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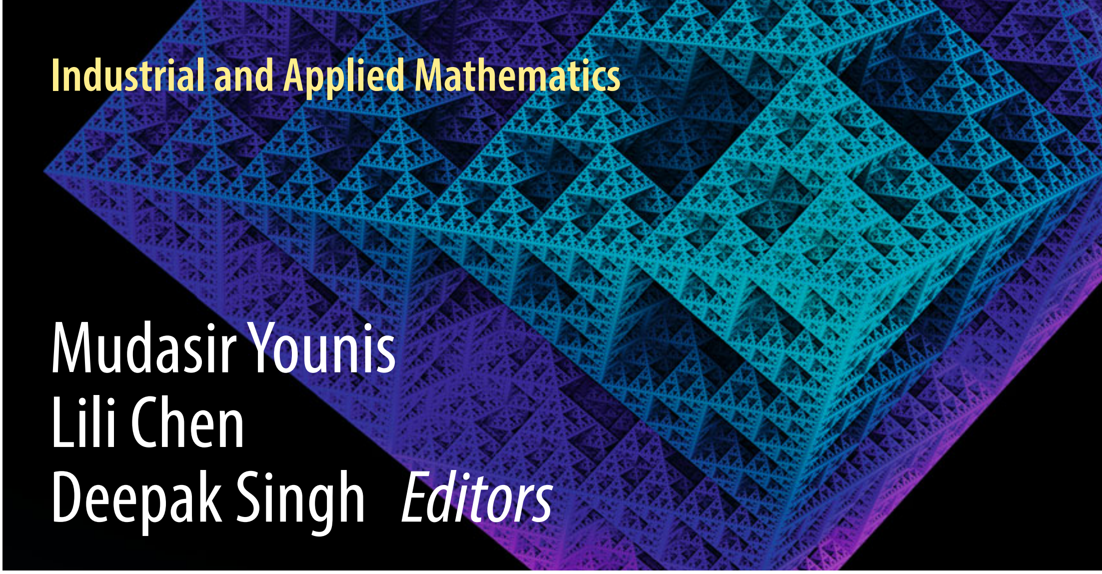

Mudasir Younis
Lili Chen
Deepak Singh *Editors*

Recent Developments in Fixed-Point Theory

Theoretical Foundations and Real-World Applications

Springer

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Theoretical Foundations and Real-World Applications

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Preface

Nonlinear functional analysis includes the essential and comprehensive field of fixed-point theory, which is used as a method to solve numerous nonlinear problems in the science and engineering. Since a couple of years ago, experts in fixed-point theory have been focusing on how to apply their theories to a variety of physical-related engineering challenges.

Fixed-point theory has entered a new phase that is inextricably linked to measurements, abstract language, space analysis, and the mining of empirical studies in engineering. By incorporating the metric fixed-point theory into a plethora of literature from the fields of computational engineering, quantum dynamics, and medical research, this was frequently maintained. In the analysis of metric spaces, fixed-point theory has briefly been mentioned as independent literature while referencing numerous other mathematical groups. The definition and generalization of the various metric spaces and the concept of contractions are common applications of metric fixed-point theory. The expected outcome of these extensions is also a deeper understanding of the geometric characteristics of generalized metric spaces, set theory, and inexpensive mappings.

This book is intended for scholars, doctoral scholars, and educators intrigued by fixed-point theory. The book will also benefit the mathematical community, professionals, and scientists. Learners of this book will need at least a basic understanding of functional analysis and topology. This book contains chapters written by several notable modern academics in fixed-point theory from around the world. Readers will discover various valuable tools and approaches to help them advance their knowledge and abilities in modern fixed-point theory. The book includes adequate theory and applications of fixed points in various fields. The book provides an overview of existing knowledge and recent cutting-edge advancement by offering fresh novel contributions by world-renowned experts of fixed-point theory.

Following are salient features of the book:

- Provides a comprehensive and up-to-date assessment of the most recent research in fixed-point theory, including major breakthroughs and applications within the environment of various types of generalized metric spaces

- Offers a unique combination of fixed-point theorems and real-world applications that benefits students and researchers interested in developing an integrated research methodology
- Emphasizes the mathematical modeling of nonlinear situation, utilizing the advantages of technology as appropriate
- Takes a fresh look at fixed-point theory, highlighting linkages and applications to fields as diverse as optimization, graph theory, and differential equations
- Stresses on the practical applications of fixed-point theory in diverse fields of research and engineering, making it ideal for scientists researching in engineering sciences
- Offers thorough descriptions of numerical methods and algorithms for addressing fixed-point problems, making it a useful resource for scholars and practitioners
- Offers a variety of examples meant to assist readers deepen their comprehension of the content and apply it to new challenges
- Emphasizes recent advances in fixed-point theory and identifies significant open problems, making it a valuable resource for academics wishing to contribute to this dynamic area of research

Chapter Organization

This book is organized into 17 chapters. Chapter "A Careful Retrospection of Metric Spaces and Contraction Mappings with Computer Simulation" gives a comprehensive outlook of the famous results based on metric fixed-point theory. Illustrative non-trivial examples, along with graphical representation, are enunciated to understand the nature of inequalities of the historical results in the context of metric spaces.

Chapter "Fixed Point Theory for Multi-valued Feng–Liu Operators in Vector–Valued Metric Spaces " extends Feng and Liu's fixed-point result for the case of a set X equipped with a vector-valued metric in the sense of Perov. The results describe the existence, localization, data dependency, and stability features of a fixed-point inclusion with a generalized multi-valued Feng–Liu operator. An extension of the Feng–Liu–Subrahmanyan type of multi-valued contractions is also examined.

In Chapter "Algorithms and Applications for Split Equality Problem with Related Problems ", the authors propose a novel study on "Algorithms and applications for split equality problem with related problems." The chapter's goal is to offer a variety of novel and well-known numerical strategies for addressing the split feasibility problem, the split equality problem, and other related problems, such as projection, inertial techniques, relaxed techniques, and so on. The algorithms' weak (strong) convergence is presented, and the linear convergence of the methods is emphasized. An algorithm is enunciated as an application to tackle signal processing and picture recovery problems.

Chapter "Some Fixed Point Theorems of Generalized Contractions with Application to Boundary Value Problem" investigates the development of an exclusive rational type generalized contraction involving three self-maps. Furthermore, several fixed-point findings matching the requirements of $a(\eta,\pi,\theta)$-generalized rational contraction are proposed within the setting of complete metric spaces with a partial order. These findings not only generalize but expand numerous well-known findings from previous research.

Chapter "Fixed Points of Coset and Orbit Space Actions: An Application of Semi-hypergroup Theory" is devoted to presenting certain families of left/right coset, double coset, and orbit spaces that arise from the category of locally compact groups. The study is concerned with looking at their behavior on compact subsets of generic locally convex spaces and specific Banach spaces. Utilizing recent discoveries in abstract harmonic analysis and the theory of semihypergroups, in particular, an overview of possible characterizations for the existence of common fixed points of such actions in terms of the amenability of the underlying spaces is presented.

A counterpart of Meir–Keeler's fixed-point result in suprametric space is proved in Chapter "Strange Chaotic Attractors and Existence Results via Nonlinear Fractional Order Systems and Fixed Points", and an application to strange attractors in the context of the Atangana–Baleanu derivative is examined.

The existence findings for the n-product of fractional nonlinear equations in Orlicz spaces are studied in Chapter "On L_ϕ-Solutions for n-Product of Fractional Integral Operators". A wide range of boundedness and continuity assumptions of the researched operators in Orlicz space are explored by using different growth conditions. Also, distinct existence theorems on the product of different n-Orlicz spaces related to the generating N-functions are investigated. The main tools for achieving the results are the fixed-point hypothesis and the measure of noncompactness.

In the context of complete b-metric spaces, Chapter "New Fixed Point Results of Multivalued Contraction Mappings in b-Metric Spaces" examines recent findings for generalized contractive type multi-valued mappings. The results add to and enhance certain recent breakthroughs announced by numerous others with fewer assumptions.

In the framework of controlled type metric spaces, Chapter "Analysis of Fixed Points in Controlled Metric Type Spaces with Application" explores the presence and uniqueness of fixed points using the Ciric-type and Reich-type contractions. The work intends to offer a more thorough knowledge of fixed-point findings by including graph theory in these contraction mappings, a contemporary technique in the current state of the art. The outcomes provided in this chapter demonstrate the versatility of contraction mappings in nonlinear differential equations and their value in mathematics.

New classes of $(G,\alpha,\phi,)$-contractions are suggested in Chapter "A Study of Fixed Point Results in G-Metric Space via New Contractions with Applications", and appropriate FP theorems are demonstrated. The uniqueness of these new contractions resides in the fact that, depending on the selection of parameters, they may be specialized in a variety of ways. In-depth comparative examples are produced to verify the presumptions behind the conclusions. In addition, the existence conditions for solving a boundary value issue utilizing one of the findings are investigated.

Chapter "A Note on the Existence of Fixed Points for Rational Type Contraction Map on Orthogonal Metric Spaces" focuses on instigating the existence of fixed points for rational type contraction maps on orthogonal metric space, a more general metric space in modern literature. Some examples from this research substantiate the presented outcomes.

Chapter "Fixed Point Results in Graphical Convex Extended b-metric Spaces" provides a historical overview of F-contractions. The chapter presents a historical overview of the new fixed-point approaches for single and multi-valued mappings in various spaces. Furthermore, certain enhancements, notably the additional conditions applied to the function F of the contractive condition, are investigated. Researchers who are interested in F-contractions will find this chapter informative.

Chapter "Existence and Computational Approximation of Fixed Points of Generalized Multivalued Mappings in Banach Space" aims to define graphical convex extended b-metric space and examine certain vital aspects of the convex structure. Iterative sequences G-contraction, T-Mann, and T-Agrawal have been addressed in the context of the instigated space. These iterative methods are used to investigate the existence of strong fixed-point theorems. Combining the convex structure of metric spaces with graphs makes the article innovative and appealing to academics interested in locating fixed points inside the graph structure.

Chapter "Common Fixed Point Results in Soft b-Metric Spaces with Application" focuses on the presence and computational approximation of fixed points of generalized multi-valued mappings in Banach spaces. This chapter introduces multi-valued generalized-nonexpansive mappings and associated results for fixed-point existence and approximation. Furthermore, the Picard–Thakur hybrid iterative scheme convergence findings are reviewed and contrasted. It is demonstrated mathematically and visually that the Picard–Thakur hybrid iterative strategy converges to the fixed-point quicker than other schemes presented in the literature. The final part provides an application to integral equations to validate the offered conclusions.

Chapter "Revisiting Darbo's Fixed Point Theory with Application to a Class of Fractional Integral Equations" discusses the novel fixed-point findings in soft b-metric spaces and an application to Volterra integral inclusions. The results of the study can be expanded to other areas in the future, and new results can be drawn from them.

In Chapter "New Topologies on Partial Metric Spaces and M-Metric Spaces", new definitions for partial metric spaces and M-metric spaces are presented. Various topological features of the aforementioned metric spaces are addressed. For partial metric spaces and M-metric spaces, the new topology is demonstrated to be weaker than the previously established topology.

Chapter "Some Recent Fixed Point Results in S_b-Metric Spaces and Applications" defines a generalized proportional (k,ρ)-fractional integral operator with a nonsingular kernel. It is a more generalized version of previously known fractional integral operators such as the Riemann–Liouville fractional integral operator, the Hadamard fractional integral operator, the Katugampola fractional integral operator, etc. Using Darbo's fixed-point theorem, it is proved that there is a solution to the generalized proportional (k,ρ)-fractional integral equation.

Finally, a relevant example is built to validate the acquired findings.

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Contents

About the Editors

Mudasir Younis is a postdoctoral research fellow at the Department of Mathematics and Statistics, Indian Institute of Technology Kanpur (IIT Kanpur). He researched diverse fixed point theorems within the graph structure of metric spaces throughout the last five years of his considerable research, which is a relatively fresh addition to the relevant topic. In this context, he obtained several novel results and worked on determining the existence of solutions to various real-world engineering science and physics problems, such as a damped Spring–Mass system, deformation of an elastic beam, vibrations of a vertical heavy hanging cable, ascending motion of a rocket, tuning circuit problem, and so on. He has received fellowships at both the national and international levels. He has more than twenty-five research papers published in SCI/ESCI/Scopus listed journals and seven other papers communicated to prestigious international publications. He recently earned the International Mathematical Union's Abel Visiting Fellowship for 2022–2023. He has served on the editorial boards of various peer-reviewed journals and presented his work at several international conferences. He is also the author of a book *Fourier Analysis*, published by the University of Kashmir Press.

Lili Chen is an associate professor and doctorate supervisor at the college of Mathematics and System Science at Shandong University of Science and Technology, Qingdao, China. In recent years, she has been primarily involved in research on geometry theory of banach spaces, and fixed point theory. She is a commentator of the American Mathematical Review, a member of the Chinese Mathematical Society, and a member of the American Mathematical Society. She is a reviewer for several international SCI academic journals and has also supervised 8 projects. She has over 30 papers published in peer-reviewed SCI journals. She has so far supervised 14 graduate students and two doctoral students.

Deepak Singh is an associate professor of mathematics, Department of Applied Science, National Institute of Technical Teachers' Training and Research (NITTTR), Bhopal India. He obtained M.Sc. and Ph.D. degrees in mathematics from Barkatullah University, Bhopal. During his teaching career of about 22 years, he has taught various courses in pure mathematics, applied mathematics, and computer applications, at under-graduate and post-graduate level. He has more than 60 research papers to his credit, many of them are published in various SCI/ISI/Scopus/ESCI, Springer, and Elsevier journals. So far, he has supervised 15 Ph.D. candidates. He has also supervised 21 M.Phil. projects. Besides, he keeps participating in various scientific meetings/conferences in India and abroad. So far, he has delivered over 40 contributed/invited talks including 17 in European and Asian countries. He is a reviewer of the American Mathematical Society and also the referee of many SCI/Springer/Elsevier journals. He has been a resource person in refreshers courses for various universities/'IIT's where he delivered a series of lectures. He has also been appointed as an external expert for Ph.D. viva and thesis valuation from various central universities. He has also coordinated/contributed for more than 130 training programs of one week/two weeks duration for technical teachers.

A Careful Retrospection of Metric Spaces and Contraction Mappings with Computer Simulation

Mudasir Younis, Deepak Singh, and Lili Chen

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1 An Introduction

Fixed-point theory is the fascinating emerging field of the twenty-first century characterized by a remarkable mixture of nonlinear functional analysis, nonlinear operator theory, topology, mathematical modeling, and applications. It is one of the major research areas in nonlinear analysis. Because of the fact that in many real-world problems, the fixed-point theory is the fundamental mathematical tool employed to ascertain the existence of solutions to problems that arise naturally in applications. As a consequence, the fixed-point theory is an essential area of study in pure and applied mathematics, and it is a blooming area of research. Its scope of inquiries not only encompasses the geometric theory of infinite dimensional function spaces and operator-theoretic real-world problems but also widens the range of interdisciplinary fields ranging from engineering to space science, hydromechanics to astrophysics, chemistry to biology, theoretical mechanics to biomechanics, and economics to stochastic game theory. The deep-rooted concepts and techniques pro-

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vide the tools for developing more realistic and accurate models for a variety of phenomena encountered in various applied fields.

Fixed-point theory sheds light on the methodologies for finding a solution to nonlinear equations of the type $Jx = x$, where J is a self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space, or some suitable space.

In the subsequent part, $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ denote the set of natural numbers, the set of all non-negative real numbers and the set of real numbers, respectively.

Definition 1 (*Fixed Point*) Suppose J is a mapping that takes a set Y into itself. A fixed point of J is just a point $x \in Y$ with $J(x) = x$.

A map J can have many fixed points (example: the identity map on a set with many elements) or no fixed points (example: the mapping of "translation-by-one" $x \to x + 1$ on the real line). The fixed points of a function, mapping a real interval into itself, can be visualized as the x-coordinates of the points at which function's graph intersects the line $y = x$. We enunciate this idea by plotting the graph for some nontrivial functions $J_i, i = 1, 2 : \mathbb{R} \to \mathbb{R}$:

$$(i)\quad J_1(x) = \sqrt{\log(\sin x^{\frac{1}{4}} + e^{x+6})} + \sin(\tfrac{1}{2} + x)$$
$$(ii)\quad J_2(x) = e^{\frac{2x+3}{1+x^2}\log(x^3+2)}$$

We determine the fixed points of functions $J_1(x)$ and $J_2(x)$ by visualizing them on graph as follows (Figs. 1 and 2):

Definition 2 (*Common Fixed Point*) Let f and g be two self-mappings on a space Y. We say that $x \in Y$ is a point of coincidence of f and g if $fx = gx$, and say that it is a common fixed point of f and g if $fx = gx = x$.

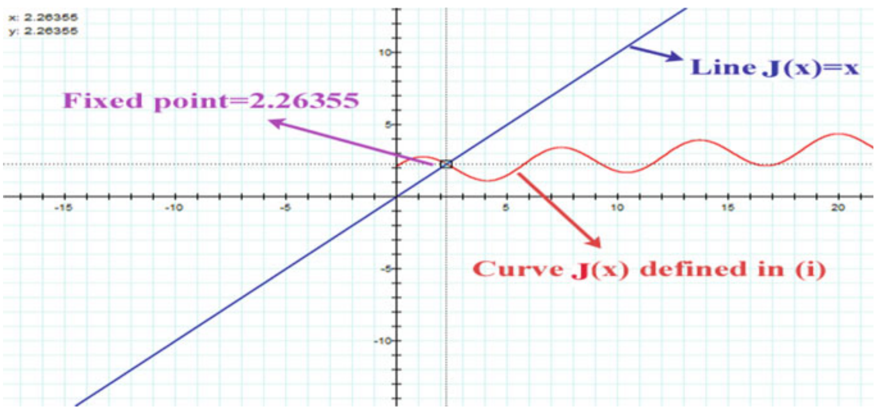

Fig. 1 Fixed point of $J_1(x) = \sqrt{\log(\sin x^{\frac{1}{4}} + e^{x+6})} + \sin(\frac{1}{2} + x)$

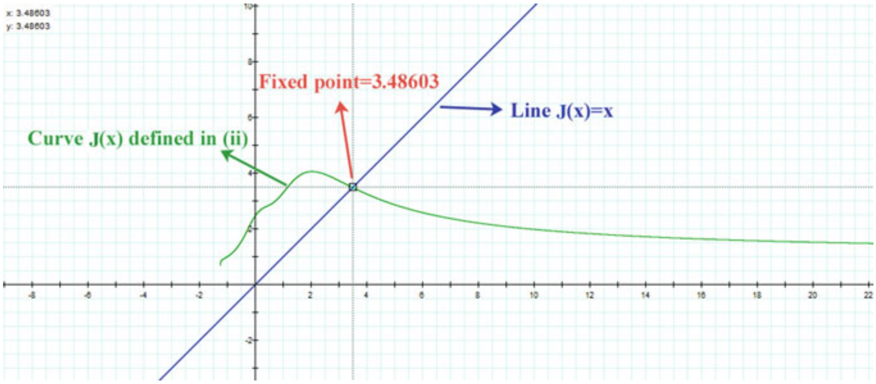

Fig. 2 Fixed point of $J_2(x) = e^{\frac{2x+3}{1+x^2}\log(x^3+2)}$

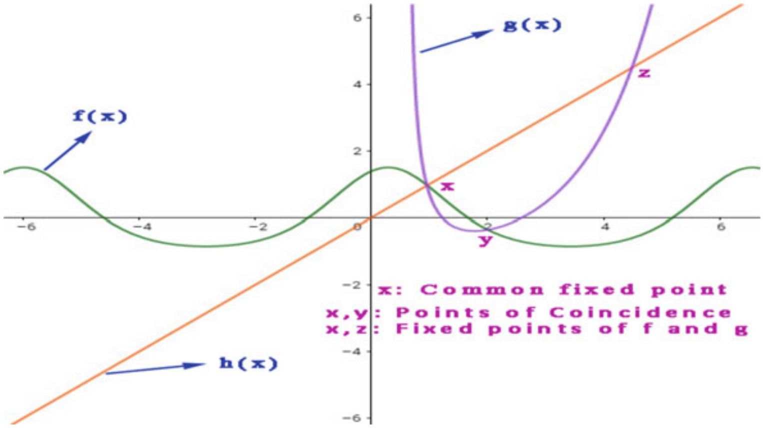

Fig. 3 Common fixed point of mappings $f(x)$ and $g(x)$

Example 1 Let $f(x) = e^{\sin(x-5)} - 1.22$ and $g(x) = \frac{\cosh(x-0.58)}{e^{\ln(x-0.58)}} - 1.9$ for all $x \in [0, \infty)$. Then $x = 0.95518$ is the common fixed point of mappings f and g (Fig. 3).

Some very important examples of fixed points, we generally go through in our study, are as follows:

Example 2 (*Initial-Value Problems*) From a continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ and a point $(x_0, y_0) \in \mathbb{R}^2$, we can generate an initial-value problem

$$y' = g(x, y), \; y(x_0) = y_0. \tag{1}$$

Geometrically, initial-value problem (1) demands a differentiable function y whose graph is a smooth "solution curve" in the plane possessing the following properties:

(i) At each of its points (x, y) the curve has slope $g(x, y)$,
(ii) The curve encompasses the point (x_0, y_0).
As a first attempt to solve the differential equation $y' = g(x, y)$, we might try integrating both sides with respect to x. This results in the following integral equation:

$$y(x) = y_0 + \int_{x_0}^{x} g(t, y(t))dt, \tag{2}$$

which is implied by initial-value problem (1) in the sense that each function y satisfying (1) for some interval of x's containing x_0, also satisfies integral Eq. (2) for that same interval. To make a concern with fixed points, let $C(\mathbb{R})$ denote the vector space of continuous, real-valued functions on $\mathbb{R}$, and consider the integral transform $J : C(\mathbb{R}) \to C(\mathbb{R})$ defined by

$$(Jy)(x) = y_0 + \int_{t=x_0}^{x} g(t, y(t))dt, \quad x \in \mathbb{R}.$$

Equation (2) can thus be rewritten as $Jy = y$, so to say $y \in C(\mathbb{R})$ satisfying (1) turns out to be the same as saying: y is a fixed point of the mapping J.

Example 3 (*Newton's Method*) Let us suppose that J is a differentiable function $\mathbb{R} \to \mathbb{R}$, with continuous and never vanishing derivative J' on $\mathbb{R}$. Consider for J its "Newton" function N, defined by

$$x - \frac{J(x)}{J'(x)} = N(x), \quad x \in \mathbb{R}.$$

One can think of $N(x)$ as the horizontal coordinate of the point at which the line tangent to the graph of J at the point $(x, J(x))$ intersects the horizontal axis. Since J' doesn't vanish, N is a continuous mapping taking $\mathbb{R}$ into itself. Fixed points of N are explicitly the roots of J (those $x \in \mathbb{R}$ such that $J(x) = 0$). Newton's method is concerned with the iteration of the Newton function in anticipation of generating approximations to the roots of J. One starts with an initial guess x_0, sets $x_1 = N(x_0)$, $x_2 = N(x_1), \ldots$, and hopes that the resulting sequence of "Newton iterates" converges to a fixed point of N. Geometrically it appears evident that if the Newton iterate sequence converges then it must converge to a root of J.

Example 4 (*The PageRank algorithm*) Google's prosperity as a web crawler comes from its calculation: the PageRank calculation algorithm. In this calculation, one processes a fixed point of a linear map on R^n, which is itself a contraction, and this fixed point (which is, in fact, a vector) yields the requesting of the pages. Further amalgamation and illustrative insights about "the PageRank calculation algorithm" can be found in the vignette on *How Google works*?.

2 Contraction Mappings: History and Development

A natural question arises that under what conditions on the set Y and the self mapping J, a fixed point exists?. Theorems which establish the existence and uniqueness of such points are called **Fixed-Point Theorems**.

There are three major branches of fixed-point theory in functional analysis and each branch has its celebrated theorems.

– Metric Fixed-Point Theory,
– Topological Fixed-Point Theory,
– Discrete Fixed-Point Theory.

Historically, the above approaches were initiated by the discovery of three major theorems:

– Banach's contraction principle [19],
– Brouwer's fixed-point theorem [33],
– Tarski's fixed-point theorem [177].

Historically, the study of fixed-point theory began in 1912 with a theorem given by famous Dutch mathematician Brouwer [33]. This is the most famous and important theorem on the topological fixed-point property. It can be formulated as:

★ The closed unit ball $\mathscr{B}^n \in \mathbb{R}^n$ has the topological fixed-point property.

He also proved the fixed-point theorems for a square, a sphere and their n-dimensional counterparts which was further extended by Kakutani [90]. Brouwer's theorem has many applications in analysis, differential equation and generally in proving all kinds of so-called existence theorems for many types of equations.

An important generalization of Brouwer's theorem was discovered in 1930 by Schauder [157] it may be stated as follows:

Theorem 1 *Let Y be a Banach Space and $\mathscr{B}$ is a compact, convex subset of Y. If the self map $J : \mathscr{B} \to \mathscr{B}$ is a continuous, then J admits a fixed point.*

The Schauder fixed-point hypothesis has various applications in scientific theory, approximation theory, and different scientific areas like the improvement theory, social science, and engineering. The compactness condition on $\mathscr{B}$ is a solid one, and most of the analysis results don't possess a compact setting. It's natural to prove the results by relaxing the condition of compactness. Schauder demonstrated the subsequent theorem:

Theorem 2 *Let Y be a Banach space and $\mathscr{B}$ be its closed and bounded convex subset. If $J(\mathscr{B})$ is compact, where $J : \mathscr{B} \to \mathscr{B}$ is a continuous map, then J admits a fixed point.*

Meanwhile, the Banach principle came into existence, considered to be one of the elementary principles in nonlinear analysis. In 1922, Banach [19] proved that a contraction mapping in the context of a complete metric space admits a unique fixed point.

In the subsequent part, we focus mainly on the first area, that is, ***metric fixed-point theory*** based on Banach's contraction.

The term *metric fixed-point theory* refers to those fixed point theoretical results in which geometric conditions on the underlying spaces and/or mappings play a crucial role. The first-ever fixed-point theorem in metric space appeared in explicit form in Banach's thesis [19], known as the "Banach's Contraction Principle", used to establish the existence of a solution to an integral equation. Before presenting this remarkable result, formal definition of the distance introduced in 1905 by Frèchet [60] is worth mentioning.

Definition 3 Let Y be a nonempty set, and let $d : Y \times Y \to [0, \infty)$ be a given mapping. We say that d is a metric on Y, if for all $x, y, z \in Y$, the following conditions are fulfilled:

$(d1)$ $d(x, y) = 0$ if and only if $x = y$;
$(d2)$ $d(x, y) = d(y, x)$;
$(d3)$ $d(x, y) \leq d(x, z) + d(z, y)$.
The pair (Y, d) is called a metric space.

Banach [19] published his fixed-point theorem, also known as "Banach's Contraction Principle" which uses the concept of Lipschitz mappings.

Definition 4 Let (Y, d) be a metric space. The self mapping $J : Y \to Y$ is said to be Lipschitzian if there exists a constant $\lambda > 0$ (called Lipschitz constant) such that

$$d(Jx, Jy) \leq \lambda\, d(x, y), \quad \text{for all } x, y \in Y. \tag{3}$$

A Lipschitzian mapping with a Lipschitz constant $\lambda < 1$ is called **contraction**.

Theorem 3 (Banach's Contraction Principle) *Let (Y, d) be a complete metric space and $J : Y \to Y$ be a contraction mapping with Lipschitz constant $\lambda < 1$. Then J has a unique fixed point $x^* \in Y$ and $\lim_{n\to+\infty} J^n x = x^*$ for all $x \in Y$.*

We verify Banach's contraction principle with an illustrative example along with its graphical representation, where the surface representing the right-hand side of the contraction mapping dominates the surface constituting the left-hand side of the contraction mapping.

Example 5 Let $Y = [0, 10]$ be a set. Define a function $d : Y \times Y \to [0, 10]$ by $d(x, y) = |x - y|$, then clearly (Y, d) is a complete metric space. Let J be a self-map on Y defined as

$$J(x) = \frac{\log(2 + e^{x+x^2})}{2 + e^x + \sin x}.$$

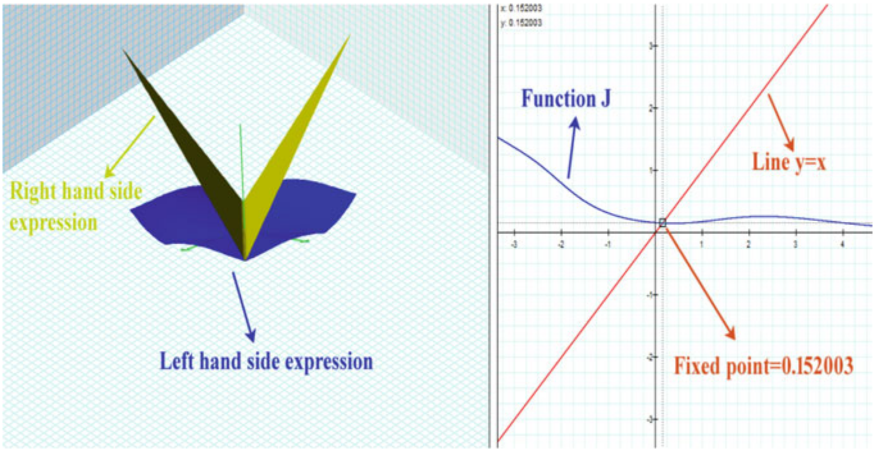

Fig. 4 Visual verification of Banach's contraction principle with fixed point

The following figure shows that surface representing the right-hand side is dominating the surface constituting the left-hand side of (3) with $J(x) = \frac{\log(2+e^{x+x^2})}{2+e^x+\sin x}$ and $\lambda = \frac{11}{12}$.
This demonstrates the validity of Banach's contraction principle, and so, J has a unique fixed point $x = 0.152003$ shown by Fig. 4.

After the establishment of this significant result, it was generalized and extended in different ways by several authors (see, e.g., [2, 4, 38, 53, 63, 64, 77, 92, 95, 96, 114, 127, 178]). Authors obtained numerous fixed-point theorems on the following two approaches.

1. Extending the contraction condition (3) to more general contraction conditions, and
2. Replacing the complete metric space (Y, d) by specific generalized metric spaces.

In the first-mentioned direction, the results due to Edelstein [54, 55] and Rakotch [141] created a new milestone in the literature of fixed-point theory. Rakotch [141] generalized Banach contraction principle in the following way.

Theorem 4 ([141]) *Let Y be a complete metric space and suppose $J : Y \to Y$ satisfies*

$$d(J(x), J(y)) \leq \alpha(d(x, y))d(x, y), \text{ for all } x, y \in Y,$$

where $\alpha : \mathbb{R}^+ \to [0, 1)$ is monotonically decreasing. Then J has a unique fixed point x^* and $\{J^n(x)\}$ converges to x^* for each $x \in Y$.

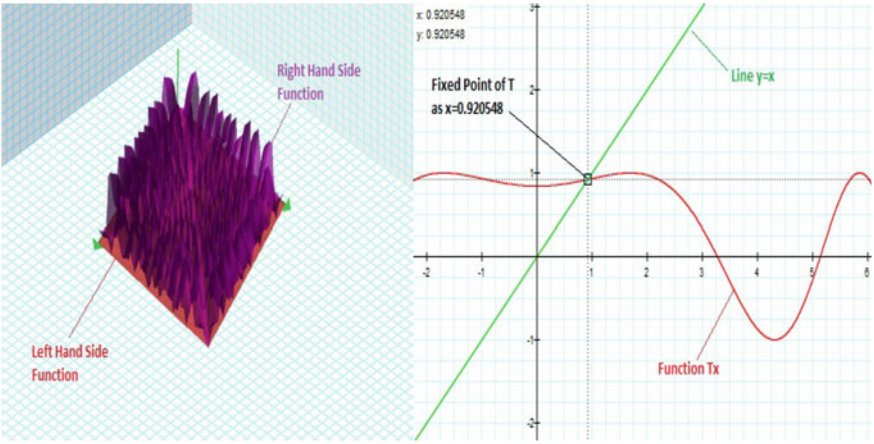

Fig. 5 Visual verification of Rakotch's theorem and the corresponding fixed point

Example 6 Let $Y = [0, \infty)$ be a set. Define a function $d : Y \times Y \to [0, \infty)$ by $d(x, y) = |x - y|$ then clearly (Y, d) is a complete metric space. Let T be a self-map on Y defined as $T(x) = \sin(1 + \frac{x^2}{5})$ with $\alpha(x) = \frac{1}{(x^2+5)^2}$.
The following figure demonstrates that on invoking function $T(x) = \sin(1 + \frac{x^2}{5})$ on (4), the surface representing the right-hand side is superimposing the surface representing the left-hand side. This demonstrates the validity of Rakotch's theorem and so x has a unique fixed point $x = 0.920548$ shown by adjacent figure (Fig. 5).

In 1968, Kannan [91] generalized Banach's contraction principle in some different way, where the map involved does not need to be continuous. This theorem gained quite more attention in the field of metric fixed-point theory.

Theorem 5 ([91]) *Let (Y, d) be a complete metric space and J be a self-map on Y such that, for all $x, y \in Y$, the following condition is satisfied*

$$d(Jx, Jy) \leq \mathscr{K}\,[d(x, Jx) + d(y, Jy)], \tag{4}$$

where $\mathscr{K} \in [0, \frac{1}{2})$. Then J has a unique fixed point $x^* \in Y$.

The following example is worked out to verify Kannan's theorem along with its visualization.

Example 7 Let $Y =]-\infty, 0]$ be a set. A function $d : Y \times Y \to [0, \infty)$ is defined as $d(x, y) = |x - y|$, then clearly (Y, d) is a complete metric space. Let $J : Y \to Y$ be defined by the following

$$J(x) = \frac{1}{2}\log(2\cos x + 1).$$

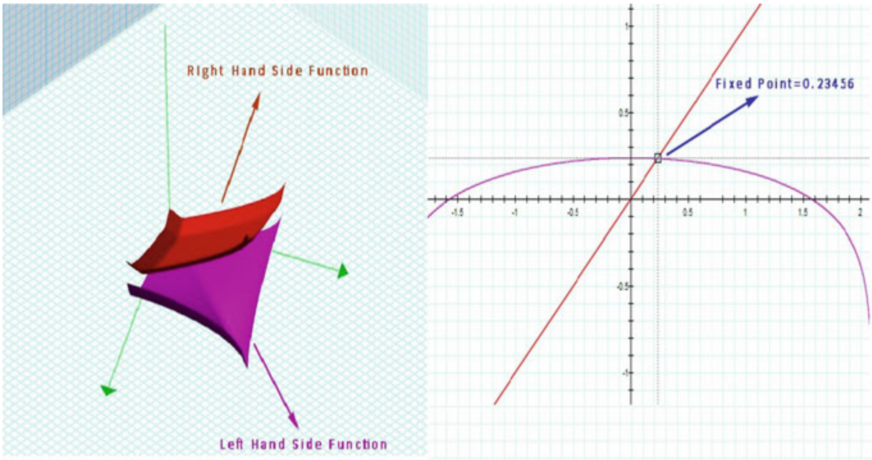

Fig. 6 Pictorial verification of Kannan's theorem and related fixed point of J

Figure 6 authenticates the domination of the right-hand side of the inequality (4) over its left-hand side by invoking the function $J(x) = \frac{1}{2}\log(2\cos x + 1)$ with $\mathscr{K} = \frac{2}{5}$. Hence, all the conditions of Kannan's result are contended and J has a unique fixed point $x = 0.23456$ shown by Fig. 6.

Kannan's theorem is also fundamental because Subrahmanyam [176] proved that Kannan's theory characterizes the metric completeness, that is, a metric space (Y, d) is complete if and only if every Kannan mapping on Y has a fixed point. However, contractions (in the sense of Banach) do not have this property.

The following examples compare Banach's and Kannan's contraction conditions.

Example 8 Let $Y = \mathbb{R}$ be a usual metric space and $J : Y \to Y$ be a mapping defined by

$$J(x) = \begin{cases} 0, & \text{if } x \in]-\infty, 2], \\ \frac{1}{2}, & \text{if } x \in]2, +\infty[. \end{cases}$$

In this example, J is not continuous on $\mathbb{R}$; therefore, Banach's contraction condition is not satisfied but it satisfies Kannan's contraction with $\mathscr{K} = \frac{1}{5}$.

Example 9 Let $Y = [0, 1]$ and J be the self mapping of non-empty set Y such that, $J(x) = \frac{x}{3}$ for $x \in [0, 1]$.

In this example, Banach's contraction condition is satisfied but, for $x = \frac{1}{3}$ and $\mathscr{K} = 0$, it does not satisfy Kannan's contractive condition.

Later, in 1971, Reich [143] established more general and innovate extension of Banach's principle for single-valued as well as multi-valued mappings. Since then Reich-type mappings have been the center of intensive research for many authors.

Theorem 6 ([143]) *Let* (Y, d) *be a complete metric space and* J *be a self-map on* Y. *There exist some real constants* $\alpha, \beta, \gamma \in \mathbb{R}^+$ *with* $\alpha + \beta + \gamma < 1$, *such that*

$$d(Jx, Jy) \leq \alpha\, d(x, y) + \beta\, d(x, Jx) + \gamma\, d(y, Jy), \tag{5}$$

for all $x, y \in Y$. *Then* J *has a unique fixed point* $x^* \in Y$.

The following example validates Reich's fixed-point theorem.

Example 10 Let $Y = [0, 1]$ be a set. A function $d : Y \times Y \to [0, \infty)$ is defined by $d(x, y) = |x - y|$, then obviously (Y, d) is a complete metric space. Let J be a self-map on Y defined by the following

$$J(x) = \frac{e^{2x^2-1} + \sqrt{x+2}}{7 + \log(x + 50)}.$$

Calculating the right-hand side and the left-hand side of the inequality (5), it follows that the inequality (5) of Reich's theorem is satisfied for $\alpha = 0.39$, $\beta = 0.29$, $\gamma = 0.27$ by invoking the function $J(x) = \frac{e^{2x^2-1}+\sqrt{x+2}}{7+\log(x+50)}$.

Figure 7 authenticates the validity of inequality (5), where the surface corresponding to the right-hand side is overlaying the surface corresponding to the left-hand side, and $x = 0.217634$ is the corresponding unique fixed point.

After that, several authors have introduced a variety of contraction-type conditions and established fixed-point theorems in the framework of complete metric spaces. Bianchini [29], Chatterjea [37], Geraghty [65] and Hardy–Roger [68] also extended Banach's result in their manner.

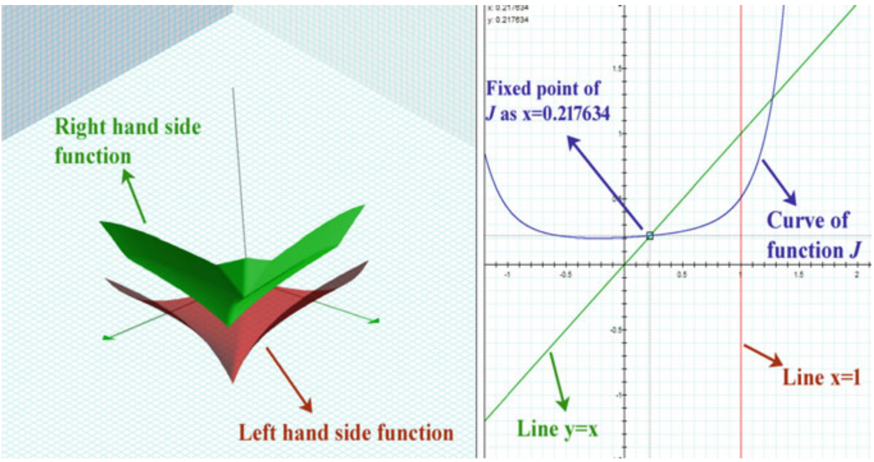

Fig. 7 Visual verification of Reich's with fixed point

In 1973, Hardy–Roger [68] gave an innovative extension of Banach's principle as follows:

Theorem 7 ([68]) *Let* (Y, d) *be a complete metric space and* J *be a self-map on* Y*. There exist some real constants* $\alpha, \beta, \gamma, \delta, \xi \in \mathbb{R}^+$ *with* $\alpha + \beta + \gamma + \delta + \xi$*, such that*

$$d(Jx, Jy) \le \alpha d(x, Jx) + \beta d(y, Jy) + \gamma d(x, Jy) + \delta d(y, Jx) + \xi d(x, y), \quad (6)$$

for all $x, y \in Y$*. Then* J *has a unique fixed point* $x^* \in J$*.*

The following example validates Hardy–Rogers's theorem.

Example 11 Let $Y = [0, 2]$ be a set. A function $d : Y \times Y \to [0, \infty[$ is defined by $d(x, y) = |x - y|$ then clearly (Y, d) is a complete metric space. Let J be a self-map on Y defined as $J(x) = \frac{e^{x-1}}{1+\log(1+x)}$.
Subsequent figure makes evident that on employing function $J(x) = \frac{e^{x-1}}{1+\log(1+x)}$ on (6) with $\alpha = 0.15, \beta = 0.15, \gamma = 0.2, \delta = 0.1, \xi = 0.3$, the surface corresponding to the right-hand side is overlaying the surface corresponding to the left-hand side. This substantiates Hardy–Rogers's theorem and so Jx has a unique fixed point $x = 0.52594$ expressed by Fig. 8.

Later, in 1974, Ćirić [40] presented a new version of Banach's contraction in a quite different way. Ćirić [40] involved all distances $d(Jx, Jy), d(x, y), d(x, Jx), d(y, fJy), d(x, Jy), d(y, Jx)$ in his contraction in a linear way, while Banach [19] used only the first two distances. More precisely, the renowned Ćirić [40] for a single-valued map is the following:

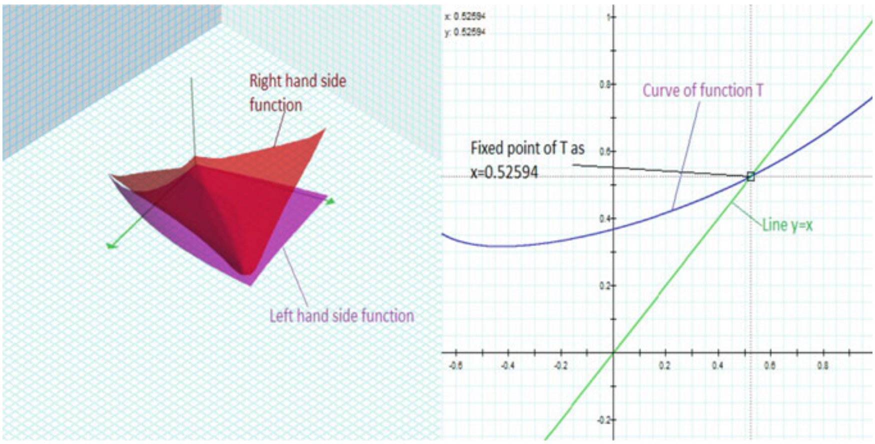

Fig. 8 Visual verification of Hardy–Rogers's result and the corresponding fixed point

Theorem 8 ([40]) *Let* (Y, d) *be a complete metric space and* J *be a self-map on* Y*. There exists a constant* $\lambda \in [0, 1)$*, such that*

$$d(Jx, Jy) \leq \lambda \mathscr{M}(x, y), \tag{7}$$

for all $x, y \in Y$*, where,*

$$\mathscr{M}(x, y) = max\{d(x, Jx), d(y, Jy), d(x, Jy), d(y, Jx), d(x, y)\}.$$

Then J *admits a unique fixed point* $x^* \in J$*.*

The validation of Ćirić's result is endorsed by the subsequent example.

Example 12 Let $Y = [0, \frac{1}{2}]$ be a set. A function $d : Y \times Y \to [0, \infty)$ is defined by $d(x, y) = |x - y|$. Then (Y, d) is a complete metric space. Now, define a self mapping $J : Y \to Y$ by the following

$$J(x) = \frac{e^{2x^2-1}}{\sqrt{1 + \sin(1 + x)}}.$$

The following figure shows that on applying the value of function $J(x) = \frac{e^{2x^2-1}}{\sqrt{1+\sin(1+x)}}$ in (7) with $\lambda = 0.91$, the surface corresponding to the right-hand side dominates the surface corresponding to the left-hand side. This substantiates Ćirić's theorem with unique fixed point $x = 0.322937$ expressed by Fig. 9.

In 1969, Boyd and Wong [30] obtained a more general result. In this result, the authors assumed that $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is upper semi-continuous from the right that is $r_n \downarrow r > 0 \Rightarrow \lim_{n\to\infty} \sup \phi(r_n) \leq \phi(r)$. In the same paper, author showed that, if the space Y is metrically convex, then the upper semi-continuity assumption on ϕ can be

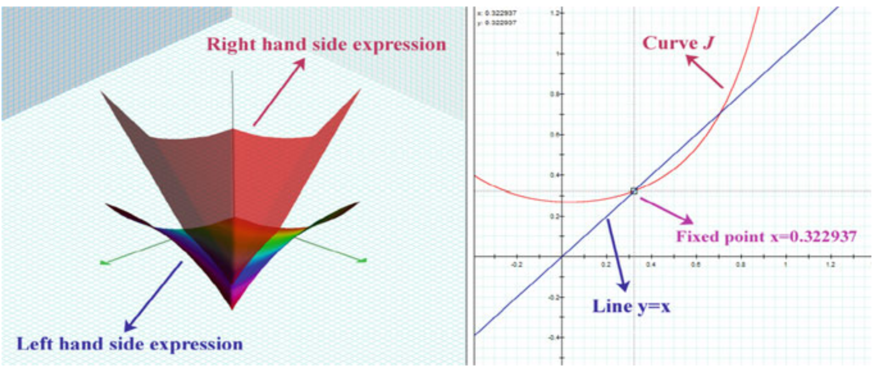

Fig. 9 Computer simulation of Ćirić's result with fixed point

dropped. On the other hand, Matkowski in [110] extended the above result claiming that, if ϕ is assumed to be continuous at 0, then there exists a sequence $t_n \downarrow 0$ for which $\phi(t_n) < t_n$.

Some other generalized results were collected and compared in the well-known article of Rhoades [145], and later one of Collaço and Silva [43]. Further escalations in the level of complexity can be found in a noteworthy paper by van An et al. [181]. In 2003, Berinde [24] introduced the notion of weak contraction (also known as almost contraction) and generalized Banach's result. He enhanced that Banach's and Kannan's mappings are weak contractions. The merit of weak contractions is that they unify large classes of contractive-type operators including quasi-contractions whose fixed points can be obtained employing the Picard iteration and for which both a priori and posteriori estimates are also available.

Berinde [24] adorned the concept of nonlinear-type weak contraction operating a comparison function in a metric space as follows:

Theorem 9 ([24]) *Let (Y, d) be a complete metric space and $J : Y \to Y$ be a self mapping such that there exist $\lambda \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Jx, Jy) \leq \lambda\, d(x, y) + L\, d(y, Jx), \tag{8}$$

for all $x, y \in Y$. Then J has one and only one fixed point $x^* \in Y$.

Due to symmetry of the distance, the weak contraction condition (8) implicitly includes the following dual condition:

$$d(Jx, Jy) \leq \lambda\, d(x, y) + L\, d(x, Jy).$$

The following example substantiates the validity of Theorem 9 due to Berinde.

Example 13 Let $Y = [0, \frac{1}{2}]$ be a set. Define a function $d : Y \times Y \to [0, \infty)$ by $d(x, y) = \{\max(x, y)\}^2$. Then (Y, d) is a complete metric space. Let J be a self map defined on Y as follows:

$$J(x) = \frac{x}{3} + x \log(1 + x^{\frac{1}{3}}).$$

Without loss of generality, we take $x > y$. The subsequent figure presents that on utilizing the value of operator $J(x) = \frac{x}{3} + x \log(1 + x^{\frac{1}{3}})$ on the inequality (8) with $\lambda = 0.8$ and $L = 2$, the surface corresponding to the right-hand side is superimposing the surface corresponding to the left-hand side. This validates Berinde's theorem and so J has a unique fixed point $x = 0$ shown by Fig. 10.

In recent investigations, Wardowski [182] considered a new type of contraction, namely, F-contraction, and proved some fixed-point results in a very general and natural setting.

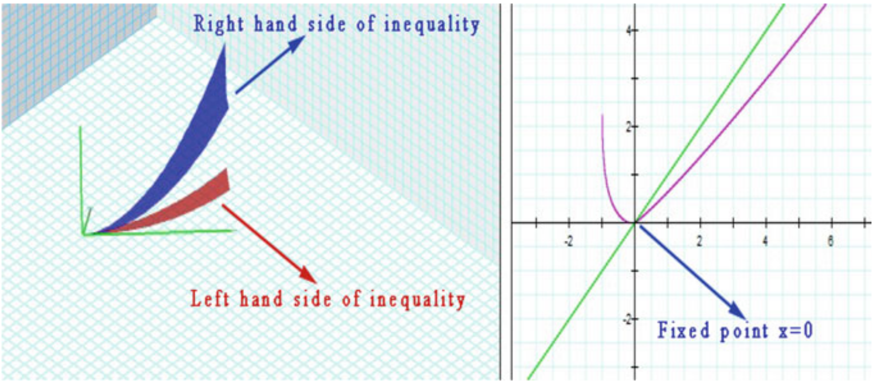

Fig. 10 Pictorial verification of Berinde's result with the fixed point

Definition 5 ([182]) Let (Y, d) be a metric space. A self-mapping $J : Y \to Y$ is said to be an F-contraction, if there exists $\tau > 0$ such that for every $x, y \in Y$

$$d(Jx, Jy) > 0 \Rightarrow \tau + F(d(Jx, Jy)) \leq F(d(x, y)), \tag{9}$$

where the function $F : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following assertions:

$(F1)$ F is strictly increasing, i.e., for all $a, b \in \mathbb{R}^+$ such that $a < b$, $F(a) < F(b)$;
$(F2)$ for all sequence $\alpha_n \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
$(F3)$ there exists $0 < k < 1$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

By considering various types of mappings F in (9), one can obtain a variety of contractions, some of them are of a type known in the literature. Wardowski [182] described the class of all functions F satisfying $F1$, $F2$ and $F3$ by $\mathscr{F}$.

Theorem 10 *Let (Y, d) be a complete metric space and $J : Y \to Y$ be F-contraction. Then J has precisely one fixed point.*

We verify Wardowski's theorem by the following non-trivial example.

Example 14 Let $Y = [0, 2]$ be a set. Define a function $d : Y \times Y \to [0, \infty)$ by $d(x, y) = |x - y|$. Then (Y, d) is a complete metric space. Let $J : Y \to Y$ be a self map defined on Y as follows

$$Jx = \sin\left(x + \frac{1}{e^{x+1}}\right).$$

Taking $F(a) = \log a$ and $\tau = 0.2$, one can easily see that the left-hand side of the inequality (9) is dominated by its right-hand side as shown in Fig. 11 along with the corresponding fixed point $x^* = 0.841525$.

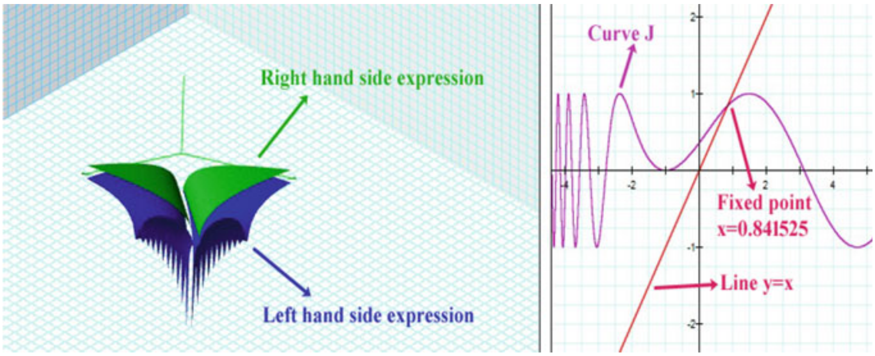

Fig. 11 Verification of inequality $\tau + F(d(Jx, Jy)) \leq F(d(x, y))$ and corresponding fixed point

Later, Secelean et al. [160] and Piri and Kumam [136] extended and refined Definition (5) by establishing some equivalent conditions over the mapping F.

Secelean et al. [160], utilized an equivalent but a more simple condition $(F2')$ instead of condition $(F2)$ as follows:

$(F2')$ $\inf F = -\infty$

or

$(F2')$ there exists a sequence α_n be a sequence of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Piri et al. [136] replaced the condition $(F3)$ by $(F3')$ in the Definition (5) as follows:

$(F3')$ F is continuous on $(0, 1)$.

Detailed description on F-contraction can be also seen in the articles [75, 78, 136, 160, 185, 188, 189] and the related references therein.

In order to prove that a fixed-point theorem is a proper generalization of the Banach contraction principle, the authors usually show the Banach contraction principle to be a direct consequence of their result and construct a map $J : Y \to Y$ to which the Banach contraction principle is not applicable, while the new one is. For more details on these counter-examples, see the proofs of [145, Theorem 1], and [43, Sect. 3].

Moreover, there is an enormous number of works that generalized the contraction on single-valued as well as multi-valued mappings or in terms of space extension. One of these generalizations was carried out recently by considering on partial contractive-type mappings in metric spaces endowed with an arbitrary binary relation, a partial order, and also with a graph.

In 2003, Kirk [105] introduced the notion of cyclic representation which are cyclic relation and cyclic contraction in metric spaces and investigated the existence and uniqueness of the fixed point for cyclical condition. Many papers considered cyclic condition for different contractions and some works introduced new class of cyclic contraction mappings and further in other spaces such as Neammanee [128] extended the concept of cyclic for single-valued to set-valued mappings, Shatanawi

[165] utilized the cyclic mapping for Ω-distance in G-metric spaces, Nashine et al. [124] presented the new formula of cyclic contractive condition for implicit relation and proved the existence and uniqueness of the fixed point for the mappings, while in 2014, Nashine [122] got some fixed-point results for cyclic contraction endowed with implicit relation. In addition, Popa [137] provided cyclic for set-valued mapping, which is more generalized than [122]. Moreover, Kumari and Panthi [107] also considered the cyclic contraction for proving fixed-point theorem in the generating spaces. Moreover, there are many works introduced the new condition of implicit function that we shall see in [25, 26, 123].

Furthermore, the notion of the pair $(\mathscr{F}, h)$ is an upper class which was introduced by Ansari and Shukla [11]. They involved this pair in a contraction condition and proved a fixed-point theorem which generalized many existing results. On the other hand, Samet et al. [156] introduced the notions of α-ψ-contractive and α-admissible mappings and proved fixed-point theorems which also unify several existing fixed point results in the setting of complete metric spaces. Many authors were inspired by the work of Samet et al. [156] and generalized many other results by using the notion of α-admissible mappings. Very recently, Karapinar et al. [93] gave a new type of rational contraction condition for set-valued mappings.

The existence of fixed-point theorems for monotone single-valued mappings in a metric space endowed with a partial ordering has been a relatively new development in metric fixed-point theory and it was initially considered by Ran and Reurings while investigating some applications of matrix equations in [142] (see also [180]). They proved the following result:

Theorem 11 *Let $(Y, \preceq)$ be a partially ordered set such that every pair $x, y \in Y$ has an upper and lower bound and d be a metric on Y such that (Y, d) is a complete metric space. Let $f : Y \to Y$ be a continuous monotone (either order-preserving or order-reversing) mapping. Suppose that the following conditions hold:*

1. *There exists a $k \in (0, 1)$ with*

$$d(fx, fy) \leq kd(x, y) \quad for\ all\ \ x \succeq y.$$

2. *There exists an $x_0 \in Y$ with $x_0 \preceq fx_0$ or $x_0 \succeq fx_0$.*

Then f is a Picard Operator(PO), that is f has a unique fixed point $x^* \in Y$ and for each $x \in Y$, $\lim_{n\to\infty} f^n(x) = x^*$.

Subsequently, Nieto [129, 131] and others [28, 135] modified and improved Ran and Reurings results. There have been so many exciting developments in the field of the existence of fixed points in partially ordered metric spaces. For more details, we can see in the papers by Turinici [179], Nieto and López [130], Agarwal et al. [3], Ćirić et al. [42], Harjani and Sadarangani [70], Jachymski [83] Bhaskar and Lakshmikantham [28], Samet et al. [154, 155], and the references therein.

In 2012, Samet and Turinici [156] extended the concepts of metric spaces and partial ordering to define and construct fixed point theorems in metric spaces with binary relation. After that Berzig [27] considered to the coincidence and common fixed points for contractive mappings which investigated on metric spaces endowed with binary relation. In 2015, Asgari and Mousavi [13] reconsidered this space for coupled fixed-point theorems while Khan et al. [100] constructed the fixed-point theorem in bimetric spaces. Recently, Ahmadullah et al. [7] proved fixed-point theorems through implicit contractive condition on metric spaces employed with an arbitrary binary relation and Ayari et al. [15] also proved this space for showing the existence of coincidence points and common fixed points.

In 2008, Jachymski [84] introduced a new approach in metric fixed-point theory by replacing the order structure with a graph structure (which subsumes the partial ordering) on a metric space. He introduced the concept of G-contraction on a metric space endowed with a graph and obtained some fixed-point results which unified most of the previous results concerning partial ordering e.g., [28, 129, 131, 135, 142]. In this way, the results proved in ordered metric spaces are generalized (see also [84] and the reference therein). After that, Beg et al. illustrated the existence of fixed points for set-valued mappings in metric spaces with a graph by defining the G-contraction which can see in [21, 22]. In 2003, Dinevari et al. [48] also considered set-valued maps which were defined on complete metric space endowed with a directed graph. They established the fixed-point result for weak G-contraction mapping. Recently, the concepts of coincidence point common fixed points and common coupled fixed points were focused on both single-valued and set-valued mappings, in metric spaces, endowed with a directed graph which see in [1, 67, 106, 175, 185, 187, 190–194] and the references therein.

The study of new space discoveries in mathematics and their basic properties are always favorite topics of interest among the mathematical research community.

In this line, multi-valued mappings are of major importance with applications to control theory, convex optimization, and economics, for example, Nadler [120] was the first one to define multi-valued contractions and proved a multi-valued version of the Banach contraction principle in complete metric spaces. Many generalizations of Nadler's fixed-point theorem were published soon after. The constructive nature of any fixed-point theorem makes it interesting for people looking for an algorithm which computes fixed points.

Let (Y, d) be a metric space. Then, following Nadler [120], we recall

(N-1) $CB(Y) = \{A \in 2^Y : A$ is nonempty closed and bounded set$\}$,
(N-2) $C(Y) = \{A \in 2^Y : A$ is nonempty compact set$\}$.
(N-3) For nonempty subsets A, B of Y and $x \in Y$ $\quad d(x, A) = \inf d(x, a) : a \in A$, $d(A, B) = \inf d(a, b) : a \in A, b \in B$, $\quad \delta(A, B) = \sup d(a, b) : a \in A, b \in B$ and $\quad H(A, B) = \max\{\{\sup d(a, B) : a \in A\}, \sup d(A, b) : b \in B\}$. Here we note that $d(A, B) \leq H(A, B) \leq \delta(A, B)$. Also
(N-4) $\delta(A, B) = 0$ if and only if $A = B = \{a\}$
(N-5) $\delta(A, B) = \delta(B, A)$
(N-6) $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$.

Multitude of generalizations of the multi-valued contraction principle can be seen in Ćirić [41], Khan [99] and Smithson [171]. Sufficient conditions for non-self mappings to ensure the fixed point by proving a result on multi-valued contraction in complete metrically convex metric spaces were given in [14]. Nonexpansive mappings are those maps, which have Lipschitz constant equals to one. These mappings can obviously be viewed as a natural extension of contraction mappings. However, the fixed-point problem for nonexpansive mappings differs sharply from that of the contraction mappings in the sense that additional structure of the domain set is needed to insure the existence of fixed points. It took almost four decades to see the first fixed-point results for nonexpansive mappings in Banach spaces following the publication in 1965 of the work of Browder [32], Göhde [66] and Kirk [104].

The nonexpansive mappings can obviously be viewed as natural extensions of the contraction mappings. However fixed-point theory for nonexpansive mappings differs sharply from that of the contraction mappings. This can be seen considering the following very simple example given in [102].

Example 15 Consider the unit ball B in the Banach space c_0 of all sequences of real numbers with zero limit and supremum norm. Thus if $x = (x_1, x_2, x_3, ...) \in c_0$, $\lim\limits_{i\to\infty} x_i = 0$, and $||x|| = \max\limits_n |x_n|$. It is easy to check that the mapping T defined by

$$T(x) = (1 - ||x||, \quad x_1, x_2, x_3, ...)$$

is nonexpansive, mappings B in to itself and has no fixed points.

With the invention of personal computer and development of recent softwares for quick and fast computing, a brand new dimension has been given to fixed-point theory. New fields of study are generated like applied mathematics, numerical analysis, and algorithms. Fixed-point theory has become the subject of scientific research, both in deterministic and fuzzy, stochastic circumstances. Because the introduction of Jungck's fixed-point theorem on commutative mappings and then relaxing the condition of commutatively by weak commutatively by the results of Sessa et al. [163] and similar ideas new flip takes place within the development of fixed point theory. Forceful probabilities came about with the work of Ciric [39] followed by the work of Rhoades [144], Krik [103] on nonexpansive mappings and work of Park [134] and Sadovski [152] have created valuable contribution by considering new types of mapping conditions.

After the work of Mann [109] and Ishikawa [82] a brand new direction came about within the field of fixed-point theory for approximating fixed points and convergence of iterative sequences. Several works were exhausted in this field by several different authors like Singh et al. [170]. The reason behind this kind of work is that once the equation might not be able to provide exact solution, some type of approximate numerical solution is desired. It is ascertained from the above studies that the fixed points are often achieved either by dynamic the character of mappings or by stressing upon the studies on the structure of the space including its topological characteristic.

In the second mentioned direction, many types of generalized metric spaces were introduced by modifying the metric axioms. Some authors showed recently that fixed-point results on some generalized metric spaces may be derived from certain results in standard metric spaces, for example, see [10, 51, 52, 59, 69, 85, 88, 101].

However, it is worth mentioning, and this applies to all mentioned methods, it is not true that all generalized fixed-point results become trivial in this way. Namely, these results utilize some contractive conditions, and it is not always clear whether the given condition remains valid when one considers the problem in the associated metric space.

In recent times, some new types of generalized metric spaces were introduced and various spaces constructed as hybrids of the previous types were considered such as D^*-metric spaces [162], b_v(s)-metric spaces [114], cone rectangular metric spaces [16], topological vector space-valued cone metric spaces or tvs-cone metric spaces [23, 89], generalized cone metric spaces or tvs-G-cone metric spaces [20], metric-type spaces [97, 98], complex-valued metric spaces [17], partial cone metric spaces [108], cone symmetric spaces [139], partial-G-metric spaces [153], 2-metric spaces [61, 195], S-metric spaces [161], metric-like spaces [9], quasi b-metric spaces [164], G^*-metric spaces [146], tvs-cone b-metric spaces [138], quasi-partial metric spaces [94], G_b-metric spaces [5], b-metric-like spaces [8], G_{pb}-metric spaces [79], non-Archimedean cone metric spaces [80], complex-valued b-metric spaces [115], quasi-metric-like spaces [196], M-metric spaces [12], quaternion-valued metric spaces [56], partial b-metric spaces [166], complex-valued fuzzy metric spaces [168], graphical b-metric spaces [36], graphical rectangular b-metric spaces [187] and others. The key in constructing these spaces is to weaken axioms of metric spaces or certain generalized metric spaces.

These papers usually have similar approaches, where the authors introduce the notion of a generalized metric space and then state some fixed-point theorems in such spaces. The topological properties of new generalized metric spaces are often not stated and the relations between fixed-point theorems in these generalized metric spaces and previously generalized metric spaces are often not considered (see [181]).

3 Some Notable Abstract Spaces

3.1 Partial b-Metric Spaces

In recent years, many works on domain theory deal with to equip the semantics domain with a notion of distance. The concept of partial metric spaces was proposed by Matthews [112] in 1992 to solve the problems of computer science, especially, to domain theory and semantics, by transferring the structure of metric space. The most important difference of partial metric rather than the standard metric is the real possibility of non-zero self-distance. In other words, in partial metric, self-distance, $p(x, x)$, does not need to be zero. The existence of several connections between partial metrics and topological aspects of domain theory has been lately pointed by other authors as Neill [133], Waszkiewicz [183, 184], Schellekens [158, 159], Imdad along with Erduan [57, 81], Escardo [58], Romaguera and Schellekens [149, 150], Oltra and Valero [132] and Romaguera [148]. See also the presentation by Bukatin et al. [34] where the motivation for introducing non-zero distance (i.e., the 'distance' p where $p(x, x) = 0$ need not hold) is explained, which is also leading to exciting research in foundations of topology.

The concept of b-metric spaces was introduced by Bakhtin [18], which was further extended by Czerwik [44]. Amalgamating these two notions (i.e., partial metric spaces and b-metric spaces), Shukla [166] introduced another generalization of metric spaces known as partial b-metric spaces. This concept was further modified by Mustafa [116] to find that each partial b-metric p_b generates a b-metric d_{p_b}. The advantage of their definition of partial b-metric is that by using it, one can define a dependent b-metric, which is called the b-metric associated with the partial b-metric. After the establishment of these spaces, many researchers generalized a series of fixed-point results in the framework of partial b-metric spaces. Some interesting results in this connection can be seen in [185, 189].

Definition 6 ([166]) Let Y be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : Y \times Y \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in Y$ the following conditions are satisfied:

(p_{b1}) $x = y$ iff $p_b(x, x) = p_b(x, y) = p_b(y, y)$;
(p_{b2}) $p_b(x, x) \leq p_b(x, y)$;
(p_{b3}) $p_b(x, y) = p_b(y, x)$;
(p_{b4}) $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.

The pair (Y, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (Y, p_b).

In the following definition, Mustafa et al. [116] modified Definition (6) in order to find that each partial b-metric p_b generates a b-metric d_{p_b}.

Definition 7 ([116]) Let Y be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : Y \times Y \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in Y$ the following conditions are satisfied:

(p_{b1}) $x = y$ iff $p_b(x, x) = p_b(x, y) = p_b(y, y)$;
(p_{b2}) $p_b(x, x) \leq p_b(x, y)$;
(p_{b3}) $p_b(x, y) = p_b(y, x)$;
(p_{b4}) $p_b(x, y) \leq s(p_b(x, z) + p_b(z, y) - p_b(z, z)) + (\frac{1-s}{2})(p_b(x, x) + p_b(y, y))$.

The pair (Y, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (Y, p_b).

Example 16 ([166]) Let $Y = \mathbb{R}^+$, $q > 1$ be a constant and $p_b : Y \times Y \to \mathbb{R}^+$ be defined by

$$p_b(x, y) = [\max\{x, y\}]^q + |x - y|^q,$$

for all $x, y \in Y$. Then (Y, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1} > 1$, but it is neither a b-metric nor a partial metric space.

Remark 1 The class of partial b-metric spaces (Y, p_b) is effectively larger than the class of partial metric spaces, since a partial metric space is a special case of a partial b-metric space (Y, p_b) when $s = 1$. Also, the class of partial b-metric spaces (Y, p_b) is effectively larger than the class of b-metric spaces, since a b-metric space is a special case of a partial b-metric space (Y, p_b) when the self distance $p(x, x) = 0$.

Proposition 1 ([166]) *Let Y be a nonempty set, and let p be a partial metric and d be a b-metric with the coefficient $s \geq 1$ on Y. Then the function $p_b : Y \times Y \to [0, \infty)$ defined by $p_b(x, y) = p(x, y) + d(x, y)$ for all $x, y \in Y$, is a partial b-metric on Y with the coefficient s.*

Proposition 2 ([166]) *Let (Y, p) be a partial metric space and $q \geq 1$. Then (Y, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1}$, where p_b is defined by $p_b(x, y) = [p(x, y)]^q$.*

Proposition 3 ([116]) *Every partial b-metric p_b defines a b-metric d_{p_b}, where $d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, x) - p_b(y, y)$ for all $x, y \in Y$.*

Definition 8 ([116]) A sequence $\{x_n\}$ in a partial b-metric space (Y, p_b) is said to be:

1. x_n convergent to a point $x \in Y$ if $p_b(x, x) = \lim_{n\to\infty} p_b(x, x_n)$;
2. a p_b-Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n, x_m)$ exists (and is finite);
3. a partial b-metric space (Y, p_b) is said to be p_b-complete if every p_b-Cauchy sequence $\{x_n\}$ in Y p_b-converges to a point $x \in Y$, such that $p_b(x, x) = \lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n\to\infty} p_b(x, x_n)$.

Lemma 1 ([116]) *Let (Y, p_b) be a partial b-metric space. Then*

1. *A sequence $\{x_n\}$ is a p_b-Cauchy sequence in (Y, p_b) if and only if it is a b-Cauchy sequence in the b-metric space (Y, d_{p_b});*
2. *(Y, p_b) is p_b-complete if and only if the b-metric space (Y, d_{p_b}) is complete. Moreover, $\lim_{n\to\infty} d_{p_b}(x_n, x) = 0$ if and only if $p_b(x, x) = \lim_{n\to\infty} p_b(x_n, x) = \lim_{n,m\to\infty} p_b(x_n, x_m)$.*

For every $x \in Y$ and $\delta > 0$, take $\mathscr{B}_{p_b}(x, \delta) = \{y \in Y : p_b(x, y) < \delta + p_b(x, x)\}$ and $\mathscr{B} = \{\mathscr{B}_{p_b}(x, \delta) : x \in Y, \delta > 0\}$. Although $\mathscr{B}$ is not a base for any topology on Y, hence not a topology defined on Y. However $\mathscr{B}$ can be sub-base for some topology τ_{p_b} on Y, which is indeed T_0 but need not be T_1. For detailed description, we refer the readers to [62].

It may be noted that the limit of a convergent sequence in a partial b-metric spaces may not be unique.

3.2 b-Metric-Like Spaces

Matthews [111] initiated the notion of a partial metric space as a part of the study of denotational semantics of a dataflow network. In this space, the usual metric is replaced by a partial metric with an appealing property that the self-distance of any point of space may not be zero. Further, Matthews illustrated that the Banach contraction principle is applicable in a partial metric space and can be applied in program verifications. Furthermore O' Neill [133] generalized the concept of a partial metric space by acknowledging negative distances. The partial metric launched by O' Neill is called the dualistic partial metric. Heckmann [71] extended it by omitting the small self-distance axiom. The partial metric defined by Heckmann is called a weak partial metric. Hitzler and Seda [73] presented the notion of dislocated metric space and projected their generalization of the Banach–Caccioppoli's theorem. Hitzler and Seda's idea was to apply this theorem in order to obtain a unique supported model for acceptable logic programs. Also, many authors developed the fixed-point theory in the setting of dislocated metric spaces. In 2012 Amini-Harandi rediscovered the notion of dislocated metric spaces in [9], as a generalization of a partial metric spaces [112]. These spaces were called metric-like spaces. The hypothesis of the smallest self distance of partial metric was replaced with a weaker version of triangular inequality. After that, in [8], Alghamdi et al. launched b-metric-like spaces, which expands the notions of partial metric spaces, b-metric spaces, and metric-like spaces. Acknowledging this concept, many authors paid attention to this space and published many research articles. Further synthesis of these spaces can be obtained in noteworthy papers [8, 125, 126, 197, 198].

Definition 9 ([8]) Let Y be a nonempty set and $s \geq 1$ be a given real number. A function $\sigma_b : Y \times Y \to [0, \infty)$ is called a b-metric-like if for all $x, y, z \in Y$ the following conditions are fulfilled:

($\sigma_b 1$) $\sigma_b(x, y) = 0$ implies $x = y$;
($\sigma_b 2$) $\sigma_b(x, y) = \sigma_b(y, x)$;
($\sigma_b 3$) $\sigma_b(x, y) \leq s[\sigma_b(x, z) + \sigma_b(z, y)]$.
The pair (Y, σ_b) is called a b-metric-like space and the number $s \geq 1$ is called the coefficient of (Y, σ_b).

Example 17 ([8]) Let $Y = \mathbb{R}^+$ and the mapping $\sigma_b : Y \times Y \to \mathbb{R}^+$ be defined by

$$\sigma_b(x, y) = [\max\{x, y\}]^2,$$

for all $x, y \in Y$. Then (Y, σ_b) is a b-metric-like space with the coefficient $s = 2 > 1$, but it is neither a b-metric nor a metric-like space.

Remark 2 The class of b-metric-like spaces (Y, σ_b) is effectively more extensive than the class of metric-like spaces since a metric-like space is a special case of b-metric-like space (Y, p_b) when $s = 1$. Also, the class of b-metric-like spaces (Y, σ_b) is effectively more extensive than the class of b-metric spaces, since a b-metric space is a particular case of a b-metric-like space (Y, σ_b) when the self distance $\sigma_b(x, x) = 0$.

Each b-metric-like σ_b on Y generalizes a topology τ_{σ_b} on Y whose base is the family of open σ_b-balls $B\text{-}\sigma_b(x, \varepsilon) = \{y \in Y : |\sigma_b(x, y) - \sigma_b(x, x)| < \varepsilon\}$ for all $x \in Y$ and $\varepsilon > 0$.

Definition 10 ([8]) A sequence $\{x_n\}$ in a b-metric-like space (Y, σ_b) is said to be:

1. x_n-convergent to a point $x \in Y$ if $\sigma_b(x, x) = \lim_{n\to\infty} \sigma_b(x, x_n)$;
2. a σ_b-Cauchy sequence if $\lim_{n,m\to\infty} \sigma_b(x_n, x_m)$ exists (and is finite).

Definition 11 ([8]) A b-metric-like space (Y, σ_b) is said to be σ_b-complete if every σ_b-Cauchy sequence $\{x_n\}$ in Y, σ_b-converges to a point $x \in Y$, such that $\sigma_b(x, x) = \lim_{n,m\to\infty} \sigma_b(x_n, x_m) = \lim_{n\to\infty} \sigma_b(x, x_n)$.

Definition 12 ([8]) Suppose that (Y, σ_b) is a b-metric-like space. A mapping $J : Y \to J$ is said to be continuous at $x \in Y$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $J(\mathscr{B}_{\sigma_b}(x, \delta)) \subset \mathscr{B}_{\sigma_b}(Jx, \varepsilon)$. We say that J is continuous on Y if J is continuous at all $x \in Y$.

Lemma 2 ([8]) *Let $\{y_n\}$ be a sequence in a b-metric like space (Y, σ_b) such that*

$$\sigma_b(y_n, y_{n+1}) \leq \lambda \sigma_b(y_{n-1}, y_n),$$

for some λ, $0 < \lambda < \frac{1}{s}$ and each $n \in N$. Then $\{y_n\}$ is a Cauchy sequence in Y and $\lim_{n,m\to\infty} \sigma_b(y_n, y_m) = 0$.

Remark 3 ([8]) Let (Y, σ_b) be a b-metric-like space with constant $s \geq 1$. Then it is clear that

$$\sigma_b^s(x, y) = |2\sigma_b(x, y) - \sigma_b(x, x) - \sigma_b(y, y)|.$$

satisfies $\sigma_b^s(x, x) = 0$, for all $x \in Y$.

Remark 4 ([35]) Let (Y, σ_b) be a b-metric-like space and let $J : Y \to Y$ be a continuous mapping. Then $\lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x) \Rightarrow \lim_{n\to\infty} \sigma_b(Jx, Jx_n) = \sigma_b(x, x)$.

3.3 G-Metric Spaces

The first modification of a 2-metric was a D-metric which was introduced in 1984 by Dhage [45–47]. While for a 2-metric d, $d(x, y, z)$ can be thought as a generalization of the area of a triangle with vertices at x, y, z in $\mathbb{R}^2$, then, for a D-metric D, $D(x, y, z)$ can be treated as a generalization of the perimeter of this triangle.

But in 2003, Mustafa and Sims [119] stated some remarks concerning D-metric spaces and presented some examples which showed that many of the basic claims concerning the topological structure of D-spaces were incorrect, thus nullifying many of the results claimed for D-spaces. After that, in 2005, various concepts of open balls in D-metric spaces were studied in the case of certain D-metric spaces and many results in the literature on such balls were shown to be false by Naidu et al. [121]. In 2006, to overcome the flaws of D-metric spaces which were pointed out, Mustafa and Sims [118] introduced the notion of a G-metric space by modifying axioms of a D-metric. Mustafa provided many examples of G-metric spaces in [117] and developed some of their properties. For example, he proved that G-metric spaces are provided with a Hausdorff topology which allows us to consider, among other topological notions, convergent sequences, limits, Cauchy sequences, continuous mappings, completeness, and compactness. He also developed further topics in G-metric spaces such as the properties of ordinary metrics derived from a G-metric, and he investigated the properties of G-metrics derived from ordinary metrics.

Definition 13 ([118]) Let Y be a nonempty set and let $G : Y \times Y \times Y \to [0, \infty)$ be a function satisfying the following properties:

(G-1) $G(x, y, z) = 0$ if $x = y = z$,

(G-2) $G(x, x, y) > 0$, for all $x, y \in Y$ with $x \neq y$,

(G-3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in Y$ with $y \neq z$,

(G-4) $G(x, y, z)=G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables),

(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in Y$.

The function G is called a generalized or a G-metric on Y and the pair (Y, G) is called a G-metric space.

The previous properties may be easily interpreted in the setting of metric spaces. Let (Y, d) be a metric space and define $G : Y \times Y \times Y \to [0, \infty)$ by

$$G(x, y, z) = d(x, y) + d(x, z) + d(y, z), \quad for \ \ all \ \ x, y, z \in Y.$$

Then (Y, G) is a G-metric space. In this case, $G(x, y, z)$ can be interpreted as the perimeter of the triangle of vertices x, y and z. For example, $(G - 1)$ means that, with one point, we cannot have a positive perimeter, and $(G - 2)$ is equivalent to the fact that the distance between two different points cannot be zero. Furthermore, as the perimeter of a triangle cannot depend on the order in which we consider its vertices, we have $(G - 4)$ and $(G - 5)$ is an extension of the triangle inequality using a fourth vertex. Maybe, the most controversial axiom is $(G - 3)$ which has an obvious geometric interpretation: the length of an edge of a triangle is less than or equal to its semiperimeter, that is,

$$d(x, y) \leq \frac{d(x, y) + d(y, z) + d(z, x)}{2}$$

Example 18 If Y is a non-empty subset of $\mathbb{R}$, then the function $G : Y \times Y \times Y \to [0, \infty)$, given by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|, \quad \text{for all} \quad x, y, z \in Y,$$

is a G-metric on Y.

Example 19 Every non-empty set Y can be provided with the discrete G-metric, which is defined, for all $x, y, z \in Y$, by

$$G(x, y, z) = \begin{cases} 0, & \text{if} \ \ x = y = z; \\ 1, & \text{otherwise.} \end{cases}$$

Example 20 Let $Y \in [0, \infty)$ be the interval of non-negative real numbers and let G be defined by:

$$G(x, y, z) = \begin{cases} 0, & \text{if} \ \ x = y = z; \\ \max\{x, y, z\}, & \text{otherwise.} \end{cases}$$

Then G is a complete G-metric on Y.

Example 21 If G is a G-metric on Y, then $G' : Y \times Y \times Y \to [0, \infty)$, given by

$$G'(x, y, z) = \frac{G(x, y, z)}{1 + G(x, y, z)}, \quad for \ \ all \ \ x, y, z \in Y,$$

is another G-metric on Y.

Basic properties of G-metric spaces:

The following lemma can be obtained easily from the definition of a G-metric space.

Lemma 3 ([118]) *Let (Y, G) be a G-metric space. Then, for any $x, y, z, a \in Y$, the following properties hold.*
1. $G(x, y, y) \leq 2G(y, x, x)$.
2. $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$.
3. $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.
4. $|G(x, y, z) - G(x; a; a)| \leq max\{G(y, a, a), G(z, a, a)$.
5. If $G(x, y, z) = 0$, *then* $x = y = z$.
6. $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.
7. $G(x, y, z) \leq \frac{2}{3}[G(x, y, a) + G(x, a, z) + G(a, y, z)]$.

Relationship between metrics and G-metrics:
Every metric on Y induces G-metrics on Y in different ways.

Lemma 4 ([118]) *If* (Y, d) *is a metric space, then the functions* G_m^d, $G_s^d : Y^3 \to [0, \infty)$ *defined by*

$$G_m^d(x, y, z) = max\{d(x, y), d(y, z), d(z, x)\},$$

and

$$G_s^d(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

for all $x, y, z \in Y$, *are* G*-metrics on* Y. *Furthermore,*

$$G_m^d(x, y, z) \le G_s^d(x, y, z) \le 3G_m^d(x, y, z)$$

for all $x, y, z \in Y$.

Conversely, a G-metric on Y also induces some metrics on Y.

Lemma 5 *If* (Y, G) *is a* G*-metric space, then the functions* $d_m^G, d_s^G : Y^2 \to [0, \infty)$ *defined by*

$$d_m^G(x, y) = max\{G(x, y, y), G(y, x, x)\}$$

and

$$d_s^G(x; y) = G(x, y, y), G(y, x, x)$$

for all $x, y \in Y$, *are metrics on* Y.

Symmetric G**-metric spaces:** A G-metric space (Y, G) is called symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in Y$.

The mappings given in Examples 18, 19 and 20 are symmetric G-metrics. Moreover, G_m^d and G_s^d are symmetric G-metrics on Y. In fact,

$$G_s^d(x, y, y) = 2G_m^d(x, y, y) = 2d(x, y), \qquad for \;\; all \;\; x, y \in Y$$

Topology of a G**-metric space:** Here, we state the canonical Hausdorff topology of a G-metric space.

Definition 14 The open ball of center $x \in Y$ and radius $r > 0$ in a G-metric space (Y, G) is the subset

$$\mathscr{B}_G(x, r) = \{y \in Y : G(x, y, y) < r\}.$$

Similarly, the closed ball of center $x \in Y$ and radius $r > 0$ in a G-metric space (Y, G) is the subset

$$\bar{\mathscr{B}}_G(x, r) = \{y \in Y : G(x, y, y) \le r\}.$$

We immediately conclude that $x \in \mathscr{B}_G(x,r) \subseteq \bar{\mathscr{B}}_G(x,r)$.

Convergent and Cauchy Sequences:

Definition 15 ([118]) Let (Y, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of Y. We say that the sequence $\{x_n\}$ is G-convergent to $x \in Y$ if

$$\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0,$$

that is, for any $\varepsilon > 0$, there exists $N \in \mathscr{N}$ such that

$$G(x, x_n, x_m) < \varepsilon,$$

for all $m, n > N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim\limits_{n,m\to+\infty} x_n = x$.

Proposition 4 ([118]) *Let (Y, G) be a G-metric space. Then the following statements are equivalent:*

(1) $\{x_n\}$ *is G-convergent to x;*
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$*;*
(3) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$*;*
(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$.

Definition 16 ([118]) Let (Y, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is $N \in \mathscr{N}$ such that

$$G(x_n, x_m, x_l) < \varepsilon,$$

for all $n, m, l \geq N$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 5 ([118]) *Let (Y, G) be a G-metric space. Then the following statements are equivalent:*

(1) $\{x_n\}$ *is G-Cauchy;*
(2) *For every* $\varepsilon > 0$, *there is* $N \in \mathscr{N}$ *such that* $G(x_n, x_n, x_m) < \varepsilon$ *for all* $n, m \geq N$.

Definition 17 ([118]) A G-metric space (Y, G) is called G-complete if every G-Cauchy sequence is G-convergent in (Y, G).

3.4 Complex-Valued Metric Spaces

In 2011, Azam et al. [17] introduced and studied complex-valued metric spaces and established some fixed-point results for maps satisfying a rational inequality. The idea of complex-valued metric spaces is simply to replace $\mathbb{R}$ with the usual order by C with a certain order. The authors asserted that the idea of complex-valued metric spaces can be exploited to define complex normed metric spaces and complex-valued Hilbert spaces. The idea of complex-valued metric spaces is intended to define rational expressions which are not meaningful in cone metric spaces and thus many such results of analysis can not be generalized to cone metric spaces but to complex-valued metric spaces.

After the establishment of complex-valued metric spaces, Rouzkard and Imdad [151] established some common fixed-point theorems satisfying certain rational expressions in these spaces to generalize the result of Azam, Fisher, and Khan [17]. Subsequently, Sintunavarat and Kumam [173] obtained common fixed-point results by replacing the constant of contractive condition to control functions. Sitthikul and Seajung [174] established some fixed-point results by generalizing the contractive conditions in the context of complex-valued metric spaces. Recently, Sintunavarat, Cho, and Kumam [172] applied it to obtain the common solution of Urysohn integral equations. Very recently, Singh et al. [169] established certain fixed-point theorems which generalized numerous preceding results in the setting of complex-valued metric spaces. Furthermore, Azam et al. [6] proved some common fixed-point results for multi-valued mappings in complex-valued metric space. Afterward, Joshi et al. [86] generalized the results of Azam et al. [6] in the same setting.

Definition 18 ([17]) Let the set C of complex numbers be equipped with the partial order $' \precsim '$ defined by $x \precsim y$ if and only if $Re\ x \leq Re\ y$ and $Im\ x \leq Im\ y$. Let Y be a non-empty set. Consider a mapping $d : Y \times Y \to C$ is such that

(CM1) $0 \precsim d(x, y)$ for all $x, y \in Y$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(CM2) $d(x, y) = d(y, x)$ for all $x, y \in Y$;
(CM3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in Y$.

Then d is called a complex-valued metric on Y and the pair (Y, d) is called a complex-valued metric space.

Example 22 Let $Y = C$ be a set of complex number. Define $d : C \times C \to C$ by

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (C, d) is a complex-valued metric space.

Example 23 Let $Y = C$. Define a mapping $d : Y \times Y \to C$ by $d(z_1, z_2) = e^{ik}|z_1 - z_2|$, where $k \in [0, \frac{\pi}{2}]$. Then (Y, d) is a complex-valued metric space.

Definition 19 ([17]) Suppose that (Y, d) is a complex-valued metric space.

1. We say that a sequence $\{x_n\}$ is a Cauchy sequence if for every $0 \prec c \in C$ there exists an integer N such that $d(x_n, x_m) \prec c$ for all $n, m \geq N$.
2. We say that $\{x_n\}$ converges to an element $x \in Y$ if for every $0 \prec c \in C$ there exists an integer N such that $d(x_n, x) \prec c$ for all $n \geq N$. In this case, we write $x_n \xrightarrow{d} x$.

3. We say that (Y, d) is complete if every Cauchy sequence in Y converges to a point in Y.

Lemma 6 ([17]) *Let (Y, d) be a complex-valued metric space and let $\{x_n\}$ be a sequence in Y. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 7 ([17]) *Let (Y, d) be a complex-valued metric space and let $\{x_n\}$ be a sequence in Y. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

3.5 Rectangular b-Metric Spaces

In 2000, Branciari [31] introduced the concept of a rectangular (generalized) metric space where the sum at the right-hand side of triangle inequality of a metric space was replaced by another inequality which involves four (or more) points instead of three and proved Banach's contraction principle in such spaces. Afterward, in 2015, George et al. [64] launched rectangular b-metric spaces, not necessarily Hausdorff, and claimed these spaces to be the generalization of other known spaces. Acknowledging the concept of George et al. [64], many authors paid attention to these spaces and published many research articles along with some applications in engineering and sciences. On the other hand, the same concept was introduced in [147] (independently of [64]) where the authors obtained some remarkable results dealing with rational-type contractions and almost generalized weakly contractive mappings. A detailed description of these spaces can be found in [49, 50, 64, 87, 113, 147].

Definition 20 ([64]) A rectangular b-metric on a nonempty set Y is a mapping $b_r : Y \times Y \to [0, \infty)$ with $s \geq 1$ satisfying the following conditions:

$(b_r M1)$ $b_r(x, y) = 0$ if and only if $x = y$, for all $x, y \in Y$;
$(b_r M2)$ $b_r(x, y) = b_r(y, x)$, for all $x, y \in Y$;
$(r_r M3)$ $b_r(x, y) \leq s\,[b_r(x, u) + b_r(u, v) + b_r(v, y)]$, for all $x, y \in Y$ and all distinct points $a, b \in Y\backslash_{\{u,v\}}$.

The pair (Y, b_r) is called a rectangular b-metric space with coefficient s on Y.

Note that every metric space is a rectangular metric space and every rectangular metric space is a rectangular b-metric space (with coefficient $s = 1$). However, the converse of this implication is not true in general.

Example 24 ([64]) Let $Y = \mathbb{N}$ be endowed with rectangular b-metric b_r defined by:

$$b_r(x, y) = \begin{cases} 0 & ; \quad x = y, \\ m & ; \quad x \quad \text{or y} \notin \{1, 2\} \text{ and } \text{x} \neq \text{y}, \\ 4m & ; \quad x, y \in \{1, 2\} \text{ and } x \neq y, \end{cases}$$

where $m > 0$ is a constant. Then (Y, b_r) is a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$, but (Y, b_r) is not a rectangular metric space, since $b_r(1, 2) = 4 > 3 = b_r(1, 3) + b_r(3, 4) + b_r(4, 2)$.

Let (Y, b_r) be a rectangular b-metric space. Let $x \in Y$ and $\alpha > 0$, then the open ball with centre x and radius α is defined by

$$\mathscr{B}_\alpha(x) = \{y \in Y : b_r(x, y) < \alpha\}.$$

The open balls in rectangular b-metric spaces are not necessarily open.

For basic definitions like convergence, Cauchy sequence, completeness, etc., in these spaces, we refer [64].

The following important remark from [50] is worth mentioning.

Remark 5 (i) Every metric space and every rectangular metric space is rectangular b-metric space;
(ii) The limit of a sequence in a rectangular b-metric space is not unique;
(iii) Every convergent sequence in a rectangular b-metric space is not necessarily a Cauchy.

Very recently, in 2019, Dung [53] enunciated a metrization theorem on rectangular b-metric spaces. He proved a necessary and sufficient for a rectangular b-metric space to be metrizable by introducing the following theorem.

Theorem 12 ([53]) *Let (Y, b_r, s) be a rectangular b-metric space such that the limit of convergent sequence is unique. Then*
(i) There exists a metric d on Y such that $\lim_{t\to\infty} x_t = x$ in (Y, b_r, s) if and only if $\lim_{t\to\infty} x_t = x$ in (Y, d). In particular, (Y, b_r, s) is metrizable by the metric d.

(ii) A sequence $\{x_t\}$ is Cauchy in (Y, b_r, s) if and only if it is Cauchy in (Y, d). In particular, (Y, b_r, s) is complete if and only if (Y, d) is complete.

3.6 Graphical Rectangular b-Metric Spaces

In recent studies, graph theory engages an influential role, especially for metric fixed-point theory in numerous aspects, and has been at the center of vigorous research activity. In the last couple of years, some manuscripts in the framework of graphical metric spaces, in the area of fixed-point theory have emerged significantly. The condition of possessing a graph on the underlying space becomes more interesting when the graph has a mathematical or practical significance (for instance, possible ways of reaching an airport through different routes, tracing a hospital in a crowded area, etc.). In this direction, here we recall some notable works done in the pioneering articles [36, 74, 76, 140, 167, 186, 187, 190] based on various applications. While working on the graph structure, note that we can move from one point to another point through a specified edge (directed edge) of the directed graph. If we repeat these movements for a finite number of times, then we talk about a path between two points (vertices). This distinctive feature allows us to instigate a new version of the triangle inequality for graphical metric spaces only on those elements which are related to each other under the underlying graph structure.

Most recently, in 2019, Younis et al. exhibited an innovative structure, called graphical rectangular b-metric spaces along with various topological properties, which extends the concepts given in [36, 64]. Under this graph structure, we can consider contractive conditions that only hold for binary relations that are not necessarily transitive. If the contractive condition is strong enough, transitivity can be avoided. Another advantage of graphical metric spaces is the fact that we have only to check that the triangle inequality is satisfied for all three points placed in a directed path of the graph. For a detailed summary of this structure, we refer the reader to the noteworthy and informative article [72].

Recently, Younis et al. [186, 187, 190] took Banach's contraction principle for giving an association with graph theory, and with this amalgamation, they established some remarkable results in existing theory. For a non-void set $\mathscr{S}$, Δ denotes the diagonal $\mathscr{S} \times \mathscr{S}$. The notions $V(\mathscr{M})$ and $E(\mathscr{M})$, respectively, denote the set of all vertices and edges for a digraph $\mathscr{M}$, where $E(\mathscr{M})$ accommodates all the loops of $\mathscr{M}$ (i.e., $\Delta \subset E(\mathscr{M})$). We represent the digraph by $\mathscr{M} = (V(\mathscr{M}), E(\mathscr{M}))$. By $\mathscr{M}^{-1}$, we represent the digraph $\mathscr{M}$ with reversed edges. Additionally the digraph $\mathscr{M}$ with symmetric edges is denoted by $\breve{\mathscr{M}}$.

Unambiguously, we write $E(\breve{\mathscr{M}}) := E(\mathscr{M}^{-1}) \cup E(\mathscr{M})$.

A sequence $\{\varkappa_j\}_{j=0}^{h}$ consisting of $(h+1)$ vertices with $r = \varkappa_0$, $r' = \varkappa_h$ and $(\varkappa_{j-1}, \varkappa_j) \in E(\mathscr{M})$ for $j = 1, 2, \ldots, h$ is called a directed path or simply a path. We say $\mathscr{M}$ to be a connected graph if there is a path between any of its vertices. However if $\mathscr{M}$ is undirected and there endures a path joining every two of its vertices, then call $\mathscr{M}$ to be a weakly connected graph. Moreover, a graph $\mathscr{M}^* = (V(\mathscr{M}^*), E(\mathscr{M}^*))$ is termed as a subgraph of $\mathscr{M} = (V(\mathscr{M}), E(\mathscr{M}))$ if $V(\mathscr{M}) \supseteq V(\mathscr{M}^*)$ and $E(\mathscr{M}) \supseteq E(\mathscr{M}^*)$.

Following are the important notions that will be carried out throughout the manuscript.

(i) $[r]^t_{\mathscr{M}} := \{r' \in \mathscr{S} : \exists$ a path directing from r to r' with length $t\}$.
(ii) $(r\mathscr{R}r')_{\mathscr{M}}$ denotes the relation $\mathscr{R}$ describing that there exists a path starting from r to r'.
(iii) If a point $\boldsymbol{r}$ lies on the path $(r\mathscr{R}r')_{\mathscr{M}}$, we denote it by $\boldsymbol{r} \in (r\mathscr{R}r')_{\mathscr{M}}$.
(iv) A sequence $\{\varkappa_h\} \in \mathscr{S}$ is called $\mathscr{M}$-term wise connected ($\mathscr{M}$-twc) if $(\varkappa_h \mathscr{R} \varkappa_{h+1})_{\mathscr{M}}$ for all $h \in \mathbb{N}$.

Using the applicative approach, Younis et al. [186, 187, 190] employed some fixed-point results based on graph structure to find the solutions of some nonlinear problems describing some physical models from science and engineering. They introduced the notion of graphical rectangular b-metric spaces [187] as an extension and generalization of b-metric spaces, graphical metric spaces, and rectangular metric spaces by the amalgamation of graph theory with metric spaces.

Following is the formal definition of graphical rectangular b-metric spaces.

Definition 21 ([187]) Let $\mathscr{M}$ be a graph endowing a non-void set $\mathscr{S}$ and let $\mathscr{M}_{b_r} : \mathscr{S} \times \mathscr{S} \to [0, \infty[$ be such that for $s \geq 1$, the following conditions are asserted:

($\mathscr{M}_1$) $\mathscr{M}_{b_r}(r, r') = 0 \iff r = r'$;
($\mathscr{M}_2$) $\mathscr{M}_{b_r}(r, r') = \mathscr{M}_{b_r}(r, r')$ for all $r, r' \in \mathscr{S}$;
($\mathscr{M}_3$) For $(r\mathscr{R}r')_{\mathscr{M}}$, $p, q \in (r\mathscr{R}r')_{\mathscr{M}}$, we have

$$\mathscr{M}_{b_r}(r, r') \leq s\left[\mathscr{M}_{b_r}(r, p) + \mathscr{M}_{b_r}(p, q) + \mathscr{M}_{b_r}(q, r')\right]$$

for all $r, r' \in \mathscr{S}$ and all distinct points (distinct from r and r') $p, q \in \mathscr{S}$.

Then the doublet $(\mathscr{M}_{b_r}, \mathscr{S})$ is termed as a graphical generalized b-metric space or graphical rectangular b-metric space ($gr_b ms$) with coefficient $s \geq 1$.

Remark 6 ([187])

(i) It may be noted that graphical rectangular b-metric spaces generalize graphical rectangular metric spaces because a graphical rectangular b-metric space reduces to a graphical rectangular metric space for $s = 1$.
(ii) A graphical b-metric space with coefficient s is a graphical rectangular b-metric space with coefficient s^2.
However, the converse need not be true in general.

Definition 22 ([187]) Let $\{\varkappa_h\}$ be a sequence in a graphical rectangular b-metric space:

(i) $\{\varkappa_h\}$ is said to be convergent iff there exists $r \in \mathscr{S}$ such that $\mathscr{M}_{b_r}(\varkappa_h, \varkappa) \to 0$ whenever $h \to \infty$;

(ii) $\{\varkappa_h\}$ is said to be Cauchy iff $\mathscr{M}_{b_r}(\varkappa_h, \varkappa_{h'}) \to 0$ as $h, h' \to \infty$. In other words, $\{\varkappa_h\}$ is a Cauchy sequence, if given $\mu > 0$, there exists $h_0 \in \mathbb{N}$ such that $\mathscr{M}_{b_r}(\varkappa_h, \varkappa_{h'}) < \mu, \forall\, h, h' > h_0$.

Example 25 ([187]) Let $\mathscr{M}_{b_r}$ be the rectangular metric endowing the set $\mathscr{S} = \{e, d, c, b, a\}$. Define the metric by the following:

$$\mathscr{M}_{b_r}(r, r') = \begin{cases} 0 & , \quad r = r', \\ \frac{\sqrt{\beta}}{5} & , \quad r \; or \; r' \notin \{b, a\} \; and \; r \neq r', \\ 3\sqrt{\beta} & , \quad r, r' \in \{b, a\} \; and \; r \neq r', \end{cases}$$

with $\beta > 0$. Then, $(\mathscr{S}, \mathscr{M}_{b_r})$ is a graphical rectangular b-metric space with coefficient $s = 5$ equipped with the graph $\mathscr{M}$, where $\mathscr{S} = V(\mathscr{M})$ and $E(\mathscr{M})$ describes all the edges shown in the adjacent figure (Fig. 12).

Definition 23 ([187]) Let $(\mathscr{S}, \mathscr{M}_{b_r})$ be a graphical rectangular b-metric space. An open ball with center $r \in \mathscr{S}$ and radius $\mu > 0$ is defined by

$$\mathscr{B}_{\mathscr{M}_{b_r}}(r, \mu) = \{r' \in \mathscr{S} : (r\mathscr{R}r')_{\mathscr{M}_{b_r}}, \mathscr{M}_{b_r}(r, r') < \mu\}.$$

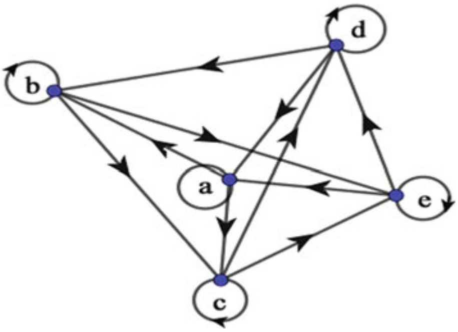

Fig. 12 Graph $\mathscr{M}$ representing graphical rectangular b-metric space $(\mathscr{S}, \mathscr{M}_{b_r}, s)$

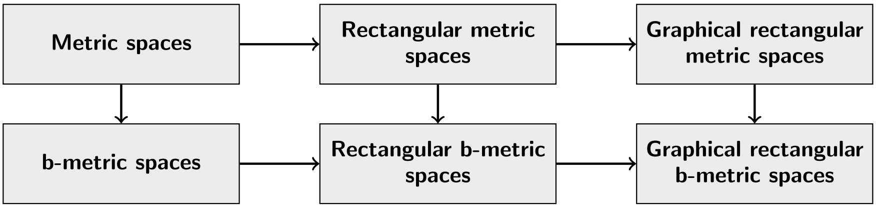

Fig. 13 Flow diagram showing relationship of various metric spaces

Remark 7 ([187]) Let $(\mathscr{S}, \mathscr{M}_{b_r})$ be a graphical rectangular b-metric space and $\mathscr{M} = (V(\mathscr{M}), E(\mathscr{M})$ be the graph related to $(\mathscr{S}, \mathscr{M}_{b_r})$.

- Every rectangular b-metric space is a graphical rectangular b-metric space.
- A graphical rectangular metric space is not necessarily a rectangular metric space.
- A graphical rectangular b-metric space is not necessarily a graphical rectangular metric space.
- A graphical rectangular b-metric can quite often be obtained from an ordered rectangular b-metric space.

The following flow diagram presents a clear picture of the aforesaid findings along with the results presented in the informative papers [36, 64, 147], where arrows describe inclusions but reverse inclusions are not true in general (Fig. 13).

There are hundreds of generalizations of contraction mappings and metric spaces, and comprehensive literature can be found on this topic. It is quite natural that there have been several attempts to extend these results, concerning metric fixed-point theory, to a more general setting. However, it is a matter of concern whether these generalizations and extensions produce new and real results.

Research and development is a non-stop process. For any research to be carried out, there is always a possibility for better chances of improvement and many approaches can be disclosed for further work.

Every thing can be optimized

4 Conclusions

The presented article gives a comprehensive outlook of the famous results based on metric fixed-point theory. Illustrative non-trivial examples, along with graphical representation, are enunciated to understand the nature of inequalities of the historical results in the context of metric spaces. Moreover, the latest developments on the said theory are discussed with comprehensive literature. In the later part, a brief survey of the well-known generalizations of metric spaces is propounded with the latest ones, along with topological aspects, and metrization of some notable metric spaces are also discussed.

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Fixed Point Theory for Multi-valued Feng–Liu Operators in Vector-Valued Metric Spaces

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1 Introduction and Preliminary Considerations

We start this section by presenting several notations and some notions which are used throughout this chapter.

Let (X, d) be a metric space and $P(X)$ be the set of all nonempty subsets of X. We denote:

$$P_{cl}(X) := \{Y \in P(X) | \; Y \text{ is closed}\},$$

$$P_{b,cl}(X) := \{Y \in P(X) | \; Y \text{ is bounded and closed}\}.$$

We recall first the following notions:
(1) The distance from a point $x \in X$ to a set $Y \in P(X)$:

$$D_d(x, Y) := \inf\{d(x, y) \mid y \in Y\};$$

(2) The excess of Y over Z (where $Y, Z \in P(X)$):

$$e_d(Y, Z) := \sup\{D_d(y, Z), y \in Y\};$$

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(3) The Pompeiu–Hausdorff distance between two sets $Y, Z \in P(X)$:

$$H_d(Y, Z) = \max\{e_d(Y, Z), e_d(Z, Y)\}.$$

Notice that we will renounce to subscript d when the metric we are working is obvious.

It is well-known that H is a generalized metric in Luxemburg–Jung' sense on $P_{cl}(X)$ (i.e., $H(Y, Z) \in \mathbb{R}_+ \cup \{\infty\}$) and becomes a classical metric on $P_{b,cl}(X)$.

For $u \in X$ and $r > 0$, we denote by

$$\tilde{B}(u; r) := \{z \in X : d(z, u) \leq r\}$$

the closed ball centered in u with radius r.

If X is a nonempty set and $F : X \to P(X)$ is a multi-valued operator, then $x \in X$ is called a fixed point for F if $x \in F(x)$. The set

$$Fix(F) := \{x \in X |\ x \in F(x)\}$$

is the fixed point set of F, while by

$$Graph(F) := \{(x, y) \in X \times X |\ y \in F(x)\}$$

the graph of the multi-valued operator F.

The following property of the multi-valued operators with closed graph is well-known.

Lemma 1 *Let (X, d) be a metric space and $F : X \to P(X)$ be a multi-valued operator such that $Graph(F)$ is closed in $X \times X$. Then $F(x) \in P_{cl}(X)$, for every $x \in X$.*

Remark 1 If X is a nonempty set and $F : X \to P(X)$, then the sequence $(x_n)_{n\in\mathbb{N}}$ satisfying

$$x_0 \in X,\ x_{n+1} \in F(x_n),\ \text{for each } n \in \mathbb{N}$$

is called an iterative sequence of Picard type for F starting from $x_0 \in X$.

In 1969, S. B. Nadler Jr. published (see [9]) the first extension, to the case of a multi-valued operator in a complete metric space, of the classical Banach–Caccioppoli Contraction Principle. His result is as follows.

Theorem 1 *Let (X, d) be a complete metric space and $x_0 \in X$ be arbitrary. Let $F : X \to P_{b,cl}(X)$ be a multi-valued operator for which there exists $\alpha \in]0, 1[$ such that*

$$H(F(x_1), F(x_2)) \leq \alpha d(x_1, x_2),\ \textit{for every pair } (x_1, x_2) \in X \times X. \tag{1}$$

Then there exists an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard type for F starting from x_0 such that $\{x_n\}_{n\in\mathbb{N}}$ converges to a fixed point of F.

Notice that a multi-valued operator satisfying the above condition (1) is usually called a multi-valued α- contraction.

One year later, H. Covitz and S. B. Nadler Jr. observed that one can remove the boundedness assumption of the values of the multi-valued operator. Their result was published in 1970 (see Corollary 3 in [3]) and is generally known as Multi-valued Contraction Principle (MCP). We recall it here.

Theorem 2 *Let (X, d) be a complete metric space and let $x_0 \in X$ be arbitrary. If $F : X \to P_{cl}(X)$ is a multi-valued α-contraction, then there exists an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard type for F starting from x_0 such that $\{x_n\}_{n\in\mathbb{N}}$ converges to a fixed point of F.*

As a matter of fact, from the proof of the above theorem we get that for each $(x_0, x_1) \in Graph(F)$ there exists an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard type for F starting from (x_0, x_1) which converges to a fixed point of F. This remark gives rise to the following general concepts; see, e.g., [13, 22, 23].

Definition 1 Let (X, d) be a metric space. Then $F : X \to P(X)$ is called a multi-valued weakly Picard operator if for each $(x, y) \in Graph(F)$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that

(i) $x_0 = x,\ x_1 = y$;
(ii) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$;
(iii) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and its limit $x^*(x, y)$ is a fixed point of F.

We remark that a sequence satisfying (i) and (ii) is an iterative sequence of Picard type for F starting from (x_0, x_1).

Definition 2 Let (X, d) be a metric space and $F : X \to P(X)$ be a multi-valued weakly Picard operator. We define the multi-valued operator $F^\infty : Graph(F) \to P(Fix(F))$ by the formula $F^\infty(x, y) = \{x^* \in Fix(F) \mid$ there exists an iterative sequence of Picard type for F starting from (x, y) which converges to $x^*\}$.

Another important concept is given by the following definition.

Definition 3 Let (X, d) be a metric space and $F : X \to P(X)$ a multi-valued weakly Picard operator. Then, F is a c-multi-valued weakly Picard operator if $c > 0$ and there exists a selection f^∞ of F^∞ (i.e., $f^\infty(x, y) \in F^\infty(x, y)$, for every $(x, y) \in Graph(F)$) such that

$$d(x, f^\infty(x, y)) \le c d(x, y), \text{ for all } (x, y) \in Graph(F).$$

There are several generalizations of the MCP. One of these generalizations is the Multi-valued Graph Contraction Principle.

Definition 4 Let (X, d) be a metric space and $F : X \to P(X)$ be a multi-valued operator. Then, F is called a multi-valued graph α-contraction if there exists $\alpha \in]0, 1[$ such that

$$H(F(x_1), F(x_2)) \le \alpha d(x_1, x_2), \text{ for every pair } (x_1, x_2) \in Graph(F). \quad (2)$$

It is easy to see that any multi-valued α-contraction is a multi-valued graph α-contraction, but not reversely, see [18]. The Multi-valued Graph Contraction Principle was proved in [18].

Theorem 3 *Let (X, d) be a complete metric space and $F : X \to P(X)$ be a multi-valued graph α-contraction such that $Graph(F)$ is a closed set in $X \times X$. Then, the following conclusions hold:*

(a) there exists $x^* \in X$ such that $x^* \in Fix(F^n)$, for each $n \in \mathbb{N}^*$;

(b) for each $(x, y) \in Graph(F)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of Picard type for F starting from (x, y) which converges to a fixed point of F;

(c) there exists a selection $f^\infty : Graph(F) \to Fix(F)$ of F^∞ such that

$$d(x, f^\infty(x, y)) \leq \frac{1}{1-\alpha} d(x, y), \text{ for all } (x, y) \in Graph(F).$$

Using the Definitions 1–3, the above Multi-valued Graph Contraction Principle can be stated as follows: If (X, d) is a complete metric space and $F : X \to P(X)$ is a multi-valued graph α-contraction with closed graph, then F is a $\frac{1}{1-\alpha}$-multi-valued weakly Picard operator.

For related results, examples and applications of the fixed point theory for the multi-valued graph contraction, see [18].

Another consistent extension of the MCP was given by Feng and Liu in 2006 (see [5]), as follows.

Definition 5 Let (X, d) be a metric space, $F : X \to P(X)$ be a multi-valued operator, $\beta \in]0, 1[$ and $x \in X$. Consider the set

$$I_\beta^x := \{y \in F(x) : \beta d(x, y) \leq D(x, F(x))\}.$$

Then, F is called a multi-valued α-contraction of the Feng–Liu type if there exists $\alpha \in]0, \beta[$, such that for each $x \in X$ there exists $y \in I_\beta^x$ satisfying the following relation:

$$D(y, F(y)) \leq \alpha d(x, y).$$

The main result of Feng and Liu in [5] is the following theorem.

Theorem 4 *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multi-valued α-contraction of the Feng–Liu type. Suppose that the functional $g : X \to \mathbb{R}_+$ defined by $g(x) = D(x, F(x))$ is lower semi-continuous. Then, $Fix(F) \neq \emptyset$.*

Notice that any multi-valued α-contraction is a multi-valued graph α-contraction and any multi-valued graph α-contraction is a multi-valued α-contraction of the Feng–Liu type.

In this work, we will present an extension of the fixed point result of Feng and Liu for the case of a set X endowed with a vector-valued metric in the sense of Perov.

Existence, localization, data dependence and different kinds of stability properties of a fixed point inclusion with a generalized multi-valued Feng–Liu operator are presented. An extension to the so-called multi-valued contractions of the Feng–Liu–Subrahmanyan type is also considered. Some applications are suggested. For the role and the importance of vector-valued metrics of Perov type in fixed point theory, see, e.g., [2, 4, 15, 21]. For complementary results, see [1, 6–8, 16, 17]. For a synthesis on metric fixed point theory for single-valued and multi-valued operators, see [24].

2 Vector-Valued Metric Spaces and Fixed Point Results

2.1 Vector-Valued Metric Spaces of Perov Type

Let us recall the concept of vector-valued metric in the sense of Perov. For this purpose, we consider the following notations.

