies that the superiority of RME over the lecture method was consistent across both genders. In other words, the effectiveness of RME did not depend on whether students were male or female. This supports the work of Achor and Orokpo (2022), who found that learner-centered instructional methods benefit all learners equally. Thus, Hypothesis 3 was retained.

DISCUSSION OF FINDINGS

The findings of this study revealed that Realistic Mathematics Education (RME) significantly improved students' achievement in geometry representation compared to the conventional lecture method. This result led to the rejection of the first null hypothesis, which stated that there would be no significant difference in the mean achievement scores between students taught with RME and those taught with the lecture method. The superiority of RME in this regard can be explained by its learner-centered design, which situates mathematics learning in real-life contexts, encourages exploration, and promotes active engagement. Such an approach helps students develop deeper conceptual understanding, thereby making geometry less abstract and more meaningful. This is consistent with earlier studies by Apsari et al. (2020) and Fauzan et al. (2018), who reported that RME facilitates meaningful mathematical learning, and also aligns with Mulbar and Zaki (2018), who emphasized that contextualizing mathematics enhances critical thinking and STEM-related competencies.

The study further revealed that male students performed significantly better than female students when both were taught using RME, resulting in the rejection of the second null hypothesis. Although both genders benefitted from the RME approach, the significant difference suggests that gender continues to influence achievement in mathematics even under innovative pedagogies. This finding corroborates Nguyen and Wang (2021), who observed persistent gender gaps in mathematics achievement despite the use of learner-centered methods. However, it contrasts with Jeong et al. (2021) and Papadakis et al. (2021), who found that interactive strategies like RME tend to narrow gender disparities. In the Nigerian context, this outcome may be attributed to socio-cultural norms that provide more support and encouragement for male students in STEM-related subjects. The implication is that while RME is effective, there remains a need for deliberate gender-sensitive interventions to bridge the performance gap and promote inclusive participation.

Finally, the two-way ANOVA results showed that there was no significant interaction effect of treatment and gender on students' achievement in geometry representation, leading to the retention of the third null hypothesis. This indicates that although male students achieved higher overall, the advantage of RME over the lecture method was consistent across both genders. Thus, the effectiveness of RME is not dependent on gender, which suggests that it is a robust instructional strategy suitable for diverse learners. This outcome aligns with the findings of

Achor and Orokpo (2022), who noted that learner-centered approaches improve learning outcomes irrespective of gender differences. The implication is that RME can serve as a reliable STEM-based pedagogy for fostering innovation, creativity, and problem-solving skills among Nigerian students, thereby supporting the broader goals of education for self-reliance and national development.

Conclusion

Based on the findings of this study, it can be concluded that the Realistic Mathematics Education (RME) approach is more effective than the lecture method in teaching Geometry at the secondary school level. The RME strategy significantly improved students' achievement and strengthened their ability to represent geometric concepts. Its emphasis on contextual learning made lessons more engaging, understandable, and applicable to real-life situations. By integrating RME, mathematics classrooms can foster creativity, problem-solving, and innovation among learners. If widely adopted, RME has the potential to transform mathematics teaching and learning in North Central Nigeria.

Recommendations

In light of the findings, the following recommendations are made:

1. Mathematics teachers should adopt the RME approach by incorporating real-life examples, visual aids, and interactive activities to improve students' representation skills in Geometry.
2. Government and educational stakeholders should organize continuous professional development programs to train teachers on effective integration of RME into classroom practice.
3. Curriculum developers should revise the mathematics curriculum to explicitly include RME principles for Geometry and other topics that demand strong representational skills.
4. Further research should explore the long-term effects of RME on students' retention, problem-solving ability, and creativity in mathematics learning.

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ADAPTING VAN HIELE THEORY IN GEOMETRIC THINKING AMONG SENIOR SECONDARY STUDENTS ACHIEVEMENT IN SOKOTO STATE, NIGERIA

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Abstract

Geometry is a fundamental component of mathematics education with broad applications in real-world contexts, yet students in Nigeria continue to perform poorly in this area. This study adapts Van Hiele theory in geometric thinking among senior secondary students achievement in Sokoto State Nigeria. Guided by three research questions and corresponding null hypotheses, the study adopts a quasi-experimental design with pre-test and post-test measures, involving an experimental group taught using the Van Hiele model and a control group taught using conventional approach. A total of 377 students were selected through proportionate and cluster sampling techniques. Data were collected using a validated Van Hiele Geometric Thinking Level Test (VHGTLT) and analyzed using descriptive statistics and the Mann-Whitney U test. Findings reveal that the Van Hiele model significantly enhances students' geometric thinking, particularly benefiting female students through its visual and interactive strategies. In contrast, students taught with conventional methods showed limited progression. The study concludes that the Van Hiele model offers a more effective framework for geometry instruction and recommends its adoption by mathematics teachers, curriculum developers, and education policymakers. Future research should explore its application in other geometric domains and educational levels for broader generalization.

Key Words: Geometry, Geometric Thinking, Achievement, Van Hiele Instructional Model

Introduction

Geometry plays a crucial role in both academic and real-world contexts, providing a framework for understanding spatial relationships, shapes, and structures, making it essential for various fields such as engineering, architecture, and design (Ozkan et al., 2018). Geometry is used daily by professionals like scientists, architects, artists, and land developers, and it plays a vital role in understanding the world and solving real-world problems. Geometry's significance extends to

natural structures and human creations, aiding in problem-solving and supporting other mathematical disciplines. To foster interest in mathematics, teachers should encourage imaginative thinking and inquiry in geometry. Proficiency in geometry helps students understand its significance: geometry offers experiences that build understanding of shapes and their properties, enabling learners to solve relevant problems and apply geometric concepts in real-world situations (Alex & Mammen, 2024). Akka (2021) emphasizes the interconnection between problem-solving skills and spatial reasoning, noting that developing spatial and visual skills enhances problem-solving abilities. Geometric concepts are intertwined with the broader meaning of space, and spatial thinking should be integrated into geometry education.

The proficiency of students in grasping geometry is observed through their cognitive processes, as learning styles vary among students (Senk et al., 2022). These challenges can be cognitive, educational, or related to how geometry is presented and practiced (Omar, 2013). Students often perceive geometry as a challenging and intimidating subject (Adolphus, 2011). They frequently encounter obstacles in comprehending various concepts within geometry. Geometric thinking is the ability to understand and work with shapes, space, and their relationships (Cho & Win, 2020). It is important for subjects like mathematics, science, engineering, and architecture (Alex, 2016). Students build this skill by moving from recognizing simple shapes to solving complex problems and creating logical proofs (Agustiningsih & Yuliati, 2019). It includes abilities like visualizing in 3D, understanding symmetry, and using reasoning (Naufal et al., 2021). The Van Hiele model explains how students grow through different levels of geometric thinking, from basic visualization to formal deduction and rigor (Cahyaningtyas, 2020). To improve these skills, teachers should use strategies like hands-on learning, group work, visual aids, and gradually increasing task complexity (Ananta et al., 2022; Mason, 2021). These methods help students become better problem-solvers in both school and real-life situations (Ananta et al., 2022).

In spite of the importance of geometric thinking skills, students continue to demonstrate a lack of basic skills in geometry and difficulties in learning school geometry in Nigeria. Atebe and Schäfer (2011) affirmed this in both Nigeria and South Africa. In a qualitative investigation, the research indicated that Nigerian senior secondary school students lack conceptual understanding of identifying rhombuses with different orientations; use imprecise reasoning to describe geometric figures, and those 9 out of 18 (50%) students were at level 0 of the van

Hiele's levels (2008). This situation indicates that Nigerian students' levels of geometric thinking skills are lower than the needed geometric reasoning as specified in the senior secondary school mathematics curriculum. The present study adapts the Van Hiele theory of geometric thinking due to its effectiveness in identifying and describing students' levels of geometric understanding. The theory is widely recognized as one of the most comprehensive frameworks for explaining how learners progress in geometry, moving through distinct cognitive levels from visualization to analysis, abstraction, deduction, and ultimately, rigorous proof. These levels are not naturally acquired with age but are developed through appropriately structured instruction (Van Hiele, 1986; Naufal et al., 2021; Sinclair & Moss, 2012).

Originating from the doctoral work of Dina van Hiele-Geldof and later expanded by Pierre van Hiele, the theory was developed in response to students' persistent difficulties in learning geometry. Instruction based on the Van Hiele model facilitates students' cognitive development by guiding them through these sequential levels, promoting deeper understanding and higher-order thinking (Sinclair & Moss, 2012). Consequently, the theory provides a solid foundation for investigating and improving students' geometric thinking skills in this study.

Statement of the Problem

Despite the importance of geometry in developing spatial reasoning and supporting careers in STEM fields, Nigerian students continue to perform poorly in geometry, as reflected in WAEC examiners report from 2017 to 2023. This persistent underachievement is largely due to the continued use of conventional, teacher-centered instructional methods that do not promote deep thinking or conceptual understanding. Many students struggle to recognize geometric shapes, understand their properties, and explain their reasoning. To address this issue, the present study investigates students' levels of geometric thinking in Sokoto State, Nigeria, using the Van Hiele theory, with the goal of informing instructional improvement and supporting deeper learning in school geometry. "

Objectives of the Study

The aim of this study is to adapt Van Hiele theory in geometric thinking among senior secondary students achievement in Sokoto State Nigeria. The specific objectives are to:

1. Investigate the difference between the level of geometric thinking of students taught using van Hiele model of instruction and those taught using conventional approach in the posttest among senior secondary school students in Sokoto State.

2. Identify the level of geometric thinking of senior secondary school students' base on gender in the experimental group in Sokoto State.
3. Identify the level of geometric thinking of senior secondary school students' base on gender in the control group in Sokoto State.

Research Questions

Base on the above stated objectives, the following research questions will be used to guide the study.

1. Is there any significance difference between the level of geometric thinking of students taught using van hiele model of instruction and those taught using conventional approach in the posttest among senior secondary school students in Sokoto state?
1. Is there any significance difference between the levels of geometric thinking of students' base on gender in the experimental group in Sokoto State?
2. Is there any significance difference between the levels of geometric thinking of students' base on gender in the control group in Sokoto State?

Research Hypotheses

The following null hypotheses will be formulated and tested to guide the study.

Ho_1: There is no significant difference between the level of geometric thinking of students taught using Van Hiele model of instructions and those taught using conventional approach among senior secondary school students in Sokoto State.

HO_2: There is no significant difference in the level of Geometric Thinking between male and female senior secondary school students in the experimental group

HO_3: There is no significant difference in the level of Geometric Thinking between male and female senior secondary school students in the control group

Significance of the Study

The outcomes of this study will be of immense benefits to the mathematics teachers, curriculum developers and ministry of education, learners, and the researchers.

Methodology

This study adopted a quasi-experimental research design with a pre-test and post-test format involving experimental and control groups to evaluate students' geometric thinking skills and address misconceptions. The target population comprised 16,488 Senior Secondary School II (SS II) students across 48 schools in Sokoto metropolis. Using cluster sampling, the Sokoto South

Educational Zone was selected, and Wammako Local Government Area was chosen, from which four schools were randomly selected to ensure gender representation. Proportionate sampling was applied, resulting in a sample of 377 students, determined using the Krejcie and Morgan (1970) table. Data were collected using the Van Hiele Geometric Thinking Test (VHGTLT), adapted from Usiskin (1982) with 25 items representing the five Van Hiele levels. The instruments were validated by experts, pilot-tested with 25 students, and recorded reliability coefficients of 0.69 (VHGTLT). The data collected was analyzed using descriptive statistic and Mann Whitney U test to test the hypothesis at 0.05 levels of significance. The tables below summarize the making and grading criteria for the instruments.

Table 1: Criteria for making and grading in van Hiele geometric thinking test

Marks	3 out 5 correct answer criteria	van Hiele Levels
1	**1-5**	**1**
2	**6-10**	**2**
4	**11-15**	**3**
8	**16-20**	**4**
16	**21-25**	**5**

Students can achieve a certain level of thinking if they answer 3 out 5 questions correctly and that one mark is given in level one and two marks are given for answering 3 out 5 questions between 6-10 correctly in level two, to determine the van Hiele's levels of thinking, the weighted sum score is used as a reference as explained in (Usiskin, 1982). Table 2 provides the weighted sum score.

Table 2: The Weighted sum score for van Hiele levels

Forced van Hiele levels	Weighted Sum Score of VHL							
0	0	2	4	8	16	18	20	24
1	1	5	9	17	21	25		
2	3	11	19	27				
3	6	7	22	23				
4	13	14	15	29	30			
5			31					
Not fit	10	12	26	28				

Source: (Usiskin, 1982)

Table 2 was developed based on the van Hiele's theory of geometric thinking, in which the levels are achieved sequentially in order for the students to move from lower to a higher level of thinking (Hassan,et al., 2023). For example, student who answers at least three or more questions in levels 1 and 3 will be given 1 and 4 marks. Adding up the score we have 1+ 4= 5 in reference to Table 2 he/she achieved level one of VHL of thinking.

Results

Based on the result obtained the analysis were done with the use of table to provide a clear picture of the finding as follows

Research Question One

Is there any difference between the level of geometric thinking of students taught using van hiele model of instruction and those taught using conventional approach in the posttest among senior secondary school students in Sokoto state?

Table 3 Showing mean rank of students levels of geometric thinking

Variable	N	Mean Rank
Experimental group	225	207.66
Control group	152	161.38

To determine the difference in geometric thinking between the two groups, the posttest scores were subjected to a Mann-Whitney U test. The analysis revealed a higher mean rank for the experimental group (207.66) compared to the control group (161.38). This result confirms that students who received instruction through the Van Hiele model developed higher geometric thinking than those who were taught using conventional methods. The figures 1 provide is the diagramed presentation of difference based on student's levels in the posttest.

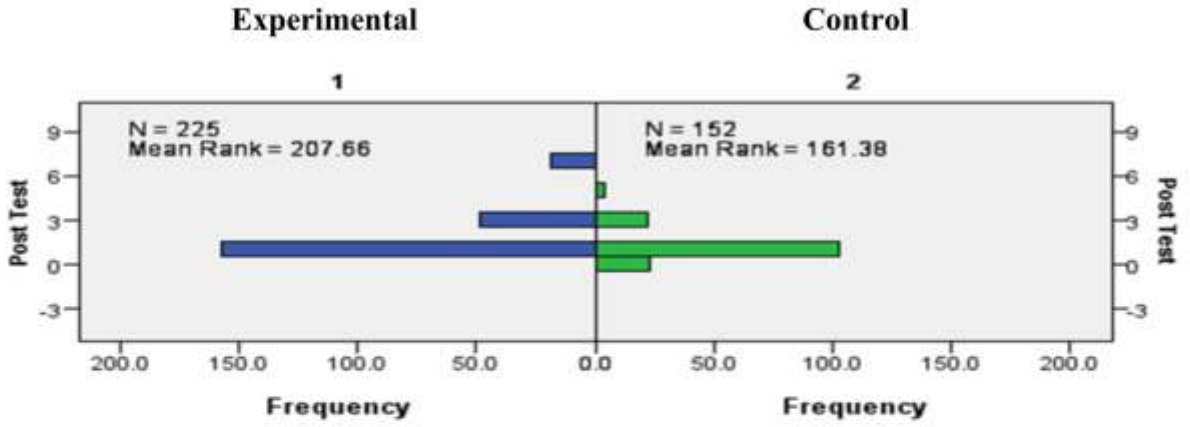


Figure 1 Chart Representation of the Mean rank of two Independent Groups (Experimental and control group).

Null Hypotheses One

There is no significant difference between the geometric thinking of students taught using Van Hiele model of instructions and those taught using conventional approach among senior

secondary school students in Sokoto State. The null hypothesis was tested at 0.05 level of significant.

Table 4 Summary of Mann-Whitney U test Analysis for Experimental and Control Groups.

levels	N	Mean Rank	Mann Whitney U	Wilcoxon W	Test statistics	Stand. Error	Standardized test statistics	Sig.
experimental	225	207.66	12,902,500	24,530,500	12,902,500	846,376	-4.959	.000
control	152	161.38						
Total	**377**							

Table 4 presents the analysis of students geometric thinking based on the H_{o1} whether if there are significant. The results reveal a statistically significant difference between the two independent groups, with U = 12,902,500 and a P-value of 0.000. Since the P-value is less than the 0.05 significance level, the null hypothesis is rejected. This leads to the conclusion that there is a significant difference in the van Hiele levels of geometric thinking skills between the experimental and control groups, in favor of experimental group that taught using Van Hiele model of instructions.

Research Question Two

Is there any difference between the levels of geometric thinking of students' base on gender in the experimental group in Sokoto State?

Analysis for the Distribution of Students' Van Hiele Levels of Thinking at Posttest for male students in the experimental Group

The analysis of 75 male students' van Hiele's levels' of geometric thinking at the posttest for the experimental group, the summary of the result is provided in Table 5

Table 5 Distribution of male students' van Hiele's levels of geometric thinking at the posttest in the experimental group

van Hiele's levels	Sum of scores	Levels					3 out of 5 Correct Answer Criteria	Total (%)
		1	2	3	4	5		
0	0						-	-
1	1	*					58	58(77.3%)
2	3	*	*				11	11(14.7%)
3	7	*	*	*			6	6 (8%)
4	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-
Total							**75**	**100%**

Table 5 shows the distributions of van Hiele's levels for male students at the posttest in the experimental group. The result indicates that a total of 58(77.3%) students in the posttest achieve 1 mark. The van Hiele's level table was assigned at the level 1. Eleven students 11 (12%) achieved 3 mark and were assigned level 2, and six students 6 (8%) scored the sum of 7 marks and were assigned level 3 of van Hiele's level of thinking as recommended in van Hiele's level table.

Analysis for the Distribution of female Students' Van Hiele Levels of Thinking in Posttest for experimental Group

The analysis of 150 female students' van Hiele's levels' of geometric thinking at the posttest for the experimental group, the summary of the result is provided in Table 6

Table 6 Distribution of female students' van Hiele's levels of geometric thinking at the posttest in the experimental group

van Hiele's levels	Sum of scores	Levels					3 out of 5 Correct Answer Criteria	Total (%)
		1	2	3	4	5		
0	0						-	-
1	1 5	*					95 4	99 (66%)
2	3	*	*				38	38 (25.3%)
3	7	*	*	*			13	13 (8.7%)
4	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-
Total							**150**	**100%**

Table 6 shows the distributions of van Hiele's levels for female students at the posttest in the experimental group. The result indicates that a total of 99(81.3%) students in the posttest achieved 1 mark and were assigned level 1, with 4 students that skipped level 1 & 2. 38 (25.3%) students scored the sum of 3 marks and were assigned level 2. 13 (8.7%) students score the sum of 7 marks level 3 were assigned to them as recommended in van Hiele's level table.

Table 7 Showing mean rank of students levels of geometric thinking in experimental group

Variable	N	Mean Rank
Female	150	118.03
Male	75	102.92

Furthermore to see the difference based on gender the result the result was subjected to analysis of Man Whitney U test to obtain a mean rank in the post-test. The analysis of data obtained

shows that the sample of female students obtained higher mean rank (118.03) than the sample of male students with a mean rank of (102.93).This shows that the sample of female students have a higher geometric thinking level. See figure 2

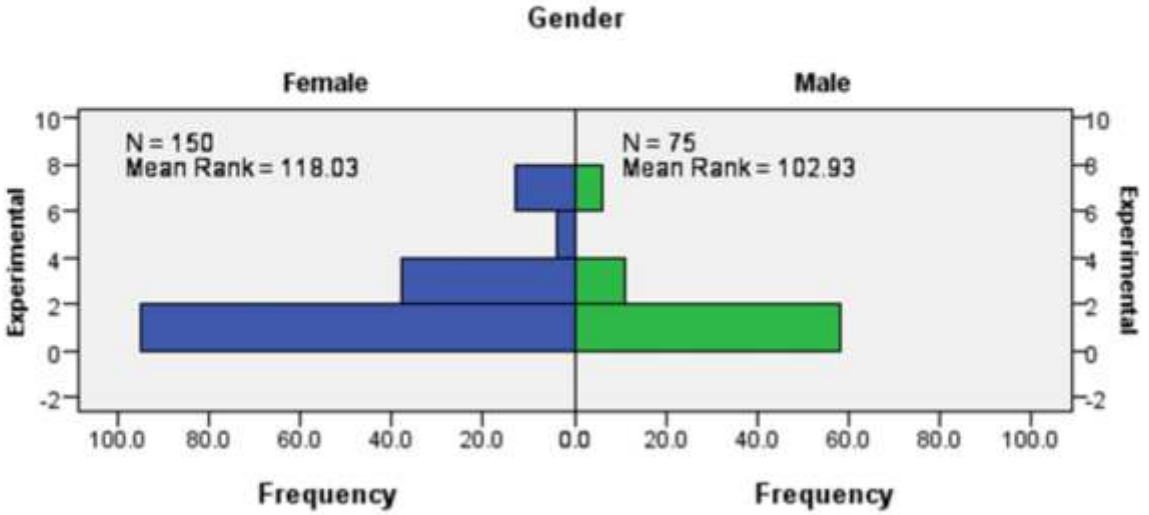


Figure 2 Chart Representation of the Mean rank of male and female in Experimental group.