If $x, y \in \mathbb{R}^m$, $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, then, by definition

$$x \preceq y \text{ if and only if } x_i \leq y_i, \text{ for each } i \in \{1, 2, \ldots, m\}.$$

Throughout this work, we will make an identification between row and column vectors in $\mathbb{R}^m$.

We can now recall the concept of vector-valued metric in the sense of Perov; see [11]. We say that (X, d) is a vector-valued metric space if X is a nonempty set and $d : X \times X \to \mathbb{R}^m_+$ satisfies all the axioms of the usual metric, where the inequalities from the axioms of the metric are in the above-mentioned sense.

We denote by $M_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by I_m the identity $m \times m$ matrix and by O_m the null $m \times m$ matrix.

Definition 6 We will say, by definition, that a matrix $L \in M_{m,m}(\mathbb{R}_+)$ is convergent to zero if $L^n \to O_m$ as $n \to \infty$.

The following result will be important for our considerations (see, e.g., [29]).

Theorem 5 *Let $L \in M_{m,m}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) $L^n \to O_m$ *as* $n \to \infty$;
(ii) The spectral radius $\rho(L)$ of L is strictly less than 1, i.e., all the eigenvalues of L are in the open unit disk;
(iii) The matrix $(I_m - L)$ is nonsingular and

$$(I_m - L)^{-1} = I_m + L + \cdots + L^n + \cdots ; \tag{3}$$

(iv) The matrix $(I_m - L)$ is nonsingular and $(I_m - L)^{-1}$ has nonnegative elements.

2.2 Fixed Point Theorems

Using the vector-valued metric space context, Perov (see [10, 11]) proved the following fixed point result for single-valued operators.

Theorem 6 *Let (X, d) be a complete vector-valued metric space and let $f : X \to X$ be an L-contraction, i.e., the matrix $L \in M_{m,m}(\mathbb{R}_+)$ converges to zero and*

$$d(f(x), f(y)) \preceq Ld(x, y), \text{ for all } x, y \in X.$$

Then:

(1) *$Fix(f) = \{x^*\}$, i.e., there exists a unique solution $x^* \in X$ of the fixed point equation $x = f(x)$;*

(2) *The sequence $(x_n)_{n\in\mathbb{N}}$, $x_n := f^n(x_0)$ of Picard iterates of f starting from any $x_0 \in X$ is convergent to x^*;*

(3) *The following estimation holds:*

$$d\left(x_n, x^*\right) \preceq L^n (I_m - L)^{-1} d(x_0, x_1), \text{ for every } n \in \mathbb{N}. \tag{4}$$

We will consider now the case of multi-valued operators.

We suppose that

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_m(x, y) \end{pmatrix}, \text{ for } x, y \in X.$$

We denote by

$$D(x, Y) := \begin{pmatrix} D_{d_1}(x, Y) \\ \cdots \\ D_{d_m}(x, Y) \end{pmatrix},$$

the vectorial distance from a point $x \in X$ to a set $Y \in P(X)$ and by

$$H(A, B) := \begin{pmatrix} H_{d_1}(A, B) \\ \cdots \\ H_{d_m}(A, B) \end{pmatrix},$$

the vectorial Pompeiu–Hausdorff distance on $P(X)$.

A multi-valued variant of Perov's Theorem was given in [15].

Theorem 7 *Let (X, d) be a complete vector-valued metric space and let $F : X \to P_{cl}(X)$ be a multi-valued L-contraction, i.e., $L \in M_{m,m}(\mathbb{R}_+)$ is convergent to zero and*

$$H(F(x), F(y)) \preceq Ld(x, y), \text{ for all } x, y \in X. \tag{5}$$

Then

(i) $Fix(F) \neq \emptyset$;
(ii) *For each* $(x, y) \in Graph(F)$, *there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *(with* $x_0 = x$, $x_1 = y$ *and* $x_{n+1} \in F(x_n)$, *for each* $n \in \mathbb{N}^*$*), such that* $(x_n)_{n\in\mathbb{N}}$ *is convergent to a fixed point* $x^* := x^*(x, y)$ *of* F, *and the following relation hold:*

$$d(x_n, x^*) \preceq L^n (I - L)^{-1} d(x_0, x_1), \text{ for each } n \in \mathbb{N}.$$

3 Main Results

In this section, we will recall first the main fixed point results for multi-valued contractions of Feng–Liu type in a complete vector-valued metric space. For details and related considerations, see [20].

We start this part by recalling the notion of multi-valued vectorial contraction of the Feng–Liu type.

Definition 7 Let (X, d) be a vector-valued metric space, $F : X \to P(X)$ be a multi-valued operator, $B \in M_{m,m}(\mathbb{R}_+)$ be a diagonal matrix with elements $b_1, \ldots, b_m \in]0, 1[$ and let $x \in X$. Consider the set

$$I_B^x := \{y \in F(x) : Bd(x, y) \preceq D(x, F(x))\}.$$

Then, F is called a multi-valued vectorial contraction of the Feng–Liu type if there exists a matrix $A \in M_{m,m}(\mathbb{R}_+)$, such that the matrix $B^{-1}A$ is convergent to zero, and for each $x \in X$ there is $y \in I_B^x$ satisfying the relation

$$D(y, F(y)) \preceq Ad(x, y).$$

The main existence and approximation result for a multi-valued vectorial contraction of the Feng–Liu type is the following theorem; see also [20].

Theorem 8 *Let* (X, d) *be a complete vector-valued metric space and* $F : X \to P(X)$ *be a multi-valued vectorial contraction of the Feng–Liu type. Suppose that* F *has closed graph. Then, for each* $x_0 \in X$, *there exists an iterative sequence* $(x_n)_{n\in\mathbb{N}}$ *of Picard type for* F *starting from* x_0 *with the following properties:*

(1) $(x_n)_{n\in\mathbb{N}}$ *converges to* $x^*(x_0) \in Fix(F)$;
(2) *the following apriori estimation holds:*

$$d(x_n, x^*(x_0)) \preceq (B^{-1}A)^n \left(I_m - B^{-1}A\right)^{-1} d(x_0, x_1), n \in \mathbb{N};$$

(3) the following retraction–displacement type condition holds:

$$d(x_0, x^*(x_0)) \preceq \left(I_m - B^{-1}A\right)^{-1} d(x_0, x_1) \preceq \left(I_m - B^{-1}A\right)^{-1} B^{-1}D(x_0, F(x_0)).$$

Proof Let $x_0 \in X$ be arbitrarily chosen. Then, there exists $x_1 \in I_B^{x_0}$ such that

$$D(x_1, F(x_1)) \preceq Ad(x_0, x_1).$$

For $x_1 \in X$, by the multi-valued vectorial contraction condition of the Feng–Liu type, there exists $x_2 \in I_B^{x_1}$, such that

$$D(x_2, F(x_2)) \preceq Ad(x_1, x_2).$$

Since $Bd(x_1, x_2) \preceq D(x_1, F(x_1))$, we obtain that

$$d(x_1, x_2) \preceq B^{-1}D(x_1, F(x_1)) \preceq B^{-1}Ad(x_0, x_1).$$

Let us denote $L := B^{-1}A$. Notice that $B^{-1}A = AB^{-1}$. By the above procedure, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X with the following properties:

(a) $x_{n+1} \in I_B^{x_n}$, for each $n \in \mathbb{N}$;
(b) $d(x_n, x_{n+1}) \preceq Ld(x_{n-1}, x_n) \preceq \cdots \preceq L^n d(x_0, x_1)$, for each $n \in \mathbb{N}^*$;
(c) $D(x_{n+1}, F(x_{n+1})) \preceq LD(x_n, F(x_n)) \preceq \cdots \preceq L^{n+1}D(x_0, F(x_0))$, for each $n \in \mathbb{N}$.

Then, by (b), the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d). Hence, it is convergent to an element $x^* := x^*(x_0) \in X$. Since $(x_n)_{n\in\mathbb{N}}$ is an iterative sequence of the Picard type of F starting from x_0 and F has a closed graph, it follows that $x^* \in Fix(F)$. Moreover, by the relation

$$d(x_n, x_{n+p}) \preceq L^n \left(I_m + L + \cdots L^{p-1}\right) d(x_0, x_1) \preceq L^n \left(I_m - L\right)^{-1} d(x_0, x_1),$$

letting $p \to \infty$, we obtain the following apriori estimation for the fixed point:

$$d(x_n, x^*(x_0)) \preceq L^n \left(I_m - L\right)^{-1} d(x_0, x_1), n \in \mathbb{N}.$$

Taking $n = 0$ in the above relation, we obtain the following retraction–displacement condition:

$$d(x_0, x^*(x_0)) \preceq (I_m - L)^{-1} d(x_0, x_1) \preceq (I_m - L)^{-1} B^{-1}D(x_0, F(x_0)).$$

The proof is complete. □

Notice that in the conditions of Definition 7 we have that $I_B^x \subset F(x)$. Thus, the following extension of the multi-valued graph contraction principle (see Theorem 3) can be obtained as a consequence of Theorem 8.

Corollary 1 *Let (X, d) be a complete metric space and $F : X \to P(X)$ be a multi-valued operator with closed graph. Suppose there exists matrix $A \in M_{m,m}(\mathbb{R}_+)$ convergent to zero, such that*

$$D(y, F(y)) \leq Ad(x, y), \text{ for all } (x, y) \in Graph(F).$$

Then $Fix(F) \neq \emptyset$.

We conclude this section with a strict fixed point theorem for a multi-valued vectorial contraction of the Feng–Liu type. Our next theorem extends a previous theorem in [1].

Theorem 9 *Let (X, d) be a complete metric space and $F : X \to P(X)$ be a multi-valued vectorial contraction condition of the Feng–Liu type with closed graph. Suppose that*

(i) $F(F(x)) \subset F(x)$, for each $x \in M$;

(ii) If $Y \in P_{cl}(X)$ with $F(Y) = Y$, then Y is a singleton.

Then, $Fix(F) = SFix(F) \neq \emptyset$.

Proof By Theorem 8, we have that $Fix(F) \neq \emptyset$. Let $x^* \in Fix(F)$. By the condition (i), we obtain that $F(x^*)$ is a fixed set for F. Hence, by (ii), we obtain that $F(x^*)$ is a singleton consisting only in $\{x^*\}$. Thus, $x^* \in SFix(F)$ and, as a consequence, $Fix(F) \subset SFix(F)$. Thus, the conclusion follows. □

We now present some stability concepts for the fixed point inclusion $x \in F(x)$ in the setting of a vector-valued metric space.

The concept of the Ulam–Hyers stability is presented in the next definition. For classical metric spaces see [12].

Definition 8 Let (X, d) be a vector-valued metric space and $F : X \to P(X)$ be a multi-valued operator. The fixed point inclusion $x \in F(x)$ is called Ulam–Hyers stable if there exists a matrix $C \in M_{m,m}(\mathbb{R}_+) \setminus \{O_m\}$, such that for every $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m)$ (with $\varepsilon_i > 0$ for each $i \in \{1, 2, \ldots, m\}$) and for each ε-fixed point $\tilde{x} \in X$ of F (i.e., $D(\tilde{x}, F(\tilde{x})) \preceq \varepsilon$), there exists $x^* \in Fix(F)$, such that

$$d(\tilde{x}, x^*) \preceq C\varepsilon.$$

The well-posedness of the fixed point inclusion $x \in F(x)$ in a vector-valued metric space is defined, as follows. For the well-posedness concept for single-valued operators in classical metric spaces see the papers of Reich and Zaslavski [25, 26]. For the multi-valued case, see also [14, 17].

Definition 9 Let (X, d) be a vector-valued metric space. Let $F : X \to P(X)$ be a multi-valued operator such that $Fix(F) \neq \emptyset$ and let $r : X \to Fix(F)$ be a set retraction. Then, the fixed point inclusion $x \in F(x)$ is called well-posed in the sense of Reich and Zaslavski if for each $x^* \in Fix(F)$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$, such that $\{D(y_n, F(y_n))\}_{n\in\mathbb{N}}$ converges to zero as $n \to \infty$; we have that

$$y_n \to x^* \text{ as } n \to \infty.$$

The data dependence property is given in our next definition. Our next theorem extends some previous theorems from [22, 23].

Definition 10 Let (X, d) be a vector-valued metric space and $F : X \to P(X)$ be a multi-valued operator. Let $G : X \to P(X)$ be a multi-valued operator satisfying the following conditions:

(i) $Fix(G) \neq \emptyset$;
(ii) There exists $\eta := (\eta_1, \dots, \eta_m)$ (with $\eta_i > 0$ for each $i \in \{1, 2, \dots, m\}$), such that $H(F(x), G(x)) \preceq \eta$, for all $x \in X$.

Then, the fixed point inclusion $x \in F(x)$ has the data dependence property if for each $u^* \in Fix(G)$ there exists $x^* \in Fix(F)$, such that

$$d(u^*, x^*) \preceq S\eta, \text{ for some matrix } S \in M_{m,m}\left(\mathbb{R}_+^*\right) \backslash \{O_m\}.$$

The notion of the Ostrowski stability property for a fixed point inclusion in the vector-valued metric space is now presented. For the case of classical metric spaces, see [17].

Definition 11 Let (X, d) be a vector-valued metric space. Let $F : X \to P(X)$ be a multi-valued operator such that $Fix(F) \neq \emptyset$ and let $r : X \to Fix(F)$ be a set retraction. Then, the fixed point inclusion $x \in F(x)$ is said to have the Ostrowski stability property if for each $x^* \in Fix(F)$ and for any sequence $\{z_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$, such that $\{D(z_{n+1}, F(z_n))\}_{n\in\mathbb{N}}$ converges to zero as $n \to \infty$; we have that

$$z_n \to x^* \text{ as } n \to \infty.$$

The following retraction–displacement type condition will be important for our next results.

Definition 12 Let (X, d) be a vector-valued metric space and let $F : X \to P(X)$ be a multi-valued operator such that $Fix(F) \neq \emptyset$. Then, we say that F satisfies the strong vectorial retraction–displacement condition if there exist a matrix $Q \in M_{m,m}\left(\mathbb{R}_+^*\right) \backslash \{O_m\}$ and a set retraction $r : X \to Fix(F)$, such that

$$d(x, r(x)) \preceq QD(x, F(x)), \text{ for all } x \in X. \tag{6}$$

An abstract result concerning the above-mentioned stability properties of a multi-valued operator is given as follows.

Theorem 10 *Let (X, d) be a vector-valued metric space and let $F : X \to P(X)$ be a multi-valued operator satisfying the strong vectorial retraction–displacement condition, such that $Fix(F) \neq \emptyset$. Then, we have the following conclusions:*

(1) the fixed point inclusion $x \in F(x)$ has the Ulam–Hyers stability property;

(2) the fixed point inclusion $x \in F(x)$ is well-posed in the sense of Reich and Zaslavski;

(3) the fixed point inclusion $x \in F(x)$ satisfies the data dependence property.

Proof Suppose that there exists a matrix $Q \in M_{m,m}\left(\mathbb{R}_+^*\right) \setminus \{O_m\}$ and a set retraction $r : X \to Fix(F)$, such that

$$d(x, r(x)) \preceq QD(x, F(x)), \text{ for all } x \in X.$$

In order to prove the Ulam–Hyers stability property, let us consider $\varepsilon := (\varepsilon_1, \dots, \varepsilon_m)$ (with $\varepsilon_i > 0$ for each $i \in \{1, 2, \dots, m\}$) and $\tilde{x} \in X$ such that $D(\tilde{x}, F(\tilde{x})) \preceq \varepsilon$. Then, by the strong vectorial retraction–displacement condition, we have

$$d(\tilde{x}, r(\tilde{x})) \preceq QD(\tilde{x}, F(\tilde{x})) \preceq Q\varepsilon.$$

Thus, the fixed point inclusion $x \in F(x)$ is Ulam–Hyers stable.

For the well-posedness property of the fixed point inclusion, let us consider the sequence $\{y_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$, such that the sequence $D(y_n, F(y_n))$ converges to zero as $n \to \infty$. Then, for each $n \in \mathbb{N}$, we have $r(y_n) = x^*$ and, again by the strong vectorial retraction–displacement condition, we conclude that

$$d(y_n, x^*) = d(y_n, r(y_n)) \preceq QD(y_n, F(y_n)) \to 0 \text{ as } n \to \infty.$$

Let us now prove the data dependence of the fixed point set. Let us consider a multi-valued operator $G : X \to P(X)$ to have the properties:

(i) $Fix(G) \neq \emptyset$;

(ii) There exists $\eta := (\eta_1, \dots, \eta_m)$ (with $\eta_i > 0$ for each $i \in \{1, 2, \dots, m\}$), such that $H(F(x), G(x)) \preceq \eta$, for all $x \in X$.

Take any $u^* \in Fix(G)$ and denote $x^* := r(u^*)$. Then, by the strong vectorial retraction–displacement condition, we have that

$$d(u^*, x^*) = d(u^*, r(u^*)) \preceq QD(u^*, F(u^*)) \preceq QH(G(u^*), F(u^*)) \leq Q\eta.$$

The proof is now complete. □

The following result shows that any multi-valued vectorial contraction of the Feng–Liu type satisfies the strong vectorial retraction–displacement condition.

Theorem 11 *Let (X, d) be a complete vector-valued metric space and $F : X \to P(X)$ be a multi-valued vectorial contraction of the Feng–Liu type. Suppose that F has a closed graph. Then, F satisfies the strong vectorial retraction–displacement condition.*

Proof By Theorem 8, we know that $Fix(F) \neq \emptyset$ and there exists an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ of the Picard type for F starting from the arbitrary $x_0 \in X$, which converges to a fixed point $x^*(x_0)$ of F. Moreover, the following relation holds:

$$d(x_0, x^*(x_0)) \preceq \left(I_m - B^{-1}A\right)^{-1} B^{-1} D(x_0, F(x_0)).$$

Thus, we can define the set retraction $r : X \to Fix(F), x \longmapsto r(x) := x^*(x)$ with the property

$$d(x, r(x)) \preceq \left(I_m - B^{-1}A\right)^{-1} B^{-1} D(x, F(x)), x \in X.$$

Hence, the strong vectorial retraction–displacement condition from Definition 12 is satisfied. □

By combining the above two theorems, we obtain the following stability properties for the multi-valued vectorial contraction of the Feng–Liu type.

Theorem 12 *Let (X, d) be a vector-valued metric space and $F : X \to P(X)$ be a multi-valued vectorial contraction of the Feng–Liu type. Suppose that F has a closed graph. Then, the fixed point inclusion $x \in F(x)$ is well-posed in the sense of Reich and Zaslavsi, has the Ulam–Hyers stability property and satisfies the data dependence property.*

Proof By Theorem 8, we have that $Fix(F) \neq \emptyset$, while Theorem 11 implies that F has the strong vectorial retraction–displacement property. The conclusions follow by Theorem 10. □

Remark 2 It is an open question to prove the Ostrowski stability property for a multi-valued vectorial contraction of the Feng–Liu type.

In the last part of this section, we will prove an existence and localization theorem for a nonself multi-valued vectorial contraction of Feng–Liu type.

Let (X, d) be a vector-valued metric space, $x_0 \in X$ and $R = (R_1, \dots, R_m)$ with $R_i > 0$ for each $i \in \{1, \dots, m\}$. Let $F : \tilde{B}(x_0, R) \subset X \to P(X)$ be a multi-valued operator, $B \in M_{m,m}(\mathbb{R}_+)$ be a diagonal matrix with elements $b_1, \dots, b_m \in]0, 1[$. For each $x \in \tilde{B}(x_0, R)$, we consider the set

$$I_B^x := \{y \in F(x) : Bd(x, y) \preceq D(x, F(x))\}.$$

Then, we have the following result.

Theorem 13 *Let (X, d) be a complete vector-valued metric space, $x_0 \in X$ and $R = (R_1, \ldots, R_m)$ with $R_i > 0$, for each $i \in \{1, \ldots, m\}$. Let $F : \tilde{B}(x_0, R) \subset X \to P(X)$ be a multi-valued operator with closed graph and $B \in M_{m,m}(\mathbb{R}_+)$ be a diagonal matrix with elements $b_1, \ldots, b_m \in]0, 1[$. We suppose that there exists a matrix $A \in M_{m,m}(\mathbb{R}_+)$ satisfying the following assumptions:*

(i) the matrix $L := B^{-1}A$ is convergent to zero;

(ii) for every $x_1 \in I_B^{x_0}$ we have that $(I_m - L)^{-1} d(x_0, x_1) \leq R$;

(iii) for each $x \in \tilde{B}(x_0, R)$ there is $y \in I_B^x$ such that

$$y \in \tilde{B}(x_0, R) \text{ implies } D(y, F(y)) \preceq Ad(x, y).$$

Then, there exists an iterative sequence $(x_n)_{n\in\mathbb{N}} \subset \tilde{B}(x_0, R)$ of Picard type for F starting from x_0 with the following properties:

(1) $(x_n)_{n\in\mathbb{N}}$ converges to $x^*(x_0) \in Fix(F) \cap \tilde{B}(x_0, R)$;

(2) the following apriori estimation holds:

$$d(x_n, x^*(x_0)) \preceq L^n (I_m - L)^{-1} d(x_0, x_1), n \in \mathbb{N};$$

(3) the following retraction–displacement type condition holds:

$$d(x_0, x^*(x_0)) \preceq (I_m - L)^{-1} d(x_0, x_1) \preceq (I_m - L)^{-1} B^{-1} D(x_0, F(x_0)).$$

Proof By (iii) there exists $x_1 \in I_B^{x_0}$ with $Bd(x_0, x_1) \preceq D(x_0, F(x_0))$. By (ii) we get that $(I_m - L)^{-1} d(x_0, x_1) \leq R$ and so $x_1 \in \tilde{B}(x_0, R)$. Hence, we also have that

$$D(x_1, F(x_1)) \preceq Ad(x_0, x_1) \preceq AB^{-1} D(x_0, F(x_0)) = LD(x_0, F(x_0)).$$

Notice that we have used the fact that $B^{-1}A = AB^{-1}$. Now, using (iii), for $x_1 \in \tilde{B}(x_0, R)$ there exists $x_2 \in I_B^{x_1}$ with $Bd(x_1, x_2) \preceq D(x_1, F(x_1))$. Thus

$$d(x_1, x_2) \preceq B^{-1} D(x_1, F(x_1)) \preceq Ld(x_0, x_1).$$

Moreover, if $x_2 \in \tilde{B}(x_0, R)$, then we can write that

$$D(x_2, F(x_2)) \preceq Ad(x_1, x_2).$$

Let us show that $x_2 \in \tilde{B}(x_0, R)$. Indeed,

$$d(x_0, x_2) \preceq d(x_0, x_1) + d(x_1, x_2) \preceq d(x_0, x_1) + B^{-1} D(x_1, F(x_1)) \preceq$$

$$d(x_0, x_1) + B^{-1} Ad(x_0, x_1) = d(x_0, x_1) + Ld(x_0, x_1) \preceq (I_m - L)^{-1} d(x_0, x_1) \leq R.$$

Hence, we also have that

$$D(x_2, F(x_2)) \preceq Ad(x_1, x_2) \preceq AB^{-1}D(x_1, F(x_1)) = LD(x_1, F(x_1)).$$

By the above procedure, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X with the following properties:

(a) $x_{n+1} \in I_B^{x_n} \cap \tilde{B}(x_0, R)$, for each $n \in \mathbb{N}$;
(b) $d(x_n, x_{n+1}) \preceq Ld(x_{n-1}, x_n) \preceq \cdots \preceq L^n d(x_0, x_1)$, for each $n \in \mathbb{N}^*$;
(c) $D(x_{n+1}, F(x_{n+1})) \preceq LD(x_n, F(x_n)) \preceq \cdots \preceq L^{n+1}D(x_0, F(x_0))$, for each $n \in \mathbb{N}$.

Then, by (b), the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d). Hence, it is convergent to an element $x^* := x^*(x_0) \in X$. Since $(x_n)_{n\in\mathbb{N}}$ is an iterative sequence of the Picard type of F starting from x_0 and F has a closed graph, it follows that $x^* \in Fix(F)$. Moreover, by the relation

$$d(x_n, x_{n+p}) \preceq L^n \left(I_m + L + \cdots L^{p-1}\right) d(x_0, x_1) \preceq L^n (I_m - L)^{-1} d(x_0, x_1),$$

letting $p \to \infty$, we obtain the following apriori estimation for the fixed point:

$$d(x_n, x^*(x_0)) \preceq L^n (I_m - L)^{-1} d(x_0, x_1), n \in \mathbb{N}.$$

Taking $n = 0$ in the above relation, we obtain the following retraction–displacement condition:

$$d(x_0, x^*(x_0)) \preceq (I_m - L)^{-1} d(x_0, x_1) \preceq (I_m - L)^{-1} B^{-1}D(x_0, F(x_0)).$$

The proof is complete. □

A more general concept is given in the last part of this section. The new concept combines the Feng–Liu contraction condition with the following concept.

Definition 13 Let (X, d) be a metric space and $S : X \to P(X)$ be a multi-valued operator. Then, S is said to be a multi-valued Subrahmanyan contraction if there exists a function $\psi : X \to [0, 1[$ such that
(i) $H(S(x), S(y)) \leq \psi(x)d(x, y)$, for all $(x, y) \in Graph(S)$;
(ii) $\psi(v) \leq \psi(u)$, for every $(u, v) \in Graph(S)$.

The above definition was introduced in [19]. Some fixed point results for this class of multi-valued operators are given in the same paper.

We are ready now to introduce the notion of multi-valued vectorial contraction of the Feng–Liu–Subrahmanyan type as follows.

Definition 14 Let (X, d) be a vector-valued metric space, $x_0 \in X$ and $R = (R_1, \ldots, R_m)$ with $R_i > 0$, for each $i \in \{1, \ldots, m\}$. Let $F : \tilde{B}(x_0, R) \subset X \to P(X)$ be a multi-valued operator with closed graph and $B \in M_{m,m}(\mathbb{R}_+)$ be a diagonal matrix with elements $b_1, \ldots, b_m \in]0, 1[$. For each $x \in \tilde{B}(x_0, R)$, let us consider the set

$$I_B^x := \{y \in F(x) : Bd(x, y) \preceq D(x, F(x))\}.$$

Then, F is called a multi-valued vectorial contraction of the Feng–Liu–Subrahmanyan type on the ball if there exists a mapping $\Psi : \tilde{B}(x_0, R) \to M_{m,m}(\mathbb{R}_+)$, such that

(i) the matrix $B^{-1}\Psi(x_0)$ is convergent to zero;

(ii) $\Psi(u) \preceq \Psi(v)$, for every $(u, v) \in Graph(F)$;

(iii) for each $x \in \tilde{B}(x_0, R)$ there is $y \in I_B^x$ such that the following implication holds:

$$y \in \tilde{B}(x_0, R) \text{ implies } D(y, F(y)) \preceq \Psi(x)d(x, y).$$

In particular, if the mapping Ψ is constant (e.g., $\Psi(x) = A$), then we get the above-mentioned notion of multi-valued vectorial contraction of the Feng–Liu.

A fixed point result for multi-valued vectorial contraction of the Feng–Liu–Subrahmanyan type on the ball is the following theorem.

Theorem 14 *Let (X, d) be a complete vector-valued metric space and $F : \tilde{B}(x_0, R) \subset X \to P(X)$ be a multi-valued vectorial contraction of the Feng–Liu–Subrahmanyan type on the ball. Suppose that F has closed graph and for every $x_1 \in I_B^{x_0}$ we have that $(I_m - B^{-1}\Psi(x_0))^{-1}d(x_0, x_1) \leq R$. Then, there exists an iterative sequence $(x_n)_{n\in\mathbb{N}} \subset \tilde{B}(x_0, R)$ of Picard type for F starting from x_0 with the following properties:*

(1) $(x_n)_{n\in\mathbb{N}}$ converges to $x^*(x_0) \in Fix(F) \cap \tilde{B}(x_0, R)$;

(2) the following apriori estimation holds:

$$d(x_n, x^*(x_0)) \preceq (B^{-1}\Psi(x_0))^n \left(I_m - B^{-1}\Psi(x_0)\right)^{-1} d(x_0, x_1), n \in \mathbb{N};$$

(3) the following retraction–displacement type condition holds:

$$d(x_0, x^*(x_0)) \preceq \left(I_m - B^{-1}\Psi(x_0)\right)^{-1} B^{-1}D(x_0, F(x_0)).$$

Proof The proof runs in a similar way to Theorem 13. We will present here the main ideas. By the assumption (iii) in Definition 14, for given x_0 there exists $x_1 \in I_B^{x_0}$ with $Bd(x_0, x_1) \preceq D(x_0, F(x_0))$. Then, since $(I_m - B^{-1}\Psi(x_0))^{-1}d(x_0, x_1) \leq R$ we get that $x_1 \in \tilde{B}(x_0, R)$. Let us denote $L(x_0) := B^{-1}\Psi(x_0)$. Notice that $B^{-1}\Psi(x_0) = \Psi(x_0)B^{-1}$. Hence, we obtain

$$D(x_1, F(x_1)) \preceq \Psi(x_0)d(x_0, x_1) \preceq B^{-1}\Psi(x_0)D(x_0, F(x_0)) = L(x_0)D(x_0, F(x_0)).$$

Now, using again (iii) in Definition 14 for $x_1 \in \tilde{B}(x_0, R)$ there exists $x_2 \in I_B^{x_1}$ with $Bd(x_1, x_2) \preceq D(x_1, F(x_1))$. Thus

$$d(x_1, x_2) \preceq B^{-1}D(x_1, F(x_1)) \preceq L(x_0)d(x_0, x_1).$$

Let us show now that $x_2 \in \tilde{B}(x_0, R)$. Indeed,

$$d(x_0, x_2) \preceq d(x_0, x_1) + d(x_1, x_2) \preceq d(x_0, x_1) + B^{-1}D(x_1, F(x_1)) \preceq$$

$$d(x_0, x_1) + B^{-1}\Psi(x_0)d(x_0, x_1) = d(x_0, x_1) + L(x_0)d(x_0, x_1) \preceq$$

$$(I_m - L(x_0))^{-1}d(x_0, x_1) \leq R.$$

Hence, we also have that

$$D(x_2, F(x_2)) \preceq \Psi(x_1)d(x_1, x_2) \preceq \Psi(x_1)B^{-1}D(x_1, F(x_1)) = L(x_1)D(x_1, F(x_1)).$$

By the above procedure, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X with the following properties:

(a) $x_{n+1} \in I_B^{x_n} \cap \tilde{B}(x_0, R)$, for each $n \in \mathbb{N}$;
(b) $d(x_n, x_{n+1}) \preceq L(x_{n-1})d(x_{n-1}, x_n) \preceq \cdots \preceq (L(x_0))^n d(x_0, x_1)$, for each $n \in \mathbb{N}^*$;
(c) $D(x_{n+1}, F(x_{n+1})) \preceq L(x_n)D(x_n, F(x_n)) \preceq \cdots \preceq (L(x_0))^{n+1}D(x_0, F(x_0))$, for each $n \in \mathbb{N}$.

Then, by (b), the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d). Hence, it is convergent to an element $x^* := x^*(x_0) \in X$. Since $(x_n)_{n\in\mathbb{N}}$ is an iterative sequence of the Picard type of F starting from x_0 and F has a closed graph, it follows that $x^* \in Fix(F)$.

The rest of the proof is similar to the last part of the proof of Theorem 14. □

4 An Application

In this section, we will present an existence result for a system of operatorial inclusion in complete metric spaces. The approach is based on the vectorial technique for multi-valued Feng–Liu operators and it follows the ideas from [20].

Let (X, d_1) and (Y, d_2) be two complete metric spaces and let $G_1 : X \times Y \to P(X)$ and $G_2 : X \times Y \to P(Y)$ be two multi-valued operators with a closed graph. We consider the following system of operatorial inclusions:

$$\begin{cases} x \in G_1(x, y) \\ y \in G_2(x, y). \end{cases} \tag{7}$$

Denote by $Z := X \times Y$ and define on Z the vectorial metric $\tilde{d} : Z \times Z \to \mathbb{R}^2_+$ given by

$$\tilde{d}((x, y), (u, v)) := \begin{pmatrix} d_1(x, u) \\ d_2(y, v) \end{pmatrix}, \text{ for each } (x, y), (u, v) \in Z.$$

Let $b_1, b_2 \in]0, 1[$ and define the following nonempty sets:

$$I_{b_1}^{(x,y)} := \{u \in G_1(x, y) : b_1 d_1(x, u) \leq D_1(x, G_1(x, y))\} \subset X$$

and

$$I_{b_2}^{(x,y)} := \{v \in G_2(x, y) : b_2 d_2(y, v) \leq D_2(y, G_2(x, y))\} \subset Y,$$

where D_1 and D_2 are the distances from a point to a set with respect to d_1 and d_2, respectively.

Denote also

$$B := \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}. \tag{8}$$

We suppose that for every $(x, y) \in X \times Y$ there exist $u \in I_{b_1}^{(x,y)}$ and $v \in I_{b_2}^{(x,y)}$, such that

$$D_1(u, G_1(u, v)) \leq a_1 d_1(x, u) + a_2 d_2(y, v) \tag{9}$$

and

$$D_2(v, G_2(u, v)) \leq a_3 d_1(x, u) + a_4 d_2(y, v), \tag{10}$$

where

$$A := \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{11}$$

is a matrix with nonnegative elements. We also suppose that the matrix

$$B^{-1}A = \begin{pmatrix} \frac{a_1}{b_1} & \frac{a_2}{b_1} \\ \frac{a_3}{b_2} & \frac{a_4}{b_2} \end{pmatrix} \tag{12}$$

is convergent to zero.

Under the above assumptions, we have the following existence and approximation result.

Theorem 15 *Let us consider the system of operatorial inclusions (7). Under the above assumptions, the system (7) has at least one solution* $(x^*, y^*) \in X \times Y$. *Moreover, for each* $(x_0, y_0) \in X \times Y$, *there exist two sequences* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *and* $\{y_n\}_{n\in\mathbb{N}} \subset Y$ *with the following properties:*

(A) $x_{n+1} \in G_1(x_n, y_n)$ *and* $y_{n+1} \in G_2(x_n, y_n)$, *for each* $n \in \mathbb{N}$;

(B) $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x^* *and* $\{y_n\}_{n\in\mathbb{N}}$ *converges to* y^* *as* $n \to \infty$;

(C) $\begin{pmatrix} d_1(x_n, x^*) \\ d_2(y_n, y^*) \end{pmatrix} \preceq (B^{-1}A)^n \left(I_m - B^{-1}A\right)^{-1} \begin{pmatrix} d_1(x_0, x_1) \\ d_2(y_0, y_1) \end{pmatrix}, n \in \mathbb{N}$;

(D) $\begin{pmatrix} d_1(x_0, x^*) \\ d_2(y_0, y^*) \end{pmatrix} \preceq \left(I_m - B^{-1}A\right)^{-1} \begin{pmatrix} d_1(x_0, x_1) \\ d_2(y_0, y_1) \end{pmatrix}$

$$\preceq \left(I_m - B^{-1}A\right)^{-1} B^{-1} \begin{pmatrix} D_1(x_0, G_1(x_0, y_0)) \\ D_2(y_0, G_2(x_0, y_0)) \end{pmatrix}.$$

Proof We denote $Z := X \times Y$ and, for $z := (x, y) \in Z$, consider the multi-valued operator $G : Z \to P(Z)$ given by $G(z) := G_1(z) \times G_2(z)$. Notice that the fixed points $z^* = (x^*, y^*)$ of G are solutions for the operatorial inclusion (7).

Let $(x, y) \in X \times Y$ and $(u, v) \in I_{b_1}^{(x,y)} \times I_{b_2}^{(x,y)}$, such that

$$D_1(u, G_1(u, v)) \leq a_1 d_1(x, u) + a_2 d_2(y, v)$$

and

$$D_2(v, G_2(u, v)) \leq a_3 d_1(x, u) + a_4 d_2(y, v).$$

We consider the vectorial gap function

$$\tilde{D}(z, Z) := \begin{pmatrix} D_1(x, X) \\ D_2(y, Y) \end{pmatrix}.$$

We denote $I_B^{(x,y)} := \{(u, v) \in G(x, y) : B\tilde{d}((x, y), (u, v)) \preceq \tilde{D}((x, y), G(x, y))\}$. Then, by our assumptions, the set $I_B^{(x,y)}$ is nonempty for each $(x, y) \in Z$. Moreover, by (9) and (10), we obtain that for each $z := (x, y) \in Z$, there exists $w := (u, v) \in I_B^{(x,y)}$, such that

$$\tilde{D}\,(w, G(w))) \preceq A\tilde{d}(z, w),$$

where

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.$$

Thus, G satisfies all the assumptions of Theorem 8. As a consequence, for each $z_0 := (x_0, y_0) \in Z$ there exists an iterative sequence of the Picard type $\{z_n := (x_n, y_n)\}_{n\in\mathbb{N}}$, which converges to a fixed point $z^*(z_0) \in Z$ of G, and the following relations hold:

(I) $\tilde{d}(z_n, z^*(z_0)) \preceq (B^{-1}A)^n \left(I_m - B^{-1}A\right)^{-1} \tilde{d}(z_0, z_1), n \in \mathbb{N}$;

(II) $\tilde{d}(z_0, z^*(z_0)) \preceq \left(I_m - B^{-1}A\right)^{-1} \tilde{d}(z_0, z_1) \preceq \left(I_m - B^{-1}A\right)^{-1} B^{-1}\tilde{D}(z_0, G(z_0))$.

Thus, the proof is complete. □

Remark 3 In the above-mentioned conditions, some stability results (such as well-posedness in the sense of Reich and Zaslavski, Ulam–Hyers stability and data dependence property) for the system of operatorial inclusions (7) can be established by applying the abstract results proved in the main results section.

In particular, an existence and approximation result for the multi-valued altering points problem can be obtained. We notice that, if (X, d_1) and (Y, d_2) are two metric spaces and $G_1 : Y \to P(X)$ and $G_2 : X \to P(Y)$ are two multi-valued operators, then the following system of operatorial inclusions is called an altering points problem for multi-valued operators:

$$\begin{cases} x \in G_1(y) \\ y \in G_2(x). \end{cases} \tag{13}$$

The above problem has important applications in the theory of multi-valued variational inequalities (see [28]) or in various systems of integral/differential inclusions.

Theorem 16 *Let (X, d_1) and (Y, d_2) be two complete metric spaces and let $G_1 : Y \to P(X)$ and $G_2 : X \to P(Y)$ be two multi-valued operators with a closed graph. Let $b_1, b_2 \in]0, 1[$ and the sets*

$$J_{b_1}^{(x,y)} := \{u \in G_1(y) : b_1 d_1(x, u) \leq D_1(x, G_1(y))\}$$

and

$$J_{b_2}^{(x,y)} := \{v \in G_2(x) : b_2 d_2(y, v) \leq D_2(y, G_2(x))\}.$$

We suppose that for every $(x, y) \in X \times Y$, there exists $(u, v) \in J_{b_1}^{(x,y)} \times J_{b_2}^{(x,y)}$, such that

$$D_1(u, G_1(v)) \leq a_1 d_1(x, u) + a_2 d_2(y, v)$$

and

$$D_2(v, G_2(u)) \leq a_3 d_1(x, u) + a_4 d_2(y, v),$$

where

$$A := \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.$$

is a matrix with nonnegative elements. We also suppose that the matrix

$$B^{-1}A = \begin{pmatrix} \frac{a_1}{b_1} & \frac{a_2}{b_1} \\ \frac{a_3}{b_2} & \frac{a_4}{b_2} \end{pmatrix}$$

is convergent to zero, where

$$B := \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Then, the altering points problem (13) has at least one solution in $X \times Y$. Moreover, for each $(x_0, y_0) \in X \times Y$, there exist two sequences $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$ with the following properties:

(A) $x_{n+1} \in G_1(y_n)$ and $y_{n+1} \in G_2(x_n)$, for each $n \in \mathbb{N}$;

(B) $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* and $\{y_n\}_{n\in\mathbb{N}}$ converges to y^* as $n \to \infty$;

(C) $\begin{pmatrix} d_1(x_n, x^*) \\ d_2(y_n, y^*) \end{pmatrix} \preceq (B^{-1}A)^n \left(I_m - B^{-1}A\right)^{-1} \begin{pmatrix} d_1(x_0, x_1) \\ d_2(y_0, y_1) \end{pmatrix}, n \in \mathbb{N};$

(D) $\begin{pmatrix} d_1(x_0, x^*) \\ d_2(y_0, y^*) \end{pmatrix} \preceq \left(I_m - B^{-1}A\right)^{-1} \begin{pmatrix} d_1(x_0, x_1) \\ d_2(y_0, y_1) \end{pmatrix}$

$$\preceq \left(I_m - B^{-1}A\right)^{-1} B^{-1} \begin{pmatrix} D_1\left(x_0, G_1(y_0)\right) \\ D_2\left(y_0, G_2(x_0)\right) \end{pmatrix}.$$

The above results generalize some altering points theorems, as given for the single-valued case in [27].

In particular, the above result is of interest for solving the following system of Fredholm–Volterra integral inclusions:

$$\begin{cases} x(t) \in \int_a^b K_1(t,s,y(s))ds + g_1(t) \\ y(t) \in \int_a^t K_2(t,s,x(s))ds + g_2(t), \end{cases} \tag{14}$$

where K_1, K_2 are multi-valued operators, while g_1, g_2 are continuous single-valued operators, satisfying some appropriate conditions.

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Algorithms and Applications for Split Equality Problem with Related Problems

Luoyi Shi, Tong Ling, Xiaolei Tong, Yu Cao, and Yishuo Peng

1 Introduction

The split feasibility problem in finite-dimensional Hilbert spaces (shortly, SFP) was introduced by Censor and Elfving [45] in 1994, for modeling an inverse problem which arises from the phase retrievals in medical image reconstruction [13]. The SFP is formulated as follows:

$$\text{Find a point } x^* \in C \text{ such that } Ax^* \in Q, \tag{1}$$

where C and Q are two nonempty, closed and convex subsets of two Hilbert spaces H_1 and H_2, respectively. In addition, $A : H_1 \to H_2$ is a bounded linear operator. Denote the set of solutions of the SFP (1) by $\Gamma = C \cap A^{-1}(Q) \neq \emptyset$.

The split equality problem (SEP) was first introduced and studied by Moudafi in [46]. This problem is formulated as follows: Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C and Q be two nonempty closed and convex subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators with adjoint operators A^* and B^*, respectively.

$$\text{Find } x \in C \text{ and } y \in Q \text{ such that } Ax = By. \tag{2}$$

Obviously, if $B = I^{H_2}$ and $H_3 = H_2$, the SEP reduces to the SFP (1).

Let H_1, H_2 and H_3 be three real Hilbert spaces. Let $\{C_i\}_{i=1}^N$ and $\{Q_j\}_{j=1}^M$ be two families of nonempty closed convex subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Then the multiple-sets split equality problem (MSSEP) is proposed:

$$\text{Find } x \in \cap_{i=1}^N C_i,\ y \in \cap_{j=1}^M Q_j \text{ such that } Ax = By, \tag{3}$$

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where N and M are positive integers. If $N = M = 1$,the multiple-sets split equality problem reduces to the SEP (2).

The above three problems have received much attention due to its broad applicability in vast application problems such as image reconstruction, signal processing and intensity-modulated radiation therapy, data compression and many more. In the last 30 years, focusing on real world applications, several iterative methods for solving the above problems have been introduced and analyzed.

The following sections are arranged as follows: in Sect. 2, we state several definitions and results that we need subsequently. In Sect. 3, we review and report the recent progresses on the fixed-point methods and optimization methods for solving the SFP. To speed up the convergence rate of the algorithms, several acceleration methods are introduced, such as inertial technique and alternated inertial technique. Subsequently, current relaxed techniques are presented, including half-space relaxed, ball relaxed and half-space-intersection relaxed techniques. In addition, we report several frequently used step-sizes. In Sect. 4, we review the original CQ-algorithm for SEP firstly. At the same time, we report some achievements in recent years, including but not limited to the new algorithm with strong convergence. Then we extend SEP to MSSEP by changing the domains and we report the corresponding algorithms and convergence analysis. Finally we introduce some special cases derived from SEP or MSSEP. In Sect. 5, we generalize some typical algorithms for solving split feasibility problems and split equality problems in p-uniformly convex, uniformly smooth Banach spaces. These algorithms are improved in terms of convergence and step size selection. In Sect. 6, we consider the linear convergence of some algorithms under the assumption of bounded linear regularity property. In Sect. 7, we present two applications of the SFP, one is to the signal processing and the other is to the image recovery.

2 Preliminaries and Basic Concepts

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, and let C be a nonempty closed convex subset of H. Let I denote the identity operator on H. Projections are an important tool for our work in this chapter. Recall that the (nearest point or metric) projection from H onto C, denoted P_C, is defined in such a way that, for each $x \in H$, $P_C x$ is the unique point in C with the property

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}.$$

The following is a useful characterization of projections.

Proposition 3.1 ([10]) *Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if*

$$\langle x - z, y - z \rangle \leq 0 \quad for\ all\ y \in C.$$

Lemma 3.1 ([10]) *Basic properties of projections are*
(i) P_C is nonexpansive, that is,

$$\|P_C x - P_C y\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(ii) $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2$ for all $x \in H$ and $z \in C$;
(iii) $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$ for all $x, y \in H$;
(iv) $\|P_C x - P_C y\|^2 \leq \|x - y\|^2 - \|(I - P_C)x - (I - P_C)y\|^2$ for all $x, y \in H$;
(v) If C is a closed subspace of a Hilbert space H, then P_C is the orthogonal projection from H onto C: $x - P_C x \perp C$ or $\langle x - P_C x, y \rangle = 0$ for all $x, y \in H$;
(vi) For any $x, v \in H$ with $v \neq 0$, let the half-space T be defined by

$$T = \{z \in H : \langle v, z - x \rangle \leq 0\}.$$

Then, for all $u \in H$, the metric projection $P_T u$ on the half-space T is computed by

$$P_T u = u - \max\left\{0, \frac{\langle v, u - x \rangle}{||v||^2}\right\} v.$$

We also need other sorts of nonlinear mappings which are introduced below. Denote by $Fix(T)$ the set of fixed points of T (i.e. $Fix(T) = \{x \in H : Tx = x\}$).

Definition 3.1 ([10]) Let $T : H \to H$ be a mapping. Then T is
(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.$$

(ii) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H.$$

(iii) averaged if $T = (1 - \alpha)I + \alpha S$, where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. In this case, we also say that T is α-averaged. A firmly nonexpansive mapping is $\frac{1}{2}$-averaged.
(iv) quasi-nonexpansive if

$$\|Tx - y\| \leq \|x - y\|, \quad \forall x \in H, y \in Fix(T).$$

(v) strictly pseudo-contractive if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - y) - (Tx - Ty)\|^2, \quad \forall x, y \in H.$$

(vi) pseudo-contractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(x - y) - (Tx - Ty)\|^2, \quad \forall x, y \in H.$$

(vii) demicontractive (or k-demicontractive) if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|Tx - x\|^2, \quad \forall x \in H, y \in Fix(T),$$

which is equivalent to

$$\langle x - Tx, x - y\rangle \geq \frac{1-k}{2}\|x - Tx\|, \quad \forall x \in H, y \in Fix(T).$$

Definition 3.2 ([10]) Let $A : H \to H$ be a mapping. Then A is
(i) strongly monotone if

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in H.$$

(ii) α-inverse strongly monotone(α-ism) (or α-cocoercive) if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in H.$$

(iii) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in H.$$

(iv) strongly pseudomonotone if there exists a number $\alpha > 0$ such that

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in H.$$

(v) pseudomonotone if

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \quad \forall x, y \in H.$$

(vi) quasi-monotone if

$$\langle Ax, y - x\rangle > 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \quad \forall x, y \in H.$$

(vii) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in H.$$

Proposition 3.2 ([85]) *We have the following assertions.*
(i) T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.
(ii) If T is v-ism and $\gamma > 0$, then γT is $\frac{v}{\gamma}$-ism.
(iii) T is averaged if and only if the complement $I - T$ is v-ism for some $v > \frac{1}{2}$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if the complement $I - T$ is $\frac{1}{2\alpha}$-ism.
(iv) If T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

(v) If T_1 and T_2 are averaged and have a common fixed point, then $Fix(T_1T_2) = Fix(T_1) \cap Fix(T_2)$.

Lemma 3.2 ([10]) *Let H be a real Hilbert space. Then the following results hold:*
(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \quad \forall x, y \in H$;
(ii) $\|x+y\|^2 \le 2\|x\|^2 + 2\|y\|^2, \quad \forall x, y \in H$;
(iii) for all $x, y \in H$ and $\alpha \in \mathbb{R}$,

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2;$$

(iv) $2\langle x-y, x-z\rangle = \|x-y\|^2 + \|x-z\|^2 - \|y-z\|^2, \quad \forall x, y, z \in H$.

Lemma 3.3 *(Demiclosedness principle, see [4]) Let X be a Banach space, C be a closed convex subset of X, and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$. In particular, if $y = 0$, then $x \in Fix(T)$.*

Definition 3.3 ([10]) A function $f : H \to \mathbb{R}$ is said to be weakly lower semi-continuous (w-lsc) at x if x_n weakly converging to x implies

$$f(x) \le \liminf_{n\to\infty} f(x_n).$$

Definition 3.4 ([10]) A function $f : H \to \mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y), \quad \forall \lambda \in (0,1), \forall x, y \in H.$$

Then a differentiable function f is convex if and only if there holds the relation:

$$f(z) \ge f(x) + \langle \nabla f, z - x\rangle, \quad \forall z \in H.$$

Definition 3.5 ([10]) An element $g \in H$ is said to be a subgradient of $f : H \to \mathbb{R}$ at x if

$$f(z) \ge f(x) + \langle g, z - x\rangle, \quad \forall z \in H.$$

If the function f has at least one subgradient at x, it is said to be subdifferentiable at x. The set of subgradients of f at the point x is called the subdifferential of f at x, and is denoted by $\partial f(x)$, i.e.,

$$\partial f(x) = \{\xi \in H : f(z) \ge f(x) + (\xi, z - x), \forall\, z \in H\}.$$

A function f is called subdifferentiable if it is subdifferentiable at all $x \in H$.

Definition 3.6 ([10]) Let C be a nonempty subset of H and $\{x_n\}$ be a sequence in H. $\{x_n\}$ is called Fejér monotone with respect to C if :

$$\|x_{n+1} - x^*\| \le \|x_n - x^*\|, \quad \forall\, x^* \in C.$$

Clearly, a Fejér monotone sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists.

Definition 3.7 ([10]) The normal cone of C at $v \in C$, denoted by $N_C(v)$, is defined as

$$N_C(v) := \{d \in H : \langle d, y - v\rangle \leq 0 \ for \ all \ y \in C\}.$$

Definition 3.8 ([10]) Let $A : H \rightrightarrows 2^H$ be a point-to-set operator defined on a real Hilbert space H. The operator A is called a maximal monotone operator if A is monotone, i.e.,

$$\langle u - v, x - y\rangle \geq 0 \text{ for all } u \in A(x) \text{ and } v \in A(y),$$

and the graph $G(A)$ of A,

$$G(A) := \{(u, v) \in H \times H : u \in A(x)\},$$

is not properly contained in the graph of any other monotone operator.

It is clearly that a monotone mapping A is maximal if and only if, for any $(x, u) \in H \times H$, if $\langle u - v, x - y\rangle \geq 0$ for all $(v, y) \in G(A)$, then it follows that $u \in A(x)$.

The following results are useful when proving weak and strong convergence of a sequence.

Proposition 3.3 ([10]) *If $T : H \to H$ is an averaged mapping with $Fix(T) \neq \emptyset$, then*
(i) T is asymptotically regular, that is

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0 \quad for \ all \ x \in H.$$

(ii) For any $x \in H$, the sequence $\{T^n x\}$ converges weakly to a fixed point of T.

Proposition 3.4 ([85]) *Let $f : H \to \mathbb{R}$ be a continuously differentiable function such that the gradient ∇f is Lipschitz continuous:*

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad x, y \in H.$$

Assume that the minimization problem

$$\min\{f(x) : x \in C\}$$

is consistent, where C is a closed convex subset of H. Then, for $0 < \gamma < \frac{2}{L}$, the sequence $\{x_n\}$ generated by the gradient-projection algorithm

$$x_{n+1} = P_C(x_n - \gamma\nabla f(x_n))$$

converges weakly to a solution of the minimization problem.

Lemma 3.4 ([85]) *Let f be a convex and differentiable function, and let C be a closed convex subset of H. Then $x \in C$ is a solution of the problem*

$$\min_{x \in C} f(x)$$

if and only if $x \in C$ satisfies the following optimality condition:

$$\langle \nabla f, v - x \rangle \geq 0, \quad \forall v \in C.$$

Moreover, if f is, in addition, strictly convex and coercive, then the minimization problem has a unique solution.

Lemma 3.5 ([51]) *Let $\{x_n\}$ be a sequence of Hilbert space H. If $\{x_n\}$ converges weakly to x, then for any $y \in H$ and $y \neq x$, we have $\liminf_{n\to\infty} \|x_n - x\| \leq \liminf_{n\to\infty} \|x_n - y\|$.*

Lemma 3.6 ([59]) *Let X be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$, and $\limsup_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 3.7 ([40]) *Let $\{x_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be sequence in $[0, +\infty)$ such that*

$$x_{n+1} \leq x_n + \alpha_n(x_n - x_{n-1}) + \delta_n,$$

for all $n \geq 0$, $\sum_{n=1}^{\infty} \delta_n < +\infty$ and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$, for all $n \in \mathbb{N}$. Then the following hold:
(i) $\sum_{n\geq 1}[\delta_n - \delta_{n-1}]_+ < +\infty$, where $[t]_+ = \max\{t, 0\}$;
(ii) there exists $x^* \in [0, +\infty)$ such that $\lim_{n\to\infty} x_n = x^*$.

Proposition 3.5 ([51]) *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{x_n\}$ be a bounded sequence which satisfies the following properties:*
(i) every weak limit point of $\{x_n\}$ lies in C;
(ii) $\lim_{n\to\infty} \|x_n - x\|$ exists for every $x \in C$.
Then $\{x_n\}$ converges weakly to a point in C.

Lemma 3.8 ([67]) *Let $\{x_n\}$, $\{z_n\}$ be two bounded sequences in a Banach space, and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Suppose that $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z_n$ and $\limsup_{n\to\infty} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq 0$, then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 3.9 ([3]) *Let* $\{x_n\}$ *be a sequence of nonnegative real numbers,* $\{\alpha_n\}$ *be a sequence of real numbers in* $[0, 1]$ *with* $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\{u_n\}$ *be a sequence of nonnegative real numbers with* $\sum_{n=1}^{\infty} u_n < \infty$, *and* $\{z_n\}$ *be a sequence of real numbers with* $\limsup_{n\to\infty} z_n \le 0$. *Suppose that*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z_n + u_n, \quad \forall n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} x_n = 0$.

3 The Split Feasibility Problem

In this section, we will review and report the recent progresses on the fixed point and optimization methods for solving the split feasibility problem. Several improvement techniques are introduced, including inertial techniques, relaxed techniques and step-size improvements.

The original algorithm introduced by Censor and Elfving [45] for solving the split feasibility problem is generated by the following form:

$$x_{n+1} = A^{-1} P_Q(P_{A(C)}(Ax_n)),$$

which involves the computations of the inverse A^{-1} (assuming the existence of the inverse of A) and thus does not become popular.

A more popular algorithm that solves the SFP (1) seems to be the CQ algorithm of Byrne [7, 9], which is defined as follows:

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \tag{4}$$

where P_C and P_Q are the metric projections onto C and Q, respectively, $A^* : H_2^* \to H_1^*$ is the adjoint of A and the stepsize $\gamma \in (0, \frac{2}{\|A\|^2})$. Compared with Censor and Elfving's algorithm, where the matrix inverse A^{-1} is involved, the CQ algorithm seems more easily executed since it only deals with metric projections with no need to compute matrix inverses.

We can look at the CQ algorithm from two different but equivalent ways: optimization and fixed point. The CQ algorithm is found to be a gradient-projection method (GPM) in convex minimization (it is also a special case of the proximal forward-backward splitting method). Since the CQ algorithm can be viewed as a fixed point algorithm for averaged mappings, we can apply Krasnosel'skiĭ-Mann algorithm to the SFP.

Next, we will review and report the recent progresses on the fixed point and optimization methods for solving the SFP.

3.1 Fixed Point Methods

Approximating fixed point problems for nonexpansive mappings has a variety of specific applications since many problems can be seen as a fixed point problem of nonexpansive mappings such as convex feasibility problems, monotone variational inequalities. We can use fixed point algorithms to solve the SFP based upon the following analysis.

Let $\gamma > 0$ and assume that $x^* \in \Gamma$. Thus $Ax^* \in Q$ which implies the equation $(I - P_Q)Ax^* = 0$ which in turns implies the equation $\gamma A^*(I - P_Q)Ax^* = 0$, hence the fixed point equation $(I - \gamma A^*(I - P_Q)A)x^* = x^*$. Requiring that $x^* \in C$, we consider the fixed point equation

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*. \tag{5}$$

We will see that solutions of the fixed point equation are exactly solutions of the SFP. The following proposition is due to Byrne and Xu.

Proposition 3.6 ([85]) *Given $x^* \in H_1$. Then x^* solves the SFP if and only if x^* solves the fixed point equation (5).*

This fact motivates the use of fixed point algorithms for solving the SFP. In particular, we recover the CQ algorithm as the following fixed point algorithm:

$$x_{n+1} = Tx_n, \ \forall n \geq 0, \tag{6}$$

where the operator T is given by

$$T = P_C(I - \gamma A^*(I - P_Q)A). \tag{7}$$

When T is an averaged mapping, Byrne proved that the sequence $\{T^n x\}$ converges weakly to a point in $Fix(T)$, where x is any initial point. Therefore, when $\gamma \in (0, \frac{2}{\|A\|^2})$, $T = P_C(I - \gamma A^*(I - P_Q)A)$ is an averaged mapping, and the convergence of CQ algorithm can be obtained.

In order to find a fixed point of a nonexpansive operator $T : C \to C$, many researchers introduced various methods. One of the famous iterative methods is the Picard iteration process, which was proposed by Picard in 1890, defined by $x_{n+1} = Tx_n$.

Next, the generalizations of the Picard iteration are as follows:

(1) The Mann iteration (Mann [43], 1953): The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \ \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence satisfying $0 \leq \alpha_n \leq 1$. Mann proved that if $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, then $\{x_n\}$ converges weakly to a fixed point of T.

(2) The Krasnosel'skiĭ iteration (Krasnosel'skiĭ [35], 1955): The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = T_{\frac{1}{2}} x_n = \frac{1}{2} x_n + \frac{1}{2} T x_n = \frac{1}{2}(x_n + T x_n), \ \forall n \geq 0, \end{cases}$$

(3) The Krasnosel'skiĭ-Mann (KM) approach to finding fixed points of a nonexpansive (ne) operator N is quite simple, yet remarkably useful. Given a ne operator N, let

$$T = (1 - \alpha)I + \alpha N$$

for some $\alpha \in (0, 1)$. The operator T is then said to be averaged (av); note that T is then also ne. The KM theorem tells us that the sequence defined by equation $x_{n+1} = T x_n$ then converges (weakly) to a fixed point of N whenever such points exist. As far as we know, the operator P_C is av, as is the operator $(I - \gamma \nabla f)$ if ∇f is Lipschitz continuous and the parameter γ is appropriately chosen; the product of finitely many av operators is av, so the operator $P_C(I - \gamma A^*(I - P_Q)A)$ is also av. Therefore, the CQ algorithm is a KM algorithm.

(4) The Schäefer iteration (Schäefer, 1957): The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \ \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 1)$.

As we know, the sequences $\{x_n\}$ generated by the above methods converge weakly to a solution of the SFP. But, in the problems, economics (Khan and Yannelis 1991), image recovery (Combettes 1996), mechanics (Dautray and Lions 1993), electromagnetics (Dautray and Lions 1993), quantum physics (Dautray and Lions 1993), and control theory (Fattorini 1999), the norm convergence is often much more desirable than the weak convergence in infinite dimensional spaces, for it translates the physically tangible property that the energy

$$\|x_n - x\|^2$$

of the error between the iterate x_n and a solution x eventually becomes arbitrarily small (see [4]). In order to get the strong convergence, some authors considered the following methods: for example, Xu [84] presented a modified KM algorithm with strong convergence, Wang and Xu [68] proposed a modified CQ algorithm with strong convergence, which is defined as follows:

$$x_{n+1} = P_C[(1 - \alpha_n)(x_n - \gamma A^*(I - P_Q)Ax_n)], \tag{8}$$

where the parameter $\{\alpha_n\}$ and the stepsize γ satisfy the following conditions: (1) $0 < \gamma < \frac{2}{\|A\|^2}$; (2) $\lim_{k \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$; (3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \to \infty} |\alpha_{n+1} - \alpha_n| / \alpha_n = 0$. The main idea of this algorithm is to use contractions to approximate the nonexpansive mapping $I - \gamma A^*(I - P_Q)A$. They proved that $\{x_n\}$ defined by (8) converges strongly to the minimum-norm solution of the SFP. In 2012, Yu et al. [69] proved that $\{x_n\}$ generated by (8) is strongly convergent without the condition (3).

Dang and Gao [19] presented a strongly convergent KM-CQ-like algorithm which combines the KM algorithm with the CQ algorithm by introducing two parameter sequences. It has the following iterative process:

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n T_n(x_n),$$

where $T_n(x_n) = P_C[(1 - \alpha_n)Ux_n]$, $U = I - \gamma A^*(I - P_Q)A$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in (0, 1).

Several strongly convergent fixed point algorithms are also described below.

(5) The Halpern iteration (Halpern [31], 1967): for any fixed $u \in C$, the sequence $\{x_n\}$ is defined by

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)Tx_n, \ \forall n \geq 0,$$

where $\{\lambda_n\}$ is a real sequence satisfying $0 \leq \lambda_n \leq 1$. Halpern proved the strong convergence of the iterative sequence in a Hilbert space.

(6) The Ishikawa iteration (Ishikawa [34], 1974): The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in C, \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \ \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \alpha_n, \beta_n \leq 1$. He proved that if $0 \leq \alpha_n, \beta_n \leq 1$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, then $\{x_n\}$ converges strongly to a fixed point of T.

(7) The Noor iteration (Noor [50], 2000): The sequence $\{x_n\}$ is defined by, for any $x_0 \in C$,

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Tz_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \ \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences of real numbers satisfying $0 \leq \alpha_n, \beta_n, \gamma_n < 1$. Clearly, Mann and Ishikawa iterations are special cases of Noor iteration, and Mann iteration is a special case of Ishikawa iteration.

Based on Noor iteration, Dang et al. [20] proposed a three-step KM-CQ-like algorithm: The algorithm can be viewed as an extension of KM-CQ-like algorithm. When $\gamma_n = 0, \beta_n \neq 0$, it is a two-step KM-CQ-like algorithm; When $\gamma_n = 0, \beta_n = 0$, it is a KM-CQ-like algorithm.

Algorithm 1 Three-step KM-CQ-like algorithm

For any arbitrary initial point x_0, $\{x_n\}_{n\geq 0}$ is generated by the iteration

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n P_C[(1-\lambda_n)Ux_n], \\ y_n = (1-\beta_n)x_n + \beta_n P_C[(1-\lambda_n)Uz_n], \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C[(1-\lambda_n)Uy_n], \ \forall n \geq 0, \end{cases}$$

where $U = I - \gamma A^*(I - P_Q)A$, $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$, and $\{\alpha_n\}$ are four real sequences in $(0, 1)$.

Theorem 3.1 ([20]) *Let* $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$, *and* $\{\alpha_n\}$ *are four real sequences in* $(0, 1)$. *Suppose they satisfy the following conditions:*
(i) $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$;
(ii) $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0$, $\lim_{n\to\infty} |\gamma_n - \gamma_{n+1}| = 0$, $\lim_{n\to\infty} |\beta_n - \beta_{n+1}| = 0$;
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
Then, $\{x_n\}$ *generated by Algorithm 1 converges strongly to a point in* Γ.

(8) The Nakajo and Takahashi iteration (Nakajo and Takahashi [48], 2003): Nakajo and Takahashi introduced the following hybrid projection method to study the fixed point problem of nonexpansive mappings: The sequence $\{x_n\}$ is defined by, for any $x_0 \in C$,

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)x_n, \\ C_n = \{u \in C \,|\, \|y_n - u\| \leq \|x_n - u\|\}, \\ Q_n = \{u \in C \,|\, \langle x_n - u, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \qquad n \geq 0. \end{cases}$$

This algorithm constructs two closed convex sets with dominant structures and computes the projection on the intersection of the two closed convex sets per iteration, resulting in a sequence $\{x_n\}$ with strong convergence.

(9) The Takahashi iteration (Takahashi [70], 2008): Takahashi et al. proposed the following smooth projection algorithm to solve fixed point problem: The sequence $\{x_n\}$ is defined by, for any $x_0 \in C$ and $C_0 = C$,

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)x_n, \\ C_{n+1} = \{u \in C_n \,|\, \|y_n - u\| \leq \|x_n - u\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \qquad n \geq 0. \end{cases}$$

Compared with the hybrid projection algorithm (8), the closed convex set sequence generated iteratively by this algorithm also has a shrinking structure, that is, $C_{n+1} \subset C_n \subset \ldots \subset C_0$. Under appropriate conditions, the iterative sequence $\{x_n\}$ constructed by hybrid projection algorithm and smooth projection algorithm converges strongly to $x^* = P_{Fix(T)} x_0$. Because the closed convex half-space makes the projection mapping easy to calculate and does not involve contractive mapping and strongly monotone mapping, the two projection algorithms are also widely used in many problems.

(10) The Moudafi viscosity iteration (Moudafi [44], 2000): The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n)Tx_n, \ \forall n \geq 0, \end{cases}$$

where $f : H_1 \to H_1$ is a contractive mapping and $\{\lambda_n\}$ is a real sequence satisfying $0 \leq \lambda_n \leq 1$. Moudafi proved the strong convergence of the iterative sequence in a Hilbert space.

(11) The SP-iteration (Suantai [52], 2011): The sequence $\{x_n\}$ is defined by, for any $x_0 \in C$,

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n, \ \forall n \geq 0, \end{cases}$$

where $\{\gamma_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences of real numbers satisfying $0 \leq \gamma_n, \alpha_n, \beta_n \leq 1$.

(12) The Feng iteration (Feng [18], 2019): The sequence $\{x_n\}$ is defined by, for any $x_0 \in C$,

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\ x_{n+1} = (1-\alpha_n)Tz_n + \alpha_n Ty_n, \ \forall n \geq 0, \end{cases}$$

where $T = P_C(I - \gamma A^*(I - P_Q)A)$, $\gamma \in (0, \frac{2}{\|A\|^2})$, and $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences of real numbers satisfying $0 < \gamma_n, \alpha_n, \beta_n < 1$. She proved that the sequence $\{x_n\}$ converges weakly to a point in Γ.

Theorem 3.2 *Let $\{x_n\}$ be the sequence generated by Feng iteration. Then $\{x_n\}$ converges weakly to a point in Γ.*

Proof Taking a point $x^* \in F(T)$. Since T is nonexpansive, we have

$$\begin{aligned} \|z_n - x^*\| &= \|(1-\gamma_n)x_n + \gamma_n Tx_n - x^*\| \\ &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|Tx_n - x^*\| \\ &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{9}$$

i.e.,

$$\|z_n - x^*\| \leq \|x_n - x^*\|. \tag{10}$$

Similarly, we obtain

$$\|y_n - x^*\| \leq \|x_n - x^*\|. \tag{11}$$

Combining (10) and (11), we get

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1-\alpha_n)Tz_n + \alpha_n Ty_n - x^*\| \\ &\leq (1-\alpha_n)\|Tz_n - x^*\| + \alpha_n\|Ty_n - x^*\| \\ &\leq (1-\alpha_n)\|z_n - x^*\| + \alpha_n\|y_n - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^*\| \\ &= \|x_n - x^*\|.\end{aligned} \tag{12}$$

Since x^* is chosen arbitrarily in $F(T)$, one deduces that $\{\|x_n - x^*\|\}_n$ is decreasing, then $\lim_{n\to\infty}\|x_n - x^*\|$ exists for any $x^* \in F(T)$.

Suppose that

$$\lim_{n\to\infty}\|x_n - x^*\| = a(a > 0). \tag{13}$$

By (10) and (11), we have

$$\limsup_{n\to\infty}\|z_n - x^*\| \leq a \text{ and } \limsup_{n\to\infty}\|y_n - x^*\| \leq a. \tag{14}$$

Since T is nonexpansive mapping, we obtain

$$\|Tx_n - x^*\| \leq \|x_n - x^*\|, \|Tz_n - x^*\| \leq \|z_n - x^*\|, \|Ty_n - x^*\| \leq \|y_n - x^*\|. \tag{15}$$

Taking the superior limit on both sides, we get

$$\limsup_{n\to\infty}\|Tx_n - x^*\| \leq a, \limsup_{n\to\infty}\|Tz_n - x^*\| \leq a, \limsup_{n\to\infty}\|Ty_n - x^*\| \leq a. \tag{16}$$

Since

$$a = \lim_{n\to\infty}\|x_{n+1} - x^*\| = \lim_{n\to\infty}\|(1-\alpha_n)(Tz_n - x^*) + \alpha_n(Ty_n - x^*)\|, \tag{17}$$

combining (16) and (17), from Lemma 3.6, we infer that

$$\lim_{n\to\infty}\|Tz_n - Ty_n\| = 0.$$

Now

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(Tz_n - x^*) - \alpha_n(Tz_n - Ty_n)\| \\ &\leq \|Tz_n - x^*\| + \alpha_n\|Tz_n - Ty_n\|.\end{aligned} \tag{18}$$

which implies that

$$a \leq \liminf_{n\to\infty}\|Tz_n - x^*\|. \tag{19}$$

From (16) and (19), we obtain

$$\lim_{n\to\infty} \|Tz_n - x^*\| = a.$$

Moreover,

$$\begin{aligned} \|Tz_n - x^*\| &\le \|Tz_n - Ty_n\| + \|Ty_n - x^*\| \\ &\le \|Tz_n - Ty_n\| + \|y_n - x^*\|, \end{aligned} \tag{20}$$

which implies that

$$a \le \liminf_{n\to\infty} \|y_n - x^*\|. \tag{21}$$

Combining (14) and (21), we obtain

$$\lim_{n\to\infty} \|y_n - x^*\| = a.$$

Similarly, we can obtain $\lim_{n\to\infty} \|z_n - x^*\| = a$. Hence,

$$\begin{aligned} a &= \lim_{n\to\infty} \|z_n - x^*\| \\ &= \lim_{n\to\infty} \|(1-\gamma_n)x_n + \gamma_n Tx_n - x^*\| \\ &= \lim_{n\to\infty} \|(1-\gamma_n)(x_n - x^*) + \gamma_n(Tx_n - x^*)\|. \end{aligned} \tag{22}$$

Combining (13), (16) and (22), from Lemma 3.6, we infer that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Next, we show that the subsequences of $\{x_n\}$ only have a weak limit in $F(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two sequences of $\{x_n\}$, the weak limits of $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are denoted by u and v, respectively. Then we have $\lim_{n\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0$. By 3.3, $I - T$ is demiclosed at zero. Hence, we gain $Tu = u$, i.e., $u \in F(T)$. Similarly, we can prove that $v \in F(T)$.

Next, we show the uniqueness of weak limit. Suppose that $u \ne v$, according to Lemma 3.5, we have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - u\| = \lim_{n_i\to\infty} \|x_{n_i} - u\| &< \lim_{n_i\to\infty} \|x_{n_i} - v\| \\ &= \lim_{n\to\infty} \|x_n - v\| = \lim_{n_j\to\infty} \|x_{n_j} - v\| \\ &< \lim_{n_j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|. \end{aligned} \tag{23}$$

This is clearly contradictory, hence, $u = v$. Therefore, $\{x_n\}$ converges weakly to a point in $F(T)$, that is, the sequence $\{x_n\}$ converges weakly to a point in Γ. The proof is completed.

3.2 Optimization Methods

In this section, we will introduce some optimization methods for solving the split feasibility problem.

3.2.1 Gradient-Projection Method

It is well known that the Gradient-projection method (GPM) is one of the powerful methods for solving constrained optimization problems. Consider the constrained optimization problem:

$$\min_{x\in K} f(x).$$

where $f : H \to \mathbb{R}$ is a continuously differentiable function and K is a closed convex subset of a real Hilbert space H. The GPM takes the following iterative form:

$$x_{n+1} = P_K(x_n - \alpha_n \nabla f(x_n)). \tag{24}$$

We will reformulate the SFP as an optimization problem so that the GPM is applicable. Indeed, $x \in \Gamma$ means that there is an $x \in C$ such that $Ax - q = 0$ for some $q \in Q$. This motivates us to consider the distance function $d(Ax, q) = \|Ax - q\|$ and the minimization problem

$$\min_{\substack{x\in C\\ q\in Q}} \frac{1}{2}\|Ax - q\|^2.$$

Minimizing with respect to $q \in Q$ first makes us consider the minimization

$$\min_{x\in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2. \tag{25}$$

The objective function f is continuously differentiable with gradient given by

$$\nabla f(x) = A^*(I - P_Q)Ax. \tag{26}$$

(Here A^* is the adjoint of A). Due to the fact that $I - P_Q$ is (firmly) nonexpansive, we find that ∇f is Lipschitz continuous with Lipschitz constant $L := \|A\|^2$. The GPM (24) thus applies to solve (25). This method with gradient ∇f given as in (26) is referred to as the CQ algorithm and generates a sequence $\{x_n\}$ via the procedure

$$x_{n+1} = P_C(I - \gamma\nabla f)x_n, \ \forall n \geq 0, \tag{27}$$

where the initial guess $x_0 \in H_1$ and $\gamma > 0$ is a parameter.

3.2.2 Gradient Method

We now consider the minimization of a convex and differential function f and assume that the minimum point x^* and $f(x^*) = f^*$ exist. One of the most popular methods for solving this problem is the gradient method that takes the following iterative form:

$$x_{n+1} = x_n - \alpha_n \nabla f(x_n). \tag{28}$$

Here, the stepsize $\alpha_n \geq 0$ may be selected using various ways. Among them, Polyak [54] suggested the following way to select the stepsize:

$$\alpha_n = \lambda_n \frac{f(x_n) - f^*}{\|\nabla f(x_n)\|^2} \tag{29}$$

with $\lambda_n \in (0, 2)$. Then the sequence $\{x_n\}$ is weakly convergent to some minimum point if $\nabla f(x_n)$ is bounded on the set $\{x \in H : \|x - x_0\| \leq \|x^* - x_0\|\}$.

It is clear that solution of the SFP amounts to unconstrained minimization of

$$f(x) := \frac{1}{2}\|(I - P_C)x\|^2 + \frac{1}{2}\|(I - P_Q)Ax\|^2.$$

Since $\nabla f = I - P_C + A^*(I - P_Q)A$ and $f^* = 0$ in this specific case, the method is thus reduced to the following algorithm:

Algorithm 2 Polyak's gradient algorithm with weak convergence

Choose an arbitrary initial guess x_0. Assume x_n has been constructed. If

$$\|x_n - P_C x_n + A^*(I - P_Q)Ax_n\| = 0,$$

then stop; otherwise, continue and construct x_{n+1} via the formula:

$$x_{n+1} = x_n - \tau_n[(x_n - P_C x_n) + A^*(I - P_Q)Ax_n], \tag{30}$$

where the stepsize τ_n is given by

$$\tau_n = \lambda_n \frac{\|x_n - P_C x_n\|^2 + \|(I - P_Q)Ax_n\|^2}{2\|(x_n - P_C x_n) + A^*(I - P_Q)Ax_n\|^2}. \tag{31}$$

Theorem 3.3 ([77]) *If $0 < \epsilon \leq \lambda_n \leq 4 - \epsilon$, then the sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to a solution x^* of the SFP.*

Since Algorithm 2 has only weak convergence, Wang [77] also established a strong convergent algorithm.

Algorithm 3 Polyak's gradient algorithm with strong convergence

Let $u \in H_1$ and start an initial guess $x_0 \in H_1$. Assume x_n has been constructed. If

$$\|x_n - P_C x_n + A^*(I - P_Q)Ax_n\| = 0,$$

then stop; otherwise, continue and construct x_{n+1} via the formula:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)[x_n - \tau_n((x_n - P_C x_n) + A^*(I - P_Q)Ax_n)], \tag{32}$$

where the stepsize τ_n is given by

$$\tau_n = \lambda_n \frac{\|x_n - P_C x_n\|^2 + \|(I - P_Q)Ax_n\|^2}{2\|(x_n - P_C x_n) + A^*(I - P_Q)Ax_n\|^2}. \tag{33}$$

Theorem 3.4 ([77]) *If* $0 < \epsilon \leq \lambda_n \leq 4 - \epsilon$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *then the sequence* $\{x_n\}$ *generated by Algorithm 3 converges strongly to a solution* $z = P_\Gamma u$ *of the SFP.*

3.2.3 Douglas-Rachford Splitting Method

The Douglas-Rachford splitting method introduced in [26] is a popular method to solve structured optimization problems. Let g and h be proper lower semicontinuous convex functions from a Hilbert space H to $(-\infty, +\infty)$. Consider the structured optimization problem:

$$\min_{x\in H} g(x) + h(x).$$

Given the initial guess x_0, the Douglas-Rachford splitting method generates the iterative sequence via the following scheme:

$$\begin{cases} y_{n+1} = \arg\min_{x\in H}\{g(y) + \frac{1}{2\lambda_n}\|y - x_n\|^2\}, \\ z_{n+1} = \arg\min_{z\in H}\{h(z) + \frac{1}{2\lambda_n}\|z - (2y_{n+1} - x_n)\|^2\}, \\ x_{n+1} = x_n + \alpha_n(z_{n+1} - y_{n+1}), \end{cases} \tag{34}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0, 2)$ is a parameter sequence and $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0, \infty)$ is the proximal parameter sequence of the regularization terms. Note that the scheme (34) with $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ was called the general splitting method in [36] and becomes the Peaceman-Rachford splitting method when $\{\alpha_n\}_{n\in\mathbb{N}} \subset [2, +\infty)$.

To use the general splitting method to solve the SFP, we rewrite the SFP as the following unconstrained minimization:

$$\min_{x\in H_1}\{\iota_C(x) + f(x)\}, \tag{35}$$

where $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ and $\iota_C(x)$ is the indicator function of the set C defined by

$$\iota_C(x) = \begin{cases} 0, & x \in C; \\ \infty, & \text{otherwise}. \end{cases} \tag{36}$$

Recall that the gradient of $f(x)$ is $\nabla f(x) = A^*(I - P_Q)Ax$. Letting $g = f$ and $h = \iota_C$ in (34) and using the linearization technique, in [22], Dong et al. recently proposed the following general splitting methods with linearization to solve the SFP.

Algorithm 4 General splitting method with linearization

Generate $(y_{n+1}, z_{n+1}, x_{n+1})$ by

$$\begin{cases} y_{n+1} = x_n - \lambda_n \nabla f(x_n), \\ z_{n+1} = P_C(2y_{n+1} - x_n), \\ x_{n+1} = x_n + \alpha_n(z_{n+1} - y_{n+1}), \end{cases}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers and

$$\lambda_n := \begin{cases} \gamma \frac{f(x_n)}{\|\nabla f(x_n)\|^2}, & \text{if } \nabla f(x_n) \neq 0, \\ 0, & \text{otherwise}. \end{cases}$$

and $\gamma \in (0, 2)$.

The weak convergence of the Algorithm 4 was established under the standard conditions.

Based on Algorithm 4, Dong et al. also proposed an alternated inertial general splitting method with linearization for the SFP (see [23] for details).

3.2.4 Regularization Method

We know that most applied problems are ill-posed problems in the sense, they do not satisfy at least one of the following three requirements: (1) The problem is solvable; (2) Its solution is unique; (3) The problem is stable in the sense, any small change in the input data leads to only small changes in the output data (the solution of the problem) (see, e.g., [30]). Thus SFP and its related problems are also ill-posed problems. One popular effective method for solving the ill-posed problems is the Tikhonov regularization method which is introduced by Tikhonov (see, e.g., [1, 75]).

The minimization problem

$$\min_{x\in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2.$$

is, in general, ill-posed. So regularization is needed. We consider Tikhonov's regularization

$$\min_{x\in C} f_\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{37}$$

where $\alpha > 0$ is the regularization parameter. The regularized minimization (37) has a unique solution which is denoted as x_α. The following result is easily proved and thus its proof is omitted.

Proposition 3.7 ([85]) *If the SFP (1) is consistent, then the strong* $\lim_{\alpha\to\infty} x_\alpha$ *exists and is the minimum-norm solution of the SFP (1).*

First, observing that the gradient

$$\nabla f_\alpha = \nabla f + \alpha I = A^*(I - P_Q)A + \alpha I$$

is $\|A\|^2 + \alpha$-Lipschitz and α-strongly moontone, the mapping $P_C(I - \gamma)\nabla f_\alpha$ is a contraction with coefficient

$$\sqrt{1 - \gamma(2\alpha - \gamma(\|A\|^2 + \alpha))} \leq 1 - \frac{1}{2}\alpha\gamma,$$

where $0 < \gamma \leq \frac{\alpha}{\|A\|^2+\alpha}$.

Xu [85] proposed the following regularization method for solving the SFP:

Algorithm 5 Regularization algorithm

Generate $\{x_{n+1}\}$ by

$$x_{n+1} = P_C(I - \gamma_n \nabla f_{\alpha_n})x_n = P_C((1 - \alpha_n\gamma_n)x_n - \gamma_n A^*(I - P_Q)Ax_n).$$

Theorem 3.5 ([85]) *Assume that* $\{\alpha_n\}$ *and* $\{\gamma_n\}$ *satisfy the following conditions:*
(i) $0 < \gamma_n \leq \frac{\alpha_n}{\|A\|^2+\alpha_n}$ *for all (large enough) n;*
(ii) $\alpha_n \to 0$ *and* $\gamma_n \to 0$*;*
(iii) $\sum_{n=0}^{\infty} \alpha_n\gamma_n = \infty$*;*
(iv) $(|\gamma_{n+1} - \gamma_n| + \gamma_n|\alpha_{n+1} - \alpha_n|)/(\alpha_{n+1}\gamma_{n+1})^2 \to 0$.
Then $\{x_n\}$ *converges in norm to the minimum-norm solution of the SFP.*

3.3 Acceleration Methods

In this section, we will report several acceleration methods, including inertial techniques and alternated inertial techniques.

3.3.1 Inertial Technique

To accelerate the convergence rate of the algorithms, in 1964, Polyak [53] considered the second-order dynamical system for solving smooth convex minimization problem (SCMP):

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nabla f(x(t)) = 0,$$

where $\gamma > 0$ represents a friction parameter and ∇f is the gradient of f. This dynamic system is called the Heavy Ball with Friction (HBF).

Next, consider the discretization of this dynamic system (HBF):

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} + \gamma\frac{x_n - x_{n-1}}{h} + \nabla f(x_n) = 0, \ \forall n \geq 0,$$

which can be written as follows:

$$x_{n+1} = x_n + \beta(x_n - x_{n-1}) - \alpha\nabla f(x_n), \ \forall n \geq 0,$$

where $\beta = 1 - \gamma h$ and $\alpha = h^2$, then it can be written as follows:

$$\begin{cases} y_n = x_n + \beta(x_n - x_{n-1}), \\ x_{n+1} = y_n - \alpha\nabla f(x_n), \ \forall n \geq 0. \end{cases}$$

Remark 3.1 (a) This iteration is called the inertial extrapolation algorithm (ITEA) or the heavy ball method (HBM);
(b) the term $x_n - x_{n-1}$ is called the momentum;
(c) the term $\beta(x_n - x_{n-1})$ is called the extrapolation point, it gives inertia to the motion, will lead to motion along the 'essential' direction, thus acting as an acceleration;
(d) $\beta \in [0, 1)$ is called an extrapolation factor.

As far as we know, most previous algorithms only use the current point to get the next iteration and converge more slowly. The main idea of inertial techniques (also known as inertial algorithms or momentum algorithms) is to use the first two iterations to update the next. Inertia techniques are mainly used to accelerate the learning of high curvature, noise, or small but consistent gradients. The inertial algorithm accumulates the moving average of the previous exponential decay of the gradient and continues to move in this direction. In doing so, it mimics the inertia (the concept and nature of inertia in physics) that occurs when an object moves, that is, to some extent considers the direction of previous updates and uses the gradient of the current iteration to fine-tune the final result, so that some stability can be increased and faster learning can be promoted. Inertial algorithm can accelerate the learning of parameters with the same direction and reduce the updating of parameters with gradient change direction. Therefore, inertia can accelerate learning in related directions, suppress oscillations, and thus accelerate convergence. Because of the existence of inertia term in the algorithm, the convergence speed of the algorithm is accelerated, inertial algorithm has been widely studied by many scholars.

In order to improve the convergence rate, Nesterov [49] introduced a modified heavy ball method as follows:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = y_n - \lambda_n \nabla f(y_n), \ \forall\, n \geq 1. \end{cases} \tag{38}$$

where $\theta_n \in [0, 1)$ is an extrapolation factor and $\{\lambda_n\}$ is a sequence of positive numbers.

In [2], Alvarez and Attouch employed the idea of the heavy ball method to the setting of a general maxmimal monotone operator using the framework of the proximal point algorithm. This method is called the inertial proximal point algorithm, which is of the following form:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \lambda F)^{-1}(y_n), \ \forall n \geq 1, \end{cases} \tag{39}$$

where F is a maximal monotone operator. It was proved that, if $\{\lambda_n\}$ is nondecreasing and $\theta_n \in [0, 1)$ is chosen such that

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty,$$

then $\{x_n\}$ generated by (39) converges weakly to a zero point of F.

In a consequential work, Maingé [41] introduced the inertial technique for solving the fixed point problem in Hilbert spaces. He introduced the inertial Mann algorithm $\{x_n\}$ for nonexpansive mappings, which is generated as follows: For any $x_0, x_1 \in H$,

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n T(y_n), \ \forall n \geq 1, \end{cases} \tag{40}$$

It was proved that the sequence $\{x_n\}$ generated by (40) converges weakly to a fixed point of T under the following conditions:

(i) $\theta_n \in [0, \theta)$;

(ii) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$;

(iii) $0 < \inf_{n\geq 1} \alpha_n \leq \sup_{n\geq 1} \alpha_n < 1$. where T is a nonexpansive mapping on H, $\theta \in [0, 1)$ and $\alpha_n \in (0, 1)$.

In [21], Dang et al. applied the inertial technique of Alvarez and Attouch to the CQ algorithm and proposed an inertial relaxed CQ algorithm, which is as follows: For any $x_0, x_1 \in H$,

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = P_{C_n}(I - \lambda_n A^T(I - P_{Q_n} A y_n)), \end{cases} \tag{41}$$

It was proved that, if $\gamma \in (0, \frac{2}{\|A\|^2})$ and $\theta_n \in [0, \overline{\theta_n}]$ with

$$\overline{\theta_n} = \min\{\theta, \frac{1}{\max\{n^2\|x_n - x_{n-1}\|^2, n^2\|x_n - x_{n-1}\|\}}\},$$

where $\theta \in [0, 1)$. Dang proved that the algorithm $\{x_n\}$ converges weakly to a point of Γ.

3.3.2 Alternated Inertial Technique

As important observation regarding the above inertial methods, is that the sequence $\{x_n\}$ generated by these inertial-type methods does not have a monotonic behaviour with respect to $x^* \in \Gamma$ and can move or swing back and forth around Γ, see, for example, [6, 39]. This could explain why such inertial extrapolation step does not converge faster than its counterpart non-inertial methods, see, e.g., [42]. In a direction to resolve the above issue, an alternated inertial method was introduced recently in [47]. This alternated inertia method shown to exhibit attractive performances in practice including monotonicity of $\{\|x_{2n} - x^*\|\}$, see [32, 33] for more details.

Shehu et al. [61] proposed an alternated inertial CQ algorithm for the first time:

Algorithm 6 Alternated inertial CQ algorithm

Let x_0, x_1 be arbitrary. Compute

$$y_n = \begin{cases} x_n, & \text{n=even,} \\ x_n + \theta_n(x_n - x_{n-1}), & \text{n=odd.} \end{cases}$$

Construct x_{n+1} via the formula

$$x_{n+1} = P_C(y_n - \tau_n \nabla f(y_n)),$$

where $f(y_n) = \frac{1}{2}\|(I - P_Q)Ay_n\|^2$, $\nabla f(y_n) = A^*(I - P_Q)Ay_n$ and

$$\tau_n = \begin{cases} \frac{\rho_n f(y_n)}{\|\nabla f(y_n)\|^2}, & \|\nabla f(y_n)\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.6 ([61]) *Assume that $\rho_n \subset (0, 4)$ is non-decreasing and sequence $\{\theta_n\} \subset [0, \infty)$ such that $0 < \alpha \leq \rho_n(1 + \theta_n) \leq \beta < 4$. Then the sequence $\{x_n\}$ generated by Algorithm 6 weakly converges to a solution of the SFP.*

3.4 Relaxed Techniques

In this section, we will give three relaxed techniques, which make the projections have closed-form expressions.

3.4.1 Half-Space-Relaxed

The computation of a projection onto a general close convex subset is generally difficult. To overcome this difficulty, Fukushima suggested a so-called relaxed projection method to calculate the projection onto a level set of a convex function by computing a sequence of projections onto half-spaces containing the original level set. In the setting of finite-dimensional Hilbert spaces, this idea was followed by Yang [88], who introduced the relaxed CQ algorithms for solving SFP where the closed convex subsets C and Q are level sets of convex functions given as follows:

$$C = \{x \in H_1 : c(x) \leq 0\} \text{ and } Q = \{y \in H_2 : q(y) \leq 0\}, \tag{42}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex functions. Yang assumes that both c and q are subdifferentiable on H_1 and H_2, respectively, and that ∂c and ∂q are bounded operators (i.e., bounded on bounded sets). It is known that every convex function defined on a finite-dimensional Hilbert space is subdifferentiable and its sundifferential operator is a bounded operator.

Define the halfspaces C_n and Q_n as follows:

$$C_n = \{x \in H_1 : c(x_n) \leq \langle \xi_n, x_n - x\rangle\}, \tag{43}$$

where $\xi_n \in \partial c(x_n)$, and

$$Q_n = \{y \in H_2 : q(Ax_n) \leq \langle \eta_n, Ax_n - y\rangle\}, \tag{44}$$

where $\xi_n \in \partial q(Ax_n)$. By the definition of the subgradient, it is clear that $C \subseteq C_n$ and $Q \subseteq Q_n$. The projections onto C_n and Q_n are easy to compute since C_n and Q_n are half-spaces. Yang's relaxed CQ algorithm is of the form:

$$x_{n+1} = P_{C_n}(x_n - \gamma A^*(I - P_{Q_n})Ax_n),$$

where $\gamma \in (0, \frac{2}{\|A\|^2})$. The weak convergence of the above algorithm is proved.

Define

$$f_n(x) = \frac{1}{2}\|(I - P_{Q_n})Ax\|^2, \; n \geq 0. \tag{45}$$

Then we have

$$\nabla f_n(x) = A^*(I - P_{Q_n})Ax. \tag{46}$$

López et al. [38] improved this relaxed CQ algorithm as follows:

$$x_{n+1} = P_{C_n}(x_n - \tau_n A^*(I - P_{Q_n})Ax_n), \tag{47}$$

where

$$\tau_n = \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2},\ 0 < \rho_n < 4. \tag{48}$$

This algorithm has no need to know a prior any information of $\|A\|^2$.

3.4.2 Ball-Relaxed

In 2018, Yu et al. [93] proposed the ball-relaxed CQ algorithms for the SFP.

The set C is given by

$$C = \{x \in H_1 | c(x) \leq 0\}, \tag{49}$$

where $c : H_1 \to (-\infty, +\infty]$ is an α-strongly convex lower semi-continuous function. The set Q is given by

$$Q = \{y \in H_2 | q(y) \leq 0\}, \tag{50}$$

where $q : H_2 \to (-\infty, +\infty]$ is an β-strongly convex lower semi-continuous function. For any $x \in H_1$, at least one subgradient $\xi \in \partial c(x)$ can be calculated. For any $y \in H_2$, at least one subgradient $\eta \in \partial q(y)$ can be calculated. Both $\partial c(x)$ and $\partial q(y)$ are bounded on bounded sets.

Since c is a strongly convex function with constant α, it follows that

$$c(x) \geq c(x_n) + \langle \xi_n, x - x_n \rangle + \frac{\alpha}{2}\|x - x_n\|^2, \tag{51}$$

where $\xi_n \in \partial c(x_n)$. The set C_n^b is defined as

$$C_n^b = \{x \in H_1 | c(x_n) + \langle \xi_n, x - x_n \rangle + \frac{\alpha}{2}\|x - x_n\|^2 \leq 0\}. \tag{52}$$

Similarly, q is also a strongly convex function with constant β, then

$$q(y) \geq q(Ax_n) + \langle \eta_n, y - Ax_n \rangle + \frac{\beta}{2}\|y - Ax_n\|^2, \tag{53}$$

where $\eta_n \in \partial q(Ax_n)$. The set Q_n^b is defined as

$$Q_n^b = \{y \in H_2 | q(Ax_n) + \langle \eta_n, y - Ax_n \rangle + \frac{\beta}{2}\|y - Ax_n\|^2 \leq 0\}. \tag{54}$$

It is obvious that $C \subseteq C_n^b$ for all $n \geq 0$ from (51). In the following, Yu et al. proved that C_n^b is a ball. Observe that

$$\begin{aligned} C_n^b &= \{x \in H_1 | c(x_n) + \langle \xi_n, x - x_n \rangle + \frac{\alpha}{2}\|x - x_n\|^2 \leq 0\} \\ &= \{x \in H_1 | c(x_n) + \frac{\alpha}{2}(\|x - x_n\|^2 + \frac{2}{\alpha}\langle \xi_n, x - x_n \rangle) \leq 0\} \\ &= \{x \in H_1 | \|x - (x_n - \frac{1}{\alpha}\xi_n)\|^2 \leq \frac{1}{\alpha^2}\|\xi_n\|^2 - \frac{2}{\alpha}c(x_n)\}. \end{aligned} \tag{55}$$

Since C is nonempty, then there is $\hat{x} \in C$, which implies $\hat{x} \in C_n^b$. So we have

$$\frac{1}{\alpha^2}\|\xi_n\|^2 - \frac{2}{\alpha}c(x_n) \geq \|\hat{x} - (x_n - \frac{1}{\alpha}\xi_n)\|^2 \geq 0.$$

Therefore, C_n^b is a ball whose centre and radius are $x_n - \frac{1}{\alpha}\xi_n$ and $\sqrt{\frac{1}{\alpha^2}\|\xi_n\|^2 - \frac{2}{\alpha}c(x_n)}$, respectively. Similarly, we can conclude that $Q \subseteq Q_n^b$ for all $n \geq 0$ and

$$Q_n^b = \{y \in H_2 | \|y - (Ax_n - \frac{1}{\beta}\eta_n)\|^2 \leq \frac{1}{\beta^2}\|\eta_n\|^2 - \frac{2}{\beta}q(Ax_n)\}$$

is also a ball whose centre and radius are $Ax_n - \frac{1}{\beta}\eta_n$ and $\sqrt{\frac{1}{\beta^2}\|\eta_n\|^2 - \frac{2}{\beta}q(Ax_n)}$, respectively.

In what follows, define

$$f_n^b(x) = \frac{1}{2}\|(I - P_{Q_n^b})Ax\|^2, \ n \geq 0.$$

We then have

$$\nabla f_n^b(x) = A^*(I - P_{Q_n^b})Ax,$$

where Q_n^b is given as (54).

Yu et al. [93] proposed three ball-relaxed CQ algorithms for solving the SFP. The first algorithm is given as follows:

Algorithm 7 The first ball-relaxed CQ algorithm

Let x_0 be arbitrary. Given x_n, construct x_{n+1} via the formula

$$x_{n+1} = P_{C_n^b}(x_n - \tau_n^b A^*(I - P_{Q_n^b})Ax_n), \tag{56}$$

where C_n^b and Q_n^b are given as (52) and (54), respectively, the stepsize τ_n^b is defined as follows:

$$\tau_n^b = \frac{\rho_n f_n^b(x_n)}{\|\nabla f_n^b(x_n)\|^2}, \ 0 < \rho_n < 4. \tag{57}$$

Theorem 3.7 ([93]) *Assume that* $\inf_n \rho_n(4-\rho_n) > 0$. *Then the sequence* $\{x_n\}$ *generated by Algorithm 7 weakly converges to a solution of the SFP.*

In Algorithm 7, Yu et al. assumed that c and q are both strongly convex functions. But in practice, c and q are not all strongly convex functions. When one is strongly convex, the other is convex, Yu et al. gave the the following two algorithms for solving the SFP.

When c is strongly convex and q is only convex, they modify Algorithm 7 to the following Algorithm 8. If c is convex and q is strongly convex, then we use the following Algorithm 9 to solve the SFP.

Algorithm 8 The second ball-relaxed CQ algorithm

Let x_0 be arbitrary. Given x_n, construct x_{n+1} via the formula

$$x_{n+1} = P_{C_n^b}(x_n - \tau_n A^*(I - P_{Q_n})Ax_n), \tag{58}$$

where C_n^b, τ_n and Q_n are given as (52), (48) and (44), respectively.

Algorithm 9 The third ball-relaxed CQ algorithm

Let x_0 be arbitrary. Given x_n, construct x_{n+1} via the formula

$$x_{n+1} = P_{C_n}(x_n - \tau_n^b A^*(I - P_{Q_n^b})Ax_n), \tag{59}$$

where C_n, τ_n^b and Q_n^b are given as (43), (57) and (54), respectively.

3.4.3 Halfspace-Intersection-Relaxed

In 2022, Yu and Wang [92] proposed several new relaxed CQ algorithms. The main idea of the algorithms is to replace the projections to the half-spaces C_n and Q_n with the projections to the intersection of C_n and C_{n-1}, and the intersection of Q_n and Q_{n-1}, respectively. The convergence rate of the algorithms is improved by making full use of the previous half-spaces. Their algorithms are of the forms:

Remark 3.2 (1) Since C_{n-1}, C_n and Q_{n-1}, Q_n are both half-spaces, C_n^2 and Q_n^2 are both the intersection of two half-spaces.
(2) It is clear that $C \subseteq C_{n-1}$, $C \subseteq C_n$ and $Q \subseteq Q_{n-1}$, $Q \subseteq Q_n$ from the subdifferentiable inequality. Hence we have $C \subseteq C_n^2$ and $Q \subseteq Q_n^2$.
(3) The projection to the intersection of two half-spaces has a explicit formula (see ([5], p. 420–422) for details). Therefore, Algorithm 10 is easily implemented.

Algorithm 10 The first half-space-intersection relaxed CQ algorithm

Let x_0, x_1 be arbitrary. Given x_n, construct x_{n+1} via the formula

$$x_{n+1} = P_{C_n^2}(x_n - \tau_n A^*(I - P_{Q_n^2})Ax_n), \tag{60}$$

where C_n^2, Q_n^2 and τ_n are respectively defined as follows:

$$C_n^2 = C_n \cap C_{n-1},\ Q_n^2 = Q_n \cap Q_{n-1}$$

and

$$\tau_n = \frac{\rho_n \|(I - P_{Q_n^2})Ax_n\|^2}{\|A^*(I - P_{Q_n^2})Ax_n\|^2}$$

with $\rho_n \in (0, 2)$.

Theorem 3.8 ([92]) *Assume that* $\inf_n \rho_n(2 - \rho_n) > 0$. *Then the sequence* $\{x_n\}$ *generated by Algorithm 10 weakly converges to a solution of the SFP.*

Algorithm 11 The second half-space-intersection relaxed CQ algorithm

Let x_0, x_1 be arbitrary. Given x_n, construct x_{n+1} via the formula

$$x_{n+1} = P_{C_n} P_{C_{n-1}}(x_n - \tau_n(\beta A^*(I - P_{Q_n})Ax_n + (1 - \beta)A^*(I - P_{Q_{n-1}})Ax_n)), \tag{61}$$

where $\beta \in (0, 1)$ and the stepsize τ_n is chosen as

$$\tau_n = \frac{\rho_n(\beta\|(I - P_{Q_n})Ax_n\|^2 + (1 - \beta)\|(I - P_{Q_{n-1}})Ax_n\|^2)}{\beta\|A^*(I - P_{Q_n})Ax_n\|^2 + (1 - \beta)\|A^*(I - P_{Q_{n-1}})Ax_n\|^2}$$

with $\rho_n \in (0, 2)$.

Algorithm 12 The third half-space-intersection relaxed CQ algorithm

Let x_0, x_1 be arbitrary. Given x_n, construct x_{n+1} via the formula

$$\begin{cases} y_n = x_n - \tau_n(\beta A^*(I - P_{Q_n})Ax_n + (1 - \beta)A^*(I - P_{Q_{n-1}})Ax_n), \\ x_{n+1} = \alpha P_{C_n}(y_n) + (1 - \alpha)P_{C_{n-1}}(y_n), \end{cases} \tag{62}$$

where $\alpha, \beta \in (0, 1)$ and the stepsize τ_n is defined as Algorithm 11.

The proposed algorithms are all weakly convergent to the solution of SFP.

3.5 *Step-Sizes*

In this section, we will introduce three ways to determine the stepsize γ_n in CQ algorithm and other algorithms: fixed stepsize, Armijo-like searches, self-adaptive stepsize.

3.5.1 Fixed Stepsize

The first one is to take the stepsize $\gamma_n \in (0, \frac{2}{\|A\|^2})$ which depends on the operator (matrix) norm $\|A\|$ (or the largest eigenvalue of A^*A). This means that, in order to implement the CQ algorithm, one has first to compute (or, at least, estimate) the matrix norm of A, which is in general not an easy work in practice.

In fact, to overcome this difficulty, Byrne [9] presented a helpful method for estimating matrix norms. But the condition that Byrne put on his method seems restrictive. His alternative way is to construct another step that has no connection with matrix norms.

3.5.2 Line Search Rule

The second one is to select the stepsize $\gamma_n > 0$ self-adaptively by adopting Armijo-like searches:

$$\gamma_n \|\nabla f(x_n) - \nabla f(y_n)\| \leq \mu \|x_n - y_n\|, \ \forall \mu \in (0, 1),$$

where $y_n = P_C(x_n - \gamma_n \nabla f(x_n))$.

In [55], Qu and Xiu developed the relaxed CQ algorithm by adopting Armijo-like searches, the modified relaxed algorithm needs not compute the largest eigenvalue of the matrix $A^T A$, and the objective function can sufficiently decrease at each iteration. The algorithm is shown below:

Algorithm 13 Relaxed CQ algorithm with Armijo-like searches

Given constants $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$. Let x_0 be arbitrary. For $n = 0, 1, \ldots$, let

$$y_n = P_{C_n}(x_n - \gamma_n \nabla f_n(x_n)),$$

where ∇f_n is given by (46), the setpsize $\gamma_n = \gamma l^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\|\nabla f_n(x_n) - \nabla f_n(y_n)\| \leq \mu \frac{\|x_n - y_n\|}{\gamma_n}. \tag{63}$$

Set

$$x_{n+1} = P_{C_n}(x_n - \gamma_n \nabla f_n(y_n)).$$

The algorithm presented above is weakly convergent to the solution of SFP. Note that to obtain γ_n, one generally requires to compute y_n several times which may cost much since it involves the projections onto C and Q and matrix-vector product.

3.5.3 Self-adaptive Stepsize

Yang [89] used the way proposed by Byrne [9] and considered the following step-size:

$$\gamma_n = \frac{\rho_n}{\|\nabla f(x_n)\|}, \tag{64}$$

where $\{\rho_n\}$ is a sequence of positive real numbers such that

$$\sum_{n=0}^{\infty} \rho_n = \infty, \sum_{n=0}^{\infty} \rho_n^2 < \infty. \tag{65}$$

Then Yang proved the convergence of the algorithm (GPA) (27), where the sequence $\{\gamma_n\}$ of step-sizes satisfies the conditions (64) and (65) with two additional conditions:

(a) Q is a bounded subset;

(b) A is a matrix with a full column rank.

Yang's conditions are restrictive in the sense that some important cases may be excluded if we consider the following: the choice of the step-size sequence $\{\gamma_n\}$, the set Q and the matrix A.

In order to remove the two additional conditions (a) and (b) of Yang, López [38] introduced the following choice of the step-size sequence $\{\gamma_n\}$ as follows:

$$\gamma_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \tag{66}$$

where $\{\rho_n\}$ is a sequence of positive real numbers bounded from below by 0 and from above by 4.

They showed the convergence of the CQ algorithm under this choice of the step-size sequence $\{\gamma_n\}$ without assuming any additional conditions on the set Q and the matrix A or prior information about the matrix norm A. Since its inception, (66) has attracted widespread attention due to its good numerical performance and simple form.

4 The Split Equality Problem

In this section, we will review and report the development of the split equality problems. Then we extend SEP to more questions, including but not limited the Multiple-Set Split Equality problems. Let H_1, H_2, H_3 be real Hilbert spaces. Let A: $H_1 \to H_3$,

$B: H_2 \to H_3$ be two bounded linear operators. C and Q are the non-empty closed convex subsets of real Hilbert spaces H_1 and H_2. Then the split equality problem is proposed :

$$\text{Find } x \in C,\ y \in Q \text{ such that } Ax = By. \tag{67}$$

By using the fact (67), Moudafi et al. [12] obtained the corresponding algorithms and the convergence theorems of the SEP. In this situation, $\{\gamma_n\}$ is a positive not decreasing sequence such that $\gamma_n \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$ for a small enough $\varepsilon > 0$. And λ_A, λ_B stand for the spectral radius of A^*A and B^*B.

Algorithm 14 The alternating CQ-algorithm

Generate (x_{n+1}, y_{n+1}) by

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \gamma_n B^*(Ax_{n+1} - By_n)). \end{cases} \tag{68}$$

Theorem 3.9 ([12]) *The sequence (x_n, y_n) generated by Algorithm 14 weakly converges to a solution $(\bar{x}, \bar{y})$ of SEP (67). Moreover $\{Ax_n - By_n\}$ strongly converges to* 0 *and both $\{x_n\}$ and $\{y_n\}$ are asymptotically regular.*

Remark 3.3 Algorithm 14 gives us a solution about SEP. But there are some difficulty, for example we cannot usually obtain two projections P_C and P_Q. And we also need the prior information about bounded linear operator norms.

In order to ease the inconvenience of projections in the Algorithm 14, Moudafi et al. [46] proposed the relaxed alternating CQ-algorithm. Also, $\{\gamma_n\}$ is a positive not decreasing sequence such that $\gamma_n \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$ for a small enough $\varepsilon > 0$. And λ_A, λ_B stand for the spectral radius of A^*A and B^*B.

Algorithm 15 The relaxed alternating CQ-algorithm

$$\begin{cases} x_{n+1} = P_{C_n}(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_{Q_n}(y_n + \beta_n B^*(Ax_{n+1} - By_n)). \end{cases} \tag{69}$$

In this situation, (C_n), (Q_n) are two sequences of closed convex sets defined by

$$C_n = \{x \in H_1 : c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0\},$$

where $\xi_n \in \partial c(x_n)$ and

$$Q_n = \{y \in H_2 : q(y_n) + \langle \eta_n, x - x_n \rangle \leq 0\},$$

where $\eta_n \in \partial c(x_n)$.

Theorem 3.10 ([46]) *The sequence (x_n, y_n) generated by Algorithm 15 weakly converges to a solution $(\bar{x}, \bar{y})$ of SEP (67). Moreover $\{Ax_n - By_n\}$ strongly converges to 0 and both $\{x_n\}$ and $\{y_n\}$ are asymptotically regular.*

Remark 3.4 Algorithm 15 gives us a new solution about SEP. At the same time, it avoids the case that there may not be able to get P_C and P_Q on the basis of Algorithm 14. But we also need the prior information about bounded linear operator norms.

Observed that in the Algorithm 14 and the Algorithm 15 mentioned above, the determination of the stepsize γ_n depends on the operator matrix norms $\|A\|$ and $\|B\|$ (or the largest eigenvalues of A^*A and B^*B). This means that in order to implement the alternating CQ-algorthm, one has first to compute (or, at least estimate) operator norms of A and B, which is in general not easy work in practice.

To overcome this difficulty, López et al. [38] and Zhao et al. [96] presented a helpful method for estimating the stepsizes which do not need prior knowledge of the operator norms for solving the split feasibility problems and multiple-set split feasibility problems, respectively.

Inspired by them, Dong et al. [24] introduced a new choice of the stepsize sequence (γ_n) and gave the convergence analyse.

Algorithm 16 The alternating CQ-algorithm with adaptive stepsize

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \gamma_n B^*(Ax_n - By_n)), \end{cases} \tag{70}$$

where the stepsize γ_n is chosen in such a way that

$$\gamma_n = \sigma_n min\{\frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2}, \frac{\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2}\}, \tag{71}$$

where $0 < \sigma_n < 1$.

Theorem 3.11 ([24]) *Assume $\sigma_n \in (\varepsilon, 1-\varepsilon)$. Then the sequence (x_n, y_n) generated by Algorithm 16 weakly converges to a solution of the SEP (67) and both $\{x_n\}$ and $\{y_n\}$ are asymptotically regular.*

At the same time, Dong et al. [24] presented a relaxed projection algorithm.

Theorem 3.12 ([24]) *Assume $\sigma_n \in (\varepsilon, 1-\varepsilon)$. Then the sequence (x_n, y_n) generated by Algorithm 17 weakly converges to a solution of the SEP (67) and both $\{x_n\}$ and $\{y_n\}$ are asymptotically regular.*

As an acceleration process, the inertial extrapolation algorithms were widely studied. The researchers constructed many iterative algorithms by using inertial extrapolation. The main feature of the inertial extrapolation algorithms is that the next iterate is defined by making use of the previous two iterates. In this part, by using the inertial extrapolation, Dong et al. [25] introduced an inertial projection algorithm to accelerate the iterative process.

Algorithm 17 The relaxed alternating CQ-algorithm with an adaptive stepsize

$$\begin{cases} x_{n+1} = P_{C_n}(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_{Q_n}(y_n + \gamma_n B^*(Ax_{n+1} - By_n)). \end{cases} \tag{72}$$

$$C_n = \{x \in H_1 : c(x_n) + \langle \xi_n, x - x_n \rangle \le 0\},$$

where $\xi_n \in \partial c(x_n)$ and

$$Q_n = \{y \in H_2 : q(y_n) + \langle \eta_n, x - x_n \rangle \le 0\},$$

where $\eta_n \in \partial q(y_n)$. And the stepsize is chosen as follows:

$$\gamma_n = \sigma_n min\{\frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2}, \frac{\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2}\},$$

where $\sigma_n \in (0, 1)$.

Algorithm 18 The alternating CQ-algorithm with the inertial extrapolation

$$\begin{cases} (\bar{x_n}, \bar{y_n}) = (x_n, y_n) + \alpha_k(x_n - x_{n-1}, y_n - y_{n-1}), \\ x_{n+1} = P_C(\bar{x_n} - \gamma_n A^*(A\bar{x_n} - B\bar{y_n})), \\ y_{n+1} = P_Q(\bar{y_n} + \gamma_n B^*(A\bar{x_n} - B\bar{y_n})), \end{cases} \tag{73}$$

where $\alpha_n \in (0, 1)$ and the stepsize γ_n is chosen as:

$$\gamma_n = \sigma_n min\{\frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2}, \frac{\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2}\}.$$

Theorem 3.13 ([25]) *Assume $\sigma_n \in [\varepsilon, 1 - \varepsilon], \varepsilon \in (0, \frac{1}{2}]$ and suppose the assumptions following hold: (i) $0 \le \alpha_n \le \alpha$, where $\alpha \in [0, 1)$;*
(ii) $\sum_{n=1}^{+\infty} \alpha_n(\|x_n - x_{n-1}\|^2 + \|y_n - y_{n-1}\|^2) < +\infty$;
(iii) $\lim_{n\to\infty} \alpha_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|) = 0$.
Then the sequence (x_n, y_n) generated by Algorithm 18 weakly converges to a solution of the SEP (67). Furthermore, both $\{x_n\}$ and $\{y_n\}$ are asymptotically regular.

Observed that the above Algorithm 14, Algorithm 15, Algorithm 16 Algorithm 17 and Algorithm 18 only have the weak convergence. Thus Shi et al. [64] proposed the strong convergence of interative algorithms for the split equality problem. In this situation, we use Γ to denote the solution of SEP, i.e.,

$$\Gamma = \{(x, y) \in H_1, Ax = By, x \in C, y \in Q\}, \tag{74}$$

and assume the consistency of SEP so that Γ is convex, closed and nonempty.

Let $S = C \times Q$ in $H = H_1 \times H_2$, defined $G : H \to H_3$ by $G = [A, -B]$, then $G^*G : H \to H$ has the matrix form:

$$\begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix}. \tag{75}$$

The original problem can be reformulated as finding $w = (x, y) \in S$ with $Gw = 0$.

Algorithm 19 The alternating CQ-algorithm with strong convergence

$$w_{n+1} = P_S\{(1-\alpha_n)[I - \gamma G^* G]w_n\}, \tag{76}$$

i.e.,

$$\begin{cases} x_{n+1} = P_C\{(1-\alpha_n)[x_n - \gamma A^*(Ax_n - By_n)]\}, \\ y_{n+1} = P_Q\{(1-\alpha_n)[y_n + \gamma B^*(Ax_n - By_n)]\}, \end{cases}$$

where $\alpha_n > 0$ is a sequence in (0, 1) such that
(i) $\lim_n \alpha_n = 0$;
(ii) $\sum_0^\infty \alpha_n = \infty$;
(iii) $\sum_0^\infty |\alpha_{n+1} - \alpha_n| < \infty$ or $lim_n \frac{|\alpha_{n+1}-\alpha_n|}{\alpha_n} = 0$.

Theorem 3.14 ([64]) *The sequence $\{w_n\}$ generated by Algorithm 19 strongly converges to the minimum-norm solution $\tilde{w}$ of SEP(67).*

Proof Let R_n and R be defined by

$$\begin{cases} R_n w = P_S\{(1-\alpha_n)[I - \gamma G^* G]\}w = P_S[(1-\alpha_n)Tw], \\ Rw = P_S(I - \gamma G^* G)w = P_S(Tw), \end{cases} \tag{77}$$

where $T = I - \gamma G^* G$. By proposition 1.2 it is easy to see that R_n is a contraction with contractive constant $1 - \alpha_n$ and algorithm 19 can be written as $w_{n+1} = R_n w_n$.

For any $\widehat{w} \in \Gamma$, we have

$$\begin{aligned} \|R_n \widehat{w} - \widehat{w}\| &= \|P_S[(1-\alpha_n)T\widehat{w}] - \widehat{w}\| \\ &= \|P_S[(1-\alpha_n)T\widehat{w}] - P_S(T\widehat{w})\| \\ &\le \|(1-\alpha_n)T\widehat{w} - T\widehat{w}\| \\ &= \alpha_n \|T\widehat{w}\| \le \alpha_n \|\widehat{w}\|. \end{aligned} \tag{78}$$

Hence,

$$\begin{aligned} \|w_{n+1} - \widehat{w}\| &= \|R_n w_n - \widehat{w}\| \\ &\le \|R_n w_n - R_n \widehat{w}\| + \|R_n \widehat{w} - \widehat{w}\| \\ &\le (1-\alpha_n)\|w_n - \widehat{w}\| + \alpha_n \|\widehat{w}\| \\ &\le \max\{\|w_n - \widehat{w}\|, \|\widehat{w}\|\}. \end{aligned} \tag{79}$$

It follows that $\|w_n - \widehat{w}\| \le \max\{\|w_n - \widehat{w}\|, \|\widehat{w}\|\}$. So $\{w_n\}$ is bounded. Next we prove that $\lim_n \|w_{n+1} - w_n\| = 0$. Indeed,

$$\begin{aligned}\|w_{n+1} - w_n\| &= \|R_n w_n - R_{n-1} w_{n-1}\| \\ &\leq \|R_n w_n - R_n w_{n-1}\| + \|R_n w_{n-1} - R_{n-1} w_{n-1}\| \\ &\leq (1 - \alpha_n)\|w_n - w_{n-1}\| + \|R_n w_{n-1} - R_{n-1} w_{n-1}\|.\end{aligned} \tag{80}$$

Notice that

$$\begin{aligned}\|R_n w_{n-1} - R_{n-1} w_{n-1}\| &= \|P_S[(1 - \alpha_n)T w_{n-1}] - P_S[(1 - \alpha_{n-1})T w_{n-1}]\| \\ &= |\alpha_n - \alpha_{n-1}|\|T w_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}|\|w_{n-1}\|.\end{aligned} \tag{81}$$

Hence,

$$\|w_{n+1} - w_n\| \leq (1 - \alpha_n)\|w_n - w_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|w_{n-1}\|. \tag{82}$$

By virtue of assumptions of α_n and Lemma 1.7, we have

$$\lim_n \|w_{n+1} - w_n\| = 0. \tag{83}$$

Therefore,

$$\begin{aligned}\|w_n - R w_n\| &\leq \|w_{n+1} - w_n\| + \|R_n w_n - R w_n\| \\ &\leq \|w_{n+1} - w_n\| + \|(1 - \alpha_n)T w_n - T w_n\| \\ &\leq \|w_{n+1} - w_n\| + \alpha_n \|w_n\| \to 0.\end{aligned} \tag{84}$$

The demiclosedness principle ensures that each weak limit point of $\{w_n\}$ is a fixed point of the nonexpansive mapping $R = P_S T$, that is, a point of the solution set Γ of SEP (67).

At last, we will prove that $lim_n \|w_{n+1} - \tilde{w}\| = 0$.

Choose $0 < \beta < 1$ such that $\frac{\gamma}{1-\beta} < \frac{2}{\rho(G^*G)}$, then $T = I - \gamma G^* G = \beta I + (1 - \beta)V$ where $V = I - \frac{\gamma}{1-\beta} G^* G$ is a nonexpansive mapping. Taking $z \in \Gamma$, we deduce that

$$\begin{aligned}\|w_{n+1} - z\|^2 &= \|P_S[(1 - \alpha_n)T w_n] - z\|^2 \\ &\leq \|(1 - \alpha_n)T w_n - z\|^2 \\ &\leq \|\beta(w_n - z) + (1 - \beta)(V w_n - z)\|^2 + \alpha_n \|z\|^2 \\ &\leq \beta\|w_n - z\|^2 + (1 - \beta)\|V w_n - z\|^2 - \beta(1 - \beta)\|w_n - V w_n\|^2 + \alpha_n \|z\|^2 \\ &\leq \|w_n - z\|^2 - \beta(1 - \beta)\|w_n - V w_n\|^2 + \alpha_n \|z\|^2.\end{aligned} \tag{85}$$

Then

$$\begin{aligned}&\beta(1-\beta)\|w_n - Vw_n\|^2\\ \le&\|w_n - z\|^2 - \|w_{n+1} - z\|^2 + \alpha_n\|z\|^2\\ =&(\|w_n - z\| + \|w_{n+1} - z\|)(\|w_n - z\| - \|w_{n+1} - z\|) + \alpha_n\|z\|^2\\ \le&(\|w_n - z\| + \|w_n + z\|)\|w_n - w_{n+1}\| + \alpha_n\|z\|^2 \to 0.\end{aligned} \tag{86}$$

Note that $T = I - \gamma G^*G = \beta I + (1-\beta)V$, it follows that $\lim_n \|Tw_n - w_n\| = 0$.

Take a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\limsup_n \langle w_n - \tilde{w}, -\tilde{w}\rangle = \lim_k \langle w_{n_k} - \tilde{w}, -\tilde{w}\rangle$.

By virtue of the boundedness of w_n, we may further assume, with no loss of generality, that w_{n_k} converges weakly to a point $\bar{w}$. Since $\|Rw_n - w_n\| \to 0$, using the demiclosedness principle, we know that $\bar{w} \in Fix(R) = Fix(P_S T) = \Gamma$. Noticing that $\bar{w}$ is the projection of the origin onto Γ, we get that

$$\limsup_n \langle w_n - \tilde{w}, -\tilde{w}\rangle = \lim_k \langle w_{n_k} - \tilde{w}, -\tilde{w}\rangle = \langle \bar{w} - \tilde{w}, -\tilde{w}\rangle \le 0. \tag{87}$$

Finally, we compute

$$\begin{aligned}\|w_{n+1} - \tilde{w}\|^2 &= \|P_S[(1-\alpha_n)Tw_n] - \tilde{w}\|^2\\ &= \|P_S[(1-\alpha_n)Tw_n] - P_S T\tilde{w}\|^2\\ &\le \|(1-\alpha_n)Tw_n - T\tilde{w}\|^2\\ &= \|(1-\alpha_n)Tw_n - \tilde{w}\|^2\\ &= \|(1-\alpha_n)(Tw_n - \tilde{w}) + \alpha_n(-\tilde{w})\|^2\\ &= (1-\alpha_n)^2\|Tw_n - \tilde{w}\|^2 + \alpha_n^2\|\tilde{w}\|^2 + 2\alpha_n(1-\alpha_n)\langle Tw_n - \tilde{w}, -\tilde{w}\rangle\\ &\le (1-\alpha_n)\|Tw_n - \tilde{w}\|^2 + \alpha_n[\alpha_n\|\tilde{w}\|^2 + 2(1-\alpha_n)\langle Tw_n - \tilde{w}, -\tilde{w}\rangle].\end{aligned} \tag{88}$$

Since $\limsup_n \langle w_n - \tilde{w}, -\tilde{w}\rangle \le 0$, $\|w_n - Tw_n\| \to 0$, we know that $\limsup[\alpha_n\|\tilde{w}\|^2 + 2(1-\alpha_n)\langle Tw_n - \tilde{w}, -\tilde{w}\rangle] \le 0$. By Lemma 1.7, we conclude that $\lim_n \|w_{n+1} - \tilde{w}\| = 0$. This completes the proof.

Inspired by Algorithm 19, Shi et al. [64] also establishd two KM-CQ-like algorithms converging strongly to a solution of SEP.

Theorem 3.15 ([64]) *The sequence $\{w_n\}$ generated by Algorithm 20 converges strongly to a solution of SEP (3.1).*

Theorem 3.16 ([64]) *The sequence $\{w_n\}$ generated by Algorithm 21 converges strongly to a solution of SEP (67).*

Remark 3.5 By defining the operator G, Algorithm 19, Algorithm 20 and Algorithm 21 give us iterative algorithms which converges strongly to a solution $\tilde{w}$ of SEP. But we also need the prior information about bounded linear operator norms.At the same time, we cannot ease the inconvenience of the projections P_C and P_Q.

Algorithm 20 The KM-CQ-like algorithm

$$w_{n+1} = (1-\beta_n)w_n + \beta_n P_S\{(1-\alpha_n)[I-\gamma G^*G]w_n\}, \tag{89}$$

i.e.,

$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_C\{(1-\alpha_n)[x_n - \gamma A^*(Ax_n - By_n)]\}, \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_Q\{(1-\alpha_n)[y_n + \gamma B^*(Ax_n - By_n)]\}, \end{cases}$$

where $\alpha_n > 0$ is a sequence in (0, 1) such that
(i) $\lim_n \alpha_n = 0, \sum_0^\infty \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Algorithm 21 The another KM-CQ-like algorithm

$$w_{n+1} = (1-\beta_n)(1-\alpha_n)w_n + \beta_n P_S\{(1-\alpha_n)[I-\gamma G^*G]w_n\}, \tag{90}$$

i.e.,

$$\begin{cases} x_{n+1} = (1-\beta_n)(1-\alpha_n)[x_n - \gamma A^*(Ax_n - By_n)] + \beta_n P_C\{(1-\alpha_n)[x_n - \gamma A^*(Ax_n - By_n)]\}, \\ y_{n+1} = (1-\beta_n)(1-\alpha_n)[y_n + \gamma B^*(Ax_n - By_n)] + \beta_n P_Q\{(1-\alpha_n)[y_n + \gamma B^*(Ax_n - By_n)]\}, \end{cases}$$

where $\alpha_n > 0$ is a sequence in (0, 1) such that
(i) $\lim_n \alpha_n = 0, \sum_0^\infty \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Motivated by the work of SEP, there are more and more researches getting involved in the multiple-sets split equality problem(MSSEP) as a special case of SEP.

Let $\{C_1, C_2, ..., C_N\}$ and $\{Q_1, Q_2, ..., Q_M\}$ be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Then the multiple-sets split equality problem(MSSEP) is proposed:

$$\text{Find } x \in \cap_{i=1}^N C_i,\, y \in \cap_{j=1}^M Q_j, \text{ such that } Ax = By, \tag{91}$$

where N and M are positive integers. If N=M=1,the multiple-sets split equality problem reduces to the well-known split equality problem. At the same time, if B=I, the multiple-sets split equality problem reduces to the split feasibility problem.

Let $S_i = C_i \times Q_i$ in $H = H_1 \times H_2$, defined $G : H \to H_3$ by $G = [A, -B]$, then $G^*G : H \to H$ has the matrix form

$$\begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix}. \tag{91}$$

The original problem can be reformulated as finding $w = (x, y) \in \cap_{i=1}^N S_i$ with $Gw = 0$.

In this situation, we use Γ to denote the solution set of MSSEP, i.e.,

$$\Gamma = \{(x, y) \in H_1 \times H_2, Ax = By, x \in \cap_{i=1}^{N} C_i, y \in \cap_{j=1}^{M} Q_j\}, \tag{92}$$

and assume the consistency of MSSEP so that Γ is convex, closed and nonempty.

Shi et al. [65] noted that the convergence of the their algorithms depended on the exacting requirements of the iterative coefficient. They aimed to solve the MSSEP in the framework of infinite-dimensional Hilbert spaces under some more mild conditions for the iterative coefficient. Thus they proposed the following algorithm.

Algorithm 22 The iterative algorithm of MSSEP

$$w_{n+1} = (1 - \alpha_n)[I - \gamma G^* G]w_n + \alpha_n P_{S_{i(n)}}[I - \gamma G^* G]w_n, \tag{94}$$

where i(n)=n(modN)+1, $\alpha_n > 0$ is a sequence in (0,1) and $0 < \gamma < \lambda = \frac{2}{\rho(G^*G)}$ with $\rho(G^*G)$ being the spectral radius of the self-adjoint operator G^*G on H.

Theorem 3.17 ([65]) *If* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, *then the sequence* $\{w_n\}$ *generated by Algorithm 22 weakly converges to a solution of MSSEP (91).*

Remark 3.6 From SEP to MSSEP, it is a new problem which attracts many author's attention due to its applications by changing the domains. But we cannot ease the inconvenience of the projections $P_{S_{i(n)}}$. Thus it need many improvements still.

Observed that the above Algorithm 22 cannot ease the inconvenience of the projections $P_{S_{i(n)}}$. Thus Wu et al. [83] proposed the iterative algorithm for MSSEP with quasi-nonexpanisve multi-valued mappings.

Let $C_j = FixR_1^j$, $Q_j = FixR_2^j$ and $S_j = C_j \times Q_j$, $j = 1, 2, ..., N$, $S = \cap_j^N S_j$. The original problem can be reformulated as finding $w = (x, y) \in \cap_{i=1}^N S_i$ with $Gw = 0$.

Algorithm 23 The iterative algorithm of MSSEP with quasi-nonexpanisve multi-valued mappings

$$w_{n+1} = \alpha_n(I - \gamma G^* G)w_n + (1 - \alpha_n)v_n, v_n \in R_{i(n)}(w_n - \gamma G^* G w_n), \tag{95}$$

where i(n)=n(modN)+1, $\alpha_n > 0$ is a sequence in (0,1) and $0 < \gamma < \lambda = \frac{2}{\rho(G^*G)}$, $R_{i(n)} : H_1 \times H_2 \to H_1 \times H_2$ by

$$\begin{bmatrix} R_1^{i(n)} & 0 \\ 0 & R_2^{i(n)} \end{bmatrix} \tag{96}$$

and $R_1^{i(n)}$, $R_2^{i(n)}$ are a family of quasi-nonexpanisve multi-valued mappings on H_1, H_2, respectively.

Theorem 3.18 ([83]) *If* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, *then the sequence* $\{w_n\}$*generated by Algorithm 23 weakly converges to a solution of MSSEP(91). In addition, if there exists* $1 \le j \le N$ *such that* R_1^j, R_2^j *are semi-compact, then* $\{w_n\}$ *converges to strongly a solution of MSSEP (91).*

Remark 3.7 We denote by CB(H) the collection of all nonempty closed and bounded subsets of H and $d(x, K) = \inf_{y\in K} d(x, y)$. Then a multi-valued mapping $R : H \to CB(H)$ is said to be semi-compact if, for any bounded sequence $\{x_n\} \subseteq H$ with x_n converges weakly to x and $d(x_n, Rx_n) \to 0$, we have $x \in Rx$.

Observed that the above Algorithm 23 already eases the inconvenience of the projections $P_{S_{i(n)}}$. However, the Algorithm 23 still need the prior information about bounded linear operator norms. Thus, Tian et sl. [72] improved Algorithm 23 and reported a new algorithm with split self-adaptive step size.

Algorithm 24 The iterative algorithm of MSSEP with split self-adaptive step size

$$w_{n+1} = w_n + \frac{\rho_1^n \sum_{i=1}^t \alpha_i \|P_{S_i} w_n - w_n\|^2}{\|\sum_{i=1}^t \alpha_i (P_{S_i} w_n - w_n)\|^2} \sum_{i=1}^t \alpha_i (P_{S_i} w_n - w_n) - \frac{\rho_2^n \|G w_n\|^2}{\|G^* G w_n\|^2} G^* G w_n, \quad (97)$$

or component-wise

$$\begin{cases} x_{n+1} = x_n + \frac{\rho_1^n \sum_{i=1}^t \alpha_i \|P_{C_i} x_n - x_n\|^2}{\|\sum_{i=1}^t \alpha_i (P_{C_i} x_n - x_n)\|^2} \sum_{i=1}^t \alpha_i (P_{C_i} x_n - x_n) - \frac{\rho_2^n \|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2} A^*(Ax_n - By_n), \\ y_{n+1} = y_n + \frac{\rho_1^n \sum_{i=1}^t \alpha_i \|P_{Q_i} y_n - y_n\|^2}{\|\sum_{i=1}^t \alpha_i (P_{Q_i} y_n - y_n)\|^2} \sum_{i=1}^t \alpha_i (P_{Q_i} y_n - y_n) + \frac{\rho_2^n \|Ax_n - By_n\|^2}{B^*(Ax_n - By_n)\|^2} B^*(Ax_n - By_n), \end{cases} \quad (98)$$

where $0 < \underline{\rho_1} \le \rho_1^n \le \bar{\rho}_1 < 1, 0 < \underline{\rho_2} \le \rho_2^n \le \bar{\rho}_2 < 1, \{\alpha_i\}_{i=1}^t > 0$.

Theorem 3.19 ([72]) *Let* $\Gamma \ne \emptyset$ *be the solution set of the problem MSSEP (91). Choose an initial point* $w_0 \in H$ *arbitrarily, the iterative sequence* $\{w_n\}$ *with split self-adaptive step size converges weakly to a solution of the problem MSSEP (91).*

Observed that the sequence generated by Algorithm 22 and Algorithm 23 has weak convergence only. Thus Zhao et al. [95] proposed an iterative algorithm solving the problem MSSEP with strong convergence.

Theorem 3.20 ([95]) *Let* $\Gamma \ne \emptyset$ *be the solution set of MSSEP(91), for an arbitrary initial point* $w_0 \in S$, *then the iterative sequence* $\{w_n\}$ *generated by Algorithm 25 converges strongly to a solution of MSSEP(91).*

Remark 3.8 Ceng et al. [11] presented an extragradient method and Yao et al. [90] proposed a subgradient extragradient method to solve the SFP. However, all their methods to solve the problem have only weak convergence in a Hilbert space. On the other hand, a variant extragradient-type method and a subgradient extragradient method introduced by Censor et al. [14, 15] possess strong convergence for solving the variational inequality.

Algorithm 25 An extragradient-type algorithm for MSSEP

$$\begin{cases} v_n = P_S\{(1-\alpha_n)w_n - \gamma_n G^* G w_n\}, \\ w_{n+1} = P_S\{w_n - \mu_n G^* G v_n + \lambda_n(v_n - w_n)\}, \end{cases} \tag{99}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in [0,1] such that $\lim_{n\to\infty}\alpha_n = 0$, and $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\{\gamma_n\}_{n=0}^{\infty}$, $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$ are sequences in H satisfying the following conditions:
(i) $\gamma_n \in (0, \frac{2}{\rho(G^*G)})$, $\lim_{n\to\infty}(\gamma_{n+1} - \gamma_n) = 0$;
(ii) $\lambda_n \in (0, 1)$, $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$;
(iii) $\mu_n \le \frac{2}{\rho(G^*G)\lambda_n}$, $\lim_{n\to\infty}(\mu_{n+1} - \mu_n) = 0$;
(iv) $\sum_{n=1}^{\infty}\frac{\gamma_n}{\lambda_n} < \infty$.

Assume the problem(91) is consistent and let Γ denote its solution set, that is, Γ is not empty. We consider the proximity function

$$f(w) = \frac{1}{2}\sum_{i=1}^{t}\alpha_i\|w - P_{S_i}w\| + \frac{1}{2}\|Gw\|^2, \tag{100}$$

where α_i, $i = 1, ..., t$ are positive real numbers and P_{S_i} are metric projections from H onto S_i. Since C_i and Q_i are closed convex, so are S_i and then P_{S_i} are well defined. Then the problem(91) can be transformed into the minimization problem:

$$min_{w\in\cap_{i=1}^{t}S_i} f(w). \tag{101}$$

Note that the proximity function $f(w)$ is convex and differentiable with gradient

$$\nabla f(w) = \sum_{i=1}^{t}\alpha_i(I - P_{S_i})w + G^*Gw, \tag{102}$$

where I is the identity operator on H. The gradient function $\nabla f(w)$ is L-Lipschitz continuous with Lipschitz constant

$$L = \sum_{i=1}^{t}\alpha_i + \|G\|^2 \tag{103}$$

to solve the minimization problem(4.23), a classical method is the gradient algorithm, which takes the iterative issue

$$w_{n+1} = w_n - \gamma_n\nabla f(w_n), \tag{104}$$

where γ_n is the iterative step size in step n.

Note that in the iteration(4.26), we need to calculate projections for t times in each step. On the other hand, notice that $w^* \in \Gamma$ if and only if $g(w^*) = 0$, where

$$g(w) = \frac{1}{2}\|w - P_{S_{i(n)}}w\| + \frac{1}{2}\|Gw\|^2, \tag{105}$$

in which

$$i(n) \in \{i|max_{1\leq i\leq t}\|w - P_{S_{i(n)}}w\|\}.$$

Then we consider the iterative issue

$$w_{n+1} = w_n - \gamma_n \nabla g(w_n). \tag{106}$$

In the iteration (4.28), we only need to implement a projection once in each step. Motivated by this point, Tian et al. [71] presented the following algorithms to solve problem (91).

Algorithm 26 The Gradient method 1

Take $w_0 \in H$ arbitrarily and compute

$$\begin{cases} z_n = P_{S_{i(n)}}w_n, \\ q_n = G^*Gw_n, \end{cases} \tag{107}$$

where $n \geq 0$ and

$$i(n) \in \{i : max_{i\in\Lambda}\|w_n - P_{S_i}w_n\|, \Lambda = \{1, 2, ..., t\}\}.$$

If

$$\|w_n + q_n - z_n\| = 0, \tag{108}$$

then stop. w_n is the solution. Otherwise, calculate

$$w_{n+1} = w_n - \tau_n(w_n + q_n - z_n), \tag{109}$$

where

$$\tau_n = \lambda_n \frac{\|w_n - z_n\|^2 + \|Gw_n\|^2}{\|2w_n + q_n - z_n\|^2},$$

in which $\lambda_n \in (0, 4)$.

Theorem 3.21 ([71]) *If* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 4$, *taking initial point* $w_0 \in H$ *arbitrarily, then the sequence* $\{w_n\}$ *generated by Algorithm 26 converges weakly to a solution of the problem (91).*

Theorem 3.22 ([71]) *Suppose that* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 4$. *Taking* $u \in H$ *and initial point* $w_0 \in H$ *arbitrarily, then the sequence* $\{w_n\}$ *generated by the Algorithm 27 converges strongly to* $z = P_\Gamma u$.

As we have known, the SEP and the MSSEP have attracted much attention due to its application in various disciplines. At the same time, there are many branch problems as special cases derived from the SEP. Now, we will introduce the algorithms and theorems generated by these problems.

Algorithm 27 The Gradient method 2

Take $u \in H$ and initial point $w_0 \in H$. Compute

$$\begin{cases} z_n = P_{S_{i(n)}} w_n, \\ q_n = G^* G w_n, \end{cases} \tag{110}$$

where

$$i(n) = \{i : \max_{i \in \Lambda} \|w_n - P_{S_i} w_n\|, \Lambda = \{1, 2, ..., t\}\}.$$

If

$$\|w_n + q_n - z_n\| = 0, \tag{111}$$

then stop. w_n is the solution. Otherwise, calculate

$$w_{n+1} = \alpha_n u + (1 - \alpha_n)(w_n - \tau_n(w_n + q_n - z_n)), \tag{112}$$

where $\alpha_n \in (0, 1), n \geq 0$ and

$$\tau_n = \lambda_n \frac{\|w_n - z_n\|^2 + \|G w_n\|^2}{\|2w_n + q_n - z_n\|^2},$$

in which $\lambda_n \in (0, 4)$.

In this part, we are still concerned with the split equality problem (SEP). Chen et al. [16] proposed an algorithm about SEP with κ-asymptotically strictly pseudo-nonspreading mappings and proved its weak convergence.

Let H_1, H_2, H_3 be three real Hilbert spaces. Let $T_i : H_1 \to H_1$ be a κ-asymptotically strictly pseudo-nonspreading mappings. Denote by $Fix(T_i)$ the set of fixed points of $T_i (i = 1, 2, ..., m)$. Set $C = \cap_{i=1}^m Fix(T_i)$. Let A: $H_1 \to H_3$, B: $H_2 \to H_3$ be two bounded linear operators. Let Q be a nonempty closed convex subset of H_2. Then the split equality problem is proposed :

$$\text{finding } x \in C, y \in Q \text{ such that } Ax = By.$$

We use Γ to denote the solution of SEP, i.e.,

$$\Gamma = \{(x, y) \in H_1, Ax = By, x \in C, y \in Q\}$$

and assume the consistency of SEP so that Γ is convex, closed and nonempty.

Theorem 3.23 ([16]) *Let H_1,H_2 be two real Hilbert spaces. Let $T_i : H_1 \to H_1$ be a κ-asymptotically strictly pseudo-nonspreading mappings. Let $C = \cap_{i=1}^m Fix(T_i)$ and Q be any nonempty closed convex set of H_2. Let $S = C \times Q$ and P_S be the metric projection of $H = H_1 \times H_2$ to S. Then the sequence $\{w_n\}$ generated by Algorithm 28 converges weakly to a point $w \in \Gamma$.*

Algorithm 28 The iterative algorithm with κ-asymptotically strictly pseudo-nonexpansive mappings for SEP

$$\begin{cases} w_1 \in H, \\ u_n = (I - \gamma G^* G) w_n, \\ w_{n+1} = P_S[(1 - \alpha_n) u_n + \alpha_n T_{[n]}^n u_n], \end{cases} \tag{113}$$

where $[n] = nmodm$, $0 < \gamma < \lambda = \frac{2}{\rho(G^*G)}$, and $\rho(G^*G)$ being the spectral radius of the self-adjoint operator G^*G, $\{\alpha_n\}$ is a sequence in$(0, 1 - \kappa]$, and $\sum_{n=0}^{\infty} \alpha_n < \infty$ and we remember T_i as $T_i \oplus I$.

Remark 3.9 Let H be a Hilbert space and K be a nonempty closed convex subset of H. We denote by $Fix(T)$ the fixed points set of set of T. T is said to be κ-asymptotically strictly pseudo-nonspreading mappings, if there exists a constant $\kappa \in [0, 1)$ and a sequence $k_n \geq 1$ and $\lim_{n\to\infty} = 1$, such that

$$\|T_n x - T_n y\|^2 \leq k_n \|x - y\| + \kappa \|x - T_n x - (y - T_n y)\|^2 + 2\langle x - T_n x, y - T_n y\rangle \tag{114}$$

for all $x, y \in D(T)$.

Let H_1, H_2, H_3 be three real Hilbert spaces. And we denote by $FixT$ the set of fixed points of T. Now we will introduce two branch problems.

Firstly, the multiple-set split feasibility problem(MSSFP) is formulated as follows:

$$\text{Find a point} x \in C = \cap_{i=1}^{r_1} C_i \text{ such that } Ax \in Q = \cap_{j=1}^{r_2} Q_i, \tag{115}$$

where $r_1, r_2 \in N$, $C_1, ..., C_{r_1}$ are closed convex subsets of H_1, $Q_1, ..., Q_{r_2}$ are closed convex subsets of H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

Next, we are concerned with the split equality fixed point problem (SEFPP):

$$\text{find } x \in C = F(U),\ y \in Q = F(T) \text{ such that } Ax = By, \tag{116}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two firmly quasi-nonexpansive operators.

We use Γ to denote the solution set of SEFPP and MSSFP, i.e.,

$$\begin{aligned} \Gamma = \{(x, y) \in H_1 \times H_2 : x \in \cap_{i=1}^{t_1} C_i, y \in \cap_{j=1}^{r_1} Q_j, A_1 x \in \cap_{i=1}^{t_2} D_i, B_1 y \in \cap_{j=1}^{r_2} \Theta_j, \\ A_2 x = B_2 y, x \in F(T_1), y \in F(T_2)\}, \end{aligned} \tag{117}$$

and assume the consistence of SEFPP and MSSFP, so that Γ is closed, convex and nonempty.

Let the sets C_i, Q_j, D_i, Θ_j be defined as

$$\begin{cases} C_i = \{x \in H_1, c_i(x) \le 0\}, \\ Q_j = \{y \in H_2, q_j(y) \le 0\}, \\ D_i = \{u \in H_1, d_i(u) \le 0\}, \\ \Theta_j = \{v \in H_2, \phi_j(v) \le 0\}, \end{cases} \tag{118}$$

where $c_i : H_1 \to R, i = 1, 2, ..., t_1; q_j : H_2 \to R, j = 1, 2, ..., r_1; d_i : H_3 \to R, i = 1, 2, ..., t_2; \phi_j : H_3 \to R, j = 1, 2, ..., r_2$ are convex functions.

Based on two above problems, Xu et al. [86] reported an algorithm to solve the MSSFP and SEFPP at the same time and reported the convergence analyse.

Algorithm 29 The iterative algorithm of SEFFP and MSSFP

$$w_{n+1} = P_{Fix(T)}\{(1-\tau_n)[I - \sigma_1 M - \sigma_2\lambda_1 N - \sigma_3\lambda_2 G^* G]w_n\}, \tag{119}$$

i.e.,

$$\begin{cases} x_{n+1} = P_{Fix(T_1)}\{(1-\tau_n)[x_n - \sigma_1 \sum_{i=1}^{t_1} \alpha_i (I - P_{C_i})x_n - \sigma_2\lambda_1 \sum_{i=1}^{t_2} \beta_i A_1^*(I - P_{D_i})A_1 x_n \\ \qquad -\sigma_3\lambda_2 A_2^*(A_2 x_n - B_2 y_n)]\}, \\ y_{n+1} = P_{Fix(T_2)}\{(1-\tau_n)[y_n - \sigma_1 \sum_{j=1}^{r_1} \gamma_j (I - P_{Q_j})y_n - \sigma_2\lambda_1 \sum_{j=1}^{r_2} \delta_j B_1^*(I - P_{\Theta_j})B_1 y_n \\ \qquad +\sigma_3\lambda_2 B_2^*(A_2 x_n - B_2 y_n)]\}, \end{cases}$$

where $\tau_n > 0$ is a sequence in (0,1) such that
(i) $\lim_n \tau_n = 0$;
(ii) $\sum_{n=0}^{\infty} \tau_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\tau_{n+1} - \tau_n| < \infty$ or $\lim_n \frac{|\tau_{n+1}-\tau_n|}{\tau_n} = 0$.

Theorem 3.24 ([86]) *The sequence $\{w_n\}$ generated by Algorithm 29 converges strongly to a solution $\bar{w}$ of MSSFP and SEFPP.*

Remark 3.10 Guan et al. [29] proposed a new iterative scheme to solve the above problem:

$$\begin{cases} x_{n+1} = T_1[x_n - \lambda_n \sum_{i=1}^{t_1} \alpha_i (x_n - P_{C_{i,n}} x_n) - \xi_n \sum_{i=1}^{t_2} \beta_i A_1^*(A_1 x_n - P_{D_{i,n}} A_1 x_n) \\ \qquad -\tau A_2^*(A_2 x_n - B_2 y_n)], \\ y_{n+1} = T_2[y_n - \sigma_n \sum_{j=1}^{r_1} \gamma_j (y_n - P_{Q_{j,n}} y_n) - \zeta_k \sum_{j=1}^{r_2} \delta_j B_1^*(B_1 y_n - P_{\Theta_{j,n}} B_1 y_n) \\ \qquad -\tau B_2^*(B_2 y_n - A_2 x_{n+1})]. \end{cases} \tag{120}$$

Further, he proved a weak convergence theorem under some mild restrictions on the parameters.

5 Banach Space

In this section, we will introduce some results on the iterative algorithms of the split feasibility problem and split equality problem in Banach spaces. These algorithms are improved by convergence and step size selection.

It is noted that all the above mentioned results in the literature are confined to the framework of Hilbert spaces. However, many important problems related to practical problems are generally defined in Banach spaces. For instance, Zhang et al. [94] remarked that in machine learning, Banach spaces possess much richer geometric structures, which are potentially useful for developing learning algorithms. In addition, numerous problems such as membrane bending problems, obstacle problems, and a number of filtration equations in PDEs are often formulated in Banach spaces which are not necessarily Hilbert spaces.

The results of the research in Hilbert space are based on the structure and geometric interpretability provided by the concept of an inner product. If X is not a Hilbert space, then it may happen that the metric projections P_C do not exist and, even if they exist (as in the case where C is Chebychev subset of X), convergence of the sequence can be guaranteed under restrictive conditions concerning set C. The Bregman projection method is very effective in solving problems in Banach spaces without involving metric projections. The use of Bregman technology provides more flexible choices of the distance for projecting onto feasible sets.

The notions of uniform smoothness and uniform convexity play a central role in the structure theory of Banach spaces. The uniformly convex spaces have some good stability properties, one of which is examined that all non-expansive mappings have a fixed point on uniformly convex spaces; similarly with uniformly smooth spaces. Throughout Sect. 1.5, all our spaces are assumed to be in p-uniformly convex, uniformly smooth Banach spaces. We first introduce the modulus of convexity and of smoothness.

5.1 Preliminaries

Let $1 \leq q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let E be a real Banach space with dual E^* and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E^*. The modulus of convexity $\delta_E(\epsilon) : [0, 2] \to [0, 1]$ is defined as

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\}. \tag{121}$$

The space E is called uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$, strictly convex if $\delta_E(2) = 1$. E is called p-uniformly convex ($1 < p < \infty$) if there exist a constant $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p \epsilon^p$ for any $\epsilon \in (0, 2]$.

The modulus of smoothness $\rho_E(\tau):[0,\infty)\to[0,\infty)$ is defined by

$$\rho_E(\tau)=\sup\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2}-1:\|x\|=\|y\|=1\}. \tag{122}$$

The space E is called uniformly smooth if $\lim\limits_{\tau\to 0}\frac{\rho_E(\tau)}{\tau}=0$. E is called q-uniformly smooth if there exists a positive real number $C_q>0$ such that $\rho_E(\tau)\le C_q\tau^q$ for any $\tau>0$. It is know that E is p-uniformly convex if and only if its dual E^* is q-uniformly smooth [45]. The duality mapping $J_E^p: E\to 2^{E^*}$ is defined by

$$J_E^p(x)=\left\{x^*\in E^*:\left\langle x,x^*\right\rangle=\|x\|^p,\|x^*\|=\|x\|^{p-1}\right\}, \tag{123}$$

for every $x\in E$. In this situation, it is known that the duality mapping J_E^p is one-to-one, single-valued and satisfies $J_E^p=(J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the duality mapping of E^* [13]. When $p=2$, the (123) becomes

$$J_2(x)=\{x^*\in E^*:\langle x^*,x\rangle=\|x\|^2=\|x^*\|^2\},$$

for every $x\in E$. J_2 is also called the normalized duality mapping.

Lemma 3.10 ([43]) *Let $x,y\in E$. If E is a q-uniformly smooth Banach space, then there exists a $c_q>0$ such that*

$$\|x-y\|^q\le\|x\|^q-q\left\langle J_{E^*}^q(x),y\right\rangle+c_q\|y\|^q. \tag{124}$$

5.1.1 Facts About Bregman Distance.

Definition 3.9 ([7]) Let $f:E\to(-\infty,+\infty]$ be a *Gâteaux* differentiable function, the function $D_f: domg\times int(domg)\to[0,+\infty)$ between x and y in E is defined by

$$D_f(x,y)=f(x)-f(y)-\langle f'(y),x-y\rangle,\ \forall x,\ y\in E. \tag{125}$$

Let the functions $f(x)=\frac{1}{p}\|x\|^p$ and let $1<q\le 2\le p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$ and denote by J_E^p and $J_{E^*}^q$ the duality mappings of a smooth Banach space X and its dual space, respectively. The Bregman distance with respect to the function $f(x)=\frac{1}{p}\|x\|^p$ can be written as

$$\begin{aligned} D_p(x,y)&:=\frac{\|x\|^p}{p}-\frac{\|y\|^p}{p}-\left\langle J_E^p(y),x-y\right\rangle\\ &=\frac{\|x\|^p}{p}+\frac{\|y\|^p}{q}-\left\langle J_E^p(y),x\right\rangle. \end{aligned} \tag{126}$$

Note that the Bregman distance D_f does not satisfy the properties of metric and the triangular inequality properties. However it has the following important properties:

$$D_p(x, y) + D_p(y, x) = \left\langle J_E^p(x) - J_E^p(y), x - y \right\rangle, \ \forall x, y \in E.$$

$$D_p(x, y) + D_p(y, z) - D_p(x, z) = \left\langle J_E^p(z) - J_E^p(y), x - y \right\rangle, \ \forall x, y, z \in E.$$

For p-uniformly convex space, the metric and Bregman distance have the following relation [9]:

$$\tau \|x - y\|^p \leq \Delta_p(x, y) \leq \left\langle J_E^p(x) - J_E^p(y), x - y \right\rangle, \tag{127}$$

where $\tau > 0$ is some fixed number.

5.1.2 Facts About Bregman Projection

Definition 3.10 ([7]) Let E be a reflexive, smooth and strictly convex Banach space and C be nonempty closed convex subset of E. Then for every $x \in C$, there exists a unique element $\Pi_C^p(x) \in C$ such that

$$D_p(x, \Pi_C^p(x)) = \min_{y \in C}\{D_p(x, y) : y \in C\},$$

where $\Pi_C^p(x)$ is called the Bregman projection of x onto C. The following characterization of the Bregman projectin can be characterized by [35]:

$$\left\langle J_E^p(x) - J_E^p(\Pi_C x), y - \Pi_C x \right\rangle \leq 0, \ \forall y \in C, \tag{128}$$

or equivalently

$$D_p(y, \Pi_C x) \leq D_p(y, x) - D_p(\Pi_C x, x), \ \forall y \in C. \tag{129}$$

In Hilbert spaces the Bregman projection with respect to the function $f(x) = \frac{1}{2}\|x\|^2$ coincides with the metric projection.

The following result describes the definition and properties of the function $V(\cdot, \cdot)$. The function $V_p : E \times E^* \to [0, +\infty)$ with respect to f_p is defined by

$$V_p(x, x^*) = \frac{1}{p}\|x\|^p - \left\langle x, x^* \right\rangle + \frac{1}{q}\|x^*\|^q, \ \forall x \in E, \ x^* \in E^*. \tag{130}$$

It is easy to see that V_p is nonnegative and

$$V_p(x, x^*) = D_p(x, J_{E^*}^q(x^*)), \ \forall x \in E, \ x^* \in E^*. \tag{131}$$

In addition, V_p satisfies the following inequality [4]:

$$V_p(x, x^*) + \left\langle y^*, J_{E^*}^q(x^*) - x \right\rangle \leq V_p(x, x^* + y^*), \ \forall x \in E, x^*, y^* \in E^*. \quad (132)$$

5.2 Problems and Algorithms

The split feasibility problem (SFP) is formulated in Banach space as follows:

$$\text{Find a point } x \in C \text{ such that } Ax \in Q, \quad (133)$$

where E_1 and E_2 are Banach spaces, $A : E_1 \to E_2$ is a bounded linear operator and C and Q be two nonempty, closed and convex subsets of E_1 and E_2 respectively.

For the extension of SFP to Banach spaces, Schöpfer et al. [9] first introduced a generalization of CQ algorithm via duality mappings, metric projections and Bregman projections. The authors proved the weak convergence of Algorithm 134 under the condition that E is p-uniformly convex, uniformly smooth Banach spaces. The proposed algorithm is formulated as follows: For $x_1 \in E$,

$$x_{n+1} = \Pi_C J_{E^*}^q [J_p(x_n) - \lambda_n A^* J(Ax_n - P_Q(Ax_n))], \quad (134)$$

where J_p, $J_{E^*}^q$, J are duality mappings, $\lambda_n > 0$, and Π_C denotes the Bregman projection on C and P_Q denotes the metric projection on Q. The step size $\lambda_n \in (0, (\frac{q}{C_q \|A\|^q})^{\frac{1}{q-1}})$ where $\frac{1}{p} + \frac{1}{q} = 1$ and C_q is the uniform smoothness coefficient of E.

To achieve strong convergence, Takahashi (2014) and Shehu (2015) have proposed separate methods to prove the SFP on Banach spaces and proved the strong convergence of their proposed algorithm. Takahashi [68] suggested a hybrid projection algorithm for the SFP:

Algorithm 30 A hybrid projection algorithm for the SFP

$$\begin{cases} z_n = x_n - \lambda J_X^{-1} A^* J_Y (Ax_n - P_Q(Ax_n)), \\ D_n = \{z \in C : \langle z_n - z, J_X(x_n - z_n) \rangle \geq 0\}, \\ E_n = \{z \in C : \langle x_n - z, J_X(x_0 - x_n) \rangle \geq 0\}, \\ x_{n+1} = \Pi_{D_n \cap E_n} x_0. \end{cases} \quad (135)$$

And later using the shrinking projection method, Takahashi [69] prove two strong convergence theorems for finding a solution of the split feasibility problem in Banach spaces.

Shehu [19] proposed an iterative algorithm for solving SFP in Banach spaces:

Theorem 3.25 ([19]) *Suppose the following conditions are satisfied:*
(1) $\lim_{n\to\infty} \alpha_n = 0$;
(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

Algorithm 31 An iterative algorithm of the SFP

For a fixed $u \in C$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$, let sequences $\{y_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be generated by $x_1 \in C$,

$$\begin{cases} y_n = J_{E_1^*}^q[J_{E_1}^p(x_n) - \lambda_n A^* J_{E_2}^p(Ax_n - P_Q Ax_n)], \\ x_{n+1} = \Pi_C J_{E_1^*}^q[\alpha_n J_{E_1}^p u + (1-\alpha_n) J_{E_1}^p(y_n)], \quad n \geq 1. \end{cases} \tag{136}$$

(3) λ_n *satisfies* $0 < a \leq \lambda_n \leq b < (\frac{q}{C_q\|A\|^q})^{\frac{1}{q-1}}$ *for* $a, b > 0$.
Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to an element* $x \in \Gamma$, *where* $x = \Pi_\Gamma u$.

Wang [22] modified the above Algorithm 30 and proved strong convergence for the following multiple-sets split feasibility problem (MSSFP): find an element $x \in E$ satisfying

$$x \in \bigcap_{i=1}^{r} C_i, \ Ax \in \bigcap_{i=r+1}^{r+s} Q_i, \tag{137}$$

where E_1 and E_2 are two Banach spaces, r, s are two given integers, $A : E_1 \to E_2$ is a bounded linear operator, $C_i, i = 1, ..., r$ is a closed convex subset in E_1 , and $Q_i, i = r+1, ..., r+s$, is a closed convex subset in E_2. In the particular case $r = s = 1$, problem (137) is known as the split feasibility problem (SFP).

Algorithm 32 An algorithm for solving the multiple-sets split feasibility problem

For each $n \in \mathbb{N}$, $T_n x$ is defined by

$$T_n x = \begin{cases} \Pi_{C_{i(n)}(x)}, & i \leq i(n) \leq r, \\ J_{E^*}^q[J_E^p x - \lambda_n A^* J_E^p(I - P_{Q_{i(n)}})Ax], & r+1 \leq i(n) \leq r+s, \end{cases} \tag{138}$$

where $i : \mathbb{N} \to I$ is the cyclic control mapping $i(n) = n \, mod(r+s) + 1$, and λ_n satisfies $0 < \lambda \leq \lambda_n \leq \frac{q}{C_q||A||^q})^{\frac{1}{q-1}}$, C_q is a constant.
For any initial guess x_1, define $\{x_n\}$ recursively by

$$\begin{cases} y_n = T_n x_n, \\ D_n = \{u \in E : D_p(y_n, u) \leq D_p(x_n, u)\}, \\ E_n = \{u \in E : \langle x_n - u, J_p x^* - J_p x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{D_n \cap E_n}(x^*). \end{cases} \tag{139}$$

Theorem 3.26 ([22]) *The sequence* $\{x_n\}$, *generated by Algorithm 32, converges strongly to the Bregman projection of* x *onto the solution set* Γ.

We note that the above result is that the choice of step size is depends on the operators norm $\|A\|$ which usually slows down the convergence rate of the method.

To overcome this difficulty, Wang [31] proposed two iterative algorithms in Banach spaces in 2018. When the spaces involved are smooth and uniformly convex, then these two algorithms converge strongly to a solution of the SFP.

Algorithm 33 A iterative algorithm for solving the SFP

Choose an arbitrary initial guess $x_0 \in E_1$. Update x_{n+1} by the iteration formula:

$$\begin{cases} z_n = x_n - \lambda_n J_{E_1}^{-1}[J_X(x_n - P_C x_n) + A^* J_{E_2}(Ax_n - P_Q Ax_n)], \\ D_n = \{z \in E : \langle z_n - z, J_{E_1}(x_n - z_n)\rangle \geq 0\}, \\ E_n = \{z \in E : \langle x_n - z, J_{E_1}(x_0 - x_n)\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{140}$$

with $\{\lambda_n\}$ satisfying $0 < a \leq \lambda_n \leq \frac{1}{1+\|A\|^2}$.

Theorem 3.27 ([31]) *Let* $\{x_n\}$ *be generated by Algorithm 33. Then the following hold:*
(i) $\lim_n \|x_{n+1} - x_n\| = 0$;
(ii) $\omega_w\{x_n\} \subseteq \Gamma$;
(iii) $x_n \to \hat{z} = P_\Gamma(x_0)$, *where* $\hat{z} = P_\Gamma(x_0)$.

Next, Algorithm 34 introduces an alternative choice of λ_n, which is ultimately unrelated to $\|A\|$.

Algorithm 34 A iterative algorithm with adaptive step size for solving the SFP

Choose an arbitrary initial guess $x_0 \in E_1$. If

$$\|J_{E_1}(x_n - P_C x_n) + A^* J_{E_2}(Ax_n - P_Q(Ax_n))\| = 0,$$

then stop, and x_n is a solution; otherwise, go on to the next step. Update x_{n+1} by the iteration formula:

$$\begin{cases} z_n = x_n - \lambda_n J_{E_1}^{-1}[J_X(x_n - P_C x_n) + A^* J_{E_2}(Ax_n - P_Q Ax_n)], \\ D_n = \{z \in E : \langle z_n - z, J_{E_1}(x_n - z_n)\rangle \geq 0\}, \\ E_n = \{z \in E : \langle x_n - z, J_{E_1}(x_0 - x_n)\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{141}$$

where the parameter $\{\lambda_n\}$ is chosen as

$$\lambda_n = \frac{\|x_n - P_C x_n\|^2 + \|Ax_n - P_Q(Ax_n)\|^2}{\|J_{E_1}(x_n - P_C x_n) + A^* J_{E_2}(Ax_n - P_Q(Ax_n))\|^2}.$$

Theorem 3.28 ([31]) *Let* $\{x_n\}$ *be generated by Algorithm 33. Then* $\{x_n\}$ *converges strongly to* $\hat{z}$, *where* $\hat{z} = P_\Gamma(x_0)$.

In the first algorithm, the stepsize $\{\lambda_n\}$ generated by the iterative sequence $\{x_n\}$ satisfies the following conditions $0 < a \leq \lambda_n \leq \frac{1}{1+\|A\|^2}$. In another algorithm, the stepsize $\{\lambda_n\}$ satisfies

$$\lambda_n = \frac{\|x_n - P_C x_n\|^2 + \|Ax_n - P_Q(Ax_n)\|^2}{\|J_X(x_n - P_C x_n) + A^* J_Y(Ax_n - P_Q(Ax_n))\|^2}.$$

We noted that the parameter in the second algorithm is chosen in such a way that no priori knowledge of the operator norms is required. And these two algorithms are strongly convergent provided that the involved Banach spaces are smooth and uniformly convex.

To overcome this difficulty, Yang [88] used this way and considered the following variable stepsize:

$$\lambda_n = \frac{\rho_n}{\|\nabla f(x_n)\|}, \tag{142}$$

where ρ_n is a sequence of positive real numbers such that

$$\sum_{n=0}^{\infty} \rho_n = \infty, \ \sum_{n=0}^{\infty} \rho_n^2 = \infty. \tag{143}$$

Then, Yang proves the convergence, where the sequence λ_n of stepsizes satisfies (142) and (143), and where two additional conditions are satisfied: (i) Q is a bounded subset and (ii) A is a matrix with a full column rank.

In order to remove the two additional conditions (i) and (ii) of Yang, López et al. [38] suggested the more applicable self-adaptive method, which permits step-size λ_n being selected self-adaptively in such a way:

$$\lambda_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \ n \geq 1,$$

where $\rho_n \in (0, 4)$, $\inf_n \rho_n(4 - \rho_n) > 0$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|$ and $\nabla f(x_n) = A^*(I - P_Q)Ax_n$ for all $n \geq 1$. It was proved that the sequence $\{x_n\}$ converges weakly to a solution of SFP.

Inspired by the adaptive step size above, Suantai et al. [20] introduce a Halpern-type iteration in the framework of Banach spaces:

$$x_{n+1} = \Pi_C J_{E^*}^q[\alpha_n J_E^p(u) + (1 - \alpha_n)(J_E^p(x_n) - \rho_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p} \nabla f(x_n))], \ n \geq 1,$$

where $f(x_n) = \frac{1}{p}\|(I - P_Q)Ax_n\|^p$. They prove the strong convergence of the proposed sequence for solving the SFP without prior knowledge of the operator norm.

In 2019, Shehu et al. [48] propose a hybrid projection algorithm for solving the solution of SFP in the framework of p-uniformly convex and uniformly smooth Banach spaces and prove a strong convergence theorem.

Algorithm 35 An inertial hybrid projection algorithm of SFP

Let $\{\theta_n\} \subset \mathbb{R}$ be a bounded set. Set $x_0, x_1 \in C$. Define a sequence $\{x_n\}$ by the following manner:

$$\begin{cases} w_n = J_{E^*}^q[J_E^p(x_n) + \theta_n(J_E^p(x_n) - J_E^p(x_{n-1}))], \\ y_n = \Pi_C J_{E^*}^q[J_E^p(w_n) - \lambda_n \frac{f^{p-1}(x_n)}{\|\nabla f(x_n)\|^p} \nabla f(w_n)], \\ C_n = \{u \in E : D_p(u, y_n) \leq D_p(u, w_n)\}, \\ Q_n = \{u \in E : \langle x_n - u, J_E^p(x_0) - J_E^p(x_n), x_n - u \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \ \forall\, n \geq 0, \end{cases} \tag{144}$$

where $f(w_n) := \frac{1}{p}\|(I - P_Q)Aw_n\|^p$, $f^{p-1}(x_n) := (\frac{1}{p}\|(I - P_Q)Aw_n\|^p)^{p-1}$, $\{\rho_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} \rho_n(p - c_q \frac{\rho_n^{q-1}}{q}) > 0$.

Theorem 3.29 ([48]) *Suppose* $\{x_n\}$ *be generated by Algorithm 35 converge strongly to the Bregman projection of* x_0 *onto the solution set* Γ.

Recently, Wang et al. ([44, 70]) the two self-adaptive algorithm to solve the split equality problem in Banach spaces. Subsequently, the strong convergence of the proposed algorithms are analyzed and established. The split equality problem (SEP) is formulated in Banach space as follows:

$$\text{Find a point } x \in C,\ \ y \in Q,\ \text{ such that } Ax = By, \tag{145}$$

where E_1, E_2 and E_3 are p-uniformly convex and uniformly smooth real Banach spaces, $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are two bounded linear operators, and C and Q are nonempty closed convex subsets of E_1 and E_2, respectively. The solution set Γ of SEP is nonempty:

$$\Gamma = \{(x, y) \in E_1 \times E_2, Ax = By, x \in C, y \in Q\} \neq \emptyset.$$

Let $S = C \times Q$ in $E = E_1 \times E_2$, $w = (x, y) \in S$, define $G : E \to E_3$ by $G = [A, -B]$. Then, the original SEP becomes finding $w = (x, y) \in S$ with $Gw = 0$. The following is a self-adaptive algorithm with the inertial technique. The step size selection of the proposed algorithm does not require a prior estimation of operator norm, and the inertial term improves the performance of the algorithm.

Theorem 3.30 ([70]) *Suppose* $\{w_n\}$ *be generated by Algorithm 36 converge strongly to a solution* $\Pi_\Gamma w_0$.

Algorithm 36 An inertial hybrid algorithm with adaptive step size for solving SEP

Let $\{\theta_n\} \subset \mathbb{R}$ be a bounded set. Set $w_0, w_1 \in S$.
The sequence $\{w_n\}$ is defined by the following iteration:

$$\begin{cases} u_n = J_{E^*}^q[J_E^p(w_n) + \alpha_n(J_E^p(w_n) - J_E^p(w_{n-1}))], \\ z_n = \Pi_S J_{E^*}^q[J_E^p(u_n) - \rho_n G^* J_E^p G(u_n)], \\ D_n = \{u \in E : D_p(u, z_n) \le D_p(u, u_n)\}, \\ Q_n = \{u \in E : \langle w_n - u, J_E^p(w_0) - J_E^p(w_n), w_n - u\rangle \ge 0\}, \\ w_{n+1} = \Pi_{D_n \cap E_n} w_0, \forall\, n \ge 0, \end{cases} \tag{146}$$

where $\lambda_n^{q-1} \in (\epsilon, \frac{q\|Gu_n\|^p}{C_q\|G^* J_{E_3}^p Gu_n\|^q} - \epsilon)$.

Proof *Step 1.* The sequence $\{w_n\}$ generated by Algorithm 36 is well-defined.

In order to prove that $\{w_n\}$ is well-defined, first of all, we need to prove that $D_n \cap E_n$ is nonempty closed and convex for all $n \ge 1$. Obviously, D_n is closed and E_n is closed and convex. To prove the convexity of D_n, note that

$$D_p(u, z_n) \le D_p(u, u_n),$$

then, using (121) we have

$$\frac{||u||^p}{p} + \frac{||z_n||^p}{q} - \langle J_E^p(z_n), u\rangle \le \frac{||u||^p}{p} + \frac{||u_n||^p}{q} - \langle J_E^p(u_n), u\rangle,$$

that is,

$$\langle J_E^p(u_n) - J_E^p(z_n), u\rangle \le \frac{1}{q}(||u_n||^p - ||z_n||^p), \quad \forall u \in E,$$

so D_n is a half-space, which means D_n is convex. Hence, $D_n \cap E_n$ is closed and convex. Secondly, we show that $D_n \cap E_n \ne \emptyset$. To do this, it suffices to prove that

$$\Gamma \subset D_n \cap E_n. \tag{147}$$

If (147) holds, we notice that $\Gamma \ne \emptyset$, so $D_n \cap E_n \ne \emptyset$. Next we show $\Gamma \subset D_n$. Let $z \in \Gamma$, $m_n = J_E^p(u_n) - \rho_n G^* J_{E_3}^p G(u_n), \quad \forall n \ge 1$. From Lemma 3.10, we get

$$\begin{aligned} ||m_n||_{E^*}^q &= ||J_E^p(u_n) - \rho_n G^* J_{E_3}^p G(u_n)||_{E^*}^q \\ &\le ||u_n||^p - q\rho_n\langle G^* J_{E_3}^p G(u_n), u_n\rangle + c_q \rho_n^q ||G^* J_{E_3}^p G(u_n)||^q. \end{aligned} \tag{148}$$

From (127) and (148), we have

$$\begin{aligned}
\Delta_p(z, z_n) &\le \Delta_p(z, J^p_{E^*}(m_n)) \\
&= \frac{||z||^p}{p} - \langle m_n, z\rangle + \frac{||J^p_{E^*}(m_n)||^p}{q} \\
&= \frac{||z||^p}{p} - \langle m_n, z\rangle + \frac{1}{q}||m_n||^{(q-1)p} \\
&= \frac{||z||^p}{p} - \langle m_n, z\rangle + \frac{1}{q}||m_n||^{q} \\
&\le \frac{||z||^p}{p} - \langle m_n, z\rangle + \frac{1}{q}||u_n||^p - \rho_n\langle G^* J^p_{E_3} G(u_n), u_n\rangle \\
&\quad + \frac{c_q\rho_n^q}{q}||G^* J^p_{E_3} G(u_n)||^q \\
&= \frac{||z||^p}{p} - \langle J^p_E(u_n), z\rangle + \frac{1}{q}||u_n||^p - \rho_n\langle J^p_{E_3} G(u_n), Gu_n - Gz\rangle \\
&\quad + \frac{c_q\rho_n^q}{q}||G^* J^p_{E_3} G(u_n)||^q \\
&= D_p(z, u_n) - \rho_n\langle J^p_{E_3} G(u_n), Gu_n\rangle + \frac{c_q\rho_n^q}{q}||G^* J^p_{E_3} G(u_n)||^q \\
&= D_p(z, u_n) - \rho_n(||Gu_n||^p - \frac{c_q\rho_n^{q-1}}{q}||G^* J^p_{E_3} G(u_n)||^q). \qquad (149)
\end{aligned}$$

By using the value of $\{\rho_n^{q-1}\}$, we have

$$D_p(z, z_n) \le D_p(z, u_n).$$

This implies that $\Gamma \subset D_n$.

Finally, we show that $\Gamma \subset E_n$. For $n = 0$, we have $E_0 = E$, so $\Gamma \subseteq E_0$. Given w_n and suppose $\Gamma \subseteq D_n \cap E_n$ for some $n \in \mathbb{N}$. Then, there exists w_{n+1} such that

$$w_{n+1} = \Pi_{D_n \cap E_n}(w_0).$$

Using (126), we have

$$\langle J^p_E(w_0) - J^p_E(w_{n+1}), w_{n+1} - z\rangle \ge 0.$$

Therefore, $\Gamma \subset E_{n+1}$. By induction, we can get that $\Gamma \subset E_n \ \ \forall n \in \mathbb{N}$.

Step 2. Let $\{w_n\}$ be generated by Algorithm 36. Then

(i) $\lim_{n\to\infty} ||u_n - w_n|| = 0$;

(ii) $\lim_{n\to\infty} ||w_n - z_n|| = 0$.

The definition of E_n actually implies that $w_n = \Pi_{E_n}(w_0)$. Combined with the fact that $\Gamma \subset E_n$ and the definition of Bregman projection, we get

$$D_p(w_n, w_0) \leq D_p(z, w_0), \quad \forall z \in \Gamma.$$

And since $v := \Pi_\Gamma(w_0) \in \Gamma$, we obtain

$$D_p(w_n, w_0) \leq D_p(v, w_0), \tag{150}$$

which means that $\{D_p(w_n, w_0)\}$ is bounded. Hence, we know from (139) that $\{w_n\}$ is bounded. On the other hand, according to $w_{n+1} \in E_n$ and (126), we have $\langle J_E^p(w_0) - J_E^p(w_n), w_{n+1} - w_n\rangle \leq 0$ and by (127)

$$D_p(w_{n+1}, w_n) \leq D_p(w_{n+1}, w_0) - D_p(w_n, w_0), \quad \forall n \geq 0. \tag{151}$$

Which means that

$$\begin{aligned} D_p(w_n, w_0) &\leq D_p(w_{n+1}, w_0) - D_p(w_{n+1}, w_n) \\ &\leq D_p(w_{n+1}, w_0). \end{aligned}$$

Thus, $\{D_p(w_n, w_0)\}$ is nondecreasing and since $\{D_p(w_n, w_0)\}$ is bounded, we get $\lim\limits_{n\to\infty} D_p(w_n, w_0)$ exists. And then from (151) we have

$$\lim_{n\to\infty} D_p(w_{n+1}, w_n) = 0.$$

Hence, we obtain from (139) that

$$\lim_{n\to\infty} ||w_{n+1} - w_n|| = 0. \tag{152}$$

Since J_E^p is norm-to-norm uniformly continuous, we get

$$\lim_{n\to\infty} ||J_E^p(w_{n+1}) - J_E^p(w_n)|| = 0.$$

According to the definition of $\{u_n\}$ that

$$J_E^p(u_n) - J_E^p(w_n) = \alpha_n(J_E^p(w_n) - J_E^p(w_{n-1})).$$

Therefore,

$$||J_E^p(u_n) - J_E^p(w_n)|| = \alpha_n ||J_E^p(w_n) - J_E^p(w_{n-1})|| \to 0, \quad n \to \infty.$$

Since $J_{E^*}^p$ is also norm-to-norm uniformly continuous, we have

$$||u_n - w_n|| \to 0, \quad n \to \infty.$$

This completes (i).

In addition,

$$||w_{n+1} - u_n|| \leq ||w_{n+1} - w_n|| + ||w_n - u_n|| \to 0, \quad n \to \infty.$$

This shows that,

$$||J_E^p(u_n) - J_E^p(w_{n+1})|| \to 0.$$

From (139), we have

$$\begin{aligned} D_p(w_{n+1}, u_n) &\leq \langle J_E^p(w_{n+1}) - J_E^p(u_n), w_{n+1} - u_n \rangle \\ &\leq ||J_E^p(w_{n+1}) - J_E^p(u_n)|| ||w_{n+1} - u_n|| \to 0, \quad n \to \infty. \end{aligned}$$

Since $w_{n+1} \in D_n$, we have that

$$D_p(w_{n+1}, z_n) \leq D_p(w_{n+1}, u_n) \to 0, \quad n \to \infty.$$

This implies that

$$||w_{n+1} - z_n|| \to 0, \quad n \to \infty. \tag{153}$$

From (152) and (153) we get

$$||w_n - z_n|| \leq ||w_n - w_{n+1}|| + ||w_{n+1} - z_n|| \to 0, \quad n \to \infty.$$

Step 3. Let $\{w_n\}$ be generated by Algorithm 36. Then the sequence $\{w_n\}$ has a weak cluster point and $\omega_w(w_n) \subseteq \Gamma$.

We know $\{w_n\}$ is bounded. Since E is a reflexive Banach space, $\omega_w(w_n)$ is nonempty. Therefore, we take a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $w_{n_j} \rightharpoonup z \in \omega_w(w_n)$. Since $||w_n - z_n|| \to 0, \ \ n \to \infty$, we can get $z_{n_j} \rightharpoonup z$. Obviously we have $z \in S$. And since $||w_n - u_n|| = 0$, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightharpoonup z$. From (149), we have

$$\rho_n(||Gu_n||^p - \frac{c_q \rho_n^{q-1}}{q} ||G^* J_{E_3}^p G(u_n)||^q) \leq D_p(z, u_n) - D_p(z, z_n). \tag{154}$$

By (122), we get

$$D_p(z, z_n) + D_p(z_n, u_n) - D_p(z, u_n) = \langle J_E^p(u_n) - J_E^p(z_n), z - z_n\rangle,$$

combine this with (139) we get

$$\begin{aligned} D_p(z, u_n) - D_p(z, z_n) &= D_p(z_n, u_n) + \langle J_E^p(z_n) - J_E^p(u_n), z - z_n\rangle \\ &\le \langle J_E^p(z_n) - J_E^p(u_n), z_n - u_n\rangle + \langle J_E^p(z_n) - J_E^p(u_n), z - z_n\rangle \\ &\le ||J_E^p(z_n) - J_E^p(u_n)||||z - u_n|| \to 0, \ \ n \to \infty. \end{aligned}$$

Therefore, we have

$$||Gu_n||^p - \frac{c_q \rho_n^{q-1}}{q} ||G^* J_{E_3}^p G(u_n)||^q \to 0, \ \ n \to \infty. \tag{155}$$

Since $\rho_n^{q-1} < \frac{q||Gu_n||^p}{c_q||G^* J_{E_3}^p Gu_n||^q} - \epsilon$, we get

$$\frac{\epsilon c_q}{q} ||G^* J_{E_3}^p Gu_n||^q < ||Gu_n||^p - \frac{c_q \rho_n^{q-1}}{q} ||G^* J_{E_3}^p Gu_n||^q \to 0, \ \ n \to \infty.$$

Thus,

$$\lim_{n\to\infty} ||G^* J_{E_3}^p Gu_n|| = 0. \tag{156}$$

From (155) and (156), we get $\lim_{n\to\infty} ||Gu_n|| = 0$, so $\lim_{n\to\infty} ||Gu_{n_j}|| = 0$. By the continuity of G, we obtain $Gw_{n_j} \rightharpoonup Gz$ and

$$||Gw_{n_j}|| - ||Gu_{n_j}|| \le ||G||||w_{n_j} - z_{n_j}|| \to 0, \ \ j \to \infty.$$

Hence, we have that $||Gw_{n_j}|| = 0$.
Therefore,

$$\begin{aligned} 0 \le ||Gz||^p &= \langle J_{E_3}^p Gz, Gz\rangle \\ &= \lim_{j\to\infty} \langle J_{E_3}^p Gz, Gw_{n_j}\rangle \\ &\le \lim_{j\to\infty} ||J_{E_3}^p Gz||||Gw_{n_j}|| \\ &= 0. \end{aligned}$$

Thus $Gz = 0$ and hence $z \in \Gamma$.

Step 4. $\{w_n\}$ converges strongly to a point $\Pi_\Gamma(w_0)$.

We know that $w_{n_j} \rightharpoonup z$. From Step 3 it follows that $z \in \Gamma$. Since $w_{n+1} \in E_n$ and $\Pi_{E_n}(w_0) = \arg\min_{w\in E} D_p(w_0, w)$, then we get

$$\begin{aligned} D_p(w_n, w_0) &= D_p(\Pi_{E_n}(w_0), w_0) \\ &\le D_p(w_{n+1}, w_0). \end{aligned}$$

In Step 1, we know $\Pi_\Gamma(w_0) \in \Gamma \subseteq E_{n+1}$. So

$$\begin{aligned} D_p(w_{n+1}, w_0) &= D_p(\Pi_{E_{n+1}}(w_0), w_0) \\ &\le D_p(\Pi_\Gamma(w_0), w_0). \end{aligned}$$

Therefore,

$$D_p(w_n, w_0) \le D_p(w_{n+1}, w_0) \le D_p(\Pi_\Gamma(w_0), w_0).$$

From (122) and (123), we can obtain

$$\begin{aligned} D_p(w_{n_j}, \Pi_\Gamma(w_0)) &= D_p(w_{n_j}, w_0) + D_p(w_0, \Pi_\Gamma(w_0)) \\ &\quad + \langle J_E^p(\Pi_\Gamma(w_0)) - J_E^p(w_0), w_0 - w_{n_j}\rangle \\ &\le D_p(\Pi_\Gamma(w_0), w_0) + D_p(w_0, \Pi_\Gamma(w_0)) \\ &\quad + \langle J_E^p(\Pi_\Gamma(w_0)) - J_E^p(w_0), w_0 - \Pi_\Gamma(w_0)\rangle \\ &\quad + \langle J_E^p(\Pi_\Gamma(w_0)) - J_E^p(w_0), \Pi_\Gamma(w_0) - w_{n_j}\rangle \\ &= \langle J_E^p(w_0) - J_E^p(\Pi_\Gamma(w_0)), w_{n_j} - \Pi_\Gamma(w_0)\rangle. \end{aligned} \tag{157}$$

Taking lim sup, we get

$$\begin{aligned} \limsup_{j\to\infty} D_p(w_{n_j}, \Pi_\Gamma(w_0)) &\le \limsup_{j\to\infty} \langle J_E^p(w_0) - J_E^p(\Pi_\Gamma(w_0)), w_{n_j} - \Pi_\Gamma(w_0)\rangle \\ &= \langle J_E^p(w_0) - J_E^p(\Pi_\Gamma(w_0)), z - \Pi_\Gamma(w_0)\rangle \\ &\le 0. \end{aligned}$$

Therefore, $\lim_{j\to\infty} D_p(w_{n_j}, \Pi_\Gamma(w_0)) = 0$ and $w_{n_j} \to \Pi_\Gamma(w_0)$. From the arbitrariness of $\{w_{n_j}\}$ and the uniqueness of $\Pi_\Gamma(w_0)$, we have $w_n \rightharpoonup \Pi_\Gamma(w_0)$. Using (139), it follows from (157) that

$$\tau||w_n - \Pi_\Gamma(w_0)||^p \le D_p(w_n, \Pi_\Gamma(w_0)) \le \langle J_E^p(w_0) - J_E^p(\Pi_\Gamma(w_0)), w_n - \Pi_\Gamma(w_0)\rangle.$$

Taking limit of the above inequality, we obtain that $w_n \to \Pi_\Gamma(w_0)$.

Another algorithm that the stepsize selection is self-adaptive and no prior estimation of operator norm is required. The algorithm is as follows.

Theorem 3.31 ([44]) *Suppose $\{x_n\}$ and $\{w_n\}$ be generated by Algorithm 37. Then, $\{x_n\}$ and $\{w_n\}$ converge strongly to a solution $\hat{w}$ of the SEP, where $\hat{w} = \Pi_\Gamma u$.*

Algorithm 37 A self-adaptive iterative algorithm of SEP

Choose $\epsilon > 0$, $x_1 \in E$, pick a fixed point $u \in E$ and set $n = 1$.
The step size is as shown below:

$$\lambda_n^{q-1} \in (\epsilon, \frac{q\|Gu_n\|^p}{C_q\|G^*J_{E_3}^p Gu_n\|^q} - \epsilon).$$

Compute

$$\begin{cases} w_n = J_{E^*}^q[J_E^p(x_n) - \lambda_n G^* J_{E_3}^p Gx_n], \\ x_{n+1} = J_{E^*}^q[\alpha_n J_E^p(u) + (1-\alpha_n)(\beta_n J_E^p(w_n) + (1-\beta_n)J_E^p(\Pi_S w_n)). \end{cases} \tag{158}$$

6 Linear Convergence

The algorithms proposed above only consider the results of weak convergence or strong convergence, without considering the convergence speed. In this section, we consider the linear convergence of the algorithms. Recall that a sequence $\{x_n\}$ in H is said to converge linearly to its limit x (with rate $\sigma \in [0, 1)$) if there exist $\alpha > 0$ and a positive integer N such that

$$\|x_n - x\| \leq \alpha\sigma^n, \text{ for all } n \geq N. \tag{159}$$

To study the linear convergence property of algorithms, we introduce a notion of bounded linear regularity property for SEP (67) (MSSFP and MSSEP), and use it to prove the linear convergence property. Shi et al. [63] first introduced a notion of bounded linear regularity property for SEP and provided some mild sufficient conditions to ensure this regularity property.

For a set $S \subset H$, we denote the closure, interior, relative interior and conical hull of S by clS, intS, riS and coneS, respectively. For $w \in H$ and $r > 0$, we use $\mathbb{B}(w, r)$ and $\overline{\mathbb{B}(w, r)}$ to denote the open metric ball and closed metric ball with centre at w and radius r, respectively, that is,

$$\mathbb{B}(w, r) := \{v \in H : \|w - v\| < r\} \text{ and } \overline{\mathbb{B}(w, r)} := \{v \in H : \|w - v\| \leq r\}.$$

In particular, we use $\mathbb{B}$ and $\overline{\mathbb{B}}$ to denoted the unit open metric ball and unit closed metric ball with centre at origin, respectively. For a point w onto S and the distance of w from S, denoted by

$$P_S(w) = \arg\min\{\|w - v\| : v \in S\} \text{ and } d_S(w) = \inf\{\|w - v\| : v \in S\}.$$

Let $G : H \to H$ be a bounded linear operator on H, we use $kerG = \{x \in H : Gx = 0\}$ to denote the kernel of G, and $(kerG)^\perp = \{y \in H : \langle x, y\rangle = 0, \forall x \in kerG\}$ denotes the orthogonal complement of $kerG$. It is known that $kerG$ and $(kerG)^\perp$ are closed subspaces of H.

In order to get the linear convergence property for the projection algorithm for solving convex feasibility problems, we introduce the linear regularity for a family of closed convex subsets of a real Hilbert.

Definition 3.11 ([52]) Let $\{E_i\}_{i\in I}$ be a family of closed convex subsets of a real Hilbert space H and $E = \bigcap_{i\in I} E_i \neq \emptyset$. The family $\{E_i\}_{i\in I}$ is said to be bounded linearly regular if for each $r > 0$, there exists a constant $\gamma_r > 0$ such that $d_E(w) \leq \gamma_r \sup\{d_{E_i}(w) : i \in I\}$ for all $w \in r\mathbb{B}$.

The following lemma provides sufficient conditions for bounded linear regularity property for two closed convex subsets of H.

Lemma 3.11 ([52]) *Let E and F be closed convex subsets of H. Then $\{E, F\}$ is bounded linearly regular provided that at least one of the following conditions holds:*
(i) $ri E \cap F \neq \emptyset$ and F is a polyhedron;
(ii) $ri E \cap ri F \neq \emptyset$ and E is a finite dimensional;
(iii) $ri E \cap ri F \neq \emptyset$ and E is a finite codimensional.

We use Γ to denote the solution set of SEP, that is,

$$\Gamma = \{w \in S : Gw = 0\},$$

and assume consistency of SEP (67), that is, Γ is non-empty, closed and convex. We introduce the notion of bounded linear regularity property for SEP.

Definition 3.12 ([63]) The SEP (67) is said to satify the bounded linear regularity property if for each $r > 0$, there exists $\gamma_r > 0$ such that

$$\gamma_r d_\Gamma(w) \leq \|Gw\| \text{ for all } w \in r\mathbb{B} \cap S. \tag{160}$$

Lemma 3.12 ([18], Proposition 6.4) *Let $G : H \to H$ be a bounded linear operator on H. Then G is injective and has closed range if and only if G is bounded below (i.e. there exists a constant $\gamma > 0$ such that $\|Gw\| \geq \gamma\|w\|$ for all $w \in H$).*

Lemma 3.13 ([63]) *Let $\{S, ker G\}$ be a bounded linearly regular and G has closed range. Then the SEP (67) satisfied the bounded linear regulaeity property.*

Noted that $ker G$ is a subspace of H and $riker G = ker G$. Furthermore, it is well known that if $ker G$ is finite dimensional or finite codimensional, then the range of G is closed. By Lemma 5.2 , we get the following corollary which establishes sufficient conditions for bounded linear regularity property for SEP (67).

Corollary 3.1 ([63]) *The SEP (67) satisfies the bounded linear regularity property if one of the following conditions holds:*
(i) C and Q are polyhedrons, and G has closed range;
(ii) $ri S \cap ker G \neq \emptyset$, $ker G$ is finite dimensional;
(iii) $ri S \cap ker G \neq \emptyset$, $ker G$ is finite codimensional;
(iv) $ri S \cap ker G \neq \emptyset$, G has closed range and $S = C \times Q$ is finite dimensional;
(v) $ri S \cap ker G \neq \emptyset$, G has closed range and $S = C \times Q$ is finite codimensional.

Shi et al. [63] established the linear convergence property for the gradient projection algorithm (GAP) when using different types of stepsizes and under the assumption of bounded linear regularity property for SEP (67):

Algorithm 38 The gradient projection algorithm

For an arbitrary point $w_0 = (x_0, y_0) \in H = H_1 \times H_2$, the sequence $\{w_n\} = \{x_n, y_n\}$ is generated by the iterative algorithm

$$w_{n+1} = P_S[(I - \gamma G^* G)w_n], \tag{161}$$

i.e.,

$$\begin{cases} x_{n+1} = P_C\{x_n - \gamma_n A^*(Ax_n - By_n)\}; \\ y_{n+1} = P_Q\{y_n + \gamma_n B^*(Ax_n - By_n)\}. \end{cases}$$

Theorem 3.32 ([63]) *Assume that the SEP satisfied the bounded linear regularity property. Then the sequence $\{w_n\}$ generated by Algorithm 38 with $\gamma_n \in (0, +\infty)$ converges to a solution w^* of SEP such that*

$$\|w_n - w^*\| \leq \delta p^{\sum_{i=1}^{n} \gamma_i}, \tag{162}$$

for $\delta \geq 1$ and $0 < p < 1$, provided that one of the following conditions is assumed:
(i) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 2/\|G\|^2$;
(ii)

$$\gamma_n = \begin{cases} 0, & w_n \in \Gamma, \\ \frac{\rho\|Gw_n\|^2}{\|G^*Gw_n\|^2}, & otherwise, \end{cases}$$

and

$$0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < 2;$$

(iii) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{i=1}^{n} \gamma_i = \infty$.

Consequently, $\{w_n\}$ converges w^* linearly in the case when (1) or (2) is assumed.

Proof Without loss of generality, we assume that w_n is not in Γ for all $n \geq 1$. Otherwise, GPA (161) terminates in finite number of iterates, and then the conclusions follow trivially.

For $z \in \Gamma$, since P_S is non-expansive and $Gz = 0$, we have

$$\begin{aligned} \|w_{n+1} - z\|^2 &= \|P_S(w_n - \gamma_n G^* G w_n) - P_S z\|^2 \\ &\leq \|w_n - z - \gamma_n G^* G w_n\|^2 \\ &= \|w_n - z\|^2 - 2\gamma_n \langle w_n - z, G^* G w_n\rangle + \gamma_n^2 \|G^* G w_n\|^2 \\ &\leq \|w_n - z\|^2 - 2\gamma_n \|G w_n\|^2 + \gamma_n^2 \|G^* G w_n\|^2 \\ &= \|w_n - z\|^2 - \gamma_n \left(2 - \gamma_n \frac{\|G^* G w_n\|^2}{\|G w_n\|^2}\right) \|G w_n\|^2. \end{aligned}$$

We show that $\{w_n\}$ converges to a solution w^* of SEP and (162) holds.

Note that $\frac{\|G^*Gw_n\|^2}{\|Gw_n\|^2} \leq \|G\|^2$. It is easy to see that if (i),(ii) or (iii) holds, then

$$\|w_{n+1} - z\| \leq \|w_n - z\|.$$

Hence, $\lim_{n\to\infty} \|w_n - z\|$ exits and $\{w_n\}$ is a bounded sequence.

Since SEP satisifies the bounded linear regularity property and $w_n \in S$ for all $n \geq 1$, there exists $\alpha > 0$ such that $\alpha d_\Gamma(w_n) \leq \|Gw_n\|$ for all $n \geq 1$. It follows that

$$\|w_{n+1} - z\|^2 \leq \|w_n - z\|^2 - \alpha^2\gamma_n\left(2 - \gamma_n\frac{\|G^*Gw_n\|^2}{\|Gw_n\|^2}\right)d_\Gamma(w_n)^2, \ for\ each\ z \in \Gamma$$

Hence,

$$d_\Gamma(w_{n+1})^2 \leq \left(1 - \alpha^2\gamma_n\left(2 - \gamma_n\frac{\|G^*Gw_n\|^2}{\|Gw_n\|^2}\right)\right)d_\Gamma(w_n)^2.$$

Note that if (i),(ii) or (iii) holds, then

$$\liminf_{n\to\infty}\left(2 - \gamma_n\frac{\|G^*Gw_n\|^2}{\|Gw_n\|^2}\right) > 0.$$

Hence, there exists N such that

$$\beta := \inf_{n\geq N}\alpha^2\left(2 - \gamma_n\frac{\|G^*Gw_n\|^2}{\|Gw_n\|^2}\right) \geq 0.$$

Therefore, for all $n \geq N$,

$$d_\Gamma(w_{n+1})^2 \leq (1 - \beta\gamma_n)\, d_\Gamma(w_n)^2 \leq d_\Gamma(w_N)^2\prod_{i=N+1}^{n}(1 - \beta\gamma_i)\,.$$

Observe that, for each $z \in \Gamma$, $\|w_{n+1} - z\|$ is monotone decreasing for n, hence, for all $m \geq n$,

$$\begin{aligned}\|w_m - w_n\| &\leq \|w_m - P_\Gamma(w_n)\| + \|w_n - P_\Gamma(w_n)\|\\ &\leq 2\|w_n - P_\Gamma(w_n)\| = 2d_\Gamma(w_n),\end{aligned}$$

It follows that

$$\|w_m - w_{n+1}\| \leq 2d_\Gamma(w_N)\prod_{i=N+1}^{n}\sqrt{1 - \beta\gamma_i}, \quad for\ all\ \ m \geq n+1.$$

Note that $\ln(1 - t) \leq -t$ for any $t \in [0, 1)$, let $p := e^{-(\beta/2)} \in (0, 1)$, then

$$\prod_{i=N+1}^{n} \sqrt{1-\beta\gamma_i} = exp\left\{\frac{1}{2}\sum_{i=N+1}^{n} ln(1-\beta\gamma_i)\right\} \leq p^{\sum_{i=N+1}^{n}\gamma_i}.$$

Therefore,

$$\|w_m - w_{n+1}\| \leq 2d_\Gamma(w_N)p^{\sum_{i=N+1}^{n}\gamma_i}, \quad for\ \ all\ \ m \geq n+1.$$

Since $\sum_{n=1}^{\infty}\gamma_n = +\infty$, it follows that $\{w_n\}$ is a Cauchy sequence and converges to a solution w^* of SEP satisfying

$$\|w_{n+1} - w^*\| \leq 2d_\Gamma(w_N)p^{\sum_{i=N+1}^{n}\gamma_i}, \quad for\ \ all\ \ n \geq N.$$

Let

$$\delta = \max\left\{2d_\Gamma(w_N)p^{-\sum_{i=1}^{N}\gamma_i}, \max\left\{\|w_i - w^*\|p^{-\sum_{j=1}^{i}\gamma_j}, i = 1, 2, ..., N.\right\}\right\}.$$

Then

$$\|w_n - w^*\| \leq \delta p^{\sum_{i=1}^{n}\gamma_i}.$$

Let $\gamma = \liminf_{n\to\infty}\gamma_n$, then there exists N_1 such that $\gamma_n \geq \gamma$ for $n \geq N_1$. It follows that

$$\|w_n - w^*\| \leq \delta p^{\sum_{i=1}^{N_1}\gamma_i} p^{(n-N_1)\gamma} = \alpha\sigma^n, \quad \forall n \geq \max\{N_1, N\},$$

where $\alpha = \delta p^{\sum_{i=1}^{N_1}(\gamma_i-\gamma)}$, $\sigma = p^\gamma \in (0, 1)$. Hence, $\{w_n\}$ converges to a w^* linearly. This proof is complete.

However, the gradient projection algorithm involves projection P_S, it might be difficult to calculate in the case where the projection does not have a closed-form expression. To solve this problem, Moudafi et al. [46] proposed the relaxed alternating CQ algorithm (RACQA) by using orthogonal projection onto half-space the original closed convex sets, and established the result of weak convergence. Tian et al. [73] established the linear convergence property for the relaxed gradient projection algorithm (RGAP) under the assumption of bounded linear regularity property for SEP (67):

Algorithm 39 The relaxed gradient projection algorithm

For an arbitrary point $w_0 = (x_0, y_0) \in H = H_1 \times H_2$, the sequence $\{w_n\} = \{x_n, y_n\}$ is generated by the iterative algorithm

$$w_{n+1} = P_{S_n}[(I - \gamma G^*G)w_n], \tag{163}$$

i.e.,

$$\begin{cases} x_{n+1} = P_{C_n}\{x_n - \gamma_n A^*(Ax_n - By_n)\}; \\ y_{n+1} = P_{Q_n}\{y_n + \gamma_n B^*(Ax_n - By_n)\}. \end{cases}$$

Theorem 3.33 ([73]) *Assume that the SEP satisfied the bounded linear regularity property. Then the sequence $\{w_n\}$ generated by Algorithm 39 with $\gamma_n \in (0, +\infty)$ converges to a solution w^* of SEP such that*

$$\|w_n - w^*\| \leq \delta p^{\sum_{i=1}^{n} \gamma_i}, \tag{164}$$

for $\delta \geq 1$ and $0 < p < 1$, provided that one of the following conditions is assumed:
(i) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 2/\|G\|^2$;
(ii)

$$\gamma_n = \begin{cases} 0, & w_n \in \Gamma \\ \frac{\rho\|Gw_n\|^2}{\|G^*Gw_n\|^2}, & otherwise, \end{cases}$$

and $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < 2$;
(iii) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{i=1}^{n} \gamma_i = \infty$.
Consequently, $\{w_n\}$ converges w^* linearly in the case when (1) or (2) is assumed.

Now we investigate the following multiple-sets split feasibility problem(MSSFP):

$$\text{Find } x \in C = \bigcap_{i=1}^{t} C_i \text{ such that } Ax \in Q = \bigcap_{j=1}^{r} Q_j, \tag{165}$$

where r and t are positive integers, $\{C_i\}_{i=1}^{t}$ and $\{Q_j\}_{j=1}^{r}$ are nonempty closed and convex subsets of Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ is a bounded linear operator. Without loss of generality, assume that $t > r$, and choose $Q_{r+1} = Q_{r+2} = \cdots = Q_t = H_2$. Let $S_i = C_i \times Q_i \subseteq H = H_1 \times H_2, i = 1, 2, \cdots, t$, $\bar{S} = \bigcap_{i=1}^{t} S_i$, $\bar{G} = [A, -I] : H \to H_2$, G^* be the adjoint operator of G. Then MSSFP can be reformulated as

$$\text{Find } w = (x, y) \in \bar{S} \text{ such that } \bar{G}w = 0. \tag{166}$$

We introduce the notion of bounded linear regularity property for MSSFP.

Definition 3.13 ([74]) The MSSFP is said to satisfy the bounded linear regularity property if for each $r > 0$, there exists $\tau_r > 0$ such that

$$\tau_r d_{\bar{S}}(x) \leq d_Q(Ax) \text{ for all } x \in r\mathbb{B} \cap C. \tag{167}$$

Lemma 3.14 ([74]) *Let $\{\bar{S}, ker\bar{G}\}$ be bounded linearly regular and $\bar{G}$ has closed range. Then the MSSFP satisfies the bounded linear regularity property.*

Tian et al. [74] proposed the simultaneous sub-gradient projection algorithm (SSPA) with the dynamic step size for solving the MSSFP and construct several sufficient conditions to prove the linear convergence. In particular, the SSPA is an easily calculated algorithm that uses orthogonal projection onto half-spaces.

Algorithm 40 The simultaneous sub-gradient projection algorithm with the dynamic step size

For an arbitrarily initial point $x_0 \in H$, the sequence $\{x_{n+1}\}$ is generated by

$$x_{n+1} = x_n - \gamma_n\{\sum_{i=1}^{t} \alpha_i(x_n - P_{C_i,n}x_n) + A^*(I - P_{Q_i,n})x_n\}, \tag{168}$$

where (i) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < \min\{1, \frac{1}{\|A\|^2}\}$; (ii) $\{\alpha_i\}_{i=1}^t \subset (0, +\infty)$ and $\sum_{i=1}^t \alpha_i = 1$.

Theorem 3.34 ([74]) *Suppose the MSSFP satisfies the bounded linear regularity property, and the sequence $\{x_n\}$ be defined by Algorithm 40. If the following conditions are satisfied:*
(i) $\{Ax_n\}$ is linearly focusing, that is, there exists $\beta > 0$ such that

$$\beta d_{Q_i}(Ax_n) \le d_{Q_i,n}(Ax_n) \;\; for\ any\ \ i \in \{1, 2, \cdots, t\};$$

(ii) $Q_i \bigcap int(\bigcap_{r\in I\setminus\{i\}} Q_r) \ne \emptyset (I = \{1, 2, \cdots, t\})$.
Then, $\{x_n\}$ converges linearly to a solution of MSSFP.

Next we generalize the split equality problem to the multiple-sets split equality problem, which can be characterized mathematically as following

$$\text{Find } \ x \in \bigcap_{i=1}^{t} C_i \ \text{ and } \ y \in \bigcap_{j=1}^{r} Q_j \text{ such that } Ax = By, \tag{169}$$

where r and t are positive integers, $\{C_i\}_{i=1}^t$ and $\{Q_j\}_{j=1}^r$ are nonempty closed and convex subsets of Hilbert spaces H_1 and H_2, respectively, and H_3 is also a Hilbert space, both operator $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded and linear. Obviously, when $t = r = 1$, the MSSEP reduces to the SEP. If $t = r = 1$ and $B = I$, then the MSSEP is reduced to the SFP. Without loss of generality, we may assume $t > r$, and choose $Q_{r+1} = Q_{r+2} = \cdots = Q_t = H_2$. Let $S_i = C_i \times Q_i \subseteq H = H_1 \times H_2$, $i = 1, 2, \cdots, t$, $\hat{S} = \bigcap_{i=1}^t S_i$, $\hat{G} = [A, -B] : H \to H_3$, G^* be the adjoint operator of G. Then MSSEP can be reformulated as

$$\text{Find } \ w = (x, y) \in \hat{S} \text{ such that } \hat{G}w = 0. \tag{170}$$

We use Γ to denote the solution set of MSSEP, that is,

$$\Gamma = \{w \in \hat{S} : \hat{G}w = 0\},$$

and assume consistency of SEP, that is, Γ is non-empty, closed and convex. We introduce the notion of bounded linear regularity property for MSSEP. We introduce the notion of bounded linear regularity property for MSSEP.

Definition 3.14 ([28]) The MSSEP is said to satisfy the bounded linear regularity property if for each $r > 0$, there exists $\tau_r > 0$ such that

$$\tau_r d_\Gamma(w) \leq \max\{d_{\hat{S}}(w), \|\hat{G}w\|\} \text{ for all } w \in r\mathbb{B}. \tag{171}$$

Lemma 3.15 ([28]) *Let $\{\hat{S}, ker\hat{G}\}$ be bounded linearly regular and $\hat{G}$ has closed range. Then the MSSEP satisfies the bounded linear regularity property.*

Feng et al. [28] proposed the linear convergence property for the subgradient projection algorithm to solving the MSSEP by using projections onto half-spaces and self-adaptive step size, and under the assumption of bounded linear regularity property for MSSEP:

Algorithm 41 The subgradient projection algorithm to solving the MSSEP

For an arbitrarily initial point $w_0 = (x_0, y_0) \in H$, the sequence $\{w_{n+1}\}$ is generated by

$$w_{n+1} = w_n - \gamma_n\{\sum_{i=1}^{t} \alpha_i(w_n - P_{S_i,n}w_n) + G^*Gw_n\}, \tag{172}$$

i.e.,

$$\begin{cases} x_{n+1} = x_n - \gamma_n\{\sum_{i=1}^t \alpha_i(x_n - P_{C_i,n}x_n) + A^*(Ax_n - By_n)\}; \\ y_{n+1} = y_n - \gamma_n\{\sum_{i=1}^t \alpha_i(y_n - P_{Q_i,n}y_n) + B^*(Ax_n - By_n)\}, \end{cases}$$

where, (i) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \min\{1, \frac{1}{\|G\|^2}\}$; (ii) $\{\alpha_i\}_{i=1}^t \subset (0, +\infty)$ and $\sum_{i=1}^t \alpha_i = 1$.

Theorem 3.35 ([28]) *Let the MSSEP satisfy the bounded linear regularity property, and the sequence $\{w_n\}$ be defined by Algorithm 41. If the following conditions are satisfied:*
(i) $\{w_n\}$ is linearly focusing, that is, there exists $\beta > 0$ such that

$$\beta d_{S_i}(w_n) \leq d_{S_i,n}(w_n) \;\; for\ any \;\; i \in \{1, 2, \cdots, t\};$$

(ii) There is $w = (x, y) \in S_i$, such that $c_r(x) < 0, q_r(y) < 0, r \in \{1, 2, \cdots, t\}\backslash\{i\}$, $(i.e., S_i \bigcap int(\bigcap_{r\in I\backslash\{i\}} S_r) \neq \emptyset \ (I = \{1, 2, \cdots, t\}))$.
Then, $\{w_n\}$ converges linearly to a solution of MSSEP.

7 Applications

This section introduces some applications of the split feasibility problem, one is the signal processing and the other is image recovery.

7.1 Signal Processing

LASSO problem whose form is the following:

$$\min\{\frac{1}{2}\|Ax - b\|^2 : x \in \mathbb{R}^N, \|x\|_1 \leq \kappa\}, \tag{173}$$

where $A \in \mathbb{R}^{M\times N}$, $M < N$, $b \in \mathbb{R}^M$ and $\kappa > 0$. This problem is devoted to finding a sparse solution of SFP. The system A is generated from a standard normal distribution with mean zero and unit variance. We generate the true sparse signal z^* from uniformly distribution in the interval $[-2, 2]$ with random k position nonzero while the rest is kept zero. The sample data $b = Az^*$.

Under certain conditions on matrix A, the solution of the minimization problem is equivalent to the ℓ_0-norm solution of the underdetermined linear system. For the SFP, we define $C = \{z|\|z\|_1 \leq \kappa\}$, $\kappa = k$, and $Q = \{b\}$. Since the projection onto the closed convex set C does not have a closed form solution. So, we employ the subgradient projection. Thus, we define a convex function $c(z) = \|z\|_1 - \kappa$ and denote C_n by

$$C_n = \{z : c(w_n) + \langle \epsilon_n, z - w_n\rangle \leq 0\},$$

where $\epsilon_n \in \partial c(w_n)$. Also, the orthogonal projection of a point $z \in \mathbb{R}^N$ onto C_n can be computed via:

$$P_{C_n}(z) = \begin{cases} z, & \text{if } c(w_n) + \langle \epsilon_n, z - w_n\rangle \leq 0; \\ z - \frac{c(w_n)+\langle \epsilon_n, z-w_n\rangle}{\|\epsilon_n\|^2}\epsilon_n, & \text{otherwise.} \end{cases}$$

The subdifferential ∂c at w_n is

$$\partial c(w_n) = \begin{cases} 1, & \text{if } w_n > 0; \\ , & \text{if } w_n = 0; \\ -1, & \text{if } w_n < 0. \end{cases}$$

The following method of mean square error is used for measuring the recovery accuracy:

$$MSE = \frac{1}{N}\|x_n - z^*\|^2.$$

From the definitions, it is noted that a lower value of MSE shows a higher accuracy of restored signal.

We use López's relaxed CQ algorithm (47) to solve the signal processing problem. Set the parameter $\tau = 0.5$. In our experiment, we consider the case whenever $M = 512$, $N = 1024$ and $k = 50$. **Figure** 1 shows the comparison between the original

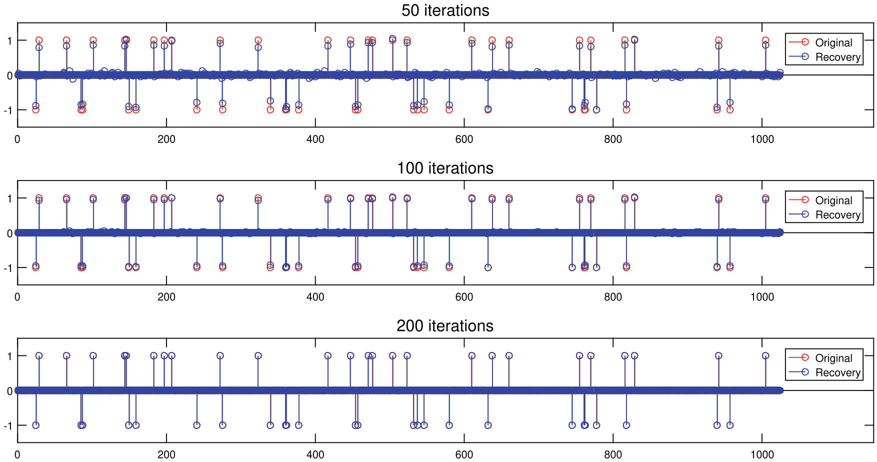

Fig. 1 Comparison of signal processing after 50, 100 and 200 iterations

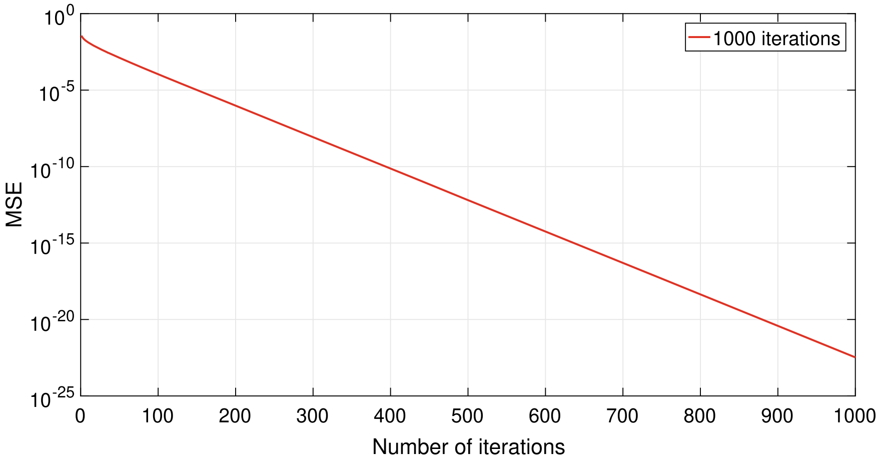

Fig. 2 The MSE value versus the iteration numbers for 1000 iterations

image and the restored image after 50, 100 and 200 iterations. **Figure** 2 shows the MSE value versus the iteration numbers for 1000 iterations. All of the numerical results show the solution of the signal recovering problem is solved by the proposed algorithm and gets good quality improvements of the observed signal.

7.2 Image Recovery

Let $C = [0, 255]^D$ such that $D = M \times N$, where M is the pixels of width and N is the pixels of height of the image. Consider the minimization problem:

$$\min_{x \in C} \frac{1}{2} \|Ax - b\|^2.$$

This problem can be solved via the SFP when $Q = \{b\}$ and $C = [0, 255]^D$.

We can consider two blur types (using the MATLAB function 'fspecial' and 'imfitler') with the 205×232 Tire Image as follows:

(i) Gaussian blur of filter size 9×9 with standard deviation 4.

(ii) Out of focus with radius 7.

To measure the restored images, we use the Peak-signal-to-noise ratio (PSNR) defined by

$$PSNR = 10 \log_{10} \frac{\|x\|^2}{\|x_n - x\|^2},$$

where x is an original image. From the definitions, it is noted that a higher value of PSNR of the same number of iteration shows a higher quality of restored image.

We use López's relaxed CQ algorithm (47) to solve the image recovery problem. Set the parameter $\tau = 0.5$. Figure 3 shows the original image and blurred images of tires. Figure 4 shows the restored image for gaussian blur after 3000 iterations. Figure 5 shows the restored image for out of focus blur after 3000 iterations. Figure 6 shows the graph of PSNR plotting for each blurs. All of the numerical results show the solution of the image recovery problem is solved by the proposed algorithm and gets good quality improvements of the restored image.

Fig. 3 The original image and blurred images of tires

Fig. 4 The restored image for gaussian blur after 3000 iterations

Fig. 5 The restored image for out of focus blur after 3000 iterations

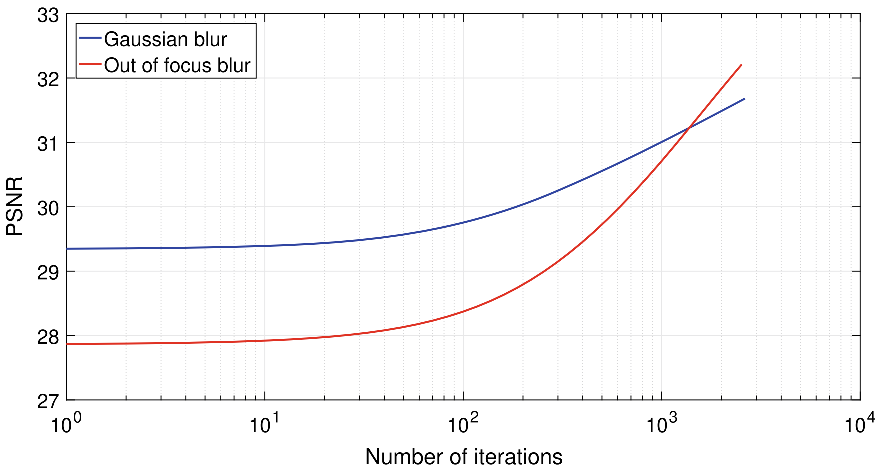

Fig. 6 Graph of PSNR plotting for each blurs

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Some Fixed Point Theorems of Generalized Contractions with Application to Boundary Value Problem

Theory, Methods and Integrative Approaches

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1 Introduction

One of the groundbreaking discoveries in nonlinear analysis was made by Banach in 1922, where he established a fixed point theorem. This theorem has found remarkable applications in various fields, including mathematical optimization, equilibrium theorems for games, graph theory, and image processing. Ever since its inception, this principle has served as a fundamental tool for solving numerous nonlinear analysis problems. Over the years, several researchers have extended and expanded upon the classical contraction theorem by introducing new types of contractions, thereby advancing the field further.

Geraghty [1] in 1973 is one of them who offered a generalized contraction mapping by considering a class of functions π defined as follows:

Definition 1 ([1]) Define $\mathfrak{W} = \{\pi | \pi : [0, \infty) \to [0, 1)\}$ which fulfills the condition

$$\pi(\kappa_p) \to 1 \quad \text{implies} \quad \kappa_p \to 0.$$

This contraction commonly known as Geraghty Contraction. Geraghty [1] then as an extension of the Banach contraction principle, deduce a fixed point result. Author(s) in [2], in 1997, extended Banach's finding in Hilbert space by introducing the concept

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of ϕ -weak contraction. Rhoades [3], in 2001, demonstrated that the result of Alber and Guerre-Delabriere [2] is not only valid in Hilbert space but also true in metric spaces. Rhoades [3] also remark that if we let $\phi(\kappa) = (1-t)\kappa, \kappa > 0$, then every contraction map M on $\mathbb{Y}$ is a weakly contractive map with contractive constant t, however, the inverse is not generally true. Zhang and Song [4], Doric [5], and Dutta and Choudhury [6], further presented some improved and generalized notion of weakly contractive mapping in complete metric spaces and derived some fixed point theorems.

One of the generalizations of Banach's result from 1922 was presented by Ran and Reuings [7] within the framework of partially ordered metric spaces. Ran and Reuings [7] were the first to establish the existence of fixed points for certain maps in this context. Subsequently, Nieto et al. [8] extended the result of Ran and Reuings [7] to non-decreasing mappings and utilized their findings to demonstrate the existence of solutions to differential equations, see also [27, 28]. Concurrently, Agarwal et al. [9] and Regan et al. [10] investigated the impact of generalized contractions in ordered spaces. Harjani and Sadarngani [11] reformulated the concept of weakly contractive mappings, incorporating the results of Rhoades [3] and Dutta and Choudhury [6], within the framework of ordered metric spaces. Following a similar line of research, Harandi and Emami [12] in 2010 presented a variation of Geraghty's result [1] in metric spaces equipped with a partial order. In the same year, Altun et al. [13] introduced the notion of weakly increasing mappings and established various fixed point results in partially ordered metric spaces for both weakly increasing and non-decreasing mappings. Altun et al. [13] provide a suitable example to emphasize that weakly increasing mappings do not necessarily imply non-decreasing behavior.

In 2008, Suzuki [14] introduced the concept of the C-contractive condition. The author proved a generalization of the Banach contraction mapping theorem and also characterized metric completeness. This simple yet powerful theorem has become a valuable tool in nonlinear analysis for ensuring the existence and uniqueness of fixed points. Furthermore, it has been subject to numerous generalizations; refer to [15, 16, 19, 29].

Definition 2 ([14]) Let $(\mathbb{Y}, d)$ be a metric space and $P : \mathbb{Y} \to \mathbb{Y}$ be a map. We say map P satisfies the $C-$ condition if,

$$\frac{1}{2}d(g, Pg) \leq d(g, e) \quad \text{implies} \quad d(Pg, Pe) \leq d(g, e),$$

for all $g, e \in \mathbb{Y}$.

Definition 3 ([25]) Denote $\Psi = \{\eta | \ \eta : [0, \infty) \to [0, \infty)\}$ such that η is continuous, non-decreasing and $\eta(\kappa) = 0$ if and only if $\kappa = 0$.

Gupta et al. [29], in 2017, generalized the result of Suzuki [14] by giving the notion of $C_\theta^\eta-$ condition and derive fixed point results both on metric as well as on partially ordered metric spaces.

Definition 4 ([29]) Let $(\mathbb{Y}, d)$ be a metric space and $P : \mathbb{Y} \to \mathbb{Y}$ be a map. We say map P satisfies $C_\theta^\eta-$ condition if

$$\frac{1}{2}d(g, Pg) \leq d(g, e) \quad \Longrightarrow \quad \eta(d(Pg, Pe)) \leq \theta(d(g, e)),$$

for all $g, e \in \mathbb{Y}$, $\eta \in \Psi$ and $\theta : [0, \infty) \to [0, \infty)$ is a continuous function.

In the mid-1970s, Jungck [21] introduced the concept of commuting maps, which extended Banach's famous result on two self-maps. This groundbreaking work by Jungck [21] provided researchers with a fresh perspective, stimulating their thoughts and encouraging them to explore, observe, and discuss various fixed point problems using a novel approach. Building upon this foundation, Sessa [22] further developed the idea in 1982 by introducing the concept of weak Commutativity in metric spaces. Jungck [23] expanded upon this notion in the mid-1980s, establishing the groundwork for weakly compatible mappings and compatible mappings. Notable results in this theory can be found in [24]. In 2019, Gupta et al. [18] presented a contraction result for two self-maps using auxiliary functions, providing a metric space result that did not rely on the compatibility and commutativity properties of maps endowed with a partial order. To proceed with the proof, let us first discuss several definitions that are crucial and necessary.

Definition 5 ([13]) Let $(\mathbb{Y}, \preceq)$ be a partially ordered set. Two mappings $M, N : \mathbb{Y} \to \mathbb{Y}$ are said to be weakly increasing if $Mg \preceq NMg$ and $Ng \preceq MNg$ for all $g \in \mathbb{Y}$.

Definition 6 ([23]) Let $M, N : \mathbb{Y} \to \mathbb{Y}$ be self-maps. Then a set of Pair (M, N) is said to be compatible on metric space $(\mathbb{Y}, d)$, if

$$\lim_{p \to \infty} d(MNg_p, NMg_p) = 0,$$

whenever $\{g_p\}$ is a sequence in $\mathbb{Y}$ such that $\lim_{p\to\infty} Mg_p = \lim_{p\to\infty} Ng_p = \kappa$ for some $\kappa \in \mathbb{Y}$.

The primary objective of this study is to establish a definition for a (η, π, θ)-generalized rational contraction involving three self-maps. Furthermore, we have presented three theorems within the context of a complete metric space equipped with a partial order that satisfies the conditions of a (η, π, θ)-generalized rational contraction. In addition, we provide two examples and several corollaries to substantiate the main findings. These findings not only generalize but also extend several well-known results from existing literature.

2 Main Results: (η, π, θ)-Generalized Rational Contraction

In this sections, we first introduce $(\eta, \pi, \theta) -$ generalized rational contraction, supported by an example.

Definition 7 Maps $M, N, L : \mathbb{Y} \to \mathbb{Y}$ are said to be $(\eta, \pi, \theta)-$ generalized rational contraction if for each $g, e \in \mathbb{Y}$

$$\eta(d(Mg, Ne)) \leq \pi(d(Lg, Le))\theta(K_M^N(Lg, Le)), \quad \forall\, g \geq e, \tag{1}$$

where

$$K_M^N(Lg, Le) = \max\left\{d(Lg, Le), \frac{d(Mg, Lg)d(Ne, Le)}{1 + d(Lg, Le)}, \frac{d(Ne, Le)[1 + d(Mg, Le)]}{1 + d(Mg, Lg)}\right\}, \tag{2}$$

$\pi \in \mathfrak{W}$, $\eta \in \Psi$ and $\theta : [0, \infty) \to [0, \infty)$ is a continuous function with condition

$$0 < \theta(\kappa) < \eta(\kappa), \forall\, \kappa > 0. \tag{3}$$

Let us start with an example.

Example

Let $\mathbb{Y} = \mathbb{N} \cup \{0\}$. Define a metric

$$d(g, e) = \begin{cases} g + e, & if\ g \neq e \\ 0, & if\ g = e. \end{cases}$$

Then $(\mathbb{Y}, d)$ is complete metric spaces.

Define $M, N, L : \mathbb{Y} \to Q^+$ as

$$Mg = \frac{g}{4}, \quad Ng = \frac{g}{6}, \quad Lg = g.$$

Define maps $\eta, \theta : [0, \infty) \to [0, \infty)$ and $\pi : [0, \infty) \to [0, 1)$ as $\eta(\kappa) = 2\kappa$, $\theta(\kappa) = \kappa$ and $\pi(\kappa) = \frac{4}{5}$. Then clearly, three maps M, N and L are said to be $(\eta, \pi, \theta)-$ generalized rational contraction.