Null Hypothesis Two

HO_2: There is no significant difference in the level of Geometric Thinking between male and female senior secondary school students in the experimental group in Sokoto state. Man Whitney U test were compared using the data collected from van Hiele test. Result was presented in the table below.

Table 8 Summary of Mann-Whitney U test Analysis for male and female in Experimental Group.

levels	N	Mean Rank	Mann Whitney U	Wilcoxon W	Test statistics	Stand. Error	Standardized test statistics	Sig.
female	150	118.03	6,380,000	17,705,000	6,380,000	378,075	1,997	.046
male	75	102.93						
Total	**225**							

Table 8 presents the analysis of students based on the H_{o2} whether if there are significant. The results reveal a statistically significant difference between male and female in the experimental group, with U = 6,380,000 and a P-value of 0.046. Since the P-value is less than the 0.05 significance level, the null hypothesis is rejected. This leads to the conclusion that there is a

significant difference in the van Hiele levels of geometric thinking skills between male and female in the experimental group.

Research Question Three

Is there any difference between the levels of geometric thinking of students' base on gender in the control group in Sokoto State?

Analysis for the Distribution of Students' Van Hiele Levels of Thinking at Posttest for male students in the control Group

The analysis of 90 male students' van Hiele's levels' of geometric thinking at the posttest for the control group, the summary of the result is provided in Table 9

Table 9 Distribution of male students' van Hiele's levels of geometric thinking at the posttest in the control group

van Hiele's levels	Sum of scores	Levels					3 out of 5 Correct Answer Criteria	Total (%)
		1	2	3	4	5		
0	0						18	18(20%)
1	1	*					57	57(63.3%)
2	3	*	*				15	15(16.7%)
3	-	-	-	-	-	-	-	-
4	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-
Total							**90**	**100%**

Table 9 shows the distributions of van Hiele's levels for male students at the posttest in the control group. The result indicates that 18(20%) students in the posttest failed to achieve level 1 of thinking the van Hiele's level table was assigned at the pre-recognition level. 57 (63.3%) students achieved 1 mark and were assigned level 1, and fifteen students 15 (16.7%) scored the sum of 3 marks and were assigned level 2 of van Hiele's level of thinking as recommended in van Hiele's level table.

Analysis for the Distribution of Students' Van Hiele Levels of Thinking at Posttest for female students in the control Group

The analysis of 62 female students' van Hiele's levels' of geometric thinking at the posttest for the control group, the summary of the result is provided in Table 10

Table 10 Distribution of female students' van Hiele's levels of geometric thinking at the posttest in the control group

van Hiele's	Sum of scores	Levels	3 out of 5	Total (%)

levels		1	2	3	4	5	Correct Criteria	Answer
0	0						5	5(8.1%)
1	1	*					50	50(80.6%)
2	3	*	*				7	7(11.3%)
3	-	-	-	-	-	-	-	-
4	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-
Total							**62**	**100%**

Table 10 shows the distributions of van Hiele's levels for female students at the posttest in the control group. The result indicates that 5(8.1%) students in the posttest failed to achieve level 1 of thinking the van Hiele's level table was assigned at the pre-recognition level. Fifty students 50 (%) achieved 1 mark and were assigned level 1, and seven students 7 (11.3%) scored the sum of 3 marks and were assigned level 2 of van Hiele's level of thinking as recommended in van Hiele's level table.

Table 11 Showing mean rank of students levels of geometric thinking in control group

Variable	N	Mean Rank
Female	62	79.04
Male	90	74.75

Furthermore to see the difference based on gender the result the result was subjected to analysis of Man Whitney U test to obtain a mean rank in the post-test. The analysis of data obtained shows that the sample of female students obtained mean rank (79.04) while the sample of male students with a mean rank of (74.75).This shows that the sample of male and female students have the same geometric thinking level. Below is the pictorial representation of student's levels of geometric thinking in the posttest.

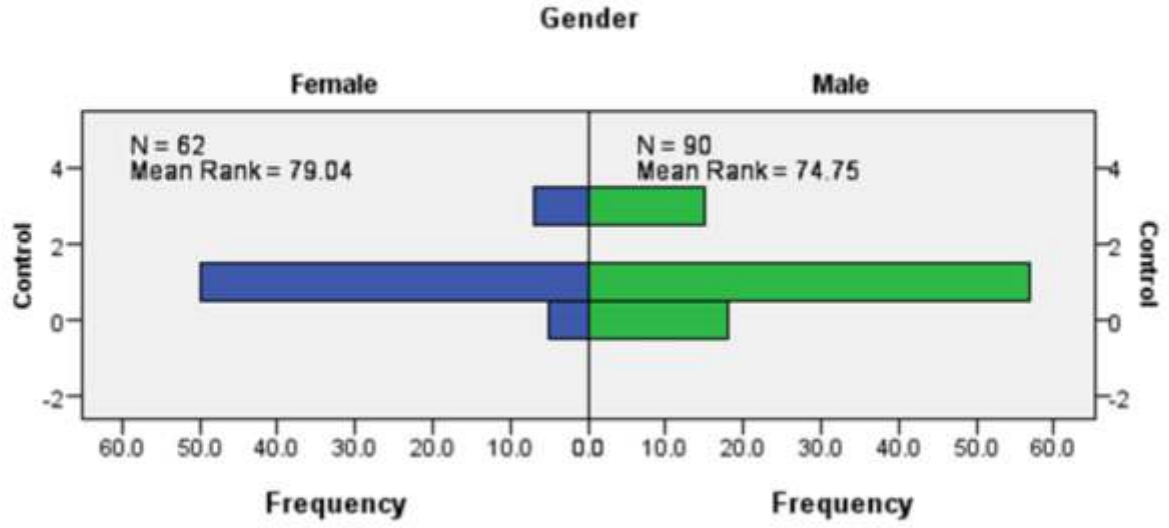


Figure 3 Chart Representation of the Mean rank of male and female in control group.

Null Hypothesis Three

HO_3: There is no significant difference in the level of Geometric Thinking between male and female senior secondary school students in the control group in Sokoto State.

Man Whitney U test were compared using the data collected from van Hiele test. The data was sorted according to the geometric thinking level of male and female in the control group. The calculation was done electronically using SPSS software version 20.0 and the result was presented in the table below.

Table 12 Summary of Mann-Whitney U test Analysis for male and female in Control Group.

levels	N	Mean Rank	Mann Whitney U	Wilcoxon W	Test statistics	Stand. Error	Standardized test statistics	Sig.
female	62	79.04	2,947,500	4,,900,500	2,947,500	214,66	.735	.462
male	90	74.75						
Total	**152**							

Table 12 presents the analysis of students based on the H_{o3} whether if there are significant. The results reveal that there is no significant difference between male and female in their geometric thinking level. With U = 2,947,500 and a P-value of 0.462. Since the P-value is greater than the 0.05 significance level, the null hypothesis is retain. This leads to the conclusion that there is no significant difference in the van Hiele levels of geometric thinking skills between male and female in the control groups.

Summary of Posttest for Students' Van Hiele Levels of Geometric Thinking Base on Gender for experimental and control group

The analysis provides the characteristics of students' van Hiele's levels of geometric thinking base on gender in experimental and control groups in the posttest, the summary of the result is provided in Table 13

Table 13 Summary Distribution of van Hiele's levels in the experimental and control groups based on gender at the posttest

van Hiele's level	Distribution for the frequency and percentage of the van Hiele levels base on gender			
	Experimental Group		Control Group	
	Male	Female	Male	Female
0			18 (20%)	5 (8.1%)
1	58 (77.3%)	99 (66%)	57 (63.3%)	50 (80.6%)

2	11 (14.7%)	38 (25.3%)	15 (16.7%)	7 (11.3%)
3	6 (8%)	13 (8.7%)	-	-
4	-		-	
5	-		-	
Total	**75**	**150**	**90**	**62**

The table 13 shows the result for the distribution of students' geometric thinking levels based on gender in both the experimental and control groups reveals notable differences in the level between male and female students, particularly how each group responded to the teaching approach used.

In the **experimental group,** which was taught using the **Van Hiele instructional model,** both male and female students showed higher levels of geometric thinking. Among male students, 58 out of 75 (77.3%) reached **Level 1 (Visualization),** 11 (14.7%) reached **Level 2 (Analysis),** and 6 (8%) advanced to **Level 3 (Informal Deduction).** Similarly, out of 150 female students, 99 (66%) attained Level 1, 38 (25.3%) reached Level 2, and 13 (8.7%) achieved Level 3. Notably, **no male or female student in the experimental group remained at Level 0,** indicating that the Van Hiele model effectively supported all learners, regardless of gender, in developing at least basic geometric thinking.

In contrast, the **control group,** which received **conventional instruction,** showed lower performance across both genders. Among the 90 male students, 18 (20%) remained at **Level 0,** 57 (63.3%) achieved only **Level 1,** and 15 (16.7%) reached **Level 2.** None reached Level 3. Among 62 female students, 5 (8.1%) remained at **Level 0,** 50 (80.6%) reached **Level 1,** and only 7 (11.3%) advanced to **Level 2.** Like their male counterparts, no female student reached Level 3.

These results suggest that while **female students in the control group** performed relatively better at the visualization level (Level 1) compared to males (80.6% vs. 63.3%), **more male students** in the same group advanced to **Level 2** (16.7% vs. 11.3%). However, the overall progression beyond Level 1 remained low for both genders in the control group.

In contrast, in the **experimental group,** a **higher percentage of females** progressed to **Level 2** (25.3%) than males (14.7%), and both genders had a nearly equal proportion reaching **Level 3** (8% for males and 8.7% for females). This indicates that the Van Hiele model provided **more equitable and effective support** for students of both genders, enabling them to advance through higher levels of geometric thinking.

Therefore, the Van Hiele instructional model proved to be more effective than the conventional method in enhancing geometric thinking across both genders, with female students showing a

slightly higher tendency to progress to the analysis and informal deduction levels when taught using the model.

Summary of the Major Findings

1. The study found that the Van Hiele model is more effective in promoting students' progression through the levels of geometric thinking than the conventional teaching approach.
2. The study also revealed that Van Hiele instructional model has a positive impact on female students' geometric thinking, due to its emphasis on visualization and interactive learning.
3. The study indicates that the conventional teaching approach did not produce significant differences in geometric thinking skills between male and female students. Both genders largely remained at lower Level.

Discussion of Findings

The study found that the Van Hiele model is more effective in promoting students' progression through the levels of geometric thinking than the conventional teaching approach. This finding aligns strongly with multiple empirical studies. **Alex and Mammen (2016)** found that students taught using the Van Hiele model in South Africa showed statistically significant improvement in their levels of geometric thinking compared to those in the control group. Similarly, **Joshi (2017)** reported that grade seven students taught geometry using the Van Hiele approach achieved significantly better results than those taught via conventional methods, confirming that the Van Hiele framework fosters deeper understanding through structured levels of reasoning. Furthermore, **Cho and Win (2020)** found that students taught with the Van Hiele model had higher mean achievement scores compared to those taught using traditional approaches, emphasizing that visualization, logical progression, and interactive learning inherent in the model are critical for conceptual understanding. The findings of **Arman (2021)** also reinforce this conclusion by showing that the Van Hiele model was effective in correcting misconceptions in geometric construction among students, whereas conventional teaching often failed to address these misconceptions. The current study's finding that conventional teaching left most students at Level 1 (analysis) resonates with **Hassan et al. (2023),** who noted that students taught traditionally in Sokoto State largely operated at Level 0, lacking basic skills and understanding in geometry.

The positive impact on female students highlighted by this study can be linked to the interactive and visual nature of the Van Hiele model, which supports diverse learning preferences. **Arman (2021),** who worked with an all-female experimental group, found that using the Van Hiele model significantly improved students' understanding and corrected misconceptions. This suggests that the model is particularly effective for learners who benefit from active engagement and step-by-step scaffolding. Additionally, Research indicates that there is no significant gender difference in Van Hiele levels when traditional teaching approaches are used, suggesting that conventional instruction does not effectively address the varied cognitive needs of male and female students (Usman, Yew, & Saleh, 2020; Anas, 2018, as cited in Bonyah & Larbi, 2021).The positive gender-related effect observed in the current study could stem from the Van Hiele model's ability to engage female students through its visual, practical, and collaborative learning activities, as supported by **Al-Ebous (2016),** who demonstrated that the model enhanced both conceptual acquisition and students' attitudes toward geometry.

The study indicates that the conventional teaching approach did not produce significant differences in geometric thinking skills between male and female students. Both genders largely remained at lower Level. This finding is consistent with research showing that traditional teaching methods often fail to advance students' geometric thinking beyond basic recognition or analysis levels. **Hassan (2015)** reported that 84.5% of students remained at Level 1 (visual), with no significant gender differences in geometric thinking levels. Similarly, **Thapa (2017)** found that most students were at Level 0 or Level 1, with very few reaching higher levels of geometric reasoning, which is indicative of the shortcomings of conventional teaching practices. The current finding also resonates with **Hassan et al. (2023),** who observed that students taught using the dominant traditional approach in Sokoto State lacked the foundational skills to progress in geometry. **Al-khateeb (2018)** further supports this view by highlighting that many misconceptions and learning difficulties in geometry stem from inadequate instructional strategies and teachers' limited awareness of students' conceptual gaps. Therefore, the lack of gender differences in the conventional group is not necessarily an indicator of equity but rather of the uniformly low effectiveness of traditional methods, which fail to promote meaningful progression in geometric reasoning for either gender.

Conclusion

The study concludes that the Van Hiele instructional model outperforms conventional methods in enhancing students' geometric thinking. Its structured approach progressing through visualization, analysis, and reasoning promotes deeper conceptual understanding. The model notably benefits female students due to its interactive and visual learning strategies. In contrast, conventional teaching fails to move students beyond basic recognition levels, leaving both genders at lower stages of geometric thinking.

Recommendations

The following recommendations were made from the study

1. Mathematics teachers should be encouraged to implement the Van Hiele instructional model and transition from conventional teacher-centered methods to interactive, student-focused approaches.
2. The Ministry of Basic and Secondary Education should organize workshops and seminars aimed at equipping mathematics teachers with the skills and knowledge required to effectively apply the Van Hiele model in classroom instruction.
3. Curriculum planners and policymakers should consider embedding the Van Hiele model within the formal education curriculum to enhance the teaching and learning of geometry.

Suggestions for Further Study

1. The study focused on plane shapes, future research should investigate other areas of geometry, such as solid geometry, using the Van Hiele instructional strategy.
2. The present study also limited to senior secondary school students, further research should extend to learners at other educational levels to obtain more generalized findings

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CURRICULUM INNOVATIONS IN MATHEMATICS EDUCATION FOR PROMOTING ENTREPRENEURIAL THINKING AND SELF-EMPLOYMENT SKILLS IN NIGERIA

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Abstract

The persistent and escalating challenge of youth unemployment in Nigeria, exacerbated by a rapidly growing young population and limited formal sector job opportunities, necessitates a fundamental paradigm shift in educational practices. This paper meticulously explores how targeted curriculum innovations in mathematics education can be strategically designed and implemented to cultivate robust entrepreneurial thinking and essential self-employment skills among Nigerian students across all educational tiers: primary, secondary, and tertiary levels. Drawing extensively upon the foundational principles articulated in Nigeria's National Policy on Education (Federal Republic of Nigeria, 2014), complemented by rigorous empirical case studies conducted within the Nigerian context, and enriched by a comprehensive review of comparative international literature on entrepreneurship education, this study unequivocally underscores the pivotal and transformative role of functional, contextualized, and application-oriented mathematics instruction. Such instruction is critical in fostering not only advanced problem-solving abilities and astute business acumen but also instilling a resilient and self-reliant mindset crucial for navigating the complexities of modern economies. Through a meticulous qualitative analysis of existing curriculum documents and an in-depth examination of documented pilot programs and innovative pedagogical initiatives, the paper identifies a suite of successful strategies. These include, but are not limited to, immersive project-based learning experiences, the integration of entrepreneurship-themed mathematics problems that mirror real-world business challenges, and the cultivation of interdisciplinary collaborations that collectively enhance students' intrinsic capacity to proficiently apply abstract mathematical concepts to concrete, real-world entrepreneurial scenarios. Furthermore, the paper candidly highlights persistent systemic challenges that impede these innovations, such such as the pervasive influence of rigid, examination-driven curricula, the critical deficit in adequately trained teachers equipped with entrepreneurship-focused pedagogy, and severe resource

constraints impacting access to modern teaching aids and technology. Concurrently, it illuminates promising opportunities for leveraging technology integration, advocating for comprehensive policy reforms, and fostering robust public-private partnerships to overcome these hurdles. The study conclusively argues that the deliberate embedding of entrepreneurial elements into the mathematics curricula not only significantly enhances students' employability and innovation skills, thereby preparing them for a dynamic job market, but also directly aligns with and actively contributes to Nigeria's broader national economic diversification and sustainable development goals. The paper culminates in a series of actionable recommendations, including urgent curriculum revisions at all levels, sustained teacher capacity-building programs focused on entrepreneurial pedagogy, and intensified collaborative stakeholder engagement, all identified as critical and indispensable steps towards realizing an education system that genuinely prepares learners for self-reliance and entrepreneurial success within the unique and evolving Nigerian socio-economic landscape.

Key words: Curriculum Innovations, Mathematics Education, Entrepreneuship, Self-Employment

Introduction

Nigeria, Africa's most populous nation, is currently grappling with an unprecedented youth unemployment crisis. With a significant portion of its burgeoning youth population either unemployed or underemployed, there is an urgent and compelling need to fundamentally rethink and reorient the nation's educational system to cultivate self-reliance, innovation, and job creation rather than merely job seeking. In this critical context, mathematics education, traditionally perceived as an abstract and theoretical discipline, emerges as a profoundly strategic and fertile platform for embedding essential entrepreneurial skills and actively promoting self-employment as a viable career pathway (Onoshakpokaiye et al., 2021).

The conventional approach to mathematics instruction in Nigeria has often been criticized for its detachment from real-world applications, leading to a perception among students that mathematics is irrelevant to their daily lives and future careers. This abstract orientation frequently results in rote learning, superficial understanding, and a failure to develop the practical problem-solving skills that are indispensable for entrepreneurial endeavors. However,

by intentionally integrating entrepreneurial thinking into the mathematics curricula, educators can significantly enhance students' capacity for critical thinking, complex problem-solving, and the acquisition of practical business competencies that are directly applicable across primary, secondary, and tertiary education levels. This integration is not merely an additive process but a transformative one, aiming to bridge the gap between theoretical knowledge and practical application, thereby empowering the next generation of Nigerian youth to become creators of wealth and employment, rather than just seekers of it. This paper will elaborate on the necessity, strategies, challenges, and opportunities inherent in this transformative approach to mathematics education in Nigeria.

Literature Review and Context

Mathematics education has long been globally recognized as a fundamental tool for fostering cognitive development and, more recently, for entrepreneurship development. Its inherent capacity to cultivate logical reasoning, analytical skills, quantitative literacy, and informed decision-making abilities makes it an indispensable discipline for nurturing entrepreneurial mindsets (Onoshakpokaiye et al., 2021; Ogunkunle & Bakare, 2022). The application of mathematical principles in real-world scenarios, such as financial planning, market analysis, and resource allocation, directly translates into competencies vital for entrepreneurial success.