Our main results are the following three theorems.

Theorem 1 *Let* $(\mathbb{Y}, \preceq, d)$ *be a partially ordered complete metric space, and let* $M, N, L : \mathbb{Y} \to \mathbb{Y}$ *are* $(\eta, \pi, \theta)-$ *generalized rational contraction mappings satisfying the following properties:*

(i) $MX \subseteq LX$*;* $NX \subseteq LX$*,*
(ii) M *and* N *are weakly increasing with respect to* L*,*
(iii) Lg *and* Le *are comparable,*
(iv) M, N *and* L *are continuous,*
(v) the pairs (M, L) *and* (N, L) *are compatible.*

Then M, N *and* L *have a coincidence point* $h \in \mathbb{Y}$*.*

Proof Let us start with the initial assumption that $g_0 \in \mathbb{Y}$ be any arbitrary point in $\mathbb{Y}$. Using Condition (1) of Theorem 1, we can find $g_1, g_2 \in \mathbb{Y}$ such that $Mg_0 = Lg_1$ and $Rg_1 = Lg_2$. In general, inductively we can define two sequences $\{g_p\}$ and $\{e_p\}$ in $\mathbb{Y}$, as

$$Lg_{2p+1} = Mg_{2p} = e_{2p}, \quad Lg_{2p+2} = Ng_{2p+1} = e_{2p+1}, \quad \forall\, p \in \mathbb{N}.$$

Condition (1) of Theorem 1 implies

$$Lg_1 = Mg_0 \preceq Ng_1 = Lg_2 = Mg_1 \preceq Ng_2 = Lg_3 \cdots \tag{4}$$

thus, we obtain

$$Lg_1 \preceq Lg_2 \preceq Lg_3 \cdots \preceq Lg_{2p+1} \preceq Lg_{2p+2} \preceq \cdots$$

which implies

$$e_0 \preceq e_1 \preceq e_2 \cdots \preceq e_{2p} \preceq e_{2p+1} \preceq \cdots$$

Claim: $\{Lg_p\}$ is a Cauchy Sequence
Proof of the claim further split into two cases
Case-1: Suppose $e_{2p-1} = e_{2p}, \forall\, p \in \mathbb{N}$.
Then from (1)

$$\eta(d(e_{2p}, e_{2p+1})) \leq \pi(d(e_{2p-1}, e_{2p}))\theta(K_M^N(e_{2p-1}, e_{2p})), \tag{5}$$

where

$$\begin{aligned} &K_M^N(e_{2p-1}, e_{2p}) \\ &= \max\left\{d(e_{2p-1}, e_{2p}), \frac{d(e_{2p}, e_{2p})d(e_{2p+1}, e_{2p})}{1 + d(e_{2p-1}, e_{2p})}, \frac{d(e_{2p+1}, e_{2p})[1 + d(e_{2p}, e_{2p})]}{1 + d(e_{2p}, e_{2p-1})}\right\}, \\ &= \max\left\{0, d(e_{2p+1}, e_{2p})\right\} = d(e_{2p+1}, e_{2p}). \end{aligned}$$

Thus from Eq. (5), we get

$$\eta(d(e_{2p}, e_{2p+1})) \leq \pi(d(e_{2p-1}, e_{2p}))\theta(d(e_{2p}, e_{2p+1})).$$

Property of $\pi \in \mathfrak{W}$, η and θ gives that $e_{2p+1} = e_{2p}$. Consequently, $e_q = e_{2p-1}$ for any $q \geq 2p$, and hence $Lg_q = Lg_{2p}$. This implies our claim.
Case-2: Suppose, in general, $e_p \neq e_{p+1}$ for any integer p.
Since e_{2p} and e_{2p-1} are comparable, then (1) gives

$$\begin{aligned}\eta(d(e_{2p+2}, e_{2p+1})) &= \eta(d(Mg_{2p+2}, Ng_{2p+1})) \\ &\leq \pi(d(e_{2p+1}, e_{2p}))\theta(K_M^N(e_{2p+1}, e_{2p})),\end{aligned} \tag{6}$$

where

$$\begin{aligned}&K_M^N(e_{2p+1}, e_{2p}) \\ &= \max\left\{d(e_{2p+1}, e_{2p}), \frac{d(e_{2p+2}, e_{2p+1})d(e_{2p+1}, e_{2p})}{1 + d(e_{2p+1}, e_{2p})}, \frac{d(e_{2p+1}, e_{2p})[1 + d(e_{2p+2}, e_{2p})]}{1 + d(e_{2p+2}, e_{2p})}\right\} \\ &= \max\left\{d(e_{2p+1}, e_{2p}), \frac{d(e_{2p+2}, e_{2p+1})d(e_{2p+1}, e_{2p})}{1 + d(e_{2p+1}, e_{2p})}.\right\}\end{aligned} \tag{7}$$

Case-2(I): Since $\frac{d(e_{2p+1}, e_{2p})}{1+d(e_{2p+1}, e_{2p})} < 1$ for any $p \in \mathbb{N}$, then if $\max\left\{d(e_{2p+1}, e_{2p}), \frac{d(e_{2p+2}, e_{2p+1})d(e_{2p+1}, e_{2p})}{1+d(e_{2p+1}, e_{2p})}\right\} = d(e_{2p+2}, e_{2p+1})$ then Eq (6) contradicts condition (3) and therefore

$$\lim_{p\to\infty} d(e_p, e_{p+1}) = 0.$$

Case-2(II): If $\max\left\{d(e_{2p+1}, e_{2p}), \frac{d(e_{2p+2}, e_{2p+1})d(e_{2p+1}, e_{2p})}{1+d(e_{2p+1}, e_{2p})}\right\} = d(e_{2p+1}, e_{2p})$
Then from Eq (6), we get

$$\eta(d(e_{2p+2}, e_{2p+1})) \leq \pi(d(e_{2p+1}, e_{2p}))\theta(d(e_{2p+1}, e_{2p})), \tag{8}$$

Consequently, condition (3), property of η and $\pi \in \mathfrak{W}$ implies

$$d(e_{2p+2}, e_{2p+1}) \leq d(e_{2p+1}, e_{2p}),$$

Continuing in this fashion, we have

$$d(e_{2p+2}, e_{2p+1}) \leq d(e_{2p+1}, e_{2p}) \leq d(e_{2p}, e_{2p-1}).$$

which is a monotonic decreasing sequence of real numbers. Thus we can find a $k \geq 0$ such that

$$\lim_{p\to\infty} d(e_p, e_{p+1}) = k. \tag{9}$$

Suppose that $k > 0$. Then on taking limit as $p \to \infty$ in (8), on making use of (9) and the fact $\pi \in \mathfrak{W}$, we get a contradiction. Therefore, $k = 0$. Imply that

$$\lim_{n\to\infty} d(e_p, e_{p+1}) = 0. \tag{10}$$

On contrary assume that, sequence $\{Lg_{2p}\}$ is not a Cauchy. Assume that there exists $\delta > 0$, for which we can get two sub-sequences q_t and p_t (of positive integers for all positive integer t) such that $p_t > q_t > t$,

$$d(Lg_{2q_t}, Lg_{2p_t}) > \delta; \quad d(Lg_{2q_t}, Lg_{2p_{t-2}}) \leq \delta. \tag{11}$$

On applying triangle inequality and using (11), we obtain

$$\begin{aligned}\delta &< d(Lg_{2q_t}, Lg_{2p_t}) \\ &\leq d(Lg_{2q_t}, Lg_{2p_{t-2}}) + d(Lg_{2p_{t-2}}, Lg_{2p_{t-1}}) + d(Lg_{2p_{t-1}}, Lg_{2p_t}).\end{aligned}$$

Consequently from above equality (on taking limit $t \to \infty$), we have

$$\lim_{t\to\infty} d(Lg_{2q_t}, Lg_{2p_t}) = \delta. \tag{12}$$

Triangle inequality again implies that

$$d(Lg_{2p_t}, Lg_{2q_{t-1}}) \leq d(Lg_{2p_t}, Lg_{2q_t}) + d(Lg_{2q_t}, Lg_{2q_{t-1}}),$$

In above equality take limit as $t \to \infty$ and make use of (10)–(12), we obtain

$$\lim_{t\to\infty} d(Lg_{2p_t}, Lg_{2q_{t-1}}) = \delta. \tag{13}$$

Since,

$$\begin{aligned}d(Lg_{2p_t}, Lg_{2q_t}) &\leq d(Lg_{2p_t}, Lg_{2p_{t+1}}) + d(Lg_{2p_{t+1}}, Lg_{2q_t}) \\ &= d(Lg_{2p_t}, Lg_{2p_{t+1}}) + d(Mg_{2p_t}, Ng_{2q_{t-1}}).\end{aligned}$$

Using (10)–(13) and letting $t \to \infty$, we have

$$\delta \leq \lim_{t\to\infty} d(Mg_{2p_t}, Ng_{2q_{t-1}}).$$

Continuity of $\eta \in \Psi$ implies that

$$\eta(\delta) \leq \lim_{t\to\infty} \eta(d(Mg_{2p_t}, Ng_{2q_{t-1}})). \tag{14}$$

Again (1) implies that

$$\eta(d(Mg_{2p_t}, Rg_{2q_{t-1}})) \leq \pi(d(Lg_{2p_t}, Lg_{2q_{t-1}}))\theta(K_M^N(Lg_{2p_t}, Lg_{2q_{t-1}})),$$

where

$$K_M^N(Lg_{2p_t}, Lg_{2q_{t-1}}) = \max\left\{\begin{array}{c} d(Lg_{2p_t}, Lg_{2q_{t-1}}), \frac{d(Mg_{2p_t}, Lg_{2p_t})d(Ng_{2q_{t-1}}, Lg_{2q_{t-1}})}{1+d(Lg_{2p_t}, Lg_{2q_{t-1}})}, \\ \frac{d(Ng_{2q_{t-1}}, Lg_{2q_{t-1}})[1+d(Mg_{2p_t}, Lg_{2q_{t-1}})]}{1+d(Mg_{2p_t}, Lg_{2p_t})} \end{array}\right\},$$

$$= \max\left\{\begin{array}{c} d(Lg_{2p_t}, Lg_{2q_{t-1}}), \frac{d(Lg_{2p_{t+1}}, Lg_{2p_t})d(Lg_{2q_t}, Lg_{2q_{t-1}})}{1+d(Lg_{2p_t}, Lg_{2q_{t-1}})}, \\ \frac{d(Lg_{2q_t}, Lg_{2q_{t-1}})[1+d(Lg_{2p_{t+1}}, Lg_{2q_{t-1}})]}{1+d(Lg_{2p_{t+1}}, Lg_{2p_t})} \end{array}\right\},$$

Taking limit $t \to \infty$ and the fact that $\pi \in \mathfrak{W}$ in above inequality, we get

$$\lim_{t\to\infty} \eta(d(Mg_{2p_t}, Ng_{2q_{t-1}}))) < \theta(\delta). \tag{15}$$

Equation (14) and (15) on using (3), gives

$$\eta(\delta) \leq \lim_{t\to\infty} \eta(d(Mg_{2p_t}, Ng_{2q_{t-1}}))) < \theta(\delta) < \eta(\delta). \tag{16}$$

Thus we arrived at contradiction. Therefore $\{Lg_{2p}\}$ is a Cauchy sequence. In general for all p, sequence $\{Lg_p\}$ is Cauchy. Thus there we have $\omega \in \mathbb{Y}$ such that

$$\lim_{n\to\infty} Lg_{2p} = \lim_{p\to\infty} Mg_{2p} = \omega. \tag{17}$$

Further, Claim that ω is a coincidence point of M, N and L.

Continuity of L and Eq. (17) gives

$$\lim_{p\to\infty} L(Lg_{2p}) = \lim_{p\to\infty} L(Lg_{2p+1}) = L\omega. \tag{18}$$

Further, continuity of M and (17) gives

$$d(M(Lg_{2p}), M\omega) \to 0. \tag{19}$$

Since pair (M, L) is compatible, then

$$d(L(Mg_{2p}), M(Lg_{2p})) \to 0. \tag{20}$$

Again triangular inequality gives

$$d(L\omega, M\omega) \leq d(L\omega, L(Lg_{2p+1})) + d(L(Mg_{2p}), M(Lg_{2p})) + d(M(Lg_{2p}), M\omega), \tag{21}$$

Using (17)–(20) (take limit $t \to \infty$) in (21) and, we get

$$d(L\omega, M\omega) \le 0,$$

this gives that $L\omega = M\omega$. In similar manner, we can deduce that $d(L\omega, N\omega) \le 0$, this gives that $L\omega = N\omega$. Therefore ω is a coincidence point of M, N and L, i.e., $L\omega = M\omega = N\omega$. This proves Theorem 1. □

Our next theorem guarantees the existence and uniqueness of fixed points for three self-maps M, N, L.

Theorem 2 *Assume that hypotheses of Theorem 1 holds for all* $(g, e) \in \mathbb{Y} \times \mathbb{Y}$, *and* $\omega \in \mathbb{Y}$ *such that*

$$Mg \preceq M\omega \text{ and } Me \preceq M\omega. \tag{22}$$

Then M, N *and* L *have unique common fixed point.*

Proof Split the proof further in two steps:
Claim: M, N and L have common fixed point.
To get the claim, suppose there exist μ and ν such that

$$L\mu = M\mu = N\mu \quad \text{and} \quad L\nu = M\nu = N\nu,$$

and we show that

$$L\mu = L\nu. \tag{23}$$

Let $(\mu, \nu) \in \mathbb{Y} \times \mathbb{Y}$, then there exists a $\omega_0 \in \mathbb{Y}$ (on using (22)) s.t.

$$M\mu \preceq M\omega_0, \quad \text{and} \quad M\nu \preceq M\omega_0. \tag{24}$$

On the line of proof of Theorem 1, we can define a sequence $\{L\omega_p\}$ as follows:

$$L\omega_{2p+1} = M\omega_{2p}, \quad L\omega_{2p+2} = N\omega_{2p+1}, \quad \forall\, p \in \mathbb{N}. \tag{25}$$

Again, we have

$$M\mu = L\mu \preceq L\omega_p, \quad M\nu = L\nu \preceq L\omega_p, \quad \forall\, p \in \mathbb{N}. \tag{26}$$

Put $g = \omega_{2p}$ and $e = \mu$ in (1), we get

$$\begin{aligned} \eta(d(L\omega_{2p+1}, L\mu)) &= \eta(d(M\omega_{2p}, N\mu)) \\ &\le \pi(d(L\omega_{2p}, L\mu))\theta(K_M^N(L\omega_{2p}, L\mu)), \end{aligned} \tag{27}$$

where

$$K_M^N(L\omega_{2p}, L\mu) = \max\left\{ \begin{array}{c} d(L\omega_{2p}, L\mu), \frac{d(M\omega_{2p}, L\omega_{2p})d(N\mu, L\mu)}{1+d(L\omega_{2p}, L\mu)}, \\ \frac{d(N\mu, L\mu)[1+d(M\omega_{2p}, L\mu)]}{1+d(M\omega_{2p}, L\omega_{2p})} \end{array} \right\},$$
$$= \max\left\{ \begin{array}{c} d(L\omega_{2p}, L\mu), \frac{d(L\omega_{2p+1}, L\omega_{2p})d(L\mu, L\mu)}{1+d(L\omega_{2p}, L\mu)}, \\ \frac{d(L\mu, L\mu)[1+d(L\omega_{2p+1}, L\mu)]}{1+d(L\omega_{2p+1}, L\omega_{2p})} \end{array} \right\},$$
$$= d(L\omega_{2p}, L\mu).$$

Hence from Eq(27), we get

$$\eta(d(L\omega_{2p+1}, L\mu)) \leq \pi(d(L\omega_{2p}, L\mu))\theta(d(L\omega_{2p}, L\mu)). \tag{28}$$

Further implies (as $\pi \in \mathfrak{W}$)

$$\eta(d(L\omega_{2p+1}, L\mu)) \leq \theta(d(L\omega_{2p}, L\mu)). \tag{29}$$

Similarly, again if we put $e = \omega_{2p+1}$ and $g = \mu$ in (1) and (2) and using the fact that $\pi \in \mathfrak{W}$, we obtain

$$\eta(d(L\omega_{2p+2}, L\mu)) \leq \theta(d(L\omega_{2p+1}, L\mu)). \tag{30}$$

$\forall\ p \in \mathbb{N}$, (29) and (30) gives

$$\eta(d(L\omega_{p+1}, L\mu)) \leq \theta(d(L\omega_p, L\mu)),$$

On using property of η and θ, we have

$$d(L\omega_{p+1}, L\mu) \leq d(L\omega_p, L\mu),$$

therefore there exist $k \geq 0$ such that

$$\lim_{p\to\infty} d(L\omega_p, L\mu) = k. \tag{31}$$

Suppose $k > 0$, then from (28) on taking limit as $p \to \infty$, we get

$$\eta(k) < \theta(k) < \eta(k).$$

we get a contradiction to (3), and so $k = 0$. Equation (31) implies

$$\lim_{p\to\infty} d(L\omega_p, L\mu) = 0. \tag{32}$$

The same way, we can prove that

$$\lim_{p\to\infty} d(L\omega_p, L\nu) = 0. \tag{33}$$

This is possible (both (32)–(33)) only if

$$\lim_{p\to\infty} L\omega_{2p} = L\mu = L\nu.$$

Thus from (25) on taking limit $p \to \infty$, we get

$$\lim_{p\to\infty} M\omega_{2p} = L\mu = L\nu, \quad \lim_{p\to\infty} N\omega_{2p+1} = L\mu = L\nu. \tag{34}$$

Since the pair $\{M, L\}$ and $\{N, L\}$ are compatible, therefore

$$\lim_{p\to\infty} d(L(M\omega_{2p}), M(L\omega_{2p})) = 0, \quad \lim_{p\to\infty} d(L(N\omega_{2p+1}), N(L\omega_{2p+1})) = 0. \tag{35}$$

Assume that,

$$h = L\mu \tag{36}$$

Consider,

$$d(Lh, Mh) \leq d(Lh, L(M\omega_{2p})) + d(L(M\omega_{2p}), M(L\omega_{2p})) + d(M(L\omega_{2p}), Mh).$$

Letting $p \to \infty$ and using the continuity of M in above inequality together with (34) and (35), we obtain

$$d(Lh, Mh) \leq 0,$$

implies that, $Lh = Mh$. The same way, we can derive that

$$d(Lh, Nh) \leq 0,$$

thus, $Lh = Nh$. Combining all together with (36), we get

$$h = L\mu = Lh = Mh = Nh.$$

This proves the claim.
Claim: Common fixed point is Unique
Assume there exist two fixed point $\alpha, h \in \mathbb{Y}$, with $\alpha \neq h$ such that

$$\alpha = L\mu = L\alpha = M\alpha = N\alpha \quad \text{and} \quad h = L\mu = Lh = Mh = Nh$$

Since α and h both are coincidence point of map L, then (from 23), we have

$$L\alpha = Lh.$$

This implies that

$$\alpha = L\alpha = Lh = h,$$

This proves our claim and also completes the proof of Theorem 2. □

If in the Definition 7 and Theorem 1, we take $L = I$, following fixed point results for two self-maps obtained.

Definition 8 Maps $M, N : \mathbb{Y} \to \mathbb{Y}$ are said to be $(\eta, \pi, \theta)-$ rational contraction if for each $g, e \in \mathbb{Y}$

$$\eta(d(Mg, Ne)) \leq \pi(d(g, e))\theta(K_M^N(g, e)), \quad \forall\, g \geq e, \tag{37}$$

where

$$K_M^N(g, e) = \max\left\{d(g, e), \frac{d(Mg, g)d(Ne, e)}{1 + d(g, e)}, \frac{d(Ne, e)[1 + d(Mg, e)]}{1 + d(Mg, g)}\right\}, \tag{38}$$

$\pi \in \mathfrak{W}$, $\eta \in \Psi$ and $\theta : [0, \infty) \to [0, \infty)$ is a continuous function with condition

$$0 < \theta(\kappa) < \eta(\kappa), \forall\, \kappa > 0. \tag{39}$$

Theorem 3 *Let $(\mathbb{Y}, \preceq, d)$ be a partially ordered complete metric space, and let $M, N : \mathbb{Y} \to \mathbb{Y}$ are continuous, weakly increasing and satisfying $(\eta, \pi, \theta)-$ rational contractive mappings. Further, suppose that for all $g, e \in \mathbb{Y}$, g and e are comparable. Moreover, if for all $(g, e) \in \mathbb{Y} \times \mathbb{Y}$, there exists $\omega \in \mathbb{Y}$ which is comparable to g and e. Then M and N have unique common fixed point $h \in \mathbb{Y}$.*

3 Corollary and Example

Here first we present some consequences of our main finding, and later furnish an example in support of our main results.

From Definition 7, we have

$$K_M^N(Lg, Le) = \max\left\{d(Lg, Le), \frac{d(Mg, Lg)d(Ne, Le)}{1 + d(Lg, Le)}, \frac{d(Ne, Le)[1 + d(Mg, Le)]}{1 + d(Mg, Lg)}\right\}.$$

In general, either

$$K_M^N(Lg, Le) = d(Lg, Le)$$

or

$$K_M^N(Lg, Le) = \frac{d(Mg, Lg)d(Ne, Le)}{1 + d(Lg, Le)}$$

or

$$K_M^N(Lg, Le) = \frac{d(Ne, Le)[1 + d(Mg, Le)]}{1 + d(Mg, Lg)}$$

On making use of the above three equalities in Theorems 1 and 2, we get the following three results.

Corollary 1 *Let* $(\mathbb{Y}, \preceq, d)$ *be a partially ordered complete metric space, and let maps* $M, N, L : \mathbb{Y} \to \mathbb{Y}$ *are continuous, M and N are weakly increasing with respect to L and satisfying:*

$$\eta(d(Mg, Ne)) \leq \pi(d(Lg, Le))\theta(d(Lg, Le)), \quad \forall\, g \geq e,$$

for each $g, e \in \mathbb{Y}$*, where* $\pi \in \mathfrak{W}$*,* $\eta \in \Psi$ *and* $\theta : [0, \infty) \to [0, \infty)$ *is a continuous function with condition* $0 < \theta(\kappa) < \eta(\kappa), \forall\, \kappa > 0$*. Further, suppose that* $MX \subseteq LX$ *and* $NX \subseteq LX$*, and for all* $g, e \in \mathbb{Y}$*, Lg and Le are comparable. Moreover, if the pairs* (M, L) *and* (N, L) *are compatible and if for all* $(g, e) \in \mathbb{Y} \times \mathbb{Y}$*, there exists* $\omega \in \mathbb{Y}$ *such that* $Mg \preceq M\omega$ *and* $Me \preceq M\omega$*. Then* M, N *and L have unique common fixed point* $h \in \mathbb{Y}$*.*

Corollary 2 *Let* $(\mathbb{Y}, \preceq, d)$ *be a partially ordered complete metric space, and let maps* $M, N, L : \mathbb{Y} \to \mathbb{Y}$ *are continuous, M and N are weakly increasing with respect to L and satisfying:*

$$\eta(d(Mg, Ne)) \leq \pi(d(Lg, Le))\theta(\frac{d(Mg, Lg)d(Ne, Le)}{1 + d(Lg, Le)}), \quad \forall\, g \geq e,$$

for each $g, e \in \mathbb{Y}$*, where* $\pi \in \mathfrak{W}$*,* $\eta \in \Psi$ *and* $\theta : [0, \infty) \to [0, \infty)$ *is a continuous function with condition* $0 < \theta(\kappa) < \eta(\kappa), \forall\, \kappa > 0$*. Further, suppose that* $MX \subseteq LX$ *and* $NX \subseteq LX$*, and for all* $g, e \in \mathbb{Y}$*, Lg and Le are comparable. Moreover, if the pairs* (M, L) *and* (N, L) *are compatible and if for all* $(g, e) \in \mathbb{Y} \times \mathbb{Y}$*, there exists* $\omega \in \mathbb{Y}$ *such that* $Mg \preceq M\omega$ *and* $Me \preceq M\omega$*. Then* M, N *and L have unique common fixed point* $h \in \mathbb{Y}$*.*

Corollary 3 *Let* $(\mathbb{Y}, \preceq, d)$ *be a partially ordered complete metric space, and let maps* $M, N, L : \mathbb{Y} \to \mathbb{Y}$ *are continuous, M and N are weakly increasing with respect to L and satisfying:*

$$\eta(d(Mg, Ne)) \leq \pi(d(Lg, Le))\theta(\frac{d(Ne, Le)[1 + d(Mg, Le)]}{1 + d(Mg, Lg)}), \quad \forall\, g \geq e,$$

for each $g, e \in \mathbb{Y}$*, where* $\pi \in \mathfrak{W}$*,* $\eta \in \Psi$ *and* $\theta : [0, \infty) \to [0, \infty)$ *is a continuous function with condition* $0 < \theta(\kappa) < \eta(\kappa), \forall\, \kappa > 0$*. Further, suppose that* $MX \subseteq LX$

and $NX \subseteq LX$, and for all $g, e \in \mathbb{Y}$, Lg and Le are comparable. Moreover, if the pairs (M, L) and (N, L) are compatible and if for all $(g, e) \in \mathbb{Y} \times \mathbb{Y}$, there exists $\omega \in \mathbb{Y}$ such that $Mg \preceq M\omega$ and $Me \preceq M\omega$. Then M, N and L have unique common fixed point $h \in \mathbb{Y}$.

On combining Definition 8, Theorem 3 and letting $N = M$, we deduce the following result.

Corollary 4 *Let $(\mathbb{Y}, \preceq, d)$ be a partially ordered complete metric space, and let $M : \mathbb{Y} \to \mathbb{Y}$ be a continuous mapping such that for all $g, e \in \mathbb{Y}$*

$$\eta(d(Mg, Me)) \leq \pi(d(g, e))\theta(K(g, e)),$$

where g and e are comparable,

$$K(g, e) = \max\left\{d(g, e), \frac{d(Mg, g)d(Me, e)}{1 + d(g, e)}, \frac{d(Me, e)[1 + d(Mg, e)]}{1 + d(Mg, g)}\right\},$$

$\pi \in \mathfrak{W}$, $\eta \in \Psi$ and $\theta : [0, \infty) \to [0, \infty)$ is a continuous function with condition

$$0 < \theta(\kappa) < \eta(\kappa), \forall\, \kappa > 0.$$

Suppose, if for all $(g, e) \in \mathbb{Y} \times \mathbb{Y}$, there exists $\omega \in \mathbb{Y}$ which is comparable to g and e. Then M has a unique fixed point $h \in \mathbb{Y}$.

Example

Let $\mathbb{Y} = \mathbb{N} \cup \{0\}$. Define a metric

$$d(g, e) = \begin{cases} g + e, & if\ g \neq e \\ 0, & if\ g = e. \end{cases}$$

Then $(\mathbb{Y}, d)$ is complete metric spaces. Consider three maps $M, N, L : \mathbb{Y} \to Q^+$ defined as

$$Mg = \frac{g}{2}, \quad Ng = \frac{g}{3}, \quad Lg = g^2.$$

Define maps $\eta, \theta : [0, \infty) \to [0, \infty)$ and $\pi : [0, \infty) \to [0, 1)$ as $\eta(\kappa) = \kappa, \quad \theta(\kappa) = \kappa/2$ and $\pi(\kappa) = \frac{9}{10}$.

It can be easily verified that the maps M, N and L are (η, π, θ)—generalized rational contraction maps, which is also continuous.

Further, $MX \subseteq LX$ and $NX \subseteq LX$ for $g \in \mathbb{Y}$ and M and N are weakly increasing with respect to L.

Moreover, 0 is the unique common fixed point of maps M, N and L.

4 Applications

The purpose of this section is to study the existence of common solution of system of integral equations, as an application of Theorem 3.

4.1 Existence Theorem for a Common Solution of Integral Equations

Consider the system of integral equations:

$$\begin{aligned} \omega(\kappa) &= \int_0^E R_1(\kappa, z, \omega(z))dz + \alpha(\kappa) \\ \omega(\kappa) &= \int_0^E R_2(\kappa, z, \omega(z))dz + \alpha(\kappa) \quad \forall\, \kappa \in [0, E], \end{aligned} \tag{40}$$

where $E > 0$. Let us consider the space $\mathbb{Y} = C([0, E])$ of continuous functions defined on $[0, E])$. Define the metric:

$$d(g, e) = \sup_{\kappa \in [0,E]} |g(\kappa) - e(\kappa)|\,, \quad \forall\, g, e \in \mathbb{Y}.$$

Clearly, the space $(\mathbb{Y}, d)$ is a complete metric space. Suppose the set $\mathbb{Y} = C([0, E])$ equipped with partial order $\preceq$ given by:

$$\forall\, g, e \in \mathbb{Y}, g \preceq e \Leftrightarrow g(\kappa) \le e(\kappa), \quad \forall\, \kappa \in [0, E].$$

Theorem 4 *Suppose the following hypotheses hold:*

(i) $R_1, R_2 : [0, E] \times [0, E] \times \mathbb{R} \to \mathbb{R}$ *and* $h : \mathbb{R} \to \mathbb{R}$ *are continuous,*
(ii) for all $\kappa, z \in [0, E]$*, we have*

$$\begin{aligned} R_1(\kappa, z, \omega(\kappa)) &\le R_2\Big(\kappa, z, \int_0^E R_1(z, h, \omega(h))dh + \alpha(z)\Big), \\ R_2(\kappa, z, \omega(\kappa)) &\le R_1\Big(\kappa, z, \int_0^E R_2(z, h, \omega(h))dh + \alpha(z)\Big), \end{aligned}$$

(iii) *there exists a continuous function* $G : [0, E] \times [0, E] \to \mathbb{R}_+$ *such that*

$$|R_1(\kappa, z, g) - R_2(\kappa, z, e)| \leq G(\kappa, z)\sqrt{\frac{log[(g-e)^2+1]}{(g-e)}},$$
$$\forall\, \kappa, z \in [0, E] \textit{ and } g, e \in \mathbb{R} \textit{ such that } e \preceq g,$$

(iv) $\sup_{\kappa \in [0,E]} \int_0^E G^2(\kappa, z)dz \leq \frac{1}{E}$.

Then the integral equation (40) have a solution $\omega^* \in C([0, E])$.

Proof Let us define $M, N : C([0, E]) \to C([0, E])$ by

$$M\omega(\kappa) = \int_0^E R_1(\kappa, z, \omega(z))dz + \alpha(\kappa)$$

and

$$N\omega(\kappa) = \int_0^E R_2(\kappa, z, \omega(z))dz + \alpha(\kappa), \quad \kappa \in [0, E],\ g \in C([0, E]).$$

Clealry, M and N are weakly increasing (refer to see [26]) Now, for all $g, e \in C([0, E])$ such that $e \preceq g$, we have

$$\begin{aligned}|Mg(\kappa) - Ne(\kappa)| &\leq \int_0^E |R_1(\kappa, z, g(z)) - R_2(\kappa, z, e(z))|\,dz\\ &\leq \int_0^E G(\kappa, z)\sqrt{\frac{log[(g(z)-e(z))^2+1]}{(g(z)-e(z))}}dz\\ &\leq \Big(\int_0^E G^2(\kappa, z)dz\Big)^{\frac{1}{2}}\Big(\int_0^E \frac{log[(g(z)-e(z))^2+1]}{(g(z)-e(z))}dz\Big)^{\frac{1}{2}}\end{aligned}$$

Now using hypothesis(IV), we get

$$\begin{aligned}|Mg(\kappa) - Ne(\kappa)| &\leq \Big(\frac{1}{E}\Big)^{\frac{1}{2}}\Big(\int_0^E \frac{log[(g(z)-e(z))^2+1]}{(g(z)-e(z))}dz\Big)^{\frac{1}{2}}\\ &\leq \Big(\frac{1}{E}\Big)^{\frac{1}{2}}\Big(\sqrt{\frac{log[d(g,e)^2+1]}{d(g,e)}}\Big)\sqrt{E}\\ &\leq \Big(\sqrt{\frac{log[d(g,e)^2+1]}{d(g,e)}}\Big).\end{aligned}$$

This implies that,

$$d(Mg, Ne) \leq \sqrt{\frac{log[d(g,e)^2 + 1]}{d(g,e)}}$$

Further can be written as

$$d(Mg, Ne)^2 \leq \frac{log[d(g,e)^2 + 1]}{d(g,e)} \leq \frac{\sqrt{log[d(g,e)^2 + 1]}}{d(g,e)}.\sqrt{log[d(g,e)^2 + 1]}. \tag{41}$$

Set the function π as

$$\pi(\kappa) = \frac{\sqrt{log[\kappa^2 + 1]}}{\kappa},$$

It is obvious that $\pi \in \mathfrak{W}$. If we let $\eta(\kappa) = \kappa^2$ and $\theta(\kappa) = \sqrt{log[\kappa^2 + 1]}$.

Therefore from (41), we have

$$\begin{aligned}\eta(d(Mg, Ne)) &\leq \pi(d(g,e))\theta(d(g,e)) \\ &\leq \pi(d(g,e))\theta(K_M^N(g,e)).\end{aligned}$$

Thus all the hypothesis of Theorem 3 are satisfied. Therefore there exists $\omega^* \in C([0, E])$, a common fixed point of M and N, that is ω^* is a solution to (40). □

5 Conclusion

In this article, we begin by introducing the concept of (η, π, θ)-generalized rational contraction for three self-maps. Subsequently, we present three significant theorems, namely Theorems 1–3. These theorems provide both the existence and uniqueness of common fixed points. Furthermore, we enhance the manuscript by including two illustrative examples that showcase the practical implications of our findings.

Moreover, we extend the existing literature by introducing novel consequences derived from our main results. Notably, these consequences generalize and expand upon previously established findings, particularly the notable works of Gupta et al. [20] and Borisut et al. [16].

Additionally, we present an application of our research by presenting an existence theorem for a common solution of integral equations.

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Fixed Points of Coset and Orbit Space Actions: An Application of Semihypergroup Theory

Theory, Methods and Integrative Approaches

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1 Introduction

The common fixed points of representations of different categories of topological and analytic objects (such as locally compact semigroups, topological groups, and semigroup of operators, among others [8, 9, 12, 13, 16–18, 20]) have been an exciting and immensely useful area of research in the fields of fixed-point theory and harmonic analysis, since their inception. In this article, we consider the existence of such structures in the setting of certain left/right coset, double-coset spaces associated to compact subgroups and orbit spaces of certain (continuous) affine actions arising from locally compact groups. These families of spaces appear frequently in several areas of research including Lie group theory, homogeneous spaces, polynomial and dynamical systems, solutions of differential equations, random walks, Brownian motion and topological graph theory, to name a few. In particular, we derive several characterizations of the existence of *common fixed points* of representations of such spaces, in terms of another central concept in the study of harmonic analysis, namely *amenability* of the underlying function spaces.

Through the lens of abstract harmonic analysis, it turns out that all these different families of objects, i.e., certain (double) coset and orbit spaces arising from locally compact groups, fall under the unified category of *Semihypergroups*. A semihypergroup is essentially a locally compact Hausdorff topological space that admits a certain associative convolution algebra on the space of complex Radon measures on itself. In particular, it turns out [1, 7, 10] that the algebraic structures retained by the (double) coset and orbit spaces from their parent category of locally compact groups, naturally give rise to such a convolution algebra via a standard averaging technique (we will discuss these structures in details in the following section). Our general aim here is to make use of the extensive literature [1–5, 7, 10] found on these

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"*hyperstructures*", namely the broader categories of semihypergroups and hypergroups, to gain further insight into the analytic behavior of these families of objects discussed above.

Till date, no systematic extensive study is found on the actions of such (double) cosets and orbits on general locally convex spaces, or even on Banach spaces. In particular, we observe that some recent developments in the study of semihypergroups [4, 6] imply that this theory can indeed be applied to derive some exciting new results regarding the common fixed points of actions of such specific objects. More importantly, this approach provides us with a fresh analytic insight into the fixed-point theory of the (double) coset and orbit spaces in question. Since classically the treatment of such topics has been more geometric and algebraic in nature, the analytic aspects and equivalences of the same remain unexplored till date (see [11, 14, 19, 21] for example), specially when it comes to the fixed-point theories of homogeneous spaces and coset spaces.

In what follows, we first provide a brief introduction to the unifying analytic category of semihypergroups and hypergroups along with concrete formulations of such structures when it comes to the families of (double) coset and orbit spaces of locally compact groups. We further introduce the concepts of amenability, translations, invariance and almost periodic functions on these spaces. Later on, we define actions on these spaces and discuss the equivalence between the existence of common fixed points of different kinds of actions and (left) amenability of the space of almost periodic functions. We conclude the article by introducing a certain fixed-point property *(FP)* of semihypergroup actions, that makes use of certain hereditary properties as well as the measure-algebraic formulation of actions in this setting. As an application of semihypergroup theory, we see that this property *(FP)* concerning induced (restricted) actions, completely characterizes the left amenability of the function space consisting of almost periodic functions on a given orbit space.

2 Preliminary

This section provides some necessary preliminary knowledge on the theory of semihypergroups, as well as elaborate on why the (double) coset and orbit spaces discussed above have a natural (semi)hypergroup structure. We begin by stating some basic notations that we will use throughout the article.

All the topologies throughout this text are assumed to be Hausdorff.

For any locally compact Hausdorff topological space X, we denote by $M(X)$ the space of all regular complex Borel measures on X, where $M_F^+(X)$, $M^+(X)$ and $P(X)$ respectively denote the subsets of $M(X)$ consisting of all non-negative measures with finite support, all finite non-negative regular Borel measures and all probability measures on X. For any measure μ on X, we denote by $supp(\mu)$ the support of the measure μ. Moreover, $B(X)$, $C(X)$ and $C_c(X)$ denote the function spaces of all bounded, bounded continuous and compactly supported continuous functions on X respectively.

Next, we introduce two very important topologies on the positive measure space $M^+(X)$ and the space of compact subsets for any locally compact topological space X. Unless mentioned otherwise, we will always assume these two topologies on the respective spaces.

The **cone topology** on $M^+(X)$ is defined as the topology induced by $C_c^+(X) \cup \{1_X\}$, i.e., the weakest topology on $M^+(X)$ for which the maps $\mu \mapsto \int_X f \, d\mu$ is continuous for any $f \in C_c^+(X) \cup \{1_X\}$ where 1_X denotes the characteristic function of X. Note that if X is compact then it follows immediately from the duality of Riesz representation theorem that the cone topology coincides with the weak*-topology on $M^+(X)$.

We denote by $\mathfrak{C}(X)$ the set of all compact subsets of X. The **Michael topology** on $\mathfrak{C}(X)$ is defined to be the topology generated by the sub-basis $\{C_U(V) : U, V \subseteq X$ are open sets in $X\}$, where

$$C_U(V) = \{C \in \mathfrak{C}(X) : C \cap U \neq \emptyset, C \subset V\}.$$

Observe that the Michael topology is one of the most intuitive and natural topologies on this space, and $\mathfrak{C}(X)$ indeed becomes a locally compact Hausdorff space with respect to this topology. Moreover if X is compact then $\mathfrak{C}(X)$ is also compact [10, 15].

For any element $x \in X$, we denote by δ_x the point-mass measure or the Dirac measure at the point x. For any three locally compact Hausdorff spaces X, Y, Z, a bilinear map $\Psi : M(X) \times M(Y) \to M(Z)$ is called *positive continuous* if the following properties hold true.

1. $\Psi(\mu, \nu) \in M^+(Z)$ whenever $(\mu, \nu) \in M^+(X) \times M^+(Y)$.
2. The map $\Psi|_{M^+(X)\times M^+(Y)}$ is continuous (w.r.t the cone topology).

Now we are ready to state the formal definition for a (topological) semihypergroup. Note that we mostly follow Jewett's notions [10] in terms of definitions and notations.

Definition 1 (*Semihypergroup*) A pair $(K, *)$ is called a (topological) semihypergroup if they satisfy the following properties:

(A1) K is a locally compact Hausdorff space and $*$ defines a binary operation on $M(K)$ such that $(M(K), *)$ becomes an associative algebra.
(A2) The bilinear mapping $* : M(K) \times M(K) \to M(K)$ is positive continuous.
(A3) For any $x, y \in K$ the measure $(\delta_x * \delta_y)$ is a probability measure with compact support.
(A4) The map $(x, y) \mapsto \text{supp}(\delta_x * \delta_y)$ from $K \times K$ into $\mathfrak{C}(K)$ is continuous.

Note that for any $A, B \subset K$ the convolution of subsets is defined as the following:

$$A * B := \cup_{x\in A, y\in B} \, supp(\delta_x * \delta_y).$$

Observe that $\mathfrak{C}(K)$ is closed under this convolution of sets, i.e., $A * B$ will always be compact whenever both $A, B \subseteq K$ are compact subsets. We say that a closed subset $H \subseteq K$ is a *sub-semihypergroup* of K if $H * H \subseteq H$. We define the concepts of left (resp. right) topological and semitopological semihypergroups, analogous to similar notions in the classical semigroup theory.

Definition 2 A pair $(K, *)$ is called a left (resp. right) topological semihypergroup if it satisfies all the conditions of Definition 1, with property $(A2)$ replaced by property $(A2')$ (resp. property $(A2'')$), given as the following:

(A2$'$) The map $(\mu, \nu) \mapsto \mu * \nu$ is positive and for each $\omega \in M^+(K)$ the map $L_\omega : M^+(K) \to M^+(K)$ given by $L_\omega(\mu) := \omega * \mu$ is continuous.

(A2$''$) The map $(\mu, \nu) \mapsto \mu * \nu$ is positive and for each $\omega \in M^+(K)$ the map $R_\omega : M^+(K) \to M^+(K)$ given by $R_\omega(\mu) := \mu * \omega$ is continuous.

A pair $(K, *)$ is called a *semitopological semihypergroup* if it is both left and right topological semihypergroup, *i.e*, if the convolution $*$ on $M(K)$ is only separately continuous.

For any Borel measurable function $f \in B(K)$ for a (semitopological) semihypergroup K and for each $x, y \in K$, we define the left translate $L_x f$ of f by x (resp. the right translate $R_y f$ of f by y) as a Borel function on K defined as

$$L_x f(y) = R_y f(x) = f(x * y) := \int_K f \, d(\delta_x * \delta_y) \, .$$

Whenever $f \in C(K)$, we have that $L_x f, R_x f \in C(K)$ for each $x \in K$. Unless mentioned otherwise, we will always assume the uniform (supremum) norm $|| \cdot ||_u$ on $C(K)$ and $B(K)$. We denote by $\mathcal{B}_1$ the closed unit ball of $C(K)^*$. Similarly, for any linear subspace $\mathcal{F}$ of $C(K)$, we denote the closed unit ball of $\mathcal{F}^*$ as

$$\mathcal{B}_1(\mathcal{F}^*) := \{\omega \in \mathcal{F}^* : ||\omega|| \leq 1\}.$$

Moreover, $\mathcal{F}$ is called left (resp. right) translation-invariant if $L_x f \in \mathcal{F}$ (resp. $R_x f \in \mathcal{F}$) for each $x \in K$, $f \in \mathcal{F}$. We simply say that $\mathcal{F}$ is translation-invariant, if it is both left and right translation-invariant.

A function $f \in C(K)$ is called left (resp. right) uniformly continuous if the map $x \mapsto L_x f$ (resp. $x \mapsto R_x f$) from K to $(C(K), || \cdot ||_u)$ is continuous. We say that f is *uniformly continuous* if it is both left and right uniformly continuous. The space consisting of all such functions is denoted by $UC(K)$, which forms a norm-closed linear subspace of $C(K)$.

The left (resp. right) orbit of a function $f \in C(K)$, denoted as $\mathcal{O}_l(f)$ (resp. $\mathcal{O}_r(f)$), is defined as $\mathcal{O}_l(f) := \{L_x f : x \in K\}$ (resp. $\mathcal{O}_r(f) := \{R_x f : x \in K\}$). A function $f \in C(K)$ is called left (resp. right) almost periodic if we have that $\mathcal{O}_l(f)$ (resp. $\mathcal{O}_r(f)$) is relatively compact in $(C(K), || \cdot ||_u)$. We showed in a previous work [1, Corollary 4.4] that a function f on K is left almost periodic if and only if it is right almost periodic. Hence we regard any left or right almost periodic function on K simply as an *almost periodic function*, and denote the space of all almost periodic functions on K as $AP(K)$. We further saw in [1] that $AP(K)$ is a norm-closed, conjugate-closed (with respect to complex conjugation), translation-invariant linear subspace of $C(K)$ containing constant functions, such that $AP(K) \subseteq UC(K)$.

Now recall [10] that for any locally compact Hausdorff space X, a map $i : X \to X$ is called a *topological involution* if i is a homeomorphism and $(i \circ i)(x) = x$ for each $x \in X$. On a semitopological semihypergroup $(K, *)$, a topological involution $i : K \to K$ given by $i(x) := x^-$ is called a *(semihypergroup) involution* if $(\mu * \nu)^- = \nu^- * \mu^-$ for any $\mu, \nu \in M(K)$. For any measure $\omega \in M(K)$, we have that

$$\omega^-(B) := \omega(B^-) := \omega(i(B)),$$

for any Borel measurable subset B of K. As expected, an involution on a semihypergroup is analogous to inverses on a semigroup. Hence a semihypergroup with an identity and an involution of the following characteristic is a hypergroup.

Definition 3 (*Hypergroup*) A semihypergroup $(H, *)$ is called a hypergroup if it satisfies the following conditions:

(A5) There exists an element $e \in H$ such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for any $x \in H$.
(A6) There exists an involution $x \mapsto x^-$ on H such that $e \in \text{supp}(\delta_x * \delta_y)$ if and only if $x = y^-$.

The element e in the above definition is called the *identity* of H. Note that the identity and involution of a hypergroup are necessarily unique [10]. A sub-semihypergroup H of a hypergroup K is called a sub-hypergroup if we have that $H^- = H$ as well.

Remark 1 Given a Hausdorff topological space K, in order to define a positive continuous bilinear map $* : M(K) \times M(K) \to M(K)$, it suffices to only define the measures $(\delta_x * \delta_y)$ for each $x, y \in K$. This is true since one can then extend the convolution '$*$' bilinearly to $M_F^+(K)$. As $M_F^+(K)$ is dense in $M^+(K)$ in the cone topology [10], one can further achieve a continuous extension of '$*$' to $M^+(K)$ and hence to the whole of $M(K)$ using bilinearity of the convolution.

Finally, the center $Z(K)$ of a (semitopological) semihypergroup $(K, *)$ is defined to be the largest semigroup included in $(K, *)$. In other words, we have that

$$Z(K) := \{x \in K : \text{ supp}(\delta_x * \delta_y) \text{ and } \text{supp}(\delta_y * \delta_x) \text{ are singleton for any } y \in K\}.$$

Note that it is possible that $Z(K) = \emptyset$, and even if $x \in Z(K)$, then $(\delta_x * \delta_y)$ and $(\delta_y * \delta_x)$ need not be supported on the same element in K, for each $y \in K$.

But if K is a hypergroup, we immediately see that $e \in Z(K)$. In fact, in that case, i.e., if K is a hypergroup, then the center $Z(K)$ is indeed the largest group included in K, and it can easily be checked [7] that the following relation holds true:

$$Z(K) := \{x \in K : \delta_x * \delta_{x^-} = \delta_{x^-} * \delta_x = \delta_e\}.$$

Hence the definition for center of a hypergroup[10] defined as above, coincides with that of the center of a semihypergroup. The following is a simple example of a (one dimensional) hypergroup with a non-trivial center.

Example (Zeuner [22])

Consider the hypergroup $(H, *)$ where $H = [0, 1]$ and the convolution is defined as

$$\delta_s * \delta_t = \frac{\delta_{|s-t|} + \delta_{1-|1-s-t|}}{2}.$$

We immediately see that $Z(H) = \{0, 1\}$.

Now we list some well-known examples [10, 22] of semihypergroups and hypergroups. See [1, Sect. 3] for details on the constructions as well as the reasons why most of the structures discussed there, although attain a semihypergroup structure, fail to be hypergroups.

Trivial Examples

If $(S, \cdot)$ is a locally compact (topological) semigroup, then $(S, *)$ has a trivial semihypergroup structure given as

$$\delta_x * \delta_y = \delta_{x.y}$$

for each $x, y \in S$.

Similarly, if $(G, \cdot)$ is a locally compact topological group with identity e_G, then $(G, *)$ is a hypergroup with the same bilinear operation $*$, identity element e_G and the involution on G defined as $x \mapsto x^{-1}$.

Observe that $Z(S) = S$, $Z(G) = G$ in the above example.

A Finite Case

Take $T = \{e, a, b\}$ and equip it with the discrete topology. Define

$$\begin{aligned}
\delta_e * \delta_a = \delta_a * \delta_e \; &= \; \delta_a \\
\delta_e * \delta_b = \delta_b * \delta_e \; &= \; \delta_b \\
\delta_a * \delta_b = \delta_b * \delta_a \; &= \; z_1\delta_a + z_2\delta_b \\
\delta_a * \delta_a &= x_1\delta_e + x_2\delta_a + x_3\delta_b \\
\delta_b * \delta_b &= y_1\delta_e + y_2\delta_a + y_3\delta_b,
\end{aligned}$$

where $x_i, y_i, z_i \in \mathbb{R}$ such that $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = z_1 + z_2 = 1$ and $y_1x_3 = z_1x_1$.

Then $(T, *)$ is a commutative hypergroup with e as the identity element and the identity map on T taken as involution. In fact, any finite set can be given several (not necessarily equivalent) semihypergroup and hypergroup structures.

Next we see how some of the the primary examples discussed in the introduction, i.e., the (double) coset and orbit spaces arising from general locally compact groups, attain a natural hyper-structure using a standard averaging technique on their measure spaces.

Coset Spaces

Let G be a locally compact group and H be a compact subgroup of G. Moreover, let μ be the normalized Haar measure of H. Consider the left quotient space $K := G/H = \{xH : x \in G\}$ and equip it with the natural quotient topology. For any $x, y \in G$, define

$$\delta_{xH} * \delta_{yH} = \int_H \delta_{(xty)H} \, d\mu(t).$$

Then it can be easily checked [10] that $(K, *)$ is a semihypergroup.

For instance, consider G to be the symmetric group S_4 and H to be the dihedral group D_8. We know that D_8 is not a normal subgroup of S_4. Consider the left coset space

$$G/H = \{H, s_1H, s_2H\},$$

where $s_1 = (1\ 2\ 4)$ and $s_2 = (1\ 4\ 2)$. Since a Haar measure on D_8 is simply a multiple of the counting measure, the above formulation implies that

$$\delta_{xH} * \delta_{yH} = \frac{1}{8} \sum_{h \in H} \delta_{(xty)H}$$

Hence a direct computation gives us that the left coset space S_4/D_8 is a discrete semihypergroup where the convolution is given by the following table:

$*$	δ_H	δ_{s_1H}	δ_{s_2H}
δ_H	δ_H	$\frac{1}{2}(\delta_{s_1H} + \delta_{s_2H})$	$\frac{1}{2}(\delta_{s_1H} + \delta_{s_2H})$
δ_{s_1H}	δ_{s_1H}	$\frac{1}{2}(\delta_H + \delta_{s_2H})$	$\frac{1}{2}(\delta_H + \delta_{s_1H})$
δ_{s_2H}	δ_{s_2H}	$\frac{1}{2}(\delta_H + \delta_{s_1H})$	$\frac{1}{2}(\delta_H + \delta_{s_2H})$

Observe that here the element $x = H$ fails to be a right identity, and hence S_4/D_8 does not attain a hypergroup structure. Thus in general, for a left coset space G/H where H is not normal, H fails to serve as a right identity.

The next example [10] of double coset spaces overcomes this barrier and becomes a hypergroup using a similar convolution product.

Double Coset Spaces

As before, let G be a locally compact topological group, H be a compact subgroup of G and μ be the normalized Haar measure of H. We equip the space of double cosets $K := G//H = \{HxH : x \in G\}$ with the usual quotient topology. For any $x, y \in G$, use the usual averaging technique to define

$$\delta_{HxH} * \delta_{HyH} = \int_H \delta_{H(xty)H} \, d\mu(t).$$

Then $(K, *)$ is a hypergroup [10] with the identity element $e = H$ and involution function $HxH \mapsto Hx^{-1}H$.

Finally, the next example of orbit spaces includes [10] the families of coset and double-coset spaces discussed above. Recall that a continuous action of a topological group G on a Hausdorff space X is a continuous map $(g, x) \mapsto x^g : G \times X \to X$ such that $x^e = x$ and

$$x^{gk} = (x^g)^k$$

for each $x \in X, g, k \in G$. Moreover, a map $\phi : G \to G$ is called affine if there exists an element $\alpha \in G$ and an automorphism Ψ of G such that

$$\phi(x) = \alpha\Psi(x),$$

for each $x \in G$. Observe that if $g, k \in G$ and Ψ is an automorphism of G, then $g\Psi k$ is affine as well (this fact is used in checking that the standard averaging technique indeed induces a convolution product on the measure space of orbits of certain actions as discussed below). Thus, a continuous action of G on X is called a *continuous affine action* if the map $g \mapsto x^g : G \to G$ is affine for each $x \in X$.

Orbit Spaces

Let G be a locally compact topological group and H be any compact group where $\pi : H \times G \to G$ is a continuous affine action of H on G given by $\pi(h, g) = g^h$ for each $g \in G, h \in H$. Consider the orbit space

$$O = G^H := \{x^H : x \in G\},$$

where $x^H = \mathcal{O}(x) = \{\pi(h, x) : h \in H\}$ is the orbit of x under the action π of H. Let μ be the normalized Haar measure of H. Consider G^H with the quotient topology and the following convolution:

$$\delta_{x^H} * \delta_{y^H} := \int_H \int_H \delta_{(\pi(s,x)\pi(t,y))^H} \, d\mu(s) \, d\mu(t).$$

Then $(\mathcal{O}, *)$ becomes a semihypergroup [10].

For any such action π of H on G, the maps $\{\pi_h : G \to G : h \in H\}$ given by

$$\pi_h(g) := \pi(h, g) \text{ for each } g \in G$$

are called the *co-efficient functions* of the respective action. It can be shown [7, 10] that $(\mathcal{O}, *)$ becomes a hypergroup if for each $h \in H$, the co-efficient map π_h is an automorphism of G.

Note that the formulation of the convolution product for coset spaces is a special case of the above formulation for orbit spaces. In particular, for a compact subgroup $H \leq G$ let us consider the action π given by $H \times G \to G : (h, x) \mapsto xh$. Then $G^H = G/H$ and using the general formulation for convolution of measures in orbit spaces we have

$$\begin{aligned} \delta_{xH} * \delta_{yH} &= \int_H \int_H \delta_{(\pi(s,x)\pi(t,y))H} \, d\mu(s) \, d\mu(t) \\ &= \int_H \int_H \delta_{((xsyt)H} \, d\mu(s) \, d\mu(t) \\ &= \int_H \int_H \delta_{((xsy)H} \, d\mu(s) \, d\mu(t) \\ &= \int_H \delta_{((xsy)H} \, d\mu(s), \end{aligned}$$

where the last equality follows since μ is a normalized measure, i.e., $\mu(H) = 1$. The formulation for double cosets follows in a similar manner.

Thus we see that the convolution structure of the coset and double-coset spaces are compatible with the general convolution structure corresponding to orbit spaces. Hence in the next section, we consider the results for orbit spaces in general. Moreover, for the rest of this chapter, whenever we mention (double) coset and orbit spaces, we assume the setting for these spaces discussed in this section.

3 Common Fixed Points

Now we proceed toward investigating certain representations of such coset and orbit spaces on a locally convex space. In particular, we use the notion of *semihypergroup actions* to explore the intrinsic connection between the common fixed points of such actions and the amenability of certain function spaces of the underlying orbit space. We use the following notational convention for the rest of the article.

A locally convex Hausdorff topological vector space E with a family Q of seminorms is simply denoted by (E, Q) or (E, τ_Q) where τ_Q is the topology on E generated by the family of seminorms Q. In rest of the chapter, the topology assumed on E (resp. on any Borel subset X of E), is the topology (resp. induced topology) τ_Q.

Definition 4 (*Semihypergroup Action*) Let $(K, *)$ be a (semitopological) semihypergroup and X be a convex subset of (E, Q). A map $\pi : K \times X \to X$ is called a (semihypergroup) action if the following conditions are satisfied:

1. For any $s, t \in K, x \in X$ we have

$$\pi(s, \pi(t, x)) = \int_K \pi(\zeta, x)\, d(\delta_s * \delta_t)(\zeta).$$

2. Whenever K has a two-sided identity e, we have that $\pi(e, x) = x$ for each $x \in X$.

Such an action π of K on X is called (separately) continuous, if the map π is (separately) continuous on $K \times X$. For each $s \in K$, we denote the *co-efficient map* $x \mapsto \pi(s, x) : X \to X$ by π_s. The following definitions are immediate consequences of the general definition of semihyperoup action, keeping in mind the convolution structure of the measure-algebras of respective spaces.

Coset and Orbit Actions

In view of the above discussion, we see that if $(G/H, *)$ is a left coset space with the natural semihypergroup structure as discussed in the previous section, then a map $\pi : G/H \times X \to X$ is called a *coset action* if for any $s, g \in G, x \in X$ we have

$$\pi(sH, \pi(gH, x)) = \int_H \pi\big((shg)H, x\big)\, d\mu(h),$$

where μ is the normalized Haar measure of the compact subgroup H of G.

Similarly, the *double-coset action* of $G//H$ on X is a map $\pi : G//H \times X \to X$ such that for any $s, g \in G, x \in X$ we have that $\pi(H, x) = x$ and the following holds true:

$$\pi(HsH, \pi(HgH, x)) = \int_H \pi\big(H(shg)H, x\big)\, d\mu(h).$$

Finally, consider the orbit space G^H for any compact group H and locally compact group G. An *orbit action* of G^H on a convex subset X of a locally convex space (E, Q) is a map $\pi : G^H \times X \to X$ such that for any $s, g \in G$, $x \in X$ we have that:

$$\pi(s^H, \pi(g^H, x)) = \int_H \int_H \pi\big((s^h g^k)^H, x\big)\, d\mu(h)\, d\mu(k).$$

Note that if G^H is a hypergroup as well, for π to be an orbit action, we would also need that $\pi(e^H, x) = \pi(H, x) = x$ for each $x \in X$, in addition to the above property.

It turns out that separate continuity of the action of a compact semihypergroup automatically implies joint continuity on the center of a sub-hypergroup. In particular, the following result is true [4] even for any left topological semihypergroup K with an identity e.

Theorem 1 *Let $(K, *)$ be a compact left topological semihypergroup with identity, and $H \subseteq K$ be a hypergroup. Then any separately continuous action σ of K on a compact convex space X is jointly continuous on $Z(H) \times X$.*

Hence in particular, we have that for a compact orbit space G^H where all the co-efficient functions of the action of H on G are automorphisms, any separately continuous action π of G^H on a compact convex space X is jointly continuous on $Z(G^H) \times X$. In particular, for a compact double-coset space $G//H$, any separately continuous action π of $G//H$ on a compact convex space X is jointly continuous on $\big(C_G(H)//H\big) \times X$, where $C_G(H)$ is the stabilizer of H defined as

$$C_G(H) := \{g \in G : gHg^{-1} \subseteq H\}.$$

Given a coset action (resp. orbit action) π of G/H (resp. G^H) on a convex subset X of a locally convex space (E, Q), we say that an element $x_0 \in X$ is a *common fixed point* of π if we have that

$$\pi(gH, x_0) = x_0 \quad (resp.\ \ \pi(g^H, x_0) = x_0)$$

for each $g \in G$.

For the rest of this chapter, we will explore different equivalent conditions for a coset (resp. orbit) action π to allow a common fixed point. In particular, we will see that the existence of such a fixed point is intrinsically related to the amenability of the function space consisting of almost periodic functions on the underlying coset (resp. orbit) space.

First, we briefly recall some definitions and results on amenability. Recall from the previous section that each coset, double coset, and orbit space admits a natural convolution algebra on its measure space, making it a semihypergroup. For any semihypergroup $(K, *)$ and a linear subspace $\mathcal{F}$ of $C(K)$ containing constant functions, a function $m \in \mathcal{F}^*$ is called a *mean* of $\mathcal{F}$ if we have that

$$||m|| = 1 = m(1),$$

where 1 denotes the constant function $\equiv 1$ on K. We denote the set of all means on $\mathcal{F}$ as $\mathcal{M}(\mathcal{F})$. If $\mathcal{F}$ is a left translation-invariant linear subspace of $C(K)$ containing constant functions, a mean m of $\mathcal{F}$ is called a *left invariant mean* (LIM) if

$$m(L_x f) = m(f)$$

for any $x \in K, f \in \mathcal{F}$. We say that $\mathcal{F}$ is left amenable if it admits a LIM. In particular, we say that the semihypergroup K is left amenable if the space $C(K)$ of bounded continuous functions on K admits a LIM.

Now finally, let X be a convex subset of (E, Q) and Y be any locally convex space. Then a continuous map $T : X \to Y$ is called *affine* if for any $\alpha \in (0, 1)$, $x_1, x_2 \in X$ we have that

$$T(\alpha x_1 + (1-\alpha)x_2) = \alpha T(x_1) + (1-\alpha)T(x_2).$$

We say that a (separately) continuous action π of K on X is affine if for each $s \in K$, the co-efficient map $\pi_s : X \to X$ is affine.The following fixed-point theorems hold true [4] even for the broader category of semihypergroups. We omit the proofs here, as they are immediate applications of the general results on the category of semihypergroups.

Theorem 2 *Let G^H be commutative, i.e., $\delta_{_{sH}} * \delta_{_{gH}} = \delta_{_{gH}} * \delta_{_{sH}}$ for each $s, g \in G$ and X be a compact convex subset of a locally convex Hausdorff vector space (E, Q). Then any separately continuous affine action of G^H on X has a common fixed point.*

Theorem 3 *If any jointly continuous affine action π of G^H on a compact convex subset X of a locally convex vector space (E, Q) has a common fixed point, then $AP(G^H)$ is left amenable.*

In fact, we see that the above theorems can be further amended [6] to prove a stronger equivalence. We provide a sketch of proof here.

Theorem 4 *The following statements are equivalent:*

1. *The function space $AP(G^H)$ is left amenable.*
2. *Any separately continuous, equicontinuous, affine action π of G^H on a compact convex subset X of a locally convex vector space (E, Q) has a common fixed point.*

Proof (2) $\Rightarrow$ (1): Consider the action π of G^H on $AP(G^H)^*$ given as

$$\begin{aligned} G^H \times AP(G^H)^* &\to AP(G^H)^* \\ (g^H, \phi) &\mapsto L^*_{_{gH}} \phi, \end{aligned}$$

where $L^*_{g^H}\phi : AP(G^H)^* \to \mathbb{C}$ is given as

$$L^*_{g^H}\phi(f) := \phi(L_{g^H} f)$$

for each $f \in AP(G^H)$. Next, consider the set $\mathcal{K}$ consisting of all means on $AP(G^H)$. Note that $\mathcal{K} \subseteq AP(G^H)^*$ is a compact convex set, when equipped with the weak*-topology [1, 4] since $\mathcal{K} \subseteq \mathcal{B}_1(AP(G^H)^*)$. Moreover, note that the action π is $\mathcal{K}$ invariant, since the elements of $\mathcal{K}$ are means on $AP(G^H)$. Here we abuse notation a little for convenience and denote $\pi|_{\mathcal{K}}$ simply as π.

Using similar techniques as in Theorem 3 we see that π is affine and separately continuous on $\mathcal{K} \times \mathcal{K}$. Now to prove that π is equicontinuous as well, for each $f \in AP(G^H)$ set p_f to be the pseudonorm on $AP(G^H)^*$ defined as

$$p_f(\phi) := \sup_{g \in G} \; |\phi(L_{g^H} f)|$$

for each $\phi \in AP(G^H)^*$. Set $Q := \{p_f : f \in AP(G^H)\}$. Now given any $\varepsilon > 0$ we can choose $\phi_1, \phi_2 \in AP(G^H)^*$ such that

$$\sup_{g \in G} \; |(\phi_1 - \phi_2)(L_{g^H} f)| = p_f(\phi_1 - \phi_2) < \varepsilon.$$

Then setting $\delta := \varepsilon$, for each $g_0 \in G$ we have

$$\begin{aligned}
&\sup_{g \in G} \; |(\pi(g_0^H, \phi_1) - \pi(g_0^H, \phi_2))(L_{g^H} f)| \\
&= \sup_{g \in G} \; |(\phi_1 - \phi_2)(L_{g_0^H} L_{g^H} f)| \\
&= \sup_{g \in G} \; \left| \int_{G^H} (\phi_1 - \phi_2)(L_{u^H} f) \, d(\delta_{g_0^H} * \delta_{g^H})(u^H) \right| \\
&\leq \sup_{g \in G} \int_{G^H} |(\phi_1 - \phi_2)(L_{u^H} f)| \, d(\delta_{g_0^H} * \delta_{g^H})(u^H) \\
&\leq \sup_{g \in G} \int_{G^H} p_f(\phi_1 - \phi_2) \, d(\delta_{g_0^H} * \delta_{g^H})(u^H) < \varepsilon,
\end{aligned}$$

as required. Now the common fixed point of this separately continuous, equicontinuous, affine action provides us with a LIM on $AP(G^H)$, following standard arguments.

(1) $\Rightarrow$ (2): Let π be an action of G^H on K as described in the hypothesis and ψ be a LIM on $AP(G^H)$. In a previous work [4], we showed that the point-evaluation maps on $AP(G^H)$ behave as extreme points, and hence there exists a net $\{\psi_\alpha\}_{\alpha \in I}$ of means such that $\lim_\alpha \psi_\alpha(f) = \psi(f)$ for each $f \in AP(G^H)$, where for each $\alpha \in I$ we have that

$$\psi_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha E_{g^H_{(i,\alpha)}}$$

where $\sum_{i=1}^{n_\alpha} \lambda_i^\alpha = 1, \lambda_i^\alpha > 0, g_{(i,\alpha)} \in G, n_\alpha \in \mathbb{N}$ for each $i = 1, 2, \ldots, n_\alpha$ and for each $s \in G$, E_{s^H} denotes the evaluation map on $AP(G^H)$ given as $E_{s^H}(f) := f(s^H)$.

Pick and fix some $y \in K$, and consider the cluster point z of the net $\{z_\alpha\}_{\alpha \in I}$ in K where we have that

$$z_\alpha := \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \pi\left(g^H_{(i,\alpha)}, y\right).$$

Using the convexity of K and passing through a subnet if necessary, one can then show that z serves as a common fixed point of the given action. □

As we confine ourselves to normed spaces instead of locally convex ones, we acquire an evolved version of the above result as the following. The result was recently proved in the smaller setting of semitopological semigroups in [13]. We provide a sketch of proof here for the broader class of orbit spaces. In an upcoming article, we consider similar characterizations in the even broader setting of general semitopological semihypergroups.

Theorem 5 *The following statements are equivalent:*

1. *The function space $AP(G^H)$ is left amenable.*
2. *Any separately continuous, equicontinuous, affine action π of G^H on a dual Banach space E^* has a common fixed point whenever there exists $f_0 \in E^*$ such that the orbit $\{\pi(g^H, f_0) : g \in G\}$ is bounded in E^*.*

Proof (2) $\Rightarrow$ (1) : Consider $X := AP(G^H)$ with the weak* topology. Then one can show that $AP(G^H)$ is a right Banach $M(G^H)$-module with the module action $AP(G^H) \times M(G^H) \to AP(G^H)$ given by the following vector integral:

$$(f, \mu) \mapsto L_\mu f := \int_{G^H} L_{x^H} f \, d\mu(x^H).$$

We immediately see that $\mathbb{C}\mathbf{1}$ is a submodule of X since for each scalar $\alpha \in \mathbb{C}$, $\mu \in M(G^H)$, $y \in G$ we have

$$\begin{aligned} L_\mu(\alpha\mathbf{1})(y^H) &= \int_{G^H} L_{x^H}(\alpha\mathbf{1})(y^H)\, d\mu(x^H) \\ &= \int_{G^H} \alpha\mathbf{1}(x^H * y^H)\, d\mu(x^H) \\ &= \big(\alpha\mu(G^H)\big)\mathbf{1}(y^H) \end{aligned}$$

where $\mathbf{1}$ denotes the constant function 1_{G^H}. Hence the quotient $E := X/\mathbb{C}\mathbf{1}$ is a right Banach $M(G^H)$-module as well. We consider the canonical left Banach $M(G^H)$-module E^* where the left module action is given by $\mu \mapsto L_\mu^*$ and use the following orthogonal identification:

$$E^* \equiv \{u \in X^* : u(\mathbf{1}) = 0\} = \mathbb{C}\mathbf{1}^\perp.$$

Pick and fix any $v_0 \in AP(G^H)^*$ such that $v_0(\mathbf{1}) = 1$. Observe that

$$L_\mu^*(v_0)(\mathbf{1}) = v_0(L_\mu \mathbf{1}) = ||\mu||$$

and hence $\left(L^*_{\delta_{x^H}} v_0 - v_0\right) \in \mathbb{C}\mathbf{1}^\perp$. Finally, consider the action π of G^H on E^* given by

$$\pi(x^H, u) := L^*_{\delta_{x^H}} u + L^*_{\delta_{x^H}} v_0 - v_0$$

for each $u \in E^*$, $x \in G$. One can show that π indeed provides a well-defined separately continuous, equicontinuous affine action of G^H on E^*. The common fixed point of π thereby provides us with a LIM on $AP(G^H)$.

The other direction can be proved using similar techniques as in [13]. □

Next, we use an equivalent formulation of coset (resp. orbit) action in terms of the convolutive measure-algebra. Observe that given a (separately) continuous action π of G^H on a convex subset X of a locally convex Hausdorff vector space (E, Q), one can naturally induce a map $\sigma_\pi : M(G^H) \times X \to X$ defined as

$$\sigma_\pi(\mu, x) := \int_{G^H} \pi(\zeta^H, x)\, d\mu(\zeta^H),$$

for each $\mu \in M(G^H)$.

Remark 2 In the above setting, since $\sigma_\pi(\delta_{g^H}, x) := \pi(g^H, x)$ for any $g \in G, x \in X$, and $M_F^+(G^H)$ is dense in $M^+(G^H)$ in the cone topology, we see that σ_π is a (separately) continuous action of $M(G^H)$ on X in the classical algebraic sense, since in general for a semihypergroup $(K, *)$ we have that

$$\mu * \nu = \int_K \int_K (\delta_s * \delta_t)\, d\mu(s)\, d\nu(t)$$

for each $\mu, \nu \in M^+(K)$ [10].

Observe that the action σ_π induced by π is linear by construction. In fact, it follows immediately that our previous definition for orbit action is structurally equivalent to the following definition using the classical notion of action in terms of homomorphisms.

Definition 5 Let (E, Q) be a separated locally convex space. Then an *orbit action* π of G^H on E is a homomorphism from the associative algebra $M(G^H)$ to the algebra $L(E)$ of linear operators on E.

In other words, an orbit action π of G^H on E is a bilinear map $(\mu, t) \mapsto \pi_\mu(t) : M(G^H) \times E \to E$ such that

$$\pi_{(\mu * \nu)} = \pi_\mu \circ \pi_\nu$$

on E for each $\mu, \nu \in M(G^H)$.

We say that an orbit action $\pi : M(G^H) \times E \to E$ is *(separately) τ-continuous* if the map π is (separately) continuous when $M(G^H)$ is equipped with a certain topology τ and E is given the usual topology τ_Q induced by the associated family of seminorms Q. This takes us to the final result of the article, providing us with another complete characterization of left amenability of the almost periodic functions, in terms of the existence of a certain fixed point. We omit the proof here, as the result is proved for the broader category of semihypergroups in [4].

Theorem 6 *Then the following statements are equivalent:*

1. *The function space $AP(G^H)$ is left amenable.*
2. *Given any action $\pi : M(G^H) \times E \to E$ of G^H on a locally convex space E, the following fixed-point property (FP) is satisfied:*
   "For any compact, convex, $P(G^H)$-invariant subset F of E, if the induced action $\tilde{\pi} := \pi|_{P(G^H)\times F} : P(G^H) \times F \to F$ is separately $\tau_{AP(G^H)}$-continuous, then $\tilde{\pi}$ has a common fixed point."

Thus we see that as we refine the structure of the space which the orbit space acts on, we keep gaining more useful characterizations for the existence of a common fixed point of such actions. In fact, one can potentially make use of the rich theory of Banach algebras and in particular, the theory of F-algebras to gain further insight into the fixed points of actions of such spaces, which is still a widely open area of research.

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Strange Chaotic Attractors and Existence Results via Nonlinear Fractional Order Systems and Fixed Points

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1 Introduction

Fixed point theory flourished from the beginning of topology, when Poincare, Lefschetz-Hopf, and Leray-Schauder, among others, contributed to the discipline. Numerous fields extensively employ fixed point theory, including control theory, logic programming, convex optimization, economics, and the study of numerical analysis. Fixed point theory is undoubtedly one of the finest and most significant techniques of contemporary mathematics, as demonstrated by Stephen Banach, one of the pioneers in developing the field of nonlinear functional analysis. The successive approximations in metric fixed point theory have their roots in the work of Peano, Cauchy, Liouville, and Picard. Experts in this field generally concur that Banach is to be credited for laying the foundation for an abstract framework that goes well beyond the realm of simple integral equations and differential equations. The fixed point theory of multi-valued mappings is just as important as the theory of single-valued mappings. Numerous authors generalized Nadler's fixed point theorem in various ways after he presented the multi-valued type of the contraction theorem. The value of the conclusion is increased by a constructive demonstration of a fixed point result since it results in a methodology for finding a fixed point [1–7].

Following the publication of Meir-Keeler's analog to the contraction theorem in metric spaces has generated some interest during the past two decades, (for example,

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[8–11]). Their theorem has been used extensively in locating the solutions to various fractal-fractionals, which frequently arise in the data structures of linear systems, adaptive analysis, theory of control, probabilistic filtration, data, and numerous other fields, serving as the inspiration for their conclusion.

On the other-side of this article, In the computational investigation of systems with dynamics in general, an attractor is a collection of states that a system tends to move toward under various initial conditions. If the attractor values are sufficiently close to the system values, they don't move much even when they are somewhat disturbed. The developing variable in finite-dimensional systems can be algebraically represented as vector with n-dimensions. A location in space with n-dimensions is the attractor. The n-dimensions in physical systems could, for instance, be two or three spatial coordinates for each of a number of distinct elements; in financial systems, they could be distinct variables like the unemployment rate and inflation rate.

1.1 Connections in Between Fixed Points and Attractors

A point that a function (transformation) maps to itself is referred to as a fixed point. Every transformation, if we consider the development of a dynamical system as a sequence of transformations, doesn't necessarily have a fixed point. A dynamical system's final state, such as the bottom center of a bowl containing a rolling marble, an attracting fixed point of the system's evolving function is represented by either the level and uninterrupted water line of sloshing glassware or the bottom center positioning of a diminished pendulum. However, a dynamic system's fixed point(s) is not always an attractor of the system. For instance, if the bowl containing a whirling marble flips over and the marble is placed on top of the bowl, the center bottom of the bowl is fixed in place but not an attractor. This corresponds to defining the difference between consistent and inconsistent equilibria in the context of equilibria.

The point at the top of the bowl in the scenario with a marble on top is a fixed point, but it is not an attractor (unstable equilibrium) [12]. Furthermore, due to the actual nature of dynamics, physical dynamical structures having at most one fixed point invariably have numerous attractors and fixed points, including the nonlinear dynamics of friction, the roughness of the surface, distortion, and even quantum mechanics itself. The shifting structure of a relatively rough marble rolling around on such tiny terrain contains several places that are regarded as fixed points or stationary points, particularly those which have been described as attractors (Fig. 1).

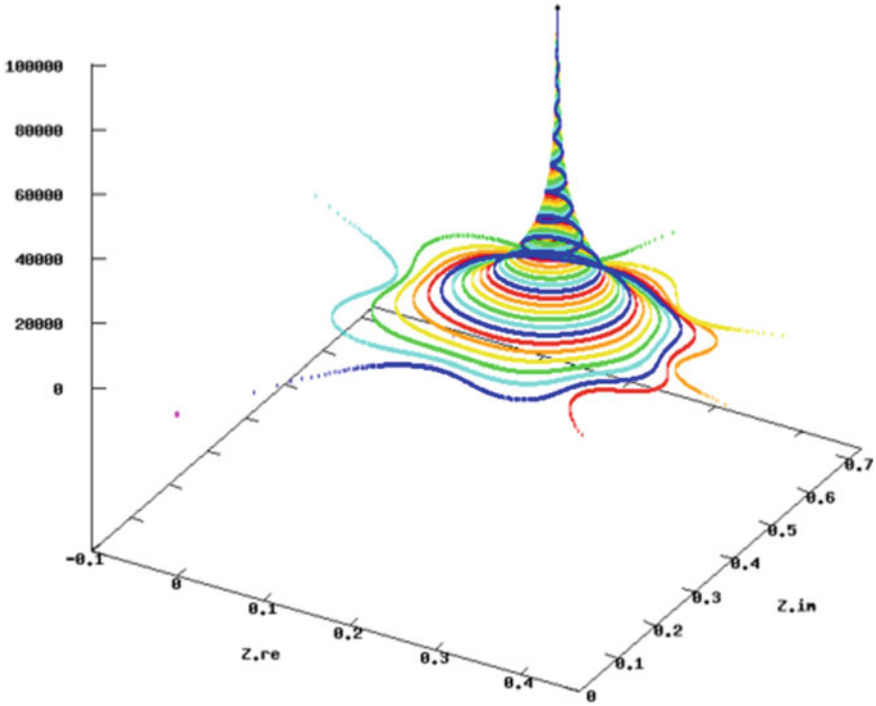

Fig. 1 A weakly attractive fixed point exists for a complex number accumulating in accordance with a complex quadrennial polynomial

2 Generalized Contractions in Suprametric Space

The following conceptual framework of *suprametric space* was introduced by M. Berzig.

Definition 2.1 ([13]) Assume that Υ is a non-empty set and $\gamma \in \mathbb{R}^+$. A function $\mathscr{S}_M : \Upsilon \times \Upsilon \to \mathbb{R}^+$ is referred to as an extended suprametric if for all $\varkappa, \mathtt{y}, \mathfrak{z} \in \Upsilon$, if $\mathscr{S}_M$ meets $(\mathscr{S}_{M1})$, $(\mathscr{S}_{M2})$ and $(\mathscr{S}_{M3})$.

$(\mathscr{S}_{M1})$. $\mathscr{S}_M(\varkappa, \mathtt{y}) = 0$ iff $\varkappa = \mathtt{y}$;
$(\mathscr{S}_{M2})$. $\mathscr{S}_M(\varkappa, \mathtt{y}) = \mathscr{S}_M(\mathtt{y}, \varkappa)$;
$(\mathscr{S}_{M3})$. $\mathscr{S}_M(\varkappa, \mathfrak{z}) \leq \mathscr{S}_M(\varkappa, \mathtt{y}) + \mathscr{S}_M(\mathtt{y}, \mathfrak{z}) + \gamma \mathscr{S}_M(\varkappa, \mathtt{y}) \mathscr{S}_M(\mathtt{y}, \mathfrak{z})$.

Then the pair $(\Upsilon, \mathscr{S}_M)$ is abbreviated as *suprametric space*. The topological definitions of this suprametric space can be found in [13].

Definition 2.2 ([14]) A mapping Θ maps from $[0, \infty)$ to $[0, \infty)$ is said to be a $(\mathcal{C})$-comparison mapping if below mentioned assertions gratifies:

1. Θ is nondecreasing;
2. there exists $g_0 \in \mathbb{N},\ \nu \in (0, 1)$, and $\sum_{g=1}^{\infty} \hbar_g$ in such a way that $\mathfrak{q}^{g+1}\Theta^{g+1}(\mathfrak{s}) \leq \nu\mathfrak{q}^g\Theta^g(\mathfrak{s}) + \nu_g$, for $g \geq g_0$ and any $\mathfrak{s} \geq 0$,

where $\mathfrak{q} \geq 1$ be a real number, and $\sum_{g=1}^{\infty} \hbar_g$ is a convergent positive series.

As the collection of $(\mathcal{C})$-comparison mappings, let's refer to this as Φ. In our subsequent discussion, we will require the ensuing fundamental characteristics.

Lemma 2.3 ([14–16]) *For a* $(\mathcal{C})$*-comparison mapping* ξ *maps from* $[0, +\infty)$ *to* $[0, +\infty)$*, the below mentioned assertions are true:*

1. *For any* $\mathfrak{s} \in [0, +\infty)$ *and* $\mathfrak{q} \geq 1$, $\sum_{g=0}^{\infty} \mathfrak{q}^g\xi^g(\mathfrak{s})$ *converges.*
2. *The mapping* $\eta_\mathfrak{q}$ *maps from* $[0, +\infty)$ *to* $[0, +\infty)$ *defined by* $\eta_\mathfrak{q}(\mathfrak{s}) = \sum_{g=0}^{\infty} \mathfrak{q}^g\xi^g(\mathfrak{s}),\ \mathfrak{s} \in [0, \infty)$*, is nondecreasing.*
3. $\eta_\mathfrak{q}$ *is continuous at 0.*
4. *Every iteration* ξ^g *of* ξ *is also a* $(\mathcal{C})$*-comparison mapping and* ξ *is continuous at 0, where* $g \geq 1$.
5. $\xi(\mathfrak{s}) < \mathfrak{s}$ *for every* $\mathfrak{s} > 0$.

Definition 2.4 ([9]) Let us take the two mappings $\mathcal{Q}$ and α, where $\mathcal{Q} : \Upsilon \to \Upsilon$ and $\alpha : \Upsilon \times \Upsilon \to [0, \infty]$. $\mathcal{Q}$ is said to be a generalized α-orbital admissible if the following assertions gratifies:

$$\alpha(\varkappa, \mathcal{Q}\varkappa) \geq 1 \text{ implies } \alpha(\mathcal{Q}\varkappa, \mathcal{Q}^2\varkappa) \geq 1,$$
$$\alpha(\varkappa, \mathcal{Q}\varkappa) < \infty \text{ implies } \alpha(\mathcal{Q}\varkappa, \mathcal{Q}\varkappa^2) < \infty.$$

Definition 2.5 For any arbitrary constant $\mathfrak{q} \geq 1$, let $(\Upsilon, \mathscr{S}_M)$ be a suprametric space and $\mathcal{Q}$ be a mapping from Υ to Υ. Then $\mathcal{Q}$ is known as an (α, ζ)-Meir-Keeler contractive mapping if there exist two auxiliary mappings $\alpha : \Upsilon \times \Upsilon \to [0, \infty]$ and $\zeta \in \Phi$ in such a way that

$$\epsilon \leq \zeta(\mathscr{S}_M(\varkappa, \mathrm{y})) < \epsilon + \gamma \Rightarrow \alpha(\varkappa, \mathrm{y})\mathscr{S}_M(\mathcal{Q}\varkappa, \mathcal{Q}\mathrm{y}) < \epsilon, \text{ for all } \varkappa, \mathrm{y} \in \Upsilon. \quad (2.1)$$

Remark 2.6 For $\varkappa \neq \mathrm{y}$ and $\mathscr{S}_M(\varkappa, \mathrm{y}) < \infty$ with $\alpha(\varkappa, \mathrm{y}) < \infty$, from 2.1, we derive that

$$\alpha(\varkappa, \mathrm{y})\mathscr{S}_M(\mathcal{Q}\varkappa, \mathcal{Q}\mathrm{y}) < \zeta(\mathscr{S}_M(\varkappa, \mathrm{y})). \quad (2.2)$$

Theorem 2.7 *Let* $(\Upsilon, \mathscr{S}_M)$ *be a complete suprametric space. Assume that a self mapping* $\mathcal{Q} : \Upsilon \to \Upsilon$ *is an* (α, ζ)*-Meir-Keeler contraction. Assume also that*

1. $\mathcal{Q}$ *is generalized* α*-orbital admissible and continuous;*
2. *there exists* $\varkappa \in \Upsilon$ *such that* $1 \leq \alpha(\varkappa, \mathcal{Q}\varkappa) < \infty$*;*

As a consequence, one of the following statements holds for such a $\varkappa$*:*

$\mathcal{A}_1$. *For each* $n \in \mathbb{N}_0$

$$\mathscr{S}_M(\mathcal{Q}^n \varkappa, \mathcal{Q}^{n+1}\varkappa) = \infty \tag{2.3}$$

$$\text{or } \alpha(\mathcal{Q}^n \varkappa, \mathcal{Q}^{n+1}\varkappa) = \infty \tag{2.4}$$

$\mathcal{A}_2$. *For all* $\varkappa, y \in Fix(\mathcal{Q})$*, we have* $1 \leq \alpha(\varkappa, y) < \infty$*, where* $Fix(\mathcal{Q})$ *denotes the set of fixed points of* $\mathcal{Q}$*.*

$\mathcal{A}_3$. *There exists* $g \in \mathbb{N}_0$ *such that* $\mathscr{S}_M(\mathcal{Q}^n \varkappa, \mathcal{Q}^{n+1}\varkappa) < \infty$ *and* $\alpha(\mathcal{Q}^n \varkappa, \mathcal{Q}^{n+1}\varkappa) < \infty$*. In the present instance, there is* $u \in \Upsilon$ *in such a way that* $\mathcal{Q}u = u$*, and also* $\mathcal{Q}$ *has atmost one fixed point in* $\{t \subset \Upsilon / \mathscr{S}_M(\mathcal{Q}^g \varkappa, t) < \infty\}$*.*

Proof On account of assumption (2), there exists $\varkappa \in \Upsilon$ such that $\alpha(\varkappa, \mathcal{Q}\varkappa) \geq 1$. According to our presumption, $\mathcal{A}_1$ is not satisfied. Therefore, we must investigate $\mathcal{A}_2$. As a result, there is $g \in \mathbb{N}_0$ in such a way that $\mathscr{S}_M(\mathcal{Q}^g \varkappa, \mathcal{Q}^{g+1}\varkappa) < \infty$ and $\alpha(\mathcal{Q}^g \varkappa, \mathcal{Q}^{g+1}\varkappa) < \infty$. If $\mathcal{Q}^g \varkappa = \mathcal{Q}^{g+1}\varkappa$, the proof is completed. Assume that $\mathscr{S}_M(\mathcal{Q}^g \varkappa, \mathcal{Q}^{g+1}\varkappa) > 0$. By the property of ζ and above remark, we have

$$\begin{aligned} \alpha(\mathcal{Q}^g \varkappa, \mathcal{Q}^{g+1}\varkappa)\mathscr{S}_M(\mathcal{Q}^{g+1}\varkappa, \mathcal{Q}^{g+2}\varkappa) &< \zeta(\mathscr{S}_M(\mathcal{Q}^g \varkappa, \mathcal{Q}^{g+1}\varkappa)) \\ &< \mathscr{S}_M(\mathcal{Q}^g \varkappa, \mathcal{Q}^{g+1}\varkappa) \\ &< \infty. \end{aligned} \tag{2.5}$$

From assumption (2) and the concept of generalized α-orbital admissible mapping $\mathcal{Q}$, we get

$$1 \leq \alpha(\varkappa, \mathcal{Q}\varkappa) < \infty \Rightarrow 1 \leq \alpha(\mathcal{Q}\varkappa, \mathcal{Q}^2\varkappa) < \infty. \tag{2.6}$$

Recursively, we obtain that

$$1 \leq \alpha(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa) < \infty, \text{ for all } n \in \mathbb{N}_0. \tag{2.7}$$

Applying Eq. 2.7 in Eq. 2.5, we get

$$\mathscr{S}_M(\mathcal{Q}^{g+1}\varkappa, \mathcal{Q}^{g+2}\varkappa) < \zeta(\mathscr{S}_M(\mathcal{Q}^g \varkappa, \mathcal{Q}^{g+1}\varkappa)) < \infty. \tag{2.8}$$

Thus

$$\mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa) < \infty, \text{ for all } n \in \mathbb{N}_0. \tag{2.9}$$

Again, on account of Eq. 2.7,Eq. 2.9 in Eq. 2.5, by induction, one gets

$$\mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa) < \zeta^n(\mathscr{S}_M(\mathcal{Q}^g \varkappa, \mathcal{Q}^{g+1}\varkappa)) \tag{2.10}$$

As a consequence, for $n, \hbar \in \mathbb{N}$, and by using triangle inequality of suprametric we have

$$\begin{aligned}
&\mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+\hbar}\varkappa)\\
&\quad\le \mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa) + \mathscr{S}_M(\mathcal{Q}^{g+n+1}\varkappa, \mathcal{Q}^{g+n+\hbar}\varkappa)\\
&\qquad + \gamma\mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa)\mathscr{S}_M(\mathcal{Q}^{g+n+1}\varkappa, \mathcal{Q}^{g+n+\hbar}\varkappa)\\
&\quad\le \mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa) + [1+\gamma\mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa)]\mathscr{S}_M(\mathcal{Q}^{g+n+1}\varkappa, \mathcal{Q}^{g+n+\hbar}\varkappa)\\
&\quad\le \mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa) + [1+\gamma\mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa)]\\
&\qquad \times[\mathscr{S}_M(\mathcal{Q}^{g+n+1}\varkappa, \mathcal{Q}^{g+n+2}\varkappa) + \mathscr{S}_M(\mathcal{Q}^{g+n+2}\varkappa, \mathcal{Q}^{g+n+\hbar}\varkappa)\\
&\qquad + \gamma\mathscr{S}_M(\mathcal{Q}^{g+n+1}\varkappa, \mathcal{Q}^{g+n+2}\varkappa)\mathscr{S}_M(\mathcal{Q}^{g+n+2}\varkappa, \mathcal{Q}^{g+n+\hbar}\varkappa)]\\
&\quad\le \mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa) + [1+\gamma\mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa)]\mathscr{S}_M(\mathcal{Q}^{g+n+1}\varkappa, \mathcal{Q}^{g+n+2}\varkappa)\\
&\qquad [1+\gamma\mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa)](1+\gamma\mathscr{S}_M(\mathcal{Q}^{g+n+1}\varkappa, \mathcal{Q}^{g+n+2}\varkappa))\mathscr{S}_M(\mathcal{Q}^{g+n+2}\varkappa, \mathcal{Q}^{g+n+\hbar}\varkappa)\\
&\quad\le \mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa) + [1+\gamma\mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa)]\mathscr{S}_M(\mathcal{Q}^{g+n+1}\varkappa, \mathcal{Q}^{g+n+2}\varkappa)\\
&\qquad [1+\gamma\mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa)](1+\gamma\mathscr{S}_M(\mathcal{Q}^{g+n+1}\varkappa, \mathcal{Q}^{g+n+2}\varkappa))\mathscr{S}_M(\mathcal{Q}^{g+n+2}\varkappa, \mathcal{Q}^{g+n+3}\varkappa)\\
&\qquad + \ldots.. + [1+\gamma\mathscr{S}_M(\mathcal{Q}^{g+n}\varkappa, \mathcal{Q}^{g+n+1}\varkappa)][1+\gamma\mathscr{S}_M(\mathcal{Q}^{g+n+1}\varkappa, \mathcal{Q}^{g+n+2}\varkappa)]\\
&\qquad \ldots\ldots[1+\gamma\mathscr{S}_M(\mathcal{Q}^{g+n+\hbar-2}\varkappa, \mathcal{Q}^{g+n+\hbar-1}\varkappa)]\mathscr{S}_M(\mathcal{Q}^{g+n+(\hbar-1)}\varkappa, \mathcal{Q}^{g+n+\hbar}\varkappa)\\
&\quad\le \zeta^n\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa) + [1+\gamma\zeta^n\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa)]\zeta^{n+1}\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa)\\
&\qquad + [1+\gamma\zeta^n\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa)][1+\gamma\zeta^{n+1}\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa)]\zeta^{n+2}\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa)\\
&\qquad + \ldots.. + [1+\gamma\zeta^n\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa)][1+\gamma\zeta^{n+1}\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa)]\ldots..\\
&\qquad [1+\gamma\zeta^{n+\hbar-2}\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa)]\zeta^{n+(\hbar-1)}\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa)\\
&\quad= \zeta^n\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa) + \sum_{n=0}^{\hbar-2}\zeta^{n+(m+1)}\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa)\prod_{j=0}^{n}[1+\gamma\zeta^{n+j}\mathscr{S}_M(\mathcal{Q}^{g}\varkappa, \mathcal{Q}^{g+1}\varkappa)]
\end{aligned}$$

for all $n, \hbar \in \mathbb{N}_0$.

Consequently, from above equation, we have $\{\mathcal{Q}^n\varkappa\}$ isa Cauchy sequence since $\zeta \in \Phi$.

By using the concept of completeness, there exists an element $\omega \in \Upsilon$ such that $\{\mathcal{Q}^n\varkappa\}$ converges to ω. Since $\mathcal{Q}$ is continuous, we get,

$$\omega = \lim_{n\to\infty}\mathcal{Q}^{n+1}\varkappa = \mathcal{Q}\left(\lim_{n\to\infty}\mathcal{Q}^n\varkappa\right) = \mathcal{Q}\omega.$$

Thus ω is a fixed point of $\mathcal{Q}$.

In order to prove the uniqueness of the fixed point, in contrary let us suppose that there is two fixed points $\omega^*$and $\omega^\star$ where $\omega^* \neq \omega^\star$ such that $\{\mathcal{Q}^n\varkappa\}$ converges to ω^*, as well as $\{\mathcal{Q}^n\varkappa\}$ converges to $\omega^\star$.

Consider,

$$\mathscr{S}_M(\omega^*, \omega^\star) \le \mathscr{S}_M(\omega^*, \mathcal{Q}^n\varkappa) + \mathscr{S}_M(\mathcal{Q}^n\varkappa, \omega^\star) + \gamma\mathscr{S}_M(\omega^*, \mathcal{Q}^n\varkappa)\mathscr{S}_M(\mathcal{Q}^n\varkappa, \omega^\star) < \infty.$$

By condition $(\mathcal{A}_3)$, $1 \le \alpha(\omega^*, \omega^\star) < \infty$ and since $0 < \mathscr{S}_M(\omega^*, \omega^\star) < \infty$, we have

$$\begin{aligned}
\mathscr{S}_M(\omega^*, \omega^\star) &= \mathscr{S}_M(\mathcal{Q}\omega^*, \mathcal{Q}\omega^\star)\\
&\le \alpha(\omega^*, \omega^\star)\mathscr{S}_M(\mathcal{Q}\omega^*, \mathcal{Q}\omega^\star)\\
&< \zeta(\mathscr{S}_M(\omega^*, \omega^\star))\\
&< \mathscr{S}_M(\omega^*, \omega^\star), \text{ a contradiction.}
\end{aligned}$$

Hence $\omega^* = \omega^\star$. This completes the Proof. □

Definition 2.8 A suprametric space is regular if $\{\varkappa_n\}$ is a sequence in Υ such that $1 \leq \alpha(\varkappa_n, \varkappa_{n+1}) < \infty$ for all n and $\varkappa_n \to \varkappa$ as $n \to \infty$; then there exists a subsequence $\{\varkappa_{n(g)}\}$ of $\varkappa_n$ such that $1 \leq \alpha(\varkappa_{n(g)}, \varkappa) < \infty$ and $0 < \mathscr{S}_M(\varkappa_{n(g)}, \varkappa) < \infty$ for all g.

Theorem 2.9 *Let $(\Upsilon, \mathscr{S}_M)$ be a complete suprametric space. Assume that a self mapping $\mathcal{Q} : \Upsilon \to \Upsilon$ is an (α, ζ)-Meir-Keeler contraction. Assume also that*

1. *$\mathcal{Q}$ is generalized α-orbital admissible;*
2. *there exists $\varkappa \in \Upsilon$ such that $1 \leq \alpha(\varkappa, \mathcal{Q}\varkappa) < \infty$;*
3. *$\mathcal{Q}$ is regular.*

Then for such $\varkappa$, one of the following statement holds:

$\mathcal{B}_1$. *For every $n \in \mathbb{N}_0$*

$$\mathscr{S}_M(\mathcal{Q}^n\varkappa, \mathcal{Q}^{n+1}\varkappa) = \infty \tag{2.11}$$

$$\text{or } \alpha(\mathcal{Q}^n\varkappa, \mathcal{Q}^{n+1}\varkappa) = \infty \tag{2.12}$$

$\mathcal{B}_2$. *For all $\varkappa, \mathrm{y} \in Fix(\mathcal{Q})$, we have $1 \leq \alpha(\varkappa, \mathrm{y}) < \infty$, where $Fix(\mathcal{Q})$ denotes the set of fixed points of $\mathcal{Q}$.*

$\mathcal{B}_3$. *There exists $g \in \mathbb{N}_0$ such that $\mathscr{S}_M(\mathcal{Q}^n\varkappa, \mathcal{Q}^{n+1}\varkappa) < \infty$ and $\alpha(\mathcal{Q}^n\varkappa, \mathcal{Q}^{n+1}\varkappa) < \infty$. In this case, there exists $u \in \Upsilon$ such that $\mathcal{Q}u = u$.*

Proof One can easily prove this. So proof is excluded.

Now we introduce an (α, ζ)-Meir-Keeler contractive mapping in the setting of suprametric space via a triangular α-admissible mapping. We prove the existence and uniqueness of a fixed point of such mapping. □

Definition 2.10 ([10]) Let $\mathcal{Q} : \Upsilon \to \Upsilon$ and $\alpha : \Upsilon \times \Upsilon \to (-\infty, +\infty)$.If $\mathcal{Q}$ is a triangular α-admissible mapping, then we claim that,

$(\Im_1)$. $\alpha(\varkappa, \mathrm{y}) \geq 1 \Rightarrow \alpha(\mathcal{Q}\varkappa, \mathcal{Q}\mathrm{y}) \geq 1,\ \varkappa, \mathrm{y} \in \Upsilon$,
$(\Im_2)$. $\alpha(\varkappa, \mathfrak{z}) \geq 1, \alpha(\mathfrak{z}, \mathrm{y}) \geq 1 \Rightarrow \alpha(\varkappa, \mathrm{y}) \geq 1,\ \varkappa, \mathrm{y}, \mathfrak{z} \in \Upsilon$.

Lemma 2.11 ([10]) *Let $\mathcal{Q}$ be a triangular α-admissible mapping. Assume that there exists $\varkappa_0 \in \Upsilon$ such that $\alpha(\varkappa_0, \mathcal{Q}\varkappa_0) \geq 1$. Define a sequence $\{\varkappa_n\}$ by $\varkappa_n = \mathcal{Q}^n\varkappa_0$. Then*

$$\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1, \text{ for all } m, n \in \mathbb{N} \text{ with } m < n. \tag{2.13}$$

Let us denote Ψ be the family of increasing functions ξ from $[0, +\infty)$ to $[0, +\infty)$ continuous in $\mathfrak{s} = 0$ in such a way that

$$\begin{cases} \xi(\mathfrak{s}) = 0 \; iff \; \mathfrak{s} = 0 \\ \xi(\mathfrak{s} + \mathfrak{q}) \leq \xi(\mathfrak{s}) + \xi(\mathfrak{q}) \\ \sum_{n=1}^{\infty} \xi^n(\mathfrak{s}) < \infty \text{ for all } \mathfrak{s} > 0. \end{cases} \tag{2.14}$$

Remark 2.12 Assume that $\mathcal{Q}$ be an $\alpha - \xi$-Meir-Keeler contractive mapping. Then

$$\alpha(\varkappa, y)\xi(\mathscr{S}_M(\mathcal{Q}\varkappa, \mathcal{Q}y)) < \xi(\mathscr{S}_M(\varkappa, y))$$

for all $\varkappa, y \in \Upsilon$ when $\varkappa \neq y$. Also, if $\varkappa = y$ then $\mathscr{S}_M(\mathcal{Q}\varkappa, \mathcal{Q}y) = 0$. Clearly, which means that

$$\alpha(\varkappa, y)\xi(\mathscr{S}_M(\mathcal{Q}\varkappa, \mathcal{Q}y)) < \xi(\mathscr{S}_M(\varkappa, y)), \text{ for all } \varkappa, y \in \Upsilon.$$

Definition 2.13 Let $(\Upsilon, \mathscr{S}_M)$ be a suprametric space and $\mathcal{Q} : \Upsilon \to \Upsilon$ be a given mapping. We say that $\mathcal{Q}$ is an (α, ζ)-triangular-Meir-Keeler type contraction if there exists two functions $\alpha : \Upsilon \times \Upsilon \to [0, \infty)$ and $\zeta \in \Psi$ such that for each $\varkappa, y \in \Upsilon$,

$$\epsilon \leq \zeta(\sigma(\varkappa, y)) < \epsilon + \delta \Rightarrow \alpha(\varkappa, y)\mathscr{S}_M(\mathcal{Q}\varkappa, \mathcal{Q}y) < \epsilon, \tag{2.15}$$

where

$$\sigma(\varkappa, y) = \max\left\{\mathscr{S}_M(\varkappa, y) + \mathscr{S}_M(\varkappa, \mathcal{Q}\varkappa) + \mathscr{S}_M(y, \mathcal{Q}y) + \frac{\mathscr{S}_M(\varkappa, \mathcal{Q}y) + \mathscr{S}_M(y, \mathcal{Q}\varkappa)}{2}\right\}.$$

Remark 2.14 Let $\mathcal{Q}$ be a (α, ζ)-triangular-Meir-Keeler type contraction. For $\varkappa \neq y$, from 2.15, we derive that

$$\alpha(\varkappa, y)\mathscr{S}_M(\mathcal{Q}\varkappa, \mathcal{Q}y) < \zeta(\sigma(\varkappa, y)). \tag{2.16}$$

Also, if $\varkappa = y$, then

$$\alpha(\varkappa, y)\mathscr{S}_M(\mathcal{Q}\varkappa, \mathcal{Q}y) \leq \zeta(\sigma(\varkappa, y)), \text{ for all } \varkappa, y \in \Upsilon.$$

Theorem 2.15 *Let $(\Upsilon, \mathscr{S}_M)$ be a complete suprametric space and $\mathcal{Q}$ be a (α, ζ)-triangular-Meir-Keeler type contraction and satisfies the following conditions:*

1. *$\mathcal{Q}$ is a triangular α-admissible mapping;*
2. *There exists $\varkappa_0 \in \Upsilon$ such that $\alpha(\varkappa_0, \mathcal{Q}\varkappa_0) \geq 1$;*
3. *$\mathcal{Q}$ is continuous;*
4. *For all $\varkappa, y \in Fix(\mathcal{Q})$, i.e., set of fixed points of $\mathcal{Q}$, there exists $\mathfrak{z} \in \Upsilon$ such that $\alpha(\varkappa, y) \geq 1$ and $\alpha(y, \mathfrak{z}) \geq 1$.*

Then $\mathcal{Q}$ has a fixed point. Moreover, if condition (4) satisfies, then $\mathcal{Q}$ has a unique fixed point.

Proof Let $\varkappa_0$ be an arbitrary point of Υ such that $\alpha(\varkappa_0, \mathcal{Q}\varkappa_0) \geq 1$. We construct an iterative sequence $\{\varkappa_n\}$ in Υ in the following way,

$$\varkappa_{n+1} = \mathcal{Q}\varkappa_n \text{ for all } n \geq 0.$$

If $\varkappa_{n_0} = \varkappa_{n_0+1}$ for some n_0, then, obviously, $\varkappa^\star = \varkappa_{n_0}$ is a fixed point of $\mathcal{Q}$ and the proof is completed.

Hence from now on, we suppose that $\varkappa_n \neq \varkappa_{n+1}$ for all n. By using $(\Im_2)$, we get

$$\alpha(\varkappa_0, \varkappa_1) = \alpha(\varkappa_0, \mathcal{Q}\varkappa_0) \geq 1 \Rightarrow \alpha(\mathcal{Q}\varkappa_0, \mathcal{Q}\varkappa_1) = \alpha(\varkappa_1, \varkappa_2) \geq 1.$$

Inductively, we obtain that

$$1 \leq \alpha(\varkappa_n, \varkappa_{n+1}), \text{ for all } n \geq 0. \tag{2.17}$$

By the property of ζ and hypothesis, we have

$$\begin{aligned}
&\mathscr{S}_M(\varkappa_{n+1}, \varkappa_{n+2})\\
&< \mathscr{S}_M(\varkappa_{n+1}, \varkappa_{n+2})\\
&\leq \alpha(\varkappa_n, \varkappa_{n+1})\mathscr{S}_M(\varkappa_{n+1}, \varkappa_{n+2})\\
&\leq \zeta(\sigma(\varkappa_n, \varkappa_{n+1}))\\
&= \zeta\left(\max\left\{\mathscr{S}_M(\varkappa_n, \varkappa_{n+1}), \mathscr{S}_M(\varkappa_n, \mathcal{Q}\varkappa_n), \mathscr{S}_M(\varkappa_{n+1}, \mathcal{Q}\varkappa_{n+1}), \frac{\mathscr{S}_M(\varkappa_n, \mathcal{Q}\varkappa_{n+1}) + \mathscr{S}_M(\varkappa_{n+1}, \mathcal{Q}\varkappa_n)}{2}\right\}\right)\\
&< \zeta\left(\max\left\{\mathscr{S}_M(\varkappa_n, \varkappa_{n+1}), \mathscr{S}_M(\varkappa_{n+1}, \varkappa_{n+2})\right\}\right)
\end{aligned}$$

Now since ζ is strictly nondecreasing, we get

$$\mathscr{S}_M(\varkappa_{n+1}, \varkappa_{n+2}) < \max\left\{\mathscr{S}_M(\varkappa_n, \varkappa_{n+1}), \mathscr{S}_M(\varkappa_{n+1}, \varkappa_{n+2})\right\} \tag{2.18}$$

If $\max\left\{\mathscr{S}_M(\varkappa_n, \varkappa_{n+1}), \mathscr{S}_M(\varkappa_{n+1}, \varkappa_{n+2})\right\} = \mathscr{S}_M(\varkappa_{n+1}, \varkappa_{n+2})$, then by 2.18, it contradicts.

Hence $\max\left\{\mathscr{S}_M(\varkappa_n, \varkappa_{n+1}), \mathscr{S}_M(\varkappa_{n+1}, \varkappa_{n+2})\right\} = \mathscr{S}_M(\varkappa_n, \varkappa_{n+1})$.

From Eq. 2.18, $\mathscr{S}_M(\varkappa_{n+1}, \varkappa_{n+2}) < \mathscr{S}_M(\varkappa_n, \varkappa_{n+1})$ for all n. Thus the sequence $\{\mathscr{S}_M(\varkappa_{n+1}, \varkappa_n)\}_{n=0}^{\infty}$ is decreasing and converges to ϵ, in other words,

$$\lim_{n\to\infty} \zeta(\mathscr{S}_M(\varkappa_{n+1}, \varkappa_n)) = \lim_{n\to\infty} \zeta(\sigma(\varkappa_{n+1}, \varkappa_n)) = \zeta(\epsilon). \tag{2.19}$$

Notice that $\epsilon = \inf\{\mathscr{S}_M(\varkappa_n, \varkappa_{n+1})/n \in \mathbb{N}\}$. Now we are going to prove $\epsilon = 0$. Contrarily, assume that $\epsilon > 0$.

Which yields $\zeta(\epsilon) > 0$. By making the use of $\mathcal{Q}$ is a (α, ζ)-triangular-Meir-Keeler type contraction and 2.19, there is a $\delta > 0$ and $m \in \mathbb{N}$ in such a way that

$$\zeta(\epsilon) \leq \zeta(\sigma(\varkappa_m, \varkappa_{m+1})) < \zeta(\epsilon) + \delta$$

which implies

$$\alpha(\varkappa_m, \varkappa_{m+1})\zeta(\mathscr{S}_M(\varkappa_{m+1}, \varkappa_{m+2})) < \zeta(\epsilon). \tag{2.20}$$

From Eq. 2.17,

$$\zeta(\varkappa_{m+1}, \varkappa_{m+2}) \leq \alpha(\varkappa_m, \varkappa_{m+1})\zeta(\mathscr{S}_M(\varkappa_{m+1}, \varkappa_{m+2})). \tag{2.21}$$

From Eqs. 2.20, 2.21 and using the fact that ζ is strictly nonincreasing, we get $\mathscr{S}_M(\varkappa_{m+1}, \varkappa_{m+2}) < \epsilon$, which is a contradiction since $\epsilon = \inf\{\mathscr{S}_M(\varkappa_n, \varkappa_{n+1})/n \in \mathbb{N}\}$. Thus $\epsilon = 0$.

So,

$$\lim_{n\to\infty} \mathscr{S}_M(\varkappa_{n+1}, \varkappa_n) = 0. \tag{2.22}$$

Now we will prove that $\{\varkappa_n\}$ is a Cauchy sequence. Suppose, on the contrary, that there exists an $\epsilon > 0$ and subsequences $m(a)$ and $n(a)$ such that $m(a) < n(a) < m(a+1)$, with

$$\mathscr{S}_M(\varkappa_{m(a)}, \varkappa_{n(a)}) \geq \epsilon,\ \mathscr{S}_M(\varkappa_{m(a)}, \varkappa_{n(a)-1}) < \frac{\epsilon}{2}. \tag{2.23}$$

From Eq. 2.22, there exist n_0 such that

$$\mathscr{S}_M(\varkappa_n, \varkappa_{n+1}) < \frac{\epsilon}{2+\theta\epsilon},\ \text{for all } n \geq n_0. \tag{2.24}$$

By using triangular α-admissible concept,

$$\alpha(\varkappa_{m(a)-1}, \varkappa_{n(a)-1}) \geq 1. \tag{2.25}$$

From Eqs. 2.16 and 2.25,

$$\begin{aligned}\mathscr{S}_M(\mathcal{Q}\varkappa_{m(a)-1}, \mathcal{Q}\varkappa_{n(a)-1}) &\leq \alpha(\varkappa_{m(a)-1}, \varkappa_{n(a)-1})\mathscr{S}_M(\mathcal{Q}\varkappa_{m(a)-1}, \mathcal{Q}\varkappa_{n(a)-1})\\ &< \zeta(\sigma(\varkappa_{m(a)-1}, \varkappa_{n(a)-1})),\end{aligned} \tag{2.26}$$

where

$$\sigma(\varkappa_{m(a)-1}, \varkappa_{n(a)-1}) = \max\Big\{\mathscr{S}_M(\varkappa_{m(a)-1}, \varkappa_{n(a)-1}), \mathscr{S}_M(\varkappa_{m(a)-1}, \varkappa_{m(a)}), \mathscr{S}_M(\varkappa_{n(a)-1}, \varkappa_{n(a)}), \frac{\mathscr{S}_M(\varkappa_{m(a)-1}, \varkappa_{n(a)}) + \mathscr{S}_M(\varkappa_{n(a)-1}, \varkappa_{m(a)})}{2}\Big\}$$

By using Eqs. 2.22 and 2.23, 2.25 reduces to

$$\begin{aligned}\mathscr{S}_M(\varkappa_{m(a)}, \varkappa_{n(a)}) &< \zeta\left(\frac{\mathscr{S}_M(\varkappa_{m(a)-1}, \varkappa_{n(a)-1})}{2}\right)\\ &< \mathscr{S}_M(\varkappa_{m(a)-1}, \varkappa_{n(a)-1})\end{aligned} \tag{2.27}$$

By using the triangle inequality of suprametric space, Eq. 2.27 reduces to

$$\mathscr{S}_M(\varkappa_{m(a)}, \varkappa_{n(a)}) < \mathscr{S}_M(\varkappa_{m(a)-1}, \varkappa_{m(a)}) + \mathscr{S}_M(\varkappa_{m(a)}, \varkappa_{n(a)-1}) + \gamma\mathscr{S}_M(\varkappa_{m(a)-1}, \varkappa_{m(a)})\mathscr{S}_M(\varkappa_{m(a)}, \varkappa_{n(a)-1}).$$

From Eqs. 2.23 and 2.24,

$$\lim \mathscr{S}_M(\varkappa_{m(a)}, \varkappa_{n(a)}) < \epsilon,\ \text{a contradiction.}$$

This implies that $\{\varkappa_n\}$ is a Cauchy sequence in $(\Upsilon, \mathscr{S}_M)$. Due to the completeness of $(\Upsilon, \mathscr{S}_M)$, there exists $\mathfrak{z} \in \Upsilon$ such that $\varkappa_n \to \mathfrak{z}$ as $n \to \infty$.