In Nigeria, the imperative for entrepreneurship education is particularly acute given the persistent challenge of youth unemployment and the need for economic diversification. The nation's educational policy frameworks, notably the National Policy on Education (NPE), have consistently emphasized the provision of functional education. The 6th edition of the NPE (Federal Republic of Nigeria, 2014) explicitly articulates the goal of equipping learners with practical life skills necessary for sustainable livelihoods and self-reliance, thereby laying a policy foundation for integrating entrepreneurial elements into the curriculum.

Recent empirical studies within Nigeria corroborate the transformative potential of functional mathematics education. For instance, Onoshakpokaiye et al. (2021) conducted a comprehensive study demonstrating that mathematics instruction, when focused on real-life applications such as profit and loss computations, budgeting, cost analysis, and resource management, serves as a powerful catalyst for fostering entrepreneurial competencies among Nigerian graduates. Their

findings suggest a direct correlation between exposure to practical mathematical concepts and an enhanced capacity for entrepreneurial engagement.

Similarly, Ogunkunle and Bakare (2022) underscored the critical role of mathematics educators in promoting entrepreneurial thinking. They argued that by deliberately contextualizing mathematical lessons within local economic activities and indigenous business practices, abstract mathematical concepts become more relevant, relatable, and engaging for students. This contextualization not only deepens understanding but also highlights the immediate applicability of mathematics in everyday entrepreneurial endeavors, such as calculating returns on agricultural produce or managing small-scale trade.

Further reinforcing these insights, a compelling case study by Kalu (2023) in Rivers State, Nigeria, provided empirical evidence of the positive impact of integrating entrepreneurial scenarios into junior secondary mathematics lessons. The study revealed a statistically significant improvement in students' entrepreneurial skills development, particularly in core areas such as cost analysis, accurate profit computation, and fundamental financial literacy. This suggests that even at the secondary level, a deliberate infusion of entrepreneurial content can yield measurable gains in practical skills. These findings collectively advocate for a pedagogical shift towards a more applied and entrepreneurship-oriented mathematics curriculum that directly addresses Nigeria's socio-economic realities.

Primary Education Innovations

At the foundational primary education level in Nigeria, innovative approaches are being implemented to introduce entrepreneurial elements into mathematics lessons, thereby instilling foundational business acumen from an early age. These innovations move beyond traditional rote learning to create engaging and practical learning experiences.

One prominent strategy involves the simulation of trading activities within the classroom. Teachers facilitate mock markets or small business scenarios where students actively participate in buying and selling goods. In these simulations, students practice essential mathematical skills such as money handling, calculating simple profit and loss, determining change, and engaging in basic business transactions (Olanrewaju, 2023). These hands-on activities serve to demystify abstract mathematical concepts by grounding them in relatable, real-world contexts. For instance,

students might set up a lemonade stand, calculate the cost of ingredients, determine a selling price, and track their earnings and expenses, all while applying arithmetic operations.

A compelling case study by Olufemi et al. (2022) provides empirical evidence of the effectiveness of such approaches. Their research observed that incorporating story problems centered on local entrepreneurs significantly improved primary students' engagement and deepened their understanding of core mathematical concepts, including arithmetic operations, measurement, and data handling. By weaving narratives around familiar local businesses (e.g., a tailor, a farmer, a market vendor), teachers make mathematics more culturally relevant and immediately applicable to the students' environment. This contextualization not only enhances comprehension but also cultivates an appreciation for entrepreneurial activities within their communities.

These pedagogical innovations at the primary level are in direct alignment with the principles of functional mathematics education, which is increasingly advocated as a crucial pathway for fostering self-reliance and entrepreneurial spirit (Onoshakpokaiye et al., 2021). By integrating practical, entrepreneurship-themed activities, primary mathematics education can effectively lay a robust groundwork for developing critical thinking, financial literacy, and a proactive mindset necessary for future entrepreneurial endeavors. The early exposure to these concepts helps to build a positive attitude towards self-employment and problem-solving using mathematical tools.

Secondary Education Innovations

At the secondary school level in Nigeria, curriculum innovations are increasingly leveraging mathematics education to cultivate more sophisticated entrepreneurial skills and foster a deeper understanding of business principles. These innovations often involve more complex applications of mathematical concepts to simulated or real-world business scenarios.

One highly effective approach gaining traction is project-based learning (PBL). In this model, students are tasked with initiating and managing simulated small-scale businesses, which requires them to apply a wide range of mathematical skills in an integrated manner. For instance, students might collaborate to design a product, conduct comprehensive market surveys to assess demand and pricing, and then utilize algebraic expressions and functions to analyze production costs, revenue projections, and potential profits (Kalu, 2023). This hands-on experience allows

them to see the direct relevance of abstract mathematical concepts like linear equations, inequalities, and optimization in a practical business context. PBL also promotes teamwork, critical thinking, and problem-solving, all essential entrepreneurial attributes.

Furthermore, some secondary schools have begun introducing specialized subjects like "Business Mathematics" as an elective or integrated component within the broader mathematics curriculum. Ogunkunle and Bakare (2022) reported that such subjects cover crucial topics including compound interest calculations, loan amortization, budgeting, financial statement analysis, and basic investment planning. This equips students with practical financial literacy skills directly applicable to managing micro-enterprises and personal finances, thereby bridging the gap between theoretical mathematics and real-world economic decision-making.

Beyond formal curriculum adjustments, inter-school mathematics competitions are being innovatively designed to incorporate entrepreneurial challenges. Eze et al. (2023) found that these competitions, which often require students to solve complex problems related to business logistics, resource allocation, or market strategy, have proven exceptionally effective in fostering advanced problem-solving skills directly linked to real-world business scenarios. Such competitive platforms not only enhance mathematical proficiency but also stimulate creative thinking and competitive spirit, mirroring the dynamics of the entrepreneurial landscape.

These innovations at the secondary level are crucial for building upon the foundational entrepreneurial concepts introduced in primary school, preparing students for more complex economic realities and potentially inspiring them to pursue self-employment or entrepreneurial ventures upon graduation.

Tertiary Education Innovations

At the tertiary level, Nigerian universities are increasingly recognizing the imperative to equip mathematics graduates with entrepreneurial competencies, moving beyond traditional academic roles to foster job creation and innovation. This involves a more sophisticated integration of entrepreneurship modules and practical application opportunities within mathematics curricula.

Universities have begun to integrate dedicated entrepreneurship modules directly into mathematics curricula. These modules often focus on advanced applications of mathematical principles to business contexts, including business modeling, optimization techniques (e.g.,

linear programming for resource allocation, calculus for maximizing profit), quantitative decision-making under uncertainty, and financial mathematics (Okoli & Ekechukwu, 2022). This ensures that mathematics students are not only proficient in theoretical concepts but also capable of translating these into actionable business insights.

Furthermore, programs such as innovation hubs and entrepreneurship fairs have become integral components of the tertiary education landscape. These platforms actively encourage mathematics students to apply their rigorous analytical skills to develop viable business ideas and prototypes. For example, a study by Abubakar et al. (2022) on STEM researchers in Nigeria revealed that targeted entrepreneurship training embedded within academic programs significantly enhanced students' readiness for self-employment. This training specifically focused on increasing their capacity to leverage advanced mathematical modeling and statistical analysis for developing robust business solutions, market forecasting, and risk assessment.

Collaborations between universities and private sector organizations are also playing a pivotal role. Partnerships with banks, financial institutions, technology firms, and established entrepreneurial ventures facilitate invaluable mentorship programs. Through these collaborations, students are guided by industry experts to translate complex mathematical concepts and theoretical models into viable entrepreneurial ventures. These mentorships provide real-world context, networking opportunities, and practical guidance on navigating the entrepreneurial ecosystem (Abubakar et al., 2022). Such initiatives bridge the gap between academia and industry, ensuring that graduates are not only academically strong but also possess the practical skills and industry exposure necessary to launch and sustain their own businesses.

These tertiary-level innovations are crucial for producing a cadre of mathematically skilled entrepreneurs who can drive economic growth and address the unemployment challenge in Nigeria.

Challenges and Opportunities

Despite the promising innovations in integrating entrepreneurial thinking into mathematics education across Nigeria, several significant challenges persist. Addressing these challenges is crucial for the widespread and sustainable success of these initiatives. Concurrently, numerous

opportunities exist that can be leveraged to overcome these hurdles and accelerate the integration process.

Challenges

i. Curriculum Rigidity: The prevailing Nigerian curriculum, particularly at the secondary level, is largely characterized by its rigid, examination-driven nature. This strong emphasis on standardized tests and content coverage often leaves little room for the flexible, project-based, and experiential learning approaches that are fundamental to fostering entrepreneurial skills (Onoshakpokaiye et al., 2021). Teachers often feel pressured to "teach to the test," prioritizing rote memorization over the development of critical thinking and practical application skills. This rigidity stifles creativity and innovation in pedagogical approaches.

ii. Teacher Preparedness: A significant bottleneck is the inadequate training and professional development of many mathematics educators in entrepreneurship-focused pedagogy. As highlighted by Ogunkunle and Bakare (2022), a substantial number of teachers lack the necessary knowledge, skills, and confidence to effectively integrate entrepreneurial concepts into their mathematics lessons. Their own educational backgrounds may not have exposed them to entrepreneurial thinking, making it challenging for them to deliver innovative and contextually relevant lessons. This deficit in teacher capacity directly limits the effective implementation of entrepreneurial mathematics curricula.

iii. Resource Constraints: Insufficient access to essential teaching aids, modern technological tools (such as computers, internet connectivity, and educational software), and adequate funding severely hampers the effective implementation of entrepreneurial projects in schools (Kalu, 2023). Many public schools, especially in rural and underserved areas, operate with dilapidated infrastructure and a scarcity of resources, making it difficult to conduct hands-on, project-based activities that require materials, technology, or even basic classroom space for collaborative work.

iv. Cultural Perceptions: Deep-seated societal attitudes and cultural perceptions in Nigeria often favor white-collar jobs and formal employment over entrepreneurship. This cultural bias can significantly diminish students' motivation to engage in entrepreneurial activities within math

classes, as they may not perceive self-employment as a prestigious or secure career path (Abubakar et al., 2022). Parental expectations and societal norms can inadvertently steer students away from entrepreneurial pursuits, even when such skills are being fostered in the curriculum.

Opportunities

Despite these challenges, Nigeria possesses significant opportunities that can be strategically leveraged to advance entrepreneurial mathematics education:

i. Policy Alignment: Nigeria's current Basic Education Curriculum, as outlined in the National Policy on Education (Federal Republic of Nigeria, 2014), strongly emphasizes the development of competencies and life skills. This policy framework creates a highly conducive environment for formally embedding entrepreneurship into the mathematics curriculum. The existing policy provides a legitimate mandate for curriculum developers and educators to integrate entrepreneurial learning outcomes, aligning educational goals with national development aspirations.

ii. Technological Integration: The rapid rise and increasing accessibility of Information and Communication Technology (ICT) in education present new and exciting avenues for interactive, entrepreneurship-themed mathematics activities (Okoli & Ekechukwu, 2022). Digital platforms, educational apps, and online simulation tools can provide engaging and scalable ways to teach financial literacy, business modeling, and data analysis. Virtual business simulations, for instance, can offer risk-free environments for students to apply mathematical concepts to complex business decisions, overcoming some resource constraints of physical materials.

iii. Public-Private Partnerships (PPPs): Fostering robust public-private partnerships can significantly bolster entrepreneurial mathematics education. Collaborations with local businesses, industry leaders, financial institutions, and tech companies can provide invaluable resources, expertise, and real-world exposure to students (Abubakar et al., 2022). These partnerships can manifest as mentorship programs, industry-sponsored contests, internships, or guest lectures, offering students practical insights and networking opportunities that classroom learning alone cannot provide. Such collaborations can also help bridge the gap between academic knowledge and industry demands.

iv. Youth Demographics and Innovation Potential: Nigeria's large and youthful population, often characterized by its dynamism and adaptability, represents an immense pool of potential entrepreneurs. By nurturing their innate creativity and problem-solving abilities through relevant mathematics education, this demographic dividend can be transformed into a powerful engine for economic growth and innovation. Investing in entrepreneurial mathematics education is an investment in the nation's future economic resilience.

v. Growing Entrepreneurial Ecosystem: While nascent, Nigeria's entrepreneurial ecosystem is growing, with increasing numbers of startups and incubators. This provides a fertile ground for students to observe, learn from, and eventually participate in real entrepreneurial ventures. Connecting mathematics education to this burgeoning ecosystem can make learning more relevant and inspiring.

Addressing the challenges while strategically capitalizing on these opportunities will be paramount for the successful and sustainable integration of entrepreneurial thinking into mathematics education in Nigeria.

Conclusion

The integration of entrepreneurial thinking into the mathematics curriculum represents a critical and timely innovation essential for addressing Nigeria's pressing socio-economic challenges, particularly the pervasive issue of youth unemployment. By deliberately embedding real-life business applications and entrepreneurial scenarios into mathematics education across all levels primary, secondary, and tertiary students are empowered to acquire not only advanced critical thinking and analytical skills but also practical competencies indispensable for self-reliance and innovation in a rapidly evolving global economy. This transformative approach moves beyond traditional pedagogical methods, fostering a dynamic learning environment where mathematical concepts are not merely abstract theories but tangible tools for real-world problem-solving and value creation.

The empirical evidence and documented pilot programs reviewed in this paper unequivocally demonstrate the efficacy of such curriculum innovations. From simulating trading activities in primary schools to engaging in complex business modeling at the tertiary level, these initiatives have shown significant promise in enhancing students' entrepreneurial skills, fostering a

proactive mindset, and bridging the critical gap between academic knowledge and practical application. The success stories highlight the potential for mathematics education to serve as a powerful catalyst for economic empowerment and national development.

However, the widespread and sustainable implementation of these curriculum reforms is contingent upon overcoming several systemic hurdles. These include the inherent rigidity of the existing examination-driven curriculum, the urgent need for comprehensive teacher training in entrepreneurship-focused pedagogy, and the persistent challenge of resource constraints. Despite these obstacles, significant opportunities abound, particularly through strategic policy alignment, leveraging the transformative potential of information and communication technology (ICT), and fostering robust public-private partnerships. These opportunities, if effectively harnessed, can create a supportive ecosystem for entrepreneurial mathematics education.

In essence, the journey towards a more entrepreneurial mathematics curriculum is a collaborative endeavor. It necessitates unwavering policy support from governmental bodies, continuous and targeted professional development for mathematics educators, and active, sustained collaboration among all key stakeholders' educators, government agencies, the private sector, and local communities. Such concerted efforts are critical steps towards realizing an education system that is not only academically rigorous but also profoundly relevant, equipping learners with the foresight, skills, and resilience to become successful entrepreneurs and contribute meaningfully to Nigeria's economic prosperity and self-sufficiency.

Recommendations

To effectively realize the vision of mathematics education as a catalyst for entrepreneurial thinking and self-employment skills in Nigeria, the following recommendations are crucial:

1.Curriculum Revision and Integration:

Action: National mathematics curricula at all educational levels (primary, secondary, and tertiary) must undergo a comprehensive and systematic revision to formally incorporate entrepreneurial learning outcomes. This involves clearly articulating specific competencies related to self-reliance, innovation, and business acumen within the mathematics syllabus.

Detail: This revision should move beyond mere theoretical inclusion to practical integration. For instance, problem sets should consistently feature real-world entrepreneurial scenarios (e.g., calculating startup costs for a small business, analyzing market trends for a product, optimizing production schedules). Learning objectives should explicitly link mathematical concepts to entrepreneurial applications, ensuring that students understand the direct relevance of their studies to wealth creation and self-employment. This could involve developing new modules or units specifically focused on business mathematics, financial literacy, and data analysis for entrepreneurial decision-making.

Rationale: A formal curriculum mandate provides the necessary framework and legitimacy for educators to prioritize and implement entrepreneurial mathematics education, ensuring consistency and widespread adoption across the nation.

2.Teacher Professional Development and Capacity Building:

Action: Implement continuous and robust capacity building workshops and training programs for all mathematics teachers, from primary to tertiary levels, focusing specifically on integrating entrepreneurship into their pedagogy.

Detail: These programs should not only cover the theoretical underpinnings of entrepreneurship but also provide practical, hands-on strategies for lesson planning, activity design, and assessment that foster entrepreneurial thinking. Training should include exposure to successful entrepreneurial case studies, opportunities for teachers to develop their own entrepreneurial projects, and guidance on utilizing technology and local resources to create engaging, contextually relevant learning experiences. Mentorship programs where experienced teachers or entrepreneurs guide their less experienced colleagues could also be highly beneficial.

Rationale: Teachers are the primary implementers of curriculum reforms. Equipping them with the necessary pedagogical skills and confidence is paramount to translating policy into effective classroom practice.

3.Project-Based and Authentic Assessments:

Action: Encourage and institutionalize the widespread use of open-ended, project-based assessments in mathematics. These assessments should evaluate students' ability to apply

mathematical concepts to solve real-world entrepreneurial problems, rather than solely relying on traditional, rote-memorization-based examinations.

Detail: Assessment tasks could include developing a business plan that requires financial projections, designing a product and calculating its profitability, analyzing market data to make strategic decisions, or creating a budget for a community project. Such assessments should emphasize critical thinking, creativity, collaboration, and the ability to communicate mathematical reasoning effectively. Rubrics should be developed to clearly define expectations for entrepreneurial competencies alongside mathematical proficiency.

Rationale: Shifting assessment paradigms will drive pedagogical change, compelling teachers to adopt more innovative, application-oriented teaching methods that align with entrepreneurial learning outcomes.

4.Enhanced Stakeholder Collaboration and Partnerships:

Action: Foster and strengthen partnerships among government agencies (e.g., Ministry of Education, Ministry of Youth Development), private sector players (e.g., banks, tech companies, established entrepreneurs), educational institutions (universities, colleges, schools), and local communities to support entrepreneurship programs in schools.

Detail: These collaborations can take various forms: private sector companies can offer internships, mentorships, and funding for entrepreneurial projects; government agencies can provide policy support and infrastructure; and community leaders can facilitate access to local business insights and resources. Jointly organized entrepreneurship fairs, hackathons, and innovation challenges can provide platforms for students to showcase their ideas and connect with potential investors or mentors. Establishing a national steering committee with representatives from all key stakeholders could ensure coordinated efforts.

Rationale: A multi-sectoral approach ensures that entrepreneurial mathematics education is well-resourced, relevant to industry needs, and supported by the broader societal ecosystem.

5. Resource Mobilization and Development of Localized Teaching Aids:

Action: Prioritize the mobilization of resources and the development of low-cost, locally relevant teaching aids that specifically facilitate entrepreneurship-themed mathematics instruction.

Detail: This includes creating culturally appropriate case studies of Nigerian entrepreneurs, developing open-source digital tools for financial modeling and data analysis, and producing instructional materials that reflect local economic activities and challenges. Emphasis should be placed on utilizing readily available materials to create manipulatives and learning tools, reducing reliance on expensive imported resources. Government and private sector grants could be channeled towards initiatives that develop and disseminate these localized resources.

Rationale: Adequate and appropriate resources are fundamental to effective teaching and learning. Localized aids make learning more relatable and accessible, especially in resource-constrained environments, ensuring that entrepreneurial mathematics education is inclusive and sustainable.

By diligently implementing these comprehensive recommendations, Nigeria can transform its mathematics education system into a powerful engine for fostering entrepreneurial thinking and self-employment skills, thereby contributing significantly to national development and youth empowerment.

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PEER INFLUENCE AND MATHEMATICS PERFORMANCE OF SENIOR SECONDARY SCHOOL STUDENTS IN SOKOTO STATE, NIGERIA

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Abstract

The Peer group is composed of equal status, where students within a given peer group are of the same age and have the same social status. The performance task is any learning activity or assessment that asks students to perform to demonstrate their knowledge, understanding, and proficiency. Performance tasks as part of the student's performance in Mathematics in senior secondary schools, probably this might be influenced by peers of the students. The study used a descriptive design to determine the influence of peer on the performance task of the students. A total of 243 senior secondary school students in Sokoto state answered the Likert scale questionnaire, an instrument used by the researcher to conduct the study. The study used weighted mean and chi-square to determine whether peer influence has an association with students' performance tasks. The results show that peers can influence students' performance in Mathematics. It was shown in the study that students get a low grade from skipping classes together with their peers. The study concluded that there is an association between peer influence and academic performance of the students. Therefore, peer plays a big role in student's academic engagement and academic success.

Keywords: Peer influence, Peer group, Performance task in Mathematics, Senior Secondary School Students

Introduction

Peers are the people in which you have a close relationship with each other and they have regular interactions. They share views, opinions, and exchange ideas with each other and do activities together. Peer group composed of equal status, that all students within a given peer group are of the same age, and came from the same social status. Somehow, peer influence affects the performance of the students either inside or outside the school. Performance tasks as part of the student's performance in senior secondary schools, probably this might be influenced by peers of the students. However, this was still in a process of inquiry.

The performance task is any learning activity or assessment that asks students to perform to demonstrate their knowledge, understanding, and proficiency. Performance task senses as evidence of learning. Some students, somehow, having a lower performance task than the others. Furthermore, the performance task is one of the basic computing student's grades that symbolize how well the students perform in school. This task is being used in every subject area and at all school levels. The Peer group is one of the factors that can affect the students' learning process [1]. Peers can offer a feeling of motivation to learners and assist learners to see how important achievements to pursue [2]. A peer group increases its value and worth if they give help and assistance in the right direction [3].

In senior secondary schools in Sokoto state, it has been observed that students belong to a specific peer group. This peer group might help the students to improve their work on the performance on exercises or assignment in mathematics. Also, it is one of the sources of motivation for some students in doing well on their Mathematical assignments. This study aims to examine the relationship between peer influence and performance tasks. This research paper specifically focuses on how peer influence is related to the performance task towards Mathematics.

Statement of the Problem

The study intends to determine the association of peer influence and performance of mathematics of senior secondary school students. Especially, the study seeks to know about the level of peer influence of senior secondary school students, the level of performance of the students in their task and the association between the peer influence and performance task of senior secondary school students.

Null Hypotheses: There is no association between the peer influence and performance of mathematics task of senior secondary school students in Sokoto state.

Alternative Hypotheses: There is an association between the peer influence and performance of Mathematics task of senior secondary school students in Sokoto state.

Review of Related Literature

The interaction in the group determined to some extent by the personal characteristics of each group member. The group is influenced only to a degree by its members regardless of behaviours exhibited by each individual; the group develops behaviour on its own. Therefore, this research work will focus on giving a clear understanding of the issue of peer influence of the students. Peer influence has factors that affect adolescent behaviour positively and negatively [4]. Peers may help students to encourage and engage their selves towards academic success [2]. Besides, Peers may also influence adolescent decision making due to the changes of their brain's reward system as they go through adolescence in contrast to the immaturity of their cognitive control which associates with each other in a complex manner [5].In the aspect of selecting schools, parents decide better than their children's decisions [3].

Students having positive engagement with peers who value education will have a positive effect on influencing students in their academic activities [6]. Concerning this, the interaction between children and their peers affects their learning capacity positively and negatively [1]. Gender differences during mid-adolescence were most pronounced when girls were more resistant to peer influence than boys [7]. Students are more susceptible to influence from close friends who are similar to them in history or school experiences, friends who have different gender and are not mutual in their choices are less influential than friends that are similar to each other [8]. A friend with the same gender influences a teenager to change his or her level of risk activity, teenagers are more likely to influence by their friends who participated in a risk activity [9]. Besides, the perceived parental and peer attachment of late adolescent college students showed considerable variation, and the two forms of attachment were significantly associated. Therefore, low levels of parental and peer attachment put students at risk of academic failure and inadequate psychosocial skills [10]. Peer influence contributed an important way to the psychosocial functioning as well as the performance of the students in the secondary schools (private and government) peer group affects adolescent students' academic performance positively and negatively [11].

A person together with his or her peers & friends performs a higher activity intensity while spending alone results in less intense activity [12]. Peer relationships can also promote a positive influence as children of different levels of ability begin to work together. Educational similarities among peers in the classroom can enhance students' academic performance [13]. Peer encouragement was linked positively to student grades. Supportive peer relationships also promote rather than discourage achievement for African American youth, they provide a more positive view of African American peer groups that are often represented concerning academic achievement in a primarily negative light [14]. On the other hand, peer influence helps reduce hyperactivity in the classroom, but it has not been investigated for its effect on academic achievement [15].

Student academic performance was substantially related to measuring competence and academic competence [16]. Many parents have a disparity between the past academic performance of students and future parental expectations. Parents with unrealistic expectations about the future performance of their children may find it difficult to know what they can do to improve the academic potential of their children [17]. In academic performance, 4 factors can affect the student's performance in school. Communication, learning facilities, proper guidance shows a positive impact on student performance. While family stress shows a negative impact on the students' performance [18]. The use of social media had affected the academic performance of the students because the majority of them have a mobile phone with internet access and having knowledge of the existence of social media sites. It also confirms that the use of social media and the academic performance of the students are strongly associated [19]. Sleepiness found no significant improvements in the ability to learn and maintain new information amidst having a few hours of sleep [20].

Students who have a high risk of procrastination are academically low average while students who are fewer procrastinators are academically above average in their performances [21]. Other predictors of academic performance include components such as age, course, and a history of failures as well as residence. Gender, and emotional intelligence are the best predictors of students' academic performance [22]. The parents' level of income on the students' academic achievement. Parents also have a vital role to play in improving their children's academic achievement by offering the social, financial and material support they need to encourage their children to succeed at school [23]. Students essentially improve their home environment and

study habits such as giving adequate study time, proper revision of lecture notes, avoiding activities interfering with the planned study schedule, then their learning skills such as presentation skills, reading material related to course content in addition to lecture notes and academic interaction with fellow students [24].

Self-esteem at a high-level lead to good academic performance. Female students are found to have higher academic performance compared to men. It was also observed that the self-esteem of male students was higher compared to female students [25]. Among 200 students, there is no statistically significant correlation among male and female anxiety and their academic performance. This means that gender has no major role to play in deciding nursing students' level of anxiety and academic performance [26]. The academic performance of peers has a powerful explanatory power for an individual's academic performance. The evidence shows the strongest associations across various channels of social interactions with calls and text messages, further reinforcing the phenomenon [27].

Methodology

Design

The study is a descriptive research study. The study design seeks descriptive information on the influence of peer influence on the mathematics performance task of senior secondary school students in Sokoto state.

Respondents

The study that the researcher made was answered by all senior secondary school students. A total of 243 respondents were chosen for the study because they are qualified to answer about peer influence and mathematics performance tasks.

Instrumentation

The main instrument used for the study survey was the questionnaire-Likert scale, which was composed of two parts. Part one contained questions on the peer influence of the students. While part two contained another set of questions about the mathematics performance task of the students.

The questionnaire was scaled on the 4-point scale of Strongly Agree (SA), Agree (A), Strongly Disagree (SD), and Disagree (D). The respondents were asked to indicate the extent of their agreement with the given items.

Procedure for Data Collection

The researcher seek permission from the schools by using consent letter and the respondents were given questionnaires. The researcher gave them enough time to answer the questions. Then, the researcher gave appreciation to the respondents for their cooperation. The data gathered by the researcher were used for the analysis, interpretation, findings, conclusion, and recommendation of the study.

Statistical Treatment

1. The weighted mean and chi-square were used in this research study. A weighted mean was used to identify the mean of each questionnaire and chi-square was used to determine the significance of each variable of the study.

Data Presentation

The data presents and showcases the data been collected, it also shows the analysis and interpretation of data. The purpose of this study was to identify if there is a significant relationship between peer influence and mathematics performance tasks of senior secondary school students in Sokoto state.

Table 1

	Indicators	W/M	Interpretation
1	I belong to a peer group	1.68	Strongly Disagree
2	I spend much time with my peer group	1.95	Disagree
3	My peers have assisted me in improving my mathematics scores	1.92	Disagree
4	I can make academic decisions easily when I am with my peers.	1.86	Disagree
5	We study together after class.	2.42	Disagree
6	We always help each other with academic difficulties.	1.78	Disagree
7	My current achievement in all my subjects is better than the previous one before I met my friends.	2.18	Disagree
8	I often skip classes to spend time with my friends.	3.22	Agree
9	The competition with my peers for grades is quite intense.	2.21	Disagree
10	I and my friends are always talking every class hour.	2.58	Agree
11	I and my friends compete for mathematical scores.	2.77	Agree
12	My peers destruct me in my studies.	2.79	Agree
	Overall Weighted Mean	**2.28**	**Disagree**

Table 1: Peer Influence

Legend: N=243 Strongly Disagree (1-1.75) Disagree (1.76-2.5) Agree (2.51-3.25) Strongly Agree (3.26-4).

The result in table 1 shows that the overall weighted mean of peer influence is 2.28 signifies that students disagreed that their peers don't help them to improve their mathematical scores (grades). The statements, I often skip classes to spend time with my friends, my peers destruct me on my studies, I and my friends compete for mathematical scores (grades), are the indicators with the highest weighted mean of 3.22, 2.79 and 2.77, respectively, which has an interpretation of agreed. While the statements, I belong to a peer group, we always help each other with academic difficulties, I can make academic decisions easily when I am with my peers, are the indicators with a lower weighted mean of 1.68, 1.78 and 1.86, respectively, with an interpretation of disagreed. A student who has a peer with high academic attainment increase their school engagement [2]. Students who are not interested in their studies become study inclined if they belong in a group that encourages them to have effective learning [1]. Peers play a big role in affecting student's decision making [3].

Peer Influence

	Frequency	Percent	Valid Percent	Cumulative Percent
Not Influential	0	0	0	0
Slightly Influential	70	28.8	28.8	28.8
Moderately Influential	158	65.0	65.0	93.8
Highly Influential	14	5.8	5.8	99.6
Total	243	100.0	100.0	

The table above shows the frequency of peer influence on the students. The level with the highest number of students is the average level consist of 158 students which is 65.0% while the level with the lowest number is not influential which is consist of 0 number of the student which is equal to 0%. This shows that most of the students were not influenced by their peers.

Table 2 Mathematics Performance Task

Indicators	Weighted Mean	Interpretation

1	I study and prepare for the best.	1.69	Strongly Disagree
2	I am performing well in school.	1.77	Disagree
3	I actively participate in school activities.	1.91	Disagree
4	I want to get a good score on every test.	1.78	Disagree
5	I study harder to improve my performance	1.78	Disagree
6	I do my tasks every day.	1.78	Disagree
7	I always prepare myself for the next subject.	1.77	Disagree
8	I exert more effort in performance.	1.69	Strongly Disagree
9	I spend most of my time doing my task.	1.79	Disagree
10	I get high grades in every performance task.	2.00	Disagree
	Overall weighted Mean	**1.80**	**Disagree**

Legend: N=243 Strongly Disagree (1-1.75) Disagree (1.76-2.5) Agree (2.51-3.25) Strongly Agree (3.26-4).

The table 2 shows that the overall weighted mean of performance task is1.80 which signifies that students disagreed that they have a high grade in their performance task. The statements, I get high grades in every performance task, I actively participate in school activities, I spend most of my time doing my performance task, are the indicators with the highest weighted mean of 2.00, 1.91 and 1.79, respectively, which has an interpretation of disagree. While the statements, I study and prepare for the best, I exert more effort when it comes to performance, I always prepare myself for the next subject, are the indicators with a lowest weighted mean of 1.69, 1.69 and 1.77, respectively, which has an interpretation of strongly disagree. Being academically competent is an important factor that associates with the high academic performance [16]. Effort is the key determinant of academic success and it can be controlled over time [17]. Students develop their study habits through proper study time, presentation skills and reading lesson-related material [24].

Performance Task

	Frequency	Percentage	Valid percent	Cumulative percent
Poor	17	7.0	7.0	7.0
Average	160	65.8	65.8	72.8
High	59	24.3	24.3	97.1
Excellent	7	2.9	2.9	100.0
Total	243	100.0	100.0	

The table above shows the frequency of the student's performance task. The level with the highest number of responses is the average with 160 responses which is equivalent to 65.8% while the level with the lowest number of responses is excellent with 7 number of responses which is equivalent to 2.9%. This shows that most of the student's grade is average.

Table 3

	Peer Influence and Mathematics Performance Task		
	Value	df	Asymp. Sig(2-sided)
Pearson Chi-Square	9.832E2[a]	700	.000
Likelihood Ratio	462.668	700	1.000
Linear-by-Linear Association	17.756	1	.000
N of Valid Cases	243		

a. 754 cells (100.0%) have expected count less than 5. The minimum expected count is .00.

There is a significant association between the variables peer influence and performance task of Senior High students because the significant value is .000 which is lesser than the alpha =0.05. A negative strong relationship between peer influence and performance task was revealed. This means that the peer influence of students is having a significant correlation with the performance task of students in the negative aspect. Peer group with which a child interacts certainly has an impact on the performance task of the students [1]. The academic accomplishments of students are due to the effect of their peer group [3]. If the peer group was interested in studying hard and doing well, their fellow teenage students would possibly be motivated to perform better [11].

Findings

The result of this study reveals the level of peer influence and performance tasks of senior secondary school students. It shows that there is a negative association between peer influence and mathematics performance tasks of senior secondary school Students. The study shows that peers can influence a student's engagement in class. This means that the peer has a negative influence on the students. It shows from the results that peer destruct the students in their studies. Students were disagreeing that they get a high score in every mathematics performance task and they are not active in class. The results show that students' mathematics performance task can be influenced by their peers. The study shows that there is a significant association

between the variables peer influence and mathematics performance task of senior secondary school students in Sokoto state. The results show that peers can influence the students' mathematics performance tasks negatively. This means that if the peers have a negative action, the students can influence, or vice versa. A strong relationship between peer influence and mathematics performance task was revealed.

Conclusion

Peers have the most influential factor that affects the students' behaviour and actions positively or negatively. Students spend a lot of time being with their peers particularly in a school that can also give an impact on their behaviour in school during class. The mathematics performance of students also reflects of what behaviour and actions they show in school. Most of the respondents disagree that they get a high score in their task because of the influence of their peer group. Also, the study concluded that peer influence truly gives a negative impact and has an association with the performance task.

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RETHINKING TEACHING AND LEARNING OF MATHEMATICS AS TOOLS FOR CREATIVITY, TECHNOLOGICAL INNOVATION, SELF-RELIANCE, AND SUSTAINABLE DEVELOPMENT IN TIMES OF ECONOMIC DEPRESSION

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Abstract

In times of economic depression, the importance of robust educational systems becomes more pronounced, particularly in the teaching and learning of mathematics. This paper explores how mathematics, beyond being a core academic subject, can serve as a catalyst for creativity, technological innovation, self-reliance, and sustainable development. The current economic climate calls for a paradigm shift in the way mathematics is taught and perceived from rote learning to dynamic, real-world problem-solving. Drawing on contemporary pedagogical theories and empirical studies, this paper proposes a reconceptualization of mathematics education to empower learners to think critically, innovate locally, and contribute sustainably to national development. Recommendations are offered for educators, policymakers, and curriculum developers to align mathematical instruction with 21st-century needs and economic realities.

Key wordsl: Teaching, Learning, Mathematics, Creativity, Technological, Innovation, Self-Reliance, Sustainable Development

Introduction

The global economic landscape has witnessed multiple downturns, with the most vulnerable sectors often being education and innovation. In such times, there is a pressing need to leverage core academic disciplines as tools for economic revitalization and social empowerment. Mathematics, often relegated to abstract theory and procedural learning, holds immense potential as a foundation for creativity, technological advancement, and sustainable development. Traditionally viewed as rigid and mechanical, mathematics must now be reimagined as an empowering discipline that promotes analytical thinking, innovation, and practical application. This paper examines how rethinking the pedagogy of mathematics can lead to greater self-reliance and national resilience, especially in periods of economic hardship. The role of mathematics in national development has long been emphasized in educational literature. According to Restivo (2017), mathematics is a socially constructed discipline that reflects the values and needs of society. In line with this, Freudenthal (2012) advocated for "mathematics as a human activity," highlighting its relevance to real-life problem solving. Recent scholarship has emphasized the integration of STEM (Science, Technology, Engineering, and Mathematics) education to foster creativity and innovation (Bybee, 2010).

Moreover, constructivist theories (Piaget, Vygotsky) underscore the importance of learner-centered approaches in mathematics education, promoting deeper conceptual understanding and autonomous thinking. D'Ambrosio's (1985) concept of ethnomathematics also supports the idea that mathematics should be rooted in the learner's cultural and social context, encouraging self-reliance and local innovation.

In times of economic depression, mathematics education becomes even more critical. Studies by UNESCO (2021) and the World Bank (2022) highlight how countries with strong math and STEM education infrastructures tend to show faster recovery from economic shocks due to a skilled, innovative workforce. Therefore, curriculum reform must focus on the application of

mathematics in entrepreneurship, environmental modeling, digital technology, and sustainable practices. Periods of economic depression are characterized by severe downturns in economic activity, marked by high unemployment, reduced consumer spending, business failures, and declining GDP (Kose, Sugawara, & Terrones, 2020). Global recessions.. These crises often expose the structural weaknesses of globalized economic systems, highlighting the overreliance on external resources, imports, and unstable international markets (Kahler, 2013). In such times, the concepts of *self-reliance* and *sustainable development* emerge as vital pathways for economic resilience, social stability, and long-term recovery.

Self-reliance refers to the ability of individuals, communities, and nations to meet their needs independently, using local resources, knowledge, and capabilities (Thomas, & Pawar, 2010). Sustainable development, as defined by the Brundtland Commission (1987), involves development that meets the needs of the present without compromising the ability of future generations to meet their own needs. When integrated, these approaches form a potent strategy for navigating economic depression by fostering local innovation, encouraging resource efficiency, and reducing dependency on volatile external systems (Song, & Zhao, 2024).

Conceptualizing Technological Innovation

Technological innovation refers to the process by which new or significantly improved technologies are developed and implemented to create value be it economic, social, or environmental. According to Bubanja, and Madzar, (2019), innovation is the engine of economic development, involving new combinations of existing knowledge, resources, and capabilities. Technological innovation can be both radical (transformative breakthroughs) and incremental (gradual improvements).

Tassey, (2016) emphasizes that technological innovation is path-dependent and often shaped by existing technological trajectories and institutional frameworks. It goes beyond invention by including diffusion, adaptation, and commercialization.

Drivers of Technological Innovation

a. Research and Development (R&D)

R&D is a primary driver of innovation. Camisón, and Forés, (2010), describe firms' "absorptive capacity" their ability to recognize, assimilate, and apply external knowledge as crucial to successful innovation. Public and private investment in R&D fosters the creation of new knowledge and supports technological advancement.

b. Market Demand and Competition

Market needs and competitive pressure drive firms to innovate for differentiation and survival (De Oliveira, & Werther, 2013).) The demand-pull model suggests that innovation is often a response to market needs, while the technology-push model views it as a product of scientific advancement (Roy, 2024).

c. Digital Transformation

Recent literature highlights the role of digital technologies such as artificial intelligence (AI), big data, cloud computing, and the Internet of Things (IoT) in reshaping innovation processes. According to Kuloğlu, (2024) these technologies accelerate innovation cycles and disrupt traditional business models.

Types and Patterns of Technological Innovation

Diaconu, (2011).categorize technological innovations into four types: product, process, organizational, and marketing innovation. Each type contributes differently to firm performance and sectoral transformation. Additionally, Choi, and Woo, (2020). theory of disruptive innovation explains how simpler, more affordable technologies can overtake established market leaders by initially serving niche or underserved segments.

Innovation Ecosystems and Collaboration

Modern innovation is increasingly collaborative. Innovation ecosystems including universities, startups, governments, and corporations play a vital role in knowledge sharing and co-creation (Kreiling, & Paunov, 2021). Open innovation, as described by Berchicci, (2013), emphasizes the use of external ideas and paths to market as firms recognize they cannot rely solely on internal R&D.

Barriers and Challenges

Despite its potential, technological innovation faces challenges such as high costs, regulatory constraints, ethical concerns (e.g., privacy and AI bias), and resistance to change (Lescrauwaet, Wagner, Yoon, & Shukla, 2022). Additionally, the digital divide creates disparities in access to innovative technologies, especially in developing regions (Drori, 2010).

Self-Reliance

The idea of self-reliance has been extensively discussed in development discourse, particularly by thinkers like Mahatma Gandhi, who emphasized local production and autonomy as a counter

to colonial economic structures (Parekh, 1997). In modern economic thought, self-reliance is often associated with endogenous development, where communities draw on internal strengths to drive growth (Toye, 1987). In times of economic crisis, self-reliance helps mitigate the impacts of global supply chain disruptions and fosters innovation in local industries (Sen, 1999).