By using condition (3) of theorem, $\mathcal{Q}$ yields that $\mathcal{Q}\varkappa_n \to \mathcal{Q}\mathfrak{z}$ as $n \to \infty$, i.e.,$\varkappa_{n+1} \to \mathcal{Q}\mathfrak{z}$ as $n \to \infty$.

By the uniqueness of limit, $\mathcal{Q}\mathfrak{z} = \mathfrak{z}$. Therefore $\mathfrak{z}$ is a fixed point of $\mathcal{Q}$. Uniqueness can be proved easily. Hence it is omitted. □

Remark 2.16 Let $(\Upsilon, \mathscr{S}_M)$ be a complete suprametric space and $\mathcal{Q}$ be a (α, ζ)-triangular-Meir-Keeler type contraction and satisfies the following conditions:

1. There exists $\varkappa_0 \in \Upsilon$ such that $\alpha(\varkappa_0, \mathcal{Q}\varkappa_0) \geq 1$;
2. If $\{\varkappa_n\}$ is a sequence in Υ such that $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$ for all n and $\varkappa_n \to \varkappa$ as $n \to +\infty$, then $\alpha(\varkappa_n, \varkappa) \geq 1$ for all n.
3. For all $\varkappa, \mathtt{y} \in Fix(\mathcal{Q})$, i.e., set of fixed points of $\mathcal{Q}$, there exists $\mathfrak{z} \in \Upsilon$ such that $\alpha(\varkappa, \mathtt{y}) \geq 1$ and $\alpha(\mathtt{y}, \mathfrak{z}) \geq 1$.

Then $\mathcal{Q}$ has a fixed point. Moreover, if condition (3) satisfies, then $\mathcal{Q}$ has a unique fixed point.

3 A Chaotic Attractor System with Atangana-Baleanu Derivative

In recent years, chaos theory has risen its importance, and numerous investigations on chaotic systems via fractional derivatives and fractal-fractional derivatives have been observed (see, [18, 19]). The purpose of this section is to identify novel chaotic behavior that is produced by the fractional operator rather than to provide a comprehensive investigation of the system with the new derivative and fractional order. The model that is the subject of this inquiry is described in [20, 21]:

$$\begin{cases} {}_0^{\mathscr{A}}\mathscr{BCS}_{M}{}_{\ell}^{\gamma}\{\mathbb{a}(\ell)\} &= \eta\mathbb{b}(\ell) - \gamma\mathbb{a}(\ell); \\ {}_0^{\mathscr{A}}\mathscr{BCS}_{M}{}_{\ell}^{\gamma}\{\mathbb{b}(\ell)\} &= \mu\mathbb{a}(\ell) - \mathbb{a}(\ell)\mathbb{c}(\ell) - \mathbb{b}(\ell); \\ {}_0^{\mathscr{A}}\mathscr{BCS}_{M}{}_{\ell}^{\gamma}\{\mathbb{c}(\ell)\} &= \mathbb{a}(\ell)\mathbb{b}(\ell) - \lambda\mathbb{c}(\ell). \end{cases} \quad (3.1)$$

With $\mathbb{a}$, $\mathbb{b}$ and $\mathbb{c}$, the system state is made up. The constants are the same as in [22–24]. The system described above is comparable to the following.

$$\begin{cases} \mathbb{a}(\ell) - \mathcal{S}_1(\ell) &= \frac{1-\gamma}{\mathscr{B}(\gamma)}\{\eta\mathbb{b}(\ell) - \gamma\mathbb{a}(\ell)\} \\ &\quad + \frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)}\int_0^\ell (\ell - \hbar)^{\gamma-1}\{\eta\mathbb{b}(\hbar) - \gamma\mathbb{a}(\hbar)\}d\hbar; \\ \mathbb{b}(\ell) - \mathcal{S}_2(\ell) &= \frac{1-\gamma}{\mathscr{B}(\gamma)}\{\mu\mathbb{a}(\ell) - \mathbb{a}(\ell)\mathbb{c}(\ell) - \mathbb{b}(\ell)\} \\ &\quad + \frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)}\int_0^\ell (\ell - \hbar)^{\gamma-1}\{\mu\mathbb{a}(\hbar) - \mathbb{a}(\hbar)\mathbb{c}(\hbar) - \mathbb{c}(\hbar)\}d\hbar; \\ \mathbb{c}(\ell) - \mathcal{S}_3(\ell) &= \frac{1-\gamma}{\mathscr{B}(\gamma)}\{\mathbb{a}(\ell)\mathbb{b}(\ell) - \lambda\mathbb{c}(\ell)\} \\ &\quad + \frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)}\int_0^\ell (\ell - \hbar)^{\gamma-1}\{\mathbb{a}(\hbar)\mathbb{b}(\hbar) - \lambda\mathbb{c}(\hbar)\}d\hbar. \end{cases} \tag{3.2}$$

The following is an iterative representation of the aforementioned:

$$\begin{cases} \mathbb{a}_0(\ell) &= \mathcal{S}_1(\ell); \\ \mathbb{b}_0(\ell) &= \mathcal{S}_2(\ell); \\ \mathbb{c}_0(\ell) &= \mathcal{S}_3(\ell). \end{cases} \tag{3.3}$$

$$\begin{cases} \mathbb{a}_{n+1}(\ell) &= \frac{1-\gamma}{\mathscr{B}(\gamma)}\{\eta\mathbb{b}_n(\ell) - \gamma\mathbb{a}_n(\ell)\} \\ &\quad + \frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)}\int_0^\ell (\ell - \hbar)^{\gamma-1}\{\eta\mathbb{b}_n(\hbar) - \gamma\mathbb{a}_n(\hbar)\}d\hbar; \\ \mathbb{b}_{n+1}(\ell) &= \frac{1-\gamma}{\mathscr{B}(\gamma)}\{\mu\mathbb{a}_n(\ell) - \mathbb{a}_n(\ell)\mathbb{c}_n(\ell) - \mathbb{b}_n(\ell)\} \\ &\quad + \frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)}\int_0^\ell (\ell - \hbar)^{\gamma-1}\{\mu\mathbb{a}_n(\hbar) - \mathbb{a}_n(\hbar)\mathbb{c}(\hbar) - \mathbb{c}_n(\hbar)\}d\hbar; \\ \mathbb{c}_{n+1}(\ell) &= \frac{1-\gamma}{\mathscr{B}(\gamma)}\{\mathbb{a}_n(\ell)\mathbb{b}_n(\ell) - \lambda\mathbb{c}_n(\ell)\} \\ &\quad + \frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)}\int_0^\ell (\ell - \hbar)^{\gamma-1}\{\mathbb{a}_n(\hbar)\mathbb{b}_n(\hbar) - \lambda\mathbb{c}_n(\hbar)\}d\hbar. \end{cases} \tag{3.4}$$

The desired exact solutions associated with the preceding system are assumed to be attained at the limit when the number of iterations approaches infinity.

3.1 Existence and Uniqueness of the Nonlinear Fractional Order System

We consider the following operator to demonstrate the complete proof of existence:

$$\begin{cases} \mathscr{P}(\ell, \mathbb{a}) &= \eta\mathbb{b}(\ell) - \gamma\mathbb{a}(\ell); \\ \mathscr{Q}(\ell, \mathbb{b}) &= \mu\mathbb{a}(\ell) - \mathbb{a}(\ell)\mathbb{c}(\ell) - \mathbb{b}(\ell); \\ \mathscr{R}(\ell, \mathbb{c}) &= \mathbb{a}(\ell)\mathbb{b}(\ell) - \lambda\mathbb{c}(\ell). \end{cases} \tag{3.5}$$

It is clear that $\mathscr{P}, \mathscr{Q}, \mathscr{R}$ are contraction in respect to $\mathbb{a}$ for the first function, $\mathbb{b}$ for the second function and $\mathbb{c}$ for the last function.

Let $\mathscr{H}_1 = \sup ||\mathscr{P}_{\mathcal{C}_{i,j_1}}(\ell, \mathbb{a})||$, $\mathscr{H}_2 = \sup ||\mathscr{Q}_{\mathcal{C}_{i,j_2}}(\ell, \mathbb{b})||$, $\mathscr{H}_3 = \sup ||\mathscr{R}_{\mathcal{C}_{i,j_3}}(\ell, \mathbb{c})||$, $\mathscr{H} = \max\{\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3\}$ and $\frac{1-\gamma(1-i^\gamma)}{\mathscr{B}(\gamma)} < \frac{\sqrt{j}}{\mathscr{H}}$,
where

$$\mathcal{C}_{i,j_1} = [\ell - i, \ell + i] \times [\mathbb{a} - j_1, \mathbb{a} + j_1] = \mathscr{K}_1 \times \mathscr{L}_1,$$

$$\mathcal{C}_{i,j_2} = [\ell - i, \ell + i] \times [\mathbb{a} - j_2, \mathbb{a} + j_2] = \mathscr{K}_2 \times \mathscr{L}_2,$$

$$\mathcal{C}_{i,j_3} = [\ell - i, \ell + i] \times [\mathbb{a} - j_3, \mathbb{a} + j_3] = \mathscr{K}_3 \times \mathscr{L}_3.$$

Let $\mathbb{X} = \mathcal{C}(\mathscr{K}_1, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ and $\mathscr{S}_M : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ defined by $\mathscr{S}_M(\mathcal{X}(\ell), \mathcal{Y}(\ell)) = |\mathcal{X} - \mathcal{Y}|^2$ is a suprametric induced by norm $||\mathscr{Q}(\ell)||_\infty = \max_{\ell \in [\ell-i, \ell+i]} |\mathscr{Q}(\ell)|^2$ and $\alpha : \mathbb{X} \times \mathbb{X} \to [0, \infty]$ defined by $\alpha(\mathcal{X}(\ell), \mathcal{Y}(\ell)) = 1$. Clearly α is a generalized α-orbital admissible and $\zeta(\ell) = \frac{\ell}{2}$.

Picard's operator, which is defined between the two functional spaces of continuous functions, is as mentioned below:

$$\mathscr{W}\mathcal{X}(\ell) = \mathcal{X}_0(\ell) + \mathcal{X}(\ell)\frac{1-\gamma}{\mathscr{B}(\gamma)} + \frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)} \int_0^\ell (\ell - \hbar)^{\gamma-1} \mathscr{F}(\hbar, \mathcal{X}(\hbar))d\hbar, \tag{3.6}$$

where $\mathscr{W} : \mathcal{C}(\mathscr{K}_1, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3) \to \mathcal{C}(\mathscr{K}_1, \mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3)$ and $\mathcal{X}$ be the matrix as below

$$\mathcal{X}(\ell) = \begin{cases} \mathbb{a}(\ell) \\ \mathbb{b}(\ell) \\ \mathbb{c}(\ell) \end{cases}, \quad \mathcal{X}_0(\ell) = \begin{cases} \mathcal{S}_1(\ell) \\ \mathcal{S}_2(\ell) \\ \mathcal{S}_3(\ell) \end{cases}, \quad \mathscr{F}(\ell, \mathcal{X}(\ell)) = \begin{cases} \mathscr{P}(\ell, \mathbb{a}) \\ \mathscr{Q}(\ell, \mathbb{a}) \\ \mathscr{R}(\ell, \mathbb{a}) \end{cases}$$

To get an adequate outcome, we presume that the physical problem under examination meets the following:

$$||\mathbb{a}(\ell)||_\infty \leq \max\{j_1, j_2, j_3\}.$$

Now consider,

$$\begin{aligned}
|\mathscr{W}\mathcal{X}(\ell) - \mathcal{X}_0(\ell)|^2 &= \left|\mathscr{F}(\ell, \mathcal{X}(\ell))\frac{1-\gamma}{\mathscr{B}(\gamma)} + \frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)} \int_0^\ell (\ell - \hbar)^{\gamma-1} \mathscr{F}(\hbar, \mathcal{X}(\hbar))d\hbar\right|^2 \\
&\leq \left(\frac{1-\gamma}{\mathscr{B}(\gamma)}\right)^2 |\mathscr{F}(\ell, \mathcal{X}(\ell))|^2 + \left(\frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)} \int_0^\ell (\ell - \hbar)^{\gamma-1} d\hbar\right)^2 |\mathscr{F}(\hbar, \mathcal{X}(\hbar))|^2 \\
&\quad + \frac{2(1-\gamma)}{\mathscr{B}(\gamma)} |\mathscr{F}(\ell, \mathcal{X}(\ell))| \frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)} \int_0^\ell (\ell - \hbar)^{\gamma-1} d\hbar |\mathscr{F}(\hbar, \mathcal{X}(\hbar))| \\
&\leq \left(\frac{1-\gamma}{\mathscr{B}(\gamma)}\right)^2 \mathscr{H}^2 + \frac{\gamma^2}{(\mathscr{B}(\gamma))^2} \mathscr{H}^2 i^{2\gamma} + \frac{2(1-\gamma)\gamma}{(\mathscr{B}(\gamma))^2} \mathscr{H}^2 i^\gamma \\
&\leq \mathscr{H}^2 \left(\frac{1-\gamma(1-i^\gamma)}{\mathscr{B}(\gamma)}\right)^2 \\
&\leq j
\end{aligned}$$

Again consider,

$$
\begin{aligned}
|\mathscr{W}\mathcal{X}_1 - \mathscr{W}\mathcal{X}_2|^2 &= \Bigg|\{\mathscr{F}(\ell, \mathcal{X}_1(\ell)) - \mathscr{F}(\ell, \mathcal{X}_2(\ell))\}\frac{1-\gamma}{\mathscr{B}(\gamma)} \\
&\quad + \frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)}\int_0^\ell (\ell-\hbar)^{\gamma-1}\{\mathscr{F}(\hbar, \mathcal{X}_1(\hbar)) - \mathscr{F}(\hbar, \mathcal{X}_2(\hbar))\}d\hbar\Bigg|^2 \\
&\le \left(\frac{1-\gamma}{\mathscr{B}(\gamma)}\right)^2 |\mathscr{F}(\ell, \mathcal{X}_1(\ell)) - \mathscr{F}(\ell, \mathcal{X}_2(\ell))|^2 \\
&\quad + \left(\frac{\gamma}{\mathscr{B}(\gamma)\Gamma(\gamma)}\right)^2 |\{\mathscr{F}(\hbar, \mathcal{X}_1(\hbar)) - \mathscr{F}(\hbar, \mathcal{X}_2(\hbar))\}|^2 \left|\int_0^\ell (\ell-\hbar)^{\gamma-1}d\hbar\right|^2 \\
&\quad + \frac{2(1-\gamma)\gamma}{\Gamma(\gamma)(\mathscr{B}(\gamma))^2}|\mathscr{F}(\ell, \mathcal{X}_1(\ell)) - \mathscr{F}(\ell, \mathcal{X}_2(\ell))| \\
&\quad \times |\mathscr{F}(\hbar, \mathcal{X}_1(\hbar)) - \mathscr{F}(\hbar, \mathcal{X}_2(\hbar))|\left|\int_0^\ell (\ell-\hbar)^{\gamma-1}d\hbar\right| \\
&\le \left(\frac{1-\gamma}{\mathscr{B}(\gamma)}\right)^2 \kappa|\mathcal{X}_1(\ell) - \mathcal{X}_2(\ell)|^2 + \left(\frac{\gamma}{\mathscr{B}(\gamma)}\right)^2 \kappa i^{2\gamma}|\mathcal{X}_1(\ell) - \mathcal{X}_2(\ell)|^2 \\
&\quad + \frac{2(1-\gamma)\gamma}{(\mathscr{B}(\gamma))^2}\kappa^2 i^\gamma |\mathcal{X}_1(\ell) - \mathcal{X}_2(\ell)|^2 \\
&\le \kappa^2\left(\frac{1-\gamma(1-i^\gamma)}{\mathscr{B}(\gamma)}\right)^2 |\mathcal{X}_1(\ell) - \mathcal{X}_2(\ell)|^2 \\
&\le \frac{\kappa^2 j}{\mathscr{H}}|\mathcal{X}_1(\ell) - \mathcal{X}_2(\ell)|^2 \\
&< \frac{1}{2}|\mathcal{X}_1(\ell) - \mathcal{X}_2(\ell)|^2 \text{ with } \frac{\kappa^2 j}{\mathscr{H}} < \frac{1}{2} \\
&= \frac{1}{2}\mathscr{S}_M(\mathcal{X}_1(\ell), \mathcal{X}_2(\ell)) \\
&= \zeta(\mathscr{S}_M(\mathcal{X}_1(\ell), \mathcal{X}_2(\ell)))
\end{aligned}
$$

which yields,

$$
\alpha(\mathcal{X}_1(\ell), \mathcal{X}_2(\ell))\mathscr{S}_M(\mathscr{W}\mathcal{X}_1(\ell), \mathscr{W}\mathcal{X}_2(\ell)) < \zeta(\mathscr{S}_M(\mathcal{X}_1(\ell), \mathcal{X}_2(\ell))).
$$

Thus, Theorem 2.15 conditions satisfied. Hence $\mathscr{W}$ has a fixed point. This shows that the system under these conditions has a solution. Including condition (4), we can say that the system will have a unique solution.

3.2 Graphical Representations

For various values of fractional order, graphical solutions have been conducted in this section. Additionally, we provided some graphical solutions of the model (3.4) with the Atangana-Baleanu derivative as γ varies from 0.95 to 0.25 using the iterative approach we presented pertinent plots for various orders as in Figs. 2, 3, 4, 5, 6, 7, 8 and 9.

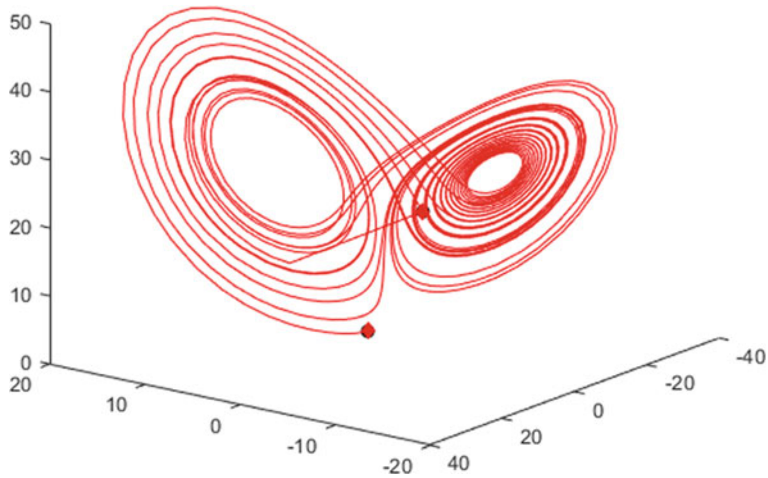

Fig. 2 The chaotic attractor 3D phase trajectories when $\gamma = 0.95$

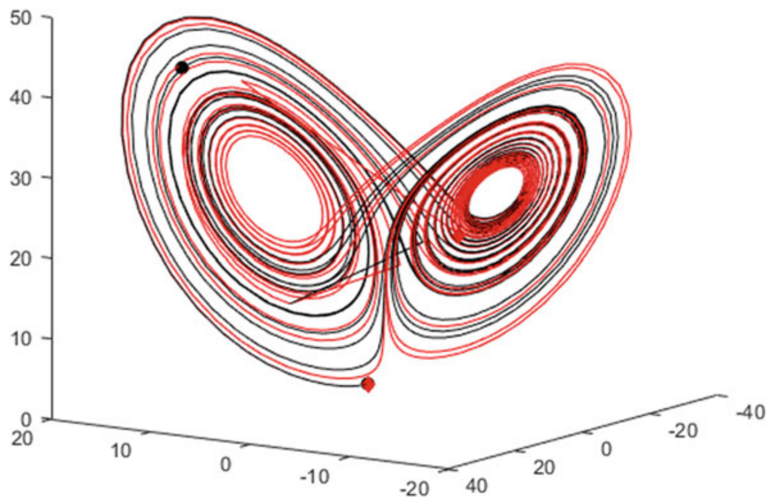

Fig. 3 The chaotic attractor 3D phase trajectories when $\gamma = 0.80$

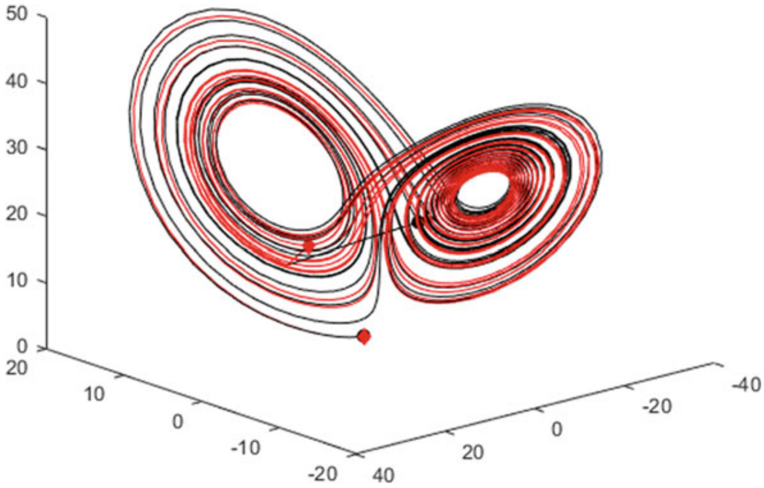

Fig. 4 The chaotic attractor 3D phase trajectories when $\gamma = 0.75$

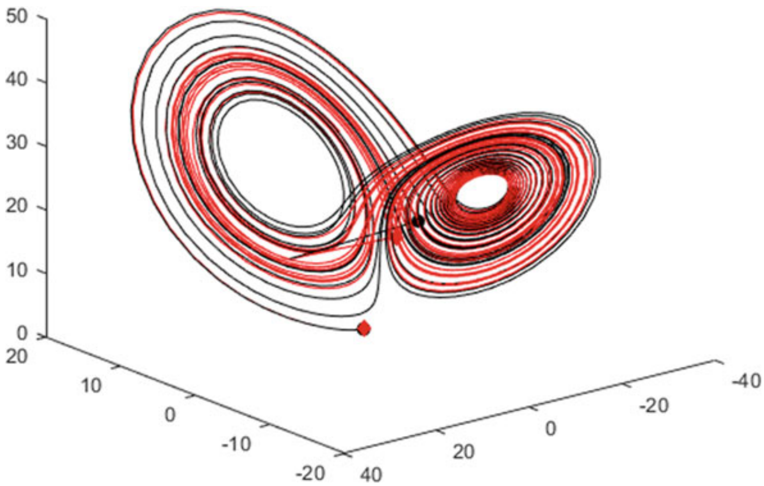

Fig. 5 The chaotic attractor 3D phase trajectories when $\gamma = 0.65$

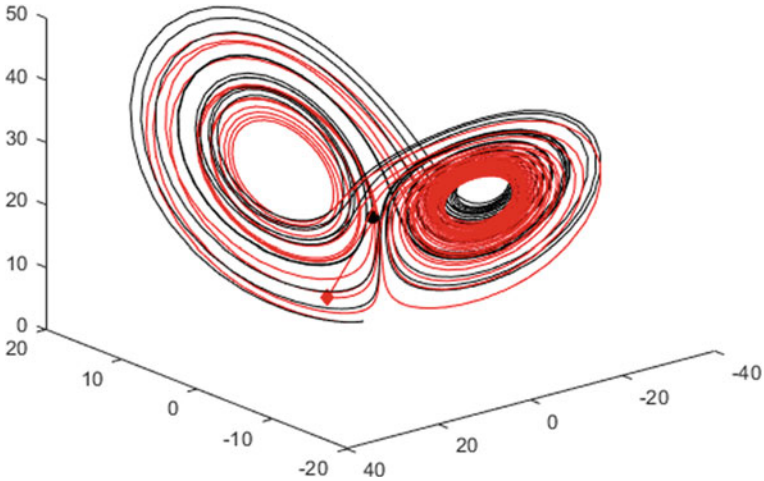

Fig. 6 The chaotic attractor 3D phase trajectories when $\gamma = 0.50$

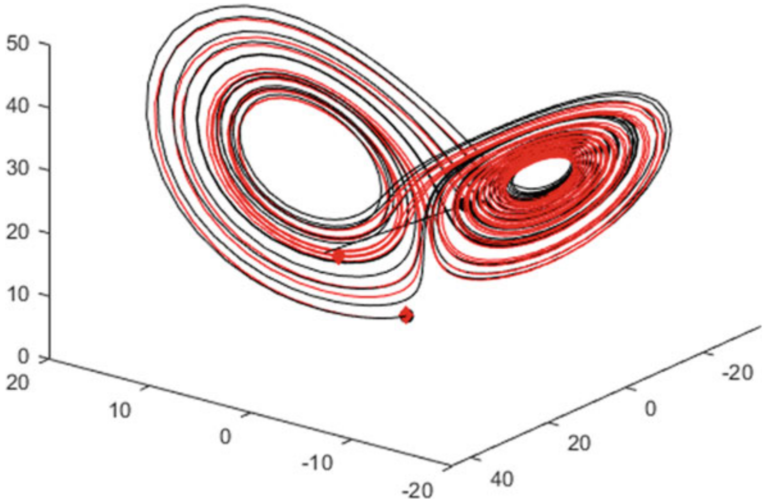

Fig. 7 The chaotic attractor 3D phase trajectories when $\gamma = 0.45$

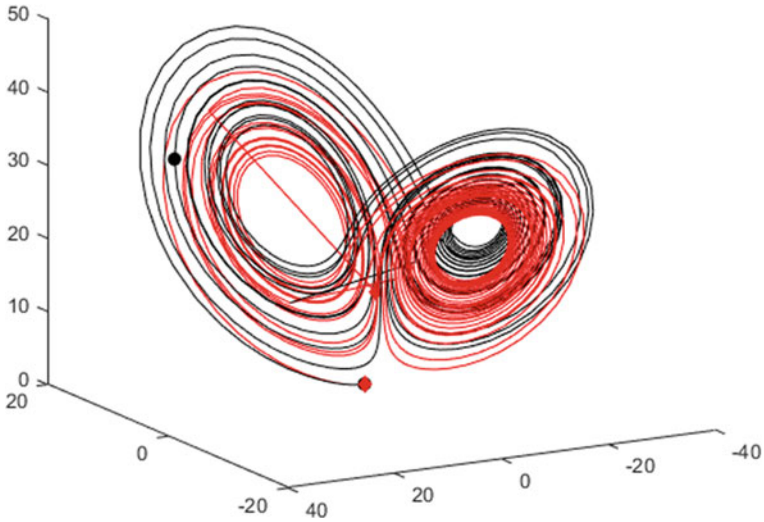

Fig. 8 The chaotic attractor 3D phase trajectories when $\gamma = 0.35$

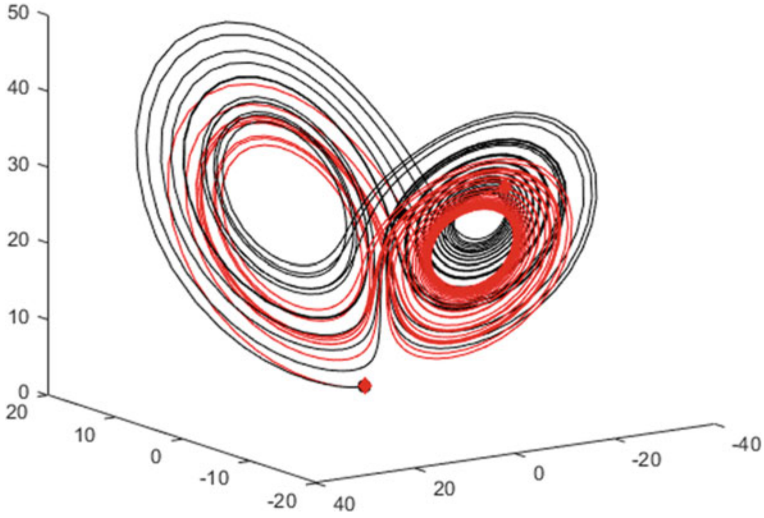

Fig. 9 The chaotic attractor 3D phase trajectories when $\gamma = 0.25$

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On L_ϕ-Solutions for n-Product of Fractional Integral Operators

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Mathematics Subject Classification (2010) 47N20, 47H30, 45G10, 46E30

1 Introduction

Fractional integral equations are a major and significant topic of scientific studies due to their applications in many fields of science. These include traffic models, fluid dynamics, statistical mechanics, radioactive transmission, acoustic scattering, and cytotoxic activity see [4, 11, 37].

These equations have been deeply analyzed by numerous authors in various types of function spaces (cf. [2, 14–16, 19, 26, 34, 36]).

O'Neil started the research on fractional integrations in Orlicz spaces in 1965 [29], and since then, there have been a lot of fascinating works on this topic (cf. [3, 28, 32]).

This paper focuses on studying the existence of the theorems for the product of n-Riemann–Liouville (R–L) fractional integral operators operating on n-various Orlicz spaces $L_{\varphi_i},\ n \geq 2$ of the structure

$$y(\theta) = \prod_{i=1}^{n} \left(g_i(\theta) + \int_0^\theta \frac{(\theta - v)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_i(v, y(v)) dv \right),\ \theta \in [0, d],\ \alpha_i \in (0, 1). \quad (1)$$

We investigate three separate existence theorems for Eq. (1), that are based on various growth conditions using either the Δ', Δ_2 or Δ_3-conditions of the generated N-functions of the analyzed Orlicz spaces. As a result, we can construct a variety of

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continuity and boundedness conditions for the investigated operators in the considered spaces. Since such spaces are not Banach algebras, we prefer using the method described in [7, 20, 22] to reach our conclusions.

The Orlicz spaces L_φ are proper spaces to study operators or kernels with strong nonlinearity (for example, exponential growth), and they represent general cases of Lebesgue spaces $L_p,\ p > 1$, (with N-function $\varphi = \frac{\theta^p}{p}$ realizes Δ_2-condition cf. [6, 21, 24]). These are supported by statistical physics and mathematical physics (cf. [5, 17, 18]). Following is the thermodynamic problem

$$y(\theta) + \int_I k(\theta, v) \cdot e^{y(v)}\, dv = 0,$$

containing exponential nonlinearities.

The Orlicz spaces and generalized Orlicz spaces have both been used to study integral equations (cf. [30, 31, 33]). The constant-sign solutions in L_φ were also investigated in [1]. Furthermore, the quadratic integral equations in L_φ have been investigated via the fixed point hypothesis (FPT) fixed point hypothesis and an appropriate fixed point hypothesis (MNC) measure of noncompactness in [7–9, 21, 23] under a distinct set of conditions.

In the case of $n = 2$, Eq. (1) covers the prototype of the Gripenberg-fractional integral equation

$$\begin{aligned} y(\theta) = &\left(g_1(\theta) + \int_0^\theta \frac{(\theta - v)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \Big(b_1(v) + \gamma_1 N^{-1} \varphi\big(y(v)\big)\Big)\, dv\right) \\ &\times \left(g_2(\theta) + \int_0^\theta \frac{(\theta - v)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} \Big(b_2(v) + \gamma_2 N^{-1} \varphi\big(y(v)\big)\Big)\, dv\right), \end{aligned}$$

which is widely used in mathematical biology to model diseases without producing lifelong immunity (SI models, cf. [12]). Infectious disease models rely on potentially discontinuous data functions, so we are interested in discontinuous solutions for those problems.

The author in [25] discussed the L_φ-results of the quadratic (R–L) fractional integral equations

$$y(\theta) \ = \ g(\theta) + G(y)(\theta) \int_0^\theta \frac{(\theta - v)^{\alpha - 1}}{\Gamma(\alpha)} f(v, y(v))\, dv, \quad \theta \in [0, d],\ 0 < \alpha < 1.$$

The existence of L_φ-results for Hadamard-type fractional integral equations

$$y(\theta) = G_2(y)(\theta) + \frac{G_1(y)(\theta)}{\Gamma(\alpha)} \int_1^\theta \left(\log \frac{\theta}{v}\right)^{\alpha - 1} G_2(y)(v)\, dv,\ \theta \in [1, e],\ 0 < \alpha < 1,$$

have been studied in [27].

In this article, multiple existence theorems for Eq. (1) in L_φ-spaces are demonstrated and discussed using the fixed point technique introduced in [22]. Finally, we provide some concrete instances that illustrate the applicability of our lemmas and theorems.

2 Notation and Auxiliary Facts

Let $I = [0, a] \subset \mathbb{R}^+ = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$. The function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called Young function if

$$\Psi(v) = \int_0^v u(\theta) d\theta, \quad \text{for} \;\; v \geq 0,$$

where $u : \mathbb{R}^+ \to \mathbb{R}^+$ is a left-continuous and increasing function that is not equal to infinite or zero on $\mathbb{R}^+$. If $N(y) = (yz - \Psi(y))$, then the functions Ψ and N are said to be complimentary Young functions.

Particularly, Ψ is a N-function, if it is finite-valued, with $\lim_{v\to 0} \frac{\Psi(v)}{v} = 0$, $\lim_{v\to\infty} \frac{\Psi(v)}{v} = \infty$ and $\Psi(v) > 0$ if $v > 0$ ($\Psi(v) = 0 \iff v = 0$).

Let $L_\Psi = L_\Psi(I)$ refers to the *Orlicz space* of all measurable functions $y : I \to \mathbb{R}$ with the norm

$$\|y\|_\Psi = \inf_{\varepsilon>0} \left\{ \int_I \Psi\left(\frac{y(v)}{\varepsilon}\right) dv \leq 1 \right\}.$$

Let the set of all bounded functions of $L_\Psi(I)$ with absolutely continuous norms be denoted by $E_\Psi(I)$.

Furthermore, we have $E_\Psi = L_\Psi$ if Ψ realizes the so-called Δ_2-condition, i.e.,

$$(\Delta_2) \;\; \text{there exist } K, \; \theta_0 \geq 0 \text{ s.t. } \Psi(2\theta) \leq K\Psi(\theta), \; \theta \geq \theta_0.$$

The N-function Ψ fulfills the Δ'-condition if $\exists\, K, \; \theta_0 \geq 0$ s.t. for $\theta, v \geq \theta_0$, we get $\Psi(\theta v) \leq K\Psi(\theta)\Psi(v)$.

Additionally, the N-function Ψ fulfills the Δ_3-condition if $\exists\, K, \; \theta_0 \geq 0$ s.t. for $\theta \geq \theta_0$, we get $\theta\Psi(\theta) \leq \Psi(K\theta)$.

Remark 2.1 It is appropriate to consider $\Psi_1(v) = (1 + |v|) \cdot \ln[(1 + |v|)] - |v|$, and $\Psi_2(u) = \exp[|v|] - |v| - 1$ as examples of complementary N-functions, where Ψ_1 realizes the Δ'-condition, and Ψ_2 realizes the Δ_3-condition. The N-functions $\Psi_3(v) = \frac{v^p}{p}, \;\; p > 1$ and $\Psi_4(v) = |v|^\alpha(|\ln|v| + 1)$ for $\alpha \geq \frac{3+\sqrt{5}}{2}$ realize the Δ_2-condition.

Definition 2.2 ([19]) The (R–L) fractional integral operator of Riemann–Liouville type of order α of a well defined function y takes the form

$$J^{\alpha} y(\theta) = \int_0^{\theta} \frac{(\theta - v)^{\alpha-1}}{\Gamma(\alpha)} y(v)\, dv, \quad \alpha > 0, \quad \theta > 0,$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-v} v^{\alpha-1}\, dv$.

Lemma 2.3 ([25]) *Suppose that Ψ_i and N_i are complementary N-functions and φ_i are N-functions $i = 1, \ldots, n$. Moreover, suppose that*

$$\frac{1}{\varepsilon^{\frac{1}{1-\alpha_i}}} \int_0^{\theta \varepsilon^{\frac{1}{1-\alpha_i}}} \Psi_i(v^{\alpha_i - 1})\, dv \in E_{\varphi_i} \ \textit{for a.e.} \ \ v \in I \ \ \textit{and} \ \ \varepsilon > 0,$$

then the (R–L) operators $J^{\alpha_i} : L_{N_i} \to L_{\varphi_i}$ and is continuous.

Proposition 2.4 ([19]) *The (R–L) operators J^{α_i}, $\alpha_i > 0$, $i = 1, \ldots, n$ map the a.e. nondecreasing and nonnegative functions into functions have the same properties.*

Definition 2.5 ([17]) Assume that the function $h : I \times \mathbb{R} \to \mathbb{R}$ realizes Carathéodory conditions, i.e., it is measurable in θ for any $y \in \mathbb{R}$ and is continuous in y for almost all $\theta \in I$. For every measurable function $y(\theta)$ on I, the superposition operator F_h is denoted as

$$F_h(y)(\theta) = h(\theta, y), \quad \theta \in I.$$

Lemma 2.6 ([17, Theorem 17.6]) *Suppose that $h : I \times \mathbb{R} \to \mathbb{R}$ realizes Carathéodory conditions. Then*

$$|h(\theta, y)| \leq b(\theta) + \gamma \Psi_2^{-1}\left[\Psi_1\left(\frac{y}{r}\right)\right], \ \theta \in I \ \ \textit{and} \ \ y \in \mathbb{R},$$

where $\gamma, r \geq 0$ and $b \in L_{\Psi_2}$. The operator $F_h : B_r(E_{\Psi_1}) \to E_{\Psi_2}$ is continuous, when the N-function Ψ_2 satisfies Δ_2-condition.

The following lemma describes the product of n-operators:

Lemma 2.7 ([13, Theorem 1]) *Let $n \geq 2$. If φ and φ_i, $i = 1, \ldots n$ be arbitrary N-functions, then the following axioms are identical:*

1. *For every $v_i \in L_{\varphi_i}$, $\prod_{i=1}^{n} v_i \in L_{\varphi}$.*
2. *There exists a constant $k > 0$ s.t.*

$$\left\| \prod_{i=1}^{n} v_i \right\|_{\varphi} \leq k \prod_{i=1}^{n} \|v_i\|_{\varphi_i},$$

for every $v_i \in L_{\varphi_i}$, $i = 1, 2, \ldots n$.

3. *There exists a constant* $C > 0$ *s.t.*

$$\prod_{i=1}^{n} \varphi_i^{-1}(\theta) \le C\varphi^{-1}(\theta)$$

for every $\theta \ge 0$.
4. *There exists a constant* $C > 0$ *such that for all* $\theta_i \ge 0, i = 1, \dots n,$

$$\varphi\left(\frac{\prod_{i=1}^{n}\theta_i}{C}\right) \le \sum_{i=1}^{n} \varphi_i(\theta_i).$$

Let "*meas*" point to the Lebesgue measure in $\mathbb{R}$ and $X = X(I)$ refers to the set of Lebesgue measurable functions on I. The set X concerns the metric

$$d(y, z) = \inf_{\varepsilon>0}[\varepsilon + meas\{v : |y(v) - z(v)| \ge \varepsilon\}]$$

generates a complete space. It appears that the convergence on d is the same as the convergence in measure on I (cf. Proposition 2.14 in [35]). We shall call compactness in X "compactness in measure".

Lemma 2.8 ([8]) *Assume that,* $Y \subset L_\Psi$ *be bounded set and there exists a family* $(\sigma_c)_{0\le c\le a} \subset I$ *s.t. meas* $\sigma_c = c$ *for every* $c \in [0, a]$. *Then for every* $y \in Y$,

$$y(\theta_1) \ge y(\theta_2), \quad (\theta_1 \in \sigma_c,\ \theta_2 \notin \sigma_c)$$

the set Y *is compact in measure in* L_Ψ.

Definition 2.9 ([7]) Let $\emptyset \ne Y \subset L_\Psi$ be a bounded set. The Hausdorff MNC $\beta_H(Y)$ is defined as

$$\beta_H(Y) = \inf\Big\{r > 0 : \exists \text{ a finite subset } S \text{ of } L_\Psi \text{ s.t. } y \subset S + B_r\Big\},$$

where $B_r = \{y \in L_\Psi : \|y\|_\Psi \le r\}$

Definition 2.10 ([35, Definition 3.9] or [10]) The measure of equiintegrability c of the set $Y \in L_\Psi$ for any $\varepsilon > 0$ is defined as

$$c(Y) = \lim_{\varepsilon\to 0} \sup_{mes T\le\varepsilon} \sup_{y\in Y} \|y \cdot \chi_T\|_{L_\Psi},$$

where χ_T refers to the characteristic function of $T \subset I$.

Lemma 2.11 ([8, 10]) *Let* $\emptyset \ne Y \subset E_\Psi$ *be a bounded set and compact in measure. Then*

$$\beta_H(Y) = c(Y).$$

Theorem 2.12 ([7]) *Let $\emptyset \neq D \subset L_\Psi$ be closed, bounded, and convex set and let $P : D \to D$ be continuous and contraction operator regarding β_H, i.e.,*

$$\beta_H(P(Y)) \leq k\beta_H(Y), \quad k \in [0, 1)$$

for any $\emptyset \neq Y \subset D$. Then P has at least one fixed point in D.

3 Main Results

Equation (1) should be presented in the following way:

$$y(\theta) = H(y)(\theta) = \prod_{i=1}^{n} H_i(y)(\theta), \quad \theta \in \mathbb{I} = [0, d],$$

where

$$H_i(y)(\theta) = g_i(\theta) + A_i(y)(\theta) \text{ and } A_i(y)(\theta) = J^{\alpha_i} F_{f_i} y(\theta)$$

s.t. F_{f_i} are the superposition operators and J^{α_i} are the (R–L) operators as in Definition 2.2.

Based on different general growth conditions, we will characterize three existence theorems.

3.1 The Case of Δ'-Condition

For $i = 1, 2, \ldots, n$ suppose that φ, φ_i are N-functions and consider the next assumptions with Ψ_i and N_i being complementary N-functions:

(C1) $g_i \in E_{\varphi_i}(\mathbb{I})$, $i = 1, \ldots n$ are a.e. nondecreasing on $\mathbb{I}$,

(C2) $f_i : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ utilize Carathéodory conditions and suppose that $f_i(\theta, y)$ be nondecreasing with regard to θ and y separately,

(C3) $|f_i(\theta, y)| \leq b_i(\theta) + R_i(|y|)$ for $\theta \in \mathbb{I}$ and $y \in \mathbb{R}$, where $b_i \in E_{N_i}(\mathbb{I})$ and R_i are nonnegative, continuous nondecreasing function on $\mathbb{R}^+$.

(C4) Suppose that N_i realize the Δ'-condition and $\exists\, \omega, \gamma, u_0 \geq 0$ s.t.

$$N_i(\omega(R_i(v))) \leq \gamma\varphi(v) \leq \gamma\Psi_i(v) \text{ for } v \geq v_0.$$

(G1) For every $v_i \in L_{\varphi_i}(\mathbb{I})$, we have $\| \prod_{i=1}^{n} v_i \|_\varphi \leq k \prod_{i=1}^{n} \|v_i\|_{\varphi_i}, \quad k > 0.$

(G2) Assume that $k_i(\theta) = \frac{1}{\varepsilon^{\frac{1}{1-\alpha_i}}} \int_0^{\theta\varepsilon^{\frac{1}{1-\alpha_i}}} \Psi_i(v^{\alpha_i - 1})\, dv \in E_{\varphi_i}(\mathbb{I})$ for a.e. $v \in \mathbb{I}$ and $\varepsilon > 0$.

Remark 3.1 By using assumption (C3), (G2) and Lemma 2.3, then for $y \in E_\varphi(\mathbb{I})$ and $T \subset \mathbb{I}$, we have

$$\|A_i(y)\cdot\chi_T\|_{\varphi_i} \le \frac{2}{\Gamma(\alpha_i)}\|k_i\cdot\chi_T\|_{\varphi_i}\cdot\left(\|b_i\cdot\chi_T\|_{N_i} + \|R_i(|y(\cdot)|\cdot\chi_T\|_{N_i}\right).$$

Then, by utilizing (C4) $\exists\,\omega,\ \gamma,\ u_0 > 0$, (cf. [17, Theorem 19.1]), s.t.

$$\begin{aligned}\|R_i(|y(\cdot)|)\|_{N_i} &\le \frac{1}{\omega}\left(1+\int_0^d N_i(\omega R_i(|y(\theta)|))d\theta\right)\\ &\le \frac{1}{\omega}\left(1+N_i(\omega R_i(u_0))+\gamma\right)\int_0^d \varphi(|y(\theta)|)d\theta.\end{aligned}$$

Theorem 3.2 *Let assumptions (C1)–(C4), (G1), and (G2) hold. If*

$$k\prod_{i=1}^{n}\left(\|g_i\|_{\varphi_i} + \frac{2}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_i}\left(\|b_i\|_{N_i} + R_i(1)\right)\right) \le 1$$

and

$$k\prod_{i=1}^{n}\left(\frac{2\|k_i\|_{\varphi_i}}{\omega\Gamma(\alpha_i)}\left(1+N_i(\omega R_i(u_0))+\gamma\right)\right) < 1,$$

then there exists a solution $y \in E_\varphi(I)$ of (1) that is a.e. nondecreasing on $I = [0, a] \subset \mathbb{I}$.

Proof **I.** Lemma 2.3 grants us that the operators $J^{\alpha_i} : L_{N_i}(\mathbb{I}) \to L_{\varphi_i}(\mathbb{I})$ and they are continuous and by assumption (C2) the operators $F_{f_i} : B_1(E_\varphi(\mathbb{I})) \to L_{N_i}(\mathbb{I})$ and they are continuous. Then $A_i = J^{\alpha_i}F_{f_i} : B_1(E_\varphi(\mathbb{I})) \to E_{\varphi_i}(\mathbb{I})$ and they are continuous. By assumption (C1), the operators $H_i : B_1(E_\varphi(\mathbb{I})) \to E_{\varphi_i}(\mathbb{I})$ are continuous. By $(G1)$, we can refer that $H = \prod_{i=1}^n H_i : B_1(E_\varphi(\mathbb{I})) \to E_\varphi(\mathbb{I})$ is continuous.

II. We shall formulate the ball $B_1(E_\varphi(I)) = \{y \in L_\varphi(I) : \|y\|_\varphi \le 1\}$.
By utilizing Lemma 2.3 and for $y \in B_1(E_\varphi(I))$, we get

$$\begin{aligned}\|H_i(y)\|_{\varphi_i} &\le \|g_i\|_{\varphi_i} + \|A_i(y)\|_{\varphi_i}\\ &\le \|g_i\|_{\varphi_i} + \|J^\alpha F_{f_i}(y)\|_{\varphi_i}\\ &\le \|g_i\|_{\varphi_i} + \frac{2}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_i}\|F_{f_i}(y)\|_{N_i}\\ &\le \|g_i\|_{\varphi_i} + \frac{2}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_i}\left(\|b_i\|_{N_i} + \|R_i(|y(\cdot)|)\|_{N_i}\right)\\ &\le \|g_i\|_{\varphi_i} + \frac{2}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_i}\left(\|b_i\|_{N_i} + R_i(1)\right),\end{aligned}$$

where $\|y\|_\varphi \le 1$. Therefore, utilizing assumption (G1), we get

$$\|H(y)\|_{\varphi} \leq k\prod_{i=1}^{n}\|H_i(y)\|_{\varphi_i}$$
$$= k\prod_{i=1}^{n}\Big(\|g_i\|_{\varphi_i} + \frac{2}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_i}\Big(\|b_i\|_{N_i} + R_i(1)\Big)\Big) \leq 1.$$

Then the operator $B : B_1(E_\varphi(I)) \to E_\varphi(I)$ is continuous on $I = [0, a] \subset \mathbb{I}$.

III. Let $Q_1 \subset B_1(E_\varphi(I))$ contain all functions which are a.e. nondecreasing on I. This set is a convex, closed, bounded, and nonempty set in $L_\varphi(I)$ see [8]. Additionally, Q_1 is compact in measure (thanks to Lemma 2.8).

IV. We shall now demonstrate that H maintains the monotonicity of functions. Select $y \in Q_1$, then y is a.e. nondecreasing on I and consequently the operators $A_i(y) = J^\alpha F_{f_i}(y)$ are a.e. nondecreasing on I (see assumption (C2) and Proposition 2.4). According to (C1), the operators $H_i = g_i + A_i(y)$ are a.e. nondecreasing on I. Finally, assumption (G1) gives that $H : Q_1 \to Q_1$ is continuous.

V. We shall now prove that H is a contraction with regard to β_H.

Suppose that $T \subset I$, with meas$T \leq \varepsilon$, $\varepsilon > 0$. Then for $y \in Y \subset Q_1$, we get

$$\begin{aligned}\|H_i(y)\cdot\chi_T\|_{\varphi_i} &\leq \|g_i\cdot\chi_T\|_{\varphi_i} + \|A_i(y)\cdot\chi_T\|_{\varphi_i}\\ &\leq \|g_i\cdot\chi_T\|_{\varphi_i} + \frac{2}{\Gamma(\alpha_i)}\|k_i\cdot\chi_T\|_{\varphi_i}\|F_{f_i}(y)\cdot\chi_T\|_{N_i}\\ &\leq \|g_i\cdot\chi_T\|_{\varphi_i} + \frac{2}{\Gamma(\alpha_i)}\|k_i\cdot\chi_T\|_{\varphi_i}\Big(\|b_i\cdot\chi_T\|_{N_i}\\ &\quad + \frac{1}{\omega}\Big(1 + N_i(\omega R_i(u_0)) + \gamma\Big)\int_T \varphi(|y(\theta)|)d\theta\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}\|H(y)\cdot\chi_T\|_{\varphi_i} &\leq k\prod_{i=1}^{n}\|H_i(y)\cdot\chi_T\|_{\varphi_i}\\ &= k\prod_{i=1}^{n}\Big(\|g_i\cdot\chi_T\|_{\varphi_i} + \frac{2\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\Big(\|b_i\cdot\chi_T\|_{N_i}\\ &\quad + \frac{1}{\omega}\Big(1 + N_i(\omega R_i(u_0)) + \gamma\Big)\int_T \varphi(|y(\theta)|)d\theta\Big)\Big),\end{aligned}$$

where $\|k_i\cdot\chi_T\|_{\varphi_i} \leq \|k_i\|_{\varphi_i}$. Since $g_i \in E_{\varphi_i}$ and , $b_i \in E_{N_i}$, we have

$$\lim_{\varepsilon\to 0}\Big\{\sup_{mes\ T\leq\varepsilon}\Big[\sup_{y\in Y}\Big\{\|g_i\cdot\chi_T\|_{\varphi_i} + \frac{2\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\|b_i\cdot\chi_T\|_{N_i}\Big\}\Big]\Big\} = 0.$$

Thus, by utilizing the definition of $c(Y)$ with

$$\Big(\int_T \varphi(|y(\theta)|)d\theta\Big)^n \le \|y\|_\varphi^{n-1}\cdot\int_T \varphi(|y(\theta)|)d\theta \le \int_T \varphi(|y(\theta)|)\,d\theta,$$

we get

$$c(H(Y)) \le k\prod_{i=1}^n\left(\frac{2\|k_i\|_{\varphi_i}}{\omega\Gamma(\alpha_i)}\Big(1+N_i(\omega R_i(u_0))+\gamma\Big)\right)\cdot c(Y).$$

Since $\emptyset \ne Y \subset Q_1$ be bounded set and compact in measure in E_φ, we shall use Lemma 2.11 and get

$$\beta_H(H(Y)) \le k\prod_{i=1}^n\left(\frac{2\|k_i\|_{\varphi_i}}{\omega\Gamma(\alpha_i)}\Big(1+N_i(\omega R_i(u_0))+\gamma\Big)\right)\cdot \beta_H(Y).$$

Since $k\prod_{i=1}^n\left(\frac{2\|k_i\|_{\varphi_i}}{\omega\Gamma(\alpha_i)}\Big(1+N_i(\omega R_i(u_0))+\gamma\Big)\right) < 1$. Once we have employed Theorem 2.12, we are done. □

3.2 The Case of Δ_3-Condition

In what follows, we study Eq. (1) using growth condition that is essentially faster than a polynomial (i.e., when the N-functions fulfilling Δ_3-condition). Let $\vartheta = \sup\Big\{\|y\|_1 : y \in B_1(L_\varphi(I))\Big\} : L_\varphi(I) \to L^1(I)$ be the norm of the identity operator.

For $i = 1, \dots, n$, take into consideration the next assumption:

(C5) a. N_i realize the Δ_3-condition.
b. There exist $\beta,\ v_0 > 0$ s.t.

$$R_i(v) \le \beta\frac{\Psi_i(v)}{v},\quad \text{for}\quad v \ge v_0.$$

c. Suppose that $r > 0$ is satisfying

$$k\prod_{i=1}^n\left(\|g_i\|_{\varphi_i} + \frac{2C_i\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\left(\|b_i\|_{N_i} + \frac{1}{\omega}\Big(1+\eta_0 u_0 a+\eta_0\Big)\vartheta r\right)\right) \le r$$

and

$$\prod_{i=1}^n\left(\frac{C_i\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\right) < \frac{\omega^n}{2^n k r^{n-1}\Big(1+\eta_0 u_0 a+\eta_0\Big)^n},$$

where $\omega, k, a > 0,\ \eta_0 > 1$ and $C_i = \Big(2 + a(1 + \varphi_i(1))\Big)$.

Remark 3.3 (a) From assumption (C5), we get that the intervals I and $\mathbb{I}$ are identical.

(b) Assumption (C5) 2. implies that $\exists\, \omega, u_0 > 0$ and $\eta_0 > 1$ s.t. $N_i(\omega R(u)) \le \eta_0 u$ for $u \ge u_0$. Thus, for $y \in L_\varphi$

$$\begin{aligned}\|R_i(|y(\cdot)|)\|_{N_i} &\le \frac{1}{\omega}\Big(1 + \int_I N_i(\omega R_i(|y(v)|)\, dv\Big)\\ &\le \frac{1}{\omega}\Big(1 + \eta_0 u_0 a + \eta_0\Big)\int_I |y(v)|\, dv.\end{aligned}$$

By assumption (G2) and [17, Lemma 15.1 and Theorem 19.2] for $y \in L_\varphi$:

$$\begin{aligned}\|A_i(y)\chi_T\|_{\varphi_i} &\le \frac{2C_i}{\Gamma(\alpha_i)}\|k_i \cdot \chi_{T\times I}\|_{\varphi_i}\Big(\|b_i\chi_{T\times I}\|_{N_i} + \|R_i(|y(\cdot)|)\chi_{T\times I}\|_{N_i}\Big)\\ &\le \frac{2C_i}{\Gamma(\alpha_i)}\|k_i\|_{\varphi_i}\left(\|b_i\chi_T\|_{N_i} + \frac{1}{\omega}\Big(1 + \eta_0 u_0 a + \eta_0\Big)\int_T |y(v)|dv\right),\end{aligned}$$

where $C_i = \Big(2 + a(1 + \varphi_i(1))\Big)$ and T is a measurable subset of I.

Theorem 3.4 *Suppose, that Ψ_i and N_i are complementary N-functions and φ, φ_i are N-functions for $i = 1, \ldots n$. Let assumptions (C1)–(C3), (C5), (G1), and (G2) be fulfilled, then there exists a solution $y \in E_\varphi$ of (1) which is a.e. nondecreasing on I.*

Proof **I'** is same as **I** but on E_φ, i.e., the operator $H : E_\varphi \to E_\varphi$ is continuous.

II'. By recalling Lemma 3.3 and for $y \in B_r(E_\varphi(I))$, we have

$$\begin{aligned}&\|H_i(y)\|_{\varphi_i}\\ &\le \|g_i\|_{\varphi_i} + \frac{2C_i\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\left(\|b_i\|_{N_i} + \frac{1}{\omega}\Big(1 + N_i(\omega R_i(u_0)) + \eta_0\Big)\int_I |y(v)|dv\right)\\ &\le \|g_i\|_{\varphi_i} + \frac{2C_i\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\left(\|b_i\|_{N_i} + \frac{1}{\omega}\Big(1 + N_i(\omega R_i(u_0)) + \eta_0\Big)\|y\|_1\right)\\ &\le \|g_i\|_{\varphi_i} + \frac{2C_i\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\left(\|b_i\|_N + \frac{1}{\omega}\Big(1 + N_i(\omega R_i(u_0)) + \eta_0\Big)\vartheta\|y\|_\varphi\right),\end{aligned}$$

where $C_i = \Big(2 + a(1 + \varphi_i(1))\Big)$. Then, by using (G1), we have

$$\|H(y)\|_\varphi \le k\prod_{i=1}^{n}\|H_i(y)\|_{\varphi_i}$$
$$\le k\prod_{i=1}^{n}\left(\|g_i\|_{\varphi_i}+\frac{2C_i\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\left(\|b_i\|_{N_i}+\frac{1}{\omega}\Big(1+\eta_0u_0a+\eta_0\Big)\vartheta r\right)\right)\le r.$$

By using assumption $(C5)_{3.}$, we deduce that $H : B_r(E_\varphi(I)) \to B_r(E_\varphi(I))$ is continuous.

III' and **IV'** are the same as **III** and **IV** on $Q_r \subset B_r(E_\varphi(I))$.

V'. Suppose that $T \subset I$, with meas$T \le \varepsilon$, $\varepsilon > 0$. Then, for $y \in Y \subset Q_r$, we get

$$\begin{aligned}\|H_i(y)\cdot\chi_T\|_{\varphi_i} &\le \|g_i\cdot\chi_T\|_{\varphi_i} + \|A_i(y)\cdot\chi_T\|_{\varphi_i}\\ &\le \|g_i\cdot\chi_T\|_{\varphi_i} + \frac{2C_i\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\Big(\|b_i\cdot\chi_T\|_{N_i}\\ &\quad+\frac{1}{\omega}\Big(1+\eta_0u_0a+\eta_0\Big)\|y\cdot\chi_T\|_\varphi\Big).\end{aligned}$$

As in Theorem 3.2, where

$$\|y\cdot\chi_T\|_\varphi^n \le \|y\cdot\chi_T\|_\varphi^{n-1}\cdot\|y\cdot\chi_T\|_\varphi \le r^{n-1}\cdot\|y\cdot\chi_T\|_\varphi,$$

we obtain

$$\beta_H(B(Y)) \le 2^nkr^{n-1}\Big(1+\eta_0u_0a+\eta_0\Big)^n\prod_{i=1}^{n}\left(\frac{C_i\|k_i\|_{\varphi_i}}{\omega\Gamma(\alpha_i)}\right)\beta_H(Y).$$

Since $\prod_{i=1}^{n}\left(\frac{C_i\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\right) < \frac{\omega^n}{2^nkr^{n-1}\left(1+\eta_0u_0a+\eta_0\right)^n}$, then by Theorem 2.12, we are done.

□

3.3 *The Case of* Δ_2*-Condition*

We will now present the existence theorem when the N-functions satisfy the Δ_2-condition. For $i = 1,\ldots,n$, take into consideration the assumptions:

(C6) Suppose that φ_i are N-functions and the functions N_i satisfy the Δ_2-condition:

a. There exist $\gamma_i \ge 0$ s.t.

$$R_i(v) \le \gamma_i N_i^{-1}(\varphi(v)) \quad \text{for } v \ge 0.$$

b. Suppose that $\exists\ r^* > 0$ on $I = [0, a] \subset [0, d]$ satisfying

$$\int_I \varphi\Bigg(\prod_{i=1}^{n} \bigg(|g_i(\theta)| + \frac{2|k_i(\theta)|}{\Gamma(\alpha_i)} \Big(\|b_i\|_{N_i} + \gamma_i r^* \Big) \bigg) \Bigg)\, d\theta \le r^*$$

and

$$\prod_{i=1}^{n} \frac{\gamma_i \|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)} < \frac{1}{2^n k(r^*)^{n-1}}.$$

Remark 3.5 Assumption $(C6)_{1.}$ with [17, Theorem 10.5 with $k = 1$] indicates for any $y \in E_\varphi$, $\gamma > 0$ that

$$\begin{aligned} \|R_i(|y \cdot \chi_{[0,\theta]}|)\|_{N_i} &\le \gamma_i \left\| N_i^{-1} \left(\varphi \left(|y \cdot \chi_{[0,\theta]}| \right) \right) \right\|_{N_i} \\ &\le \gamma_i \|y\|_\varphi \le \gamma_i + \gamma_i \int_0^\theta \varphi\left(|y(v)|\right)\, dv. \end{aligned} \tag{2}$$

By using the Hölder inequality with our assumptions, we have

$$|A_i(y)(\theta)| \le |k_i(\theta)| \bigg(\|b_i\|_{N_i} + \|R_i(|y \cdot \chi_{[0,\theta]}|)\|_{N_i} \bigg).$$

Theorem 3.6 *Suppose, that Ψ_i and N_i are complementary N-functions and φ, φ_i are N-functions for $i = 1, \ldots n$. Let assumptions (C1)–(C3), (C6), (G1) and (G2) be fulfilled, then there exists a solution $y \in E_\varphi$ of (1) which is a.e. nondecreasing on $I = [0, a] \subset \mathbb{I}$.*

Proof **I''** is the same as **I** i.e., $B : B_1(E_\varphi(I)) \to E_\varphi$ is continuous.

II''. We will demonstrate that the operator H is bounded in L_φ.

let Q be the set of all numbers $r^* > 0$ satisfying

$$\int_I \varphi\Bigg(\prod_{i=1}^{n} \bigg(|g_i(\theta)| + \frac{2|k_i(\theta)|}{\Gamma(\alpha_i)} \Big(\|b_i\|_{N_i} + \gamma_i r^* \Big) \bigg) \Bigg)\, d\theta \le r^*.$$

Let D points to the closure of the set $\{y \in E_\varphi : \int_0^a \varphi(|y(v)|)\, dv \le r^* - 1\}$. Obviously, D does not indicate a ball in E_φ, but $D \subset B_{r^*}(E_\varphi(I))$ (cf. [17, p. 222]) and $\overline{D} \subset E_\varphi$ is convex, closed, and bounded set.

For $y \in D$ and $\theta \in I$, we have

$$\begin{aligned} |H_i(y)(\theta)| &\le |g_i(\theta)| + |A_i(y)(\theta)| \\ &\le |g_i(\theta)| + \frac{2|k_i(\theta)|}{\Gamma(\alpha_i)} \bigg(\|b_i\|_{N_i} + \|R_i(|y \cdot \chi_{[0,\theta]}|)\|_{N_i} \bigg) \end{aligned}$$

$$\leq |g_i(\theta)| + \frac{2|k_i(\theta)|}{\Gamma(\alpha_i)}\left(\|b_i\|_{N_i} + \gamma_i + \gamma_i \int_0^a \varphi(|y(v)|)\ dv\right)$$
$$\leq |g_i(\theta)| + \frac{2|k_i(\theta)|}{\Gamma(\alpha_i)}\left(\|b_i\|_{N_i} + \gamma_i + \gamma_i(r^* - 1)\right).$$

Therefore,

$$\int_I \varphi(H(y)(\theta))\ d\theta \leq \int_I \varphi\left(\prod_{i=1}^n |H_i(y)(\theta)|\right) d\theta$$
$$\leq \int_I \varphi\left(\prod_{i=1}^n \left(|g_i(\theta)| + \frac{2|k_i(\theta)|}{\Gamma(\alpha_i)}\left(\|b_i\|_{N_i} + \gamma_i r^*\right)\right)\right) d\theta \leq r.$$

Then $H(D) \subset D$. Consequently, $H(\overline{D}) \subset \overline{H(D)} \subset \overline{D} = D$. Then $H : D \to D$ is continuous on $D \subset B_{r^*}(E_\varphi(I))$.

III'' and **IV''** are the same as **III** and **IV** on $Q_{r^*} \subset B_{r^*}(E_\varphi(I))$.

V''. Suppose that $T \subset I$, with meas$T \leq \varepsilon$, $\varepsilon > 0$. Then, by using Remark 2 and for $y \in Y \subset Q_{r^*}$, we get

$$\|H_i(y) \cdot \chi_T\|_{\varphi_i} \leq \|g_i \cdot \chi_T\|_{\varphi_i} + \frac{2\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)} \left\| \left(b_i + R_i(|y(\cdot)|)\right) \cdot \chi_T \right\|_{N_i}$$
$$\leq \|g_i \cdot \chi_T\|_{\varphi_i} + \frac{2\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\left(\|b_i \cdot \chi_T\|_{N_i} + \left\|R_i(|y(\cdot)|) \cdot \chi_T\right\|_{N_i}\right)$$
$$\leq \|g_i \cdot \chi_T\|_{\varphi_i} + \frac{2\|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)}\left(\|b_i \cdot \chi_T\|_{N_i} + \gamma_i \|y \cdot \chi_T\|_{\varphi}\right).$$

As done in Theorem 3.2, we have

$$\beta_H(H(Y)) \leq 2^n k(r^*)^{n-1} \prod_{i=1}^n \frac{\gamma_i \|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)} \cdot \beta_H(Y).$$

Since $\prod_{i=1}^n \frac{\gamma_i \|k_i\|_{\varphi_i}}{\Gamma(\alpha_i)} < \frac{1}{2^n k(r^*)^{n-1}}$, then by Theorem 2.12, we are done. □

4 Examples

We are in a position to provide some noteworthy examples that illustrate and support our findings.

In what follows, for $i = 1, \ldots, n$, put the N-functions $\Psi_i(u) = N_i(u) = u^2$ and $\varphi_i(u) = e^{|u|} - |u| - 1$.

Example 4.1 We need to show that, the operator $J^{\alpha_i} = \int_0^\theta \frac{(\theta-v)^{\alpha_i-1}}{\Gamma(\alpha_i)} y(v)\, dv : L_{N_i} \to L_{\varphi_i}$ is continuous and Lemma 2.3 hold.

Indeed: For $\theta \in [0, d]$ and any $\alpha_i \in (0, 1)$, we get

$$k_i(\theta) = \int_0^\theta \Psi_i\big(v^{\alpha_i-1}\big)\, dv = \int_0^\theta v^{2\alpha_i-2}\, dv = \frac{\theta^{2\alpha_i-1}}{2\alpha_i - 1}.$$

Moreover,

$$\int_0^d \varphi_i\big(k_i(\theta)\big)\, dv = \int_0^d \left(e^{\frac{\theta^{2\alpha_i-1}}{2\alpha_i-1}} - \frac{\theta^{2\alpha_i-1}}{2\alpha_i-1} - 1\right) d\theta$$

which is finite. Then, for $y \in L_{N_i}$, we have $J_i^\alpha : L_{N_i} \to L_{\varphi_i}$ is continuous.

Example 4.2 We will show that the operator $F_{f_i}(y)(v) = b_i(v) + \gamma_i N_i^{-1}\varphi\big(y(v)\big) : B_r(E_\varphi) \to L_{N_i}$, where $b_i \in L_{N_i}$ and $r, \gamma_i \geq 0$.

Indeed: For $y \in L_\varphi$, we have

$$\begin{aligned}
\|F_{f_i}(y)\|_{N_i} &\leq \|b_i\|_{N_i} + \gamma_i \left\| N_i^{-1}\Big(\varphi(y)\Big)\right\|_{N_i} \\
&\leq \|b_i\|_{N_i} + \gamma_i \int_I N_i\Big(N_i^{-1}\Big(\varphi(y(v))\Big)\Big)\, dv \\
&= \|b_i\|_{N_i} + \gamma_i \int_I \varphi(y(v))\, dv \\
&\leq \|b_i\|_{N_i} + \gamma_i \|y\|_\varphi.
\end{aligned}$$

If the N-functions N_i satisfy the Δ_2-condition, then $F_{f_i} : B_r(E_\varphi) \to E_{N_i}$ and is continuous.

For additional information and many examples of how the functions Ψ_i, N_i, φ_i and φ fulfill Lemma 2.7 (see [21, Theorem 15.4] and [13]).

Example 4.3 If $n = 1$, Eq. (1) becomes the standard fractional integral equations

$$y(\theta) = g(\theta) + \int_0^\theta \frac{(\theta - v)^{\alpha-1}}{\Gamma(\alpha)}\Big(b(v) + \gamma N^{-1}\varphi\big(y(v)\big)\Big)\, dv, \quad \theta \in [a, b],\ 0 < \alpha < 1,$$

which are logical extensions and practical applications of the theories and the results addressed in [1, 3, 17, 33].

Example 4.4 If $n = 2$, Eq. (1) forms the quadratic fractional integral equations

$$\begin{aligned}
y(\theta) = &\left(g_1(\theta) + \int_0^\theta \frac{(\theta - v)^{\alpha_1-1}}{\Gamma(\alpha_1)}\Big(b_1(v) + \gamma_1 N^{-1}\varphi\big(y(v)\big)\Big)\, dv\right) \\
&\times \left(g_2(\theta) + \int_0^\theta \frac{(\theta - v)^{\alpha_2-1}}{\Gamma(\alpha_2)}\Big(b_2(v) + \gamma_2 N^{-1}\varphi\big(y(v)\big)\Big)\, dv\right), \qquad (3)
\end{aligned}$$

for $\theta \in [a, b]$, $0 < \alpha_1, \alpha_2 < 1$. This is closer to the problems raised in [6, 8, 9, 21, 24–26].

Example 4.5 For $\theta \in [0, d]$ and $0 < \alpha_i < 1$, consider the integral equation

$$y(\theta) = \prod_{i=1}^{n} \left(g_i(\theta) + \int_0^\theta \frac{(\theta - v)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \Big(b_i(v) + \gamma_i N_i^{-1} \varphi\big(y(v)\big)\Big)\, dv \right), \quad (4)$$

which performs a special case of Eq. (1) with proper structure of the functions $g_i \in L_{\varphi_i}$ and $b_i \in L_{N_i}$.

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New Fixed Point Results of Multivalued Contraction Mappings in b-Metric Spaces

Theory, Space and Contraction Mappings

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1 Introduction and Preliminaries

In 1989, Bakhtin [5] introduced the concept of *b*-metric spaces under the name quasi-metric spaces and showed a contraction principle in these spaces, however, Czerwik [7] gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the Banach contraction mapping theorem [4]. Nadler [17] used the Pompeiu–Hausdorff metric to prove the multivalued contraction principle over the collection of nonempty closed and bounded subsets of a metric space.

After that, many generalizations of metric spaces and enormous fixed point results have been achieved for single-valued and multivalued mappings in many directions, also we refer the readers to some papers for more details of this topic ([1–3, 9, 10, 12, 13, 15]). We begin by introducing the basic definitions and properties of complete b-metric spaces and multivalued maps. We then present the concept of generalized contractive type of multivalued maps, which extends the classical notion of contractive mappings to multivalued settings. The proposed framework allows us to investigate the convergence and existence of fixed points for these maps.

Our main focus is to establish new fixed point results for generalized contractive type of multivalued maps in complete b-metric spaces. We provide sufficient conditions under which the existence and uniqueness of fixed points are guaranteed. The proofs of the main theorems rely on appropriate fixed point theorems and suitable mathematical techniques.

So, we introduce the concept of multivalued contraction mappings and prove fixed point results for such mappings in the context of b-metric spaces that generalize many preexisting results in the literature.

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To establish the main results for generalized contractive type of multivalued maps in complete b-metric spaces, we need to introduce the following definitions and preliminaries:

Definition 1 ([5]) Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}_+$ is called a b-metric on X if for all $x, y, z \in X$, the following conditions are satisfied.

(bm-1) $d(x, y) = 0 \iff x = y$,
(bm-2) $d(x, y) = d(y, x)$,
(bm-3) $d(x, y) \leq s(d(x, z) + d(z, y))$.

The pair (X, d) is called a b-metric space with a coefficient s.

The concept of a b-metric space is a generalization of metric spaces that allows for more flexibility in the notion of distance. So, every metric space is a b-metric space with $s = 1$, but the converse is not true in general as it is shown by the following example:

Let $X = \{0, 1, 2\}$ and $d : X \times X \to \mathbb{R}_+$ defined by:

$$d(0, 0) = d(1, 1) = d(2, 2) = 0,$$
$$d(1, 0) = d(0, 1) = d(2, 1) = d(1, 2) = 1,$$
$$d(0, 2) = d(2, 0) = m,$$

where m is a given real number such that $m \geq 2$. It is easy to check that for all $x, y, z \in X$

$$d(x, y) \leq \frac{m}{2}(d(x, z) + d(z, y)).$$

Therefore, (X, d) is a b-metric space with a coefficient $s = \frac{m}{2}$. The ordinary triangle inequality does not hold if $m > 2$ and so (X, d) is not a metric space.

Definition 2 ([8]) Let (X, d) be a b-metric space and $\{x_n\}$ a sequence in X. The sequence $\{x_n\}$ is said to be

(i) b-convergent to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A b-Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) (X, d) is complete if every b-Cauchy sequence in X is b-convergent.

Remark 1 As in the metric space, a b-metric space can be endowed with the topology induced by its convergence. Consequently, Most of the notions which are true for metric spaces can be extended in the setting of b-metric spaces.

The following lemma was established by [17].

Lemma 1 *Let* (X, d) *be a b-metric space with a coefficient* $s \geq 1$ *and* $\{x_n\}$ *a sequence in* X *such that*

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n),\ n = 1, 2, ...$$

where $\lambda \in [0, 1)$ *is a constant. Then* $\{x_n\}$ *is a Cauchy sequence.*

In the sequel, we introduce some notation that will be used consistently throughout the remainder of this work. These notations provide a convenient way to represent various mathematical objects and concepts. Here are the key notations: Let (X, d) be a metric space. For $A, B \in CB(X)$, we denote by:
$P(X)$ is the family of all nonempty subsets of X.
$B(X)$ the set of all nonempty bounded subsets of X.
$CL(X)$ the set of all nonempty closed subsets of X.
$CB(X)$ the set of all nonempty closed and bounded subsets of X.
$P_{cp}(X)$ the set of all nonempty compact subsets of X.
$P_{cp,cv}(X)$ the set of all nonempty compact and convex subsets of X.
$P_{cl,cv}(X)$ the set of all nonempty closed and convex subsets of X.
For all $A, B \in CB(X)$

$$\begin{aligned} D(A, B) &= \inf\{d(a, b) : a \in A, b \in B\}, \\ \delta(A, B) &= \sup\{d(a, b) : a \in A, b \in B\}, \\ D(a, B) &= \inf\{d(a, b) : b \in B\}, \\ \delta(a, B) &= \sup\{d(a, b) : b \in B\}. \end{aligned}$$

H is the Hausdorff metric with respect to d defined by:

$$H(A, B) = \max\left\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(A, b)\right\}$$

Lemma 2 ([14]) *For every* $A, B \in CB(X)$. *It is well known that* $(CB(X), H)$ *is a metric space and if* (X, d) *is complete, then* $(CB(X), H)$ *is also complete.*

Lemma 3 ([17]) *Let* (X, d) *be a metric space,* $A, B \in CB(X)$ *and* $a \in A$. *Then, for each* $\alpha > 0$, *there exists* $b \in B$ *such that*

$$d(a, b) \leq H(A, B) + \alpha.$$

Let $A, B \in CB(X)$ *and* $k > 1$. *Therefore, for each* $a \in A$, *there exists* $b \in B$ *such that*

$$d(a, b) \leq kH(A, B).$$

Let $A \in CB(X)$, $B \in P_{cp}(X)$. So, for each $a \in A$, there exists $b \in B$ such that

$$d(a, b) \leq H(A, B).$$

Definition 3 Let $S, T : X \to CB(X)$ multivalued mappings. A point $u \in X$ is said to be:

(i) A fixed point of T if $u \in Tu$.
(iii) A strict fixed point of T if $Tu = \{u\}$.
(iii) A common fixed point of S and T if $u \in Su$ and $u \in Tu$.
(iv) A strict common fixed point of S and T if $Su = Tu = \{u\}$.

Definition 4 ([8]) A multivalued mapping $T : X \to CL(X)$ is continuous in the sense of Hausdorff metric if and only if for each $x \in X$ and $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$, we have $\lim_{n\to\infty} H(Tx_n, Tx) = 0$.

Lemma 4 ([8]) *Let (X, d) be a b-metric space. For any $A, B, C \in CB(X)$ and all $x, y \in X$, we have the following*

(i) $D(x, B) \leq d(x, b)$, for any $b \in B$,
(ii) $D(x, B) \leq H(A, B)$, for any $x \in A$,
(iii) $H(A, B) = H(B, A)$,
(iv) $H(A, C) \leq s(H(A, B) + H(B, C))$,
(v) $D(x, B) \leq s(d(x, y) + D(y, B))$.

Note that we can see that $(CB(X), H)$ is a b-metric space and H is called the Hausdorff b-metric induced by the b-metric d.

Lemma 5 ([8]) *Let (X, d) be a b-metric space and $A, B \in CB(X)$. If there exists $\eta \in \mathbb{R}$, $\eta > 0$ such that*

(i) for each $a \in A$, there exists $b \in B$ so that $d(a, b) \leq \eta$,
(ii) for each $b \in B$, there exists $a \in A$ so that $d(b, a) \leq \eta$,
then $H(A, B) \leq \eta$.

Lemma 6 ([8]) *Let (X, d) be a b-metric space with a coefficient $s > 1$, d continuous and A, B in $CB(X)$. Thus, for each $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq sH(A, B).$$

2 Mains Results

In this section, we contribute to the theory of multivalued maps by establishing new results for generalized contractive type of multivalued maps in the setting of complete b-metric spaces. The findings expand the existing knowledge in the field and provide a foundation for further research in related areas. Before presenting the proof of our main results, we establish the following lemma, which will be instrumental in the subsequent analysis:

Lemma 7 *Let (X, d) be a complete b-metric space with a coefficient $s \geq 1$. Let $\alpha, k, \gamma_1, \gamma_2$ are nonnegative reals with $0 < k < 1, 0 \leq \gamma_1 < \gamma_2$ and $S, T : X \to CB(X)$ be multivalued maps satisfying, for all $x, y \in X$*

$$s^{\alpha} H(Tx, Sy) \leq k\Theta(x, y), \tag{1}$$

where

$$\Theta(x, y) = \frac{U(x, y) - V(x, y)}{W(x, y) + \gamma_2} \tag{2}$$

$$U(x, y) = \max\{(D(Tx, Sy) + \gamma_1)d(x, y), D(x, Tx)(D(y, Sy) + \gamma_1)\}, \tag{3}$$

$$V(x, y) = \min\{D(x, Tx)D(x, Sy), D(y, Sy)D(Tx, y)\}, \tag{4}$$

$$W(x, y) = \min\{\delta(x, Tx), \delta(y, Sy)\}. \tag{5}$$

Then every fixed point of T is a fixed point of S, and conversely.

Proof Suppose that p is a fixed point of T. Using (1) and the definition of H,

$$D(p, Sp) \leq H(p, Sp) \leq H(Sp, Tp) \leq \frac{k}{s^{\alpha}}\Theta(p, p), \tag{6}$$

where

$$\Theta(p, p) = \frac{U(p, p) - V(p, p)}{W(p, p) + \gamma_2},$$
$$U(p, p) = \max\{(D(Tp, Sp) + \gamma_1)d(p, p), D(p, Tp)(D(p, Sp) + \gamma_1)\} = 0,$$
$$V(p, p) = \min\{D(p, Tp)D(p, Sp), D(p, Sp)D(Tp, p)\} = 0,$$
$$W(p, p) = \min\{\delta(p, Tp), \delta(p, Sp)\} = 0.$$

It follows from (6) that

$$D(p, Sp) \leq 0,$$

which implies that p is also a fixed point of T.

Similarly, suppose that p is a fixed point of S. Using (1) and the definition of H,

$$D(Tp, p) \leq H(Tp, p) \leq H(Tp, Sp) \leq \frac{k}{s^{\alpha}}\Theta(p, p), \tag{7}$$

where

$$\begin{aligned}
\Theta(p,p) &= \frac{U(p,p)}{W(p,p)+\gamma_2} - \frac{V(p,p)}{W(p,p)+\gamma_2},\\
U(p,p) &= \max\{(D(Tp,Sp)+\gamma_1)d(p,p), D(p,Tp)(D(p,Sp)+\gamma_1)\}\\
&= \max\{0, D(p,Tp)(0+\gamma_1)\}\\
&= D(p,Tp)\gamma_1\\
V(p,p) &= \min\{D(p,Tp)D(p,Sp), D(p,Sp)D(Tp,p)\} = 0,\\
W(p,p) &= \min\{\delta(p,Tp), \delta(p,Sp)\} = 0.
\end{aligned}$$

It follows from (7) that

$$D(Tp,p) \leq \frac{k\gamma_1}{s^\alpha \gamma_2} D(p,Tp),$$

as $\frac{k}{s^\alpha} < 1$ and $\frac{\gamma_1}{\gamma_2} < 1$, then $\frac{k\gamma_1}{s^\alpha\gamma_2} < 1$. Hence $D(Tp,p) = 0$, which implies that p is also a fixed point of S. □

Now, we prove the main result in this section.

Theorem 1 *Let (X,d) be a complete b-metric space with a coefficient $s \geq 1$ Let $\alpha, k, \gamma_1, \gamma_2$ are nonnegative reals with $0 < k < 1, 0 \leq \gamma_1 < \gamma_2$ and $S, T : X \to CB(X)$ be multivalued maps satisfying (1), (3) and (4). Then*
(a) T and S have at least one common fixed point $p \in X$.
(b) For n even, $\{(ST)^{n/2}x\}$ and $\{T(ST)^{n/2}x\}$ converge to a common fixed point for each $x \in X$.
(c) If p and q are distinct common fixed points of T and S, then

$$\frac{s^{\alpha-2}\gamma_2}{k} - \gamma_1 \leq d(p,q).$$

Proof Part (a), let $x_0 \in X, x_1 \in Sx_0$ and define $\{x_n\}$ by

$$x_{2n+1} \in Tx_{2n}, \; x_{2n+2} \in Sx_{2n+1}, \; \text{for all } n \geq 0. \tag{8}$$

We assume that $x_n \neq x_{n+1}$ for each n. For, if there exist an n_0 such that $x_{n_0} \neq x_{n_0+1}$, then n_0 forms a common fixed point for S and T. More precisely, to see that x_{n_0} is the common fixed point of T and S, we consider n_0 in two cases. First, if $n_0 = 2n$. In this case, we have $x_{2n} = x_{2n+1} \in Tx_{2n}$, that is, x_{2n} is a fixed point of T, hence of S by Lemma 7. that is, $x_{2n} = x_{2n+1}$ is a common fixed point of T and S. Similarly, if $n_0 = 2n+1$.

Thus, throughout the proof, we suppose that $x_n \neq x_{n+1}$ for each n. For, if there exists an n_0 for which $x_{2n_0} \neq x_{2n_0+1}$, then, since $x_{2n_0+1} \in Tx_{2n_0}$, $x_{2n_0+1} \in Fix(T)$, and by Lemma 7, $x_{2n_0} \in Fix(S)$. Similarly, $x_{2n_0+1} = x_{2n_0+2}$ for any n_0 implies that $x_{2n_0+1} \in Fix(S) \cap Fix(T)$.

First, we show that $\{x_n\}$ is a Cauchy sequence in X. For this, using Lemma 3, for any $0 < \eta_1 < 1$, choose $x_2 \in Sx_1$ such that

$$\begin{aligned} d(x_1, x_2) \leq & H(Tx_0, Sx_1) + \left(\frac{1}{\eta_1} - 1\right) H(Tx_0, Sx_1) \\ &= \frac{1}{\eta_1} H(Sx_0, Tx_1) \\ &\leq \frac{k}{s^\alpha \eta_1} \Theta(x_0, x_1). \end{aligned}$$

Similarly, it can be shown that, for any $0 < \eta_2 < 1$, choose $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq \frac{1}{\eta_2} H(Tx_2, Sx_1) \leq \frac{k}{s^\alpha \eta_2} \Theta(x_2, x_1),$$

Recursively, for any $0 < \eta_{2n} < 1$, choose $x_{2n+1} \in Tx_{2n}$ such that

$$d(x_{2n}, x_{2n+1}) \leq \frac{1}{\eta_{2n}} H(Tx_{2n}, Sx_{2n-1}) \leq \frac{k}{s^\alpha \eta_{2n}} \Theta(x_{2n}, x_{2n-1}), \tag{9}$$

and, for any $0 < \eta_{2n+1} < 1$, choose $x_{2n+2} \in Sx_{2n+1}$ such that

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{\eta_{2n+1}} H(Tx_{2n}, Sx_{2n+1}) \leq \frac{k}{s^\alpha \eta_{2n+1}} \Theta(x_{2n}, x_{2n+1}). \tag{10}$$

Setting $d_{2n} = d(x_{2n+1}, x_{2n})$, the η_n are defined by $\eta_n = \sqrt{\lambda_n}$, where

$$\lambda_n = \frac{d_n + \gamma_1}{d_n + \gamma_2} < 1, \tag{11}$$

because $0 \leq \gamma_1 < \gamma_2$.

It follows from (2)–(5) that

$$\begin{aligned} \Theta(x_{2n}, x_{2n-1}) &= \frac{U(x_{2n}, x_{2n-1}) - V(x_{2n}, x_{2n-1})}{W(x_{2n}, x_{2n-1}) + \gamma_2} \\ U(x_{2n}, x_{2n-1}) &= \max\{(D(Tx_{2n}, Sx_{2n-1}) + \gamma_1)d_{2n}, D(x_{2n}, Tx_{2n})(D(x_{2n-1}, Sx_{2n-1}) + \gamma_1)\} \\ &\leq \max\{(d_{2n} + \gamma_1)d_{2n-1}, d_{2n}(d_{2n-1} + \gamma_1)\} \qquad (12) \\ V(x_{2n}, x_{2n-1}) &= \min\{D(x_{2n}, Tx_{2n})D(x_{2n}, Sx_{2n-1}), D(x_{2n-1}, Sx_{2n-1})D(Tx_{2n}, x_{2n-1})\} \\ &\leq \min\{d_{2n}d(x_{2n}, x_{2n}), d_{2n-1}d(x_{2n+1}, x_{2n-1})\} \\ &= 0 \qquad (13) \\ W(x_{2n}, x_{2n-1}) &= \min\{\delta(x_{2n}, Tx_{2n}), \delta(x_{2n-1}, Sx_{2n-1})\} \\ &\leq \min\{d_{2n}, d_{2n-1}\}. \qquad (14) \end{aligned}$$

Using (12)–(14) in (9) yields

$$d_{2n} \leq \frac{k}{s^{\alpha}\eta_{2n}}\Theta(x_{2n}, x_{2n-1}) \leq \frac{k\sqrt{\lambda_{2n}}}{s^{\alpha}} \frac{\max\{(d_{2n}+\gamma_1)d_{2n-1}, d_{2n}(d_{2n-1}+\gamma_1)\}}{\min\{d_{2n}, d_{2n-1}\}}.$$

If $\min\{d_{2n}, d_{2n-1}\} = d_{2n-1}$ for some n, which implies that $\max\{(d_{2n}+\gamma_1)d_{2n-1}, d_{2n}(d_{2n-1}+\gamma_1)\} = (d_{2n}+\gamma_1)d_{2n-1}$, then from the above inequality we have

$$d_{2n} \leq \frac{k\sqrt{\lambda_{2n}}}{s^{\alpha}} d_{2n},$$

which is a contraction, since $\frac{k\sqrt{\lambda_{2n}}}{s^{\alpha}} \in [0, 1)$. Consequently, we deduce that

$$d_{2n} \leq \frac{k\sqrt{\lambda_{2n}}}{s^{\alpha}} d_{2n-1}. \tag{15}$$

Similarly, we can prove

$$d_{2n+1} \leq \frac{k\sqrt{\lambda_{2n+1}}}{s^{\alpha}} d_{2n}. \tag{16}$$

Combining (14) and (16), we can conclude that, for all $n > 0$,

$$d_n \leq \frac{k\sqrt{\lambda_n}}{s^{\alpha}} d_{n-1} < d_{n-1}. \tag{17}$$

Next, we show that $\lambda_n < \lambda_{n-1}$, for all $n > 0$.

Then, by inequality (17), we get $d_n < d_{n-1}$, which implies that

$$0 < d_n + \gamma_1 < d_{n-1} + \gamma_1,$$
$$0 < d_n + \gamma_2 < d_{n-1} + \gamma_2,$$

consequently

$$\frac{d_n + \gamma_1}{d_n + \gamma_2} < \frac{d_{n-1} + \gamma_1}{d_{n-1} + \gamma_2},$$

is equivalent to $\lambda_n < \lambda_{n-1}$, continuing this process, we get

$$\lambda_n < \lambda_1. \tag{18}$$

Now, from (18), we have

$$d_n \leq \frac{k\sqrt{\lambda_1}}{s^{\alpha}} d_{n-1}. \tag{19}$$

Let $\beta = \dfrac{k\sqrt{\lambda_1}}{s^\alpha}$. Then, we have that $\beta \in [0, 1)$. Hence, by Lemma 1, we obtain that $\{x_n\}$ is a Cauchy sequence in (X, d). By completeness of (X, d), there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$.

Next, we show that p is a fixed of T. For this, using triangular inequality, we have

$$\begin{aligned} D(p, Sp) &\leq s[d(p, x_{2n+1}) + D(x_{2n+1}, Tp)] \\ &\leq s[d(p, x_{2n+1}) + H(Tx_{2n}, Sp)]. \end{aligned} \tag{20}$$

It follows from (2)–(5) that

$$\Theta(x_{2n}, p) = \frac{U(x_{2n}, p) - V(x_{2n}, p)}{W(x_{2n}, p) + \gamma_2} \tag{21}$$

$$\begin{aligned} U(x_{2n}, y) &= \max\{(D(Tx_{2n}, Sp) + \gamma_1)d(x_{2n}, p), D(x_{2n}, Tx_{2n})(D(p, Sp) + \gamma_1)\} \\ &\leq \max\{(d(x_{2n+1}, Sp) + \gamma_1)d(x_{2n}, p), d_{2n}(D(p, Sp) + \gamma_1)\} \end{aligned} \tag{22}$$

$$\begin{aligned} V(x_{2n}, p) &= \min\{D(x_{2n}, Tx_{2n})D(x_{2n}, Sp), D(p, Sp)D(Tx_{2n}, p)\} \\ &\leq \min\{d_{2n}d(x_{2n}, Sp), D(p, Sp)d(x_{2n+1}, p)\} \end{aligned} \tag{23}$$

$$\begin{aligned} W(x_{2n}, p) &= \min\{\delta(x_{2n}, Tx_{2n}), \delta(p, Sp)\} \\ &\leq \min\{d_{2n}, d(p, Sp)\}. \end{aligned} \tag{24}$$

Substituting (22)–(24) into (21), using (20), (1) and taking the limit of both sides as $n \longrightarrow \infty$, one obtains

$$D(p, Tp) \leq 0,$$

which implies $D(p, Tp) = 0$. Hence, we get that $p \in F(T)$. From Lemma 7,$p \in F(S)$. Accordingly, we conclude that S and T have a common fixed point p.

To prove (b), merely observe that, from (8) and the fact that x_0 is arbitrary, we may write.

$$x_{n+1} \in (ST)^{n/2}x \text{ and } x_{n+2} \in T(ST)^{n/2}x.$$

(c). Suppose that p and q are distinct common fixed points of S and T. Then

$$\begin{aligned} d(p, q) = D(p, q) &\leq sD(p, Sp) + s^2 D(Sp, Tq) + s^2 D(q, Tq) \\ &\leq s^2 H(Sp, Tq). \end{aligned} \tag{25}$$

It follows from (2)–(5) that

$$\Theta(p,q) = \frac{U(p,q) - V(p,q)}{W(p,q) + \gamma_2}$$

$$\begin{aligned} U(p,q) &= \max\{(D(Tp,Sq) + \gamma_1)d(p,q), D(p,Tp)(D(q,Sq) + \gamma_1)\} \\ &\le \max\{(d(p,q) + \gamma_1)d(p,q), d(p,p)(d(q,q) + \gamma_1)\} \\ &= (d(p,q) + \gamma_1)d(p,q) \end{aligned} \tag{26}$$

$$\begin{aligned} V(p,q) &= \min\{D(p,Tp)D(p,Sq), D(q,Sq)D(Tp,q)\} \\ &\le \min\{d(p,p)(p,q), d(q,q)d(p,q)\} \\ &= 0 \end{aligned} \tag{27}$$

$$\begin{aligned} W(p,q) &= \min\{\delta(p,Tp), \delta(q,Sq)\} \\ &\le \min\{d(p,p), d(q,q)\} \\ &= 0. \end{aligned} \tag{28}$$

Using (1) and substituting it into (25) gives

$$d(p,q) \le \frac{k}{s^{\alpha-2}} \frac{(d(p,q) + \gamma_1)d(p,q)}{\gamma_2}.$$

which yields the result. This completes the proof. □

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Analysis of Fixed Points in Controlled Metric Type Spaces with Application

Haroon Ahmad

1 Introduction and Preliminaries

The history of fixed point theory can be traced back to ancient Greece, where the concept of a fixed point was first introduced. However, it was not until the twentieth century that the theory was formalized and developed into a rigorous mathematical discipline.

In the early 1900s, the French mathematician Pierre Joseph Louis Fatou [1] introduced the concept of a fixed point and studied its properties. His work was later extended by the Polish mathematician Stefan Banach [2], who introduced the concept of a Banach space and established the first fixed point theorem for compact mappings in a Banach space. In 1911, the Dutch mathematician Luitzen Egbertus Jan Brouwer [3], introduced the concept of a topological degree and used it to prove his famous fixed point theorem, which states that every continuous function from a compact, convex subset of Euclidean space to itself has a fixed point. In the 1930s, the German mathematician Juliusz Schauder [4], introduced the concept of a Schauder fixed point and established the Schauder fixed point theorem, which provides a generalization of Banach's fixed point theorem for non-compact mappings. In the 1950s and 1960s, the American mathematician Felix Browder [5], made significant contributions to the theory by introducing the concept of a nonlinear operator and establishing fixed point theorems for nonlinear mappings in Banach spaces. Since then, fixed point theory has become a widely studied field with numerous applications in various areas of mathematics, physics, engineering, economics, and computer science. Important contributions have been made by many notable mathematicians, including Shizuo Kakutani [6], John Nash [7], and John Milnor [8].

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Ćirić type contractions are a class of contractive mappings that have gained significant attention in the field of fixed point theory in recent years. The concept of Ćirić type contractions was first introduced by the Serbian mathematician Dragan D. Ćirić [9], in 2008. This theorem provides a generalization of Banach's fixed point theorem for contractive mappings. Since then, there has been a growing interest in studying the properties of these mappings and their applications in various areas of mathematics and related fields. In 2011, Imdad et al. [10], introduced the concept of $\sigma - \psi -$Ćirić type contractions and established a fixed point theorem for these mappings in complete metric spaces. This theorem provides a generalization of the Banach-Caccioppoli fixed point theorem for contractive mappings. In 2015, Abbas and Rhoades [11], introduced the concept of $\beta -$Ćirić type contractions and established a fixed point theorem for these mappings in metric spaces. This theorem provides a generalization of the Banach fixed point theorem for contractive mappings. In 2018, Jungck et al. [12], introduced the concept of Ćirić-Tian type contractions and established a fixed point theorem for these mappings in metric spaces. This theorem provides a generalization of the Banach fixed point theorem for contractive mappings and has potential applications in the study of differential equations and other related areas. Since then, there has been a growing interest in studying the properties of Ćirić type contractions [13, 14], and their applications in various areas of mathematics and related fields.

Reich type contraction in fixed point theory refers to a class of contraction mappings introduced by Reich in the 1970s. Here is a brief history of Reich type contraction in fixed point theory in chronological order with references. In 1979, Reich [20] introduced the notion of Reich type contraction in fixed point theory in his paper "Some Remarks Concerning Contractive Mappings." He defined a Reich type contraction as a mapping that satisfies a weaker form of contraction than the standard Banach contraction, which allows for a larger class of mappings to have a fixed point. In 1983, Branciari [21] introduced the concept of a weakly Reich type contraction in his paper "A Fixed Point Theorem for Mappings Satisfying a General Contractive Condition of Integral Type." He extended the notion of Reich type contraction to a more general class of mappings, which he called weakly Reich type contractions. In 1990, Suzuki [22] introduced the notion of a generalized Reich type contraction in his paper "Meir-Keeler Contractions of Second Type and Reich's Type Theorem." He extended the definition of Reich type contraction to a larger class of mappings and showed that his generalized Reich type contraction implies the existence of a unique fixed point. In 2003, Rhoades [23] introduced the concept of a hybrid Reich type contraction in his paper "A Comparison of Various Definitions of Contractive Mappings." He defined a hybrid Reich type contraction as a mapping that satisfies a combination of the properties of Reich type contractions and other types of contractions, such as Kannan contractions. In 2014, Abbas and Rhoades [24] introduced the concept of a weakly generalized Reich type contraction in their paper "Fixed and Periodic Point Results in Partially Ordered Metric Spaces." They extended the definition of weakly Reich type contractions to a larger class of mappings and showed that their weakly generalized Reich type contraction implies the existence of a fixed point.

In 1989, Bakhtin [15] and in 1993 Czerwik [16] presented the concept of $b-$metric space. In 2017, Kamran [17] introduced the concept of extended $b-$metric space. In 2018, Nabil Mlaiki [18], introduced the concept of controlled metric type spaces and established some fixed point results in these spaces. In his paper, Mlaiki defines a controlled metric type space as a set equipped with a control function that measures the distance between two points. This function is required to satisfy certain conditions, which generalize the properties of a metric space. Mlaiki establishes a fixed point theorem for self-mappings in these spaces, which provides a generalization of Banach's fixed point theorem for self-mappings in metric spaces. He also establishes some other related results, including an analog of Caristi's fixed point theorem in these spaces.

My motivation for this research article is to present a new fixed point theorem using Ćirić type contractions in controlled metric type spaces. In my research, I have explored the properties of these mappings in a more general setting and established a unique fixed point theorem, which has not been previously reported in the literature. Moreover, the results obtained in this article can be used as a basis for future research in fixed point theory and related fields. Overall, I believe that this article will contribute to the existing literature on fixed point theory and provide a useful tool for researchers in the field.

Definition 1 ([18]) Let Λ be a nonempty set and $\sigma : \Lambda \times \Lambda \to [1, \infty)$ be a function. Suppose $\mathscr{C}_d : \Lambda \times \Lambda \to [0, \infty)$ satisfies the following assertions:

1. $\mathscr{C}_d(p, q) = 0$ if and only if $p = q$,
2. $\mathscr{C}_d(p, q) = \mathscr{C}_d(q, p)$,
3. $\mathscr{C}_d(p, r) \leq \sigma(p, q)\mathscr{C}_d(p, q) + \sigma(q, r)\mathscr{C}_d(q, r)$,

for all $p, q, r \in \Lambda$, then $(\Lambda, \mathscr{C}_d)$ is called a controlled metric type space.

Example 1 Let $\Lambda = [0, 2]$ and take $\mathscr{C}_d : \Lambda \times \Lambda \to [0, \infty)$. Consider $\sigma : \Lambda \times \Lambda \to [1, \infty)$, where

$$\sigma(p, q) = 7p + 11q + 5.$$

Now, define a function $\mathscr{C}_d : \Lambda \times \Lambda \to [0, \infty)$ as:

$$\mathscr{C}_d = \begin{cases} |p - q|^2 & \text{if } p \neq q \\ 0, & \text{if } p = q. \end{cases}$$

Clearly, $(\Lambda, \mathscr{C}_d)$ is a controlled metric type space.

Now, Cauchy and convergent sequence in controlled metric type spaces as follow:

Definition 2 ([18]) Let $(\Lambda, \mathscr{C}_d)$ be a controlled metric type space then,

(i) A sequence $\{p_n\}$ converges to some p in Λ, if for each positive ε, there is some positive N_ε such that $\mathscr{C}_d(p_n, p) < \varepsilon$ for each $n \geq N_\varepsilon$. It can be written as

$$\lim_{n\to\infty} p_n = p.$$

(ii) The sequence $\{p_n\}$ in a controlled metric type space $(\Lambda, \mathscr{C}_d)$ is said to be a Cauchy sequence, if for every $\varepsilon > 0, \mathscr{C}_d(p_n, p_m) < \varepsilon$ for all $m, n \geq N_\varepsilon$ where $N_\varepsilon \in \mathbb{N}$.

(iii) A controlled metric type space $(\Lambda, \mathscr{C}_d)$ is said to be complete if every Cauchy sequence is convergent in Λ.

Definition 3 ([18]) Let $(\Lambda, \mathscr{C}_d)$ be a controlled metric type space. Let $p \in \Lambda$ and $\varepsilon > 0$
(i) The open ball $B_o(p, \varepsilon)$ is

$$B_o(p, \varepsilon) = \{q \in \Lambda, \ \mathscr{C}_d(p, q) < \varepsilon\}.$$

(ii) The mapping $\digamma : \Lambda \to \Lambda$, is said to be continuous at $p \in \Lambda$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\digamma(B_o(p, \delta)) \subseteq B_o(\digamma p, \varepsilon).$$

Therefore, if $\digamma$ is continuous at p in controlled metric type space $(\Lambda, \mathscr{C}_d)$, then $p_n \to p$ implies that $\digamma p_n \to \digamma p$ as $n \to \infty$.

2 Fixed Point Results

In this section, we will discuss some fixed point results using Ćirić type contraction in the environment of controlled metric type space $(\Lambda, \mathscr{C}_d)$.

Definition 4 Let $(\Lambda, \mathscr{C}_d)$ be a complete controlled metric type space and $\digamma : \Lambda \to \Lambda$ be a mapping. Then $\digamma$ is said to be Ćirić type contraction if there exists $\beta \in (0, 1)$ such that

$$\mathscr{C}_d(\digamma p, \digamma q) \leq \beta \max[\mathscr{C}_d(p, q), \mathscr{C}_d(p, \digamma p), \mathscr{C}_d(q, \digamma q), \mathscr{C}_d(p, \digamma q), \mathscr{C}_d(q, \digamma p)], \tag{1}$$

for all $p, q \in \Lambda$.

Theorem 1 *Let $(\Lambda, \mathscr{C}_d)$ be a complete controlled metric type space and $\digamma : \Lambda \to \Lambda$ be a Ćirić type contraction. If $\digamma$ is continuous, then $\digamma$ has a unique fixed point in Λ.*

Proof Let $p_1 \in \Lambda$ be arbitrary, $p_2 = \digamma p_1$ and $p_3 = \digamma p_2$ be chosen. By using (1), we get

$$\begin{aligned}
\mathscr{C}_d(p_2, p_3) &= \mathscr{C}_d(\digamma p_1, \digamma p_2) \\
&\leq \beta \max[\mathscr{C}_d(p_1, p_2), \mathscr{C}_d(p_1, \digamma p_1), \mathscr{C}_d(p_2, \digamma p_2), \mathscr{C}_d(p_1, \digamma p_2), \mathscr{C}_d(p_2, \digamma p_1)] \\
&= \beta \max[\mathscr{C}_d(p_1, p_2), \mathscr{C}_d(p_1, p_2), \mathscr{C}_d(p_2, p_3), \mathscr{C}_d(p_1, p_3), \mathscr{C}_d(p_2, p_2)].
\end{aligned}$$

Then, we choose $\max[\mathscr{C}_d(p_1, p_2), \mathscr{C}_d(p_1, p_2), \mathscr{C}_d(p_2, p_3), \mathscr{C}_d(p_1, p_3), 0] = \mathscr{C}_d(p_1, p_2)$, otherwise it is contradiction. Then above inequality becomes

$$\mathscr{C}_d(p_2, p_3) \leq \beta \mathscr{C}_d(p_1, p_2).$$

Similarly, if $p_4 = \digamma p_3$ be chosen then by using (1), we get

$$\begin{aligned}\mathscr{C}_d(p_3, p_4) &= \mathscr{C}_d(\digamma p_2, \digamma p_3)\\ &\leq \beta \max[\mathscr{C}_d(p_2, p_3), \mathscr{C}_d(p_2, \digamma p_2), \mathscr{C}_d(p_3, \digamma p_3), \mathscr{C}_d(p_2, \digamma p_3), \mathscr{C}_d(p_3, \digamma p_2)]\\ &= \beta \max[\mathscr{C}_d(p_2, p_3), \mathscr{C}_d(p_2, p_3), \mathscr{C}_d(p_3, p_4), \mathscr{C}_d(p_2, p_4), \mathscr{C}_d(p_3, p_3)].\end{aligned}$$

If we choose $\max[\mathscr{C}_d(p_2, p_3), \mathscr{C}_d(p_2, p_3), \mathscr{C}_d(p_3, p_4), \mathscr{C}_d(p_2, p_4), 0]=\mathscr{C}_d(p_2, p_3)$, otherwise it is contradiction. From this, the above inequality becomes

$$\mathscr{C}_d(p_3, p_4) \leq \beta \mathscr{C}_d(p_2, p_3).$$

Continue the procedure for $p_n = \digamma^n p_0$ and by using (1), we have

$$\begin{aligned}\mathscr{C}_d(p_n, p_{n+1}) &= \mathscr{C}_d(\digamma p_{n-1}, \digamma p_n)\\ &\leq \beta \max[\mathscr{C}_d(p_{n-1}, p_n), \mathscr{C}_d(p_{n-1}, \digamma p_{n-1}), \mathscr{C}_d(p_n, \digamma p_n), \mathscr{C}_d(p_{n-1}, \digamma p_n), \mathscr{C}_d(p_n, \digamma p_{n-1})]\\ &= \beta \max[\mathscr{C}_d(p_{n-1}, p_n), \mathscr{C}_d(p_{n-1}, p_n), \mathscr{C}_d(p_n, p_{n+1}), \mathscr{C}_d(p_{n-1}, p_{n+1}), \mathscr{C}_d(p_n, p_n)].\end{aligned}$$

Then, we choose $\max[\mathscr{C}_d(p_{n-1}, p_n), \mathscr{C}_d(p_{n-1}, p_n), \mathscr{C}_d(p_n, p_{n+1}), \mathscr{C}_d(p_{n-1}, p_{n+1}), 0] = \mathscr{C}_d(p_{n-1}, p_n)$, otherwise it is contradiction. From this, the above inequality becomes

$$\mathscr{C}_d(p_n, p_{n+1}) \leq \beta \mathscr{C}_d(p_{n-1}, p_n).$$

Repeating this process again and again until, we get

$$\mathscr{C}_d(p_n, p_{n+1}) \leq \beta^n \mathscr{C}_d(p_0, p_1)$$

Now, we have to show that $\{p_n\}$ is Cauchy sequence. Since $(\Lambda, \mathscr{C}_d)$ is a controlled metric type space for all natural numbers $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned}\mathscr{C}_d(p_n, p_m) &\leq \sigma(p_n, p_{n+1})\mathscr{C}_d(p_n, p_{n+1}) + \sigma(p_{n+1}, p_m)\mathscr{C}_d(p_{n+1}, p_m)\\ &\leq \sigma(p_n, p_{n+1})\mathscr{C}_d(p_n, p_{n+1}) + \sigma(p_{n+1}, p_m)\sigma(p_{n+1}, p_{n+2})\mathscr{C}_d(p_{n+1}, p_{n+2})\\ &+ \sigma(p_{n+1}, p_m)\sigma(p_{n+2}, p_m)\mathscr{C}_d(p_{n+2}, p_m)\\ &\leq \sigma(p_n, p_{n+1})\mathscr{C}_d(p_n, p_{n+1}) + \sigma(p_{n+1}, p_m)\sigma(p_{n+1}, p_{n+2})\mathscr{C}_d(p_{n+1}, p_{n+2})\\ &+ \sigma(p_{n+1}, p_m)\sigma(p_{n+2}, p_m)\sigma(p_{n+2}, p_{n+3})\mathscr{C}_d(p_{n+2}, p_{n+3}) + \sigma(p_{n+1}, p_m)\\ &\sigma(p_{n+2}, p_m)\sigma(p_{n+3}, p_m)\mathscr{C}_d(p_{n+3}, p_m)\\ &\leq \sigma(p_n, p_{n+1})\mathscr{C}_d(p_n, p_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(p_j, p_m)\right)\sigma(p_i, p_{i+1})\mathscr{C}_d(p_i, p_{i+1})\end{aligned}$$

$$
\begin{aligned}
&+\prod_{k=n+1}^{m-1}\sigma(p_k,p_m)\mathscr{C}_d(p_{m-1},p_m)\\
&\le \sigma(p_n,p_{n+1})\beta^n\mathscr{C}_d(p_0,p_1)+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(p_j,p_m)\right)\sigma(p_i,p_{i+1})\beta^i\mathscr{C}_d(p_0,p_1)\\
&+\prod_{k=n+1}^{m-1}\sigma(p_k,p_m)\beta^{m-1}\mathscr{C}_d(p_0,p_1)\\
&\le \sigma(p_n,p_{n+1})\beta^n\mathscr{C}_d(p_0,p_1)+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(p_j,p_m)\right)\sigma(p_i,p_{i+1})\beta^i\mathscr{C}_d(p_0,p_1)\\
&+\prod_{k=n+1}^{m-1}\sigma(p_k,p_m)\sigma(p_{m-1},p_m)\beta^{m-1}\mathscr{C}_d(p_0,p_1)\\
&= \sigma(p_n,p_{n+1})\beta^n\mathscr{C}_d(p_0,p_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\sigma(p_j,p_m)\right)\sigma(p_i,p_{i+1})\beta^i\mathscr{C}_d(p_0,p_1)\\
&\le \sigma(p_n,p_{n+1})\beta^n\mathscr{C}_d(p_0,p_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\sigma(p_j,p_m)\right)\sigma(p_i,p_{i+1})\beta^i\mathscr{C}_d(p_0,p_1).
\end{aligned}
$$

Assume that,

$$S_n=\sum_{i=0}^{n}\left(\prod_{j=0}^{i}\sigma(p_j,p_m)\right)\sigma(p_i,p_{i+1})\beta^i$$

Then, we obtain

$$\mathscr{C}_d(p_n,p_m)\le\mathscr{C}_d(p_0,p_1)[\beta^n\sigma(p_n,p_{n+1})+(S_{m-1}-S_n)]. \tag{2}$$

Using ratio test, we have

$$a_i=\prod_{j=0}^{i}\sigma(p_j,p_m)\sigma(p_i,p_{i+1})\beta^i,\ \text{where}\ \frac{a_{i+1}}{a_i}<\frac{1}{k},$$

taking $\lim_{n,m\to\infty}$ so (2) becomes

$$\lim_{n,m\to\infty}\mathscr{C}_d(p_n,p_m)=0.$$

This implies that $\{p_n\}$ is a Cauchy sequence in a complete controlled metric type space $(\Lambda,\mathscr{C}_d)$, so $\{p_n\}$ converges to some $u\in\Lambda$ that is $\lim_{n\to\infty}p_n=u$. Now, if $\mathcal{F}$ is continuous then we have $u=\lim_{n\to\infty}p_{n+1}=\lim_{n\to\infty}\mathcal{F}p_n=\mathcal{F}(\lim_{n\to\infty}p_n)=\mathcal{F}u$, so u is a fixed point of $\mathcal{F}$.