Sustainable Development as a Crisis Response

Sustainable development has become a central theme in both policy and academic discussions on post-crisis recovery. Following the 2008 global financial crisis, numerous scholars emphasized the need for a "green recovery" that promotes environmental sustainability alongside economic revitalization (Jacobs, 2012; UNEP, 2009). Sustainable practices such as renewable energy development, sustainable agriculture, and circular economies not only reduce environmental degradation but also create jobs and enhance economic security during downturns (Meadows et al., 2004).

Synergies Between Self-Reliance and Sustainability

Recent literature suggests that self-reliance and sustainable development are mutually reinforcing. For instance, Altieri (2009) discusses agroecology as a model of food sovereignty that enhances local food systems while promoting ecological balance. Similarly, Sachs (2015) emphasizes that sustainable development goals (SDGs) are best achieved through locally-led initiatives rooted in community empowerment. During economic depressions, such integrated approaches help reduce external dependencies, increase resilience to shocks, and build adaptive capacity.

Historical and contemporary case studies provide evidence of self-reliance and sustainability strategies during economic downturns. Cuba's "Special Period" in the 1990s, triggered by the collapse of the Soviet Union, is often cited as a successful example of local adaptation through urban agriculture and energy conservation (Funes et al., 2002). Similarly, the 2008 financial crisis prompted a wave of local economic initiatives in Southern Europe, including cooperative businesses, time banks, and community-supported agriculture (Bauhardt, 2014). These cases demonstrate how grassroots innovations can contribute to both economic survival and long-term transformation.

Times of Economic Depression

A time of economic depression refers to a prolonged period of severe economic decline that is worse and longer-lasting than a recession (Forbes, & Krueger, 2019). It is marked by a

significant and sustained downturn in economic activity across multiple sectors, affecting employment, production, trade, and overall societal well-being (El Keshky, Basyouni, & Al Sabban, 2020).

Key Characteristics of Economic Depression

1. Massive Unemployment
   Large-scale job losses are a hallmark. Unemployment rates can soar into double digits and remain high for years.
2. Sharp Decline in GDP
   Gross Domestic Product (GDP) contracts sharply and may take years to recover.
3. Business Failures
   Many companies, especially small and medium enterprises, shut down due to reduced demand and financial instability.
4. Deflation or Hyperinflation
   Prices may fall significantly (deflation), or in some cases rise uncontrollably (hyperinflation), depending on the root cause of the depression.
5. Bank Failures and Financial Crisis
   Financial institutions may collapse due to bad loans or runs on banks, leading to a loss of public confidence.
6. Reduced Consumer Spending and Investment
   With low income and high uncertainty, people and businesses cut spending, deepening the economic decline.
7. Social Distress
   Rising poverty, homelessness, and mental health issues often accompany economic depressions.

According to Matés-Barco, (2023) the following are the Causes of Economic Depression

1. Stock Market Crashes
   Like the 1929 Wall Street Crash that triggered the Great Depression.
2. Banking Collapse
   When banks fail, credit dries up, paralyzing economic activity.

3. Severe Policy Mistakes

   Austerity, poor regulation, or tight monetary policy can worsen downturns.
4. Global Shocks

   Wars, pandemics, or international financial crises can trigger or deepen depressions.
5. Demand Collapse

   A sudden drop in consumer and business demand causes supply chains and production to grind to a halt.

Historical Example: The Great Depression (1929–1939)

1. Began with the U.S. stock market crash in 1929.
2. GDP fell by about 30% in the U.S.
3. Unemployment reached nearly 25%.
4. Spread worldwide, affecting Europe, Latin America, and Asia.
5. Recovery came only with large-scale public spending and World War II industrial mobilization.

Modern Concerns and Prevention

Today, most governments and central banks actively work to prevent depressions through:

1. Monetary stimulus (cutting interest rates, quantitative easing)
2. Fiscal stimulus (government spending, tax cuts)
3. Regulation to prevent financial system collapse
4. Social safety nets to cushion vulnerable populations

Mathematics plays a key role in understanding economic depressions by modeling economic trends, analyzing financial data, forecasting downturns, optimizing policies, quantifying risks, and evaluating recovery strategies through tools like statistics, calculus, and econometrics to inform sound economic decisions.

Conclusion

The traditional model of mathematics education is inadequate for addressing the multifaceted challenges of the 21st century, especially during economic downturns. Rethinking mathematics instruction to incorporate creativity, problem-solving, technological integration, and sustainable practices is not only timely but necessary. Such an approach enables learners to move beyond theoretical knowledge and engage with mathematics as a tool for innovation and societal transformation. Empowering students through relevant, contextualized, and purpose-driven

mathematics education can drive self-reliance and catalyze national development even amid economic hardship. Times of economic depression are rare but deeply damaging events in economic history. They reveal vulnerabilities in global and national economies, often leaving long-lasting scars on societies. Understanding their causes and impacts is essential for building economic systems that are resilient, inclusive, and better prepared for future shocks.

Recommendations

1. Curriculum Redesign: Integrate real-world applications of mathematics in curriculum content to enhance relevance and learner engagement.
2. Teacher Training: Provide continuous professional development for teachers to adopt innovative, student-centered pedagogies.
3. Technological Integration: Encourage the use of digital tools, simulations, and coding in math classes to foster creativity and tech-savviness.
4. Entrepreneurial Mathematics: Introduce mathematical modeling in small business, finance, and sustainability as part of classroom instruction.
5. Policy Support: National education policies should prioritize mathematics as a key driver of innovation and sustainable development.

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IMPACT OF LANGUAGE OF IMMEDIATE ENVIRONMENT (LIE) ON PERFORMANCE IN MATHEMATICS AMONG LOWER BASIC PUPILS IN NORTH-WEST ZONE, NIGERIA.

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Abstract

This study investigated the Impact of Language of .....liate Environment (LIE) on Performance in Mathematics among Lower Basic Pupils in North- West Zone, Nigeria. The study was conducted based two (2) specific objectives. Two (2) research questions were constructed and two (2) null hypotheses were formulated to guide the study. A pre-test, post-test and post-posttest quasi experimental design was adopted. From the target Population of seven million six hundred and seventy-four thousand and ninety- eight (7,674,098) Pupils from the North Western states of Nigeria. A multi stage purposive simple random sampling technique was used to select the states, local governments and the schools of the pupils. Three hundred and eighty-four (384) pupils were selected for the study in accordance with Krejci's and Morgan's method of estimating a sample size from a large population. The Mathematics Performance Test (MPT) was used to collect data on the performance of the lower basic pupils in both the experimental and the control group. The instrument was validated and the reliability coefficients of MPT was established at r=0.87. The data were analyzed using mean, standard deviation, and t-test Analysis at 0.05 level of significance using statistical package for social sciences (SPSS). Result of the study indicated that lower basic primary school pupils taught using Language of Immediate Environment (LIE) performed better in learning mathematics than those taught using English language. The study established that using Language of Immediate Environment (LIE) to teach Mathematics concepts was highly effective in enhancing the performance of lower basic primary school pupils. The study revealed among others that Language of Immediate Environment was gender friendly. The study recommended among others that mathematics teachers should endeavor to use Language of Immediate Environment in teaching lower basic pupils in the North West zone, Nigeria. The results of the findings of the study suggests for further studies in different location of the country and in different subjects or by using other dependent variable such as self-efficacy. Mathematics students should use their language of immediate environment to help them understand mathematical concepts by practicing mathematics in their mother tongue and ask for language support from teachers, if they struggle with the language of instruction. The findings of the study would also benefit the mathematics educators, curriculum planners, government and non-governmental organizations among others.

Introduction

The use of Language of Immediate Environment (LIE) as a medium of instruction is globally recommended for pupils in lower primary schools (UNESCO, 2013). Global campaign for education (2013) also reveals that, learners taught in foreign language perform poorly than those taught in Language of Immediate Environment.

Omoniyi, Israel, Olabode and Thomas (2013) stated that Mathematics taught in a child's Language of Immediate Environment has benefits such as overcoming limited knowledge of foreign mathematical vocabulary and brings children closer to Mathematics.

According to the global monitoring report (2013), in many countries, most children were taught in languages that they do not understand or they do not speak at home. Dialyn, Andrea, Jochelle, and Pailileng (2014) further asserted that using Language of Immediate Environment as a medium of instruction improve pupils'performance and strengthens their learning. Dialyn, *et al.* (2014) stated that Language of Immediate Environment enables a child to express himself/herself easily, as there is no fear of making mistakes. Learners taught in Language of Immediate Environment are more active and perform better than those taught in foreign language (Umar, 2018).

Ball (2011), asserts that many children speak a home language that differs from the language of instruction in education programmes. Consequently, when educationists discuss a child's home language, it may be the learner's Language of Immediate Environment or it may be the language the learner has learnt first and knows best. It can also be the language one uses often or most. In this literature review and the research discussions the term familiar language will mean any of these.

Hazik and Farik (2016), Beka (2016), Kazima (2014) conducted the study on using the Language of Immediate Environment or familiar language as a medium of instruction for the early primary education. Their study from two schools of thought can be discerned: Hazik and Farik (2016), Beka (2016), asserts that teaching Mathematics and science in a home language (language of play) enhances learner performance and reduces school drop outs while Kazima and Mashegoane (2016) condemns the pedagogy because it assumes that most local languages have limited mathematical and science vocabulary.

Mohammed, *et.al,* (2010), conducted a study on Mathematical Game in Hausa Culture and explained that, before the coming of Western education, Hausa people in northern Nigeria used Mathematics in sorting, ordering, measuring, timing and weighing in their day to day activities. Featuring prominently in this culture are the traditional games played by children or adults with the aim of deriving lessons. These games are meant to shape individual brain. The primary concern of this paper is on the games that involves calculations and to highlight therein the existence of algebra, set theory, coordinate geometry, arithmetic progression and geometry progression in Hausa culture. Furthermore the paper will correct the erroneous believe that there was no Mathematics in Hausa land/northern Nigeria before the coming of the Western education and address those who believe Mathematics only starts and ends in the classroom, thereby divorcing the rich cultural elements from the use of Mathematics.

Ya'u (2018), stated that Lower Basic Primary School pupils taught using ethnomathematics teaching approch performed and retained mathematical concepts better than those taught with conventional teaching method. The result also revealed no significant difference between the performance and retention of male and female Lower Basic Primary School pupils taught mathematical concepts using ethnomathematics teaching strategy. The study revealed among others that ethnomathematics teaching strategy was gender friendly.

This study was guided by social constructism learning theory in one way or the other have influenced educationist to fiercely advocate for the use of Social Constructism theory in learning mathematics.

Social constructivism is a learning theory propounded by Levy Vygotsky (1968). The theory states that language and culture are the frameworks through which humans experience, communicate, and understand reality. According to Vygotsky, language and culture play essential roles both in human intellectual development and in how humans perceive the world. This is to say that learning concepts are transmitted by means of language, interpreted and understood by experience and interactions within a cultural setting. Since it takes a group of people to have language and culture to construct cognitive structures, knowledge therefore is not only socially constructed but co-constructed. The link here is that while the constructivist sees knowledge as what students construct by themselves based on the experience they gather from their environment, the social constructivist sees knowledge as what students do in collaboration with other students, teachers and peers. Social constructivism is a variety of cognitive constructivism that emphasizes the collaborative nature of learning under the guidance of a facilitator or in collaboration with other students.

The study therefore, determined the Impact of Language of Immediate Environment on Performance in Mathematics among Lower Basic Pupils in North- West Zone, Nigeria.

Statement of the Problem

Despite National Policy on Education's recommendation that pupils in the lower primary schools in Nigeria be taught in their Language of Immediate Environment, primary school teachers in Nigeria usually teach Mathematics using English language. Across the developing world, countries face challenges in improving learning outcomes. Researchers conducting theory-based studies in the United States and other developed countries have built a body of evidence demonstrating that children learn to read most easily in their Language of Immediate Environment, and that skills they acquire while learning to read in that language transfer to learning to read in other languages, at least to some extent (Melby-Lervåg & Lervåg, 2011). One of the main reasons that stakeholders resist the implementation of Language of Immediate Environment based-instruction is their fear that it will somehow harm students' acquisition of English and other subjects seen as more valuable and prestigious for students (Jones, 2012). Even when a government's official policy mandates initial literacy instruction in a Language of Immediate Environment, as it does in Kenya, such mandates are often disregarded at the local level (Piper, *et. al*, 2014).Several of the targets for the new Sustainable Development Goals focus on education quality, or more specifically, literacy and numeracy (United Nations, 2015).

The issue of language of instruction is central to the debate around improving instructional quality and student achievement in multilingual countries. A small but growing body of research from the Global South suggests that Language of Immediate Environment based literacy outcomes can be improved at medium scale (Piper, *et al.*, 2016; Taylor & von Fintel, 2016). However, in sub-Saharan Africa, there are disconnects among theory-based recommendations, policy, and practice (Clegg & Simpson, 2016). These practices might lead to children in the lower primary schools to find it difficult to comprehend Mathematics lessons and thus, have lack of performance, perform poorly and with low retention in Mathematics. Unless appropriate measures are taken, this problem will be a barrier to the effort of the United Nations declaration to provide inclusive, quality education and lifelong educational opportunities for all by 2030, as contained in the Sustainable Development goal four (4).

According to (Kim, 2017), until the day that Africans will start practicing their own culture they will develop. Globalization has never helped any country or continent, so if any Africa must succeed, they should never take global standard as a way of life. Practice what you know best in accordance to your culture. Against this background, the study investigated the Impact of Language of Immediate Environment on Performance in Mathematics among Lower Basic Pupils in North- West Zone, Nigeria.

Objectives of the Study

The main objective of this study was to examine the Impact of Language of Immediate Environment on Performance in Mathematics among Lower Basic Pupils in North- West Zone, Nigeria. Specifically, the study seeks to:

1. find out impacts of Language of Immediate Environment on performance in Mathematics among lower basic pupils.
2. investigate gender influences on the impact of Language of Immediate Environment on performance in Mathematics among lower basic pupils.

Research Questions

The following research questions were asked to guide the study.

1. What is the impact of Language of Immediate Environment on performance in Mathematics among Lower basic pupils?
2. What is the gender influence on the impact of Language of Immediate Environment on performance in Mathematics among Lower basic pupils?

Null Hypotheses

The following null hypotheses were formulated based on the research questions and tested at $P \leq 0.05$ level of significance.

$H_{01:}$ There is no significant difference between the performance of pupils taught Mathematics using Language of Immediate Environment and those taught using English language.

$H_{02:}$ There is no significant difference between the performance of male and female pupils taught Mathematics using Language of Immediate Environment.

Methodology

This research employed pre-test, post-test and delayed (post-posttest) quasi-experimental control group design. This design was used because, it is not possible to have complete randomization of the subjects as this would disrupt school organization. Hence, intact classes were randomly assigned to the experimental and control groups. The researcher taught the experimental group using Language of Immediate Environment and the Control group pupils were taught same mathematical concepts but using English Language as the medium of instructions. The impacts of these approaches were carefully determined and recorded. Prior to administration of the treatment, pre-test was administered to the two groups to determine the comparability ability levels of the groups.The Population of the study was made up of sixteen thousand seven hundred and sixty-three (16,763) public primary schools which comprised of four million four hundred and ninety four thousand and five (4,494,005) males and four million three hundred and fourty six thousand nine hundred and sixty five ((4,346,965) females with the total population of eight million eight hundred and fourty thousand schools used for the study were co-education, nine hundred and seventy (8,840,970) Lower basic primary school pupils from the North western states. The details of the population were presented in Table 1.

Table 1: Enrollment of Lower Basic Primary Schools Pupils' (1-6) in North-West Zone, Nigeria.

S/N	STATE	NO. OF SCHOOLS	MALE	FEMALE	TOTAL
1.	Jigawa	2,345	421,645	399,830	821,475
2.	Kaduna	4,177	595,787	619,595	1,215,382
3.	Kano	2,061	1,574,321	1,739,116	3,313,437
4.	Katsina	2,772	925,620	935,991	1,861,611
5.	Kebbi	1,788	320,828	197,772	518,600
6.	Sokoto	2,047	268,998	197,445	466,443
7.	Zamfara	1,573	386,806	257,216	644,022
TOTAL		**16,763**	**4,494,005**	**4,346,965**	**8,840,970**

Source: Federal Ministry of Education, 2021.

In this study, a multi stage simple random sampling technique was used and two states (Kano and Kaduna) were selected from the North Western states in Nigeria. Furthermore, a simple random sampling technique was used to select one local government area each from the selected states (Lere LGA and Doguwa LGA from Kaduna and Kano states respectively were selected as

experimental and control groups respectively. The sampling was carried out by using simple random sampling using hat and draw method and selected two schools from each selected Local government area. Furthermore, in selecting the pupils' samples, intact classes were used for both the experimental and control groups. A sample size of 384 pupils from both experimental and control groups were used for the study. This was supported by Krejcie and Morgan (1970) who stated that with a confidence level of 95% (given an alpha level α= 0.05) a viable sample size (S) for a large population N can be calculated by using the formula as follows:

$$\mathbf{S}= \frac{X^2NP(1-P)}{d^2(N-1)+X^2P(1-P)}$$

Where S = required sample size

N = the Population Size

X^2= the table value of chi- square for one degree of freedom at the desired confidence level

P = the population proportion (assumed to be 0.50)

d= the degree of accuracy expressed as a proportion (0.05).

Table 2: Sample Selected for the Study

State	Schools	Group	No. of Males	No. of Females	Total
Kaduna	A	Experimental	110	91	201
Kano	B	Control	105	78	183
Total			**215**	**169**	**384**

Mathematics Performance Test (MPT) was developed by the researcher in order to determine the level of homogeneity between the experimental and control groups. The test consists of 20 multiple choice objectives items with options A-D, which were based on the topics covered (geometry, statistics, number and numeration and measurement). One correct answer and three distractors that made up the options were drawn from Lower Basic Primary School syllabus, 2011-2021.The Mathematics Performance Test (MPT) covered the factual, conceptual and procedural knowledge from the topics covered. The instrument was validated by the experts in the area of Mathematics, Mathematics Education and Hausa Language. The experts were given the instruments and checked their relevance, clarity of the language and any contradictions.

The reliability of the instruments was determined after the pilot testing. The reliability was established Using split-half method with the aid of Statistical Package for Social Sciences (SPSS) and determined the reliability coefficient of the instrument. Based on the data obtained from the pilot testing of the instrument, the reliability coefficient of Mathematics Performance Test (MPT) was ascertained to be r = 0.87 indicating that the research instrument was reliable. All data collected via the instrument for this study were analyzed by the use of descriptive or inferential statistic with the aid of statistical Packages for Social Sciences (SPSS). Data from the research questions were described using mean, standard deviation and mean differences. The null hypotheses were tested at 0.05 and analyzed by the use of independent t-test statistic.

Results

Question One:

What is the impact of Language of Immediate Environment on performance in Mathematics among Lower basic pupils?

To answer this research question, the mean performance scores and standard deviations were computed for both the experimental and control groups. Summary of the findings was shown in Table 3.

Table 3: Summary of the Mean and Standard Deviation of Post-test Scores for Experimental and Control groups.

Variable	N	Mean($\bar{X}$)	*SD*	Mean Diff.
Experimental	201	16.56	2.03	
				4.32
Control	183	12.24	1.24	

From Table 4.2, the result of the descriptive statistics showed that "Experimental group (m=16.56, S.D=2.03) have the higher mean than the Control group (($\bar{X}$) =12.24, S.D=1.24) and a mean difference of 4.32 between the experimental and the control group scores.

However, further analysis was conducted to ascertain whether the difference in mean was statistically significance and this led to the analysis of hypothesis one as presented in Table 4.

Null Hypothesis One: There is no significant difference between the performance of Lower Basic Primary School pupils taught mathematics using Language of Immediate Environment (LIE) and those taught using English Language. This hypothesis was analyzed using independent sample t-test statistic at $P \leq 0.05$ and the summary was presented in Table 4.