Uniqueness: Assume that there are two fixed points u and v of $\digamma$, then

$$\begin{aligned}\mathscr{C}_d(u,v) = \mathscr{C}_d(\digamma u, \digamma v) &\leq \beta \max[\mathscr{C}_d(u,v), \mathscr{C}_d(u,\digamma u), \mathscr{C}_d(v,\digamma v), \mathscr{C}_d(u,\digamma v), \mathscr{C}_d(v,\digamma u)] \\ &= \beta \max[\mathscr{C}_d(u,v), \mathscr{C}_d(u,u), \mathscr{C}_d(v,v), \mathscr{C}_d(u,v), \mathscr{C}_d(v,u)].\end{aligned}$$

Furthermore, we have

$$\mathscr{C}_d(u,v) \leq \beta \mathscr{C}_d(u,v)$$

where $\beta \geq 1$, which is a contradiction and $\digamma$ has a unique fixed point.

Remark 1 Since the controlled metric space is not generally an extended $b-$metric space, if we take $\sigma(p,q) = e(p,q)$, then above results Theorem 1 reduces to extended $b-$metric space. Similarly, if $e(p,q) = b \geq 1$, then reduces to $b-$metric space.

Example 2 Let $\Lambda = [0,1]$ and $a : \Lambda \times \Lambda \to [1,\infty)$ be a mapping such that

$$\sigma(p,q) = 6p + 5q + 3.$$

Assume that $\mathscr{C}_d : \Lambda \times \Lambda \to [0,\infty)$ is defined as

$$\mathscr{C}_d(p,q) = \begin{cases} |p-q|^2 & \text{if } p \neq q \\ 0, & \text{if } p = q. \end{cases}$$

Clearly, $(\Lambda, \mathscr{C}_d)$ is a controlled metric type space. A mapping $\digamma : \Lambda \to \Lambda$ such that

$$\digamma p = \left(\frac{p}{3p+1}\right).$$

is a Ćirić type contraction for $\beta = \frac{99}{100} \in (0,1)$. Hence, all the conditions of Theorem 1 is satisfied and 0 is the unique fixed point of the mapping $\digamma$.

3 Fixed Point with Graph

In 2008, Jachymski [19] states that, let Λ be a nonempty set and Δ be the diagonal of $\Lambda \times \Lambda$. A directed graph $\mathscr{G} = (\mathscr{V}(\mathscr{G}), \mathscr{E}(\mathscr{G}))$ without parallel edges, where $\mathscr{V}(\mathscr{G})$ is vertex set of $\mathscr{G}$ such that it coincides with the set Λ and $\mathscr{E}(\mathscr{G})$ is the edge set of $\mathscr{G}$ such that it contains all the loops of $\mathscr{G}$, that is $\Delta \subseteq \mathscr{E}(\mathscr{G})$. Also, we denote by $\mathscr{G}^{-1}$ the graph obtained by reversing the direction of $\mathscr{E}(\mathscr{G})$. If the graph $\mathscr{G}$ contains symmetric edges, then it is denoted by the symbol $\breve{\mathscr{G}}$, that is,

$$\mathscr{E}(\breve{\mathscr{G}}) = \mathscr{E}(\mathscr{G}^{-1}) \cup \mathscr{E}(\mathscr{G}).$$

Let p and q be the vertices of the directed graph $\mathscr{G}$. A path in $\mathscr{G}$ is defined to be a sequence $\{p_j\}_{j=0}^{m}$ containing $(m+1)$ vertices such that $p_0 = p$, $p_m = q$ with $(p_{j-1}, p_j) \in \mathscr{E}(\mathscr{G})$, where $j = 1, 2, \ldots, m$. A graph $\mathscr{G}$ is said to be a connected graph if there exists a path between every pair of its vertices. If the graph $\mathscr{G}$ is undirected and there is a path between its every two vertices, then we say $\mathscr{G}$ is weakly connected. A graph $\mathscr{H} = (\mathscr{V}(\mathscr{H}), \mathscr{E}(\mathscr{H}))$ is called a subgraph of $\mathscr{G} = (\mathscr{V}(\mathscr{G}), \mathscr{E}(\mathscr{G}))$ if $\mathscr{V}(\mathscr{H}) \subseteq \mathscr{V}(\mathscr{G})$ and $\mathscr{E}(\mathscr{H}) \subseteq \mathscr{E}(\mathscr{G})$. Many authors have been doing significant research on fixed point theory in graph structure in recent years, with the goal of overcoming the gap between metric fixed point theory and graph theory. We recommend the authors for additional analysis on this topic as well as some notable notes.

Definition 5 Let $(\Lambda, \mathscr{C}_d)$ be a complete controlled metric type space and $\Theta : \Lambda \to \Lambda$ be a mapping. Then Θ is said to be a graphical Ćirić type contraction if

1. there exists $p_0 \in \Lambda$, such that $(p_0, \Theta p_0) \in \mathscr{E}(\mathscr{G})$
2. Θ preserves edges of $\mathscr{G}$
3. for all $(p, q) \in \mathscr{E}(\mathscr{G})$, there exists $\tau \in (0, 1)$, such that

$$\mathscr{C}_d(\Theta p, \Theta q) \leq \tau \max[\mathscr{C}_d(p, q), \mathscr{C}_d(p, \Theta p), \mathscr{C}_d(q, \Theta q), \mathscr{C}_d(p, \Theta q), \mathscr{C}_d(q, \Theta p)], \quad (3)$$

for all $p, q \in \Lambda$.

Theorem 2 *Let $(\Lambda, \mathscr{C}_d)$ be a complete controlled metric type space and $\Theta : \Lambda \to \Lambda$ be a graphical Ćirić type contraction. If Θ is continuous, then Θ has a unique fixed point in Λ.*

Proof From (1), of definition 5 it is clear that there exists $p_0 \in \Lambda$, such that $(p_0, \Theta p_0) \in \mathscr{E}(\mathscr{G})$. Now, we define a sequence $\{p_n\}$ such that $p_{n+1} = \Theta x_n$ for all $n \in \mathbb{N}$.
Obviously, if $p_{n+1} = p_n$, the p_n is the fixed point of Θ. Suppose that for all $n \geq 0$, $p_{n+1} \neq p_n$. Since Θ preserves edges in $\mathscr{G}$ then there exists $p_n, \ p_{n+1} \in \mathscr{E}(\mathscr{G})$ by using the hypothesis (3) in the theorem, we have

$$\begin{aligned}
\mathscr{C}_d(p_n, p_{n+1}) &= \mathscr{C}_d(\Theta p_{n-1}, \Theta p_n) \\
&\leq \tau \max[\mathscr{C}_d(p_{n-1}, p_n), \mathscr{C}_d(p_{n-1}, \Theta p_{n-1}), \mathscr{C}_d(p_n, \Theta p_n), \mathscr{C}_d(p_{n-1}, \Theta p_n), \mathscr{C}_d(p_n, \Theta p_{n-1})] \\
&= \tau \max[\mathscr{C}_d(p_{n-1}, p_n), \mathscr{C}_d(p_{n-1}, p_n), \mathscr{C}_d(p_n, p_{n+1}), \mathscr{C}_d(p_{n-1}, p_{n+1}), \mathscr{C}_d(p_n, p_n)].
\end{aligned}$$

If we choose $\max[\mathscr{C}_d(p_{n-1}, p_n), \mathscr{C}_d(p_{n-1}, p_n), \mathscr{C}_d(p_n, p_{n+1}), \mathscr{C}_d(p_{n-1}, p_{n+1}), 0] = \mathscr{C}_d(p_{n-1}, p_n)$, otherwise it is contradiction. From this, the above inequality becomes

$$\mathscr{C}_d(p_n, p_{n+1}) \leq \tau \mathscr{C}_d(p_{n-1}, p_n).$$

Continue the same process until, we get

$$\mathscr{C}_d(p_n, p_{n+1}) \leq \tau^n \mathscr{C}_d(p_0, p_1).$$

Now, by using Theorem 1, $\{p_n\}$ is a Cauchy sequence in a complete controlled metric type space $(\Lambda, \mathscr{C}_d)$, so $\{p_n\}$ converges to some $u \in \Lambda$ that is $\lim_{n\to\infty} p_n = u$. Since Θ is continuous then, we have $u = \lim_{n\to\infty} p_{n+1} = \lim_{n\to\infty} \Theta p_n = \Theta(\lim_{n\to\infty} p_n) = \Theta u$, so u is a fixed point of Θ.
Uniqueness: Suppose Θ have two fixed points u and v, then

$$\begin{aligned}\mathscr{C}_d(u, v) &= \mathscr{C}_d(\Theta u, \Theta v) \leq \tau \max[\mathscr{C}_d(u, v), \mathscr{C}_d(u, \Theta u), \mathscr{C}_d(v, \Theta v), \mathscr{C}_d(u, \Theta v), \mathscr{C}_d(v, \Theta u)]\\ &= \tau \max[\mathscr{C}_d(u, v), \mathscr{C}_d(u, u), \mathscr{C}_d(v, v), \mathscr{C}_d(u, v), \mathscr{C}_d(v, u)].\end{aligned}$$

Furthermore, we have

$$\mathscr{C}_d(u, v) \leq \tau \mathscr{C}_d(u, v)$$

where $\tau \geq 1$, which is a contradiction and Θ has a unique fixed point.

Example 3 Let $\Lambda = \{1, 2, 3, 4\}$ and $\mathscr{C}_d$ be a function given as

$$\mathscr{C}_d(p, q) = \begin{cases} |p - q|^2 & \text{if } p \neq q \\ 0, & \text{if } p = q. \end{cases}$$

Given $\sigma : \Lambda \times \Lambda \to [1, \infty)$, defined as

$$\sigma(p, q) = p + q + 4,$$

since the given $(\Lambda, \mathscr{C}_d)$ is controlled metric type space. Consider a mapping $\digamma : \Lambda \to \Lambda$ such that

$$\Theta p = \begin{cases} 1, & \text{if } p = \{1, 2\}, \\ 2, & \text{if } p = 3, \\ 3, & \text{if } p = 4. \end{cases}$$

Define $\mathscr{G} = (\mathscr{V}(\mathscr{G}), \mathscr{E}(\mathscr{G}))$, where the vertex set $\mathscr{V}(\mathscr{G}) = \Lambda$ and

$$\mathscr{E}(\mathscr{G}) = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (4, 3)\},$$

as shown in the Fig. 1:

Case (i): Choose $p = 2$ and $q = 3$, then by using (3), we have

$$\begin{aligned}\mathscr{C}_d(\Theta 2, \Theta 3) &\leq \tau \max[\mathscr{C}_d(2, 3), \mathscr{C}_d(2, \Theta 2), \mathscr{C}_d(3, \Theta 3), \mathscr{C}_d(2, \Theta 3), \mathscr{C}_d(3, \Theta 2)]\\ 1 &\leq 4\tau.\end{aligned}$$

Case (ii): If $p = 2$ and $q = 4$ then, we have

$$\begin{aligned}\mathscr{C}_d(\Theta 2, \Theta 4) &\leq \tau \max[\mathscr{C}_d(2, 4), \mathscr{C}_d(2, \Theta 2), \mathscr{C}_d(4, \Theta 4), \mathscr{C}_d(2, \Theta 4), \mathscr{C}_d(4, \Theta 2)]\\ 4 &\leq 9\tau.\end{aligned}$$

Case (iii): At last, $p = 3$ and $q = 4$ then, we have

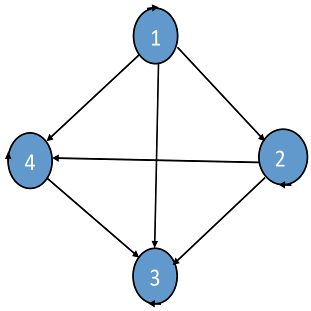

Fig. 1 Graph associated with Example 3

$$\mathscr{C}_d(\Theta 3, \Theta 4) \leq \tau \max[\mathscr{C}_d(3,4), \mathscr{C}_d(3,\Theta 3), \mathscr{C}_d(4,\Theta 4), \mathscr{C}_d(3,\Theta 4), \mathscr{C}_d(4,\Theta 3)]$$
$$1 \leq 4\tau.$$

If we choose $\lambda = \frac{1}{2} \in (0,1)$. All the hypothesis of the theorem (2) are fulfilled and 1 is the unique fixed point of the mapping Θ.

Definition 6 A mapping $\mathcal{F} : \varLambda \to \varLambda$ on a complete controlled metric type space $(\varLambda, \mathscr{C}_d)$. Then $\mathcal{F}$ is said to be graphical Reich type contraction if there exists non-negative λ, β and γ, where $\lambda + \beta + \gamma < 1$ such that

$$\mathscr{C}_d(\mathcal{F}p, \mathcal{F}q) \leq \lambda \mathscr{C}_d(p,q) + \beta \mathscr{C}_d(p, \mathcal{F}p) + \gamma \mathscr{C}_d(q, \mathcal{F}q),$$

for all $p, q \in \varLambda$.

Theorem 3 *Let $(\varLambda, \mathscr{C}_d)$ be a complete controlled metric type space and $\mathcal{F} : \varLambda \to \varLambda$ be a self-mapping satisfying the following conditions:*

1. there exists $p_0 \in \varLambda$, such that $(p_0, \mathcal{F}p_0) \in \mathscr{E}(\mathscr{G})$
2. $\mathcal{F}$ preserves edges of $\mathscr{G}$
3. for all $(p,q) \in \mathscr{E}(\mathscr{G})$, there exists nonnegative $\lambda + \beta + \gamma < 1$, such that

$$\mathscr{C}_d(\mathcal{F}p, \mathcal{F}q) \leq \lambda \mathscr{C}_d(p,q) + \beta \mathscr{C}_d(p, \mathcal{F}p) + \gamma \mathscr{C}_d(q, \mathcal{F}q). \tag{4}$$

Then $\mathcal{F}$ has fixed point.

Proof From (1), it is obvious there exists $p_0 \in \varLambda$, such that $(p_0, \mathcal{F}p_0) \in \mathscr{E}(\mathscr{G})$. Now define a sequence $\{p_n\}$ such that $p_{n+1} = \mathcal{F}x_n$ for all $n \in \mathbb{N}$.
If $p_{n+1} = p_n$, the p_n is the fixed point of $\mathcal{F}$. Suppose that for all $n \geq 0$, $p_{n+1} \neq p_n$. Since $\mathcal{F}$ preserves edges in $\mathscr{G}$ then there exists $p_n, p_{n+1} \in \mathscr{E}(\mathscr{G})$ by using the hypothesis in the theorem, we have

$$\begin{aligned}
\mathscr{C}_d(p_n, p_{n+1}) &= \mathscr{C}_d(\mathcal{F}p_{n-1}, \mathcal{F}p_n) \\
&\leq \lambda \mathscr{C}_d(p_{n-1}, p_n) + \beta \mathscr{C}_d(p_{n-1}, \mathcal{F}p_{n-1}) + \gamma \mathscr{C}_d(p_n, \mathcal{F}p_n) \\
&= \lambda \mathscr{C}_d(p_{n-1}, p_n) + \beta \mathscr{C}_d(p_{n-1}, p_n) + \gamma \mathscr{C}_d(p_n, p_{n+1}).
\end{aligned}$$

After simplification, we get

$$\mathscr{C}_d(p_n, p_{n+1}) \le \left(\frac{\lambda + \beta}{1 - \gamma}\right) \mathscr{C}_d(p_{n-1}, p_n).$$

Let $\frac{\lambda+\beta}{1-\gamma} = \eta$, obviously $\eta \in [0, 1)$ such that

$$\mathscr{C}_d(p_n, p_{n+1}) \le q\mathscr{C}_d(p_{n-1}, p_n).$$

Repeating this process, we have

$$\mathscr{C}_d(p_n, p_{n+1}) \le \eta^n \mathscr{C}_d(p_0, p_1).$$

Now, we have to show that $\{p_n\}$ is a Cauchy sequence. Since $(\Lambda, \mathscr{C}_d)$ is a controlled metric type space for all natural numbers $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned}
\mathscr{C}_d(p_n, p_m) &\le \sigma(p_n, p_{n+1})\mathscr{C}_d(p_n, p_{n+1}) + \sigma(p_{n+1}, p_m)\mathscr{C}_d(p_{n+1}, p_m)\\
&\le \sigma(p_n, p_{n+1})\mathscr{C}_d(p_n, p_{n+1}) + \sigma(p_{n+1}, p_m)\sigma(p_{n+1}, p_{n+2})\mathscr{C}_d(p_{n+1}, p_{n+2})\\
&+ \sigma(p_{n+1}, p_m)\sigma(p_{n+2}, p_m)\mathscr{C}_d(p_{n+2}, p_m)\\
&\le \sigma(p_n, p_{n+1})\mathscr{C}_d(p_n, p_{n+1}) + \sigma(p_{n+1}, p_m)\sigma(p_{n+1}, p_{n+2})\mathscr{C}_d(p_{n+1}, p_{n+2})\\
&+ \sigma(p_{n+1}, p_m)\sigma(p_{n+2}, p_m)\sigma(p_{n+2}, p_{n+3})\mathscr{C}_d(p_{n+2}, p_{n+3}) + \sigma(p_{n+1}, p_m)\\
&\sigma(p_{n+2}, p_m)\sigma(p_{n+3}, p_m)\mathscr{C}_d(p_{n+3}, p_m)\\
&\le \sigma(p_n, p_{n+1})\mathscr{C}_d(p_n, p_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(p_j, p_m)\right)\sigma(p_i, p_{i+1})\mathscr{C}_d(p_i, p_{i+1})\\
&+ \prod_{k=n+1}^{m-1}\sigma(p_k, p_m)\mathscr{C}_d(p_{m-1}, p_m)\\
&\le \sigma(p_n, p_{n+1})\eta^n\mathscr{C}_d(p_0, p_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(p_j, p_m)\right)\sigma(p_i, p_{i+1})\eta^i\mathscr{C}_d(p_0, p_1)\\
&+ \prod_{k=n+1}^{m-1}\sigma(p_k, p_m)\eta^{m-1}\mathscr{C}_d(p_0, p_1)\\
&\le \sigma(p_n, p_{n+1})\eta^n\mathscr{C}_d(p_0, p_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\sigma(p_j, p_m)\right)\sigma(p_i, p_{i+1})\eta^i\mathscr{C}_d(p_0, p_1)\\
&+ \prod_{k=n+1}^{m-1}\sigma(p_k, p_m)\sigma(p_{m-1}, p_m)\eta^{m-1}\mathscr{C}_d(p_0, p_1)\\
&= \sigma(p_n, p_{n+1})\eta^n\mathscr{C}_d(p_0, p_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\sigma(p_j, p_m)\right)\sigma(p_i, p_{i+1})\eta^i\mathscr{C}_d(p_0, p_1)\\
&\le \sigma(p_n, p_{n+1})\eta^n\mathscr{C}_d(p_0, p_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\sigma(p_j, p_m)\right)\sigma(p_i, p_{i+1})\eta^i\mathscr{C}_d(p_0, p_1).
\end{aligned}$$

Assume that,

$$S_n = \sum_{i=0}^{n} (\prod_{j=0}^{i} \sigma(p_j, p_m))\sigma(p_i, p_{i+1})\eta^i$$

Then, we obtain

$$\mathscr{C}_d(p_n, p_m) \leq \mathscr{C}_d(p_0, p_1)[\eta^n \sigma(p_n, p_{n+1}) + (S_{m-1} - S_n)]. \quad (5)$$

Using ratio test, we have

$$a_i = \prod_{j=0}^{i} \sigma(p_j, p_m)\sigma(p_i, p_{i+1})\eta^i, \text{ where } \frac{a_{i+1}}{a_i} < \frac{1}{k},$$

taking $\lim_{n,m\to\infty}$ so (5) becomes

$$\lim_{n,m\to\infty} \mathscr{C}_d(p_n, p_m) = 0.$$

This implies that $\{p_n\}$ is a Cauchy sequence in a complete controlled metric type space $(\Lambda, \mathscr{C}_d)$, so $\{p_n\}$ converges to some $u \in \Lambda$ that is $\lim_{n\to\infty} p_n = u$. Since $\digamma$ is continuous then, we have $u = \lim_{n\to\infty} p_{n+1} = \lim_{n\to\infty} \digamma p_n = \digamma(\lim_{n\to\infty} p_n) = \digamma u$, so u is a fixed point of $\digamma$.

Uniqueness: Assume that there are two fixed points u and v of $\digamma$, then

$$\begin{aligned} \mathscr{C}_d(u, v) = \mathscr{C}_d(\digamma u, \digamma v) &\leq \lambda \mathscr{C}_d(u, v) + \beta \mathscr{C}_d(u, \digamma u) + \gamma \mathscr{C}_d(v, \digamma v) \\ &= \lambda \mathscr{C}_d(u, v) + \beta \mathscr{C}_d(u, u) + \gamma \mathscr{C}_d(v, v) \end{aligned}$$

Furthermore, we have

$$\mathscr{C}_d(u, v) \leq (\lambda + \beta + \gamma)\mathscr{C}_d(u, v)$$

where $(\lambda + \beta + \gamma) \geq 1$, which is a contradiction and $\digamma$ has a unique fixed point.

Remark 2 Since the controlled metric space is not generally an extended $b-$metric space, if we take $\sigma(p, q) = e(p, q)$, then above results Theorem 3 reduces to extended $b-$metric space. Similarly, if $e(p, q) = b \geq 1$, then reduces to $b-$metric space.

Example 4 Let $\Lambda = \{0, 1, 2\}$, consider the function $\mathscr{C}_d$ given as:

$$\mathscr{C}_d(p, q) = \begin{cases} |p - q|^2 & \text{if } p \neq q \\ 0, & \text{if } p = q. \end{cases}$$

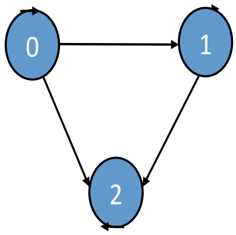

Fig. 2 Graph associated with Example 4

Given $\sigma : \Lambda \times \Lambda \to [1, \infty)$, defined as

$$\sigma(p, q) = 9p + 3q + 5,$$

since the given $(\Lambda, \mathscr{C}_d)$ is controlled metric type space. Consider a mapping $\digamma : \Lambda \to \Lambda$ such that

$$\digamma p = \begin{cases} 1, & \text{when } p = \{1, 2\}, \\ 2, & \text{when } p = 0. \end{cases}$$

Define $\mathscr{G} = (\mathscr{V}(\mathscr{G}), \mathscr{E}(\mathscr{G}))$, where the vertex set $\mathscr{V}(\mathscr{G}) = \Lambda$ and $\mathscr{E}(\mathscr{G}) = \{(0, 1), (0, 2), (1, 2)\}$, as shown in the Fig. 2,

Choose $\digamma 0 = 2$ and $\digamma 2 = 1$, then by using (4), we have

$$\begin{aligned} \mathscr{C}_d(\digamma 0, \digamma 2) &\leq \lambda \mathscr{C}_d(0, 2) + \beta \mathscr{C}_d(0, f0) + \gamma \mathscr{C}_d(2, f2) \\ \mathscr{C}_d(2, 1) &\leq \lambda(4) + \beta(4) + \gamma(1) \\ 1 &\leq \frac{239}{105} \end{aligned}$$

As we know that $\lambda + \beta + \gamma < 1$, if we choose $\lambda = \frac{1}{3}$, $\beta = \frac{1}{5}$ and $\gamma = \frac{1}{7}$, clearly $\lambda + \beta + \gamma = \frac{71}{105} < 1$. All the hypothesis of Theorem 3 are fulfilled and 0 is the unique fixed point of the mapping $\digamma$.

4 Application of Second Order Differential Equation

Let a set of all real valued continuous function $\Lambda = C[[0, 1], \mathbb{R}]$ on $[[0, 1], \mathbb{R}]$ and $\mathscr{C}_d : \Lambda \times \Lambda \to \mathbb{R}$ is defined as

$$\mathscr{C}_d(p, q) = \sup_{\mathsf{t} \in [0,1]} |p(\mathsf{t}) - q(\mathsf{t})|^2 .$$

Let $\sigma : \Lambda \times \Lambda \to [1, \infty)$, defined as

$$\sigma(p, q) = 7p(\mathsf{t}) + 9q(\mathsf{t}) + 5$$

for all $p, q \in \Lambda$ and $\mathsf{t} \in [a, b]$. Clearly, $(\Lambda, \mathscr{C}_d)$ is complete controlled metric type space.

Now, we will assume second-order differential equation as

$$\begin{cases} p''(p) = g(\mathsf{t}, p(\mathsf{t})), \\ p(0) = p_0, \;\; p(1) = p, \end{cases} \tag{6}$$

for all $\mathsf{t} \in [0, 1]$ and $g : [0, 1] \times \mathbb{R}$, is a continuous function.

The problem defined in (6) is equivalent to second kind Fredholm integral equation

$$p(\mathsf{t}) = L(\mathsf{t}) + \gamma \int_0^1 \dot{G}(\mathsf{t}, s)p(s)ds, \tag{7}$$

where $\mathsf{t} \in [0, 1]$ and $L(\mathsf{t}) = u_0 + \mathsf{t}(u_1 - u_0)$. In (7), $\dot{G}(\mathsf{t}, s)$ is Green's function that is

$$\dot{G}(\mathsf{t}, s) = \begin{cases} s(1 - s) \; 0 \le s \le \mathsf{t} \\ \mathsf{t}(1 - s) \;\; \mathsf{t} \le s \le 1 \end{cases}$$

and if $p \in \Lambda$ is a fixed point of $\digamma$ then p is a solution of (6).

Theorem 4 *Let* $\digamma : \Lambda \to \Lambda$ *be a continuous nonlinear integral operator defined by*

$$p(\mathsf{t}) = L(\mathsf{t}) + \gamma \int_0^1 \dot{G}(\mathsf{t}, s)p(s)ds,$$

for all $\mathsf{t} \in [0, 1]$. *Assume that following conditions holds*

1. For any $k > 0$, we have

$$\sup_{\mathsf{t} \in [0,1]} \left(\frac{\mathsf{t}^3}{6} - \frac{\mathsf{t}^2}{2} + \frac{\mathsf{t}}{2}\right) = \frac{1}{2k}.$$

   Then second-order differential equation (6) have solution in Λ.

Proof For any $p_0 \in \Lambda$, we define a sequence $\{p_n\}$ in Λ, by $p_{n+1} = \digamma p_n = \digamma^{n+1} p_0$, $n \ge 1$ then we obtain

$$p_{n+1}(\mathsf{t}) = \digamma p_n(\mathsf{t}) = L(\mathsf{t}) + k \int_0^1 \dot{G}(\mathsf{t}, s)p_n(s)ds,$$

for $p, q \in \Lambda$, we have

$$
\begin{aligned}
|\digamma p(\mathrm{t}) - \digamma q(\mathrm{t})| &= \left| L(\mathrm{t}) + k\int_0^1 \dot{G}(\mathrm{t}, s)p(s)ds - L(p) + k\int_0^1 \dot{G}(\mathrm{t}, s)q(s)ds \right| \\
&\le k\int_0^1 \dot{G}(\mathrm{t}, s)\,|p(s) - q(s)|\,ds \\
&\le k \sup_{\mathrm{t}\in[0,1]} |p(\mathrm{t}) - q(\mathrm{t})| \int_0^1 \dot{G}(\mathrm{t}, s)ds \\
&\le k\left(\frac{\mathrm{t}^3}{6} - \frac{\mathrm{t}^2}{2} + \frac{\mathrm{t}}{2}\right) \sup_{\mathrm{t}\in[0,1]} |p(\mathrm{t}) - q(\mathrm{t})|,
\end{aligned}
$$

which implies that

$$
\sup_{\mathrm{t}\in[0,1]} |\digamma p(\mathrm{t}) - \digamma q(\mathrm{t})|^2 \le \frac{1}{4} \sup_{\mathrm{t}\in[0,1]} |p(\mathrm{t}) - q(\mathrm{t})|^2,
$$

we know that $\beta \in (0, 1)$, since we get

$$
\mathscr{C}_d(\digamma p, \digamma q) \le \beta \mathscr{C}_d(p, q).
$$

The above inequality can be written in the manners as

$$
\mathscr{C}_d(\digamma p, \digamma q) \le \beta \max[\mathscr{C}_d(p, q), \mathscr{C}_d(p, \digamma p), \mathscr{C}_d(q, \digamma q), \mathscr{C}_d(p, \digamma q), \mathscr{C}_d(q, \digamma p)]
$$

Thus all the conditions of Theorem 1 are satisfied. Hence, $\digamma$ has a fixed point and Fredholm integral equation (7) has a solution.

Conclusion: In this manuscript, we have shown how the Ćirić type and Reich type contractions can be used to prove the existence and uniqueness of fixed points results in the setting of controlled metric type space. By incorporating graph-based techniques, we have gained deeper insights into the behavior and properties of fixed points. The visual representation provided by graphs has aided in understanding the structural aspects of fixed point mappings, leading to improved results in terms of existence, uniqueness, and convergence of solutions. The combination of contraction principles and graph theory has enhanced our understanding of fixed point theory and has the potential to further advance our knowledge in various mathematical applications. This approach holds promise for future research in fixed point theory and related fields. Through examples, we demonstrated the practical application of these results in various contexts. Furthermore, we utilized these findings to obtain the solution of a second-order differential equation. By leveraging the power of these contraction mappings, we were able to establish important theoretical results and also solve practical problems. Overall, this work highlights the usefulness of contraction mappings in mathematics and the wide range of applications they have in different fields.

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A Study of Fixed Point Results in G-Metric Space via New Contractions with Applications

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1 Introduction

Following the introduction of the well-known Banach contraction principle (BCP), investigation of the existence and uniqueness of fixed points (FPs) of contraction mappings in the framework of metric spaces (MSs) is one of the centers of interest in linear and nonlinear functional analysis, given its important applications in applied mathematics, engineering, and social sciences. Several extensions of BCP have been obtained over the years by either generalizing the contractive conditions, introducing additional algebraic structures, or altering the metric structures of the underlying space (see, e.g., [7, 8, 11, 13, 14]). In this connection, Jaggi [16] proposed a new generalization of BCP in MSs and investigated the existence and uniqueness of FP of a self-mapping in that space. Similarly, Dass and Gupta [12] obtained an extension of BCP by means of rational expression.

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By modifying the defining structures of MS, Mustafa and Sims [26] introduced a new generalization of MS called G-MS and obtained some FP theorems for contraction mappings. Subsequently, Mustafa et al. [29] and several other authors (see, e.g., [1, 5, 10, 24, 28, 31]) established notable FP results in G-MSs. In this regard, Samet et al. [34], in addition to Jleli and Samet [20], published observations that an essential number of the FP results in G-MSs are directly deducible from the known results in MSs. More so, Jleli and Samet [20] noted that any obtained FP result in G-MSs is equivalent to a related FP result in quasi-MSs, if the G-metric is reducible to a quasi-metric. After carefully studying these observations, several researchers (see, e.g., [3, 6, 9, 17, 18, 23, 36]) have developed techniques for establishing FP results in G-MSs that cannot be followed from their corresponding ones in ordinary or quasi-MSs.

In a recent paper, Karapınar et al. [22] proposed some new generalizations in MSs, which unify a few existing ones in the literature, including that of Jaggi [16], Dass and Gupta [12] and investigated some FP results for contraction mappings in that space. What sets these generalizations apart is the fact that they can be reduced in a variety of ways, depending on one's choice of parameters.

It is noted from the review of existing literature that FP results of Jaggi-type and Dass-Gupta-type mappings in the manner proposed by [22] have not been sufficiently investigated in the context of G-MSs. Taking motivation from the ideas in [9, 22, 23], therefore, we establish some novel concepts of Jaggi-type (G-α-ϕ)-contraction and Dass-Gupta-type (G-α-ϕ)-contraction in G-MSs alongside some related FP results. A constructive and vivid example is presented to demonstrate the validity of our main ideas and to show that they do not narrow to any related result in MSs. A few special cases are generated to point out that the ideas proposed in this paper are indeed extensions of some famous FP theorems in the literature. Additionally, the main result in this manuscript is adopted to initiate peculiar conditions for which the solution of a given boundary value problem (BVP) exists.

2 Preliminaries

Here, we present some essential notions and propositions which will be applied in our main results.

Throughout, the set ξ is assumed to be non-empty, the set of natural numbers, real numbers and non-negative real numbers are represented by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively.

Definition 1 ([26]) Let ξ be a nonempty set together with a function $G : \xi \times \xi \times \xi \longrightarrow \mathbb{R}_+$ such that

(G_1) $G(q, b, t) = 0$ if $q = b = t$;
(G_2) $G(q, q, b) > 0$ for every $q, b \in \xi$ where $q \neq b$;
(G_3) $G(q, q, b) \leq G(q, b, t)$, for every $q, b, t \in \xi$ where $t \neq b$;
(G_4) $G(q, b, t) = G(q, t, b) = G(b, q, t) = \cdots$ (symmetry);

(G_5) $G(q,t,b) \leq G(q,u,u) + G(u,t,b)$, for every $u,t,q,b \in \xi$ (rectangle inequality).

By that, G is a G-metric on ξ, and the pair (ξ, G) is said to be a G-MS.

Example 1 [29] Take ξ along with the Euclidean metric d. Then (ξ, G_u) and (ξ, G_v) are G-MSs, where

$$G_u(q,b,t) = d(q,b) + d(b,t) + d(q,t) \text{ for all } q,b,t \in \xi, \quad (1)$$

$$G_v(q,b,t) = \max\{d(q,b), d(b,t), d(q,t)\} \text{ for all } q,b,t \in \xi. \quad (2)$$

Definition 2 [29] A sequence $\{q_i\}_{i\in\mathbb{N}}$ in a G-MS (ξ, G) is referred to as G-convergent to a point q if $\lim_{i,j\to\infty} G(q,q_i,q_j) = 0$, that is, given any $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$ satisfying $G(q,q_i,q_j) < \epsilon$, for all $i, j \geq i_0$.

Proposition 1 ([29]) *If (ξ, G) is a G-MS, then the assertions below are equivalent:*

(i) $\{q_i\}_{i\in\mathbb{N}}$ *is G-convergent to q.*
(ii) $G(q,q_i,q_j) \longrightarrow 0$*, as* $i, j \to \infty$.
(iii) $G(q_i,q,q) \longrightarrow 0$*, as* $i \to \infty$.
(iv) $G(q_i,q_i,q) \longrightarrow 0$*, as* $i \to \infty$.

Definition 3 ([29]) A sequence $\{q_i\}_{i\in\mathbb{N}}$ in a G-MS (ξ, G) is referred to as G-Cauchy if for any $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$ satisfying $G(q_i,q_j,q_l) < \epsilon$, for all $i, j, l \geq i_0$.

Proposition 2 ([29]) *If (ξ, G) is a G-MS, then the assertions below are equivalent:*

(i) $\{q_i\}_{i\in\mathbb{N}}$ *is a G-Cauchy sequence.*
(ii) Given any $\epsilon > 0$*, we can find* $i_0 \in \mathbb{N}$ *satisfying* $G(q_i,q_j,q_j) < \epsilon$*, for all* $i, j \geq i_0$.

Definition 4 ([29]) Let $\digamma : \xi \longrightarrow \xi'$ be a function from a G-MS (ξ, G) to another, (ξ', G'). Then $\digamma$ is referred to as G-continuous at a point $r \in \xi$ iff for any $\epsilon > 0$, we can find $\delta > 0$ for which $q, b \in \xi$ and $G(q,q,b) < \delta \Rightarrow G'(\digamma q, \digamma q, \digamma b) < \epsilon$. If $\digamma$ is G-continuous on ξ, then $\digamma$ is G-continuous at every point $r \in \xi$ and conversely.

Proposition 3 ([29]) *If (ξ, G) and (ξ', G') are two G-MSs, then a function $\digamma : \xi \longrightarrow \xi'$ is G-sequentially continuous at a point $q \in \xi$ if it is G-continuous at q, and conversely. That is, $\{\digamma q_i\}$ is G-convergent to $\digamma q$ whenever $\{q_i\}_{i\in\mathbb{N}}$ is G-convergent to q.*

Definition 5 ([29]) A G-MS (ξ, G) is referred to as symmetric G-MS if

$$G(q,q,b) = G(b,q,q), \quad \text{for all } q, b \in \xi.$$

Proposition 4 ([29]) *If (ξ, G) is a G-MS, then $G(q,b,t)$ is jointly continuous in all the three points $q, b, t \in \xi$.*

Definition 6 ([29]) Let (ξ, G) be a G-MS. Then (ξ, G) is referred to as G-complete if every G-Cauchy sequence in ξ is G-convergent in ξ.

Proposition 5 ([29]) *A G-MS (ξ, G) is G-complete iff (ξ, d_G) is a complete MS.*

Let Φ be the set of all functions $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying

(i) ϕ is continuous and non-decreasing;
(ii) $\phi(v) = 0$ iff $v = 0$.

Then ϕ is called an altering distance function (see [2, 4, 5, 35]).

Definition 7 ([19]) For a given function $\alpha : \xi \times \xi \times \xi \longrightarrow \mathbb{R}_+$, a self-mapping $\digamma : \xi \longrightarrow \xi$ is called $(G\text{-}\alpha)$-orbital admissible, if for every $q \in \xi$

$$\alpha(q, \digamma q, \digamma^2 q) \geq 1 \Rightarrow \alpha(\digamma q, \digamma^2 q, \digamma^3 q) \geq 1.$$

Definition 8 [19]. For a given function $\alpha : \xi \times \xi \times \xi \longrightarrow \mathbb{R}_+$, a self-mapping $\digamma : \xi \longrightarrow \xi$ is referred to as triangular $(G\text{-}\alpha)$-orbital admissible if for every $q \in \xi$, $\digamma$ is $(G\text{-}\alpha)$-orbital admissible and

$$\alpha(q, b, \digamma b) \geq 1 \text{ and } \alpha(b, \digamma b, \digamma^2 b) \geq 1 \Rightarrow \alpha(q, \digamma b, \digamma^2 b) \geq 1.$$

Lemma 1 ([19]) *Given a triangular $(G\text{-}\alpha)$-orbital admissible mapping $\digamma : \xi \longrightarrow \xi$, if there exists $q_0 \in \xi$ such that $\alpha(q_0, \digamma q_0, \digamma^2 q_0) \geq 1$, then*

$$\alpha(q_i, q_j, q_l) \geq 1, \quad \forall i, j, l \in \mathbb{N}, \tag{3}$$

where the sequence $\{q_i\}_{i\in\mathbb{N}}$ is defined by $q_{i+1} = \digamma q_i$, $i \in \mathbb{N}$.

Definition 9 ([2]) Given a mapping $\alpha : \xi \times \xi \times \xi \longrightarrow \mathbb{R}_+$, the set ξ is said to be α-regular iff for any sequence $\{q_i\}_{i\in\mathbb{N}}$ in ξ satisfying $\alpha(q_i, q_{i+1}, q_{i+2}) \geq 1$, for every i and $q_i \to q \in \xi$ as $i \to \infty$, we get $\alpha(q_i, q, q) \geq 1$ for all i.

Karapınar et al. [22] presented the following definitions in the framework of MSs.

Definition 10 ([22]) Let (ξ, d) be a MS. Denote by $\mathcal{A}(\xi)$, the family of all functions $h : \xi \times \xi \longrightarrow [0, 1)$ such that for all sequences $\{q_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$ in ξ,

$$\lim_{i\to\infty} h(q_i, b_i) = 1 \Rightarrow \lim_{i\to\infty} d(q_i, b_i) = 0.$$

Definition 11 ([22]) Let (ξ, d) be a MS and $\digamma : \xi \longrightarrow \xi$ be a self-mapping on ξ. Suppose there exist $\phi \in \Phi$, $h \in \mathcal{A}(\xi)$ and $\alpha : \xi \times \xi \longrightarrow \mathbb{R}_+$ such that for all $q, b \in \xi$, the following inequalities are satisfied:

(I_1) $\alpha(q, b)\phi(d(\digamma q, \digamma b)) \leq h(q, b)\phi(\mathcal{R}(q, b))$;
(I_2) $\alpha(q, b)\phi(d(\digamma q, \digamma b)) \leq h(q, b)\phi(\mathcal{S}(q, b))$;
(I_3) $\alpha(q, b)\phi(d(\digamma q, \digamma b)) \leq h(q, b)\phi(\digamma(q, b))$,

where

$$\mathcal{R}(q, b) = \max\left\{\begin{array}{c} \frac{d(q,\digamma q)\cdot d(b,\digamma b)}{d(q,b)}, \frac{d(q,\digamma b)+d(b,\digamma q)}{2}, \\ d(q, b), d(q, \digamma q), d(b, \digamma b) \end{array}\right\},$$

$$\mathcal{S}(q, b) = \max\left\{\frac{d(q, \digamma q)\cdot d(b, \digamma b)}{d(q, b)}, d(q, b)\right\},$$
$$\digamma(q, b) = \max\left\{d(q, b), d(q, \digamma q), d(b, \digamma b)\right\}.$$

Then
$\digamma$ is referred to as a Jaggi-type α-h-ϕ-contraction if $\digamma$ verifies (I_1).
$\digamma$ is referred to as a generalized Jaggi-type α-h-ϕ-contraction if $\digamma$ verifies (I_2).
$\digamma$ is referred to as α-h-ϕ-contraction if $\digamma$ verifies (I_3).

Definition 12 ([22]) Let (ξ, d) be a MS and $\digamma : \xi \longrightarrow \xi$ be a self-mapping on ξ. Suppose there exist $\phi \in \Phi, h \in \mathcal{A}(\xi)$ and $\alpha : \xi \times \xi \longrightarrow \mathbb{R}_+$ such that for all $q, b \in \xi$,

(I_4) $\quad \alpha(q, b)\phi(d(\digamma q, \digamma b)) \le h(q, b)\phi(\mathcal{O}(q, b)),$

where

$$\mathcal{O}(q, b) = \max\left\{\frac{d(q, \digamma q)(1+d(b, \digamma b))}{1+d(q, b)}, d(q, b), \frac{d(b, \digamma b)(1+d(q, \digamma q))}{1+d(q, b)}\right\}.$$

Then $\digamma$ is referred to as a generalized Dass-Gupta-type α-h-ϕ-contraction.

Definition 13 Let (ξ, G) be a G-MS. We denote by $\mathcal{A}(G)$, the family of all functions $f : \xi \times \xi \times \xi \longrightarrow [0, 1)$ such that for all sequences $\{q_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$ and $\{t_i\}_{i\in\mathbb{N}}$ in ξ,

$$\lim_{i\to\infty} f(q_i, b_i, t_i) = 1 \Rightarrow \lim_{i\to\infty} G(q_i, b_i, t_i) = 0,$$

where the sequence $\{G(q_i, b_i, t_i)\}_{i\in\mathbb{N}}$ is monotonic decreasing.

Example 2 Let $f_1, f_2, f_3 : \xi \times \xi \times \xi \longrightarrow [0, 1)$ be defined as follows:

1. $f_1(q, b, t) = \lambda$ for some $q, b, t \in \mathbb{R}_+$, where $\lambda \in [0, 1)$.
2. $f_2(q, b, t) = \begin{cases} \frac{\ln(1+aq+bp+ct)}{aq+bp+ct}, & \text{if } q > 0 \text{ or } b > 0 \text{ or } t > 0 \text{ for some } q, b, t \in \mathbb{R}_+; \\ \lambda \in [0, 1), & \text{if } q = b = t = 0. \end{cases}$
3. $f_3(q, b, t) = \begin{cases} \frac{\ln(1+\max\{q,b,t\})}{\max\{q,b,t\}}, & \text{if } q > 0 \text{ or } b > 0 \text{ or } t > 0 \text{ for some } q, b, t \in \mathbb{R}_+; \\ \lambda \in [0, 1), & \text{if } q = b = t = 0. \end{cases}$

Then $f_1, f_2, f_3 \in \mathcal{A}(G)$.

We refer to [15, 22, 25] for related concepts of $\mathcal{A}(G)$.

3 Main Results

We begin this section by defining some new notions of (G-α-ϕ)-contractions in G-MSs.

Definition 14 Let (ξ, G) be a G-MS and $\digamma : \xi \longrightarrow \xi$ be a self-mapping on ξ. Suppose that there exist $\phi \in \Phi$, $f \in \mathcal{A}(G)$ and $\alpha : \xi \times \xi \times \xi \longrightarrow \mathbb{R}_+$ such that for all $q, b \in \xi$, the following inequalities are satisfied:

(θ_1) $\alpha(q, b, \digamma b)\phi(G(\digamma q, \digamma b, \digamma^2 b)) \leq f(q, b, \digamma b)\phi(\mathcal{L}(q, b, \digamma b))$;
(θ_2) $\alpha(q, b, \digamma b)\phi(G(\digamma q, \digamma b, \digamma^2 b)) \leq f(q, b, \digamma b)\phi(\mathcal{M}(q, b, \digamma b))$;
(θ_3) $\alpha(q, b, \digamma b)\phi(G(\digamma q, \digamma b, \digamma^2 b)) \leq f(q, b, \digamma b)\phi(\mathcal{N}(q, b, \digamma b))$,

where

$$\mathcal{L}(q, b, \digamma b) = \max \left\{ \begin{array}{c} \frac{G(q,\digamma q,\digamma^2 q)\cdot G(b,\digamma b,\digamma^2 b)}{G(q,b,\digamma b)}, G(q, b, \digamma b), G(q, \digamma q, \digamma^2 q), \\ G(b, \digamma b, \digamma^2 b), \frac{G(q,b,\digamma b)+G(b,\digamma b,\digamma^2 b)}{2} \end{array} \right\},$$

$$\mathcal{M}(q, b, \digamma b) = \max \left\{ \frac{G(q,\digamma q,\digamma^2 q)\cdot G(b,\digamma b,\digamma^2 b)}{G(q,b,\digamma b)}, G(q, b, \digamma b) \right\},$$

$$\mathcal{N}(q, b, \digamma b) = \max \left\{ G(q, b, \digamma b), G(q, \digamma q, \digamma^2 q), G(b, \digamma b, \digamma^2 b) \right\}.$$

Then
$\digamma$ is referred to as a Jaggi-type $(G$-α-$\phi)$-contraction if (θ_1) is satisfied.
$\digamma$ is referred to as a generalized Jaggi-type $(G$-α-$\phi)$-contraction if (θ_2) is satisfied.
$\digamma$ is referred to as a $(G$-α-$\phi)$-contraction if (θ_3) is satisfied.

The following is our main result.

Theorem 1 *Let (ξ, G) be a complete G-MS and $\digamma : \xi \longrightarrow \xi$ be a self-mapping satisfying (θ_1). Assume additionally that:*

(i) $\digamma$ is triangular $(G$-$\alpha)$-orbital admissible;
(ii) $\digamma$ is continuous;
(iii) $\alpha(q_0, \digamma q_0, \digamma^2 q_0) \geq 1$ for a particular $q_0 \in \xi$.

Then $\digamma$ has a FP in ξ.

Proof Suppose $q_0 \in \xi$ is any random point. Let $\{q_i\}_{i\in\mathbb{N}}$ be a sequence in ξ such that $q_i = \digamma^i q_0$ for each $i \in \mathbb{N}$. Assume we can find a point $j \in \mathbb{N}$ satisfying $\digamma q_j = q_{j+1} = q_j$. Then obviously, q_j is a FP of $\digamma$. So, we presume that $q_i \neq q_{i-1}$ for each $i \in \mathbb{N}$. Since $\digamma$ is a Jaggi-type $(G$-α-$\phi)$-contraction, then from (θ_1) and Lemma 1, we see that

$$\begin{aligned} \phi(G(q_i, q_{i+1}, q_{i+2})) &\leq \alpha(q_{i-1}, q_i, q_{i+1})\phi(G(q_i, q_{i+1}, q_{i+2})) \\ &= \alpha(q_{i-1}, q_i, q_{i+1})\phi(G(\digamma q_{i-1}, \digamma q_i, \digamma^2 q_i)) \\ &\leq f(q_{i-1}, q_i, q_{i+1})\phi(\mathcal{L}(q_{i-1}, q_i, \digamma q_i)) \\ &< \phi(\mathcal{L}(q_{i-1}, q_i, \digamma q_i)). \end{aligned} \tag{4}$$

Now,

$$\mathcal{L}(q_{i-1}, q_i, Fq_i) = \max\left\{\begin{array}{c} \frac{G(q_{i-1}, Fq_{i-1}, F^2q_{i-1})\cdot G(q_i, Fq_i, F^2q_i)}{G(q_{i-1}, q_i, Fq_i)}, \\ G(q_{i-1}, q_i, Fq_i), G(q_{i-1}, Fq_{i-1}, F^2q_{i-1}), \\ G(q_i, Fq_i, F^2q_i), \\ \frac{G(q_{i-1}, q_i, Fq_i)+G(q_i, Fq_i, F^2q_i)}{2} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} \frac{G(q_{i-1}, q_i, q_{i+1})\cdot G(q_i, q_{i+1}, q_{i+2})}{G(q_{i-1}, q_i, q_{i+1})}, \\ G(q_{i-1}, q_i, q_{i+1}), G(q_{i-1}, q_i, q_{i+1}), \\ G(q_i, q_{i+1}, q_{i+2}), \\ \frac{G(q_{i-1}, q_i, q_{i+1})+G(q_i, q_{i+1}, q_{i+2})}{2} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} G(q_{i-1}, q_i, q_{i+1}), G(q_i, q_{i+1}, q_{i+2}), \\ \frac{G(q_{i-1}, q_i, q_{i+1})+G(q_i, q_{i+1}, q_{i+2})}{2} \end{array}\right\}.$$

If we assume that

$$G(q_{i-1}, q_i, q_{i+1}) \leq G(q_i, q_{i+1}, q_{i+2}),$$

then (4) becomes

$$\phi(G(q_i, q_{i+1}, q_{i+2})) < \phi(\mathcal{L}(q_{i-1}, q_i, Fq_i)) = \phi(G(q_i, q_{i+1}, q_{i+2})),$$

a contradiction. Hence, for every $i \in \mathbb{N}$, we conclude that

$$\phi(G(q_i, q_{i+1}, q_{i+2})) < \phi(G(q_{i-1}, q_i, q_{i+1})). \tag{5}$$

Since $\phi \in \Phi$, then for all $i \in \mathbb{N}$, we have

$$G(q_i, q_{i+1}, q_{i+2}) < G(q_{i-1}, q_i, q_{i+1}).$$

Hence, the sequence $\{G(q_i, q_{i+1}, q_{i+2})\}_{i\in\mathbb{N}}$ is non-negative and decreasing. In that case, we can find $l \geq 0$ such that $\lim_{i\to\infty} G(q_i, q_{i+1}, q_{i+2}) = l$. We now demonstrate that $l = 0$. Assume contrary that $l > 0$. Then from (4) and (5), we have

$$0 < \frac{\phi(G(q_i, q_{i+1}, q_{i+2}))}{\phi(G(q_{i-1}, q_i, q_{i+1}))} \leq f(q_{i-1}, q_i, q_{i+1}),$$

implying that

$$\lim_{i\to\infty} f(q_{i-1}, q_i, q_{i+1}) = 1.$$

Since $f \in \mathcal{A}(G)$, then we have

$$\lim_{i\to\infty} G(q_{i-1}, q_i, q_{i+1}) = 0,$$

by which we obtain $l = 0$, a contradiction. Therefore,

$$\lim_{i\to\infty} G(q_i, q_{i+1}, q_{i+2}) = 0.$$

Next, we demonstrate that $\{q_i\}_{i\in\mathbb{N}}$ is G-Cauchy in ξ. Assume contrary that $\{q_i\}_{i\in\mathbb{N}}$ is not G-Cauchy. Then, we can find $\epsilon > 0$ such that for every $r \in \mathbb{N}$, there exist j_r, i_r with $j_r \geq i_r > r$ such that $G(q_{i_r}, q_{j_r}, q_{j_r+1}) \geq \epsilon$. We can further assume that $G(q_{i_r}, q_{j_r-1}, q_{j_r}) < \epsilon$ by choosing sufficiently small j_r. In that case, for each $r \in \mathbb{N}$, we see that

$$\begin{aligned}\epsilon \leq G(q_{i_r}, q_{j_r}, q_{j_r+1}) &\leq G(q_{i_r}, q_{j_r-1}, q_{j_r}) + G(q_{j_r-1}, q_{j_r}, q_{j_r+1})\\ &\leq \epsilon + G(q_{j_r-1}, q_{j_r}, q_{j_r+1}).\end{aligned}$$

Letting $r \to \infty$, we obtain

$$\lim_{r\to\infty} G(q_{i_r}, q_{j_r}, q_{j_r+1}) = \epsilon.$$

Given any $r \in \mathbb{N}$, we can see that

$$\begin{aligned}\phi(G(q_{i_r+1}, q_{j_r+1}, q_{j_r+2})) &\leq \alpha(q_{i_r}, q_{j_r}, q_{j_r+1})\phi(G(q_{i_r+1}, q_{j_r+1}, q_{j_r+2}))\\ &= \alpha(q_{i_r}, q_{j_r}, q_{j_r+1})\phi(G(\digamma q_{i_r}, \digamma q_{j_r}, \digamma^2 q_{j_r}))\\ &\leq f(q_{i_r}, q_{j_r}, q_{j_r+1})\phi(\mathcal{L}(q_{i_r}, q_{j_r}, \digamma q_{j_r})).\end{aligned} \tag{6}$$

Now,

$$\begin{aligned}\mathcal{L}(q_{i_r}, q_{j_r}, \digamma q_{j_r}) &= \max\left\{\begin{array}{c}\frac{G(q_{i_r}, \digamma q_{i_r}, \digamma^2 q_{i_r})\cdot G(q_{j_r}, \digamma q_{j_r}, \digamma^2 q_{j_r})}{G(q_{i_r}, q_{j_r}, \digamma q_{j_r})},\\ G(q_{i_r}, q_{j_r}, \digamma q_{j_r}), G(q_{i_r}, \digamma q_{i_r}, \digamma^2 q_{i_r}),\\ G(q_{j_r}, \digamma q_{j_r}, \digamma^2 q_{j_r}),\\ \frac{G(q_{i_r}, q_{j_r}, \digamma q_{j_r}) + G(q_{j_r}, \digamma q_{j_r}, \digamma^2 q_{j_r})}{2}\end{array}\right\}\\ &= \max\left\{\begin{array}{c}\frac{G(q_{i_r}, q_{i_r+1}, q_{i_r+2})\cdot G(q_{j_r}, q_{j_r+1}, q_{j_r+2})}{G(q_{i_r}, q_{j_r}, q_{j_r+1})},\\ G(q_{i_r}, q_{j_r}, q_{j_r+1}), G(q_{i_r}, q_{i_r+1}, q_{i_r+2}),\\ G(q_{j_r}, q_{j_r+1}, q_{j_r+2}),\\ \frac{G(q_{i_r}, q_{j_r}, q_{j_r+1}) + G(q_{j_r}, q_{j_r+1}, q_{j_r+2})}{2}\end{array}\right\}.\end{aligned}$$

Noting that $\lim_{r\to\infty} G(q_{i_r}, q_{i_r+1}, q_{i_r+2}) = 0$, we obtain

$$\lim_{r\to\infty} \mathcal{L}(q_{i_r}, q_{j_r}, \digamma q_{j_r}) = \lim_{r\to\infty} G(q_{i_r}, q_{j_r}, q_{j_r+1}). \tag{7}$$

By rectangle inequality and letting $r \to \infty$, we obtain

$$\begin{aligned}\lim_{r\to\infty} G(q_{i_r}, q_{j_r}, q_{j_r+1}) &\leq \lim_{r\to\infty} (G(q_{i_r}, q_{i_r}, q_{i_r+1}) + G(q_{i_r+1}, q_{j_r+1}, q_{j_r+2})\\ &\quad + G(q_{j_r+2}, q_{j_r}, q_{j_r+1}))\\ &= \lim_{r\to\infty} G(q_{i_r+1}, q_{j_r+1}, q_{j_r+2}).\end{aligned} \tag{8}$$

Given that ϕ is continuous, then by (6), (7) and (8), we have

$$\lim_{r\to\infty}\phi(G(q_{i_r},q_{j_r},q_{j_r+1}))\leq\lim_{r\to\infty}f(q_{i_r-1},q_{j_r-1},q_{j_r})\cdot\lim_{r\to\infty}\phi(G(q_{i_r},q_{j_r},q_{j_r+1})).$$

If $\lim_{r\to\infty}G(q_{i_r},q_{j_r},q_{j_r+1})=\epsilon>0$, then we must have that

$$\lim_{r\to\infty}f(q_{i_r},q_{j_r},q_{j_r+1})=1.$$

Since $f\in\mathcal{A}(G)$, then it implies that

$$\lim_{r\to\infty}G(q_{i_r},q_{j_r},q_{j_r+1})=0,$$

by which we obtain a contradiction. Therefore, the sequence $\{q_i\}_{i\in\mathbb{N}}$ is G-Cauchy. Hence, we can find a point $\varrho\in\xi$ satisfying $\lim_{i\to\infty}q_i=\varrho$. Noting the continuity of $\digamma$, we see that

$$\varrho=\lim_{i\to\infty}q_{i+1}=\lim_{i\to\infty}\digamma q_i=\digamma\varrho.$$

Hence, $\digamma\varrho=\varrho$, that is, ϱ is a FP of $\digamma$ in ξ.

In what follows, we define the notion of generalized Dass-Gupta-type (G-α-ϕ)-contraction in G-MSs and obtain a few relevant FP results.

Definition 15 Let (ξ,G) be a G-MS and $\digamma:\xi\longrightarrow\xi$ be a self-mapping on ξ. Suppose there exist $\phi\in\Phi$, $f\in\mathcal{A}(G)$ and $\alpha:\xi\times\xi\times\xi\longrightarrow\mathbb{R}_+$ such that for all $q,b\in\xi$,

(θ_4) $\quad\alpha(q,b,\digamma b)\phi(G(\digamma q,\digamma b,\digamma^2 b))\leq f(q,b,\digamma b)\phi(\mathcal{P}(q,b,\digamma b))$,

where

$$\mathcal{P}(q,b,\digamma b)=\max\left\{\begin{array}{c}\frac{G(q,\digamma q,\digamma^2 q)(1+G(b,\digamma b,\digamma^2 b))}{1+G(q,b,\digamma b)},\\ G(q,b,\digamma b),\\ \frac{G(b,\digamma b,\digamma^2 b)(1+G(q,\digamma q,\digamma^2 q))}{1+G(q,b,\digamma b)}\end{array}\right\}.$$

Then $\digamma$ is referred to as a generalized Dass-Gupta-type (G-α-ϕ)-contraction.

Theorem 2 *Let (ξ,G) be a complete G-MS and $\digamma:\xi\longrightarrow\xi$ be a self-mapping satisfying (θ_4). Assume additionally that:*

(i) $\digamma$ is triangular (G-α)-orbital admissible;
(ii) $\digamma$ is continuous;
(iii) $\alpha(q_0,\digamma q_0,\digamma^2 q_0)\geq 1$ for a particular $q_0\in\xi$.

Then $\digamma$ has a FP in ξ.

Proof Let $q_0 \in \xi$ be a random point and let $\{q_i\}_{i\in\mathbb{N}}$ be a sequence in ξ such that $q_i = F^i q_0$ for all $i \in \mathbb{N}$. Assume we can find a point $j \in \mathbb{N}$ satisfying $Fq_j = q_{j+1} = q_j$. Then obviously, q_j is a FP of F. So, we presume that $q_i \neq q_{i-1}$ for all $i \in \mathbb{N}$. Since F is a generalized Dass-Gupta-type (G-α-ϕ)-contraction, then from (θ_4) and Lemma 1, we obtain

$$\begin{aligned}\phi(G(q_i, q_{i+1}, q_{i+2})) &\leq \alpha(q_{i-1}, q_i, q_{i+1})\phi(G(q_i, q_{i+1}, q_{i+2}))\\ &= \alpha(q_{i-1}, q_i, q_{i+1})\phi(G(Fq_{i-1}, Fq_i, F^2 q_i))\\ &\leq f(q_{i-1}, q_i, q_{i+1})\phi(\mathcal{P}(q_{i-1}, q_i, Fq_i))\\ &< \phi(\mathcal{P}(q_{i-1}, q_i, Fq_i)).\end{aligned} \tag{9}$$

Now,

$$\begin{aligned}\mathcal{P}(q_{i-1}, q_i, Fq_i) &= \max\left\{\begin{array}{c}\frac{G(q_{i-1},Fq_{i-1},F^2q_{i-1})(1+G(q_i,Fq_i,F^2q_i))}{1+G(q_{i-1},q_i,Fq_i)},\\ G(q_{i-1}, q_i, Fq_i),\\ \frac{G(q_i,Fq_i,F^2q_i)(1+G(q_{i-1},Fq_{i-1},F^2q_{i-1}))}{1+G(q_{i-1},q_i,Fq_i)}\end{array}\right\}\\ &= \max\left\{\begin{array}{c}\frac{G(q_{i-1},q_i,q_{i+1})(1+G(q_i,q_{i+1},q_{i+2}))}{1+G(q_{i-1},q_i,q_{i+1})},\\ G(q_{i-1}, q_i, q_{i+1}),\\ \frac{G(q_i,q_{i+1},q_{i+2})(1+G(q_{i-1},q_i,q_{i+1}))}{1+G(q_{i-1},q_i,q_{i+1})}\end{array}\right\}\\ &= \max\left\{\begin{array}{c}\frac{G(q_{i-1},q_i,q_{i+1})(1+G(q_i,q_{i+1},q_{i+2}))}{1+G(q_{i-1},q_i,q_{i+1})},\\ G(q_{i-1}, q_i, q_{i+1}), G(q_i, q_{i+1}, q_{i+2})\end{array}\right\}.\end{aligned}$$

If we assume that

$$G(q_{i-1}, q_i, q_{i+1}) \leq G(q_i, q_{i+1}, q_{i+2}),$$

then (9) becomes

$$\phi(G(q_i, q_{i+1}, q_{i+2})) < \phi(\mathcal{P}(q_{i-1}, q_i, Fq_i)) = \phi(G(q_i, q_{i+1}, q_{i+2})),$$

a contradiction. Hence, for each $i \in \mathbb{N}$, we obtain

$$\phi(G(q_i, q_{i+1}, q_{i+2})) < \phi(G(q_{i-1}, q_i, q_{i+1})). \tag{10}$$

Since $\phi \in \Phi$, then for all $i \in \mathbb{N}$, we have

$$G(q_i, q_{i+1}, q_{i+2}) < G(q_{i-1}, q_i, q_{i+1}).$$

Hence, the sequence $\{G(q_i, q_{i+1}, q_{i+2})\}_{i\in\mathbb{N}}$ is non-negative and decreasing. Therefore, there exists $l \geq 0$ such that $\lim_{i\to\infty} G(q_i, q_{i+1}, q_{i+2}) = l$. In similar manner as in Theorem 1, we demonstrate that $l = 0$. Therefore,

$$\lim_{i\to\infty} G(q_i, q_{i+1}, q_{i+2}) = 0.$$

Next, we show that the sequence $\{q_i\}_{i\in\mathbb{N}}$ is G-Cauchy in ξ. Assume contrary that $\{q_i\}_{i\in\mathbb{N}}$ is not G-Cauchy. Then we can find $\epsilon > 0$ such that for all $r \in \mathbb{N}$, there exist j_r, i_r with $j_r \geq i_r > r$ such that $G(q_{i_r}, q_{j_r}, q_{j_r+1}) \geq \epsilon$. We can further assume that $G(q_{i_r}, q_{j_r-1}, q_{j_r}) < \epsilon$ by choosing sufficiently small j_r. In that case, for each $r \in \mathbb{N}$, we see that

$$\begin{aligned}\epsilon \leq G(q_{i_r}, q_{j_r}, q_{j_r+1}) &\leq G(q_{i_r}, q_{j_r-1}, q_{j_r}) + G(q_{j_r-1}, q_{j_r}, q_{j_r+1})\\ &\leq \epsilon + G(q_{j_r-1}, q_{j_r}, q_{j_r+1}).\end{aligned}$$

Letting $r \to \infty$, we obtain

$$\lim_{r\to\infty} G(q_{i_r}, q_{j_r}, q_{j_r+1}) = \epsilon.$$

Notice that for any $r \in \mathbb{N}$,

$$\begin{aligned}\phi(G(q_{i_r+1}, q_{j_r+1}, q_{j_r+2})) &\leq \alpha(q_{i_r}, q_{j_r}, q_{j_r+1})\phi(G(q_{i_r+1}, q_{j_r+1}, q_{j_r+2}))\\ &= \alpha(q_{i_r}, q_{j_r}, q_{j_r+1})\phi(G(\digamma q_{i_r}, \digamma q_{j_r}, \digamma^2 q_{j_r}))\\ &\leq f(q_{i_r}, q_{j_r}, q_{j_r+1})\phi(\mathcal{P}(q_{i_r}, q_{j_r}, \digamma q_{j_r})),\end{aligned} \tag{11}$$

from which

$$\begin{aligned}\mathcal{P}(q_{i_r}, q_{j_r}, \digamma q_{j_r}) &= \max\left\{\begin{array}{c}\frac{G(q_{i_r}, \digamma q_{i_r}, \digamma^2 q_{i_r})(1+G(q_{j_r}, \digamma q_{j_r}, \digamma^2 q_{j_r}))}{1+G(q_{i_r}, q_{j_r}, \digamma q_{j_r})},\\ G(q_{i_r}, q_{j_r}, \digamma q_{j_r}),\\ \frac{G(q_{j_r}, \digamma q_{j_r}, \digamma^2 q_{j_r})(1+G(q_{i_r}, \digamma q_{i_r}, \digamma^2 q_{i_r}))}{1+G(q_{i_r}, q_{j_r}, \digamma q_{j_r})}\end{array}\right\}\\ &= \max\left\{\begin{array}{c}\frac{G(q_{i_r}, q_{i_r+1}, q_{i_r+2})(1+G(q_{j_r}, q_{j_r+1}, q_{j_r+2}))}{1+G(q_{i_r}, q_{j_r}, q_{j_r+1})},\\ G(q_{i_r}, q_{j_r}, q_{j_r+1}),\\ \frac{G(q_{j_r}, q_{j_r+1}, q_{j_r+2})(1+G(q_{i_r}, q_{i_r+1}, q_{i_r+2}))}{1+G(q_{i_r}, q_{j_r}, q_{j_r+1})}\end{array}\right\}.\end{aligned}$$

Noting that $\lim_{r\to\infty} G(q_{i_r}, q_{i_r+1}, q_{i_r+2}) = 0$, we get

$$\lim_{r\to\infty} \mathcal{P}(q_{i_r}, q_{j_r}, \digamma q_{j_r}) = \lim_{r\to\infty} G(q_{i_r}, q_{j_r}, q_{j_r+1}). \tag{12}$$

By rectangle inequality and letting $r \to \infty$, we obtain

$$\begin{aligned}\lim_{r\to\infty} G(q_{i_r}, q_{j_r}, q_{j_r+1}) &\leq \lim_{r\to\infty}(G(q_{i_r}, q_{i_r}, q_{i_r+1}) + G(q_{i_r+1}, q_{j_r+1}, q_{j_r+2})\\ &\quad + G(q_{j_r+2}, q_{j_r}, q_{j_r+1}))\\ &= \lim_{r\to\infty} G(q_{i_r+1}, q_{j_r+1}, q_{j_r+2}).\end{aligned} \tag{13}$$

Given that ϕ is continuous, then by (11), (12) and (13), we have

$$\lim_{r\to\infty}\phi(G(q_{i_r},q_{j_r},q_{j_r+1}))\leq\lim_{r\to\infty}f(q_{i_r},q_{j_r},q_{j_r+1})\cdot\lim_{r\to\infty}\phi(G(q_{i_r},q_{j_r},q_{j_r+1})).$$

If $\lim_{r\to\infty} G(q_{i_r},q_{j_r},q_{j_r+1})=\epsilon>0$, then we must have that

$$\lim_{r\to\infty}f(q_{i_r},q_{j_r},q_{j_r+1})=1.$$

Since $f\in\mathcal{A}(G)$, then it implies that

$$\lim_{r\to\infty}G(q_{i_r},q_{j_r},q_{j_r+1})=0,$$

by which we obtain a contradiction. Therefore, the sequence $\{q_i\}_{i\in\mathbb{N}}$ is G-Cauchy. Hence, we can find a point $\varrho\in\xi$ satisfying $\lim_{i\to\infty}q_i=\varrho$. Noting the continuity of $\digamma$, we deduce

$$\varrho=\lim_{i\to\infty}q_{i+1}=\lim_{i\to\infty}\digamma q_i=\digamma\varrho.$$

Hence, $\digamma\varrho=\varrho$, that is, ϱ is a FP of $\digamma$ in ξ.

In the following results, we investigate the existence of FP of the self-mapping $\digamma$ when the continuity condition of $\digamma$ is replaced by the regularity of the G-MS (ξ,G).

Theorem 3 *Let (ξ,G) be a complete G-MS and let $\digamma:\xi\longrightarrow\xi$ be a self-mapping satisfying (θ_1) for all $q,b\in\xi$, $\phi\in\Phi$ and $f\in\mathcal{A}(G)$, under the condition that*

$$\lim_{i\to\infty}f(q_i,b_i,\digamma b_i)=1\Rightarrow\lim_{i\to\infty}G(\digamma q_i,\digamma b_i,\digamma^2 b_i)=0.$$

Assume additionally that:

(i) $\digamma$ is triangular $(G$-$\alpha)$-orbital admissible;
(ii) (ξ,G) is α-regular;
(iii) $\alpha(q_0,\digamma q_0,\digamma^2 q_0)\geq 1$ for a particular $q_0\in\xi$.

Then $\digamma$ has a FP in ξ.

Proof Let $q_0\in\xi$ be an arbitrary and let $\{q_i\}_{i\in\mathbb{N}}$ be a sequence in ξ such that $q_i=\digamma^i q_0$ for all $i\in\mathbb{N}$. In Theorem 1, we have obtained a point $\varrho\in\xi$ for which $q_i\longrightarrow\varrho$ as $i\to\infty$. Given that (ξ,G) is regular with respect to α, then $\alpha(q_i,\varrho,\digamma\varrho)\geq 1$ for all $i\in\mathbb{N}$. Hence, by (θ_1), we see that

$$\begin{aligned}\phi(G(q_{i+1},\digamma\varrho,\digamma^2\varrho))&=\phi(G(\digamma q_i,\digamma\varrho,\digamma^2\varrho))\\&\leq\alpha(q_i,\varrho,\digamma\varrho)\phi(G(\digamma q_i,\digamma\varrho,\digamma^2\varrho))\\&\leq f(q_i,\varrho,\digamma\varrho)\phi(\mathcal{L}(q_i,\varrho,\digamma\varrho)),\end{aligned}\tag{14}$$

where

$$\mathcal{L}(q_i, \varrho, \digamma\varrho) = \max\left\{\begin{array}{c} \frac{G(q_i, \digamma q_i, \digamma^2 q_i)\cdot G(\varrho, \digamma\varrho, \digamma^2\varrho)}{G(q_i, \varrho, \digamma\varrho)}, \\ G(q_i, \varrho, \digamma\varrho), G(q_i, \digamma q_i, \digamma^2 q_i), \\ G(\varrho, \digamma\varrho, \digamma^2\varrho), \\ \frac{G(q_i, \varrho, \digamma\varrho)+G(\varrho, \digamma\varrho, \digamma^2\varrho)}{2} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} \frac{G(q_i, q_{i+1}, q_{i+2})\cdot G(\varrho, \digamma\varrho, \digamma^2\varrho)}{G(q_i, \varrho, \digamma\varrho)}, \\ G(q_i, \varrho, \digamma\varrho), G(q_i, q_{i+1}, q_{i+2}), \\ G(\varrho, \digamma\varrho, \digamma^2\varrho), \\ \frac{G(q_i, \varrho, \digamma\varrho)+G(\varrho, \digamma\varrho, \digamma^2\varrho)}{2} \end{array}\right\}.$$

Since $\lim_{i\to\infty} G(q_i, q_{i+1}, q_{i+2}) = 0$, then

$$\begin{aligned} \lim_{i\to\infty} \mathcal{L}(q_i, \varrho, \digamma\varrho) &= \max\left\{G(\varrho, \varrho, \digamma\varrho), G(\varrho, \digamma\varrho, \digamma^2\varrho), \frac{G(\varrho, \varrho, \digamma\varrho) + G(\varrho, \digamma\varrho, \digamma^2\varrho)}{2}\right\} \\ &= G(\varrho, \digamma\varrho, \digamma^2\varrho). \end{aligned}$$

By the continuity of ϕ, inequality (14) yields $\lim_{i\to\infty} f(q_i, \varrho, \digamma\varrho) = 1$, implying that

$$G(\varrho, \varrho, \digamma\varrho) \leq G(\varrho, \digamma\varrho, \digamma^2\varrho) = \lim_{i\to\infty} G(q_{i+1}, \digamma\varrho, \digamma^2\varrho) = \lim_{i\to\infty} G(\digamma q_i, \digamma\varrho, \digamma^2\varrho) = 0.$$

Therefore, $\digamma\varrho = \varrho$.

Theorem 4 *Let (ξ, G) be a complete G-MS and let $\digamma : \xi \longrightarrow \xi$ be a self-mapping satisfying (θ_4) for all $q, b \in \xi$, $\phi \in \Phi$ and $f \in \mathcal{A}(G)$, under the condition*

$$\lim_{i\to\infty} f(q_i, b_i, \digamma b_i) = 1 \Rightarrow \lim_{i\to\infty} G(\digamma q_i, \digamma b_i, \digamma^2 b_i) = 0.$$

Assume additionally that:

(i) $\digamma$ is triangular $(G$-$\alpha)$-orbital admissible;
(ii) (ξ, G) is α-regular;
(iii) $\alpha(q_0, \digamma q_0, \digamma^2 q_0) \geq 1$ for a particular $q_0 \in \xi$.

Then $\digamma$ has a FP in ξ.

Proof Following the same arguments as in Theorem 3 and noting (θ_4), we obtain

$$\begin{aligned} \phi(G(q_{i+1}, \digamma\varrho, \digamma^2\varrho)) &= \phi(G(\digamma q_i, \digamma\varrho, \digamma^2\varrho)) \\ &\leq \alpha(q_i, \varrho, \digamma\varrho)\phi(G(\digamma q_i, \digamma\varrho, \digamma^2\varrho)) \\ &\leq f(q_i, \varrho, \digamma\varrho)\phi(\mathcal{P}(q_i, \varrho, \digamma\varrho)), \end{aligned} \tag{15}$$

where

$$\mathcal{P}(q_i, \varrho, \digamma\varrho) = \max\left\{\begin{array}{c} \frac{G(q_i, \digamma q_i, \digamma^2 q_i)(1+G(\varrho, \digamma\varrho, \digamma^2\varrho))}{1+G(q_i, \varrho, \digamma\varrho)}, \\ G(q_i, \varrho, \digamma\varrho), \\ \frac{G(\varrho, \digamma\varrho, \digamma^2\varrho)(1+G(q_i, \digamma q_i, \digamma^2 q_i))}{1+G(q_i, \varrho, \digamma\varrho)} \end{array}\right\}$$

$$= \max \left\{ \begin{array}{c} \frac{G(q_i, q_{i+1}, q_{i+2})(1+G(\varrho, F\varrho, F^2\varrho))}{1+G(q_i, \varrho, F\varrho)}, \\ G(q_i, \varrho, F\varrho), \\ \frac{G(\varrho, F\varrho, F^2\varrho)(1+G(q_i, q_{i+1}, q_{i+2}))}{1+G(q_i, \varrho, F\varrho)} \end{array} \right\}.$$

Since $\lim_{i\to\infty} G(q_i, q_{i+1}, q_{i+2}) = 0$, then

$$\begin{aligned} \lim_{i\to\infty} \mathcal{P}(q_i, \varrho, F\varrho) &= \max \left\{ G(\varrho, \varrho, F\varrho), \frac{G(\varrho, F\varrho, F^2\varrho)}{1 + G(\varrho, \varrho, F\varrho)} \right\} \\ &\leq \max\{G(\varrho, \varrho, F\varrho), G(\varrho, F\varrho, F^2\varrho)\} \\ &= G(\varrho, F\varrho, F^2\varrho). \end{aligned}$$

By similar analysis as in Theorem 3, we have $F\varrho = \varrho$, that is, ϱ is a FP of F.

In the following results, we examine the uniqueness of FP of F under certain supplementary assumptions.
We denote by $Fix(F)$, the set of all FPs of F in ξ, that is,

$$Fix(F) = \{q \in \xi : Fq = q\}.$$

Theorem 5 *Suppose in Theorem 1, we have the additional hypothesis that* $\alpha(q, b, Fb) \geq 1$ *for all* $q, b \in Fix(F)$*. Then the* FP *of* F *is unique.*

Proof Let $v, \varrho \in Fix(F)$ be such that $v \neq \varrho$. Then from (4) and the supplementary assumptions, we obtain

$$\begin{aligned} \phi(G(v, \varrho, F\varrho)) &= \phi(G(Fv, F\varrho, F^2\varrho)) \\ &\leq \alpha(v, \varrho, F\varrho)\phi(G(Fv, F\varrho, F^2\varrho)) \\ &\leq f(v, \varrho, F\varrho)\phi(\mathcal{L}(v, \varrho, F\varrho)) \\ &< \phi(\mathcal{L}(v, \varrho, F\varrho)), \end{aligned}$$

from which

$$\mathcal{L}(v, \varrho, F\varrho) = \max \left\{ \begin{array}{c} \frac{G(v, Fv, F^2v)\cdot G(\varrho, F\varrho, F^2\varrho)}{G(v, \varrho, F\varrho)}, \\ G(v, \varrho, F\varrho), G(v, Fv, F^2v), \\ G(\varrho, F\varrho, F^2\varrho), \\ \frac{G(v, \varrho, F\varrho)+G(\varrho, F\varrho, F^2\varrho)}{2} \end{array} \right\} = G(v, \varrho, F\varrho),$$

implying that

$$\phi(G(v, \varrho, F\varrho)) < \phi(G(v, \varrho, F\varrho)),$$

a contradiction. Hence, $v = \varrho$, implying uniqueness of the FP of F.

Theorem 6 *Suppose in Theorem 2, we have the additional hypothesis that* $\alpha(q, b, Fb) \geq 1$ *for all* $q, b \in Fix(F)$*. Then the* FP *of* F *is unique.*

The proof is similar to that of Theorem 5.

Remark 1 Note that Theorems 1, 3 and 5 are still valid if we replace condition (θ_1) with either of conditions (θ_2) and (θ_3).

Example 3 Let $\xi = [-1, 1]$ and define $G : \xi \times \xi \times \xi \longrightarrow \mathbb{R}_+$ by

$$G(q, b, t) = |q - b| + |q - t| + |b - t|, \quad \forall b, q, t \in \xi.$$

Then (ξ, G) is a complete G-MS. Let $\digamma : \xi \longrightarrow \xi$ be a self-mapping on ξ defined by

$$\digamma q = \begin{cases} \frac{q}{5}, & q \in \{-1, 1\}, \\ \frac{1}{5}, & q \in (-1, 1), \end{cases}$$

for all $q \in \xi$. Define $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $\phi(b) = \frac{b}{2}$ for all $b \geq 0$, $\alpha : \xi \times \xi \times \xi \longrightarrow \mathbb{R}_+$ by

$$\alpha(q, b, t) = \begin{cases} 1, & \text{if } q, b, t \in \{-1\} \cup [0, 1], \\ 0, & \text{otherwise}, \end{cases}$$

and $f : \xi \times \xi \times \xi \longrightarrow [0, 1)$ by

$$f(q, b, t) = \begin{cases} \frac{\ln(1+|q|+|b|+|2t|)}{|q|+|b|+|2t|}, & \text{if } q \neq 0 \text{ or } b \neq 0 \text{ or } t \neq 0; \\ 0, & \text{if } q = b = t = 0. \end{cases}$$

Then obviously, $\digamma$ is continuous for all $q \in \xi$, $\phi \in \Phi$, $\digamma$ is triangular $(G$-$\alpha)$-orbital admissible and $f \in \mathcal{A}(G)$. Also, there exists $q_0 = \frac{1}{2} \in \xi$ such that $\alpha(\frac{1}{2}, \digamma(\frac{1}{2}), \digamma^2(\frac{1}{2}))$ $= \alpha(\frac{1}{2}, \frac{1}{5}, \frac{1}{5}) \geq 1$. Hence, conditions (i)-(iii) of Theorem 1 are satisfied.

To see that $\digamma$ satisfies (θ_1), let $\phi(b) = \frac{b}{2}$ for all $b \geq 0$. Observe that $\alpha(q, b, \digamma b) = 0$ for all $q, b \in (-1, 0)$ and $G(\digamma q, \digamma b, \digamma^2 b) = 0$ for all $q, b \in (-1, 1)$. Hence, condition (θ_1) is satisfied for all $q, b \in (-1, 1)$.

Now for $q, b \in \{-1, 1\}$, if $q = b = 1$, then $G(\digamma q, \digamma b, \digamma^2 b) = 0$. If $q = b = -1$, then

$$\begin{aligned}
\alpha(q, b, \digamma b)\phi(G(\digamma q, \digamma b, \digamma^2 b)) &= \alpha\left(-1, -1, \frac{-1}{5}\right)\phi\left(G\left(\frac{-1}{5}, \frac{-1}{5}, \frac{1}{5}\right)\right) \\
&= \frac{1}{2}\left(\frac{4}{5}\right) = \frac{2}{5} \\
&< \frac{9}{10} = \frac{1}{2} \cdot \frac{1}{2}\left(\frac{18}{5}\right) \\
&= \frac{1}{2} \cdot \frac{1}{2}\left(\max\left\{\frac{18}{5}, \frac{12}{5}, \frac{12}{5}, \frac{8}{5}, 2\right\}\right) \\
&= f\left(-1, -1, \frac{-1}{5}\right)\phi\left(\mathcal{L}\left(-1, -1, \frac{-1}{5}\right)\right) \\
&= f(q, b, \digamma b)\phi(\mathcal{L}(q, b, \digamma b)).
\end{aligned}$$

If $q \neq b$, then

$$\begin{aligned}\alpha(q,b,Fb)\phi(G(Fq,Fb,F^2b)) &= \alpha\left(1,-1,\frac{-1}{5}\right)\phi\left(G\left(\frac{1}{5},\frac{-1}{5},\frac{1}{5}\right)\right)\\ &= \alpha\left(-1,1,\frac{1}{5}\right)\phi\left(G\left(\frac{-1}{5},\frac{1}{5},\frac{1}{5}\right)\right)\\ &= \frac{1}{2}\left(\frac{4}{5}\right)=\frac{2}{5}\\ &< 1 = \frac{1}{2}\cdot\frac{1}{2}(4)\\ &= f\left(1,-1,\frac{-1}{5}\right)\phi\left(\mathcal{L}\left(1,-1,\frac{-1}{5}\right)\right)\\ &= f\left(-1,1,\frac{1}{5}\right)\phi\left(\mathcal{L}\left(-1,1,\frac{1}{5}\right)\right)\\ &= f(q,b,Fb)\phi(\mathcal{L}(q,b,Fb)).\end{aligned}$$

This implies that condition (θ_1) is satisfied for all $q, b \in \{-1, 1\}$. Hence, condition (θ_1) is verified for all $q, b \in \xi$. Therefore, F is a Jaggi-type (G-α-ϕ)-contraction that verifies all the hypotheses of Theorem 1. Consequently, $q = \frac{1}{5}$ is the FP of F.

To see that our obtained theorem is indeed a generalization of [22], let ξ be endowed with the Euclidean metric d. Then (ξ, d) is a complete MS. However, $\mathcal{R}(q, b)$ is undefined for all $q, b \in \xi$, whenever $q = b$. Hence, condition (I_1) cannot be applied under these assumptions.

Therefore, Jaggi-type (G-α-ϕ)-contraction is not Jaggi-type α-h-ϕ-contraction defined by [22]. Subsequently, Theorem 2.2 of [22] cannot be applied in this example.

4 Consequences

In this section, some immediate deductions from our main results are obtained and analyzed.

Let Ψ be the family of all right upper semi-continuous functions $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\psi^{-1}\{0\} = \{0\}$ and $\psi(b) < b$ for all $b > 0$.

Also, denote by $\mathcal{G}$ the family of all functions $\rho : \mathbb{R}_+ \longrightarrow [0, 1)$ such that

$$\rho(t_i) \longrightarrow 1 \;\Rightarrow\; t_i \longrightarrow 0.$$

The function $\rho \in \mathcal{G}$ is called the Geraghty function [15, 21, 32, 33].

Definition 16 Let (ξ, G) be a G-MS and $F : \xi \longrightarrow \xi$ be a self-mapping on ξ such that

(θ_5) $\alpha(q,b,Fb)\phi(G(Fq,Fb,F^2b)) \leq \rho(\phi(\mathcal{L}(q,b,Fb)))\phi(\mathcal{L}(q,b,Fb))$;
(θ_6) $\phi(G(Fq,Fb,F^2b)) \leq \phi(G(q,b,Fb)) - \varphi(G(q,b,Fb))$;
(θ_7) $G(Fq,Fb,F^2b) \leq \psi(G(q,b,Fb))$,

for all $q, b \in \xi$, $\rho \in \mathcal{G}$, $\phi, \varphi \in \Phi$ and $\psi \in \Psi$, where $\mathcal{L}(q, b, \digamma b)$ is as given in Definition 14.

Corollary 1 *Let (ξ, G) be a complete G-MS and let $\digamma : \xi \longrightarrow \xi$ be a self-mapping satisfying (θ_5). Assume supplementary that:*

(i) $\digamma$ is triangular $(G$-$\alpha)$-orbital admissible;
(ii) $\digamma$ is continuous or (ξ, G) is α-regular;
(iii) $\alpha(q_0, \digamma q_0, \digamma^2 q_0) \geq 1$ for a particular $q_0 \in \xi$.

Then $\digamma$ has a FP in ξ.

Proof Let $f : \xi \times \xi \times \xi \longrightarrow \mathbb{R}_+$ be defined by

$$f(q, b, \digamma b) = \rho(\phi(\mathcal{L}(q, b, \digamma b))), \quad \forall q, b \in \xi.$$

If there exist sequences $\{q_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$ in ξ such that $\lim_{i\to\infty} f(q_i, b_i, \digamma b_i) = 1$, then $\lim_{i\to\infty} \phi(\mathcal{L}(q_i, b_i, \digamma b_i)) = 0$. The fact that ϕ is continuous and $\phi^{-1}\{0\} = \{0\}$ implies that $\lim_{i\to\infty} \mathcal{L}(q_i, b_i, \digamma b_i) = 0$. Therefore,

$$\lim_{i\to\infty} G(q_i, b_i, \digamma b_i) = \lim_{i\to\infty} G(q_i, \digamma q_i, \digamma^2 q_i) = \lim_{i\to\infty} G(b_i, \digamma b_i, \digamma^2 b_i) = 0. \quad (16)$$

Hence, $f \in \mathcal{A}(G)$. Thus, (θ_5) resolves to

$$\alpha(q, b, \digamma b)\phi(G(\digamma q, \digamma b, \digamma^2 b)) \leq f(q, b, \digamma b)\phi(\mathcal{L}(q, b, \digamma b)), \quad \forall q, b \in \xi.$$

From (16) and rectangle inequality, we see that

$$\lim_{i\to\infty} G(\digamma q_i, \digamma b_i, \digamma^2 b_i) = 0.$$

Therefore, all the hypotheses of Theorems 1 and 3 are satisfied. It follows that there exists a point $t \in \xi$ such that $\digamma t = t$.

In the following, we deduce a Dutta-Choudhury-type FP result from our main result.

Corollary 2 (see [13], Theorem 2.1) *Let (ξ, G) be a complete G-MS and $\digamma : \xi \longrightarrow \xi$ be a continuous self-mapping satisfying (θ_6). Then $\digamma$ has a unique FP in ξ.*

Proof Assume in condition (θ_1) that $\alpha(q, b, \digamma b) = 1$ for all $b, q \in \xi$ and let

$$f(q, b, \digamma b) = \begin{cases} 0, & \text{if } q = b \text{ and } b, q \in Fix(\digamma), \\ \frac{\phi(G(q,b,\digamma b)) - \varphi(G(q,b,\digamma b))}{\phi(G(q,b,\digamma b))}, & \text{otherwise.} \end{cases} \quad (17)$$

Suppose $\{G(q_i, b_i, \digamma b_i)\}_{i\in\mathbb{N}}$ is decreasing for any sequences $\{q_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$ in ξ. Assume further that $\lim_{i\to\infty} f(q_i, b_i, \digamma b_i) = 1$. To see that $\lim_{i\to\infty} G(q_i, b_i, \digamma b_i) = 0$, suppose contrary that $\lim_{i\to\infty} G(q_i, b_i, \digamma b_i) = l > 0$. By the continuity of ϕ and φ, we have

$$\begin{aligned}\lim_{i\to\infty} f(q_i, b_i, \digamma b_i) &= \lim_{i\to\infty} \frac{\phi(G(q_i, b_i, \digamma b_i)) - \varphi(G(q_i, b_i, \digamma b_i))}{\phi(G(q_i, b_i, \digamma b_i))} \\ &= \frac{\phi(l) - \varphi(l)}{\phi(l)} = 1,\end{aligned}$$

implying that $\varphi(l) = 0$, and so $l = 0$, a contradiction. Therefore, we have $\lim_{i\to\infty} G(q_i, b_i, \digamma b_i) = 0$, from which we have $f \in \mathcal{A}(G)$. It follows from (θ_6) and (17) that

$$\begin{aligned}\phi(G(\digamma q, \digamma b, \digamma^2 b)) &\le f(q, b, \digamma b)\phi(G(q, b, \digamma b)) \\ &\le f(q, b, \digamma b)\phi(\mathcal{L}(q, b, \digamma b)), \quad \text{for all } b, q \in \xi. \end{aligned} \tag{18}$$

Given the above inequality (18), we can deduce from Corollary 1 and Theorem 5 that there exists a unique point $t \in \xi$ such that $\digamma t = t$.

The following is a generalization of the result of Boyd and Wong (see [22]) in the framework of G-MSs.

Corollary 3 *Let (ξ, G) be a complete G-MS and $\digamma : \xi \longrightarrow \xi$ be a continuous self-mapping satisfying (θ_7). Then $\digamma$ has a unique FP in ξ.*

Proof Let $f : \xi \times \xi \times \xi \longrightarrow [0, 1)$ be defined by

$$f(q, b, \digamma b) = \begin{cases} 0, & \text{if } q = b \text{ and } q, b \in Fix(\digamma), \\ \frac{\psi(G(q,b,\digamma b))}{G(q,b,\digamma b)}, & \text{otherwise.} \end{cases}$$

Suppose $\{G(q_i, b_i, \digamma b_i)\}_{i\in\mathbb{N}}$ is decreasing for any sequences $\{q_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$ in ξ. Assume further that $\lim_{i\to\infty} f(q_i, b_i, \digamma b_i) = 1$. To see that $\lim_{i\to\infty} G(q_i, b_i, \digamma b_i) = 0$, suppose contrary that $\lim_{i\to\infty} G(q_i, b_i, \digamma b_i) = l > 0$. Since $\psi \in \Psi$, we have

$$1 = \lim_{i\to\infty} f(q_i, b_i, \digamma b_i) = \lim_{i\to\infty} \frac{\psi(G(q_i, b_i, \digamma b_i))}{G(q_i, b_i, \digamma b_i)} \le \frac{\psi(l)}{l},$$

implying that $l \le \psi(l)$, a contradiction. Hence, we have $\lim_{i\to\infty} G(q_i, b_i, \digamma b_i) = 0$, from which we have $f \in \mathcal{A}(G)$. Taking $\phi(b) = b$ for all $b \ge 0$, it follows from (θ_7) that

$$\phi(G(\digamma q, \digamma b, \digamma^2 b)) \le f(q, b, \digamma b)\phi(G(q, b, \digamma b)), \quad \forall q, b \in \xi.$$

This satisfies the assumptions of Corollary 1. Hence, the proof is completed.

5 Applications to BVPs

In this section, Theorem 3 is used to study the criteria for which the BVP

$$\begin{cases} \frac{d^2x}{dp^2} = \lambda(p, x(p)), \quad p \in [0, 1],\ x \in \mathbb{R}_+ \\ \\ x(0) = x(1) = 0, \end{cases} \tag{19}$$

where $\lambda : [0, 1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function, has a solution. Ideas in this section are motivated by [19, 22, 27, 30, 37].

Note that (19) corresponds to the integral equation

$$x(p) = \int_0^1 \Omega(p, q)\lambda(q, x(q))ds, \quad p \in [0, 1] \tag{20}$$

where $\Omega(p, q)$ is called the Green function, defined by

$$\Omega(p, q) = \begin{cases} p(1-q), & \text{if } 0 \le q < p \le 1, \\ \\ q(1-p), & \text{if } 0 \le p < q \le 1. \end{cases}$$

Let $\xi = C([0, 1], \mathbb{R})$ be the set of all continuous real-valued functions defined on $[0, 1]$. We equip ξ with the mapping, $\forall x, v, w \in \xi$,

$$G(x, v, w) = \max_{p \in [0,1]} \left(|x(p) - v(p)| + |x(p) - w(p)| + |v(p) - w(p)|\right). \tag{21}$$

Then (ξ, G) is a complete G-MS. Consider the self-mapping $\digamma : \xi \longrightarrow \xi$ defined by

$$\digamma x(p) = \int_0^1 \Omega(p, q)\lambda(q, x(q))ds, \quad p \in [0, 1]. \tag{22}$$

Then, any solution x^* to (19) is a FP of $\digamma$ and conversely.
Now, we study existence conditions of the BVP (19) under the following hypotheses.

Theorem 7 *Let $\psi \in \Psi$ and $\digamma : \xi \longrightarrow \xi$ be a self-mapping on ξ. Assume further that the hypotheses below are satisfied:*

(C_1) *there exists a function $\gamma : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that for all $p \in [0, 1]$ and for all $x, y, z \in \mathbb{R}$ with $\gamma(x, y, z) \ge 0$, we have*
$|\lambda(p, x) - \lambda(p, y)| + |\lambda(p, x) - \lambda(p, z)| + |\lambda(p, y) - \lambda(p, z)| \le \psi(|x - y| + |x - z| + |y - z|)$;

(C_2) *there exists $x' \in \xi$ such that for all $p \in [0, 1]$,*
$\gamma\left(x'(p), \int_0^1 \Omega(p, q)\lambda(q, x'(q))ds, \int_0^1 \Omega(p, q)\lambda(q, \digamma x'(q))ds\right) \ge 0$;

(C_3) *for all* $p \in [0, 1]$ *and for all* $x, v \in \xi$,
$\gamma(x(p), v(p), \digamma v(p)) \geq 0$
$\Rightarrow \gamma\left(\int_0^1 \Omega(p,q)\lambda(q, x(q))ds, \int_0^1 \Omega(p,q)\lambda(q, v(q))ds, \int_0^1 \Omega(p,q)\right.$
$\left.\lambda(q, \digamma v(q))ds\right) \geq 0$*;*

(C_4) *if* $\{x_n\}_{n\in\mathbb{N}}$ *is a sequence in* ξ *such that* $x_n \longrightarrow x$ *in* ξ*, then for all* $p \in [0, 1]$ *and* $n \in \mathbb{N}$*, we have* $\gamma(x_n(p), x_{n+1}(p), x_{n+2}(p)) \geq 0 \Rightarrow \gamma(x_n(p), x(p), x(p)) \geq 0$.

Then, the BVP (19) *has a solution in* ξ.

Proof Taking (21) and (22) into account, let $x, v \in \xi$ such that $\gamma(x(p), v(p), \digamma v(p)) \geq 0$ for all $p \in [0, 1]$. Then,

$$\begin{aligned}
G(\digamma x, \digamma v, \digamma^2 v) &= \max_{p\in[0,1]}\left[\,|\digamma x(p) - \digamma v(p)| + \left|\digamma x(p) - \digamma^2 v(p)\right| + \left|\digamma v(p) - \digamma^2 v(p)\right|\,\right] \\
&\leq \max_{p\in[0,1]}\int_0^1 \Omega(p,q)\left[\,|\lambda(q, x(q)) - \lambda(q, v(q))| + |\lambda(q, x(q)) - \lambda(q, \digamma v(q))|\right. \\
&\quad \left.+ |\lambda(q, v(q)) - \lambda(q, \digamma v(q))|\,\right]ds \\
&\leq \max_{p\in[0,1]}\int_0^1 \Omega(p,q)\left[\psi\left(|x(q) - v(q)| + |x(q) - \digamma v(q)| + |v(q) - \digamma v(q)|\right)\right]ds \\
&\leq \max_{p\in[0,1]}\int_0^1 \Omega(p,q)ds\,\psi(G(x, v, \digamma v)) \\
&\leq \psi(G(x, v, \digamma v)).
\end{aligned} \tag{23}$$

Let $\alpha : \xi \times \xi \times \xi \longrightarrow \mathbb{R}_+$ be defined by

$$\alpha(x, v, \digamma v) = \begin{cases} 1, & \text{if } \gamma(x(p), v(p), \digamma v(p)) \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and $f : \xi \times \xi \times \xi \longrightarrow [0, 1)$ by

$$f(x, v, \digamma v) = \begin{cases} 0, & \text{if } x = v \text{ and } x, v \in Fix(\digamma), \\ \frac{\psi(G(x,v,\digamma v))}{G(x,v,\digamma v)}, & \text{otherwise.} \end{cases}$$

Then by (23), we infer that

$$\begin{aligned}
\alpha(x, v, \digamma v)G(\digamma x, \digamma v, \digamma^2 v) \leq \psi(G(x, v, \digamma v)) &= \frac{\psi(G(x, v, \digamma v))}{G(x, v, \digamma v)} \cdot G(x, v, \digamma v) \\
&= f(x, v, \digamma v)G(x, v, \digamma v) \\
&\leq f(x, v, \digamma v)\mathcal{L}(x, v, \digamma v).
\end{aligned} \tag{24}$$

Let $\phi(p) = p$ for all $p \geq 0$. Then (24) corresponds with (θ_1). Now, let $\{x_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ be any two sequences in ξ such that $\lim_{n\to\infty} f(x_n, v_n, \digamma v_n) = 1$ and $\alpha(x_n, v_n, \digamma v_n) \neq 0$ for all $n \in \mathbb{N}$. Then for all $p \in [0, 1]$, we have $\gamma(x_n(p), v_n(p), \digamma v_n(p)) \geq 0$, implying that

$$G(\digamma x_n(p), \digamma v_n(p), \digamma^2 v_n(p)) \leq \psi(G(x_n, v_n, \digamma v_n)).$$

This further implies that

$$G(\digamma x_n, \digamma v_n, \digamma^2 v_n) \leq \psi(G(x_n, v_n, \digamma v_n)).$$

Since $\lim_{n\to\infty} G(x_n, v_n, \digamma v_n) = 0$, then $\lim_{n\to\infty} G(\digamma x_n, \digamma v_n, \digamma^2 v_n) = 0$. By further applying $(C_2) - (C_4)$, we see that $\digamma$ is a Jaggi-type (G-α-ϕ)-contraction that verifies all the hypotheses of Theorem 3. It follows that $\digamma$ has a FP x^* in ξ, which corresponds to a solution of the BVP (19).

Conversely, assume that x^* is a solution of (19), then x^* must also be a solution of (22). This means that x^* is a FP of $\digamma$.

6 Conclusion

The BCP has led to many generalizations of MSs in an effort to explore more FP results. One of such generalizations was proposed by Mustafa and Sims [26], called G-MS and a wealth of FP results were obtained in G-MSs. However, observations were published by some authors, noting that the majority of FP results generated in G-MSs are immediate from their analogs in MSs. To counter these observations, many authors have proposed new ways of studying FP results in G-MSs that do not collapse to corresponding ones in MSs. In line with this, some new classes of (G-α-ϕ)-contractions are proposed in this paper and relevant FP theorems are proved. The novelty of these new contractions lies in the fact that they can be specialized in a few directions, depending on the choice of parameters. Substantial relative examples are established to authenticate the assumptions of our obtained results. Consequently, a handful of corollaries which include some current results in metric fixed point theory are presented and discussed. Furthermore, we examined existence conditions for the solution of a BVP using one of our main results.

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A Note on the Existence of Fixed Points for Rational Type Contraction Map on Orthogonal Metric Spaces

G. Poonguzali, R. Sri Bharathi, and Stojan Radenovic

1 Introduction

Stefan Banach, the founder of functional analysis in 1921, brought in a revolutionizing result known as the principle of Banach contraction, which provides a unique fixed point on complete metric space in his thesis [1], post which the result turned out to be an important tool for the development of fixed point theory. Later, many researchers started working on the existence of fixed points for various maps that include all types of contraction map, all types of nonexpansive maps, the sum of two operators, etc., see [8–13]. One such extension was provided by Singh [6] where the author introduced the rational type of contraction and proved the existence of a fixed point.

From the reference works mention above, we get to understand that the researchers have focused only on only on complete metric space and partially ordered complete metric space. Recently, Gordji et al. [2, 5] introduced a new type of metric space called orthogonal metric space and extended Banach fixed point result to O-contraction map.

A research on idea of Orthogonal metric space is worthy of analyzing as it has more general space where it cannot be compared with partial metric space. Upcoming two examples will explain the necessity to have Orthogonal metric space.

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Example

Consider $M = \mathbb{R}^2$. Define $\perp$ as $u \perp v$ if $< u, v >= 0$ on M. Then $(M, \perp)$ is an O-set, since $u = (0, 0) \perp v$, for all $v \in M$. But $(M, \perp)$ is not a partial order set. Choose $u = (1, 0),\ v = (0, 1)$, $r = (-1, 0)$, it is clear that, $u \perp v$, $v \perp r$ but $u \not\perp r$.

Example

Consider $(M = \mathbb{R}, \leq)$. Then M is partially ordered set but not O- set with $\leq$ relation, because we cannot find any $u \in M$ such that $u \leq p$ or $p \leq u$ for all $p \in \mathbb{R}$.

In [7], Rao, Kalyani established the existence of a fixed point by using Singh, Badshah, and Rathore contraction. In this research paper, we strive to prove. the existence of a fixed point in O-metric space, which is an extension of the fixed point result given by Rao, Seshagiri, Kalyani, and Gemechu, which states.

Theorem 1 ([7]) *Let $(M,\ d,\ \preceq)$ be a complete, partially ordered metric space. Suppose that $\Lambda : M \to M$ is a continuous, non-decreasing function satisfying the following:*

$$d(\Lambda p, \Lambda q) \leq \alpha \left[\frac{d(p, \Lambda p)[1 + d(q, \Lambda q)]}{1 + d(p, q)}\right] + \beta\ [d(p, \Lambda p) + d(q, \Lambda q)] + \gamma\ [d(p, \Lambda q) + d(q, \Lambda p)] + \delta\ d(p, q)$$

for all distinct $p, q \in M$, where α, β, γ and δ are non-negative reals with $0\ \leq\ \alpha + 2(\beta + \gamma) + \delta\ <\ 1$. Then Λ has a fixed point in M.

2 Preliminaries

To provide our result, we need the following basic definitions and results:

Definition 1 ([5]) Let $(M,\ \perp)$ be O-set. A sequence $\{p_n\}_{n\in\mathbb{W}}$ is called orthogonal sequence if, for all $n,\ p_n \perp p_{n+1}$ or, for all $n,\ p_{n+1} \perp p_n$

Example

Let $M = [0, +\infty)$ and define $p \perp q$ if $pq \in \{p, q\}$. Then, by setting

$$p_n = \begin{cases} 1, & \text{if } n \text{ is odd;} \\ \frac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

Here, for all $n,\ p_n \perp p_{n+1}$. Thus $\{p_n\}_{n\in\mathbb{N}}$ is an O-sequence.

Definition 2 If $(M, \perp)$ is an O-set and (M, d) is a metric space, then $(M, \perp, d)$ is called an orthogonal metric space.

Definition 3 ([5]) Let $(M, \perp, d)$ be an orthogonal metric space. Then $\Lambda : M \to M$ is orthogonal continuous or $\perp$-continuous at $p \in M$ if for each O-sequence $\{p_n\}_{n\in\mathbb{N}}$ in M with $p_n \to p$, then $\Lambda(p_n) \to \Lambda(p)$. Also, Λ is $\perp$-continuous on M if Λ is $\perp$-continuous at each $p \in M$.

It is easy to see that every continuous mapping is $\perp$-continuous, and the converse is not true.

Example

Let $M = [0, 1]$ and d be the Euclidean metric, and $p \perp q$ if $pq \in \{p, q\}$ and the function $\Lambda : M \to M$ defined by

$$\Lambda(p) = \begin{cases} \frac{p}{2}, & p \in \mathbb{Q} \cap M, \\ 0 & p \in \mathbb{Q}^c \cap M. \end{cases}$$

Obviously, Λ is not continuous. If $\{p_m\}$ is an arbitrary O-sequence in M, then for all $m,\ p_m = 0$ or 1. Here, only convergent sequences are eventually constant sequences.
Case: 1 There exists $k \in \mathbb{N}$ such that $p_m = 0$ for all $m > k$. Since $\{p_m\}$ converges to $p = 0$, then $\Lambda(p_m) = 0$ for all $m > k$. Since $\{\Lambda(p_m)\}$ is eventually constant, $\{\Lambda(p_m)\}$ converges to $\Lambda(p) = 0$.
Case: 2 There exists $k \in \mathbb{N}$ such that $p_m = 1$ for all $m > k$. Since $\{p_m\}$ converges to $p = \frac{1}{2}$, then $\Lambda(p_m) = 0$, for all $m > k$. Since $\{\Lambda(p_m)\}$ is eventually constant, $\{\Lambda(p_m)\}$ converges to $\Lambda(p) = \frac{1}{2}$. So, Λ is $\perp$-continuous.

Definition 4 ([5]) Let $(M, \perp, d)$ be an orthogonal set with metric d. Then M is orthogonal complete, if every Cauchy O-sequence is convergent.

It is easy to see that every complete metric space is O-complete, and the converse is not true.

Example

Let $M = [-2, 2)$ and suppose that $p \perp q$ if and only if $p \leq q \leq 1$, or $p = 0$. Then $(M, \perp)$ is an O-set. Clearly, M with the Euclidean metric is not a complete metric space. Now, let us take $\{p_n\}$ is a Cauchy O-sequence in M, then there exists a subsequence $\{p_{n_k}\}$ of $\{p_k\}$ for which $p_{n_k} = 0$ for all $n \geq 1$ or there exists a subsequence $\{p_{n_k}\}$ of $\{p_k\}$ for which $p_{n_k} \leq 1$ for all $n \geq 1$, which implies $\{p_{n_k}\}$ converges to a point $p \in [0, 1] \subseteq M$. Since we know that every Cauchy sequence with a convergent subsequence is convergent, $\{p_n\}$ is convergent. Hence $(M, \perp, d)$ is an O-complete metric space.

Definition 5 ([5]) Let $(M, \perp, d)$ be an orthogonal metric space and $0 < \lambda < 1$. A mapping $\Lambda : M \to M$ is said to be orthogonal contraction or $\perp$-contraction with contraction constant λ if $d(\Lambda p, \Lambda q) \leq \lambda\, d(p, q)$ for every $p \perp q$.

Remark 1 Every contraction is an O- contraction, but an O-contraction map need not be a contraction map.

Definition 6 ([5]) Let $(M, \perp)$ be an O-set. A mapping $\Lambda : M \to M$ is said to be $\perp$-preserving if $\Lambda(p) \perp \Lambda(q)$ whenever $p \perp q$. Also, $\Lambda : M \to M$ is said to be weakly $\perp$-preserving if $\Lambda(p) \perp \Lambda(q)$ or $\Lambda(q) \perp \Lambda(p)$ whenever $p \perp q$.

Example

If $M = \mathbb{N}$ and define $\Lambda : M \to M$ as $\Lambda(p) = p^2$ and also define $p \perp q$ as $p - 1 < q$. It is clear that whenever $p \perp q$, then $\Lambda(p) \perp \Lambda(q)$. Hence, Λ is $\perp$-preserving.

It is easy to see that every $\perp$-preserving mapping is weakly $\perp$-preserving. But the converse is not true.

Example

Let $M = \mathbb{N}$ and define $\Lambda : M \to M$ as $\Lambda(p) = \frac{1}{p}$ and also define $p \perp q$ as $p \leq q$. For $p \leq q$, $\frac{1}{q} \leq \frac{1}{p}$ which implies, $\Lambda(q) \perp \Lambda(p)$. Therefore, Λ is weakly $\perp$-preserving, but it is not $\perp$ preserving.

Definition 7 Let $(M, \perp, d)$ be any metric space with an orthogonal relation $\perp$. Then, M becomes M orthogonally complete (O-complete) if for any O-sequence (u_n), which is also Cauchy, should converge in M.

Remark 2 If M is any complete metric space, and if there exists any $\perp$ on M, then it becomes O-complete metric space. The converse of above statement is false.

3 Main Results

In this section, the existence and uniqueness of a fixed point of a self-mapping satisfying the following contraction condition are proved in O-metric space.

3.1 Results Under Generalized Rational Type Contractions

Theorem 2 *Let $(M, \perp, d)$ be any O-complete metric space. Let $\Lambda : M \to M$ be any $\perp$-continuous, $\perp$-preserving map satisfying*

$$d(\Lambda p, \Lambda q) \leq \begin{cases} \lambda d(p,q) + \eta[d(p,\Lambda q) + d(q,\Lambda p)] \\ +\mu \dfrac{d(p,\Lambda p)d(p,\Lambda q) + d(q,\Lambda p)d(q,\Lambda q)}{d(q,\Lambda p) + d(p,\Lambda q)}, & \text{if } A \neq 0; \\ 0, & \text{if } A = 0, \end{cases} \quad (1)$$

for all distinct p, q with $p \perp q$ where $A = d(q, \Lambda p) + d(p, \Lambda q)$ and $\lambda, \eta, \mu \in \mathbb{R}^+$ such that $0 \leq \lambda + 2\eta + \mu < 1$. Then Λ has a unique fixed point.

Proof Since $(M, \perp)$ is an O-set, there exists $p_0 \in M$ such that $p_0 \perp q$ or $q \perp p_0$, for all $q \in M$. It follows that $p_0 \perp \Lambda(p_0)$ or $\Lambda(p_0) \perp p_0$. Without loss of generality, assume that $p_0 \perp \Lambda(p_0)$. Now, construct a sequence by choosing $p_1 = \Lambda(p_0)$, $p_2 = \Lambda(p_1) = \Lambda^2(p_0), \cdots, p_{n+1} = \Lambda(p_n) = \Lambda^{n+1}(p_0)$ for all $n \in \mathbb{W}$. If there exists $n_0 \in \mathbb{W}$ such that $p_{n_0} = p_{n_0+1}$, then from $p_{n_0} = p_{n_0+1} = \Lambda p_{n_0}$. Thus, p_{n_0} is the required fixed point of Λ. Suppose that $p_n \neq p_{n+1}$, for all $n \in \mathbb{W}$. Since $p_1 = \Lambda(p_0)$, we get $p_0 \perp p_1$ and since Λ is $\perp$- preserving, $\Lambda(p_0) \perp \Lambda(p_1)$, which implies $p_1 \perp p_2$. By proceeding we get $p_n \perp p_{n+1}$, for all $n \in \mathbb{W}$. Therefore, $\{p_n\}_{n\in\mathbb{W}}$ is O-sequence. For $p_n, p_{n+1} \in M$ with $p_n \perp p_{n+1}$, we have the following two cases.

Case 1: If there exists $n \in \mathbb{W}$ with $A = d(p_{n-1}, \Lambda p_n) + d(p_n, \Lambda p_{n-1}) = 0$, then $d(p_{n+1}, p_n) = 0$. This implies that $p_{n+1} = p_n$, a contradiction to $p_{n+1} = \Lambda p_n$,. Thus, there exists a fixed point p of Λ.

Case 2: Suppose $A = d(p_{n-1}, \Lambda p_n) + d(p_n, \Lambda p_{n-1}) \neq 0$, for all $n \in \mathbb{W}$, then $d(p_{n+1}, p_n) = d(\Lambda(p_n), \Lambda(p_{n-1}))$. By using the condition (1), we get

$$\begin{aligned} d(p_{n+1}, p_n) \leq\ & \lambda d(p_n, p_{n-1}) + \eta[d(p_n, \Lambda p_{n-1}) + d(p_{n-1}, \Lambda p_n)] \\ & + \mu \frac{d(p_n, \Lambda p_n)d(p_n, \Lambda p_{n-1}) + d(p_{n-1}, \Lambda p_n)d(p_{n-1}, \Lambda p_{n-1})}{d(p_{n-1}, \Lambda p_n) + d(p_n, \Lambda p_{n-1})} \\ \leq\ & \lambda d(p_n, p_{n-1}) + \eta[d(p_n, p_n) + d(p_{n-1}, p_{n+1})] \\ & + \mu \frac{d(p_n, p_{n+1})d(p_n, p_n) + d(p_{n-1}, p_{n+1})d(p_{n-1}, p_n)}{d(p_{n-1}, p_{n+1}) + d(p_n, p_n)} \end{aligned}$$

which implies that

$$d(p_{n+1}, p_n) \leq \lambda d(p_n, p_{n-1}) + \eta d(p_{n-1}, p_{n+1}) + \mu d(p_{n-1}, p_n).$$

By triangular inequality,

$$\begin{aligned} d(p_{n+1}, p_n) &\leq \lambda d(p_n, p_{n-1}) + \eta[d(p_{n-1}, p_n) + d(p_n, p_{n+1})] + \mu d(p_{n-1}, p_n) \\ (1-\eta) d(p_{n+1}, p_n) &\leq (\lambda + \eta + \mu)\, d(p_{n-1}, p_n) \\ d(p_{n+1}, p_n) &\leq \left[\frac{\lambda + \eta + \mu}{1-\eta}\right] d(p_{n-1}, p_n). \end{aligned}$$

By repeating this process, we get

$$d(p_{n+1}, p_n) \leq \left[\frac{\lambda + \eta + \mu}{1-\eta}\right]^n d(p_0, p_1).$$

Thus $d(p_{n+1}, p_n) \leq h^n\, d(p_0, p_1)$, where $h = \dfrac{\lambda + \eta + \mu}{1-\eta} < 1$.

Without loss of generality, assume $m > n$

$$\begin{aligned} d(p_m, p_n) &\leq d(p_m, p_{m-1}) + d(p_{m-1}, p_{m-2}) + ... + d(p_{n+1}, p_n) \\ &\leq (h^{m-1} + h^{m-2} + ... + h^n) d(p_0, p_1) \\ &\leq \frac{h^n}{1-h} d(p_0, p_1), \end{aligned}$$

as $m, n \to +\infty$, $d(p_m, p_n) \to 0$. Thus $\{p_n\}_{n \in \mathbb{W}}$ is Cauchy O-sequence. Since $(M, \perp, d)$ is O-complete metric space, there exists $p \in M$ such that $\lim\limits_{n \to +\infty} p_n = p$. Also we have $\perp$-continuous of Λ implies that

$$\Lambda p = \Lambda \left[\lim_{n \to +\infty} p_n\right] = \lim_{n \to +\infty} \Lambda p_n = \lim_{n \to +\infty} p_{n+1} = p.$$

Thus, p is a fixed point of Λ in M.

Uniqueness: Suppose that $r, s \in M$ is any two fixed point of Λ. Then we have $\Lambda^n(r) = r$ and $\Lambda^n(s) = s$ for all $n \in \mathbb{W}$. By the choice of p_0, we have

$$p_0 \perp r \text{ and } p_0 \perp s \text{ (or) } r \perp p_0 \text{ and } s \perp p_0.$$

Since Λ is $\perp$-preserving, we have

$$\begin{gathered} \Lambda^n(p_0) \perp \Lambda^n(r) \text{ and } \Lambda^n(p_0) \perp \Lambda^n(s) \\ \text{(or)} \\ \Lambda^n(r) \perp \Lambda^n(p_0) \text{ and } \Lambda^n(s) \perp \Lambda^n(p_0) \end{gathered}$$

for all $n \in \mathbb{W}$.

Case (1): If $d(p_0, \Lambda^n r) + d(r, \Lambda^n p_0) \neq 0$ and $d(p_0, \Lambda^n s) + d(s, \Lambda^n p_0) \neq 0$ for all $n \in \mathbb{W}$. Consider,

$$\begin{aligned} d(\Lambda r, \Lambda p_0) &\leq \lambda d(r, p_0) + \eta[d(r, \Lambda p_0) + d(p_0, \Lambda r)] \\ &\quad + \mu \left[\frac{d(r, \Lambda r)d(r, \Lambda p_0) + d(p_0, \Lambda r)d(p_0, \Lambda p_0)}{d(p_0, \Lambda r) + d(r, \Lambda p_0)} \right] \\ &\leq \lambda d(r, p_0) + \eta[d(r, \Lambda p_0) + d(p_0, r)] + \mu \left[\frac{d(p_0, r)d(p_0, \Lambda p_0)}{d(p_0, r) + d(r, \Lambda p_0)} \right] \\ &\leq \lambda d(r, p_0) + \eta[d(r, \Lambda p_0) + d(p_0, r)] + \mu \left[\frac{d(p_0, r)d(p_0, \Lambda p_0)}{d(p_0, \Lambda p_0)} \right] \end{aligned}$$

which concludes,

$$d(\Lambda r, \Lambda p_0) \leq \left[\frac{\lambda + \mu + \eta}{1 - \eta} \right] d(r, p_0).$$

Using the above, we get the following inequality,

$$d(\Lambda^n r, \Lambda^n p_0) \leq \left[\frac{\lambda + \mu + \eta}{1 - \eta} \right]^n d(r, p_0).$$

Similarly, we can get

$$d(\Lambda^n s, \Lambda^n p_0) \leq \left[\frac{\lambda + \mu + \eta}{1 - \eta} \right]^n d(s, p_0).$$

Using triangular inequality, we have

$$\begin{aligned} d(r, s) &= d(\Lambda^n(r), \Lambda^n(s)) \\ &\leq d(\Lambda^n(r), \Lambda^n(p_0)) + d(\Lambda^n(p_0), \Lambda^n(s)) \\ &\leq \left[\frac{\lambda + \eta + \mu}{1 - \eta} \right]^n [d(r, p_0) + d(p_0, s)], \end{aligned}$$

where $\frac{\lambda + \eta + \mu}{1 - \eta} < 1$. This implies that $d(r, s) \to 0$ as $n \to +\infty$. Thus $r = s$.

Case (2): If $d(p_0, \Lambda^n r) + d(r, \Lambda^n p_0) = 0$ and $d(p_0, \Lambda^n s) + d(s, \Lambda^n p_0) = 0$ for some $n \in \mathbb{W}$. Then, it is easy to observe that $d(r, s) = 0$. Hence $r = s$.

Case (3): If for some $n \in \mathbb{W}$, $d(p_0, \Lambda^n r) + d(r, \Lambda^n p_0) = 0$ or $d(p_0, \Lambda^n s) + d(s, \Lambda^n p_0) = 0$. Without loss of generality, assume $d(p_0, \Lambda^n r) + d(r, \Lambda^n p_0) = 0$ and $d(p_0, \Lambda^n s) + d(s, \Lambda^n p_0) \neq 0$. Then $d(p_0, \Lambda^n p_0) = 0$ which implies $p_0 = \Lambda^n p_0$. Using that, we have $d(p_0, p_0) \leq d(\Lambda^n p_0, s) + d(s, \Lambda^n p_0) \neq 0$ which is a contradiction. So case (3) is not possible. □

By replacing the role of continuity in the above said theorem with added conditions, this paper proves the existence of a fixed point and the rest will dealt with in the upcoming theorems.

Theorem 3 *Let $(M, \perp, d)$ be an O-complete metric space. Assume that M satisfies the following:*

$$\textit{if for any } O-\textit{sequence } \{p_n\} \textit{ with } p_n \to p,\ p \in M,\ \textit{then } p_n \perp p \textit{ for all } n \in \mathbb{W}. \tag{2}$$

Let $\Lambda : M \to M$ be $\perp$-preserving map which satisfying the condition (1). Then Λ has a unique fixed point in M.

Proof If there exists $n_0 \in \mathbb{W}$ such that $p_{n_0} = p_{n_0+1} = \Lambda p_{n_0}$. Thus, p_{n_0} is a fixed point, and hence the proof is completed. Suppose that $p_n \neq p_{n+1}$, for all $n \in \mathbb{W}$. By applying Theorem 2, there exists an O-sequence $\{p_n\} \in M$ such that $p_n \to p$, which implies $p_n \perp p$. Since Λ is $\perp$-preserving, we have $\Lambda p_n \perp \Lambda p$.

Consider,

$$d(p, \Lambda p) \leq d(p, p_n) + d(p_n, \Lambda p_n) + d(\Lambda p_n, \Lambda p). \tag{3}$$

Case 1: If $A = d(p, \Lambda p_n) + d(p_n, \Lambda p) \neq 0$, then by using the condition (1), we get

$$\begin{aligned} d(\Lambda p_n, \Lambda p) \leq \lambda d(p_n, p) &+ \eta[d(p_n, \Lambda p) + d(p, \Lambda p_n)] \\ &+ \mu \left[\frac{d(p_n, \Lambda p_n) d(p_n, \Lambda p) + d(p, \Lambda p_n) d(p, \Lambda p)}{d(p, \Lambda p_n) + d(p_n, \Lambda p)} \right]. \end{aligned}$$

As $n \to +\infty$, we obtain,

$$d(p_n, p) \to 0,\ d(p, \Lambda p_n) \to 0,\ d(p_n, \Lambda p_n) \to 0.$$

Subcase 1: If $\lim_{n\to+\infty} d(p_n, \Lambda p) \neq 0$, then we have $d(\Lambda p_n, \Lambda p) \leq \eta\, d(p, \Lambda p)$.

From (3) we have $d(p, \Lambda p)(1-\eta) \leq 0$. Since $(1-\eta) > 0$, we get $d(p, \Lambda p) = 0$. Thus, p is a fixed point of Λ in M.

Subcase 2: Suppose $\lim_{n\to+\infty} d(p_n, \Lambda p) = 0$, then $d(p_n, \Lambda p) = 0$ which implies, $p = \Lambda p$, that is Λ has a fixed point.

Case 2: If $A = d(p, \Lambda p_n) + d(p_n, \Lambda p) = 0$, then by using the condition (1).

As $n \to +\infty$, $d(p, p_n) \to 0,\ d(p_n, \Lambda p_n) \to 0$. From (3) $d(p, \Lambda p) \leq 0$. Thus, we have $d(p, \Lambda p) = 0$. Therefore p is a fixed point of Λ. The proof of uniqueness is same as uniqueness proof of Theorem 2. □

Theorem 4 *Let $(M, \perp, d)$ be an O-complete metric space. Let $\Lambda : M \to M$ be $\perp$-continuous, $\perp$-preserving satisfying*

$$d(\Lambda p, \Lambda q) \leq \begin{cases} \eta[d(p, \Lambda q) + d(q, \Lambda p)] \\ +\mu \dfrac{d(p, \Lambda p) d(p, \Lambda q) + d(q, \Lambda p) d(q, \Lambda q)}{d(q, \Lambda p) + d(p, \Lambda q)}, & \textit{if } A \neq 0 \\ 0, & \textit{if } A = 0 \end{cases}$$

for all distinct p, q *with* $p \perp q$ *where* $A = d(q, \Lambda p) + d(p, \Lambda q)$ *and* $\eta, \mu \geq 0$ *such that* $0 \leq 2\eta + \mu < 1$. *If* $p_0 \in M$ *with* $p_0 \perp \Lambda(p_0)$, *then* Λ *has a unique fixed point in* M.

Proof Letting $\lambda = 0$ in (1). Then the proof is trivial from Theorem 2. □

3.2 Results Under Singh, Badshah and Rathore Contraction

Definition 8 A map $\Lambda : M \to M$ is said to be orthogonal almost Singh, Badshah, and Rathore contraction on an O-metric space $(M, \perp, d)$ if it satisfies the condition

$$\begin{aligned} d(\Lambda p, \Lambda q) \leq \alpha \left[\frac{d(p, \Lambda p)[1 + d(q, \Lambda q)]}{1 + d(p, q)} \right] &+ \beta[d(p, \Lambda p) + d(q, \Lambda q)] \\ &+ \gamma[d(p, \Lambda q) + d(q, \Lambda p)] + \delta d(p, q) \\ &+ L\, min\, \{d(p, \Lambda q), d(q, \Lambda p), d(p, \Lambda p), d(q, \Lambda q)\} \end{aligned} \tag{4}$$

for all distinct p, q with $p \perp q$, where $L \geq 0$ and there exists $\alpha, \beta, \gamma, \delta \in [0, 1)$ such that $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$.

Example

Let $M = [0, 1)$ and the metric d be the Euclidean metric. Define $p \perp q$ if $pq \leq p$ or q for all $p, q \in M$. Let $\Lambda : M \to M$ be a mapping defined by

$$\Lambda(p) = \begin{cases} pe^{-3}, & \text{if } 0 \leq p \leq \frac{1}{2} \\ 0, & \text{if } \frac{1}{2} < p < 1. \end{cases}$$

Choose $\alpha = 1/10$, $\beta = 1/10$, $\gamma = 1/10\, \delta = e^{-3}$. Then it is easy to observe the following

$$\begin{aligned} \mid \Lambda(p) - \Lambda(q) \mid \leq \alpha \left[\frac{\mid p - pe^{-3} \mid [\, 1 + \mid q - qe^{-3} \mid]}{1 + \mid p - q \mid} \right] &+ \beta\, [\, \mid p - pe^{-3} \mid + q - qe^{-3} \mid] \\ &+ \gamma\, [\, \mid p - qe^{-3} \mid + \mid q - pe^{-3} \mid] + \delta\, \mid p - q \mid \\ &+ L\, min\, \{\, \mid p - qe^{-3} \mid, \mid q - pe^{-3} \mid, \mid p - pe^{-3} \mid, \mid q - qe^{-3} \mid \} \end{aligned}$$

for all distinct p, q with $p \perp q$, where $L \geq 0$ and $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$. Therefore, Λ is an orthogonal almost Singh, Badshah and Rathore contraction.

Theorem 5 *Let* $(M, \perp, d)$ *be any O-complete metric space. Suppose that* $\Lambda : M \to M$ *is an orthogonal almost Singh, Badshah and Rathore contraction on an O-metric space,* $\perp$*-continuous and* $\perp$*-preserving map. Then* Λ *has a fixed point in* M.

Proof Since $(M, \perp)$ is an O-set, there exists $p_0 \in M$ such that

$$p_0 \perp q \text{ or } q \perp p_0, \text{ for all } q \in M.$$

It follows that $p_0 \perp \Lambda(p_0)$ *or* $\Lambda(p_0) \perp p_0$. Without loss of generality, assume that $p_0 \perp \Lambda(p_0)$. Let $p_1 = \Lambda(p_0)$, $p_2 = \Lambda(p_1) = \Lambda^2(p_0), ..., p_{n+1} = \Lambda(p_n) = \Lambda^{n+1}(p_0)$, for all $n \in \mathbb{W}$ then from $p_{n_0} = p_{n_0+1} = \Lambda p_{n_0}$. Thus, p_{n_0} is a fixed point, and hence the proof is completed. Otherwise, we have $p_1 = \Lambda(p_0)$, which implies, $p_0 \perp p_1$. Since Λ is $\perp$-preserving, $\Lambda(p_0) \perp \Lambda(p_1)$, which implies $p_1 \perp p_2$. Similarly we get $p_n \perp p_{n+1}$. Therefore, $\{p_n\}_{n\in\mathbb{W}}$ is O-sequence. For $p_n, p_{n+1} \in M$ with $p_n \perp p_{n+1}$, we get $d(p_{n+1}, p_n) = d(\Lambda(p_n), \Lambda(p_{n-1}))$. By (4),

$$\begin{aligned} d(p_{n+1}, p_n) \leq \alpha & \left[\frac{d(p_n, \Lambda p_n)[1 + d(p_{n-1}, \Lambda p_{n-1})]}{1 + d(p_n, p_{n-1})}\right] \\ & + \beta\, [d(p_n, \Lambda p_n) + d(p_{n-1}, \Lambda p_{n-1})] \\ & + \gamma\, [d(p_n, \Lambda p_{n-1}) + d(p_{n-1}, \Lambda p_n)] + \delta\, d(p_n, p_{n-1}) \\ & + L\, min\, \{d(p_n, \Lambda p_{n-1}), d(p_{n-1}, \Lambda p_n), d(p_n, \Lambda p_n), d(p_{n-1}, \Lambda p_{n-1})\}. \end{aligned}$$

$$\begin{aligned} d(p_{n+1}, p_n) \leq & \,\alpha\, d(p_n, p_{n+1}) + \beta\, [d(p_n, p_{n+1}) + d(p_{n-1}, p_n)] \\ & + \gamma\, [d(p_n, p_n) + d(p_{n-1}, p_{n+1})] + \delta\, d(p_n, p_{n-1}) \\ & + L\, min\, \{d(p_n, p_n), d(p_{n-1}, p_{n+1}), d(p_n, p_{n+1}), d(p_{n-1}, p_n)\}. \\ d(p_{n+1}, p_n) \leq & \,\alpha\, d(p_n, p_{n+1}) + \beta\, [d(p_n, p_{n+1}) + d(p_{n-1}, p_n)] + \gamma\, d(p_{n-1}, p_{n+1}) \\ & + \delta\, d(p_n, p_{n-1}). \end{aligned}$$

$$\begin{aligned} (1 - \alpha - \beta - \gamma)\, d(p_{n+1}, p_n) \; & \leq \beta\, d(p_{n-1}, p_n) + \gamma\, d(p_{n-1}, p_{n+1}) + \delta\, d(p_n, p_{n-1}) \\ & \leq (\beta + \gamma + \delta)\, d(p_{n-1}, p_n). \end{aligned}$$

$$d(p_{n+1}, p_n) \; \leq \left[\frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma}\right] d(p_{n-1}, p_n).$$

By repeating this process, we get

$$d(p_{n+1}, p_n) \; \leq \left[\frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma}\right]^n d(p_0, p_1).$$

Using triangular inequality, for $m \geq n$,

$$\begin{aligned} d(p_m, p_n) \leq & \; d(p_m, p_{m-1}) + d(p_{m-1}, p_{m-2}) + ... + d(p_{n+1}, p_n) \\ \leq & \; (k^{m-1} + k^{m-2} + ... + k^n)\, d(p_0, p_1) \\ \leq & \left[\frac{k^n}{1-k}\right] d(p_0, p_1). \end{aligned}$$

This implies, $d(p_m, p_n) \leq \left[\frac{k^n}{1-k}\right] d(p_0, p_1)$, where $k = \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma}$.

If $m, n \to +\infty$, then $d(p_m, p_n) \to 0$. Thus, the O-sequence $\{p_n\}_{n\in\mathbb{W}}$ is a Cauchy O-sequence. Since M is an O-complete, then there exists a point $p \in M$ such that $p_n \to p$. Also, we have $\perp$-continuous of Λ implies that

$$\Lambda p = \Lambda\left[\lim_{n\to+\infty} p_n\right] = \lim_{n\to+\infty} \Lambda p_n = \lim_{n\to+\infty} p_{n+1} = p.$$

Thus, p is a fixed point of Λ in M. □

In the said theorem, we don't have any sort of uniqueness; in order to get uniqueness we need to assume another condition together with the assumption mentioned above. The below theorem provides us with the uniqueness of a fixed point.

Theorem 6 *Let $(M, \perp, d)$ be any O-complete metric space. Suppose that $\Lambda : M \to M$ is an orthogonal almost Singh, Badshah, and Rathore contraction on an O-metric space, $\perp$-continuous and $\perp$-preserving map. Further, if $r, s \in Fix(\Lambda)$ (The set of all fixed points of Λ) with $r \perp s$. Then Λ has a unique fixed point in M.*

Proof From Theorem 5, it is clear Λ has a fixed point. To see uniqueness, consider,

$$\begin{aligned} d(r, s) &= d(\Lambda r, \Lambda s) \\ &\leq \alpha\left[\frac{d(r, \Lambda r)(1 + d(s, \Lambda s))}{1 + d(r, s)}\right] + \beta[d(r, \Lambda r) + d(s, \Lambda s)] \\ &+ \gamma[d(r, \Lambda s) + d(s, \Lambda r)] + \delta d(r, s) \\ &+ L \min\{d(r, \Lambda s), d(s, \Lambda r), d(r, \Lambda r), d(s, \Lambda s)\} \\ &\leq \gamma[d(r, s) + d(s, r)] \\ &\leq 2\gamma d(r, s), \end{aligned}$$

which implies that $(1 - 2\gamma)d(r, s) \leq 0$. Since, $(1 - 2\gamma) \geq (1 - \alpha - 2\beta - 2\gamma - \delta) \geq 0$, $d(r, s) = 0$. It concludes that $r = s$. □

Example

Let $(M = [0, 1], \perp, d)$ be any O-complete metric space and the metric d be the Euclidean metric. Define $p \perp q$ if $pq \in \{p, q\}$ for all $p, q \in M$. Let $\Lambda : M \to M$ be a mapping defined by

$$\Lambda(p) = \begin{cases} pe^{-3}, & \text{if } 0 \leq p \leq \frac{1}{2}; \\ 0, & \text{if } \frac{1}{2} < p \leq 1, \end{cases}$$

which is an orthogonal almost Singh, Badshah, and Rathore contraction on an O-metric space, $\perp$-continuous and $\perp$-preserving map. If $\Lambda p = p$, then $pe^{-3} = p$. From that, it is clear that 0 is the only fixed point for Λ.

Theorem 7 *Let $(M, \perp, d)$ be an O-complete metric space. Assume that M satisfies the following:*

$$\text{if for any } O-\text{sequence } \{p_n\} \text{ with } p_n \to p,\ p \in M,\ \text{then } p_n \perp p \text{ for all } n \in \mathbb{W}. \tag{5}$$

If $\Lambda : M \to M$ is a $\perp$—preserving map with orthogonal almost Singh, Badshah and Rathore contraction. Then Λ has a fixed point in M.

Proof By applying Theorem 5, there exists an O-sequence $\{p_n\} \in M$ such that $p_n \to p$, which implies $p_n \perp p$. Since Λ is $\perp$-preserving, we have $\Lambda p_n \perp \Lambda p$.

Consider,

$$d(p, \Lambda p) \leq d(p, p_n) + d(p_n, \Lambda p_n) + d(\Lambda p_n, \Lambda p). \tag{6}$$

Using the contraction condition, we get

$$\begin{aligned} d(\Lambda p_n, \Lambda p) \leq \alpha \left[\frac{d(p_n, \Lambda p_n)[1 + d(p, \Lambda p)]}{1 + d(p_n, p)} \right] &+ \beta[d(p_n, \Lambda p_n) + d(p, \Lambda p)] + \\ &\gamma[d(p_n, \Lambda p) + d(p, \Lambda p_n)] + \delta d(p_n, p) + \\ &L\, min\{d(p_n, \Lambda p), d(p, \Lambda p_n), d(p_n, \Lambda p_n), d(p, \Lambda p)\}. \end{aligned}$$

Making $n \to +\infty$, we obtain, $d(p_n, p) \to 0,\ d(p, \Lambda p_n) \to 0,\ d(p_n, \Lambda p_n) \to 0$. Thus we get $d(\Lambda p_n, \Lambda p) \leq (\beta + \gamma)\ d(p, \Lambda p)$. From (6) we have $(1 - \beta - \gamma)\ d(p, \Lambda p) \leq 0$. Since $(1 - \beta - \gamma) > 0$, we get $d(p, \Lambda p) = 0$. Thus, p is a fixed point of Λ in M. □

Definition 9 A map $\Lambda : M \to M$ is said to be orthogonal Singh, Badshah and Rathore contraction on an O-metric space $(M, \perp, d)$ if it satisfies the condition

$$\begin{aligned} d(\Lambda p, \Lambda q) \leq \alpha \left[\frac{d(p, \Lambda p)[1 + d(q, \Lambda q)]}{1 + d(p, q)} \right] &+ \beta[d(p, \Lambda p) + d(q, \Lambda q)] + \\ &\gamma[d(p, \Lambda q) + d(q, \Lambda p)] + \delta d(p, q) \end{aligned} \tag{7}$$

for all distinct p, q with $p \perp q$, there exist $\alpha, \beta, \gamma, \delta \in [0, 1)$ such that $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$.

Theorem 8 *Let $(M, \perp, d)$ be an O-complete metric space. Suppose that $\Lambda : M \to M$ is a Singh, Badshah and Rathore contraction on an O-metric space, $\perp$-continuous and $\perp$-preserving. Then Λ has a fixed point in M.*

Proof Letting $L = 0$ in (4). Then the proof is trivial from Theorem 5. □

Theorem 9 *Let $(M, \perp, d)$ be an O-complete metric space. Assume that M satisfies*

$$\textit{if for any } O - \textit{sequence } \{p_n\} \textit{ with } p_n \to p,\ p \in M,\ \textit{then } p_n \perp p \textit{ for all } n \in \mathbb{W}. \tag{8}$$

Let $\Lambda : M \to M$ be $\perp$-preserving satisfying the contraction condition (7). Then Λ has a fixed point in M.

Proof If $L = 0$ in (4). Thus the proof is trivial from Theorem 7. □

Theorem 10 *Let $(M, \perp, d)$ be any O-complete metric space. Suppose that $\Lambda : M \to M$ is an almost Singh, Badshah and Rathore contraction, $\perp$-continuous and $\perp$-preserving. Also, assume that if $\{p_n\}$ is any convergent O-sequence converging to $p \in M$, then $p \perp \Lambda p$. Further, if the operator Λ^l is $\perp$-continuous for some positive integer l, then Λ has a fixed point in M.*

Proof By Theorem 5, we construct an O-sequence by using $\perp$-preserving in M such that $p_n \to u$ for some $u \in M$. Let $\{p_{n_{2l}}\}$ be the subsequence of $\{p_n\}_{n\in\mathbb{W}}$ which converges to the same point u.

$$\Lambda^l u = \Lambda^l \left[\lim_{n\to+\infty} p_{n_l}\right] = \lim_{n\to+\infty} p_{n_{2l}} = u.$$

We have to show that u is a fixed point of Λ. Let m be the smallest positive integer such that $\Lambda^m u = u$ but $\Lambda^n u \neq u$ $(n = 1, 2, 3, ..., m-1)$, $m > 1$. For $u,\ \Lambda u \in M$ with $u \perp \Lambda u$, we get $d(\Lambda u, u) = d(\Lambda u, \Lambda^m u)$. By (4),

$$\begin{aligned} d(\Lambda u, u) = d(\Lambda u, \Lambda^m u) \leq \alpha &\left[\frac{d(u, \Lambda u)[1 + d(\Lambda^{m-1}u, \Lambda^m u)]}{1 + d(u, \Lambda^{m-1}u)}\right] \\ &+ \beta[d(u, \Lambda u) + d(\Lambda^{m-1}u, \Lambda^m u)] \\ &+ \gamma[d(u, \Lambda^m u) + d(\Lambda^{m-1}u, \Lambda u)] + \delta d(u, \Lambda^{m-1}u) \\ &+ L\, min\, \{d(u, \Lambda^m u), d(\Lambda^{m-1}u, \Lambda u), d(u, \Lambda u), d(\Lambda^{m-1}u, \Lambda^m u)\} \end{aligned}$$

$$\begin{aligned} d(\Lambda u, u) = d(u, \Lambda u) \leq \alpha d(u, \Lambda u) &+ \beta[d(u, \Lambda u) + d(\Lambda^{m-1}u, u)] \\ &+ \gamma[d(u, \Lambda u) + d(\Lambda^{m-1}u, u)] + \delta d(u, \Lambda^{m-1}u). \end{aligned}$$

$$(1 - \alpha - \beta - \gamma)\, d(u, \Lambda u) \ \leq\ (\beta + \gamma + \delta)\, d(u, \Lambda^{m-1}u).$$

$$d(u, \Lambda u) \ \leq \left[\frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma}\right] d(u, \Lambda^{m-1}u).$$

Consider,

$$\begin{aligned} d(u, \Lambda^{m-1}u) &= d(\Lambda^m u, \Lambda^{m-1}u) \\ &\le \alpha \left[\frac{d(\Lambda^{m-1}u, \Lambda^{m-2}u)[1 + d(\Lambda^{m-2}u, \Lambda^{m-1}u)]}{1 + d(\Lambda^{m-2}u, \Lambda^{m-1}u)} \right] \\ &+ \beta[d(\Lambda^{m-1}u, \Lambda^m u) + d(\Lambda^{m-2}u, \Lambda^{m-1}u)] \\ &+ \gamma[d(\Lambda^{m-1}u, \Lambda^{m-1}u) + d(\Lambda^{m-2}u, \Lambda^m u)] + \delta d(\Lambda^{m-1}u, \Lambda^{m-2}u) \\ &+ L\, min\, \{d(\Lambda^{m-1}u, \Lambda^m u), d(\Lambda^{m-2}u, \Lambda^{m-1}u), d(\Lambda^{m-1}u, \Lambda^{m-1}u), d(\Lambda^{m-2}u, \Lambda^m u)\}. \end{aligned}$$

$$d(u, \Lambda^{m-1}u) \le \left[\frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} \right] d(\Lambda^{m-1}u, \Lambda^{m-2}u).$$

By repeating this process, finally we get

$$\begin{aligned} d(\Lambda u, u) &= d(\Lambda u, \Lambda^m u) \\ &\le c\, d(u, \Lambda^{m-1}u) = c\, d(\Lambda^m u, \Lambda^{m-1}u) \le c^2\, d(\Lambda^{m-1}u, \Lambda^{m-2}u) \\ &\le ... \le c^m\, d(\Lambda u, u), \end{aligned}$$

where $c = \dfrac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} < 1$. Hence, $d(\Lambda u, u) \le c^m d(\Lambda u, u) < d(\Lambda u, u)$, which is a contradiction. Therefore, u is a fixed point of Λ. □

Theorem 11 *Let $(M, \perp, d)$ be any O-complete metric space. Suppose that $\Lambda : M \to M$ is a Singh, Badshah, and Rathore contraction, $\perp$-continuous and $\perp$-preserving. Also, assume that if $\{p_n\}$ is any convergent O-sequence converging to $p \in M$, then $p \perp \Lambda p$. If the operator Λ^l is $\perp$-continuous for some positive integer l, then Λ has a fixed point in M.*

Proof When $L = 0$ in Theorem 10, then the proof is trivial. □

Theorem 12 *Let $(M, \perp, d)$ be any O-complete metric space. Suppose that $\Lambda : M \to M$ is a $\perp$-preserving map. Also, assume that if $\{p_n\}$ is any convergent O-sequence converging to $p \in M$, then $p \perp \Lambda p$ and for some positive integer l, Λ satisfies*

$$\begin{aligned} d(\Lambda^l p, \Lambda^l q) \le\; & \alpha \left[\frac{d(p, \Lambda^l p)[1 + d(q, \Lambda^l q)]}{1 + d(p, q)} \right] + \beta[d(p, \Lambda^l p) + d(q, \Lambda^l q)] \\ & + \gamma[d(p, \Lambda^l q) + d(q, \Lambda^l p)] + \delta d(p, q) \\ & + L\, min\, \{d(p, \Lambda q), d(q, \Lambda p), d(p, \Lambda p), d(q, \Lambda q)\} \end{aligned}$$

for all distinct p, q with $p \perp q$, where $L \ge 0$ and there exists $\alpha, \beta, \gamma, \delta \in [0, 1)$ such that $0 \le \alpha + 2(\beta + \gamma) + \delta < 1$. If Λ^l is $\perp$-continuous, then Λ has a fixed point in M.

Proof It can be easily proved from Theorems 5 and 10. □

Theorem 13 *Let* $(M, \perp, d)$ *be any O-complete metric space. Suppose that* $\Lambda : M \to M$ *is a* $\perp$*-preserving map on an O-metric space and for some positive integer* l, Λ *satisfies*

$$d(\Lambda^l p, \Lambda^l q) \leq \alpha \left[\frac{d(p, \Lambda^l p)[1 + d(q, \Lambda^l q)]}{1 + d(p, q)}\right] + \beta[d(p, \Lambda^l p) + d(q, \Lambda^l q)] + \gamma[d(p, \Lambda^l q) + d(q, \Lambda^l p)] + \delta d(p, q)$$

for all distinct p, q *with* $p \perp q$, *where* $\alpha, \beta, \gamma, \delta \in [0, 1)$ *such that* $0 \leq \alpha + 2(\beta + \gamma) + \delta < 1$. *Also, assume that if* $\{p_n\}$ *is any convergent* O*-sequence converging to* $p \in M$, *then* $p \perp \Lambda p$. *If* Λ^l *is* $\perp$*-continuous, then* Λ *has a fixed point in* M.

Proof When $L = 0$ in Theorem 12, the proof follows. □

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Fixed Point Results in Graphical Convex Extended b-metric Spaces

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1 Introduction

Fixed point theory is an essential technique in modern mathematics. It is not only widely utilized in pure and practical mathematics but also served as a bridge between topology and analysis. Fixed point theory is a developing research subject with numerous applications in various fields. It is concerned with the discovery that a self-mapping on a nonempty set permits one or more fixed points under specific conditions. Fixed point theorems concern the existence and uniqueness of fixed points. Fixed point theory is divided into three primary areas. Metric fixed point theory, topological fixed point theory, and discrete fixed point theory are all examples of fixed point theories. Many scholars are working on the topic of fixed point theory. They are pursuing several avenues and generalizing their discoveries in this field.

Poincare was the first to take the initiative in this area. Later on, in 1922, Banach [1] discovered a remarkable result known as the Banach contraction principle discussed by the authors in [2, 3]. The Banach contraction principle has various extensions. In [4], Gahler offered an extension by introducing $2-$metric spaces, while Halpern et al. [5] proposed the convex compact subset of topological vector space by applying an inward and outward mapping. Kannan [6] enhanced Banach contraction by analyzing the completeness of the metric space, and he also produced certain fixed points using various conditions. In [7], Geraghty developed these findings by offering generalized contractive mappings, familiarly known as Geraghty contractions

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Following the same methodology, Ciric deduced a generalized contraction, which includes the results of Banach contraction and quasi-contraction in [8, 9]. Furthermore, Mizoguchi et al. [10] established a multivalued mapping of Caristi's fixed point theorem, and Czerwik [13] developed a new contraction mapping in the setting of $b-$metric space and also provided some fixed point theorems.

While working on fixed point theory, one of the working aspects is having a common fixed point for two or more mappings. Mappings should have some features other than continuity. Jungck [16] proposed the idea of commuting mappings. Further, Cho et al. introduced a common fixed point theorem for weakly commuting mapping in [14]. Assad et al. [17] presented a fixed point theorem for set-valued mappings.

Graph theory has important applications in the study of fixed point theory. Shukla et al. [11] put forward a novel concept about graphical metric spaces in 2017, while Chuensupantharat et al. [12] expanded this idea and offered a notion about the graphical b-metric space in 2018. Chifu et al. [18] applied a fixed point theorem for an appropriate operator on the cartesian product of a given b-metric space endowed with a graph. Recently, Younis et al. gave the notion of graphical rectangular $b-$metric [23] space and graphical extended $b-$metric space [27]. For further synthesis on graphical approach of fixed points with applications, we refer the reader to [15, 25, 26, 28].

In 1970, Takahashi [19] proposed the concepts of the convex structure and the convex metric space; he also acquired some fixed point theorems for non-expansive mappings in the convex metric spaces. Besides, Goebel and Kirk [20] investigated some iterative processes for non-expansive mappings in hyperbolic metric space. In 1990, Reich and Shafrir [22] established non-expansive iterations in hyperbolic spaces. Recently, in 2020 Chen discussed fixed point theorems in graphical convex metric space [29] and convex graphical rectangular b-metric space [30].

Motivated by the works propounded in the noteworthy articles [26, 29, 30], we put up the concept of the convex graphical extended b-metric spaces by means of the convex structure. Moreover, the definitions of G-contraction mappings, T-Mann sequences, and T-Agrawal sequences are employed to present strong convergence theorems for these mappings.

2 Preliminaries

To begin, we will clarify some essential concepts and notations stated in the articles [23, 24].

Let $\mathbb{N}$ be the set of natural numbers. Graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the binary relation on $V(G)$. The items that belong to $E(G)$ are referred to as edges. If each edge of the graph G has a direction, the graph is referred to as a directed graph. When each graph's edge has no direction, the graph is deemed undirected. Assume that the graph G has no parallel edges. We denote by G^{-1} the conversion of a directed graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus, we have

$$E(G^{-1}) = \{(x, y), \quad X \times X : (y, x) \in E(G)\}.$$

We denote by Ĝ a directed graph with symmetric edges and define

$$E(\hat{G}) = E(G) \cup E(G^{-1}),$$

then Ĝ is a symmetric directed graph. The directed graph G is said to be reflexive, if the set $E(G)$ contains all loops, that is, $(x, x) \in E(G)$ for each $x \in V(G)$. Moreover, a directed graph G is transitive if

$$(x, y) \in E(G) \quad \text{and} \quad (y, z) \in E(G) \quad \Rightarrow \quad (x, z) \in E(G).$$

Furthermore, the directed graph G can be treated as a weighted graph if the distance between its edges is allocated to each edge.

A relation $\mathscr{R}$ on $V(G)$ is such that $(p\mathscr{R}q)_G$ if there exists a path directing from p to q in G and $r \in p\mathscr{R}q)_G$ if r is contained in the path $(p\mathscr{R}q)_G$. A sequence $\{x_m\}_{m\in\mathbb{N}} \in V(G)$ is named a $G-$termwise connected $(G - TWC)$ if $(x_m\mathscr{R}x_{m+1})_G$ for all $m \in \mathbb{N}$.

Definition 1 [27] Let X be a nonempty set equipped with a graph G and $e : X \times X \to [1, \infty)$. A mapping $d : X\times X \to [0, \infty)$ is termed a graphical extended b-metric if for all $u, v, w \in X$ ensuing assertions are met:

i. $d(u, v) = 0$ if and only if $u = v$;
ii. $d(u, v) = d(v, u)$ for all $u, v \in X$;
iii. $(uPv), w \in (uPv) \Longrightarrow d(u, v) \leq e(u, v)[d(u, w) + d(w, v)]$.

The pair (X, d) is identified as a graphical extended b-metric space.

Definition 2 [27] Consider (X, d) be a graphically extended b-metric space. A sequence $\{w_n\}_{n\in\mathbb{N}}$ converges to some w in X, if for each positive ϵ, there is some positive N_ϵ such that $d\,(w_n, w) < \epsilon$ for each $n \geq N_\epsilon$. The following describes the circumstance

$$\lim_{n\to\infty} w_n = w.$$

Definition 3 [27] The sequence $\{w_n\}_{n\in\mathbb{N}}$ in a graphically extended b-metric space (X, d) is said to be a Cauchy sequence, if $d\,(w_n, w_m) \to 0$ as $n, m \to \infty$.

Definition 4 [27] A graphically extended b-metric space (X, d) is said to be $G-$complete if every Cauchy sequence is convergent in X with respect to graph G.

Definition 5 [29] Let (G, d) be a graphical metric space. If a mapping $W : V(G) \times V(G) \times [0, 1] \to V(G)$ provides

$$d(u, W(x, y; \alpha)) \leq (1-\alpha)d(u, x) + \alpha d(u, y),$$

for all $x, y, u \in V(G)$ and $\alpha \in (0, 1)$, then the space (G, d, W) is said to be graphical convex metric space.

Definition 6 ([30]) Let (Y, d) be a graphical rectangular $b-$metric space where Y is the nonempty subset of $V(G)$. If a mapping $W : Y \times Y \times [0, 1] \to V(G)$ provides

$$d(u, W(x, y; \alpha)) \leq (1-\alpha)d(u, x) + \alpha d(u, y),$$

for all $x, y, u \in X$ and $\alpha \in [0, 1]$, then the space (X, d, W) is known as a graphical rectangular convex $b-$metric space.

3 Graphically Convex Extended *b*—metric Spaces

This section aims to discuss the existence and uniqueness of fixed points for the set-valued $G-$contraction mappings in a $G-$complete GCE_bMS.

Definition 7 Let (G, d) be a graphical extended b-metric space. If a mapping $W : V(G) \times V(G) \times [0, 1] \to V(G)$ provides

$$d(u, W(x, y; \alpha)) \leq (1-\alpha)d(u, x) + \alpha d(u, y),$$

for all $x, y, u \in V(G)$ and $\alpha \in (0, 1)$, then the space is referred to as graphical convex extended $b-$metric space which will be abbreviated as (GCE_bMS).
A set to be utilized in the study is assigned below:

$$\mathbb{C}(V(G)) = \{K \subseteq V(G) : \ K \text{ is closed subset of } V(G)\}.$$

Definition 8 Let (G, d, W) be GCE_bMS and $T : V(G) \to \mathbb{C}(V(G))$ be a set-valued mapping. Assume that $x_0 \in V(G)$ is the initial value. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is referred to as a $T-$Mann sequence if $x_{n+1} = W(x_n, u_n; \alpha_n)$, where $u_n \in Tx_n$ and $\alpha_n \in (0, 1)$.

Proposition 1 *Let (G, d, W) be GCE_bMS and $T : V(G) \to \mathbb{C}(V(G))$ be a set-valued mapping. For any $x_n \in V(G)$, $u_n \in Tx_n$ and $\alpha_n \in (0, 1)$, we have*

$$d(x_n, x_{n+1}) = \alpha_n d(x_n, u_n) \text{ and } d(x_{n+1}, u_n) = (1-\alpha_n)d(x_n, u_n).$$

Definition 9 (G, d, W) is considered to fulfill the property (P) if for any $G-TWC$ and $T-$Mann sequence $\{x_n\}_{n\in\mathbb{N}}$ which converges to some $x \in V(G)$, a positive integer n_0 exists such that $(x_n, x) \in E(G)$ for any $n_0 \leq n$.

Definition 10 (G, d, W) is said to satisfy the property (Q) if for any $(x, z) \in E(G)$ and $y = W(x, y; \alpha)$, we have $(x, y) \in E(G)$ and $(y, z) \in E(G)$.

Theorem 1 *Let (G, d, W) be a $G-$complete GCE_bMS, which satisfies the property (P) and the property (Q) and $T : V(G) \to \mathbb{C}(V(G))$ be a $G-$contraction. If the sequence $\{\alpha_n\} \subset (0, 1)$ converges to α $(\alpha \neq 0)$ and the set*

$$E_T = \{x \in V(G) : \text{ there exists } u \in Tx \text{ such that } (x, u) \in E(G)\},$$

is nonempty, then T has a fixed point in G.

Proof Let $x_0 \in E_T$, $u_0 \in Tx_0$ exists such that $(x_0, u_0) \in E(G)$. Utilizing from the property (Q), it entails that $(x_1, x_0) \in E(G)$ and $(x_1, u_0) \in E(G)$. Using Definition (7), we procure

$$d(x_1, x_0) = d(W(x_0, u_0; \alpha_0), x_0) \leq \alpha_0 d(x_0, u_0),$$

and

$$d(x_1, u_0) = d(W(x_0, u_0; \alpha_0), u_0) \leq (1 - \alpha_0)d(x_0, u_0).$$

As T is a $G-$contraction and $(x_1, x_0) \in E(G)$, thereby for $u_0 \in Tx_0$ then $u_1 \in Tx_1$ exists such that

$$(u_0, u_1) \in E(G) \text{ and } d(u_0, u_1) \leq k(x_0, x_1).$$

Due to the fact that G is transitive, we deduce that $(x_1, u_1) \in E(G)$. The sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ can be obtained with the properties $x_{n+1} = W(x_n, u_n; \alpha_n)$ and $u_n \in Tx_n$, by the induction. We retrieve from the property (Q),

$$(x_{n+1}, x_n) \in E(G) \text{ and } (x_{n+1}, u_n) \in E(G).$$

By Definition (7), we achieve

$$d(x_{n+1}, x_n) = d(W(x_n, u_n; \alpha_n), x_n) \leq \alpha_n d(x_n, u_n),$$

and

$$d(x_{n+1}, u_n) = d(W(x_n, u_n; \alpha_n), u_n) \leq (1 - \alpha_n)d(x_n, u_n).$$

Then, for $u_n \in Tx_n$, $u_{n+1} \in Tx_{n+1}$ exists such that

$$(u_n, u_{n+1}) \in E(G) \text{ and } d(u_n, u_{n+1}) \leq k(x_n, x_{n+1}).$$

Thereby, it is concluded that $T-$Mann sequence $\{x_n\}_{n\in\mathbb{N}}$ is $G-$termwise connected. We shall now demonstrate that the sequence $\{d(x_n, u_n)\}$ is decreasing.

$$\begin{aligned}
d(x_n, u_n) &\leq e(x_n, u_n)\left[d(x_n, u_{n-1}) + d(u_{n-1}, u_n)\right] \\
&\leq e(x_n, u_n)\left[d(W(x_{n-1}, u_{n-1}; \alpha_{n-1}), u_{n-1}) + kd(x_{n-1}, x_n)\right] \\
&\leq e(x_n, u_n)[(1 - \alpha_{n-1})d(x_{n-1}, u_{n-1}) + k\alpha_{n-1}d(x_{n-1}, u_{n-1})] \\
&= e(x_n, u_n)(1 - \alpha_{n-1} + k\alpha_{n-1})d(x_{n-1}, u_{n-1}).
\end{aligned}$$

Let $(1 - \alpha_{i-1} + k\alpha_{i-1}) = P_{i-1}$ for $i \in N$, then

$$\lim_{i \to \infty} P_{i-1} = 1 - \alpha + k\alpha < 1.$$

Therefore, we procure

$$d(x_n, u_n) \leq e(x_n, u_n) P_{n-1} d(x_{n-1}, u_{n-1}) \leq \ldots \leq \prod_{j=0}^{n} e(x_j, u_j) \prod_{i=0}^{n-1} P_i d(x_0, u_0).$$

As $0 < \alpha_n, k < 1$, we deduce that $P_i < 1$, which indicates that$\{d(x_n, u_n)\}$ is a decreasing sequence. In addition, we possess

$$\begin{aligned} d(x_{n+1}, x_n) &= d(W(x_n, u_n; \alpha_n), x_n) = \alpha_n d(x_n, u_n) \\ &\leq \alpha_n \prod_{j=0}^{n} e(x_j, u_j) \prod_{i=0}^{n-1} P_i d(x_0, u_0). \end{aligned}$$

For any $p \in N$, we get

$$\begin{aligned} d(x_n, x_{n+p}) &\leq e(x_n, x_{n+p}) d(x_n, u_{n+1}) + e(x_n, x_{n+p}) e(x_{n+1}, x_{n+p}) d(x_{n+1}, x_{n+2}) \\ &+ \ldots + e(x_n, x_{n+p}) e(x_{n+1}, x_{n+p}) \ldots e(x_{n+p-1}, x_{n+p}) d(x_{n+p-1}, x_{n+p}) \\ &\leq e(x_n, x_{n+p}) \alpha_n d(x_n, u_n) + e(x_n, x_{n+p}) e(x_{n+1}, x_{n+p}) \alpha_{n+1} d(x_{n+1}, u_{n+1}) \\ &+ \ldots + e(x_n, x_{n+p}) e(x_{n+1}, x_{n+p}) \ldots e(x_{n+p-1}, x_{n+p}) \alpha_{n+p-1} d(x_{n+p-1}, u_{n+p-1}) \\ &\leq e(x_n, x_{n+p}) \alpha_n \prod_{j=0}^{n} e(x_j, u_j) \prod_{i=0}^{n-1} P_i d(x_0, u_0) \\ &+ e(x_n, x_{n+p}) e(x_{n+1}, x_{n+p}) \alpha_{n+1} \prod_{j=0}^{n+1} e(x_j, u_j) \prod_{i=0}^{n} P_i d(x_0, u_0) + \ldots \\ &+ e(x_n, x_{n+p}) \ldots e(x_{n+p-1}, x_{n+p}) \alpha_{n+p-1} \prod_{j=0}^{n+p-1} e(x_j, u_j) \prod_{i=0}^{n+p-2} P_i d(x_0, u_0). \end{aligned}$$

Owing to fact that $e(x, y) > 0$, we derive

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \prod_{k=0}^{n} e(x_k, x_{k+p}) \alpha_n \prod_{j=0}^{n} e(x_j, u_j) \prod_{i=0}^{n-1} P_i d(x_0, u_0) \\ &+ \prod_{k=0}^{n} e(x_k, x_{k+p}) \alpha_{n+1} \prod_{j=0}^{n+1} e(x_j, u_j) \prod_{i=0}^{n} P_i d(x_0, u_0) + \ldots \\ &+ \prod_{k=0}^{n} e(x_k, x_{k+p}) \alpha_{n+p-1} \prod_{j=0}^{n+p-1} e(x_j, u_j) \prod_{i=0}^{n+p-2} P_i d(x_0, u_0) \end{aligned}$$

$$\leq \left(\alpha_n \prod_{j=0}^{n} e(x_j, u_j) \prod_{i=0}^{n-1} P_i + \alpha_{n+1} \prod_{j=0}^{n+1} e(x_j, u_j) \prod_{i=0}^{n} P_i + \ldots \right.$$
$$\left. + \alpha_{n+p-1} \prod_{j=0}^{n+p-1} e(x_j, u_j) \prod_{i=0}^{n+p-2} P_i \right) \prod_{k=0}^{n} e(x_k, x_{k+p}) d(x_0, u_0).$$

Presume that

$$Q_n = \alpha_n \prod_{j=0}^{n} e(x_j, u_j) \prod_{i=0}^{n-1} P_i, \text{ where } n = 0, 1 \ldots p-1.$$

As a result, we infer that

$$d(x_n, x_{n+p}) \leq \left(Q_n + Q_{n+1} \ldots + Q_{n+p-1}\right) \prod_{k=0}^{n} e(x_k, x_{k+p}) d(x_0, u_0).$$

We also notice that

$$\lim_{i \to \infty} \sup \frac{Q_{n+i+1}}{Q_{n+i}} < 1. \tag{1}$$

Employing D'Alembert's test, we see that $\sum_{i=0}^{\infty} Q_i$ is convergent which implies $\lim_{i \to \infty} \sum_{i=n}^{n+m-1} Q_i = 0$. Hence, we conclude that

$$\lim_{n \to \infty} d(x_n, x_{n+p}) = 0, \tag{2}$$

which demonstrates that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. As G is $G-$complete, $z \in V(G)$ exists such that $\lim_{n \to \infty} d(x_n, z) = 0$. The property (P) provides that $(x_n, z) \in E(G)$ for large enough n. Then there exists $z_n \in Tz$ such that

$$d(u_n, z_n) \leq kd(x_n, z),$$

which implies $d(u_n, z_n) \to 0$ as $n \to \infty$. It is necessary to indicate that $\lim_{n \to \infty} d(x_n, u_n) = 0$, so

$$\begin{aligned} d(x_n, u_n) \leq\ & e(x_n, u_n) d(x_n, x_{n+p}) + e(x_n, u_n) e(x_{n+p}, u_n) d(x_{n+p}, u_{n+p}) \\ & + e(x_n, u_n) e(x_{n+p}, u_n) d(u_{n+p}, u_n), \end{aligned} \tag{3}$$

and

$$\begin{aligned} d(u_n, u_{n+p}) &\le e(u_n, u_{n+p})d(u_n, u_{n+1}) + e(u_n, u_{n+p})e(u_{n+1}, u_{n+p})d(u_{n+1}, u_{n+2}) \\ &+ \ldots + e(u_n, u_{n+p})e(u_{n+1}, u_{n+p}) \ldots e(u_{n+p-1}, u_{n+p})d(u_{n+p-1}, u_{n+p}) \\ &\le e(u_n, u_{n+p})kd(x_n, x_{n+1}) + e(u_n, u_{n+p})e(u_{n+1}, u_{n+p})kd(x_{n+1}, x_{n+2}) \\ &+ \ldots + \prod_{i=0}^{n+p-1} e(u_i, u_{i+p})kd(x_{n+p-1}, u_{n+p-1}). \end{aligned}$$

Due to the fact that $e(x, y) > 0$, we attain

$$\begin{aligned} d(u_n, u_{n+p}) &\le \prod_{i=0}^{n+p-1} e(u_i, u_{i+p})kd(x_n, x_{n+1}) + \prod_{i=0}^{n+p-1} e(u_i, u_{i+p})kd(x_{n+1}, x_{n+2}) \\ &+ \ldots + \prod_{i=0}^{n+p-1} e(u_i, u_{i+p})kd(x_{n+p-1}, u_{n+p-1}) \\ &\le k \prod_{i=0}^{n+p-1} e(u_i, u_{i+p}) \left(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots d(x_{n+p-1}, u_{n+p-1}) \right) \\ &\le k \prod_{i=0}^{n+p-1} e(u_i, u_{i+p}) \left(Q_n + Q_{n+1} \ldots + Q_{n+p-1} \right) d(x_0, u_0). \end{aligned}$$

Also, making use of (1) and (2), we assert

$$\lim_{n\to\infty} d(u_n, u_{n+p}) = 0. \tag{4}$$

We presently have

$$d(x_{n+p}, u_{n+p}) \le \prod_{i=n}^{n+p} e(x_i, u_i) P_{n+p-1\ldots} P_n d(x_n, u_n). \tag{5}$$

Taking $\lim_{n\to\infty}$ in (3) and using (2, 4, 5), we get

$$d(x_n, u_n) \le 0 + e(x_n, u_n)e(x_{n+p}, u_n) \prod_{i=n}^{n+p} e(x_i, u_i) P_{n+p-1\ldots} P_n d(x_n, u_n) + 0.$$

Since

$$(1 - e(x_n, u_n)e(x_{n+p}, u_n) \prod_{i=n}^{n+p} e(x_i, u_i) P_{n+p-1\ldots} P_n d(x_n, u_n))d(x_n, u_n) \le 0.$$

We understand this by taking

$$\lim_{n\to\infty}(1-e(x_n,u_n)e(x_{n+p},u_n)\prod_{i=n}^{n+p}e(x_i,u_i)P_{n+p-1\ldots}P_n d(x_n,u_n))\neq 0,$$

we obtain

$$\lim_{n\to\infty} d(x_n,u_n)=0,$$

and combining with $\lim_{n\to\infty} d(u_n,z_n)=0$, we have $\lim_{n\to\infty} d(x_n,z_n)=0$. This amounts to say that $\lim_{n\to\infty} d(x_n,z)=0$. Hence, we conclude that

$$\lim_{n\to\infty} d(z_n,z)=0,$$

as Tz is closed, then we have $z\in Tz$, which indicates that z is a fixed point of T.

We need to demonstrate the uniqueness of a fixed point for a $G-$contraction in a $G-$complete GCE_bMS.

Theorem 2 *Presume that all the hypotheses of the Theorem 1 are fulfilled. The following expressions*

$$x_{n+1}=W(x_n,u_n;\alpha_n) \text{ and } x'_{n+1}=W(x'_n,u'_n;\alpha'_n)$$

are generated as in Theorem 1, where $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x^* *and* $\{x'_n\}_{n\in\mathbb{N}}$ *converges to* y^*, $\lim_{n\to\infty}\alpha_n=\alpha\neq 0$ *and* $\lim_{n\to\infty}\alpha'_n=\alpha'\neq 0$. *Then* $x^*=y^*$ *is provided that* $(x_n,x'_n)\in E(G)$ *for each* $n\in\mathbb{N}$.

Proof By Theorem 1, we can conclude that x^* and y^* are fixed points of T. Suppose that $x^*\neq y^*$, because T is a $G-$contraction and $(x_n,x'_n)\in E(G)$, thus for $u_n\in Tx_n$, there exists $u'_n\in T'x_n$ such that

$$(u_n,u'_n)\in E(G) \text{ and } d(u_n,u'_n)\leq kd(x_n,x'_n).$$

The proof of Theorem 1 ensures that $\lim_{n\to\infty} d(x_n,u_n)=0$. Combining this with the fact

$$\lim_{n\to\infty} d(x_n,x^*)=0,$$

we procure $\lim_{n\to\infty} d(u_n,x^*)=0$. By using the property (P), we gain $(x_n,x^*)\in E(G)$, $(x'_n,y^*)\in E(G)$, $(u_n,x^*)\in E(G)$ and $(u'_n,x^*)\in E(G)$ for large enough n. Moreover, by using property (Q), we have $(x_{n+1},x_n)\in E(G)$, $(x_{n+1},u_n)\in E(G)$, $(x'_{n+1},x'_n)\in E(G)$ and $(x'_{n+1},u'_n)\in E(G)$ for large enough n. Therefore,

$$\begin{aligned} d(x^*,y^*) &\leq e(x^*,y^*)d(x^*,x_{n+1})+e(x^*,y^*)e(x_{n+1},y^*)d(x_{n+1},x'_{n+1})\\ &\quad + e(x^*,y^*)e(x_{n+1},y^*)d(x'_{n+1},y^*), \end{aligned}$$

as we know that $e(x, y) > 0$,

$$\begin{aligned} d(x^*, y^*) &\le e(x^*, y^*)e(x_{n+1}, y^*)d(x^*, x_{n+1}) + e(x^*, y^*)e(x_{n+1}, y^*)d(x_{n+1}, x'_{n+1}) \\ &+ e(x^*, y^*)e(x_{n+1}, y^*)d(x'_{n+1}, y^*), \end{aligned}$$

then we attain

$$d(x^*, y^*) \le e(x^*, y^*)e(x_{n+1}, y^*)\left[d(x^*, x_{n+1}) + d(x_{n+1}, x'_{n+1}) + d(x'_{n+1}, y^*)\right], \tag{6}$$

and

$$\begin{aligned} d(x_{n+1}, x'_{n+1}) &= d(W(x_n, u_n; \alpha_n), W(x'_n, u'_n; \alpha'_n)) \\ &\le (1-\alpha_n)d(x_n, W(x'_n, u'_n; \alpha'_n)) + \alpha_n d(u_n, W(x'_n, u'_n; \alpha'_n)) \\ &\le (1-\alpha_n)\left[(1-\alpha'_n)d(x_n, x'_n) + \alpha'_n d(x_n, u'_n)\right] \\ &+ \alpha_n\left[(1-\alpha'_n)d(u_n, x'_n) + \alpha'_n d(u_n, u'_n)\right] \\ &\le (1-\alpha_n)(1-\alpha'_n)d(x_n, x'_n) + (1-\alpha_n)\alpha'_n d(x_n, u'_n) \\ &+ \alpha_n(1-\alpha'_n)d(u_n, x'_n) + \alpha_n\alpha'_n d(u_n, u'_n) \\ &\le (1-\alpha_n)(1-\alpha'_n)d(x_n, x'_n) \\ &+ \alpha'_n(1-\alpha_n)e(x_n, u'_n)\left(d(x_n, u_n) + d(u_n, u'_n)\right) \\ &+ \alpha_n(1-\alpha'_n)d(u_n, x'_n) + \alpha_n\alpha'_n d(u_n, u'_n), \\ &= (1-\alpha_n)(1-\alpha'_n)d(x_n, x'_n) + \alpha'_n(1-\alpha_n)e(x_n, u'_n)d(x_n, u_n) \\ &+ \alpha'_n(1-\alpha_n)e(x_n, u'_n)d(u_n, u'_n) + \alpha_n(1-\alpha'_n)d(u_n, x'_n) \\ &+ \alpha_n\alpha'_n d(u_n, u'_n), \end{aligned}$$

using the fact $e(x, y) > 0$, we deduce that

$$\begin{aligned} d(x_{n+1}, x'_{n+1}) &\le e(x_n, u'_n)[((1-\alpha_n)(1-\alpha'_n)d(x_n, x'_n) \\ &+ \alpha'_n(1-\alpha_n)d(x_n, u_n) + \alpha'_n(1-\alpha_n)kd(x_n, x'_n) \\ &+ \alpha_n(1-\alpha'_n)d(u_n, x'_n) + \alpha_n\alpha'_n kd(x_n, x'_n)] \\ &\le e(x_n, u'_n)[((1-\alpha_n)(1-\alpha'_n) + \alpha'_n(1-\alpha_n)k \\ &+ \alpha_n\alpha'_n k)d(x_n, x'_n) + \alpha'_n(1-\alpha_n)d(x_n, u_n) \\ &+ \alpha_n(1-\alpha'_n)d(u_n, x'_n)] \\ &\le e(x_n, u'_n)[(\alpha'_n k + (1-\alpha_n)(1-\alpha'_n))d(x_n, x'_n) \\ &+ \alpha'_n(1-\alpha_n)d(x_n, u_n) + \alpha_n(1-\alpha'_n)d(u_n, x'_n)]. \end{aligned}$$

Furthermore, if we use $e(x, y) > 0$, we yield that

$$
\begin{aligned}
d(x_n, x'_n) &\leq e(x_n, x'_n)d(x_n, x^*) + e(x_n, x'_n)e(x^*, x'_n)d(x^*, y^*) \\
&\quad + e(x_n, x'_n)e(x^*, x'_n)d(y^*, x'_n) \\
&\leq e(x_n, x'_n)e(x^*, x'_n)[d(x_n, x^*) + d(x^*, y^*) + d(y^*, x'_n)].
\end{aligned}
$$

Similarly, it can be written that

$$
d(x_n, x'_n) \leq e(u_n, x'_n)e(x^*, x'_n)[d(u_n, x^*) + d(x^*, y^*) + d(y^*, x'_n)].
$$

Thereby,

$$
\begin{aligned}
d(x_{n+1}, x'_{n+1}) \leq\ & e(x_n, u'_n)[(\alpha'_n k + (1-\alpha_n)(1-\alpha'_n))e(x_n, x'_n)e(x^*, x'_n)[d(x_n, x^*) \\
&+ d(x^*, y^*) + d(y^*, x'_n)] + \alpha'_n(1-\alpha_n)d(x_n, u_n) \\
&+ \alpha_n(1-\alpha'_n)e(u_n, x'_n)e(x^*, x'_n)[d(u_n, x^*) + d(x^*, y^*) + d(y^*, x'_n)]].
\end{aligned}
$$

Together with (6), and letting $\lim\limits_{n\to\infty}$, we get the ensuing conclusion

$$
\begin{aligned}
d(x^*, y^*) &\leq e(x_n, u'_n)[(\alpha' k + (1-\alpha)(1-\alpha'))e(x_n, x'_n)e(x^*, x'_n)d(x^*, y^*) \\
&\quad + \alpha(1-\alpha')e(u_n, x'_n)e(x^*, x'_n)d(x^*, y^*)] \\
&= e(x_n, u'_n)e(x^*, x'_n)[(\alpha' k + (1-\alpha)(1-\alpha'))e(x_n, x'_n)d(x^*, y^*) \\
&\quad + \alpha(1-\alpha')e(u_n, x'_n)d(x^*, y^*)].
\end{aligned}
$$

As we know that $e : X \times X \to [1, \infty)$, so

$$
e(x_n, x'_n) > 0 \text{ and also } e(u_n, x'_n) > 0.
$$

If $\max\{e(x_n, x'_n), e(u_n, x'_n)\} = e(x_n, x'_n)$, or $\max\{e(x_n, x'_n), e(u_n, x'_n)\} = e(u_n, x'_n)$ by choosing any of the statements given above, the following statement is derived

$$
\begin{aligned}
d(x^*, y^*) &\leq e(x_n, u'_n)e(x^*, x'_n)e(x_n, x'_n)[(\alpha' k + (1-\alpha)(1-\alpha')) \\
&\quad + \alpha(1-\alpha')]d(x^*, y^*) \\
&\leq e(x_n, u'_n)e(x^*, x'_n)e(x_n, x'_n)(\alpha' k + (1-\alpha))d(x^*, y^*).
\end{aligned}
$$

Since

$$
[1 - (\alpha' k + (1-\alpha))e(x_n, u'_n)e(x^*, x'_n)e(x_n, x'_n)]d(x^*, y^*) \leq 0,
$$

we infer

$$
d(x^*, y^*) = 0.
$$

Therefore, we get $x^* = y^*$.

4 Fixed Point Theorem for T-Agrawal Sequences

This section proposes establishing some fixed point theorems for $T-$Agrawal sequences in a $G-$complete GCE_bMS.

Definition 11 Suppose (G, d, W) is a GCE_bMS and $T : V(G) \to \mathbb{C}(V(G))$ is a set-valued mapping. Presume that $x_0 \in V(G)$ is initial value. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is named $T-$Agrawal sequences if the ensuing statements are provided

$$y_n = W(x_n, z_n; \beta_n), \;\; x_{n+1} = W(z_n, z'_n; \alpha_n),$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$, $\alpha_n, \beta_n \in (0, 1)$.

Theorem 3 *Let (G, d, W) be a $G-$complete GCE_bMS, which satisfies the property (P) and the property (Q) and $T : V(G) \to \mathbb{C}(V(G))$ be a $G-$contraction. Assume that the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ provide $0 < \alpha_n\beta_n < 1$ for $n = 0, 1, 2 \ldots$. If the set*

$$E_T = \{x \in V(G) : \text{ there exists } y \in Tx \text{ such that } (x, y) \in E(G)\},$$

is nonempty, then T enjoys a fixed point in G.

Proof For any $x_0 \in E_T$, $z_0 \in Tx_0$ exists such that $(x_0, z_0) \in E(G)$. Let $y_0 = W(x_0, z_0; \beta_0)$, from the property (Q), it ensures that $(y_0, x_0) \in E(G)$ and $(y_0, z_0) \in E(G)$. By using Definition (7), we procure

$$d(y_0, z_0) = d(W(x_0, z_0; \beta_0), z_0) \leq (1 - \beta_0)d(x_0, z_0).$$

As T is a $G-$contraction and $(y_0, x_0) \in E(G)$, thus for $z_0 \in Tx_0$, there exists $z'_0 \in Ty_0$ such that

$$(z_0, z'_0) \in E(G) \text{ and } d(z_0, z'_0) \leq kd(x_0, y_0).$$

By the transitivity of G, we obtain $(z'_0, y_0) \in E(G)$ and $(z'_0, x_0) \in E(G)$. Let $x_1 = W(z_0, z'_0; \alpha_0)$, the property (Q) yields that $(x_1, z_0) \in E(G)$ and $(x_1, z'_0) \in E(G)$. Definition (7) produces that

$$d(x_1, z_0) = d(W(x_0, z'_0; \alpha_0), z_0) \leq \alpha_0 d(z_0, z'_0),$$

and

$$d(x_1, z'_0) = d(W(x_0, z'_0; \alpha_0), z'_0) \leq (1 - \alpha_0)d(z_0, z'_0).$$

Since $(x_1, z_0) \in E(G)$ and $(z_0, y_0) \in E(G)$, we gain $(y_0, x_1) \in E(G)$. Owing to fact that T is a $G-$contraction mapping and $(y_0, x_1) \in E(G)$, thus for $z'_0 \in Ty_0$, there exists $z_1 \in Ty_1$ such that

$$(z'_0, z_1) \in E(G) \text{ and } d(z'_0, z_1) \leq kd(y_0, x_1).$$

The transitivity property of G assists us to claim that $(z_1, x_1) \in E(G)$ and $(z_1, z_0) \in E(G)$. We obtain the sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$, $\{z_n\}_{n\in\mathbb{N}}$ and $\{z'_n\}_{n\in\mathbb{N}}$, with the properties

$$\begin{aligned} y_n &= W(x_n, z_n; \beta_n), \\ x_{n+1} &= W(z_n, z'_n; \alpha_n), \end{aligned}$$

by induction, where $z_n \in Tx_n$ and $z'_n \in Ty_n$ and it is not difficult to see that

$$\begin{aligned} d(y_n, x_n) &= d(W(x_n, z_n; \beta_n), x_n) \leq \beta_n d(x_n, z_n), \\ d(y_n, z_n) &= d(W(x_n, z_n; \beta_n), z_n) \leq (1-\beta_n)d(x_n, z_n), \\ d(x_{n+1}, z_n) &= d(W(z_n, z'_n; \alpha_n), z_n) \leq \alpha_n d(z_n, z'_n), \\ d(x_{n+1}, z'_n) &= d(W(z_n, z'_n; \alpha_n), z'_n) \leq (1-\alpha_n)d(z_n, z'_n), \end{aligned}$$

and

$$\begin{aligned} (z_n, z'_n) \in E(G) &\text{ and } d(z_n, z'_n) \leq kd(x_n, y_n), \\ (z'_n, z_{n+1}) \in E(G) &\text{ and } d(z'_n, z_{n+1}) \leq kd(y_n, x_{n+1}). \end{aligned}$$

Furthermore, it is concluded that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is $G-TWC$. Next we claim that $\{d(x_n, z_n)\}$ is a decreasing sequence such that

$$\begin{aligned} d(x_{n+1}, z_{n+1}) &\leq e(x_{n+1}, z_{n+1})\left[d(x_{n+1}, z'_n) + d(z'_n, z_{n+1})\right] \\ &\leq e(x_{n+1}, z_{n+1})\left[(1-\alpha_n)d(z_n, z'_n) + kd(y_n, x_{n+1})\right] \\ &\leq e(x_{n+1}, z_{n+1})\left[(1-\alpha_n)kd(x_n, y_n) + kd(y_n, x_{n+1})\right] \\ &\leq e(x_{n+1}, z_{n+1})\left[(1-\alpha_n)\beta_n kd(x_n, z_n) + kd(y_n, x_{n+1})\right], \end{aligned} \tag{7}$$

and

$$\begin{aligned} d(y_n, x_{n+1}) &\leq e(y_n, x_{n+1})\left[d(y_n, z_n) + d(z_n, x_{n+1})\right] \\ &\leq e(y_n, x_{n+1})\left[(1-\beta_n)d(x_n, z_n) + \alpha_n d(z_n, z'_n)\right] \\ &\leq e(y_n, x_{n+1})\left[(1-\beta_n)d(x_n, z_n) + \alpha_n kd(x_n, y_n)\right] \\ &\leq e(y_n, x_{n+1})\left[(1-\beta_n)d(x_n, z_n) + \alpha_n\beta_n kd(x_n, z_n)\right] \\ &\leq (1-\beta_n+\alpha_n\beta_n k)e(y_n, x_{n+1})d(x_n, z_n). \end{aligned} \tag{8}$$

Putting equation (8) in (7), then we procure

$$\begin{aligned} d(x_{n+1}, z_{n+1}) \leq\ & e(x_{n+1}, z_{n+1})[(1-\alpha_n)\beta_n kd(x_n, z_n) \\ &+ k(1-\beta_n+\alpha_n\beta_n k)e(y_n, x_{n+1})d(x_n, z_n)]. \end{aligned}$$

As we know that $e(x, y) > 0$, then the ensuing statement is yielded

$$\begin{aligned}
d(x_{n+1}, z_{n+1}) &\leq e(x_{n+1}, z_{n+1})[(1-\alpha_n)\beta_n k e(y_n, x_{n+1}) d(x_n, z_n) \\
&+ k(1-\beta_n+\alpha_n\beta_n k) e(y_n, x_{n+1}) d(x_n, z_n)] \\
&\leq e(x_{n+1}, z_{n+1}) e(y_n, x_{n+1})[(1-\alpha_n)\beta_n k d(x_n, z_n) \\
&+ k(1-\beta_n+\alpha_n\beta_n k) d(x_n, z_n)] \\
&\leq e(x_{n+1}, z_{n+1}) e(y_n, x_{n+1})[(1-\alpha_n)\beta_n k \\
&+ k(1-\beta_n+\alpha_n\beta_n k)] d(x_n, z_n), \\
&\leq k e(x_{n+1}, z_{n+1}) e(y_n, x_{n+1})[1-(1-k)\alpha_n\beta_n] d(x_n, z_n).
\end{aligned}$$

Presume that the expression $e(x_n, z_n) > e(x_{n+1}, z_{n+1})e(y_n, x_{n+1})$ exists and we own $\gamma_n = (1-(1-k)\alpha_n\beta_n) \in [0, 1)$, we attain

$$d(x_{n+1}, z_{n+1}) \leq k\gamma_n e(x_n, z_n) d(x_n, z_n) < d(x_n, z_n),$$

which also indicates that the sequence $\{d(x_n, z_n)\}$ is decreasing.
Moreover,

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(x_n, W(z_n, z_n'; \alpha_n)) \\
&\leq (1-\alpha_n) d(x_n, z_n) + \alpha_n d(x_n, z_n') \\
&\leq (1-\alpha_n) d(x_n, z_n) + \alpha_n e(x_n, z_n')[d(x_n, z_n) + d(z_n, z_n')],
\end{aligned}$$

and by using $e(x, y) > 0$, we have

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq e(x_n, z_n')(1-\alpha_n) d(x_n, z_n) + \alpha_n e(x_n, z_n')[d(x_n, z_n) + d(z_n, z_n')] \\
&\leq e(x_n, z_n')[(1-\alpha_n) d(x_n, z_n) + \alpha_n d(x_n, z_n) + k\alpha_n d(x_n, y_n)] \\
&\leq e(x_n, z_n')[d(x_n, z_n) + k\alpha_n\beta_n d(x_n, z_n)] \\
&\leq e(x_n, z_n')(1 + k\alpha_n\beta_n) d(x_n, z_n).
\end{aligned}$$

Let $e(x_n, z_n')(1 + k\alpha_n\beta_n) = t_n$, for $p \in N$. We conclude that

$$\begin{aligned}
d(x_n, x_{n+p}) &\leq e(x_n, x_{n+p}) d(x_n, x_{n+1}) + e(x_n, x_{n+p}) e(x_{n+1}, x_{n+p}) d(x_{n+1}, x_{n+2}) \\
&+ \ldots + e(x_n, x_{n+p}) e(x_{n+1}, x_{n+p}) \ldots e(x_{n+p-1}, x_{n+p}) d(x_{n+p-1}, x_{n+p}) \\
&\leq \prod_{i=0}^{n+p-1} e(x_i, x_{i+p}) \left[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots d(x_{n+p-1}, x_{n+p})\right] \\
&\leq \prod_{i=0}^{n+p-1} e(x_i, x_{i+p})[t_n d(x_n, z_n) + t_{n+1} d(x_{n+1}, z_{n+1}) + \ldots t_{n+p-1} d(x_{n+p-1}, z_{n+p-1})] \\
&\leq \prod_{i=0}^{n+p-1} e(x_i, x_{i+p})[t_n k^n \prod_{i=0}^{n} \gamma_i e(x_i, z_i) + t_{n+1} k^{n+1} \prod_{i=0}^{n} \gamma_i e(x_i, z_i) + \ldots \\
&+ t_{n+p-1} k^{n+p-1} \prod_{i=0}^{n+p-2} \gamma_i e(x_i, z_i)] d(x_0, z_0)
\end{aligned}$$

$$= \prod_{i=0}^{n+p-1} e(x_i, x_{i+p})k^n[t_n \prod_{i=0}^{n} \gamma_i e(x_i, z_i) + t_{n+1}k \prod_{i=0}^{n} \gamma_i e(x_i, z_i) + \dots$$
$$+ t_{n+p-1}k^{p-1} \prod_{i=0}^{n+p-2} \gamma_i e(x_i, z_i)]d(x_0, z_0).$$

Let $M_{n+i} = t_n k^i \prod_{i=0}^{n+i-1} \gamma_i e(x_i, z_i)$, where $i = 0, 1, 2 \dots p-1$, we obtain the following

$$d(x_n, x_{n+p}) \leq k^n[M_n + M_{n+1} + \dots M_{n+p-1}] \prod_{i=0}^{n+p-1} e(x_i, x_{i+p})d(x_0, z_0).$$

Since $0 < \alpha_n \beta_n < 1$, for $n = 0, 1, 2 \dots$, we obtain

$$\lim_{i\to\infty} \sup \frac{M_{n+i+1}}{M_{n+i}} = \lim_{i\to\infty} \sup \frac{t_{n+i+1}k\gamma_{n+i}}{t_{n+i}} < 1.$$

By using D' Alembert's test, it is evident that $\sum_{i=0}^{\infty} M_i$ is convergent. Hence, we come to the conclusion that

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0,$$

which indicates that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. $G-$completeness of the space provides that there exists $p \in V(G)$ such that $\lim_{n\to\infty} d(x_n, p) = 0$. In view of the property (P), one has $(x_n, p) \in E(G)$ for larger values n, then $p_n \in Tp$ exists such that

$$d(x_n, p_n) \leq kd(x_n, p),$$

which implies $d(x_n, p_n) \to 0$ as $n \to \infty$. Let $n \to +\infty$, then

$$d(p_n, p) \leq e(p_n, p)[d(p_n, x_n) + d(x_n, p)] \to 0,$$

which entails that $p \in Tp$, owing to the closedness of Tp.

Remark 1 In the preceding proof procedure, we may acquire

$$\lim_{n\to\infty} d(x_n, z_n) = 0 \text{ and } \lim_{n\to\infty} d(x_n, y_n) = 0.$$

Proof From the definition of $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, it ensures that

$$d(x_n, y_n) = d(x_n, W(x_n, z_n; \beta_n)) \leq \beta_n d(x_n, z_n).$$

Hence, we only need to prove that $\lim_{n\to\infty} d(x_n, z_n) = 0$, since $\beta_n \in (0, 1)$. Combining with the proof of Theorem (3), we notice that

$$d(x_n, z_n) \le k^n \prod_{i=0}^{n-1} \gamma_i e(x_i, z_i) d(x_0, z_0),$$

which entails $\lim_{n\to\infty} d(x_n, z_n) = 0$ as $\gamma_i \in [0, 1)$ and $k \in (0, 1)$.

Now, we demonstrate that the uniqueness of the fixed point for $G-$contraction with respect to $T-$Agrawal sequences.

Theorem 4 *Suppose all the hypotheses of Theorem 3 hold, set*

$$y_n = W(x_n, z_n; \beta_n), x_{n+1} = W(z_n, z_n'; \alpha_n),$$

where $z_n \in Tx_n$ *and* $z_n' \in Ty_n$, $\alpha_n, \beta_n \in (0, 1)$, *and*

$$v_n = W(u_n, w_n; \delta_n), u_{n+1} = W(w_n, w_n'; \eta_n),$$

where $w_n \in Tu_n$ *and* $w_n' \in Tv_n$, $\eta_n, \delta_n \in (0, 1)$. *The sequences* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{u_n\}_{n\in\mathbb{N}}$ *are generated as in the above, where* x_n *converges to* x *and* u_n *converges to* u, *the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\eta_n\}$ *and* $\{\delta_n\}$ *fulfill* $0 < \alpha_n\beta_n, \eta_n\delta_n < 1$. *Then* $u = v$, *provided that* $(x_n, u_n) \in E(G)$ *for each* $n \in N$.

Proof By Theorem 3, it is concluded that x and u are fixed points of T. As T is a $G-$contraction, $(x_n, u_n) \in E(G)$ and $(x_n, y_n) \in E(G)$, thus for all $z_n \in Tx_n$, there exist $w_n \in Tu_n$, and $z_n' \in Ty_n$, such that

$$(z_n, w_n) \in E(G), \ (z_n, z_n') \in E(G),$$

and

$$d(z_n, w_n) \le kd(x_n, u_n), \ d(z_n, z_n') \le kd(x_n, y_n).$$

From Remark 1, we deduce that $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(u_n, v_n) = 0$. Noticing that $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(u_n, u) = 0$, we attain

$$\lim_{n\to\infty} d(y_n, x) = 0, \quad \lim_{n\to\infty} d(v_n, u) = 0.$$

By using the property (P), we have $(x_n, x) \in E(G)$, $(u_n, u) \in E(G)$, $(y_n, x) \in E(G)$ and $(v_n, u) \in E(G)$ for large enough n. Moreover, by using property (Q), we acquire $(x_{n+1}, z_n) \in E(G)$ and $(z_n, y_n) \in E(G)$. By transitivity of the graph G, we conclude $(x_{n+1}, y_n) \in E(G)$.

Similarly, we have $(u_{n+1}, v_n) \in E(G)$. Combining with $(x_{n+1}, u_{n+1}) \in E(G)$ and $(x_{n+1}, y_n) \in E(G)$, we get $(y_n, v_n) \in E(G)$. For all $z_n' \in Ty_n$, there exists $w_n' \in Tv_n$ such that

$$(z_n', w_n') \in E(G) \text{ and } d(z_n', w_n') \le kd(y_n, v_n).$$

Observe that

$$d(x,u) \le e(x,u)d(x,x_{n+1}) + e(x,u)e(x_{n+1},u)d(x_{n+1},u_{n+1}) + e(x,u)e(x_{n+1},u)d(u_{n+1},u),$$

with the fact $e(x, y) > 0$, we deduce that

$$\begin{aligned} d(x,u) &\le e(x,u)e(x_{n+1},u)d(x,x_{n+1}) + e(x,u)e(x_{n+1},u)d(x_{n+1},u_{n+1}) \\ &+ e(x,u)e(x_{n+1},u)d(u_{n+1},u)] \\ &\le e(x,u)e(x_{n+1},u)[d(x,x_{n+1}) + d(x_{n+1},u_{n+1}) + d(u_{n+1},u)], \end{aligned} \tag{9}$$

and

$$\begin{aligned} d(x_{n+1},u_{n+1}) &= d(W(z_n,z'_n;\alpha_n), W(w_n,w'_n;\eta_n)) \\ &\le (1-\alpha_n)(1-\eta_n)d(z_n,w_n) + (1-\alpha_n)\eta_n d(z_n,w'_n) + \alpha_n(1-\eta_n)d(z'_n,w_n) \\ &+ \alpha_n\eta_n d(z'_n,w'_n) \\ &\le (1-\alpha_n)(1-\eta_n)d(z_n,w_n) + (1-\alpha_n)\eta_n e(z_n,w'_n)[d(z_n,z'_n) + d(z'_n,w'_n)] \\ &+ \alpha_n(1-\eta_n)e(z'_n,w_n)[d(z'_n,z_n) + d(z_n,w_n)] + \alpha_n\eta_n d(z'_n,w'_n). \end{aligned}$$

Similarly, by (9), the ensuing expression can be written

$$\begin{aligned} d(x_{n+1},u_{n+1}) &\le e(z_n,w'_n)e(z'_n,w_n)[(1-\alpha_n)(1-\eta_n)d(z_n,w_n) + (1-\alpha_n)\eta_n[d(z_n,z'_n) \\ &+ d(z'_n,w'_n)] + \alpha_n(1-\eta_n)[d(z'_n,z_n) + d(z_n,w_n)] + \alpha_n\eta_n d(z'_n,w'_n)] \\ &\le e(z_n,w'_n)e(z'_n,w_n)[(1-\eta_n)d(z_n,w_n) + (\alpha_n+\eta_n-2\alpha_n\eta_n)d(z_n,z'_n) \\ &+ \eta_n d(z'_n,w'_n)] \\ &\le e(z_n,w'_n)e(z'_n,w_n)[(1-\eta_n)kd(x_n,u_n) + (\alpha_n+\eta_n-2\alpha_n\eta_n)kd(x_n,y_n) \\ &+ \eta_n kd(y_n,v_n)]. \end{aligned} \tag{10}$$

Using expression (9), we yield

$$d(x_n,u_n) \le e(x_n,u_n)e(x,u_n)[d(x_n,x) + d(x,u) + d(u,u_n)], \tag{11}$$

and

$$d(y_n,v_n) \le e(y_n,u_n)e(x,v_n)[d(y_n,x) + d(x,u) + d(u,v_n)]. \tag{12}$$

By using (11) and (12) in (10), we get

$$\begin{aligned} d(x_{n+1},u_{n+1}) &\le e(z_n,w'_n)e(z'_n,w_n)[(1-\eta_n)ke(x_n,u_n)e(x,u_n)[d(x_n,x) \\ &+ d(x,u) + d(u,u_n)] + (\alpha_n+\eta_n-2\alpha_n\eta_n)kd(x_n,y_n) \\ &+ \eta_n ke(y_n,u_n)e(x,v_n)[d(y_n,x) + d(x,u) + d(u,v_n)]], \end{aligned}$$

if we choose $\max\{e(x_n,u_n)e(x,u_n), e(y_n,u_n)e(x,v_n)\} = e(x_n,u_n)e(x,u_n)$, then the above inequality provides

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\le e(z_n, w_n')e(z_n', w_n)[(1-\eta_n)ke(x_n, u_n)e(x, u_n)[d(x_n, x) \\ &\quad + d(x, u) + d(u, u_n)] + (\alpha_n + \eta_n - 2\alpha_n\eta_n)kd(x_n, y_n) \\ &\quad + \eta_n ke(x_n, u_n)e(x, u_n)[d(y_n, x) + d(x, u) + d(u, v_n)]]. \end{aligned}$$

Using the fact $e(x, y) > 0$, we achieve

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\le e(z_n, w_n')e(z_n', w_n)[(1-\eta_n)ke(x_n, u_n)e(x, u_n)[d(x_n, x) + d(x, u) \\ &\quad + d(u, u_n)] + (\alpha_n + \eta_n - 2\alpha_n\eta_n)ke(x_n, u_n)e(x, u_n)d(x_n, y_n) \\ &\quad + \eta_n ke(x_n, u_n)e(x, u_n)[d(y_n, x) + d(x, u) + d(u, u_n)]], \\ &\le e(z_n, w_n')e(z_n', w_n)e(x_n, u_n)e(x, u_n)[(1-\eta_n)k[d(x_n, x) + d(x, u) \\ &\quad + d(u, u_n)] + (\alpha_n + \eta_n - 2\alpha_n\eta_n)kd(x_n, y_n) + \eta_n k[d(y_n, x) \\ &\quad + d(x, u) + d(u, v_n)]]. \end{aligned} \tag{13}$$

On the account of (9) and (13) and $\lim_{n\to\infty}$, we deduce that

$$\lim_{n\to\infty}[(1 - ke(z_n, w_n')e(z_n', w_n)e(x_n, u_n)e(x, u_n))d(x, u)] \le 0.$$

Furthermore, $\lim_{n\to\infty}(1 - ke(z_n, w_n')e(z_n', w_n)e(x_n, u_n)e(x, u_n)) \neq 0$, we deduce

$$d(x, u) = 0,$$

which asserts that $x = u$, endorsing the uniqueness of the fixed point.

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Existence and Computational Approximation of Fixed Points of Generalized Multivalued Mappings in Banach Space

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1 Introduction

Fixed point theory is a domain of mathematics that addresses the study of solutions of the form $\mathbb{T}(x) = x$, where $\mathbb{T}$ is a mapping from a set X to itself. The basic idea is to find conditions under which such an equation has a solution and to study the properties of the solutions when they exist.

In Banach spaces, fixed point theory has a remarkable contribution in many areas of mathematics, including functional analysis, differential equations, and optimization theory. The main result concerning fixed point theory area is the well-known Banach fixed point theorem, which states that each contraction mapping on a complete metric space has a unique fixed point.

More generally, fixed point theorems in Banach spaces can be used to prove the existence of solutions of nonlinear equations, to establish the convergence of iterative algorithms, and to study the stability of dynamical systems. The basic approach in fixed point theory is to construct a suitable mapping $\mathbb{T} : X \to X$, and to show that it satisfies the necessary conditions for a fixed point theorem to hold.

The main techniques used in fixed point theory include metric space theory, functional analysis, and topological methods. One of the key ideas is to use compactness arguments to extract a convergent subsequence from a sequence of iterates of a given mapping. Another important tool is the use of the Brouwer degree, which is a topological invariant that measures the degree of a mapping from a compact domain to itself.

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In 1922, Banach [1] launched a new study of constructive theory in metric space. Banach Contraction Principle (BCP) is one of the most useful and relevant theorems in classical functional analysis. The BCP states that: "Let $(\mathcal{B}, d)$ be a metric space and $\mathbb{T} : \mathcal{B} \to \mathcal{B}$ be a self map such that $d(\mathbb{T}(\vartheta), \mathbb{T}(\zeta^*)) \leq \vartheta d(\vartheta, \zeta^*)$ for some $0 \leq \vartheta < 1$ and for all $\vartheta, \zeta^* \in \mathcal{B}$, then $\mathbb{T}$ has a unique fixed point". In 1969, Nadler [2] proved BCP for a multivalued mapping. Various other extensions of this principle can be found in [3–8].

Let $\mathcal{E}$ be a nonempty subset of a real Banach space $\mathcal{B}$ with norm $\|.\|$. A self-mapping $\mathbb{T}$ from $\mathcal{E}$ to $\mathcal{E}$ is nonexpansive if

$$\|\mathbb{T}\vartheta - \mathbb{T}\nu\| \leq \|\vartheta - \nu\|.$$

Let $\mathbb{T}$ be a mapping, and we define the set of all fixed points of $\mathbb{T}$ as $\mathbb{F}(\mathbb{T}) := \{\vartheta \in \mathcal{E} : \vartheta = \mathbb{T}\vartheta\}$. In the literature, Browder [9] and Gohde [10] have established well-known results. These results state that when $\mathcal{E}$ is a closed, bounded, and convex subset of a uniformly convex Banach space, the set of fixed points $\mathbb{F}(\mathbb{T})$ for a nonexpansive mapping $\mathbb{T}$ is nonempty. These findings have been widely recognized and accepted in the field. The notion of nonexpansive mappings has found significant applications, leading various mathematicians to extend and generalize this concept.

In 2008, Suzuki [11] gave the concept of $Condition\ C$ and obtained the fixed point results for such mapping. A mapping $\mathbb{T} : \mathcal{E} \to \mathcal{E}$ satisfy $Condition\ C$ if

$$\frac{1}{2}\|\vartheta - \mathbb{T}\vartheta\| \leq \|\vartheta - \nu\| \implies \|\mathbb{T}\vartheta - \mathbb{T}\nu\| \leq \|\vartheta - \nu\|, \ \forall\, \vartheta, \nu \in \mathcal{E}.$$

Aoyama and Kohshaka [12] introduced a more general class of mappings known as $\tilde{\alpha}$-nonexpansive mappings. They extensively studied the fixed point results for such mappings, providing a broader framework than the previously studied mappings. Their work expanded the understanding of nonexpansive mappings and their fixed point properties.

A self-mapping $\mathbb{T} : \mathcal{E} \to \mathcal{E}$ is $\tilde{\alpha}$-nonexpansive mapping if for $\tilde{\alpha} \in [0, 1)$ such that for all $\vartheta, \nu \in \mathcal{E}$,

$$\|\mathbb{T}\vartheta - \mathbb{T}\nu\|^2 \leq \tilde{\alpha}\|\nu - \mathbb{T}\vartheta\|^2 + \tilde{\alpha}\|\vartheta - \mathbb{T}\nu\|^2 + (1 - 2\tilde{\alpha})\|\vartheta - \nu\|^2.$$

Certainly, each nonexpansive mapping is 0-nonexpansive mapping.

Impelled from above, Pant and Shukla [13] propose a generalized $\tilde{\alpha}$-nonexpansive for a self-mapping $\mathbb{T} : \mathcal{E} \to \mathcal{E}$, such that for all $\vartheta, \nu \in \mathcal{E}$ and $\tilde{\alpha} \in [0, 1)$

$$\|\mathbb{T}\vartheta - \mathbb{T}\nu\| \leq \tilde{\alpha}\|\nu - \mathbb{T}\vartheta\| + \tilde{\alpha}\|\vartheta - \mathbb{T}\nu\| + (1 - 2\tilde{\alpha})\|\vartheta - \nu\|,$$

and studied the fixed point result for such generalized mapping.

Harandi et al. [14], propose a new kind of mapping—$(\tilde{\alpha}, \tilde{\beta})$-nonexpansive mapping and studied the existence and approximation fixed point result for such mapping.

Let $\mathbb{T} : \mathcal{E} \to \mathcal{E}$ be a generalized $(\tilde{\alpha}, \tilde{\beta})$-nonexpansive mapping, if, for $\tilde{\alpha}, \tilde{\beta} \in [0, 1)$ such that for all $\vartheta, \nu \in \mathcal{E}$

$$\|\mathbb{T}\vartheta - \mathbb{T}\nu\|^2 \leq \tilde{\alpha}\|\mathbb{T}\vartheta - \nu\|^2 + \tilde{\alpha}\|\mathbb{T}\nu - \vartheta\|^2\tilde{\beta}\|\vartheta - \mathbb{T}\vartheta\|^2 + \tilde{\beta}\|\nu - \mathbb{T}\nu\|^2 \\ + (1 - 2\tilde{\alpha} - 2\tilde{\beta})\|\vartheta - \nu\|^2.$$

For more discussion about the above mappings, we refer to the reader [12, 15–17, 20] and references therein.

Recently, in 2023, Ullah et al. [18] propose a new kind of mapping more generally than above mappings which is known as $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-nonexpansive mapping and they studied the fixed point results for such mapping.

Let $\mathbb{T} : \mathcal{E} \to \mathcal{E}$ be a generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-nonexpansive mapping, if $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in [0, 1)$ with $\tilde{\alpha} + \tilde{\gamma} \leq 1$ such that for all $\vartheta, \nu \in \mathcal{E}$

$$\|\mathbb{T}\vartheta - \mathbb{T}\nu\| \leq \tilde{\alpha}\|\vartheta - \nu\| + \tilde{\beta}\|\vartheta - \mathbb{T}\vartheta\| + \tilde{\gamma}\|\vartheta - \mathbb{T}\nu\|.$$

A set $\mathcal{E}$ is said to be proximinal if, for every $\vartheta \in \mathcal{B}$, there exists $\nu \in \mathcal{E}$ such that $d(\vartheta, \nu) = d(\vartheta, \mathcal{E}) = \inf\{d(\vartheta, \nu) : \nu \in \mathcal{E}\}$. In other words, $\mathcal{E}$ is proximinal if there exists an element in $\mathcal{E}$ that is closest to any given point in $\mathcal{B}$, achieving the minimum distance to $\mathcal{E}$.

In this chapter, the notion $\mathcal{P}(\mathcal{E})$, $\mathcal{P}_{C\mathcal{P}}(\mathcal{E})$, $\mathcal{P}_{\rho x}(\mathcal{E})$, and $\mathcal{P}_{C\mathcal{B}}(\mathcal{E})$ represents the families of nonempty subsets, compact subsets, proximinal subsets, and closed bounded subsets of a Banach space $\mathcal{B}$, respectively.

For $\mathcal{K}, \mathcal{L} \in \mathcal{P}_{C\mathcal{B}}(\mathcal{E})$, define:

$$\mathbb{H}(\mathcal{K}, \mathcal{L}) = \min\{\sup_{k \in K} d(k, \mathcal{L}), \sup_{l \in \mathcal{L}} d(\mathcal{K}, l)\},$$

$\mathbb{H}$ is called the Hausdorff–Pompeiu metric on $\mathcal{P}_{C\mathcal{B}}(\mathcal{E})$ induced with distance d.

A point $\vartheta \in \mathcal{E}$ is said to be a fixed point of a multivalued mapping $\mathbb{T} : \mathcal{E} \to \mathcal{P}(\mathcal{E})$ if $\vartheta \in \mathbb{T}\vartheta$. We denote the fixed point set by $\mathbb{F}(\mathbb{T})$.

Akbar and Eslamian [19] gave the multivalued version of Suzuki's condition C. In 2017, Iqbal et al. [20] modified another type of nonexpansive mapping from single-valued to multivalued mapping which they called the multivalued generalized $\tilde{\alpha}-$nonexpansive mapping and prove the existence and approximation result of the mapping. Recently, in 2021, Abbas et.al [21] gave another mapping which is the multivalued generalized $(\tilde{\alpha}, \tilde{\beta})-$nonexpansive mapping. These mappings are defined as follows:

Let $\mathcal{E}$ be a nonempty subset of a Banach space $\mathcal{B}$ and $\mathbb{T} : \mathcal{E} \to \mathcal{P}(\mathcal{E})$, then $\mathbb{T}$ is multivalued

1. nonexpansive mapping if

$$\mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\nu) \leq \|\vartheta - \nu\|,$$

for all $\vartheta, \nu \in \mathcal{E}$.

2. Suzuki nonexpansive if

$$\frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) \leq \|\vartheta - \nu\| \implies \mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\nu) \leq \|\vartheta - \nu\|, \ (Condition\ C)$$

for all $\vartheta, \nu \in \mathcal{E}$.

3. quasi-nonexpansive if

$$\mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\zeta^*) \leq \|\vartheta - \zeta^*\|,$$

for all $\vartheta \in \mathcal{E}$ and $\zeta^* \in \mathbb{F}(\mathbb{T})$.

4. generalized $\tilde{\alpha}$-nonexpansive if there is an $\tilde{\alpha} \in [0, 1)$ such that

$$\mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\nu) \leq \tilde{\alpha}d(\nu, \mathbb{T}\vartheta) + \tilde{\alpha}d(\nu, \mathbb{T}d) + (1 - 2\tilde{\alpha})\|\vartheta - \nu\|,$$

for all $\vartheta, \nu \in \mathcal{E}$.

5. Multivalued generalized $(\tilde{\alpha}, \tilde{\beta})$-nonexpansive if there exists $\tilde{\alpha}, \tilde{\beta} \in [0, 1)$ such that

$$\mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\nu) \leq \tilde{\alpha}d(\nu, \mathbb{T}\vartheta) + \tilde{\beta}d(\nu, \mathbb{T}\nu) + (1 - \tilde{\alpha} - \tilde{\beta})\|\vartheta - \nu\|,$$

for all $\vartheta, \nu \in \mathcal{E}$.

In the current century, many researchers studied and proposed different iterative schemes for fixed point approximation of a nonexpansive mapping for both single-valued and multivalued mappings. Mann [22], Ishikawa [23], Thakur [24], and the hybrid scheme of these with Picard scheme are few popular schemes in a number of iterative schemes [25–28, 30] and therein. Shahzad and Zegeye [29] presented a set $\mathcal{P}_{\mathbb{T}}(\vartheta) = \{\nu \in \mathbb{T}\vartheta : d(\vartheta, \mathbb{T}\vartheta)\} \leq \|\vartheta - \nu\|$ for a multivalued mapping $\mathbb{T} : \mathcal{E} \to \mathcal{P}(\mathcal{E})$. They proved the convergence of Mann and Ishikawa iterative schemes and showed that these iterative schemes are well defined for multivalued mapping in uniformly convex Banach space.

We list some of the leading and faster iterative schemes such as Mann, Ishikawa, Thakur, K, Picard-S, and Picard-S^* for multivalued version of the mapping $\mathbb{T} : \mathcal{E} \to \mathcal{P}(\mathcal{E})$. In the above schemes, the rate of convergence of Picard–Thakur iterative is faster than others for fixed point. The multivalued versions of the above schemes are defined as follows:

$$\text{Mann Iterative scheme:} \quad \begin{cases} \vartheta_1 \in \mathcal{E}, \\ \vartheta_{n+1} = (1 - \tau_n)\vartheta_n + \tau_n\zeta_n, \ n \in \mathbb{Z}^+, \end{cases}$$

where $\{\tau_n\} \in (0, 1)$ and $\zeta_n \in \mathcal{P}_{\mathbb{T}}(\vartheta_n)$.

$$\text{Thakur's Iterative Schemes:} \quad \begin{cases} \vartheta_1 \in \mathcal{E}, \\ \vartheta_{n+1} = (1-\tau_n)\theta_n + \tau_n \vartheta_n, \\ \nu_n = (1-\tau_n)\mu_n + \tau_n \theta_n, \quad n \in \mathbb{Z}^+, \\ \mu_n = (1-\sigma_n)\vartheta_n + \sigma_n \zeta_n, \end{cases}$$

where $\zeta_n \in \mathcal{P}_{\mathbb{T}}(\vartheta_n), \theta_n \in \mathcal{P}_{\mathbb{T}}(\mu_n), \vartheta_n \in \mathcal{P}_{\mathbb{T}}(\nu_n)$ and $\{\tau_n\}$, $\{\tau_n\}$ and $\{\sigma_n\}$ are the sequences of parameters in $(0, 1)$.

$$\text{Picard-}S^* \text{ Iterative scheme:} \quad \begin{cases} \vartheta_1 \in \mathcal{E}, \\ \vartheta_{n+1} = u_n \\ \nu_n = (1-\tau_n)\zeta_n + \tau_n t_n \\ \mu_n = (1-\tau_n)\vartheta_n + \tau_n \upsilon_n \\ \eta_n = (1-\sigma_n)\vartheta_n + \sigma_n \zeta_n, \quad n \in \mathbb{Z}^+, \end{cases}$$

where $\zeta_n \in \mathcal{P}_{\mathbb{T}}(\vartheta_n), \upsilon_n \in \mathcal{P}_{\mathbb{T}}(\eta_n), t_n \in \mathcal{P}_{\mathbb{T}}(\mu_n), u_n \in \mathcal{P}_{\mathbb{T}}(\nu_n)$ and $\{\tau_n\}, \{\tau_n\}, \{\sigma_n\} \in [0, 1)$.

Recently, Jie Jia et al. [30] introduced a four-step iterative scheme and claim that this four-step iterative scheme converges faster than the above discussed iterative schemes which is defined for single-valued as follows:

$$\begin{cases} \vartheta_1 \in \mathcal{E}, \\ \vartheta_{n+1} = \mathbb{T}(\nu_n) \\ \nu_n = (1-\tau_n)\mathbb{T}(\eta_n) + \tau_n \mathbb{T}(\mu_n), \\ \mu_n = (1-\tau_n)\eta_n + \tau_n \mathbb{T}(\eta_n), \\ \eta_n = (1-\sigma_n)\vartheta_n + \sigma_n \mathbb{T}(\vartheta_n), \quad n \in \mathbb{Z}^+, \end{cases} \tag{13.1}$$

where $\{\tau_n\}, \{\tau_n\}, \{\sigma_n\} \in [0, 1)$.

Once an iterative scheme is formulated, it becomes essential to examine the stability and convergence properties of the scheme. The importance of stability in an iterative scheme was thoroughly investigated by Harder and Hicks in 1988 [31, 32]. Their work focused on studying the significance and implications of stability in iterative schemes.

Inspired by the works of Suzuki [11], Aoyama [12], and the research on stability conducted by Harder and Hicks [32], we propose a new class of mappings called multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})-$nonexpansive mappings. This class of mappings extends the existing notions and offers a broader framework for studying fixed point theory and related properties. This work aims to establish the properties and existence results for this new type of mapping.

In this study, we provide a numerical example that pertains to the multivalued version of the mapping. In addition, we introduce the multivalued version of the iterative process (13.1) and provide both weak and strong convergence theorems for this iterative process in a uniformly convex Banach space. These convergence theorems shed light on the convergence properties of the iterative process specifically when applied to multivalued mappings. These convergence theorems serve to highlight the convergence properties of our iterative process.

By combining theoretical analysis, numerical examples, and practical applications, this work sheds light on the novel class of multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})-$nonexpansive mappings. It contributes to the understanding of their properties, existence results, convergence behavior, and potential applications.

2 Preliminaries

This section deals with some basic definitions, lemmas, and other related results which is useful for our work.

We say that a Banach space $\mathcal{B}$ is uniformly convex, if for each $\varepsilon \in (0, 2]$, there exists a number $\zeta > 0$, such that for all $\theta, \vartheta \in \mathcal{B}$,

$$\left\{ \begin{array}{l} \|\theta\| \leq 1 \\ \|\vartheta\| \leq 1 \\ \|\theta - \vartheta\| > \varepsilon \end{array} \right\} \text{implies} \quad \left\| \frac{\theta - \vartheta}{2} \right\| \leq \zeta. \tag{13.2}$$

Suppose that $\mathcal{B}$ is a real Banach space and $\mathcal{E}$ be a nonempty subset of $\mathcal{B}$, then $\vartheta_n \to e$, and $\vartheta_n \rightharpoonup e$ represents the strong and weak convergence of a sequence $\{\vartheta_n\}$ to some point e in $\mathcal{B}$.

Suppose we have a bounded sequence ϑ_n in $\mathcal{B}$. Then,

- The asymptotic radius of $\{\vartheta_n\}$ at a point ϑ in $\mathcal{B}$ is

$$\Re(\vartheta, \{\vartheta_n\}) = \limsup_{n\to\infty} \|\vartheta_n - \vartheta\|.$$

- The asymptotic radius of $\{\vartheta_n\}$ with respect to $\mathcal{E}$ is

$$\Re(\vartheta, \mathcal{E}) = \inf\{\Re(\vartheta, \{\vartheta_n\}) : \vartheta \in \mathcal{E}\}.$$

- The asymptotic center of $\{\vartheta_n\}$ with respect to $\mathcal{E}$ is defined as

$$\wp(\mathcal{E}, \{\vartheta_n\}) = \{\vartheta \in \mathcal{E} : Re(\vartheta, \{\vartheta_n\}) = \Re(\vartheta, \mathcal{E})\}.$$

Definition 1 ([33]) Consider the Banach space $\mathcal{B}$. Then, $\mathcal{B}$ has Opial property iff for each weakly convergent sequence $\{\vartheta_n\}$ in $\mathcal{B}$ with weak limit $\vartheta \in \mathcal{B}$, there is

$$\limsup_{n\to\infty} \|\vartheta_n, \vartheta\| < \limsup_{n\to\infty} \|\vartheta_n, \nu\|,$$

for each $\nu \in \mathcal{B} - \{\vartheta\}$.

The following result is a description of Schu's proof of uniform convexity [34].

Lemma 1 ([34]) *Consider the Banach space $\mathcal{B}$ which is a uniformly convex and $0 < i = \tilde{\gamma}_n = j < 1, \forall n \in \mathbb{N}$. If $\{\vartheta_n\}$ and $\{\nu_n\}$ are any two sequences in $\mathcal{B}$ such that $\limsup_{n\to\infty} \|\vartheta_n\| \le \ell$, $\limsup_{n\to\infty} \|\nu_n\| \le \ell$ and $\lim_{n\to\infty} \|\tilde{\gamma}_n\vartheta_n + (1-\tilde{\gamma}_n)\nu_n\| = \ell$ for some $\ell \ge 0$, then $\lim_{n\to\infty} \|\vartheta_n - \nu_n\| = 0$.*

Definition 2 Let $\mathbb{T} : \mathcal{E} \to \mathcal{P}_{\mathcal{CB}}(\mathcal{E})$. A sequence $\{\vartheta_n\}$ in $\mathcal{E}$ is named as an approximate fixed point sequence $(a.f.p.s.)$ for $\mathbb{T}$ provided that $d(\vartheta_n, \mathbb{T}(\vartheta_n)) \to 0$ as $n \to \infty$.

Definition 3 A multivalued mapping $\mathbb{T} : \mathcal{E} \to \mathcal{P}(\mathcal{E})$ is called demiclosed at $\nu \in \mathcal{E}$, if for any sequence $\{\vartheta_n\}$ in $\mathcal{E}$ weakly converges to $\vartheta \in \mathcal{E}$ and a sequence $\tau_n \in \mathbb{T}\vartheta_n$ strongly converges to ν, then we have $\nu \in \mathbb{T}\vartheta$.

Sentor and Dotson [35] gave the definition of $Condition\ I$ which is defined as

Definition 4 ([35]) A mapping $\mathbb{T} : \mathcal{E} \to \mathcal{P}_{\mathcal{CB}}(\mathcal{E})$ meets $Condition\ I$ if there is a nondecreasing function $\Psi : [0, \infty) \to [0, \infty)$ with properties $\Psi(0) = 0$ and $\Psi(\upsilon) > 0$ for all $\upsilon \in (0, \infty)$, such that $d(\varpi, \mathbb{T}\varpi) \ge \Psi(d(\varpi, \mathbb{F}(\mathbb{T})))$, for all $\varpi \in \mathcal{E}$, where $d(\varpi, \mathbb{F}(\mathbb{T})) = \inf\{d(\varpi, \zeta^*) : \zeta^* \in \mathbb{F}(\mathbb{T})\}$.

Lemma 2 ([19]) *Let $\mathcal{E}$ be a bounded, closed convex subset of a uniformly convex Banach space $\mathcal{B}$. If $\mathbb{T} : \mathcal{E} \to \mathcal{P}_{\mathcal{CB}}(\mathcal{E})$ is multivalued nonexpansive, then $\mathbb{T}$ has a fixed point.*

Lemma 3 ([29]) *Let $\mathbb{T} : \mathcal{E} \to \mathcal{P}_{\rho x}(\mathcal{E})$ and $\mathcal{P}_{\mathbb{T}}(\pi) = \{\varpi \in \mathbb{T}\pi : d(\pi, \mathbb{T}\pi) = \|\pi - \varpi\|\}$. Then, the next statements are equivalent:*

1. $\pi \in \mathbb{F}(\mathbb{T})$.
2. $\mathcal{P}_{\mathbb{T}}(\pi) = \{\pi\}$.
3. $\pi \in \mathbb{F}(\mathcal{P}_{\mathbb{T}})$.

Moreover, $\mathbb{F}(\mathbb{T}) = \mathbb{F}(\mathcal{P}_{\mathbb{T}})$.

3 Multivalued Generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-nonexpansive Mapping

We will give next the definition of the multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})-$ nonexpansive mapping and we will provide an example concerning this new type of mappings.

Definition 5 Let $\mathcal{E}$ be a nonempty subset of a uniformly convex Banach space $\mathcal{B}$. A mapping $\mathbb{T} : \mathcal{E} \to \mathcal{P}_{C\mathcal{B}}(\mathcal{E})$ is called multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})-$ nonexpansive mapping if there exists $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in [0, 1)$ such that, for each $\vartheta, \nu \in \mathcal{E}$,

$$\frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) \leq \|\vartheta - \nu\| \implies \mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\nu) \leq \tilde{\alpha}\|\vartheta - \nu\| + \tilde{\beta}d(\vartheta, \mathbb{T}\vartheta) + \tilde{\gamma}d(\vartheta, \mathbb{T}\nu), \tag{13.3}$$

with $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} \leq 1$, such that $0 < \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} \leq 1 - \tilde{\beta}$.