Table 4: Summary of Independent sample t-test Statistic of Mean Performance Scores in Mathematics for Experimental and Control Groups

Group	N	Mean($\bar{X}$)	*SD*	SED	Df	t-value.	P-value
Experimental	201	16.56	2.03	0.174			
					382	24.89	0.00
Control	183	12.24	1.24	0.170			

*** Significant at $P \leq 0.05$, df = 382**

From Table 4, an independent sample t-test showed that the difference in test mean scores between the experimental group (n=201, $\bar{X}$=16.56, S.D=2.03) and the control group (n=183, $\bar{X}$=12.24, S.D=1.24) were significant. From Table 4.3, A P-value of 0.000 was observed which was less than $P \leq 0.05$ at df = 382. Thus, the null hypothesis which stated that; there is no significant difference between the performance scores of pupils taught mathematics using Language of Immediate Environment (LIE) and those taught using English Language was rejected. This means that there is significant difference between the performance of Lower Basic Primary School pupils in the experimental group and control group in favored of the experimental group.

Question Two: What is the gender influence on the impact of Language of Immediate Environment on performance in Mathematics among Lower basic pupils?

To answer this research question, the mean performance scores and standard deviations were computed for both the experimental and control groups. Summary of the findings was shown in Table 5.

Table 5: Summary of Mean and Standard Deviation of Performance Scores for Male and Female Pupils in the Experimental Group.

Gender	N	Mean($\bar{X}$)	*SD*	Mean Diff.
Male	110	16.97	2.02	0.9
Female	91	16.07	1.93	

Result of the descriptive statistics showed that, in the experimental group male ($\bar{X}$= 16.97, *SD* =2.02) and female ($\bar{X}$=16.07, *SD*=1.93). This indicated that, there is a mean difference of 0.9 between the male and female performance scores.

However, further analysis was conducted to ascertain whether the difference in mean was statistically significance and this led to the analysis of hypothesis two as presented in Table 4.5.

Null Hypothesis Two: There is no significant difference between the performance scores of Male and Female Lower Basic Primary School pupils taught mathematics using Language of

Immediate Environment (LIE). This hypothesis was analyzed using independent sample t-test statistic at $P \leq 0.05$ and the summary was presented in Table 4.5.

Table 6: Summary of Independent sample t-test Statistic of Mean Performance Scores for Male and Female Pupils in the Experimental group.

Gender	N	Mean	*SD*	SED	df	t-value	P-value
Male	110	16.97	2.02	0.28			
					199	0.417	0.417
Female	91	16.07	1.93	0.27			

From Table 4.5, an independent sample t-test showed that the difference in test scores between the male (n=110,$\bar{X}$ =16.97, *SD*=2.02) and female (n=91,$\bar{X}$ =16.07, *SD*=1.93) in the experimental group were significant. From the Table, the P-value observed (0.417) was greater than $P \leq 0.05$ at df = 199. Thus, the null hypothesis which stated that; there is no significant difference between the performance of male and female taught mathematics using Language of Immediate Environment (LIE) was accepted. This implies that the use of LIE is gender friendly on the performance of pupils towards mathematics learning.

Discussion of results

The results from Table 3 and 4 showed that, the experimental group had a higher mean performance score than those in the control group. The difference was significant as indicated in the Table. This signifies that pupil taught mathematics using LIE had a significantly higher mean performance scores than those taught using English Language. By implication, the Language of Immediate Environment was able to foster a significantly higher performance than the control group. The significance difference implies rejection of the null hypothesis that states that there is

no significant difference between the performance of Lower Basic Primary School pupils taught mathematics using Language of Immediate Environment (LIE) and those taught using English Language is hereby rejected.

The significance difference indicates that pupils in experimental group recorded the highest mean scores than those in the control group. The findings of this study is in agreement with Kimizi (2012) added that children taught in Language of Immediate Environment (LIE) have better learning comprehension than those taught in a foreign language. Imoko 2009, Kurumeh 2012 and Onudioku 2013, Similarly, in support of the findings of Fiji, Hazik and Farik (2016), who assert that children learn mathematical concepts better using their home language. This is because teachers relate the concepts to children's everyday environment. The results revealed among others that using Language of Immediate Environment (LIE) in teaching was effective in improving pupils' performance in mathematics than the used of English language as the medium of instruction.

The results from Table 5 and 6, revealed that, there was no significant difference in the performance scores of male and female pupils in the experimental group. But, the difference was not statistically significant. The result was confirmed with that of Paden and Dereshiwsky (2007) and Omenka (2010), who attributed the low performance in mathematics of students particularly among sex to instructional modality adopted by teachers and the language of communication. The result disagreed with that of Usman (2007) who compared the performance between male and female in practical activities, which revealed that male outperformed female in rigorous work. In addition, the findings disagreed with that of Caplan (2005), who confirmed gender difference in favor of male after investigating the effect of modern mathematics on performance of pupils.

Conclusion

1. The study established that using Language of Immediate Environment (LIE) to teach Mathematics concepts was highly effective in fostering the acquisition of higher performance ability among lower basic primary school pupils.
2. The study established that using Language of Immediate Environment (LIE) to teach Mathematics concepts was gender friendly.

Recommendations

Based on the findings of this study, the following recommendations were made:

1. Mathematics Teachers should use language support strategies such as bilingual instruction to help pupils who speak different languages and develop teaching materials that reflect the language and culture of the pupils. Similarly, they should provide support to the pupils who struggle with the language of instruction. Mathematics Teachers and school guidance counselors should work together in order to guide the pupils to use Language of Immediate Environment (LIE).
2. Journal Publishers should publish more research on the impact of language of immediate environment (LIE) on mathematics education in Nigeria. They should encourage researchers to explore the impact of language on pupils' performance, performance and retention in mathematics and also, provide a platform for researchers to share their findings on language and mathematics education.
3. Researchers should conduct more research on language and mathematics education in Nigeria and explore the impact of language of immediate environment on pupils' performance, performance and retention in mathematics. Researchers should develop language support strategies that can be used by teacher and pupils.

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EFFECTS OF COMPUTER-ASSISTED INSTRUCTION ON GEOMETRY PERFORMANCE AND RETENTION AMONG SENIOR SECONDARY STUDENTS IN KANO: IMPLICATIONS FOR SELF-RELIANCE IN TIMES OF ECONOMIC DEPRESSION

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Abstract

This study examined the Effects of Computer-Assisted Instruction on Geometry Performance and Retention among Senior Secondary Students in Kano: Implications for Self-Reliance in Times of Economic Depression. The population of the study was all the 77,616 Senior Secondary students in Kano State. A sample size of 144 SSII students was used in the study. Two instruments, the Pre-Geometry Performance Test (PRE-GPT), Geometry Performance Test (GPT) and Geometry Retention Test (GRT) with reliability of 0.87 indexes were developed and used by the researchers to test the students' performance and retention ability in four areas of Geometry - angles, triangles, polygons and circles. Three hypotheses were formulated and t-test was used in their analyses. Experimental group was taught the selected geometry concepts using CAI for six weeks while control the group was taught same geometry concepts for the same duration using the conventional method. Result of the post-test and post post-test favored the experimental group against control group. Also, post test result favored male students against their female counterpart in the Computer environment. Findings from the study revealed that CAI significantly improves students' geometry performance and retention, which could promote self-reliance in mathematics education over the Conventional method of Instruction. The implications deduced from this study were that Experimental Group performed better than the Control Group in the areas of learning and retention, while boys performed significantly higher than girls in the experimental group. It is recommended that stakeholders in mathematics education should integrate CAI in mathematics teaching, text books and trainings in order to aid learning and retention of Geometry concepts to promote entrepreneurial skills for self-reliance in times of economic depression.

Key Words: Computer Assisted Instruction, Academic Performance, Retention, Geometry

1.1 Introduction

In today's economic hardship, Nigeria needs self-reliant citizens who can think critically and solve problems innovatively. Mathematics and Geometry are essential skills for entrepreneurial success, but many students struggle with these subjects. This study investigates the effect of CAI on Geometry performance and retention, promoting self-reliance and entrepreneurial mindset among Senior Secondary School students in Kano.

However, mathematics performance in secondary schools attracts the attention of many mathematics educators for many years. This is because performance in mathematics is a major pre-requisite for the attainment of scientific and technological development nationwide. Imuko (2004) put it that mathematics has continued to play a key role in the development of any nation. He also reported that importance is attached to mathematics teaching because of its role in national development and it serves as the bedrock of technology. Mathematics has various branches. One of the branches is Geometry.

Geometry occupies a very important position in mathematics and as such is placed to be a core topic in the Nigerian curriculum of mathematics and yet many students have problem with the concept. In science, geometry provides the ideas for modeling astronomical phenomena, the action of physical forces, and forms of living objects. In the art, geometric concepts of symmetry and projection have been fundamental in creation such as Renaissance paintings, (Haruna 2008).

Despite the importance of mathematics and Geometry in entrepreneurship and economic development, many students in Kano state struggle with performance and retention in this subject, hindering their self-reliance and economic empowerment. Some of the causes of students' poor performance in mathematics and geometry are attributed to poor method of teaching, lack of motivation, teachers' incompetency etc. On the students' part, it can be attributed to lack of interest etc. Other reasons advanced include dearth of instructional materials and the pedagogical approaches used in imparting knowledge to the learners (Iji, 2005). One of the common methods preferred by mathematics teachers today is the traditional approaches to the teaching (Lecture method) which emphasizes what the teacher did in the classroom and the students are passive in the learning process. Therefore, as mathematics educators, we need to develop teaching methods that our student can enjoy, experience, understand, and appreciate this fascinating subject.

Performance and retention in Geometry can influence entrepreneurial innovation through problem-solving skills, critical thinking, and analytical reasoning. These skills are essential for entrepreneurs to innovate and navigate complex business challenges. It enhances spatial reasoning and visualization abilities, which can benefit entrepreneurs in designing innovative products, services or solutions. It also fosters critical thinking and creativity, allowing entrepreneurs to approach problems from unique angles and develop innovative solutions. Studies have shown that entrepreneurial orientation, which includes aspects like innovativeness, risk-taking and pro-activeness significantly enhances innovation performance in small and medium-sized enterprises (SMEs), Shuja Iqbal, et al. (2021).

Performance encompasses the demonstration of knowledge, skills and competencies acquired through learning experiences. It can be evaluated through various means, including. It is also the level of success or accomplishment a student attains in their academic pursuits, typically measured through grades, test scores and other assessments. While retention is the ability to keep and consequently remember things experienced or what is learnt by an individual at a later time (Bichi 2002). Retention can also be described as a form of reaction to what has been presented in the past. So many reasons have been advanced for students' difficulty and poor retention ability in mathematics and Geometry in particular. However, the common reason cited by scholars is the use of poor teaching strategy. For instance, Inekwe (2003) described the magnanimity of students' geometrical difficulties as a strong curriculum and pedagogical problem which needs the attention of mathematics educators. Also, Uyuota (2006) maintained that if concepts learnt must be retained for problem solving, then, methods by which the learner and the teacher will interact with materials should be given adequate consideration.

According to Microsoft Encarta (2004), "Computer is an electronic device that can receive a set of instructions, or programme and then carry out these programmes by performing calculations on numerical data or by manipulating other forms of information". Computer literacy on the other hand is the knowledge and relationship of computer combined with the ability to use it effectively. CAI or Computer Based Instruction (CBI), on the other hand, is a method of instruction in which the computer is used as a tool to facilitate, guide and evaluate for improved instruction an (Ebhomien, 2017). It refers to instruction or remediation presented on a computer.

Many educational computer programmes are available online and from computer software shops and they enhance teacher instruction in several ways. Recently, some excellent dynamic geometry programs that support students in developing mathematics concepts were developed such as the Geometry sketching software, (Furner & Marinas 2007), logo & Geometry, (Clements, Battista &Sarama 2001), Geometer's Sketchpad (Hannafin, Burruss& Little, 2001), 3D software application (Christou & James 2007), web-based learning in Geometry, (Chan, Tsai & Huang 2006) and many more in desk & lap tops. These programs allow users to construct 'classical' geometric objects such as points, segments, lines, circles etc., measure distances, angles, areas, and manipulate shapes on screen. The programs let users to change the objects which are displayed on screen dynamically by dragging and re-sizing them. Clement (2001) states that ''geometry software provides a very good environment for teaching geometry because

they can be used to perform geometric experiments''. Maragos (2004) also notes that dynamic geometry software has a profound effect on classroom teaching and can help students see what is meant by a general fact.

Since the problem of poor performance in mathematics has been observed to have originated from the methods by which knowledge of Mathematics/Geometry is imparted to the learners, and the situation in Kano is not an exception with regards to the teaching and learning mathematics as a school subject, there is a need to try and explore the effectiveness of other teaching methods which have been tried with success in other environments and other subject areas. Yusuf (2014) worked on CAI in Zamfara state, Nigeria, and established the efficacy of CAI in learning fractions among JSS students. Ebhomien (2017) found that CAI enhances learning of simultaneous equations in Edo state, Nigeria. Ayuba and Timayi (2018) also found that CAI was more effective in improving the learning of algebraic word problems in Kaduna state, Nigeria. None of these studies on efficacy of CAI was conducted in Kano state, particularly in geometry.

The integration of technology in education has transformed teaching and learning. Computer-Assisted Instruction (CAI) has emerged as a promising tool to enhance student outcomes in various subjects, including mathematics. The use of Computer Assisted Instruction (CAI) has been shown to improve student performance and retention in mathematics, including geometry (Akinsola & Fagbemi, 2017). In Nigeria, where economic challenges are prevalent, entrepreneurial innovation and self-reliance are essential for economic growth and development (Adeyemi & Adeyemi, 2014).

Despite the importance of geometry in mathematics and its applications, many senior secondary school students in Kano State, Nigeria struggle with the subject, leading to poor performance and low retention rates (WAEC, 2020). In Kano State, failure in Mathematics in the Qualifying Examinations (SS 2 Qualifying Exams) is very common. It was noticed that the performance in Mathematics at this level normally falls between 30-35% (KERC, 2024). Students' attempts in answering the Geometry sections of the yearly questions papers depict the need to undertake this study. This unfortunate development trail the student to W.A.E.C and N.E.C.O Examinations.

Therefore, the study fundamentally explored and evaluated the effects of CAI on academic Performance, retention and gender differences in Geometry among Senior Secondary Schools in Kano: Implications for Self-Reliance in times of Economic Depression

1.2 Objectives of the Study

The objectives of the study were to:

1. Investigate the effect of Computer-Assisted Instruction (CAI) on performance and retention in Geometry among Senior Secondary School students in Kano, fostering self-reliance in problem solving and critical thinking.
2. Determine the effects of CAI on students' academic performance in Geometry, compared to conventional methods, to promote entrepreneurial skills in adapting to new technologies.
3. Ascertain whether differences exist in students' academic performance in Geometry due to gender, ensuring equal opportunities for self-reliance and economic empowerment.
4. Recommend steps to improve students' performance in Geometry and mathematics, emphasizing entrepreneurial mindset and innovative problem-solving.

1.3 Research Questions

To guide the study, the following research questions were raised:

1. Is there a significant difference in academic performance among students exposed to CAI and those exposed to conventional methods, impacting their self-reliance in learning and problem-solving?
2. Does CAI enhance students' retention in Geometry, promoting long-term learning and entrepreneurial skills?
3. What are the effects of CAI on gender differences in Geometry performance, and how can it promote self-reliance and economic empowerment for all students?

1.4 Null Hypotheses:

HO1: There is no significant difference between the mean scores of post-test of the experimental group (CAI) and the control group (conventional method), affecting their self-reliance in problem-solving.

HO2: There is no significant difference in mean retention scores of the experimental group and the control group, impacting their ability to apply entrepreneurial skills in real-world scenarios.

HO3: There is no significant difference between the mean performance scores of male and female students after exposure to CAI, ensuring equal opportunities for self-reliance and economic empowerment.

2.1 Methodology

The research adopted Quasi-experimental design. The research used two groups; Experimental Group (EG) and Control Group (CG), both groups were given the same pre-test before the experiment and the same post-test after the instruction, this was to find out if there was any significant difference between the two groups with respect to the academic performance in the selected Geometry concepts.

The population for this research was all public Senior Secondary School Students in Kano State. There were 626 Public Senior Secondary Schools in Kano State, with 77,616 students' population (Kano State Senior Secondary Schools Management Board 2023). The schools were distributed within the 23 Education zones comprising of 44 local governments across the State.

One education zone was purposively selected out of the twenty three (23) zones. The selection was based on the fact that it has some common characteristic of urban and rural with students from different background.

The selected zone has a total of 30 public Senior Secondary Schools (girls and boys secondary school). Two schools out of these (one girls' school and one boys' school) were selected. Two intact classes in each school were selected through flip of coin random sampling techniques. One class in each school was randomly assigned to experimental group and the other to control group. Thus, a total of 144 students formed the sample of the research, where 74 (38 boys, 36 girls) stand for experimental group and 36 boys and 34 girls stand for the control group.

The instruments used for the study were Pre-Geometry Performance Test (PRE-GPT). PRE-GPT contained twenty (20) short answer type questions. Each question carried 5 marks, giving a total of 100 marks. This instrument served as a pre-test and administered before the experiment started so as to measure the group equivalence. After the pre-test, the experimental group was taught the selected Geometry topics for six weeks, (two periods per week) using CAI approach. The researchers used computer graphics, Encarta Premium, CD ROM and shapes as CAI software because computer graphics and shapes enable one to draw object of any shape, while Encarta Premium provides so many examples in each of the topic areas. On the other hand, the control group was taught the same Geometry topics for the same duration but using the

conventional method of teaching. Regular mathematics teachers of the selected schools, with comparable teaching experience and qualification, were trained and used as research assistants in teaching the two groups. After the exposure to the different methods of teaching, both groups took Geometry Performance Test (GPT). GPT contained 25 multiple choice questions. Each question carried 4 marks, giving a total of 100 marks. The reliability of GPT was found to be 0.87. Two weeks after taking the posttest (GPT), GRT was administered. The choice of two weeks interval to measure retention was based on the recommendation of Tuckman (1975) that a minimum of two weeks interval between posttest and retention test was acceptable. Items in GRT were developed by swapping the items of GPT. Mean, standard deviation and t-test were employed to analyze the collected data. The null hypotheses were tested at $\alpha = 0.05$ level of significance.

3.1 Result

The answer to the research question number 1 was sought by testing the hypothesis number 1.

Is there a significant difference in academic performance among students exposed to CAI and those exposed to conventional methods, impacting their self-reliance in learning and problem-solving?

H_{O1}: There is no significant difference between the mean performance scores of the experimental group and those of the control group as measured by post-test, affecting their self-reliance in problem-solving.

This hypothesis was about differences in the mean performance scores of Experimental and Control Group in post-test (GPT), therefore t-test statistics was used to test it

Table 1: Summary of t-test Analysis of Scores of Experimental and Control Group in Geometry Performance Test (GPT)

GROUP	N	Mean	SD	Mean Diff.	df	t-cal	t-crit	P	Decision
Experimental	74	54.00	13.02	4.44	142	2.18	1.99	0.00	significant
Control	70	49.56	11.35						

P = 0.00 < 0.05, t computed = 2.18 > 1.96 t-crit at df 142

According to the Independent t test statistics from table 1, significant difference exists between the mean performance scores of the experimental group and those of the control groups after the instruction. Reasons being that the p-value 0.00 is below the 0.05 alpha level of significance and

the computed t value of 2.18 is higher than the 1.96 t-critical at df 142. Their mean performance scores are 54.00 and 49.56 for the experimental group and control group respectively, with a mean performance difference of 4.44 in favour of experimental group. Therefore the null hypothesis which states that there is no significant difference between the mean performance scores of post-test of students exposed to Computer Assisted Instruction and those exposed to Conventional Method is hereby rejected.

The answer to the research question number 2 was sought by testing the Hypothesis number 2 as Follows

Does CAI enhance students' retention in Geometry, promoting long-term learning and entrepreneurial skills?

HO2: There is no significant difference in the mean retention scores of the experimental group and those of the control group as measured by post post-test (GRT), impacting their ability to apply entrepreneurial skills in real-world scenarios.

The hypothesis was about differences in the mean retention scores of Experimental and Control Group in post post-test (GRT), therefore t-test statistics was used to test it

Table 2: Summary of t-test Analysis for Experimental and Control Group in Geometry Retention Test

GROUP	N	Mean	SD	Mean Diff.	df	t-cal	t-crit	P	Decision
Experimental	74	51.11	13.61	4.55	142	2.12	1.99	0.00	significant
Control	70	46.56	12.16						

P = 0.00 < 0.05, t computed = 2.12 > 1.96 t-crit at df 142

From table 2, the t test statistics showed that significant difference exist between the mean retention scores of the students exposed to CAI and those exposed to conventional method. Reasons being that the p-calculated value of 0.00 is below the 0.05 alpha level of significance and the computed t value of 2.12 is higher than the 1.96 t-critical at df 142. Their mean retention scores are 51.11 and 45.56 for students exposed to CAI and those exposed to conventional method, respectively with a mean performance difference of 4.55 in favour of those students exposed to CAI. Therefore the null hypothesis which states that there is no significant difference between the mean retention scores of the students exposed to CAI and those exposed to conventional method is hereby rejected.

The answer to the research question number was sought by testing the hypothesis number 3, i.e.

What are the effects of CAI on gender differences in Geometry performance, and how can it promote self-reliance and economic empowerment for all students?

HO3: There is no significant difference between the mean performance scores of male and female students after exposure to CAI, ensuring equal opportunities for self-reliance and economic empowerment.

This hypothesis was about differences in the mean performance scores of male and female students after exposure to CAI in post-test (GPT), therefore t-test statistics was used to test it

Table 3: Summary of t-test Analysis of Scores of Males and Females in the Experimental Group in Geometry Performance Test

Variable	N	Mean	SD	Mean Diff.	df	t-cal	t-crit	P	Decision
Males	38	53.22	13.70						
				3.44	70	2.89	1.99	0.00	significant
Females	34	49.78	10.69						

P = 0.00 < 0.05, t computed = 2.89 > 1.96 t-crit at df 70

According to the Independent t test statistics from table 3, significant difference exist between the mean performance scores of the males and females students exposed to CAI. Reasons being that the p-value of 0.00 is below the 0.05 alpha level of significance and the computed t value of 2.89 is higher than the 1.96 t-critical at df 70. Their mean performance scores are 53.22 and 49.78 for males and females students exposed to CAI, respectively with a mean performance difference of 3.44 in favour of the male students in the CAI environment. Therefore the null hypothesis which states that there is no significant difference between the mean performance scores of male and female students after exposure to CAI is hereby rejected.

3.2 Summary of Findings

This study investigated the impact of Computer-Assisted Instruction (CAI) on students' Performance and retention in Geometry, with a focus on fostering entrepreneurial innovation for self-reliance in times of economic challenges. The key findings are:

1. Students in the experimental group taught using CAI demonstrated significantly higher performance in Geometry compared to those in the control group taught using conventional methods. This suggests that CAI can be a powerful tool for promoting entrepreneurial innovation and self-reliance in mathematics education.

2. The experimental group taught using CAI retained Geometry concepts better than the control group, indicating that CAI can help students develop a stronger foundation in mathematics and enhance their ability to apply mathematical concepts to real-world problems.
3. The study revealed a significant difference in performance between male and female students in the experimental group after exposure to CAI, with males performing better. This finding highlights the need for educators to ensure equal opportunities and support for female students in developing entrepreneurial skills and self-reliance in mathematics.

4. Discussion

This study investigated the effectiveness of Computer-Assisted Instruction (CAI) in enhancing entrepreneurial innovation and self-reliance among senior secondary school students in Geometry, amidst economic challenges. Three research questions and hypotheses guided this investigation, focusing on academic performance, retention and gender differences in the computer environment.

This study's findings, as presented in Table 1, reveal a significant difference in mean performance scores between students exposed to Computer-Assisted Instruction (CAI) and those taught using the Conventional Method (CM). The experimental group (CAI) outperformed the control group (CM) with a mean performance difference of 4.44 (54.00 vs 49.56), indicating a statistically significant difference ($p = 0.00$, $t = 2.18 > 1.96$, $df = 142$). This led to the rejection of the null hypothesis and the results align with previous studies (Ayuba & Timayi, 2018; Ebhomien, 2017; Yusuf, 2014), highlighting the effectiveness of CAI in enhancing students' self-reliance in problem-solving.

Furthermore, Table 2 shows a significant difference in mean retention scores between students exposed to CAI and those exposed to CM. Students in the CAI group retained more information, with a mean retention score of 51.11, compared to 45.56 for the CM group, resulting in a mean performance difference of 4.55. The statistical analysis confirmed this difference ($p = 0.00$, $t = 2.12 > 1.96$, $df = 142$), leading to the rejection of the null hypothesis and the findings support previous research (Bunkure, 2007; Ayuba & Timayi, 2018), underscoring the potential of CAI to enhance students' ability to apply entrepreneurial skills in real-world scenarios.

Additionally, Table 3 presents a significant difference in mean performance scores between male and female students exposed to CAI, with male students outperforming their female counterparts

by a mean difference of 3.44 (53.22 vs 49.78). The statistical analysis verified this difference ($p = 0.00$, $t = 2.89 > 1.96$, $df = 70$), resulting in the rejection of the null hypothesis, and the results align with some previous studies (Clark & Gorski, 2004; Bashman, 2007), but contradict others (Yusuf, 2014), highlighting the complexity of gender differences in CAI and the need to ensure equal opportunities for self-reliance and economic empowerment.

These findings have implications for educators and policymakers seeking to integrate technology-enhanced instructional approaches into their teaching practices. The results suggest that CAI can be an effective tool for improving students' learning outcomes, but also emphasize the importance of addressing potential differences in student performance and experiences. By embracing CAI, educators can foster entrepreneurial innovation and self-reliance among students, empowering them to navigate economic challenges and thrive in an ever-changing world.

5. Conclusion:

The study's findings provided insights into the effectiveness of CAI in improving student performance and retention in geometry. The results will inform the development of mathematics education interventions and policies that promote entrepreneurial innovation and self-reliance. The study concludes that CAI is an effective tool for enhancing students' performance and retention in Geometry, can promote self-reliance and entrepreneurial mindset. The findings support the integration of CAI in mathematics education, empowering students to become innovative problem-solvers and entrepreneurs.

6. Recommendations:

1. Stakeholders in Mathematics Education should integrate CAI in mathematics teaching, text books and trainings to promote self-reliance and entrepreneurial skills.
2. Provide training and resources for teachers to effectively use CAI in teaching Geometry and mathematics.
3. Teachers of Mathematics should encourage students to develop an entrepreneurial mindset by applying mathematical concepts to real-world problems.
4. Foster equal opportunities for self-reliance and economic empowerment among male and female students.

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IMPACT OF LANGUAGE OF IMMEDIATE ENVIRONMENT (LIE) ON INTEREST IN MATHEMATICS AMONG LOWER BASIC PUPILS IN NORTH-WEST ZONE, NIGERIA.

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Abstract

This study investigated the Impact of Language of Immediate Environment (LIE) on Interest in Mathematics among Lower Basic Pupils in North- West Zone, Nigeria. The study was conducted based two (2) specific objectives. Two (2) research questions were constructed and two (2) null hypotheses were formulated to guide the study. A pre-test, post-test and post-posttest quasi experimental design was adopted. From the target Population of seven million six hundred and seventy-four thousand and ninety- eight (7,674,098) Pupils from the North Western states of Nigeria. A multi stage purposive simple random sampling technique was used to select the states, local governments and the schools of the pupils. Three hundred and eighty-four (384) pupils were selected for the study in accordance with Krejci's and Morgan's method of estimating a sample size from a large population. The Interest in Learning Mathematics Questionnaire (ILMQ) was used to collect data on the interest level of the lower basic pupils in both the experimental and the control group. The instruments were validated and the reliability coefficients of ILMQ was established at r=0.87. The data were analyzed using mean, standard deviation, and Mann-Whitney U test Analysis at 0.05 level of significance using statistical package for social sciences (SPSS). Result of the study indicated that lower basic primary school pupils taught using Language of Immediate Environment (LIE) developed high interest in learning mathematics better than those taught using English language. The result also revealed no significant difference between the interest level of male and female primary lower basic pupils in the North West zone, Nigeria taught mathematical concepts using Language of Immediate Environment. The study established that using Language of Immediate Environment (LIE) to teach Mathematics concepts was highly effective in fostering the acquisition of high positive interest levels among lower basic primary school pupils. The study revealed among others that Language of Immediate Environment was gender friendly. The study recommended among others that mathematics teachers should endeavor to use Language of Immediate Environment in teaching lower basic pupils in the North West zone, Nigeria for higher interest, better performance and enhanced retention ability of mathematical concepts in our primary schools. The results of the findings of the study suggests for further studies in different location of the country and in different subjects or by using other dependent variable such as self-efficacy. Mathematics students should use their language of immediate environment to help them understand mathematical concepts by practicing mathematics in their mother tongue and ask for language support from teachers, if they struggle with the language of instruction. The findings of the study would also benefit the mathematics educators, curriculum planners, government and non-governmental organizations among others.

Introduction

Competency in Mathematics learning is vital to any individual and nation in domestic and business deals, scientific discoveries, technological breakthrough, problem-solving and decision making in different situations in life. Akinoso (2011) viewed Mathematics as the basis for science and technology and the tool for achieving scientific and technological development. Mathematics is a science of magnitude and number that is very useful virtually in all subject areas. Despite its great importance, it is the only subject that is most dreaded to learners among all subjects offered in schools (Akinoso, 2018). Students therefore tend to respond to it with less self-confidence, negative feeling and anxiety. This situation is worsened by the compulsory nature of the subject at primary and post primary schools' levels, leading to students' poor performance in the subject. Language of Immediate Environment (LIE) can be defined as the language that most of the people in the community understand better and can communicate with it effectively (Abiam and Odok 2006). The use of Language of Immediate Environment enables the young learners to immediately construct and explain without fear of making mistakes, articulate their thoughts and add new concepts to that which they already knew. In turn, the teachers can more accurately assess what has been learned and identify the areas where they need help (Nolasco, 2010). Language of Immediate Environment-based education has a positive impact on educational and learning outcomes. The child's home language can effectively be used as a language of instruction in the early years of schooling as a bridge to learning. Appropriate language in education enables the teachers to instruct on the language a child speaks most at home and understands well enough to learn academic content through Language of Immediate Environment. A child's Language of Immediate Environment should be used as a medium of instruction in learning in Primary 1-6, because the fundamental Mathematics and Science concepts are introduced at this level (Nalasco, 2010). Kimizi (2012) added that children taught in Language of Immediate Environment have better learning comprehension than those taught in a

foreign language. The Nigerian National Policy on Education (2013) recommended that the language of the immediate environment should be used as medium of instruction in primary schools for the first four years. However, the document also states that English language shall be taught as subject during this period. The challenges of every educator for the new curriculum being implemented by the Department of Education contribute benefits to the learning of the pupils. It provides long-term benefits like higher self-esteem, greater confidence and high aspirations for schooling and life (UNESCO, 2013).

The use of Language of Immediate Environment (LIE) as a medium of instruction is globally recommended for pupils in lower primary schools (UNESCO, 2013). Global campaign for education (2013) also reveals that, learners taught in foreign language perform poorly than those taught in Language of Immediate Environment.

The study therefore, determined the Impact of Language of Immediate Environment on Interest in Mathematics among Lower Basic Pupils in North- West Zone, Nigeria.

Social Constructivism

This study was guided by several learning theories in one way or the other have influenced educationist to fiercely advocate for the use of Social Constructism theory and Interest-Driven Creator (IDC) theory in learning mathematics.

Social constructivism is a learning theory propounded by Levy Vygotsky (1968). The theory states that language and culture are the frameworks through which humans experience, communicate, and understand reality. According to Vygotsky, language and culture play essential roles both in human intellectual development and in how humans perceive the world. This is to say that learning concepts are transmitted by means of language, interpreted and understood by experience and interactions within a cultural setting. Since it takes a group of people to have language and culture to construct cognitive structures, knowledge therefore is not only socially constructed but co-constructed. The link here is that while the constructivist sees knowledge as what students construct by themselves based on the experience they gather from their environment, the social constructivist sees knowledge as what students do in collaboration with other students, teachers and peers. Social constructivism is a variety of cognitive constructivism that emphasizes the collaborative nature of learning under the guidance of a facilitator or in collaboration with other students.

In social constructivism children's understanding is shaped not only through adaptive encounters with the physical world but through interactions between people in relation to the world that is not merely physical and apprehended by the senses, but cultural, meaningful and significant, and made so primarily by language of the immediate environment. Hein (1991) puts it in his own way that the level of potential development (academic achievement) is the level of development that the learner is capable of reaching under the guidance of teachers or in collaboration with peers. He sees learning as a social activity associated with other human beings like the peers, family members as well as casual acquaintances, including the people that existed before. Social Constructivism recognizes the social aspect of learning and the use of conversation, interaction with others, and the application of knowledge as an essential aspect of learning and a means to achieving learning objectives.

Vygotsky believed that life long process of development is dependent on social interaction and that social learning actually leads to cognitive development. In other words, all learning tasks (irrespective of the level of difficulty), can be performed by learners under adult guidance or with peer collaboration. This theory helps to give a backup to the establishment of opportunities for students to collaborate with the teacher and peers in constructing knowledge and understanding. Kapur (2018), observed that social construction of knowledge takes place in various ways and at different locations.

Interest-Driven Creator (IDC) Theory

Many studies have put effort in intervening an approach to promote and develop students' interest in mathematics learning. The IDC theory suggests that students can be nurtured as creators after they have engaged in interest-driven learning activities regularly with technology support (*Chan, et al.,* 2018). There are three anchored concepts in the IDC theory, namely, interest, creation, and habit, whereas each of them will go through to a continuum learning activity that subsequently forms a loop (*Chan, et al.,* 2018).

Firstly, the interest loop comprises three coherent components which are triggering interest, immersing interest, and extending interest (Wong, Chan, Chen, King, & Wong, 2015). In the IDC theory, stimulating curiosity is one of the processes in triggering situational interest; however, presenting attractive learning subject matters is inadequate for this purpose. To provoke situational interest, teachers or instructional designers should scaffold knowledge

deficit, which subsequently can help students be immersed or fully engaged in the learning process by providing optimum levels of challenging learning tasks. As for the final component in the interest loop, it explains that utility implications of the learning content are crucial for sustaining students' interest (Wong, *et al.*, 2015). In relation to mathematics learning, students' interest can be promoted by firstly presenting a mathematical problem that is able to provoke and confront students' prior knowledge, and scaffold students to tackle challenges that help students gain successful experiences and finally present the practical value of the learning content.

Next in the creation loop, there are three components which are imitating, combining, and staging (*Chan, et al.,* 2018). According to Chan, Looi, and Chang (2015), the early stage of learning process includes imitation from a model, as an attempt to understand the model's ideas, methods, or ways of doing things. Then, students will choose what to retain or remove, and come up with their own interpretation or ideas as a result of 'combination'. To complete the creation process, students should be provided with a platform to present their product (*Chan, et al.,* 2015). In the context of mathematics learning, the creation loop can be operationalised as simply as a mathematics problem-solving situation in a classroom. Mathematics teachers usually demonstrate a step-by-step method to approach a mathematics question, and in return, students are asked to imitate the process for a similar question. When students are able to solve the question on their own, it can be known as a completed 'combination' process as they have acquired the skills and knowledge to answer a mathematics question. Staging can be as straightforward as showing their working solution on the board in the classroom to their peers.

Finally, the habit loop consists of three components: cueing environment, routine, and satisfaction (Chen*, et al.,* 2015). In the view of the IDC theory, it is important and possible to develop a positive learning habit to nurture a lifelong interest-driven creator (Chen*, et al.,* 2015). Although the habit formation process takes time and is highly related to students' affective characteristics and cognitive behaviours, teachers should begin with easy and simple habits; therefore, it comes down to the question of what is the learning habit teachers would like students to form (Chen*, et al.,* 2015). Regarding learning mathematics, there are a few fundamental and important learning habits like memorisation of the multiplication table, the accurate and systematic way of writing mathematics equation, or an analytic approach in solving mathematics questions. In the process of developing students' interest towards mathematics, it is inevitable that students are required to be innately involved in solving a mathematics problem

regularly, and that is when habits comes into play, to help them be comfortable, familiar, and most importantly, answer a mathematics question correctly as guided by the teacher. In the long term, as conceptualised in the IDC theory, students' satisfaction towards learning can be increased with increasing successful experiences (*Chan, et al.,* 2018; Chen*, et al.,* 2015).

Interest and Academic Achievement in Mathematics

Many factors have been identified in literature as reasons associated with students' lack of interest in learning Mathematics. These include Students' factor, teachers' factor, Mathematics anxiety, class size, government factor, infrastructural problem, and instructional strategy, among others (Okonkwo, 2007; Akinoso, 2011; and Goolsby, 2013).

Students' factor include lack of interest to learn Mathematics caused partly by Mathematics phobia and distractions from handsets they carry about even in the Mathematics classrooms as perceived by the researcher while teaching them Mathematics. These have lead probably to students' lack of reading culture especially Mathematics textbooks.

Onudiaku (2013) attributed factors of academic achievement among secondary school students in Mathematics to lack of interest. More so, the failure of students in Mathematics achievement was also supported by some authors to be associated with lack of interest in studying the subject (Idigo; 2010 and Goolsby, 2013). Specifically, Goolsby (2013) attributed factors influencing students' Mathematics interest to attitude towards success in Mathematics, confidence in learning Mathematics, perception of teacher attitude, Mathematics anxiety, and Locus of control. According to Idigo (2010), factors associated with Mathematics interest include, students' factor, teachers factor, Mathematics anxiety, government, lack of infrastructural facilities, lack of instructional materials and problem of large class size.

Lessons Learnt from the International Scene on Using Language of Immediate Environment as Medium of Instruction

Ball (2011), asserts that many children speak a home language that differs from the language of instruction in education programmes. Consequently, when educationists discuss a child's home language, it may be the learner's Language of Immediate Environment or it may be the language the learner has learnt first and knows best. It can also be the language one uses often or most. In

this literature review and the research discussions the term familiar language will mean any of these.

Hazik and Farik (2016), Beka (2016), Kazima (2014) conducted the study on using the Language of Immediate Environment or familiar language as a medium of instruction for the early primary education. Their study from two schools of thought can be discerned: Hazik and Farik (2016), Beka (2016), asserts that teaching Mathematics and science in a home language (language of play) enhances learner performance and reduces school drop outs while Kazima and Mashegoane (2016) condemns the pedagogy because it assumes that most local languages have limited mathematical and science vocabulary.

Consequently, this paper investigates the impact of Language of Immediate Environment (LIE) on interest in learning mathematics among lower basic pupils in North-West zone, Nigeria.

Results of Related Studies on the use of Language of Immediate Environment (LIE)

David (2008), examined the influence of Language of Immediate Environment on Students performance in English language in Junior School Certificate Examination. The study investigated if Language of Immediate Environment is solely the cause of the students' woeful performance in English Language in Junior School certificate Examination or if there are other complementing factors. The subjects for the study were one hundred male and female SSI students drawn from various government schools in western Nigeria. Using simple percentage descriptive statistic, the research questions that were raised analyzed. The findings reveal that Language of Immediate Environment influences the students' poor performance in English language in Junior School Certificate examination and that there are other factors contributing to students' poor performance in English language. These other factors are poor method of teaching, lack of textbooks, language background and lack of professional growth and development of teachers. Measures that could be taken to enhance students' achievement in every aspect of English language are suggested also. The respondents' age ranged between 10.52 and 15.17. The participants were those who have sat for the Junior Secondary School Certificate examination. They were randomly selected across diverse religious and socio-economic background.

In a study with relevance to the Kenyan context, De Sousa *et al.* (2010) examined language skill transfer among 60 South African second graders, including 30 native Zulu-speaking children

who were learning English. The latter group were not literate in Zulu. However, several aspects of phonological awareness, including rime detection, letter sound awareness, and syllable segmentation was correlated with English spelling, using both real words and non-words. In another study of 60 third graders in South Africa, De Sousa *et al.* (2011) found a relationship between spelling in English and Afrikaans, using a test of words and non-words. Other work from outside the developed world has drawn attention to the additional complexity of language transfer issues in multilingual environments. The numeracy skills of the participants were neglected by the researchers.English instrument had a Cronbach's Alpha of 0.832. Both instruments were highly reliable. The descriptive statistics, t-test for paired samples, and Analysis of Covariance (ANCOVA) were used to analyze the data.

Mohammed, *et.al,* (2010), conducted a study on Mathematical Game in Hausa Culture and explained that, before the coming of Western education, Hausa people in northern Nigeria used Mathematics in sorting, ordering, measuring, timing and weighing in their day to day activities. Featuring prominently in this culture are the traditional games played by children or adults with the aim of deriving lessons. These games are meant to shape individual brain. The primary concern of this paper is on the games that involves calculations and to highlight therein the existence of algebra, set theory, coordinate geometry, arithmetic progression and geometry progression in Hausa culture. Furthermore the paper will correct the erroneous believe that there was no Mathematics in Hausa land/northern Nigeria before the coming of the Western education and address those who believe Mathematics only starts and ends in the classroom, thereby divorcing the rich cultural elements from the use of Mathematics.

Ya'u (2018), conducted the study on the impacts of using ethnomathematics in teaching on performance and retention in Mathematics among Lower Basic Primary School pupils in Lere education district, Kaduna State, Nigeria. A pre-test, post-test and post-posttest quasi experimental design was adopted. From a target population of two hundreds and ninety five 295 public primary Schools with 11,289 pupils (5,862 males and 5,427 females) simple random sampling techniques were used to select the schools and the class of the pupils respectively. 207 pupils were selected for the study in accordance with Central Limit Theorem that regarded a minimum of 30 sample size appropriate for experimental research. Mathematics Performance Test 1 (MPT1), Mathematics Performance Test 2 (MPT2) and Mathematics Retention Test (MRT) were the instruments used in collecting data as Pretest, Posttest and Post-posttest

respectively. A Spearman-Brown coefficient of r = 0.89 was found. The experimental group was taught using the Ethnomathematics (Hausa Language). The data were analyzed using mean, standard deviation and t-test Analysis at 0.05 level of significance using statistical package for social sciences (SPSS),version 2016.Results of the study indicated that Lower Basic Primary School pupils taught using ethnomathematics teaching performed and retained mathematical concepts better than those taught with conventional teaching method. The result also revealed no significant difference between the performance and retention of male and female Lower Basic Primary School pupils taught mathematical concepts using ethnomathematics teaching strategy. The study revealed among others that ethnomathematics teaching strategy was gender friendly while in this study the researcher would specifically concentrate on the impact of the Language of Immediate Environment rather than other cultural practices.

Statement of the Problem

Despite National Policy on Education's recommendation that pupils in the lower primary schools in Nigeria be taught in their Language of Immediate Environment, primary school teachers in Nigeria usually teach Mathematics using English language. Across the developing world, countries face challenges in improving learning outcomes. Researchers conducting theory-based studies in the United States and other developed countries have built a body of evidence demonstrating that children learn to read most easily in their Language of Immediate Environment, and that skills they acquire while learning to read in that language transfer to learning to read in other languages, at least to some extent (Melby-Lervåg & Lervåg, 2011). One of the main reasons that stakeholders resist the implementation of Language of Immediate Environment based-instruction is their fear that it will somehow harm students' acquisition of English and other subjects seen as more valuable and prestigious for students (Jones, 2012). Even when a government's official policy mandates initial literacy instruction in a Language of Immediate Environment, as it does in Kenya, such mandates are often disregarded at the local level (Piper, *et. al*, 2014).Several of the targets for the new Sustainable Development Goals focus on education quality, or more specifically, literacy and numeracy (United Nations, 2015).

The issue of language of instruction is central to the debate around improving instructional quality and student achievement in multilingual countries. A small but growing body of research from the Global South suggests that Language of Immediate Environment based literacy

outcomes can be improved at medium scale (Piper, *et al.*, 2016; Taylor & von Fintel, 2016). However, in sub-Saharan Africa, there are disconnects among theory-based recommendations, policy, and practice (Clegg & Simpson, 2016). These practices might lead to children in the lower primary schools to find it difficult to comprehend Mathematics lessons and thus, have lack of interest, perform poorly and with low retention in Mathematics. Unless appropriate measures are taken, this problem will be a barrier to the effort of the United Nations declaration to provide inclusive, quality education and lifelong educational opportunities for all by 2030, as contained in the Sustainable Development goal four (4).

According to (Kim, 2017), until the day that Africans will start practicing their own culture they will develop. Globalization has never helped any country or continent, so if any Africa must succeed, they should never take global standard as a way of life. Practice what you know best in accordance to your culture. Against this background, the study investigated the Impact of Language of Immediate Environment on Interest in Mathematics among Lower Basic Pupils in North- West Zone, Nigeria.

Objectives of the Study

The main objective of this study was to examine the Impact of Language of Immediate Environment on Interest in Mathematics among Lower Basic Pupils in North- West Zone, Nigeria. Specifically, the study seeks to:

1. find out impacts of Language of Immediate Environment on interest in Mathematics among lower basic pupils.
2. investigate gender influences on the impact of Language of Immediate Environment on interest in Mathematics among lower basic pupils.

Research Questions

The following research questions were asked to guide the study.

1. Is there any impact of Language of Immediate Environment on interest in Mathematics among Lower basic pupils?
2. What is the gender influence on the impact of Language of Immediate Environment on interest in Mathematics among Lower basic pupils?

Null Hypotheses

The following null hypotheses were formulated based on the research questions and tested at $P \leq 0.05$ level of significance.

$H_{01:}$ There is no significant difference between the interest of pupils taught Mathematics using Language of Immediate Environment and those taught using English language.

$H_{02:}$ There is no significant difference between the interest of male and female pupils taught Mathematics using Language of Immediate Environment.

Methodology

This research employed pre-test, post-test and delayed (post-posttest) quasi-experimental control group design. This design was used because, it is not possible to have complete randomization of the subjects as this would disrupt school organization. Hence, intact classes were randomly assigned to the experimental and control groups. The researcher taught the experimental group using Language of Immediate Environment and the Control group pupils were taught same mathematical concepts but using English Language as the medium of instructions. The impacts of these approaches were carefully determined and recorded. Prior to administration of the treatment, pre-test was administered to the two groups to determine the comparability ability levels of the groups. The Population of the study was made up of sixteen thousand seven hundred and sixty-three (16,763) public primary schools which comprised of four million four hundred and ninety four thousand and five (4,494,005) males and four million three hundred and fourty six thousand nine hundred and sixty five ((4,346,965) females with the total population of eight million eight hundred and fourty thousand nine hundred and seventy (8,840,970) Lower basic primary school pupils from the North western states. The schools used for the study were co-education.

Table 1: Enrollment of Lower Basic Primary Schools Pupils' (1-6) in North-West Zone, Nigeria.

S/N	STATE	NO. OF SCHOOLS	MALE	FEMALE	TOTAL
1.	Jigawa	2,345	421,645	399,830	821,475
2.	Kaduna	4,177	595,787	619,595	1,215,382
3.	Kano	2,061	1,574,321	1,739,116	3,313,437
4.	Katsina	2,772	925,620	935,991	1,861,611

5.	Kebbi	1,788	320,828	197,772	518,600
6.	Sokoto	2,047	268,998	197,445	466,443
7.	Zamfara	1,573	386,806	257,216	644,022
	TOTAL	**16,763**	**4,494,005**	**4,346,965**	**8,840,970**

Source: Federal Ministry of Education, 2021.

In this study, a multi stage simple random sampling technique was used and two states (Kano and Kaduna) were selected from the North Western states in Nigeria. Furthermore, a simple random sampling technique was used to select one local government area each from the selected states (Lere LGA and Doguwa LGA from Kaduna and Kano states respectively were selected as experimental and control groups respectively. The sampling was carried out by using simple random sampling using hat and draw method and selected two schools from each selected Local government area. Furthermore, in selecting the pupils' samples, intact classes were used for both the experimental and control groups. A sample size of 384 pupils from both experimental and control groups were used for the study. This was supported by Krejcie and Morgan (1970) who stated that with a confidence level of 95% (given an alpha level α= 0.05) a viable sample size (S) for a large population N can be calculated by using the formula as follows:

$$S = \frac{X^2 NP(1-P)}{d^2(N-1) + X^2 P(1-P)}$$

Where S ***= required sample size***

N ***= the Population Size***

X^2 ***= the table value of chi- square for one degree of freedom at the desired confidence level***

P ***= the population proportion (assumed to be 0.50)***

d ***= the degree of accuracy expressed as a proportion (0.05)***

Table 2: Sample Selected for the Study

State	Schools	Group	No. of Males	No. of Females	Total
Kaduna	A	Experimental	110	91	201
Kano	B	Control	105	78	183
Total			**215**	**169**	**384**

Interest in Learning Mathematics Questionnaire (ILMQ) was a 15 items questionnaire adapted by the researcher from Umar (2018)" Interest towards learning mathematics''. The questionnaire has response as: Strongly Agree (SA); Agree (A); Undecided (U); Strongly Disagree (SD); and Disagree (D). Each statement carries weight in descending order from 5-1 points and vice-versa. Each questionnaire has a maximum points of (5x15) = 75 points while the minimum (1x15) = 15. This questionnaire was used to find out the Impact of Language of Immidiate Environment (LIE) on interest level among Lower Basic Primary School Pupils.

The instrument was validated by the experts in the area of Mathematics, Mathematics Education and Hausa Language. The experts were given the instruments and checked their relevance, clarity of the language and any contradictions.

The reliability of the instruments was determined after the pilot testing. The reliability was established Using split-half method with the aid of Statistical Package for Social Sciences (SPSS) and determined the reliability coefficient of the instrument. Based on the data obtained from the pilot testing of the instrument, the reliability coefficient of Interest in Learning Mathematics Questionnaire (ILMQ) was ascertained to be r = 0.89 indicating that the research instrument was reliable. All data collected via the instrument for this study were analyzed by the use of descriptive or inferential statistic with the aid of statistical Packages for Social Sciences (SPSS). Data from the research questions were described using mean, standard deviation and mean differences. The null hypotheses were tested at 0.05 and analyzed by the use of independent Mann-whitney U-test statistic.

Results

Question One:

What is the impact of Language of Immediate Environment on interest in Mathematics among Lower basic pupils?

To answer this research question, the mean rank and sum of ranks were computed for both the experimental and control groups using Mann-whitney U-Test statistic. Summary of the findings was presented in Table 3.

Table 3: Summary of Mean Rank and Sum of Ranks Statistic of Interest Level for the Pupils in the Experimental and Control Groups.

GROUP	N	Mean Rank (MR)	Sumof Ranks(	Mean Rank *D*iff.

			SR)	
Experimental	201	271.80	55030.5	
				96.06
Control	183	98.78	19289.00	

From Table 3, Results of Non Parametric test showed that, the difference exist in interest level between the experimental group (n=201, MR=271.80, SR=55030.50) taught using LIE and the control group (n=183, MR=98.78, SR=19289.00) taught using English Language indicating a Mean Rank difference of 96.06 in favor of the pupils in the experimental group. This implies that the use of Language of Immediate Environment has high positive impact on pupils' interest towards learning mathematics.

However, further analysis was conducted to ascertain whether the difference in mean rank was statistically significance and this led to the analysis of hypothesis one as presented in Table 4.

Null Hypothesis One:

There is no significant difference between the interest level of pupils taught Mathematics using Language of Immediate Environment (LIE) and those taught using English language. This hypothesis was tested by the use Mann-Whitney U-test statistic and the summary was presented in Table 4.

Table 4.: Summary of Mann-Whitney U-Test Analysis on Pupils' Interest Levels for Experimental and Control Groups.

GROUP	N	Mean Rank	Sum of Ranks	P
Experimental	201	271.80	55030.5	
				0. 000
Control	183	98.78	19289.00	

P=0.000<0.05 alpha level of significance

From Table 4., Results of Mann-Whitney Non Parametric test showed that difference in interest level between the experimental group (n=201, MR=271.80, SR=55030.50) and the control group (n=183,MR=98.78 ,SR=19289.00) were significant. From Table 4, the P-value observed (0.000)

was less than $P \leq 0.05$ at df = 382. Thus, the null hypothesis which stated that; there is no significant difference between the interest level of pupils taught mathematics using Language of immediate environment and those taught using English Language was rejected. This means that there is significant difference between the interest level of Lower Basic Primary School pupils in the experimental group and control group in favored of the experimental group.

Question Two:

What is the gender influence on the impact of Language of Immediate Environment on interest in Mathematics among Lower basic pupils?

To answer this research question, the mean rank and sum of ranks were computed for both male and female pupils in the experimental and control groups using Mann-Whitney U-Test statistic. Summary of the findings was shown in Table 5.

Table 5: Summary of Mean Rank and Sum of Ranks Statistic on Interest Levels for the Male and Female in the Experimental Group.

GENDER	N	Mean Rank	Sum of Ranks	Mean rank. diff.
Male	110	92.22	9683.00	
				0.52
Female	291	91.70	7152.90	

From Table 5, Results of Mann-Whitney Non Parametric test showed that the difference in interest level between the male (n=110, MR=92.22, SR=9683.00) and the female (n=91, MR=91.70, SR=9400.00) were not significant. The mean rank difference between male and female pupils taught using Language of immediate Environment was 0.52. This implies that the use of LIE is gender friendly on the interest levels of pupils towards mathematics learning.

However, further analysis was conducted to ascertain whether the difference in mean rank was statistically significance and this led to the analysis of hypothesis two as presented in Table 6.

Null Hypothesis Two:

There is no significant difference between the interest of male and female pupils taught Mathematics using Language of Immediate Environment. This hypothesis was tested by the use

Mann-Whitney U-test statistic and the Summary of the Mann-Whitney U-Test was presented in Table 6.

Table 6: Summary of Mann-Whitney U-Test Analysis on Pupils' Interest Levels for Male and Female in the Experimental Group.

GENDER	N	Mean Rank	Sum of Ranks	p-value
Male	110	92.22	9683.00	0.661
Femal	91	91.70	7152.90	

From Table 6, Results of Non -Parametric test showed that difference in interest level between (U=3967, Z= -0.551 and P=0.193), the P-value observed (0.193) was greater than $P \leq 0.05$ at df = 199. This means that there is no significant difference between the interest levels of male and female Lower Basic Primary School pupils in the experimental group. Thus, the null hypothesis which stated that; there is no significant difference between the interest level of male and female pupils taught mathematics using Language of Immediate Environment (LIE) was accepted.

Discussion of results

From the results in Table 3 and 4, showed that there is significant difference between the interest level of pupils in the experimental group and the control group. The difference observed showed that experimental group had higher interest level in learning mathematical concepts than the control group. The result was in agreement with the findings of Kurumeh (2005) and Omenka (2010), which showed that pupils taught with the Language of Immediate Environment, could have higher interest level than those taught with foreign language.

Tables 5 and 6, showed the difference in interest level between male and female in the experimental group. The difference was not statistically significant. This was due to the treatment received using Language of Immediate Environment in teaching. The finding was in line with that of the other researchers such as Buhmann and Trudell (2008), who confirmed the effectiveness of a home language as a medium of instruction in numeracy and literacy. Similarly,

Beka (2016) who assert that it is widely accepted that language plays an important role in thinking and learning. This is central to learning and teaching school Mathematics.

Conclusion

1. The study established that using Language of Immediate Environment (LIE) to teach Mathematics concepts was highly effective in fostering the acquisition of high positive interest levels among lower basic primary school pupils.
2. The study established that using Language of Immediate Environment (LIE) to teach Mathematics concepts was gender friendly.
3.

Recommendations

Based on the findings of this study, the following recommendations were made:

1. Mathematics Teachers should use language support startegies such as bilingual instruction to help pupils who speak different languages and develop teaching materials that reflect the language and culture of the pupils. Similarly, they should provide support to the pupils who struggle with the language of instruction. Mathematics Teachers and school guidance counselors should work together inorder to guide the pupils to use Language of Immediate Environment (LIE).
2. Journal Publishers should publish more research on the impact of language of immediate environment (LIE) on mathematics education in Nigeria. They should encourage researchers to explore the impact of language on pupils' interest, performance and retention in mathematics and also, provide a platform for researchers to share their findings on language and mathematics education.
3. Researchers should conduct more research on language and mathematics education in Nigeria and explore the impact of language of immediate environment on pupils' interest, performance and retention in mathematics. Researchers should develop language support strategies that can be used by teacher and pupils.

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MATHEMATICAL ASSOCIATION OF NIGERIA (M.A.N)

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The Communiqué of the 61st Annual National Conference of the Mathematical Association of Nigeria (MAN) held at Usmanu Danfodiyo University, Sokoto from 31st August to 5th September, 2025

INTRODUCTION

The Mathematical Association of Nigeria (MAN) is a non-profit and non-governmental academic organization. As part of its activities, it organizes an annual national conference where, in addition to discussions on ideas in Mathematics, students, young researchers, and educators are trained through workshops, quizzes, and Olympiads. It is a gathering of Mathematical Scientists and Educators from across the country.

This year's conference (2025) was held at Usmanu Danfodiyo University, Sokoto, with the theme: **"Rethinking Teaching and Learning of Mathematics as Tools for Creativity, Technological Innovation, Self-Reliance, and Sustainable Development in Times of Economic Depression."**

The opening ceremony, chaired by Professor Bashiru Garba, MFR, Vice-Chancellor of Usmanu Danfodiyo University, Sokoto, was formally declared open. In his welcome address, Professor Muhammad Awwal Yusha'u, FMAN, FAAN, President of MAN, emphasized the importance of Mathematics in solving real-life problems.

The keynote address was delivered by Professor Isah Garba Abor, a renowned Mathematician, on the importance of Mathematics in addressing real-life situations. Professor Garba Shaaibu also led participants in discussions on the conference theme.

From the keynote, lead papers, and presentations by participants, the following observations and recommendations were made.

OBSERVATIONS

1. The conference theme is timely. Nigeria is at a critical juncture where Mathematics must drive national transformation amid economic hardship, youth unemployment, and rapid technological change.
2. Various speakers described Mathematics as an optimizer, the silent engine of progress that powers technology, frames innovation, and guides the nation through uncertainty toward a sustainable future.
3. The conference stressed that the collective responsibility of educators, researchers, policymakers, and stakeholders in Mathematics is to promote creativity, technological skills, and self-reliance to drive sustainable development and growth in times of economic depression.
4. Both physical and virtual presentations of papers were delivered by participants and delegates nationwide. Notably, for the first time, the Association launched a textbook for tertiary education.
5. The Association also conferred fellowship and honorary awards on past presidents, and presented prizes to the top three (3) pupils/students in its Olympiad and Quiz competitions.
6. It was observed that there is a pressing need to engage members of the Association in training on effective presentation techniques.
7. The Association expressed appreciation to the management and staff of Usmanu Danfodiyo University, under the leadership of Professor Bashiru Garba, Vice-Chancellor, for their hospitality and support in successfully hosting the 61st MAN Conference.
8. Special appreciation was extended to the keynote and lead paper presenters, Professors Garba Abor Isa and Garba Shaaibu, respectively, for sharing their wealth of experience with participants.
9. The Association also acknowledged Pastor Enoch Adeboye, General Overseer of the Redeemed Christian Church of God and a Fellow of the Association, for his donation of Two Million Naira (₦2,000,000.00) towards cash prizes for the national Olympiad and Quiz competitions. This donation greatly encouraged and inspired the student winners.
10. The conference concluded with the 2025 Annual General Meeting (AGM), during which there was a change of leadership. Professor Emmanuel O. Oghre, FMAN, University of Benin, Edo State, assumed office as President of MAN for the next two (2) years. Elections were also conducted, producing the following officers:

    1. Dr. Muhammad Sadi Surajo (FMAN) – Vice President
    2. Magaji Yunana (LMAN) – Publicity Secretary
    3. Mrs. Igbokoyi Adeola Olukesoyi (LMAN) – National Treasurer
    4. Mr. Umar Musa Ohimege – Business Manager

RECOMMENDATIONS

1. Mathematics educators should integrate technology into teaching and learning to promote critical thinking, creativity, self-reliance, and sustainable development.
2. Government should harness the potential of Nigeria's youth—particularly students—by adopting transformative approaches that leverage STEM education. This should include pedagogical innovation, teacher development, technology integration, and a contextualized learner-centered curriculum. By adopting these measures, Nigeria can empower its youth with creativity, critical thinking, and problem-solving skills to drive socio-economic progress.
3. The government is further advised to collaborate with Mathematics stakeholders to transform the teaching and learning of creativity, technological skills, and self-reliance for sustainable national growth during economic challenges.
4. MAN should organize regular training sessions on effective presentation skills for its members.
5. With the successful introduction of virtual presentations, it is recommended that the conference should now evolve into an international event.

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