Proposition 1 *Let* $\mathbb{T} : \mathcal{E} \to \mathcal{P}_{C\mathcal{B}}(\mathcal{E})$*, then*

1. *If* $\mathbb{T}$ *satisfies the condition C, then* $\mathbb{T}$ *satisfies* (13.3).
2. *If* $\mathbb{T}$ *satisfies* (13.3) *with* $\mathbb{F}(\mathbb{T}) \neq \emptyset$*, then* $\mathbb{T}$ *is multivalued quasi-nonexpansive mapping.*

Proof If $\mathbb{T}$ satisfies the $Condition\ C$, then $\mathbb{T}$ satisfies (13.3) with $\tilde{\alpha} = 1, \tilde{\beta} = 0 = \tilde{\gamma}$.
For (2), let $\zeta^* \in \mathbb{F}(\mathbb{T}) \neq \emptyset$, then

$$\frac{1}{2}d(\zeta^*, \mathbb{T}\zeta^*) = 0 \leq d(\vartheta, \zeta^*), \quad \text{for all } \vartheta \in \mathcal{E}. \tag{13.4}$$

As $\mathbb{T}$ satisfies (13.3), then there exists $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in [0, 1)$, and $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} \leq 1$, we have

$$\mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\nu) \leq \tilde{\alpha}\|\vartheta - \nu\| + \tilde{\beta}d(\vartheta, \mathbb{T}\vartheta) + \tilde{\gamma}d(\vartheta, \mathbb{T}\nu)$$

holds. Then

$$\begin{aligned}\mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\zeta^*) \leq & \tilde{\alpha}\|\vartheta - \zeta^*\| + \tilde{\beta}d(\vartheta, \mathbb{T}\vartheta) + \tilde{\gamma}d(\vartheta, \mathbb{T}\zeta^*) \\ \leq & \tilde{\alpha}\|\vartheta - \zeta^*\| + \tilde{\beta}[\|\vartheta - \zeta^*\| + d(\zeta^*, \mathbb{T}\vartheta)] + \tilde{\gamma}[\|\vartheta - \zeta^*\| + d(\zeta^*, \mathbb{T}\zeta^*)] \\ \leq & (\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma})\|\vartheta - \zeta^*\| + \tilde{\beta}\mathbb{H}(\mathbb{T}\zeta^*, \mathbb{T}\vartheta).\end{aligned}$$

Using the properties of the metrics, we get

$$\begin{aligned}(1 - \tilde{\beta})\mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\zeta^*) \leq & (\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma})\|\vartheta - \zeta^*\| \\ \mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\zeta^*) \leq & \frac{(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma})}{(1 - \tilde{\beta})}\|\vartheta - \zeta^*\|.\end{aligned}$$

Since $0 < \frac{(\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})}{(1-\tilde{\beta})} \leq 1$, it follows that

$$\mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\zeta^*) \leq \|\vartheta - \zeta^*\|.$$

Hence, $\mathbb{T}$ is a multivalued quasi-nonexpansive mapping. □

Remark 1 The reverse of (1) in Proposition 1 in generally sense, it is not true. In fact, if a multivalued mapping accomplishes the condition (13.3), it doesn't have to result that the mapping accomplishes the $Condition\ C$.

Lemma 4 *For each $\vartheta, \nu \in \mathcal{E}$ and $\zeta^* \in \mathbb{T}\vartheta$, we have the following:*

1. *For any $\vartheta \in \mathcal{E}$ and $\mu \in \mathbb{T}\vartheta$, $d(\mu, \mathbb{T}\mu) \leq \|\vartheta - \mu\|$.*
2. *Either $\frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) \leq \|\vartheta - \nu\|$ or $\frac{1}{2}d(\zeta^*, \mathbb{T}\zeta^*) \leq \|\vartheta - \zeta^*\|$.*
3. *Either $\mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\nu) \leq \tilde{\alpha}\|\vartheta - \nu\| + \tilde{\beta}d(\vartheta, \mathbb{T}\vartheta) + \tilde{\gamma}d(\vartheta, \mathbb{T}\nu)$ or $\mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\zeta^*) \leq \tilde{\alpha}\|\vartheta - \zeta^*\| + \tilde{\beta}d(\vartheta, \mathbb{T}\vartheta) + \tilde{\gamma}d(\vartheta, \mathbb{T}\zeta^*)$, with $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} \leq 1$, such that $0 < \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} \leq 1 - \tilde{\beta}$.*

Proof Since $\frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) \leq \|\vartheta - \mu\|$, for any $\mu \in \mathbb{T}\vartheta$, we have

$$\begin{aligned} d(\mu, \mathbb{T}\mu) \leq& \mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\mu) \\ \leq& \tilde{\alpha}\|\vartheta - \mu\| + \tilde{\beta}d(\vartheta, \mathbb{T}\vartheta) + \tilde{\gamma}d(\vartheta, \mathbb{T}\mu) \\ \leq& \tilde{\alpha}\|\vartheta - \mu\| + \tilde{\beta}(\|\vartheta - \mu\| + d(\mu, \mathbb{T}\vartheta)) + \tilde{\gamma}(\|\vartheta - \mu\| + d(\mu, T\mu)) \\ \leq& (\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma})\|\vartheta - \mu\| + \tilde{\gamma}d(\mu, \mathbb{T}\mu) \\ \leq& \frac{(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma})}{(1 - \tilde{\gamma})}\|\vartheta - \mu\|. \end{aligned}$$

As $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} \leq 1$ and also $\tilde{\alpha} + \tilde{\gamma} \leq 1 \implies \tilde{\alpha} \leq 1 - \tilde{\gamma}$ which implies that $d(\mu, \mathbb{T}\mu) \leq \|\vartheta - \mu\|$.

Now, for (2): Suppose that

$$\frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) \geq \|\vartheta - \nu\| \text{ and } \frac{1}{2}d(\zeta^*, \mathbb{T}\zeta^*) \geq \|\vartheta - \zeta^*\|.$$

From (13.4), we have

$$\begin{aligned} \frac{1}{2}d(\zeta^*, \mathbb{T}\zeta^*) \leq& d(\vartheta, \mathbb{T}\zeta^*) \leq \|\vartheta - \zeta^*\| \\ d(\zeta^*, \mathbb{T}\zeta^*) \leq& \|\vartheta - \nu\| + \|\nu - \zeta^*\| \\ \leq& \frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) + \frac{1}{2}d(\zeta^*, \mathbb{T}\zeta^*). \\ d(\zeta^*, \mathbb{T}\zeta^*) - \frac{1}{2}d(\zeta^*, \mathbb{T}\zeta^*) <& \frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) \\ \frac{1}{2}d(\zeta^*, \mathbb{T}\zeta^*) <& \frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) \\ d(\zeta^*, \mathbb{T}\zeta^*) <& d(\vartheta, \mathbb{T}\vartheta). \end{aligned} \tag{13.5}$$

Also, we have

$$\begin{aligned} d(\vartheta, \mathbb{T}\vartheta) \leq& \|\vartheta - \zeta^*\| \\ \leq& \|\vartheta - \nu\| + \|\nu - \zeta^*\| \\ \leq& \frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) + \frac{1}{2}d(\zeta^*, \mathbb{T}\zeta^*) \end{aligned}$$

$$\begin{aligned} d(\vartheta, \mathbb{T}\vartheta) - \frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) &< \frac{1}{2}d(\zeta^*, \mathbb{T}\zeta^*) \\ \frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) &< \frac{1}{2}d(\zeta^*, \mathbb{T}\zeta^*) \\ d(\vartheta, \mathbb{T}\vartheta) &< d(\zeta^*, \mathbb{T}\zeta^*). \end{aligned} \tag{13.6}$$

Combining the inequalities (13.5) and (13.6), we get

$$d(\vartheta, \mathbb{T}\vartheta) < d(\zeta^*, \mathbb{T}\zeta^*).$$

which is contradiction. Hence, (2) holds.

For (3): The condition (3) is directly follows from condition (2). □

Lemma 5 *Let $\mathcal{E}$ be a nonempty subset of a Banach space $\mathcal{B}$ and $\mathbb{T} : \mathcal{E} \to \mathcal{P}_{C\mathcal{B}}(\mathcal{E})$ a generalized multivalued $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})-$nonexpansive mapping. Then, for any $\vartheta, \nu \in E$ and $\mu \in \mathbb{T}e$, we have*

$$d(\vartheta, \mathbb{T}\nu) \leq \frac{(1+\tilde{\beta})}{(1-\tilde{\gamma})}d(\vartheta, \mathbb{T}\vartheta) + \frac{\tilde{\alpha}}{(1-\tilde{\gamma})}\|\vartheta - \nu\|. \tag{13.7}$$

Proof From the above Lemma 4, we deduce the next cases:

Case: 1. If $\frac{1}{2}d(\mu, \mathbb{T}\mu) \leq \|\vartheta - \mu\|$ for any $\mu \in \mathbb{T}\vartheta$. It follows from Lemma 4 (1) that

$$\begin{aligned} d(\vartheta, \mathbb{T}\nu) &\leq \|\vartheta - \mu\| + d(\mu - \mathbb{T}\nu) \\ &\leq \|\vartheta - \mu\| + d(\mu, \mathbb{T}\mu) + d(\mathbb{T}\mu, \mathbb{T}\nu) \\ &\leq 2\|\vartheta - \mu\| + \mathbb{H}(\mathbb{T}\mu, \mathbb{T}\nu) \\ &\leq 2\|\vartheta - \mu\| + \tilde{\alpha}\|\mu - \nu\| + \tilde{\beta}d(\mu, \mathbb{T}\mu) + \tilde{\gamma}d(\mu, \mathbb{T}\nu) \\ &\leq 2\|\vartheta - \mu\| + \tilde{\alpha}(\|\vartheta - \mu\| + \|\vartheta - \nu\|) + \tilde{\beta}(\|\vartheta - \mu\| + \tilde{\gamma}(\|\vartheta - \mu\| + d(\vartheta, \mathbb{T}\nu)) \\ &\leq (2 + \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma})\|\vartheta - \mu\| + \tilde{\gamma}d(\vartheta, \mathbb{T}\nu) + \tilde{\alpha}\|\vartheta - \nu\| \\ &\leq \frac{(2 + \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma})}{(1-\tilde{\gamma})}\|\vartheta - \mu\| + \frac{\tilde{\alpha}}{(1-\tilde{\gamma})}\|\vartheta - \nu\|. \end{aligned}$$

The above holds for any $\mu \in \mathbb{T}\vartheta$; therefore, we have

$$d(\vartheta, \mathbb{T}\nu) \leq \frac{(2 + \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma})}{(1-\tilde{\gamma})}d(\vartheta, \mathbb{T}\vartheta) + \frac{\tilde{\alpha}}{(1-\tilde{\gamma})}\|\vartheta - \nu\|.$$

Case: 2. For any $\vartheta, \nu \in \mathcal{E}$, we have

$$\begin{aligned} d(\vartheta, \mathbb{T}\nu) \leq & d(\vartheta, T\vartheta) + d(\mathbb{T}\vartheta, \mathbb{T}\nu) \\ \leq & d(\vartheta, \mathbb{T}\vartheta) + \mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\nu) \\ \leq & d(\vartheta, \mathbb{T}\vartheta) + \tilde{\alpha}\|\vartheta - \nu\| + \tilde{\beta}d(\vartheta, \mathbb{T}\vartheta) + \tilde{\gamma}d(\vartheta, \mathbb{T}\nu) \\ d(\vartheta, \mathbb{T}\nu) - \tilde{\gamma}d(\vartheta, \mathbb{T}\nu) \leq & (1 + \tilde{\beta})d(\vartheta, \mathbb{T}\vartheta) + \tilde{\alpha}\|\vartheta - \nu\| \\ (1 - \tilde{\gamma})d(\vartheta, \mathbb{T}\nu) \leq & (1 + \tilde{\beta})d(\vartheta, \mathbb{T}\vartheta) + \tilde{\alpha}\|\vartheta - \nu\| \\ d(\vartheta, \mathbb{T}\nu) \leq & \frac{(1 + \tilde{\beta})}{(1 - \tilde{\gamma})}d(\vartheta, \mathbb{T}\vartheta) + \frac{\tilde{\alpha}}{(1 - \tilde{\gamma})}\|\vartheta - \nu\|. \end{aligned}$$

Therefore, in both scenarios, we obtain the desired inequality. □

Lemma 6 *Let $\mathcal{E}$ be a nonempty closed and convex subset of a Banach space $\mathcal{B}$ and $\mathbb{T} : \mathcal{E} \to \mathcal{P}_{C\mathcal{B}}(\mathcal{E})$ be a mapping satisfies multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-nonexpansive mapping (13.3). Suppose $\{\vartheta_n\}$ is a bounded approximate fixed point sequence for $\mathbb{T}$ in $\mathcal{E}$. Then, $\wp(\mathcal{E}, \{\vartheta_n\})$ is $\mathbb{T}$ invariant.*

Proof Let $g \in \wp(\mathcal{E}, \{\vartheta_n\})$. As $\mathbb{T}$ satisfies (13.3), we obtain

$$d(\vartheta_n, \mathbb{T}\nu) \leq \frac{(1 + \tilde{\beta})}{(1 - \tilde{\gamma})}d(\vartheta_n, \mathbb{T}\vartheta_n) + \frac{\tilde{\alpha}}{(1 - \tilde{\gamma})}\|\vartheta_n - \nu\|.$$

Using the definition of asymptotic center, we have

$$\begin{aligned} \wp(\mathbb{T}\nu, \{\vartheta_n\}) = & \limsup_{n\to\infty} d(\vartheta_n, \mathbb{T}\nu) \\ \leq & \frac{(1 + \tilde{\beta})}{(1 - \tilde{\gamma})} \limsup_{n\to\infty} d(\vartheta_n, \mathbb{T}\vartheta_n) + \frac{\tilde{\alpha}}{(1 - \tilde{\gamma})} \limsup_{n\to\infty} \|\vartheta_n - \nu\| \\ = & \frac{\tilde{\alpha}}{(1 - \tilde{\gamma})} \limsup_{n\to\infty} \|\vartheta_n - \nu\| \end{aligned}$$

As $\tilde{\alpha} + \tilde{\gamma} \leq 1$ implies $\tilde{\alpha} \leq 1 - \tilde{\gamma}$, it follows that

$$\begin{aligned} \wp(\mathbb{T}\nu, \{\vartheta_n\}) = & \limsup_{n\to\infty} \|\vartheta_n - \nu\| \\ = & \wp(\nu, \{\vartheta_n\}). \end{aligned}$$

Therefore, $\mathbb{T}\nu \in \wp(\mathcal{E}, \{\vartheta_n\})$. □

Lemma 7 *Consider $\mathbb{T}$, $\mathcal{E}$, and $\mathcal{B}$ as defined in Lemma 6. Let ϑ_n be a suitable sequence of fixed point for $\mathbb{T}$ in $\mathcal{E}$. Then,*

$$\limsup_{n\to\infty} d(\vartheta_n, \mathbb{T}\nu) \leq \limsup_{n\to\infty} \|\vartheta_n - \nu\|.$$

Proof From Lemma 5, we get

$$d(\vartheta_n, \mathbb{T}\nu) \leq \frac{(1+\tilde{\beta})}{(1-\tilde{\gamma})} d(\vartheta_n, \mathbb{T}\vartheta_n) + \frac{\tilde{\alpha}}{(1-\tilde{\gamma})} \|\vartheta_n - \nu\|.$$

Since $\tilde{\alpha} + \tilde{\gamma} \leq 1$, such that $\tilde{\alpha} + \tilde{\gamma} < 1 - \tilde{\beta}$, implies that $\tilde{\alpha} \leq 1 - \tilde{\gamma}$, it follows that

$$\limsup_{n\to\infty} d(\vartheta_n, \mathbb{T}\nu) \leq \lim_{n\to\infty} \|\vartheta_n - \nu\|.$$

So, the underlying space satisfies Opial's property, which implies that $\nu \in \mathbb{F}(\mathbb{T})$. □

Next, we present a result concerning the existence of fixed points for a multivalued mapping that satisfies condition (13.3).

Theorem 1 *Consider $\mathbb{T}$, $\mathcal{E}$, and $\mathcal{B}$ as defined in Lemma 6. Suppose the asymptotic center is nonempty and compact for each approximate fixed point sequence of $\mathbb{T}$. Then, $\mathbb{T}$ possesses a fixed point.*

Proof By using Lemma 6 of the $\mathbb{T}$ invariant, Lemma 7 of Opial's property and Theorem 1. of [17], we get our conclusion. □

4 Convergence Analysis of Fixed Point

This section deals with the approximation of the fixed point of multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-nonexpansive mappings by a four-step iterative scheme of Jia Jie et al. [30] which is (13.1) and the multivalued version of that Picard–Thakur hybrid iterative scheme is as follows:

$$\begin{cases} \vartheta_1 \in \mathcal{E}, \\ \vartheta_{n+1} = d_n, \\ \nu_n = (1-\tau_n)b_n + \tau_n c_n, \quad n \in \mathbb{Z}^+, \\ \mu_n = (1-\tau_n)\eta_n + \tau_n b_n, \\ \eta_n = (1-\sigma_n)e_n + \sigma_n a_n, \end{cases} \tag{13.8}$$

where $a_n \in \mathcal{P}_{\mathbb{T}}(\vartheta_n)$, $b_n \in \mathcal{P}_{\mathbb{T}}(\eta_n)$, $c_n \in \mathcal{P}_{\mathbb{T}}(\mu_n)$, $d_n \in \mathcal{P}_{\mathbb{T}(\nu_n)}$ and $\{\eta_n\}, \{\tau_n\}, \{\sigma_n\} \in [0, 1)$.

Lemma 8 *Let $\mathcal{B}$ be a uniformly convex Banach space, $\mathcal{E}$ be a subset of it, and $\mathbb{T} : \mathcal{E} \to \mathcal{P}_{\rho x}(\mathcal{E})$ be a multivalued mapping such that $\mathbb{F}(\mathbb{T}) \neq \emptyset$ and $\mathcal{P}_{\mathbb{T}}$ is multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-nonexpansive mapping (13.3). Let $\{\vartheta_n\}$ be the sequence defined by (13.8). Then $\lim_{n\to\infty} \|\vartheta_n - \zeta^*\|$ exists for all $\zeta^* \in \mathbb{F}(\mathbb{T})$ and $\lim_{n\to\infty} d(\vartheta_n, \mathcal{P}_{\mathbb{T}}(\vartheta_n)) = 0$.*

Proof As $\zeta^* \in \mathbb{F}(\mathbb{T})$, using Lemma 3, then $\mathcal{P}_{\mathbb{T}}(\zeta^*) = \{\zeta^*\}$ and $\mathbb{F}(\mathbb{T}) = \mathbb{F}(\mathcal{P}_{\mathbb{T}})$. As $\mathcal{P}_{\mathbb{T}}$ is multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-nonexpansive mapping, then

$$\frac{1}{2}d(\zeta^*, \mathcal{P}_{\mathbb{T}}(\zeta^*)) = 0 \leq \|\vartheta_n - \zeta^*\|.$$

Thus,

$$\begin{aligned}
\mathbb{H}(\mathcal{P}_{\mathbb{T}}(\vartheta_n, \mathcal{P}_{\mathbb{T}}(\zeta^*) \leq & \tilde{\alpha}\|\vartheta_n - \zeta^*\| + \tilde{\beta}d(\vartheta_n, \mathcal{P}_{\mathbb{T}}(\vartheta_n)) + \tilde{\gamma}d(\vartheta_n, \mathcal{P}_{\mathbb{T}}(\zeta^*)) \\
\leq & \tilde{\alpha}\|\vartheta_n - \zeta^*\| + \tilde{\beta}(\|\vartheta_n - \zeta^*\| + d(\zeta^*, \mathcal{P}_{\mathbb{T}}(\vartheta_n))) \\
& + \tilde{\gamma}(\|\vartheta_n - \zeta^*\| + d(\zeta^*, \mathcal{P}_{\mathbb{T}}(\zeta^*))) \\
\leq & (\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma})\|\vartheta_n - \zeta^*\| + \tilde{\beta}d(\zeta^*, \mathcal{P}_{\mathbb{T}}(\vartheta_n)) \\
\leq & \frac{(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma})}{(1 - \tilde{\beta})}\|\vartheta_n - \zeta^*\|.
\end{aligned}$$

As $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} \leq 1$ implies $\tilde{\alpha} + \tilde{\gamma} \leq 1 - \tilde{\beta}$, which implies that

$$\mathbb{H}(\mathcal{P}_{\mathbb{T}}(\vartheta_n), \mathcal{P}_{\mathbb{T}}(\zeta^*)) \leq \|\vartheta_n - \zeta^*\|. \tag{13.9}$$

Similarly,

$$\begin{aligned}
\mathbb{H}(\mathcal{P}_{\mathbb{T}}(\nu_n), \mathcal{P}_{\mathbb{T}}(\zeta^*)) \leq & \|\nu_n - \zeta^*\|, \\
\mathbb{H}(\mathcal{P}_{\mathbb{T}}(\mu_n), \mathcal{P}_{\mathbb{T}}(\zeta^*)) \leq & \|\mu_n - \zeta^*\|, \\
\mathbb{H}(\mathcal{P}_{\mathbb{T}}(\eta_n), \mathcal{P}_{\mathbb{T}}(\zeta^*)) \leq & \|\eta_n - \zeta^*\|.
\end{aligned}$$

Next, by Picard–Thakur hybrid iterative scheme (13.8), we have

$$\begin{aligned}
\|\eta_n - \zeta^*\| = & \|(1 - \sigma_n)\vartheta_n + \sigma_n a_n - \zeta^*\| \\
\leq & (1 - \sigma_n)\|\vartheta_n - \zeta^*\| + \sigma_n\|a_n - \zeta^*\| \\
\leq & (1 - \sigma_n)\|\vartheta_n - \zeta^*\| + \sigma_n\mathbb{H}(\mathcal{P}_{\mathbb{T}}(\vartheta_n), \mathcal{P}_{\mathbb{T}}(\zeta^*)) \\
\leq & (1 - \sigma_n)\|\vartheta_n - \zeta^*\| + \sigma_n\|\vartheta_n - \zeta^*\| \\
\leq & \|\vartheta_n - \zeta^*\|.
\end{aligned} \tag{13.10}$$

$$\begin{aligned}
\|\mu_n - \zeta^*\| = & \|(1 - \varrho_n)\eta_n + \varrho_n b_n - \zeta^*\| \\
\leq & (1 - \varrho_n)\|\eta_n - \zeta^*\| + \varrho_n\|b_n - \zeta^*\| \\
\leq & (1 - \varrho)\|\vartheta_n - \zeta^*\| + \varrho_n\mathbb{H}(\mathcal{P}_{\mathbb{T}}(\eta_n), \mathcal{P}_{\mathbb{T}}(\zeta^*)) \\
\leq & (1 - \sigma_n)\|\eta_n - \zeta^*\| + \sigma_n\|\eta_n - \zeta^*\| \\
\leq & \|\eta_n - \zeta^*\|.
\end{aligned}$$

From (13.10), we have

$$\|\mu_n - \zeta^*\| \leq \|\vartheta_n - \zeta^*\|. \tag{13.11}$$

Next,

$$\begin{aligned}\|\nu_n - \zeta^*\| =&\|(1-\tau_n)b_n + \tau_n c_n - \zeta^*\|\\ \le&(1-\tau_n)\|b_n - \zeta^*\| + \tau_n\|c_n - \zeta^*\|\\ \le&(1-\tau)\mathbb{H}(\mathcal{P}_{\mathbb{T}}\eta_n - \zeta^*\| + \tau_n\mathbb{H}(\mathcal{P}_{\mathbb{T}}(\mu_n), \mathcal{P}_{\mathbb{T}}(\zeta^*))\\ \le&(1-\tau_n)\|\eta_n - \zeta^*\| + \tau_n\|\mu_n - \zeta^*\|.\end{aligned}$$

From (13.11) and (13.10), we have

$$\|\nu_n - \zeta^*\| \le \|\vartheta_n - \zeta^*\|. \tag{13.12}$$

Now, for the fourth step of (13.8), we get

$$\begin{aligned}\|\vartheta_{n+1} - \zeta^*\| =&\|d_n - \zeta^*\|\\ \le&\mathbb{H}(\mathcal{P}_{\mathbb{T}}(\nu_n), \mathcal{P}_{\mathbb{T}}(\zeta^*))\\ \le&\|\nu_n - \zeta^*\|.\end{aligned}$$

From (13.12), we get

$$\|\vartheta_{n+1} - \zeta^*\| \le \|\vartheta_n - \zeta^*\|. \tag{13.13}$$

From (13.13), we can say that $\{\|\vartheta_n - \zeta^*\|\}$ is bounded and nonincreasing. Hence, $\lim_{n\to\infty}\|\vartheta_n - \zeta^*\|$ exists for all $\zeta^* \in \mathbb{F}(\mathbb{T})$.

Now, we have to show that

$$\lim_{n\to\infty} d(\vartheta_n, P_T(\vartheta_n)) = 0 = \lim_{n\to\infty}\|\vartheta_n - a_n\|.$$

Assume that

$$\lim_{n\to\infty}\|\vartheta_n - \zeta^*\| = \ell. \tag{13.14}$$

From (13.10) and (13.11), we have

$$\limsup_{n\to\infty}\|\eta_n - \zeta^*\| \le \ell \tag{13.15}$$

and

$$\limsup_{n\to\infty}\|\nu_n - \zeta^*\| \le \ell. \tag{13.16}$$

Also, we know that

$$\begin{aligned}\|a_n - \zeta^*\| \le&\mathbb{H}(\mathcal{P}_{\mathbb{T}}(\vartheta_n), \mathcal{P}_{\mathbb{T}}(\zeta^*)) \le \|e_n - \zeta^*\|.\\ \|d_n - \zeta^*\| \le&\mathbb{H}(\mathcal{P}_{\mathbb{T}}(\nu_n), \mathcal{P}_{\mathbb{T}}(\zeta^*)) \le \|\nu_n - \zeta^*\|.\end{aligned}$$

Applying lim sup as $n \to \infty$ on both sides of the above expressions, we get

$$\limsup_{n\to\infty} \|a_n - \zeta^*\| \leq \ell, \tag{13.17}$$

$$\limsup_{n\to\infty} \|d_n - \zeta^*\| \leq \ell, \tag{13.18}$$

and

$$\begin{aligned} \ell = &\liminf_{n\to\infty} \|\vartheta_{n+1} - \zeta^*\| \leq \liminf_{n\to\infty} \|d_n - \zeta^*\| \\ \leq &\liminf_{n\to\infty} \mathbb{H}(\mathcal{P}_{\mathbb{T}}(\nu_n), \mathcal{P}_{\mathbb{T}}(\zeta^*)) \leq \liminf_{n\to\infty} \|\nu_n - \zeta^*\|. \end{aligned} \tag{13.19}$$

From (13.16) and (13.19), we get

$$\lim_{n\to\infty} \|\nu_n - \zeta^*\| = l. \tag{13.20}$$

Also, from (13.13), (13.12) and (13.11), we can say that

$$\|\nu_n - \zeta^*\| \leq \|\eta_n - \zeta^*\|.$$

Applying lim inf as $n \to \infty$ on both sides, we obtain

$$\liminf_{n\to\infty} \|\nu_n - \zeta^*\| \leq \liminf_{n\to\infty} \|\eta_n - \zeta^*\|,$$

which implies that

$$\ell \leq \liminf_{n\to\infty} \|\eta_n - \zeta^*\|. \tag{13.21}$$

From (13.21) and (13.15), we get

$$\lim_{n\to\infty} \|\eta_n - \zeta^*\| = \ell. \tag{13.22}$$

By (13.22), we have

$$\begin{aligned} \ell &= \lim_{n\to\infty} \|\eta_n - \zeta^*\| \\ &= \lim_{n\to\infty} \|(1-\sigma_n)\vartheta_n + \sigma_n a_n - \zeta^*\| \\ &\leq [(1-\sigma_n) \lim_{n\to\infty} \|\vartheta_n - \zeta^*\| + \sigma_n \|a_n - \zeta^*\|] \\ &\leq \ell. \end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \|(1-\sigma_n)(\vartheta_n - \zeta^*) + \sigma_n(\overline{a}_n - \zeta^*)\| = \ell. \tag{13.23}$$

Thus, from (13.23), (13.17), (13.14), and Lemma 1, we have

$$\lim_{n\to\infty} \|\vartheta_n - a_n\| = \lim_{n\to\infty} d(\vartheta_n, P_{\mathbb{T}}(\vartheta_n)) = 0. \tag{13.24}$$

Theorem 2 *Let $\mathcal{E}$, $\mathcal{B}$ and $\mathbb{T}$, $\mathcal{P}_{\mathbb{T}}$ $\mathbb{F}(\mathbb{T})$ and $\{\vartheta_n\}$ be as in Lemma 8. Then $\{\vartheta_n\} \to \mathbb{F}(\mathbb{T})$.*

Proof In the above Lemma 8, we have already been shown that $\{\vartheta_n\}$ is bounded and $\lim_{n\to\infty} d(\vartheta_n, P_{\mathbb{T}}(\vartheta_n)) = 0$. Since $\mathcal{E}$ is compact then there exists a subsequence $\{\vartheta_{n_i}\}$ of a sequence $\{\vartheta_n\}$ such that $\{\vartheta_{n_i}\}$ converges to some $\nu \in \mathcal{E}$. Since $\mathcal{P}_{\mathbb{T}}$ is multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})-$nonexpansive mapping, which satisfies (13.7) and

$$d(\vartheta_{n_i}, \mathcal{P}_{\mathbb{T}}(\nu)) \leq \frac{(1+\tilde{\beta})}{(1-\tilde{\gamma})} d(\vartheta_{n_i}, \mathbb{T}(\vartheta_{n_i}) + \frac{\tilde{\alpha}}{(1-\tilde{\gamma})} \|\vartheta_{n_i} - \nu\|.$$

Since $\mathbb{F}(\mathbb{T}) = \mathbb{F}(\mathcal{P}_{\mathbb{T}})$, taking $\liminf$ as $i \to \infty$, we get $\nu \in \mathbb{T}(\nu)$. Therefore, $\{\vartheta_n\} \to \nu \in \mathbb{F}(\mathbb{T})$. □

Theorem 3 *Let $\mathcal{E}$, $\mathcal{B}$ and $\mathbb{T}$, $\mathcal{P}_{\mathbb{T}}$ $\mathbb{F}(\mathbb{T})$ and $\{\vartheta_n\}$ be as in Lemma 8. $\mathbb{T}$ also satisfies Opial's property and $0 = \liminf_{n\to\infty} d(\vartheta_n, \mathcal{F}(\overline{\mathbb{T}}))$. Then $\{\vartheta_n\} \to \mathbb{F}(\mathbb{T})$.*

Proof By Lemma 8, we have $\lim_{n\to\infty} \|\vartheta_n - \zeta^*\|$ exists for all $\zeta^* \in \mathbb{F}(\mathcal{P}_{\mathbb{T}}) = \mathbb{F}(\mathbb{T})$. Thus, $\lim_{n\to\infty} d(\vartheta_n, \mathbb{F}(\mathbb{T}))$ exists. Now, $\liminf_{n\to\infty} d(\vartheta_n, \mathbb{F}(\mathbb{T})) = 0$ gives that $\lim_{n\to\infty} d(\vartheta_n, \mathbb{F}(\mathbb{T})) = 0$. Then, there exists $\{\vartheta_{n_i}\}$ a subsequence of $\{\vartheta_n\}$ and $\nu_i \in \mathbb{F}(\mathbb{T})$ such that $\|\vartheta_{n_i} - \nu_i\| \leq \frac{1}{2^i}$ for all $i \in \mathbb{N}$. Based on the $\{\vartheta_n\}$ is nonincreasing, we have

$$\|\vartheta_{n_i+1} - \nu_i\| \leq \|\vartheta_{n_i} - \nu_i\| \leq \frac{1}{2^i}.$$

Consequently,

$$\begin{aligned}
\|\nu_{i+1} - \nu_i\| &\leq \|\nu_{i+1} - \vartheta_{i+1}\| + \|\vartheta_{i+1} - \nu_i\| \\
&\leq \frac{1}{2^{i+1}} + \frac{1}{2^i} \\
&= \frac{1}{2^{i-1}} \to 0 \text{ as } i \to \infty.
\end{aligned}$$

Then, $\{\vartheta_n\}$ is a Cauchy sequence in $\mathcal{E}$ which converges to ζ^*. Since $\mathcal{P}_{\mathbb{T}}$ fulfill (13.7), we have

$$\begin{aligned}
d(\zeta^*, \mathcal{P}_{\mathbb{T}}(\zeta^*) &\leq \|\vartheta_n - \zeta^*\| + d(\vartheta_n, \mathcal{P}_{\mathbb{T}}(\zeta^*)) \\
&\leq \|\vartheta_n - \zeta^*\| + \frac{(1+\tilde{\beta})}{(1-\tilde{\gamma})} d(\vartheta_n, \mathcal{P}_{\mathbb{T}}(\vartheta_n)) + \frac{\tilde{\alpha}}{(1-\tilde{\gamma})} \|\vartheta_n - \zeta^*\| \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Result $\zeta^* \in \mathcal{P}_{\mathbb{T}}(\zeta^*)$ and $\zeta^* \in \mathbb{F}(\mathcal{P}_{\mathbb{T}})$. Using Lemma 3, we get $\zeta^* \in \mathbb{F}(\mathbb{T})$. Then, the sequence $\{\vartheta_n\} \to \mathbb{F}(\mathbb{T})$. □

Theorem 4 *Let $\mathcal{E}$, $\mathcal{B}$ and $\mathbb{T}$, $\mathcal{P}_{\mathbb{T}}$ $\mathbb{F}(\mathbb{T})$ and $\{\vartheta_n\}$ be as in Theorem 3. $\mathbb{T}$ also satisfies the condition (I) such that $\mathbb{F}(\mathbb{T}) \neq \emptyset$. Then, $\{\vartheta_n\} \to \mathbb{F}(\mathbb{T})$.*

Proof As Lemma 8, implies $\{\vartheta_n\}$ is nonincreasing. Also, $\lim_{n\to\infty} \|\vartheta_n - \zeta^*\|$ exists for all $\zeta^* \in \mathbb{F}(\mathbb{T})$. Let $\ell = \lim_{n\to\infty} \|\vartheta_n - \zeta^*\|$, for some $\ell \geq 0$. If $\ell = 0$, then we are done.

If $\ell > 0$. Then,

$$\|\vartheta_{n+1} - \zeta^*\| \leq \|\vartheta_n - \zeta^*\|$$

implies that

$$\liminf_{n\to\infty} \|\vartheta_{n+1} - \zeta^*\| \leq \liminf_{n\to\infty} \|\vartheta_n - \zeta^*\|.$$

Therefore,

$$d(\vartheta_{n+1}, \mathbb{F}(\mathbb{T})) \leq d(\vartheta_n, \mathbb{F}(\mathbb{T})).$$

Consequently, $\lim_{n\to\infty} d(\vartheta_n, \mathbb{F}(\mathbb{T}))$ exists and

$$\mathbb{F}(\mathbb{T}) = \mathbb{F}(\mathcal{P}_{\mathbb{T}}).$$

By Lemma 8 and the given Condition (I), we obtain

$$\lim_{n\to\infty} \Psi(d(\vartheta_n, \mathbb{F}(\mathbb{T}))) \leq \lim_{n\to\infty} d(\vartheta_n, \mathbb{F}(\mathcal{P}_{\mathbb{T}}(\vartheta_n))) = 0.$$

As Ψ is nondecreasing and $\Psi(0) = 0$, we get $\lim_{n\to\infty} d(\vartheta_n, \mathbb{F}(\mathbb{T})) = 0$.
We get the conclusion following similar steps as in Theorem 3. □

Theorem 5 *Let $\mathcal{E}$, $\mathcal{B}$ and $\mathbb{T}$, $\mathbb{F}(\mathbb{T})$ be as in Theorem 3. Let $\mathcal{P}_{\mathbb{T}}$ be a multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-nonexpansive mapping and let $I - \mathcal{P}_{\mathbb{T}}$ be a demiclosed mapping w. r. t zero. If a sequence $\{\vartheta_n\}$ is given by (13.8), then, $\{\vartheta_n\}$ converges to $\mathbb{F}(\mathbb{T})$.*

Proof Suppose $\zeta^* \in \mathbb{F}(\mathbb{T}) = \mathbb{F}(\mathcal{P}_{\mathbb{T}})$. By Lemma 8, is true that $\{\vartheta_n\}$ is bounded and there exists $\lim_{n\to\infty} \|\vartheta_n - \zeta^*\|$, for each $\zeta^* \in \mathbb{F}(\mathbb{T})$. We know that $\mathcal{B}$ is uniformly convex and then, reflexive. Thus, there exists a subsequence $\{\vartheta_{n_i}\}$ of the sequence $\{\vartheta_n\}$ which it converges weakly to some $\nu \in \mathcal{E}$. As we have that $I - \mathcal{P}\mathbb{T}$ is demiclosed at zero. Moreover, $\nu_1 \in \mathbb{F}(\mathcal{P}_{\mathbb{T}}) = \mathbb{F}(\mathbb{T})$. If $\vartheta_n \not\rightharpoonup \nu_1$, then, there exists $\{\vartheta_{n_j}\}$ a subsequence of $\{\vartheta_n\}$ such that $\vartheta_{n_j} \rightharpoonup \nu_2$, where $\nu_1 \neq \nu_2$. Clearly, $\nu_2 \in \mathbb{F}(\mathcal{P}_{\mathbb{T}}) = \mathbb{F}(\mathbb{T})$.
Using Opial's property, we get the following:

$$\begin{aligned}\lim_{n\to\infty}\|\vartheta_n-\nu_1\| &= \lim_{i\to\infty}\|\vartheta_{n_i}-\nu_1\| \\ &< \lim_{i\to\infty}\|\vartheta_{n_i}-\nu_2\| \\ &= \lim_{n\to\infty}\|\vartheta_n-\nu_2\| \\ &< \lim_{j\to\infty}\|\vartheta_{n_k}-\nu_2\| \\ &< \lim_{j\to\infty}\|\vartheta_{n_k}-\nu_1\| \\ &= \lim_{n\to\infty}\|\vartheta_n-\nu_1\|.\end{aligned}$$

Contradiction. Consequently, since $\nu_1 = \nu_2$, it follows that $\vartheta_n \rightharpoonup \mathbb{F}(\mathbb{T})$. □

5 Numerical Analysis of Convergence

Example

Let $(\mathbb{R}, \|.\|)$ be a normed space with usual norm and $\mathcal{E} = [1, 3]$. Define $\mathbb{T}:\mathcal{E} \to \mathcal{P}(\mathcal{E})$ by

$$\mathbb{T}\vartheta = \begin{cases}[1, \frac{\vartheta+2}{3}], \text{ if } \vartheta \in [1, 2) \\ \{1\}, \text{ if } \vartheta \in [2, 3].\end{cases}$$

Then,

1. $\mathbb{T}$ does not satisfy the $Condition C$.
2. $\mathbb{T}$ is multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})-$ nonexpansive mapping.

First, we have to show that $\mathbb{T}$ does not satisfies the $Condition\ C$.

For this, let take $\vartheta = \frac{8}{5}$ and $\nu = \frac{5}{2}$, then

$$\frac{1}{2}d(\vartheta, \mathbb{T}\vartheta) = \frac{1}{2}d(\frac{8}{5}, \mathbb{T}(\frac{5}{2})) = \frac{1}{5},$$

and

$$\|\vartheta - \nu\| = |\frac{8}{5} - \frac{5}{2}| = |\frac{16-25}{10}| = |\frac{-9}{10}| = \frac{9}{10}$$

which implies that $\frac{1}{2}d(\vartheta, \mathbb{T}(\vartheta)) \leq \|\vartheta - \nu\|$.

On the other hand

$$\mathbb{H}(\mathbb{T}\vartheta, \mathbb{T}\nu) = \frac{13}{10} > \frac{9}{10} = \|\vartheta - \nu\|.$$

Thus, $\mathbb{T}$ does not satisfy the $Condition\ C$.

Now, we have to show that $\mathbb{T}$ satisfies (13.8). For this we have the following cases: Let take $\tilde{\alpha} = \frac{1}{2}$, $\tilde{\beta} = \frac{1}{4}$ and $\tilde{\gamma} = \frac{1}{6}$.

Case: 1. Let $\vartheta, \nu \in [1, 2)$, then

$$\begin{aligned}
&\tilde{\alpha}\|\vartheta - \nu\| + \tilde{\beta}d(\vartheta, \mathbb{T}(\vartheta)) + \tilde{\gamma}d(\vartheta, \mathbb{T}(\nu)) \\
&= \frac{1}{2}|\vartheta - \nu| + \frac{1}{4}|\vartheta - \frac{\vartheta + 2}{3}| + \frac{1}{6}|\vartheta - \frac{\nu + 2}{3}| \\
&\geq \frac{1}{2}|\vartheta - \nu - \frac{1}{2}(\vartheta - \frac{\vartheta + 2}{3}) - \frac{1}{3}(\vartheta - \frac{\nu + 2}{3})| \\
&= \frac{1}{2}|\vartheta - \frac{1}{2}\vartheta - \frac{1}{3}\vartheta + \frac{1}{6}\vartheta + \frac{1}{9}\nu - \nu + \frac{2}{6} + \frac{2}{9}| \\
&\geq \frac{1}{2}|(\frac{6 - 3 - 2 + 1}{6})\vartheta + \frac{-8}{15}\nu + \frac{10}{18}| \\
&\geq \frac{1}{2}|\frac{2}{6}\vartheta - \frac{8}{9}\nu + \frac{10}{18}| \\
&\geq \frac{1}{2}|\frac{6\vartheta - 16\nu + 10}{18}| \\
&\geq \frac{1}{3}|\vartheta - \nu| = \mathbb{H}(\mathbb{T}(\vartheta), \mathbb{T}(\nu)).
\end{aligned}$$

Case: 2. The case becomes trivial when both ϑ and ν belong to the interval [2, 3]

Case: 3. Let $\vartheta \in [1, 2)$ and $\nu \in [2, 3]$, then

$$\begin{aligned}
&\tilde{\alpha}\|\vartheta - \nu\| + \tilde{\beta}d(\vartheta, \mathbb{T}(\vartheta)) + \tilde{\gamma}d(\vartheta, \mathbb{T}(\nu)) \\
&= \frac{1}{2}|\vartheta - \nu| + \frac{1}{4}|\vartheta - \frac{\vartheta + 2}{3}| + \frac{1}{6}|\vartheta - 1| \\
&\geq \frac{1}{2}|\vartheta - \nu - \frac{1}{2}(\vartheta - \frac{\vartheta + 2}{3}) - \frac{1}{3}(\vartheta - 1)| \\
&= \frac{1}{2}|\vartheta - \nu - \vartheta + 2 - \frac{1}{3}\vartheta + \frac{1}{3}| \\
&\geq \frac{1}{2}|\frac{-1}{3}\vartheta - \nu + \frac{1}{3}| \\
&\geq \frac{1}{2}|\frac{1}{3}\vartheta - \nu - \frac{1}{3}| \\
&\geq \frac{1}{2}|\frac{\vartheta + 3\nu - 1}{3}| \\
&\geq \frac{1}{3}|\vartheta - 2| = \mathbb{H}(\mathbb{T}(\vartheta), \mathbb{T}(\nu)).
\end{aligned}$$

Hence, $\mathbb{T}$ is multivalued generalized $(\frac{1}{2}, \frac{1}{4}, \frac{1}{6})$-nonexpansive mapping.

Now, our objective is to demonstrate that $\mathcal{P}_{\mathbb{T}}$ is a multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-nonexpansive mapping For this we consider we have the following cases:

1. If $\vartheta \in [1, 2)$, then

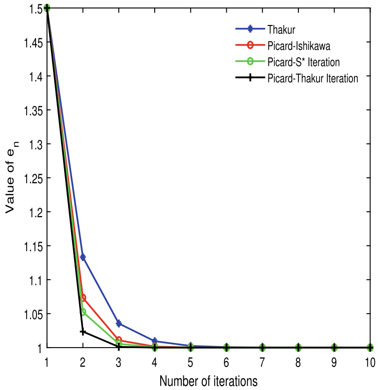
$$\begin{aligned}
\mathcal{P}_{\mathbb{T}} =& \big\{\nu \in \mathbb{T}(\vartheta) : \big|e - \nu\big| = d(\vartheta, [1, \frac{\vartheta + 2}{3}])\big|\big\} \\
=& \big\{\nu \in \mathbb{T}(\vartheta) : \big|\vartheta - \nu\big| = \big|\vartheta - \frac{\vartheta + 2}{3}\big|\big\} \\
=& \big\{\nu \in \mathbb{T}(\vartheta) : \big|\vartheta - \nu\big| = \big|\vartheta - \frac{e}{3} - \frac{2}{3}\big|\big\} \\
=& \big\{\nu \in \mathbb{T}(\vartheta) : \big|\vartheta - \nu\big| = \big|\frac{2\vartheta}{3} - \frac{2}{3}\big|\big\} \\
=& \big\{\nu \in \mathbb{T}(\vartheta) : \big|\vartheta - \nu\big| = \big|2\frac{(\vartheta - 1)}{3}\big|\big\} \\
=& \big\{\nu \in \mathbb{T}(\vartheta) : \vartheta - \nu = 2\big|\frac{\vartheta - 1}{3}\big|\big\} \text{ Since } \vartheta \geq \nu \in \mathbb{T}(\vartheta) \\
=& \big\{\nu \in \mathbb{T}(\vartheta) : \nu = \big|\frac{\vartheta + 1}{3}\big|\big\}.
\end{aligned}$$

2. If $\vartheta \in [2, 3]$, then

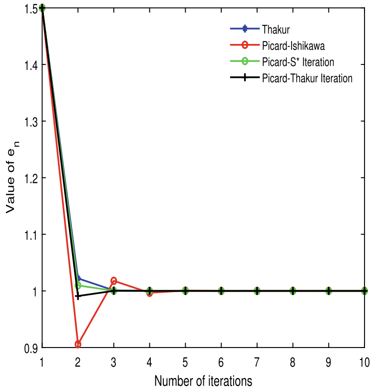
$$\begin{aligned}
\mathcal{P}_{\mathbb{T}} =& \big\{\nu \in \mathbb{T}(\vartheta) : \big|\vartheta - \nu\big| = d(\vartheta, \{1\})\big|\big\} \\
=& \big\{\nu \in \mathbb{T}(\vartheta) : \big|\vartheta - \nu\big| = \big|\vartheta - 1\big|\big\} \\
=& \big\{\nu \in \mathbb{T}(\vartheta) : \vartheta - \nu = \vartheta - 1\big\} \text{ Since } \vartheta \geq \nu \in \mathbb{T}(\vartheta) \\
=& \big\{\nu \in \mathbb{T}(\vartheta) : \nu = 1\big\}.
\end{aligned}$$

By following the same argument as in the above example, we prove that $\mathcal{P}_{\mathbb{T}}$ is multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-nonexpansive mapping.

For the approximation of the fixed point, let us consider the initial point $\vartheta_1 = \frac{3}{2}$, and the parameter $\tau_n = \frac{n+3}{n+6}$, $\varrho_n = \frac{n^2+3}{n^2+n+5}$ and $\sigma_n = \frac{n+2}{n+5}$. The following Table and Figures prove that the sequence $\{\vartheta_n\}$ generated by (13.8) converges faster than the other compared schemes. Therefore, the fixed point of $\mathbb{T}$ is $\zeta^* = 1$. In each iteration, we begin with a specific chosen value of $\vartheta 1$ and terminate the iteration process once



(a) $\tau_n = \frac{n+3}{n+6}$, $\varrho_n = \frac{n^2+3}{n^2+n+5}$, $\sigma_n = \frac{n+2}{n+5}$



(b) $\tau_n = \sqrt{\frac{n}{n+1}}$, $\varrho_n = \sqrt{\frac{n+3}{n^3+n+5}}$, $\sigma_n = \sqrt{\frac{n+4}{n^2+5}}$

Fig. 1 Comparison of iteration processes with various parameter choices

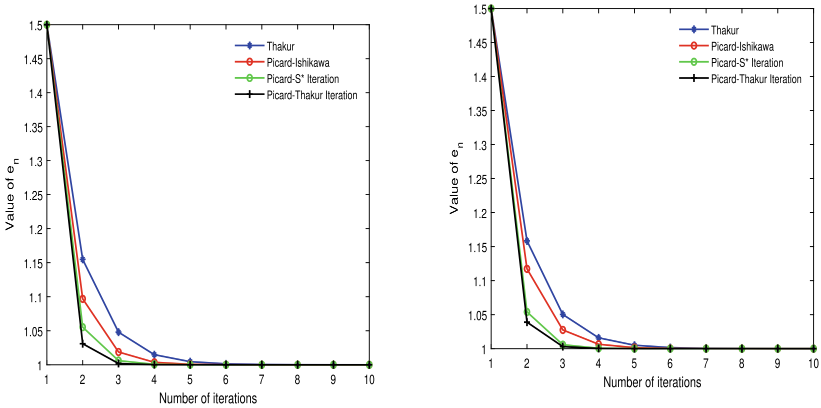
the condition $|\vartheta n - \zeta^*| < 10^{-10}$ is satisfied, where $\zeta^* = 1$ represents the fixed point of the mapping $\mathbb{T}$.

Initial Values	Thakur Scheme	Picard-Ishikawa	Picard$-S^*$	Picard–Thakur
1.22	22	18	12	08
1.55	22	18	12	08
1.99	23	19	13	09
2.01	23	19	13	09
2.91	24	19	13	09

Next, we analyze and compare the convergence behavior of the iteration (13.8) using different parameter choices. Specifically, we examine the convergence of the Picard–Thakur hybrid scheme (13.8). The following graphs shows the convergence comparison of the iterative schemes for different choice of parameter and upon observation, it becomes evident that the iteration process (13.8) exhibits a faster convergence rate towards the fixed point in comparison to the other iteration processes (Figs. 1 and 2).

6 Application to Integral Equations

In this section, we investigate the application of our iterative scheme, defined as in (13.8), for solving integral equations. We explore the capability of the scheme to provide a solution to the given integral equation. It is well known the contribution



(a) $\tau_n = \frac{n}{n+6}$, $\varrho_n = \frac{1}{n^2}$, $\sigma_n = \frac{1}{n}$ (b) $\tau_n = \sqrt{\frac{n+9}{n^2+3}}$, $\varrho_n = \sqrt{\frac{n}{n^3+4}}$, $\sigma_n = \sqrt{\frac{n}{n^2+2}}$

Fig. 2 Comparison of iteration processes with various parameter choices

of the integral equations in the modeling of real world processes as from physics, biology, engineering, etc.

Suppose the space $\mathcal{B} = \mathcal{L}^2([0, 1], \mathbb{R})$ of real valued functions on [0, 1] implies $\int_{-\infty}^{+\infty} |h(\vartheta)|^2 < +\infty$. Since all the Hilbert spaces are uniformly convex Banach spaces, then $\mathcal{L}^2([0, 1], \mathbb{R})$ is an uniformly convex Banach space.

Consider the integral equation:

$$f(q) = m(q) + \int_0^1 J(q, w, f(w))dw, \tag{13.25}$$

where $q \in [0, 1]$. Assume that (13.25) satisfies following conditions:

(i) $m \in \mathcal{L}^2([0, 1], \mathbb{R})$,

(ii) $J : [0, 1] \times [0, 1] \times \mathcal{L}^2([0, 1], \mathbb{R}) \to \mathbb{R}$ is measurable and satisfies the condition:

$$0 \leq \left|J(q, w, f) - J(q, w, g)\right| \leq \left|f - g\right|,$$

for $q, w \in [0, 1]$ and $f, g \in \mathcal{L}^2([0, 1], \mathbb{R})$ such that $f \leq g$.

Next, we assume that there exists a nonnegative function $h(., .) \in \mathcal{L}^2([0, 1] \times [0, 1])$ and $G < \frac{1}{2}$ such that

$$\left|J(q, w, f)\right| \leq f(q, w) + G|f|,$$

for $q, w \in [0, 1]$ and $f \in \mathcal{L}^2([0, 1], \mathbb{R})$.

Moreover, it is true

$$\mathcal{E} = \{f \in \mathcal{L}^2([0,1],\mathbb{R}) \textit{such that} \|f\|_{\mathcal{L}^2([0,1],\mathbb{R})} \leq \sigma\},$$

where σ is sufficiently large, that is, $\mathcal{E}$ is a closed ball of $\mathcal{L}^2([0,1],\mathbb{R})$ centered at 0 with radius σ.

Define the operator $\mathbb{T} : \mathcal{L}^2([0,1],\mathbb{R}) \to \mathcal{L}^2([0,1],\mathbb{R})$ by

$$\mathbb{T}(f(q)) = m(q) + \int_0^1 J(q, w, f(w))dw. \tag{13.26}$$

It is easy to prove that $\mathbb{T}(\mathcal{E}) \subset \mathcal{E}$ and it is nonexpansive map.

As $\mathcal{L}^2([0,1],\mathbb{R})$ is an uniformly convex Banach space and taking $\mathcal{B} = \mathcal{L}^2([0,1],\mathbb{R})$ and $\mathbb{T}$ as in (13.26) in Theorem 5, we get the following result.

Theorem 6 *Under the aforementioned assumptions, it can be shown that the sequence generated by the iteration scheme* (13.8) *converges to a solution of the integral equation* (13.25).

7 Conclusion

In this chapter, we introduced the multivalued generalized $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$-nonexpansive mapping and related result for existence and approximation of fixed point. Moreover, we present the convergence results for Picard–Thakur hybrid iterative scheme (13.8).

We compared the Picard–Thakur hybrid iterative scheme. We have shown numerically and graphically that the Picard–Thakur hybrid iterative scheme converges faster to the fixed point than other schemes as given in the literature. In the last section, to validate our results we gave an application to integral equations.

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Common Fixed Point Results in Soft b-Metric Spaces with Application

Vishal Gupta, Mohammad Saeed Khan, Aanchal, and Ishit Sehgal

1 Introduction

The concept of b-metric space was given by Czerwik [1]. Many types of sets have been introduced by various researchers to deal with uncertainties. In the case of data consisting of parameters, Molodstov [2] gave soft sets to deal with the uncertainties. Das and Samanta [3, 4] applied the concept of soft sets to metric spaces and hence presented soft metric spaces utilizing soft points of soft sets. The concept of soft sets was then extended to fuzzy sets and fuzzy metric spaces by Maji et al. [5] and Beaula and Gunaseli [6] respectively and therefore introducing soft fuzzy sets and soft fuzzy metric spaces. Afterward, the concept of soft b-metric space came into existence.

A new type of contractions called JS contractions was given by Hussain, Parvaneh, Samet, and Vetro [11], and we will use these new type of contractions to prove fixed point theorems.

In this paper, we are going to extend the fixed point theorems given by Al-Mazrooei and Ahmad [8] to soft b-metric space.

2 Preliminaries

In this section, we have given some definitions and results already present in the literature that will be used to formulate our new results.

In our results $\bar{\Upsilon}$ is the absolute soft set and U is the set of parameters.

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Definition 1 ([2]) Soft set is a pair (S, U) on a universe Υ where U represents the parameter set and S defines the map from U to power set of υ, i.e., $S : U \to P(\upsilon)$.

Definition 2 ([3, 4]) A Soft Metric Space is a 3-tuple $(\bar{\Upsilon}, \bar{\rho}, U)$, where the soft metric is $\bar{\rho} : SP(\bar{\Upsilon}) \times SP(\bar{\Upsilon}) \to R(U)$ here $R(U)$, is the set containing non-negative soft real numbers and $\bar{\rho}$ satisfies the given conditions for all $\bar{u}_{e_1}, \bar{v}_{e_2}, \bar{w}_{e_3} \in SP(\bar{\Upsilon})$:

(i) $\bar{\rho}(\bar{u}_{e_1}, \bar{v}_{e_2}) > \bar{0}$;
(ii) $\bar{\rho}(\bar{u}_{e_1}, \bar{v}_{e_2}) = \bar{0}$ iff $\bar{u}_{e_1} = \bar{v}_{e_2}$;
(iii) $\bar{\rho}(\bar{u}_{e_1}, \bar{v}_{e_2}) = \bar{\rho}(\bar{v}_{e_2}, \bar{u}_{e_1})$;
(iv) $\bar{\rho}(\bar{u}_{e_1}, \bar{v}_{e_2}) \leq \bar{\rho}(\bar{u}_{e_1}, \bar{w}_{e_3}) + \bar{\rho}(\bar{w}_{e_3}, \bar{v}_{e_2})$.

Definition 3 ([5]) Fuzzy soft set is a pair (S, U) over a universe Υ, where U represents the parameter set and S defines the map from U to $F(\Upsilon)$ which is the set containing fuzzy subsets in universe X, i.e., $S : U \to F(\Upsilon)$.

Definition 4 ([6]) A soft fuzzy metric space is the 3-tuple $(\bar{\Upsilon}, S, *)$, where soft fuzzy metric on $\bar{\Upsilon}$ is given by map $S : SP(\bar{\Upsilon}) \times SP(\bar{\Upsilon}) \times (0, \infty) \to [0, 1]$ satisfying the given conditions for all $\bar{u}_{e_1}, \bar{v}_{e_2}, \bar{w}_{e_3} \in SP(\bar{\Upsilon})$ and $s, t > 0$:

(i) $S(\bar{u}_{e_1}, \bar{v}_{e_2}, t) > 0$;
(ii) $S(\bar{u}_{e_1}, \bar{v}_{e_2}, t) = 1$ iff $\bar{u}_{e_1} = \bar{v}_{e_2}$;
(iii) $S(\bar{u}_{e_1}, \bar{v}_{e_2}, t) = S(\bar{v}_{e_2}, \bar{u}_{e_1}, t)$;
(iv) $S(\bar{u}_{e_1}, \bar{v}_{e_2}, t + s) \geq S(\bar{u}_{e_1}, \bar{w}_{e_3}, t) * S(\bar{w}_{e_3}, \bar{v}_{e_2}, s)$;
(v) $S(\bar{u}_{e_1}, \bar{v}_{e_2}, .) : (0, \infty) \to [0, 1]$ is continuous.

Definition 5 Consider $\bar{\Upsilon} \neq \phi$. The map $S_b : \bar{\Upsilon} \times \bar{\Upsilon} \to R^+$ is called a soft b-metric if the following properties holds for all $\bar{\iota}, \bar{\kappa}, \bar{\gamma} \in \bar{\Upsilon}$, where $\bar{k} \geq \bar{1}$,

Sb_i $S_b(\bar{\iota}, \bar{\kappa}) = 0$ if and only if $\bar{\iota} = \bar{\kappa}$;
Sb_{ii} $S_b(\bar{\iota}, \bar{\kappa}) = S_b(\bar{\kappa}, \bar{\iota})$;
Sb_{iii} $S_b(\bar{\iota}, \bar{\gamma}) \leq \bar{k}(S_b(\bar{\iota}, \bar{\kappa}) + S_b(\bar{\kappa}, \bar{\gamma}))$.

Then, $(\bar{\Upsilon}, S_b, \bar{k})$ is known as a soft b-metric space.

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Example 1 Consider, $\bar{\Upsilon} = \mathbb{R}$ and soft b-metric $S_b : \bar{\Upsilon} \times \bar{\Upsilon} \to R$ defined as

$$S_b(\bar{\iota}, \bar{\kappa}) = | \bar{\iota} - \bar{\kappa} |^2$$

for all $\bar{\iota}, \bar{\kappa} \in \bar{\Upsilon}$ and $\bar{k} = 2$. Then, $(\bar{\Upsilon}, S_b, \bar{k})$ is a soft b-metric space but it is not a soft metric space.

Consider Ψ as a set of functions from $[0, \infty)$ to $[1, \infty)$ which satisfies the given assertions:

1. ψ is nondecreasing and $\psi(\varrho) = 1$ for $\varrho = 0$;
2. $lt_{n\to\infty}\psi(\varrho_n) = 1$ if and only if $lt_{n\to\infty}\varrho_n = 0$ where $\{\varrho_n\}$ is a positive real number sequence;
3. $\psi(h + k) \leq \psi(h) + \psi(k)$ for $h, k > 0$;
4. for $0 < \delta < 1$ and $0 \leq K < \infty$ we have $lt_{\varrho\to 0^+} \frac{\psi(\varrho)-1}{\varrho^k} = K$.

Definition 6 ([11]) Consider (Υ, μ) as a metric space and a self-map S, then S is a JS contraction if it implies the existence of a function $\psi \in \Psi$ and positive real numbers h_1, h_2, h_3, h_4 with $0 < h_1 + h_2 + h_3 + 2h_4 < 1$ so that

$$\psi(\mu(S\iota, S\kappa)) \leq [\psi(\mu(\iota, \kappa)]^{h_1}[\psi(\mu(\iota, S\iota)]^{h_2}][\psi(\mu(\kappa, S\kappa)]^{h_3}]$$
$$[\psi(\mu(\iota, S\kappa)]^{h_2} + [\psi(\mu(\kappa, S\iota)]^{h_4}],$$

for every $\iota, \kappa \in \Upsilon$.

Consider $\Sigma_{cb}(\bar{\Upsilon})$ as the class of all non-empty subsets of $\bar{\Upsilon}$ that are bounded and closed. Let $P_1, P_2, P_3 \in \Sigma_{cb}(\bar{\Upsilon})$ and define $M_b : \Sigma_{cb}(\bar{\Upsilon}) \times \Sigma_{cb}(\bar{\Upsilon}) \to \mathbb{R}^+$ as

$$M_b(P_1, P_2) = max\{m_b(P_1, P_2), m_b(P_2, P_1)\},$$

where

$$m_b(P_1, P_2) = sup\{S_b(\bar{\iota}, \bar{\kappa}) : \bar{\iota} \in P_1, \bar{\kappa} \in P_2\},$$
$$\delta_b(\bar{\iota}, P_3) = \delta_b(\{\bar{\iota}\}, P_3) = inf\{S_b(\bar{\iota}, \bar{\gamma}) : \bar{\gamma} \in P_3\}.$$

Here, M_b is known as the Hausdorff b-metric space induced by soft b-metric S_b.

Lemma 1 *Consider $(\bar{\Upsilon}, S_b, \bar{k})$ as soft b-metric space, then the following conditions holds for $P_1, P_2, P_3 \in \Sigma_{cb}(\bar{\Upsilon})$ and $\bar{\iota}, \bar{\kappa} \in \bar{\Upsilon}$,*

(a) $\delta_b(\bar{\iota}, P_2) \leq S_b(\bar{\iota}, \bar{\gamma})$ *where* $\bar{\gamma} \in P_2$;
(b) $m_b(P_1, P_2) \leq M_b(P_1, P_2)$;
(c) $\delta_b(\bar{\mu}, P_2) \leq M_b(P_1, P_2)$, *where* $\bar{\mu} \in P_1$;
(d) $M_b(P_1, P_2) = 0$;
(e) $M_b(P_1, P_2) = M_b(P_2, P_1)$;
(f) $M_b(P_1, P_3) \leq \bar{k}[M_b(P_1, P_2) + M_b(P_2, P_3)]$;
(g) $\delta_b(\bar{\iota}, P_1) \leq \bar{k}[S_b(\bar{\iota}, \bar{\kappa}) + S_b(\bar{\kappa}, P_1)]$.

Proof The proof is on similar lines as given in [9]. □

Definition 7 Consider $\chi_{\bar{k}}(\bar{k} \geq \bar{1})$ as the family of maps $\varphi : R(U) \to (1, \infty)$, where $R(U)$ is the set containing non-negative soft real numbers, that satisfies the following properties:

φ_i $\varphi(\bar{\iota}) < \varphi(\bar{\kappa})$, for $\bar{\iota} < \bar{\kappa}$;
φ_{ii} $\varphi(\bar{\iota}_{e_n}) = 1$, if and only if $lt_{n\to\infty}(\bar{\iota}_{e_n}) = \bar{0}$ where $\{(\bar{\iota}_{e_n})\} \subseteq R(U)$;
φ_{iii} $lt_{\bar{\iota}\to\bar{0}}(\varphi(\bar{\iota}) - \frac{1}{\bar{\iota}_{e_i}}) = j$, for $i \in (0, 1)$ and $j \in (0, \infty)$;

φ_{iv} $\varphi(inf P_1) = inf \varphi(P_1)$ with $inf P_1 > \bar{0}$;
φ_v if $\varphi(\bar{k}\bar{\iota}_{e_n}) \leq [\varphi(\bar{\iota}_{e_{n-1}})]^k$, where $n \in \mathbb{N}$ and $0 < k < 1$, then $\varphi(\bar{k_n}^n \bar{\iota}) \leq [\varphi(\bar{k}^{n-1} \bar{\iota}_{e_{n-1}})]^k$.

Now, we are giving an important lemma that will be used in our new results.

Lemma 2 ([10]) *Consider* $(\bar{\Upsilon}, S_b, \bar{k})$ *as soft b-metric space and* $P_1, P_2 \in \Sigma_{cb}(\bar{\Upsilon})$, *then* (1) *holds for all* $\bar{\iota} \in P_1$,

$$S_b(\bar{\iota}, P_2) \leq M_b(P_1, P_2). \tag{1}$$

Proof The proof is on similar lines as given in [10].

In the next section, we are going to extend common α-fuzzy fixed point theorems in a soft b-metric space.

3 Main Results

Theorem 1 *Consider* $(\bar{\Upsilon}, S_b, \bar{k})$ *as complete soft b-metric space, where* $\bar{k} \geq \bar{1}$ *and* S_b *is continuous. Let* $\eta_1, \eta_2 : \bar{\Upsilon} \to \bar{\Upsilon}$, *such that for every* $\bar{\iota}, \bar{\kappa} \in \bar{\Upsilon}$, *there exists* $\beta_{\eta_1}(\bar{\iota}), \beta_{\eta_2}(\bar{\kappa}) \in (\bar{0}, \bar{1}]$ *so that* $[\eta_1\bar{\iota}]_{\beta_{\eta_1}(\bar{\iota})}, [\eta_2\bar{\kappa}]_{\beta_{\eta_2}(\bar{\kappa})} \in \Sigma_{cb}(\bar{\Upsilon})$. *Now if for all* $\bar{\iota}, \bar{\kappa} \in \bar{\Upsilon}$ *with* $M_b([\eta_1\bar{\iota}]_{\beta_{\eta_1}(\bar{\iota})}, [\eta_2\bar{\kappa}]_{\beta_{\eta_2}(\bar{\kappa})}) > \bar{0}$ *and* $0 < k < 1$, *there exists* $\varphi \in \chi_{\bar{k}}$ *so that* (2) *holds*

$$\varphi(\bar{k}M_b([\eta_1\bar{\iota}]_{\beta_{\eta_1}(\bar{\iota})}, [\eta_2\bar{\kappa}]_{\beta_{\eta_2}(\bar{\kappa})})) \leq [\varphi(S_b(\bar{\iota}, \bar{\kappa}))]^k. \tag{2}$$

Then, there exists $\bar{r}^* \in \bar{\Upsilon}$ *with* $\bar{r}^* \in [\eta_1\bar{r}^*]_{\beta_{\eta_1}(\bar{r}^*)} \cap [\eta_2\bar{r}^*]_{\beta_{\eta_2}(\bar{r}^*)}$.

Proof Consider $\bar{\iota}_{e_o} \in \bar{\Upsilon}$, thus there exists $\beta_{\eta_1}(\bar{\iota}_{e_o}) \in (0, 1]$, so that $[\eta_1\bar{\iota}_{e_o}]_{\beta_{\eta_1}(\bar{\iota}_{e_o})} \in \Sigma_{cb}(\bar{\Upsilon})$. Now, let $\bar{\iota}_{e_1} \in \bar{\Upsilon}$. Then, for $\bar{\iota}_{e_o}$ there exists $[\eta_1\bar{\iota}_{e_1}]_{\beta_{\eta_1}(\bar{\iota}_{e_1})} \in \Sigma_{cb}(\bar{\Upsilon})$. By Lemma 2 and (2), we have

$$\varphi(\bar{k}S(\bar{\iota}_{e_1}, [\eta_2\bar{\iota}_{e_1}]_{\beta_{\eta_2}(\bar{\iota}_{e_1})})) \leq \varphi(\bar{k}M_b([\eta_1\bar{\iota}_{e_o}]_{\beta_{\eta_1}(\bar{\iota}_{e_o})}, [\eta_2\bar{\iota}_{e_1}]_{\beta_{\eta_2}(\bar{\iota}_{e_1})})) \leq [\varphi(S_b(\bar{\iota}_{e_o}, \bar{\iota}_{e_1}))]^k. \tag{3}$$

Hence,

$$\varphi(\bar{k}S(\bar{\iota}_{e_1}, [\eta_2\bar{\iota}_{e_1}]_{\beta_{\eta_2}(\bar{\iota}_{e_1})})) \leq [\varphi(S_b(\bar{\iota}_{e_o}, \bar{\iota}_{e_1}))]^k. \tag{4}$$

Thus, we have

$$\varphi(\bar{k}S(\bar{\iota}_{e_1}, [\eta_2\bar{\iota}_{e_1}]_{\beta_{\eta_2}(\bar{\iota}_{e_1})})) = inf_{\bar{y} \in [\eta_2\bar{\iota}_{e_1}]_{\beta_{\eta_2}(\bar{\iota}_{e_1})}} \varphi(\bar{k}S_b(\bar{\iota}_{e_1}, \bar{y})), \tag{5}$$

and

$$inf_{\bar{y}\in[\eta_2\bar{\iota}_{e_1}]_{\beta_{\eta_2}(\bar{\iota}_{e_1})}}\varphi(\bar{k}S_b(\bar{\iota}_{e_1},\bar{y}))\leq[\varphi(S_b(\bar{\iota}_{e_o},\bar{\iota}_{e_1}))]^k. \tag{6}$$

Thus, there exists $\bar{\iota}_{e_2}\in[\eta_2\bar{\iota}_{e_1}]_{\beta_{\eta_2}(\bar{\iota}_{e_1})}$ with $\bar{\iota}_{e_2}\neq\bar{\iota}_{e_1}$, so that $\varphi(\bar{k}S_b(\bar{\iota}_{e_1},\bar{\iota}_{e_2}))\leq[\varphi(S_b(\bar{\iota}_{e_o},\bar{\iota}_{e_1}))]^k$.

Now, for $\bar{\iota}_{e_2}$ there exists $\bar{0}<\beta_{\eta_1}(\bar{\iota}_{e_1})\leq\bar{1}$, so that $[\eta_1\bar{\iota}_{e_2}]_{\beta_{\eta_1}(\bar{\iota}_{e_2})}\in\Sigma_{cb}(\bar{\Upsilon})$. Then, we have

$$\varphi(\bar{k}S(\bar{\iota}_{e_2},[\eta_1\bar{\iota}_{e_2}]_{\beta_{\eta_1}(\bar{\iota}_{e_2})}))\leq\varphi(\bar{k}M_b([\eta_2\bar{\iota}_{e_1}]_{\beta_{\eta_2}(\bar{\iota}_{e_1})},[\eta_1\bar{\iota}_{e_2}]_{\beta_{\eta_1}(\bar{\iota}_{e_2})}))\leq[\varphi(S_b(\bar{\iota}_{e_1},\bar{\iota}_{e_2}))]^k. \tag{7}$$

Hence,

$$\varphi(\bar{k}S(\bar{\iota}_{e_2},[\eta_1\bar{\iota}_{e_2}]_{\beta_{\eta_1}(\bar{\iota}_{e_2})}))\leq[\varphi(S_b(\bar{\iota}_{e_1},\bar{\iota}_{e_2}))]^k. \tag{8}$$

Thus, we have

$$\varphi(\bar{k}S(\bar{\iota}_{e_2},[\eta_1\bar{\iota}_{e_2}]_{\beta_{\eta_1}(\bar{\iota}_{e_2})}))=inf_{\bar{y}\in[\eta_1\bar{\iota}_{e_2}]_{\beta_{\eta_1}(\bar{\iota}_{e_2})}}\varphi(\bar{k}S_b(\bar{\iota}_{e_2},\bar{y})), \tag{9}$$

and

$$inf_{\bar{y}\in[\eta_1\bar{\iota}_{e_2}]_{\beta_{\eta_1}(\bar{\iota}_{e_2})}}\varphi(\bar{k}S_b(\bar{\iota}_{e_2},\bar{y}))\leq[\varphi(S_b(\bar{\iota}_{e_1},\bar{\iota}_{e_2}))]^k. \tag{10}$$

Thus, there exists $\bar{\iota}_{e_3}\in[\eta_1\bar{\iota}_{e_2}]_{\beta_{\eta_1}(\bar{\iota}_{e_2})}$ with $\bar{\iota}_{e_3}\neq\bar{\iota}_{e_2}$, so that $\varphi(\bar{k}S_b(\bar{\iota}_{e_2},\bar{\iota}_{e_3}))\leq[\varphi(S_b(\bar{\iota}_{e_1},\bar{\iota}_{e_2}))]^k$.

Going this way, we have a sequence $\{\bar{\iota}_{e_n}\}\in\bar{\Upsilon}$ so that (11) holds for every $n\in N$,

$$\begin{aligned}\bar{\iota}_{e_{2n+1}}&\in[\eta_1\bar{\iota}_{e_{2n}}]_{\beta_{\eta_1}(\bar{\iota}_{e_{2n}})},\\ \bar{\iota}_{e_{2n+2}}&\in[\eta_2\bar{\iota}_{e_{2n+1}}]_{\beta_{\eta_2}(\bar{\iota}_{e_{2n+1}})},\end{aligned} \tag{11}$$

$$\varphi(\bar{k}S_b(\bar{\iota}_{e_{2n+1}},\bar{\iota}_{e_{2n+2}}))\leq[\varphi(S_b(\bar{\iota}_{e_{2n}},\bar{\iota}_{e_{2n+1}}))]^k, \tag{12}$$

$$\varphi(\bar{k}S_b(\bar{\iota}_{e_{2n+2}},\bar{\iota}_{e_{2n+3}}))\leq[\varphi(S_b(\bar{\iota}_{e_{2n+1}},\bar{\iota}_{e_{2n+2}}))]^k. \tag{13}$$

Thus, we have

$$\varphi(\bar{k}^nS_b(\bar{\iota}_{e_n},\bar{\iota}_{e_{n+1}}))\leq[\varphi(\bar{k}^{n-1}S_b(\bar{\iota}_{e_{n-1}},\bar{\iota}_{e_n}))]^k,$$

that implies

$$\begin{aligned}\varphi(\bar{k}^nS_b(\bar{\iota}_{e_n},\bar{\iota}_{e_{n+1}}))&\leq[\varphi(\bar{k}^{n-1}S_b(\bar{\iota}_{e_{n-1}},\bar{\iota}_{e_n}))]^k\leq[\varphi(\bar{k}^{n-2}S_b(\bar{\iota}_{e_{n-2}},\bar{\iota}_{e_{n-1}}))]^{k^2}\\&\leq...\leq[\varphi(S_b(\bar{\iota}_{e_o},\bar{\iota}_{e_1}))]^{k^n}.\end{aligned}$$

Hence,

$$\varphi(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}})) \leq [\varphi(S_b(\bar{\iota}_{e_o}, \bar{\iota}_{e_1}))]^{k^n}.$$

Now, taking $n \to \infty$, we get $lt_{n\to\infty}\varphi(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}})) = 1$.

Therefore, $lt_{n\to\infty}\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}}) = 0$. Hence, there exists $0 < s < 1$ and $\alpha \in (0, \infty)$, so that

$$lt_{n\to\infty}\frac{\varphi(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}})) - 1}{(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}}))^s} = \alpha.$$

Consider $\alpha < \infty$ and let $\Theta_2 = \frac{\alpha}{2}$. Thus, there exists $m_o \in N$ such that (14) holds for all $n > m_o$,

$$\left[\frac{\varphi(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}})) - 1}{(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}}))^s} - \alpha\right] \leq \Theta_2. \tag{14}$$

Thus, we have (15) for all $n > m_o$,

$$\frac{\varphi(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}})) - 1}{(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}}))^s} \geq \alpha - \Theta_2 = \frac{\alpha}{2} = \Theta_2. \tag{15}$$

Therefore, for $\Theta_1 = \frac{1}{\Theta_2}$, we have

$$n(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}}))^s \leq \Theta_1 n[\varphi(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}})) - 1].$$

Now, consider $\alpha = \infty$ and $\Theta_2 > 0$. Thus, for $m_o \in N$ we have (16) for all $n > m_o$,

$$\begin{aligned} \Theta_2 &\leq \frac{\varphi(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}})) - 1}{(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}}))^s}, \\ n(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}}))^s &\leq \Theta_1 n[\varphi(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}})) - 1]. \end{aligned} \tag{16}$$

Thus, overall we have

$$n(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}}))^s \leq \Theta_1 n[\varphi(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}})) - 1].$$

Hence, we get

$$n(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}}))^s \leq \Theta_1 n([\varphi(S_b(\bar{\iota}_{e_o}, \bar{\iota}_{e_1}))]^{s^n} - 1).$$

Now, taking $n \to \infty$, we have $lt_{n\to\infty} n(\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}}))^s = 0$.

Thus, $lt_{n\to\infty} n^{(\frac{1}{s})}\bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}}) = 0$; that implies $\sum_{n=1}^{\infty} \bar{k}^n S_b(\bar{\iota}_{e_n}, \bar{\iota}_{e_{n+1}})$ converges. Hence, $\{\bar{\iota}_{e_n}\}$ is a Cauchy sequence in $\bar{\Upsilon}$. As $(\bar{\Upsilon}, S_b, \bar{k})$ is complete, thus there exists $\bar{\iota}^* \in \bar{\Upsilon}$, so that,

$$lt_{n\to\infty}\bar{\iota}_{e_n} = \bar{\iota}^*.$$

Now, claim that $\bar{\iota}^* \in [\eta_2\bar{\iota}^*]_{\beta_{\eta_2}(\bar{\iota}^*)}$.

Consider the opposite, then there exists $m_k \in \mathbb{N}$ and subsequence $\{\bar{\iota}_{e_{m_k}}\}$ of $\{\bar{\iota}_{e_n}\}$ so that, $S_b(\bar{\iota}_{e_{2m_k+1}}, [\eta_2\bar{\iota}^*]_{\beta_{\eta_2}(\bar{\iota}^*)}) > \bar{0}$, where $m_k \geq m_o$. Now, from (2) with $\bar{\iota} = \bar{\iota}_{2m_k+1}$ and $\bar{\kappa} = \bar{\iota}^*$, we have

$$\begin{aligned}
\varphi(S_b(\bar{\iota}_{e_{2m_k+1}}, [\eta_2\bar{\iota}^*]_{\beta_{\eta_2}(\bar{\iota}^*)})) &\leq \varphi[\bar{k}S_b(\bar{\iota}_{e_{2m_k+1}}, [\eta_2\bar{\iota}^*]_{\beta_{\eta_2}(\bar{\iota}^*)})] \\
&\leq \varphi[\bar{k}M_b([\eta_1\bar{\iota}_{e_{2m_k+1}}]_{\beta_{\eta_1}(\bar{\iota}_{e_{2m_k}})}, [\eta_2\bar{\iota}^*]_{\beta_{\eta_2}(\bar{\iota}_*)})] \\
&\leq [\varphi(S_b(\bar{\iota}_{e_{2m_k}}, \bar{\iota}^*))]^k.
\end{aligned}$$

Now, as $0 < k < 1$, we have

$$S_b(\bar{\iota}_{e_{2m_k+1}}, [\eta_2\bar{\iota}^*]_{\beta_{\eta_2}(\bar{\iota}^*)}) < S_b(\bar{\iota}_{e_{2m_k}}, \bar{\iota}^*).$$

Taking $n \to \infty$,

$$S_b(\bar{\iota}^*, [\eta_2\bar{\iota}^*]_{\beta_{\eta_2}(\bar{\iota}^*)}) \leq \bar{0}.$$

Thus, $\bar{\iota}^* \in [\eta_2\bar{\iota}^*]_{\beta_{\eta_2}(\bar{\iota}^*)}$. Similarly, we can easily prove that $\bar{\iota}^* \in [\eta_1\bar{\iota}^*]_{\beta_{\eta_1}(\bar{\iota}^*)}$. Hence, $\bar{\iota}^* \in [\eta_1\bar{\iota}^*]_{\beta_{\eta_1}(\bar{\iota}^*)} \cap [\eta_2\bar{\iota}^*]_{\beta_{\eta_2}(\bar{\iota}^*)}$.

Now, we are giving corollaries to our main result by taking $\varphi(\bar{\iota}^*) = e^{\sqrt{\bar{\iota}^*}}$, where $\bar{\iota}^* > \bar{0}$.

Corollary 1 *Consider* $\eta_1, \eta_2 : \bar{\Upsilon} \to G(\bar{\Upsilon})$*, so that for every* $\bar{\iota}, \bar{\kappa} \in \bar{\Upsilon}$*, there exists* $\beta_{\eta_1}(\bar{\iota}), \beta_{\eta_2}(\bar{\kappa}) \in (\bar{0}, \bar{1}]$ *and* $[\eta_1\bar{\iota}]_{\beta_{\eta_1}(\bar{\iota})}, [\eta_2\bar{\kappa}]_{\beta_{\eta_2}(\bar{\kappa})} \in \Sigma_{cb}(\bar{\Upsilon})$

$$\bar{k}M_b([\eta_1\bar{\iota}]_{\beta_{\eta_1}(\bar{\iota})}, [\eta_2\bar{\kappa}]_{\beta_{\eta_2}(\bar{\kappa})}) \leq kS_b(\bar{\iota}, \bar{\kappa}) \tag{17}$$

where $0 < k < 1$*, then there exists* $\bar{\iota}^* \in \bar{\Upsilon}$ *so that* $\bar{\iota}^* \in [\eta_1\bar{\iota}^*]_{\beta_{\eta_1}(\bar{\iota}^*)} \cap [\eta_2\bar{\iota}^*]_{\beta_{\eta_2}(\bar{\iota}^*)}$.

Corollary 2 *Consider* $\eta : \bar{\Upsilon} \to G(\bar{\Upsilon})$*, so that for all* $\bar{\iota}, \bar{\kappa} \in \bar{\Upsilon}$*, there exists* $\beta_{\eta_1}(\bar{\iota})$, $\beta_{\eta_2}(\bar{\kappa}) \in (\bar{0}, \bar{1}]$ $[\eta\bar{\iota}]_{\beta_\eta(\bar{\iota})}, [\eta\bar{\kappa}]_{\beta_\eta(\bar{\kappa})} \in \Sigma_{cb}(\bar{\Upsilon})$

$$\bar{k}M_b([\eta\bar{\iota}]_{\beta_\eta(\bar{\iota})}, [\eta\bar{\kappa}]_{\beta_\eta(\bar{\kappa})}) \leq kS_b(\bar{\iota}, \bar{\kappa}) \tag{18}$$

where $0 < k < 1$*, then there exists* $\bar{\iota}^* \in \bar{\Upsilon}$ *so that* $\bar{\iota}^* \in [\eta\bar{\iota}^*]_{\beta_\eta(\bar{\iota}^*)}$.

Now, we will give an example to validate our new result.

Qy

Example 2 Consider $\bar{\Upsilon} = \{0, 1, 2\}$ and $S_b : \bar{\Upsilon} \times \bar{\Upsilon} \to \mathbb{R}$ defined as

$$S_b(\bar{\iota}, \bar{\kappa}) = \begin{cases} 0, & for\, \bar{\iota} = \bar{\kappa}, \\ \frac{1}{6}, & for\, \bar{\iota} \neq \bar{\kappa} \cap \bar{\iota}, \bar{\kappa} \in \{0, 1\}, \\ \frac{1}{2}, & for\, \bar{\iota} \neq \bar{\kappa} \cap \bar{\iota}, \bar{\kappa} \in \{0, 2\}, \\ 1, & for\, \bar{\iota} \neq \bar{\kappa} \cap \bar{\iota}, \bar{\kappa} \in \{1, 2\}. \end{cases}$$

Thus, $(\bar{\Upsilon}, S_b, \bar{k})$ is a soft b-metric space where $\bar{k} = \dfrac{3}{2}$.

Now, define

$$(\eta 0)(\zeta) = (\eta 1)(\zeta) = \begin{cases} \dfrac{1}{2}, & for\, \zeta = 0, \\ 0, & for\, \zeta = \{1, 2\}; \end{cases}$$

$$(\eta 2)(\zeta) = \begin{cases} 0, & for\, \zeta = \{0, 2\}, \\ \dfrac{1}{2}, & for\, \zeta = 1. \end{cases}$$

Now, consider $\beta : \bar{\Upsilon} \to (0, 1]$ defined as $\beta(\bar{\iota}) = \dfrac{1}{2}$ for all $\bar{\iota} \in \bar{\Upsilon}$. Thus, we have

$$[\eta\bar{\iota}]_{(\frac{1}{2})} = \begin{cases} 0, & for\, \bar{\iota} = \{0, 1\}, \\ 1, & for\, \bar{\iota} = 2. \end{cases}$$

Now, for $\bar{\iota}, \bar{\kappa} \in \bar{\Upsilon}$, we have

$$M_b([\eta 1]_{(\frac{1}{2})}, [\eta 2]_{(\frac{1}{2})}) = M_b(0, 1) = \frac{1}{6}.$$

Now, considering $\varphi(\bar{\iota}^*) = e^{\sqrt{\bar{\iota}^*}}$, where $\bar{\iota}^* > \bar{0}$ and $\bar{k} = \frac{1}{2}$, we have

$$\varphi(\bar{k} M_b([\eta 0]_{(\frac{1}{2})}, [\eta 2]_{(\frac{1}{2})})) = e^{(\frac{1}{4})^{(\frac{1}{2})}} < e^{(\frac{1}{2})^{(\frac{1}{4})}} = [\eta(S_b(1, 2))]^k.$$

Thus, all the conditions of Theorem 1 holds and hence η has 0 as its soft α-fuzzy fixed point.

4 Application

Now, we will give an application to our new results.

Let the Volterra integral inclusion defined as

$$\bar{\Lambda}(r) \in \bar{\eta}(r) + \int_0^r \zeta(r, H, \bar{\Lambda}(H))dH, \tag{19}$$

where $0 < r < 1$ and $\zeta : [0, 1] \times [0, 1] \times R \to \Delta_{cb}(R)$, where $\Delta_{cb}(R)$ is the class of all non-empty subsets of $\bar{\Gamma}$, be a given set-valued mapping and $\bar{\eta}, \bar{\Lambda} \in C[0, 1]$, where $\bar{\eta}$ is given and $\bar{\Lambda}$ is unknown.

Consider soft b-metric S_b on $C[0, 1]$ defined as (20) for all $\bar{\iota}, \bar{\kappa} \in C[0, 1]$ and $k \geq 1$,

$$S_b(\bar{\iota}, \bar{\kappa}) = (max_{r\in[0,1]} \mid \bar{\iota}(r) - \bar{\kappa}(r) \mid)^k = max_{r\in[0,1]} \mid \bar{\iota}(r) - \bar{\kappa}(r) \mid^k . \tag{20}$$

Then, $(C[0, 1], S_b, 2^{k-1})$ is a complete soft b-metric space. Now, consider the following assertions:

(i) $\zeta(r, H, \bar{\Lambda}(H))$ is lower semi-continuous in $[0, 1] \times [0, 1]$ for each $\bar{\Lambda} \in C[0, 1]$.

(ii) There exists a continuous map $m : [0, 1] \to [0, +\infty]$ so that (21) holds for all $r, H \in [0, 1]$ and $\bar{\iota}, \bar{\kappa} \in C[0, 1]$,

$$\mid \zeta(r, H, \bar{\iota}) - \zeta(r, H, \bar{\kappa}) \mid^k \leq m(H) \mid \bar{\iota}(H) - \bar{\kappa}(H) \mid . \tag{21}$$

(iii) There exists $0 < r < 1$ such that,

$$\left(\int_0^r m(H)dH\right)^k \leq \frac{r}{2^{k-1}}.$$

Theorem 2 *The integral inclusion* (19) *possesses a solution in* $C[0, 1]$ *if it satisfies the assumptions* $(i)-(iii)$.

Proof Consider $\bar{\Upsilon} = C[0, 1]$, define $\eta : \bar{\Upsilon} \to G(\bar{\Upsilon})$ as (22) for all $0 \leq r \leq 1$,

$$[\eta\bar{\iota}]_{\beta_\eta(\bar{\iota})} = \{\bar{\kappa} \in \bar{\Upsilon} : \bar{\kappa}(r) \in \bar{\eta}(r) + \int_0^r \zeta(r, H, \bar{\iota}(H))dH\}. \tag{22}$$

Consider $\bar{\iota} \in \bar{\Upsilon}$ be arbitrary, thus there exists $\beta_\eta(\bar{\iota}) \in (0, 1]$.

Now, for $\zeta_{\bar{\iota}}(r, H) : [0, 1] \times [0, 1] \to \Delta_{cv}(R)$, there exists $I_{\bar{\iota}}(r, H) : [0, 1] \times [0, 1] \to R$, so that $I_{\bar{\iota}}(r, H) \in [0, 1]$. Thus, we have $\bar{\eta}(r) + \int_0^r I_{\bar{\iota}}(r, H)dH \in [\eta(\bar{\iota})]_{\beta_\eta(\bar{\iota})}$. Therefore, $[\eta(\bar{\iota})]_{\beta_\eta(\bar{\iota})} \neq \phi$. Now, it is easy to show that $[\eta(\bar{\iota})]_{\beta_\eta(\bar{\iota})}$ is closed.

Further, as $\bar{\eta}$ and $\zeta_{\bar{\iota}}(r, H)$ are continuous, they have bounded ranges, i.e., $[\eta(\bar{\iota})]_{\beta_\eta(\bar{\iota})}$ is bounded. Hence, $[\eta(\bar{\iota})]_{\beta_\eta(\bar{\iota})} \in CB(\bar{\Upsilon})$.

Consider $\bar{\iota}_{e_1}, \bar{\iota}_{e_2} \in \bar{\Upsilon}$, then there exists $\beta_\eta(\bar{\iota}_{e_1}), \beta_\eta(\bar{\iota}_{e_2}) \in CB(\bar{\Upsilon})$. Take $\bar{\kappa}_{e_1} \in [\eta(\bar{\iota}_{e_1})]_{\beta_\eta(\bar{\iota}_{e_1})}$, so that (23) holds for $0 \leq r \leq 1$,

$$\bar{\iota}_{e_1}(r) \in \bar{\eta}(r) + \int_0^r \zeta(r, H, \bar{\Lambda}(H))dH. \tag{23}$$

This implies that for all $r, H \in [0, 1]$, there exists $I_{\bar{\iota}_{e_1}}(r, H) \in \zeta_{\bar{\iota}_{e_1}}(r, H) = \zeta_{\bar{\iota}_{e_1}}(r, H, \bar{\iota}_{e_1}(H))$, so that (24) holds,

$$\bar{\kappa}_{e_1}(r) = \bar{\eta}(r) + \int_0^r I_{\bar{\iota}_{e_1}}(r, H)dH. \tag{24}$$

Thus, for all $\bar{\iota}_{e_1}, \bar{\iota}_{e_2} \in \bar{\Upsilon}$, we have

$$\mid \zeta(r, H, \bar{\iota}_{e_1}) - \zeta(r, H, \bar{\iota}_{e_2}) \mid^k \leq m(H) \mid \bar{\iota}_{e_1}(H) - \bar{\iota}_{e_2}(H) \mid .$$

That is there exists $U(r, H) \in \zeta_{\bar{\iota}_{e_2}}(r, H)$, so that (25) holds

$$\mid I_{\bar{\iota}_{e_1}}(r, H) - U(r, H) \mid^k \leq m(H) \mid \bar{\iota}_{e_1}(H) - \bar{\iota}_{e_2}(H) \mid . \tag{25}$$

Now, take Z as a set-valued operator defined as

$$Z(r, H) = \zeta_{\bar{\iota}_{e_2}}(r, H) \cap \{z \in R : \mid I_{\bar{\iota}_{e_1}}(r, H) - z \mid \leq m(H) \mid \bar{\iota}_{e_1}(H) - \bar{\iota}_{e_2}(H) \mid\}.$$

Therefore, Z is lower semi-continuous. Thus, there exists a continuous operator $I_{\bar{\iota}_{e_2}}(r, H) : [0, 1] \times [0, 1] \to R$ so that $I_{\bar{\iota}_{e_2}}(r, H) \in Z(r, H)$, then $\bar{\kappa}_{e_2}(r) = \bar{\eta}(r) + \int_0^r I_{\bar{\iota}_{e_1}}(r, H)dH$ that satisfies

$$\bar{\kappa}_{e_2}(r) \in \bar{\eta}(r) + \int_0^r \zeta(r, H, \bar{\iota}_{e_2}(H))dH.$$

Thus, $\bar{\kappa}_{e_2} \in [\eta\bar{\iota}_{e_2}]_{\beta_\eta(\bar{\iota}_{e_2})}$ and we have,

$$\begin{aligned}
\mid \bar{\kappa}_{e_1}(r) - \bar{\kappa}_{e_2}(r) \mid^k &\leq (\int_0^r \mid I_{\bar{\iota}_{e_1}}(r, H) - I_{\bar{\iota}_{e_2}}(r, H) \mid dH)^k \\
&\leq \left(\int_0^r m(H) \mid \bar{\iota}_{e_1}(r) - \bar{\iota}_{e_2}(r) \mid dH\right)^k \\
&\leq max_{H\in[0,1]} \mid \bar{\iota}(H) - \bar{\kappa}(H) \mid^k (\int_0^r m(H)dH) \\
&\leq \frac{r^2}{2^{k-1}} S_b\left(\bar{\iota}_{e_1}, \bar{\iota}_{e_2}\right).
\end{aligned}$$

Hence, we get $2^{k-1}S_b(\bar{\kappa}_{e_1}, \bar{\kappa}_{e_2}) \leq r^2 S_b(\bar{\iota}_{e_1}, \bar{\iota}_{e_2})$.

On interchanging $\bar{\iota}_{e_1}$ and $\bar{\iota}_{e_2}$, we get

$$\bar{k}H_b([\eta\bar{\iota}_{e_1}]_{\beta_\eta(\bar{\iota}_{e_1})}, [\eta\bar{\iota}_{e_2}]_{\beta_\eta(\bar{\iota}_{e_2})}) \leq r^2 S_b(\bar{\iota}_{e_1}, \bar{\iota}_{e_2}).$$

Hence, we have

$$\sqrt{\bar{k}H_b([\eta\bar{\iota}_{e_1}]_{\beta_\eta(\bar{\iota}_{e_1})}, [\eta\bar{\iota}_{e_2}]_{\beta_\eta(\bar{\iota}_{e_2})})} \leq r\sqrt{S_b(\bar{\iota}_{e_1}, \bar{\iota}_{e_2})}$$
$$e^{\sqrt{\bar{k}H_b([\eta\bar{\iota}_{e_1}]_{\beta_\eta(\bar{\iota}_{e_1})}, [\eta\bar{\iota}_{e_2}]_{\beta_\eta(\bar{\iota}_{e_2})})}} \leq e^{r\sqrt{S_b(\bar{\iota}_{e_1}, \bar{\iota}_{e_2})}}.$$

Taking $\nu \in \Lambda_{\bar{k}}$ defined as $\nu(\bar{r}) = e^{\sqrt{\bar{r}}}$, we get the solution of (19). □

5 Conclusion

This work of us has introduced new fixed point results in soft b-metric spaces along with applications to Volterra integral inclusions. Our new results can be further extended to other spaces as well as new results can be derived from them.

6 Declarations

Competing Interests

The authors declare that they do not have any competing interests **Conflict of Interest** The authors declare that there is no conflict of interest.
Ethical Approval This research complies with ethical standards.
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Revisiting Darbo's Fixed Point Theory with Application to a Class of Fractional Integral Equations

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1 Introduction and Preliminaries

Fractional calculus is a very emerging concept in applied mathematics for the modeling of various problems in the engineering sciences and has attracted the attention of a large number of authors from multiple fields. Authors developed various types of $\mathscr{FIE}$ using various types of kernels (see [1–8]). The concept of fractional calculus was adopted in the 16th century. And now, it has a broad area in the field of numerical analysis. Nowadays there are many generalized $\mathscr{FIE}$ developed containing variable orders [9]. Moreover, several authors have used fixed point theory to investigate the existence of solutions to these $\mathscr{FIE}$.

Fixed point theory is a very versatile tool in various fields of mathematical analysis. In fixed point theory, the $\mathscr{MNC}$ plays a very important role to develop a new generalization of fixed point theory, especially, $\mathscr{DFPT}$. The $\mathscr{MNC}$ was the concept, which was given by Kuratowski [10] in 1930. There are many authors, who used various types of fixed point theory to prove the existence of solutions to many types of integral or differential equations via $\mathscr{MNC}$.

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In traditional calculus, all of the real phenomena cannot be modeled using the operators. That's why Numerous researchers try to find generalizations of these operators [11, 12]. So, we are concerned with the further generalization of Riemann-Liouville and Hadamard $\mathscr{F}\mathscr{I}\mathscr{E}$. To get this, the authors of [13] have introduced the notion of the generalized proportional integral operator, which specifies the probability density functions. With the help of [13], we have generalized a $\mathscr{F}\mathscr{I}\mathscr{E}$ known as the generalized proportional (k, ρ)-$\mathscr{F}\mathscr{I}\mathscr{E}$ of $y(h) \in L^1[a, b]$ of order $\varsigma > 0$ and defined as

$$\left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}y\right](h) = \frac{(\rho+1)^{1-\frac{\varsigma}{k}}}{\theta^{\frac{\varsigma}{k}}k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\int_a^h e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]}{\theta}\right]}\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1}\ell^{\rho}(\gamma)\ell\prime(\gamma)y(\gamma)d\gamma, \quad (1)$$

where $\theta \in (0, 1],\ \rho \in \mathbb{R} - \{-1\},\ \Gamma(k) = \int\limits_0^\infty e^{-h}h^{k-1}dt,\ h,\ \gamma \in I = [a, b],\ a,\ k > 0$ and $\ell : I \to \mathbb{R}_+$ is continuously differentiable and nondecreasing.

There are several possible cases, which are obtained by taking the particular values of ℓ, k, ρ and θ.

If $\theta = 1$ in Eq. 1, we get the following $\mathscr{F}\mathscr{I}\mathscr{E}$, which is studied by Bezziou et al. [14] for the various type of Gruss inequalities,

$$\left[{}^k_\rho\mathcal{I}^{\varsigma}_{a,\ell}y\right](h) = \frac{(\rho+1)^{1-\frac{\varsigma}{k}}}{k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\int_a^h \left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1}\ell^{\rho}(\gamma)\ell\prime(\gamma)y(\gamma)d\gamma.$$

If $\ell(h) = h$ in Eq. 1, then we get the following $\mathscr{F}\mathscr{I}\mathscr{E}$ and studied by Rahul et al. [6] for the existence of solution,

$$\left[{}^k_\rho\mathcal{I}^{\varsigma}_{a}y\right](h) = \frac{(\rho+1)^{1-\frac{\varsigma}{k}}}{\theta^{\frac{\varsigma}{k}}k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\int_a^h e^{\left[\frac{(\theta-1)\left[h^{\rho+1}-\gamma^{\rho+1}\right]}{\theta}\right]}\left[h^{\rho+1}-\gamma^{\rho+1}\right]^{\frac{\varsigma}{k}-1}\gamma^{\rho}y(\gamma)d\gamma.$$

If $\ell(h) = h$ and $\theta = 1$ in Eq. 1, then, we get the following $\mathscr{F}\mathscr{I}\mathscr{E}$, which is introduced by Mehmet et al. [4] and studied for the various type of inequalities,

$$\left[{}^k_\rho\mathcal{I}^{\varsigma}_{a}y\right](h) = \frac{(\rho+1)^{1-\frac{\varsigma}{k}}}{k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\int_a^h \left[h^{\rho+1}-\gamma^{\rho+1}\right]^{\frac{\varsigma}{k}-1}\gamma^{\rho}y(\gamma)d\gamma.$$

If $\ell(h) = h,\ \theta = 1$ and $\rho = 0$ in Eq. 1, then, we get the following $\mathscr{F}\mathscr{I}\mathscr{E}$ defined by Mubeen and Habibulah [15] and discuss some other k-type $\mathscr{F}\mathscr{I}\mathscr{E}$,

$$\left[{}^k\mathcal{I}^{\varsigma}_{a}y\right](h) = \frac{1}{k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\int_a^h [h-\gamma]^{\frac{\varsigma}{k}-1}y(\gamma)d\gamma.$$

If $\ell(h) = h,\ \theta = 1,\ \rho = 0$ and $k = 1$ in Eq. 1, then, we get the classical Riemann-Liouville $\mathscr{F}\mathscr{I}\mathscr{E}$,

$$\left[\mathcal{I}_a^{\varsigma} y\right](h) = \frac{1}{\Gamma\left[\varsigma\right]} \int_a^h \left[h-\gamma\right]^{\varsigma-1} y(\gamma) d\gamma.$$

If $\ell(h) = \log(h)$, where $h \in I = [a, b],\ a > 0$ and $\rho = 0$ in Eq. 1, then, we get the following $\mathscr{FIE}$ and studied for the existence of solution,

$$\left[{}^k\mathcal{I}_a^{\varsigma,\theta} y\right](h) = \frac{1}{\theta^{\frac{\varsigma}{k}} k^{\frac{\varsigma}{k}} \Gamma\left[\frac{\varsigma}{k}\right]} \int_a^h e^{\frac{\theta-1}{\theta}\left[\log\frac{h}{\gamma}\right]} \left[\log\frac{h}{\gamma}\right]^{\frac{\varsigma}{k}-1} \frac{y(\gamma)}{\gamma} d\gamma,$$

the terms $\varsigma, a,\ k > 0$, and $\theta \in (0, 1]$.

If $\ell(h) = \log(h),\ k = 1$ and $\rho = 0$ in Eq. 1, then, we get the following $\mathscr{FIE}$, which is introduced by Barakat et al. [16] for some related inequalities,

$$\left[\mathcal{I}_a^{\varsigma,\theta} y\right](h) = \frac{1}{\theta^{\varsigma} \Gamma\left[\varsigma\right]} \int_a^h e^{\frac{\theta-1}{\theta}\left[\log\frac{h}{\gamma}\right]} \left[\log\frac{h}{\gamma}\right]^{\varsigma-1} \frac{y(\gamma)}{\gamma} d\gamma.$$

If $\ell(h) = \log(h),\ k = 1,\ \theta = 1$ and $\rho = 0$ in Eq. 1, then, we get the following $\mathscr{FIE}$, which is defined by Hadamard [17],

$$\left[\mathcal{I}_a^{\varsigma} y\right](h) = \frac{1}{\Gamma(\varsigma)} \int_a^h \left[\log\frac{h}{\gamma}\right]^{\varsigma-1} \frac{y(\gamma)}{\gamma} d\gamma.$$

Hence, we can see that Eq. 1 is a more general form, while others are particular cases. These equations are very useful in real-life problems of engineering sciences, modeling, statistics, etc. ([25–33]). Hence, all the above equations are very useful for various purposes, like their existence solutions, modeling, some Gruss inequalities, and various phenomena that have been developed in another discipline.

On the other hand, it is observed that some fractional operators found singularities which make some obstacles in modeling [34, 35]. Numerous authors proposed a new type of nonsingular fractional operator that contains either an exponential kernel or Mittag Leffler functions [36, 37] and is helpful in modeling. Our goal in this paper is to generalize a fractional operator that contains a nonsingular kernel (exponential kernel) which will be helpful in modeling. So first, we generalized a $\mathscr{DFPT}$ and then study the existence solution of generalized proportional (k, ρ)-$\mathscr{FIE}$, using $\mathscr{DFPT}$.

The structure of this article is as follows: In Sect. 2, we present some preliminary remarks and definitions. In Sect. 3, we check the existence and some properties of generalized proportional (k, ρ)-$\mathscr{FIE}$. In Sect. 4, we generalized a $\mathscr{DFPT}$. In Sect. 5, we prove the existence solution of generalized proportional (k, ρ)-$\mathscr{FIE}$, using $\mathscr{DFPT}$, and give an example that verified the obtained results. Section 6 is devoted to the conclusion.

2 Definitions and Preliminaries

The following notations are used throughout this paper.

- $\mathbb{E}$: so called, Banach space having a norm $\| \, . \, \|_{\mathbb{E}}$;
- $\bar{\mathscr{B}}$: closure of $\mathscr{B}$;
- $Conv\mathscr{B}$: The convex closure of $\mathscr{B}$;
- $\mathscr{M}_{\mathbb{E}}$: a subset having all $\mathbb{E}$ that are bounded and nonempty;
- $\mathscr{N}_{\mathbb{E}}$: a subset of $\mathscr{M}_{\mathbb{E}}$, that also includes all relatively compact sets;
- $\mathbb{R}_+ := [0, \infty)$;
- $C(I)$: the set of all continuously differentiable functions.

Banas and Lecko [18] provided the definition of a $\mathscr{MNC}$.

Definition 1 An $\mathscr{MNC}$ defined on $\mathbb{E}$ is a mapping $\mathscr{A} : \mathscr{M}_{\mathbb{E}} \to \mathbb{R}_+$, if it meets the following assertions, for all $\mathscr{B}, \ \mathscr{B}_1, \ \mathscr{B}_2 \in \mathscr{M}_{\mathbb{E}}$.

(N_1) The family $\ker \mathscr{A} = \{\mathscr{B} \in \mathscr{M}_{\mathbb{E}} : \mathscr{A}[\mathscr{B}] = 0\} \neq \emptyset$ and $\ker \mathscr{A} \subset \mathscr{N}_{\mathbb{E}}$.
(N_2) $\mathscr{B}_1 \subset \mathscr{B}_2 \implies \mathscr{A}[\mathscr{B}_1] \leq \mathscr{A}[\mathscr{B}_2]$.
(N_3) $\mathscr{A}\left[\bar{\mathscr{B}}\right] = \mathscr{A}[\mathscr{B}]$.
(N_4) $\mathscr{A}[Conv\mathscr{B}] = \mathscr{A}[\mathscr{B}]$.
(N_5) $\mathscr{A}[k\mathscr{B}_1 + [1-k]\mathscr{B}_2] \leq k\mathscr{A}[\mathscr{B}_1] + [1-k]\mathscr{A}[\mathscr{B}_2]$ for all k in $[0, 1]$.
(N_6) If $\mathscr{B}_n \in \mathscr{M}_{\mathbb{E}}, \ \mathscr{B}_n = \bar{\mathscr{B}}_n, \ \mathscr{B}_{n+1} \subset \mathscr{B}_n$ and $\lim_{n\to\infty} \mathscr{A}[\mathscr{B}_n] = 0$; where $n = 1, 2, ...$, thus $\mathscr{B}_\infty = \cap_{n=1}^{\infty} \mathscr{B}_n \neq \emptyset$ and precompact.

Remark 1 Subsequently, $\mathscr{A}(\mathscr{B}_\infty) = \mathscr{A}\left[\cap_{n=1}^{\infty}\mathscr{B}_n\right] \subseteq \mathscr{A}(\mathscr{B}_n), \ \mathscr{A}(\mathscr{B}_\infty) = 0$, then $\mathscr{B}_\infty \in ker\mathscr{A}$.

Theorem 1 (SFPT) [19] *A mapping $G : \mathscr{B} \to \mathscr{B}$ which is compact and continuous has a fixed point property(FP), where $\mathscr{B}$ is a convex, closed, bounded and nonempty ($\mathscr{NBCC}$) subset of a Banach space $\mathbb{E}$.*

Theorem 2 ($\mathscr{DFPT}$) [20] *Assume that $G : \mathscr{B} \to \mathscr{B}$ is continuous and $\mathscr{A}$ is an $\mathscr{MNC}$. If D is a nonempty subset of $\mathscr{B}$, there is a k in* $[0, 1]$ *satisfy*

$$\mathscr{A}[GD] \leq k\,\mathscr{A}(D),$$

thus the mapping G has a FP in $\mathscr{B}$.

Definition 2 ([21]) The continuous functions $\mathscr{F} : \mathbb{R}_+^2 \to \mathbb{R}$, that are reputed as $\mathscr{C}$-class functions if they gratify,

$(\mathscr{F}_1)$ $\mathscr{F}(\mu, \varsigma) \leq \mu$.
$(\mathscr{F}_2)$ $\mathscr{F}(\mu, \varsigma) = \mu \implies \mu = 0$ *or* $\varsigma = 0$ for all $\mu, \varsigma \in \mathbb{R}$.

Example: (i) $\mathscr{F}(\mu, \varsigma) = \mu - \varsigma$.
(ii) $\mathscr{F}(\mu, \varsigma) = k\mu, \quad 0 \leq k < 1$.

Let Δ be the set of all continuous mapping $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ gratify below assertions [23]:

(Υ_1) $\Upsilon(h) = 0 \Leftrightarrow h = 0$.
(Υ_2) Υ is nondecreasing function.
(Υ_3) $\Upsilon(h) < h$, for $h > 0$.

3 Existence and Some Properties of Generalized Proportional (k, ρ)-$\mathscr{F}\mathscr{I}\mathscr{E}$

First, we have checked the existence of generalized proportional (k, ρ)-$\mathscr{F}\mathscr{I}\mathscr{E}$ (1).

Theorem 3 *If $y(h)$ is any Lebesgue measurable function defined on $[a, b]$ and $k > 0$. Then $\left[{}^{k}_{\rho}\mathcal{I}^{\varsigma,\theta}_{a,\ell}\right] y(h)$ exists for any $h \in [a, b],\ a,\ \alpha > 0,\ \varsigma \in (0, 1]$ and ℓ is monotonically increasing function on $[a, b]$.*

Proof Suppose $W = [a, b] \times [a, b]$ and $Y : W \to \mathbb{R}$ is defined by

$$Y(h, \gamma) = \begin{cases} \left(e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]}{\theta}\right]} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1} \ell^{\rho}(\gamma)\ell\prime(\gamma)\right), & a \leq \gamma \leq h \leq b \\ 0, & \text{otherwise.} \end{cases}$$

Since Y is measurable on W, then we have

$$\begin{aligned} \int_a^b Y(h, \gamma)d\gamma &= \int_a^h e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]}{\theta}\right]} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1} \ell^{\rho}(\gamma)\ell\prime(\gamma)d\gamma \\ &\leq \frac{k}{\varsigma(\rho+1)} e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(h)\right]}{\theta}\right]} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(a)\right]^{\frac{\varsigma}{k}}. \end{aligned}$$

Again by using the integration, we have

$$\begin{aligned} &\left| \int_a^b \left[\int_a^b Y(h, \gamma)y(\gamma)d\gamma \right] d\gamma \right| \\ &= \left| \int_a^b y(\gamma) \left[\int_a^b Y(h, \gamma)d\gamma \right] d\gamma \right| \end{aligned}$$

$$\leq \frac{k}{\varsigma(\rho+1)} \int_a^b e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(h)\right]}{\theta}\right]} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(a)\right]^{\frac{\varsigma}{k}} |y(\gamma)| d\gamma$$

$$\leq \frac{k}{\varsigma(\rho+1)} e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(b)\right]}{\theta}\right]} \left[\ell^{\rho+1}(b) - \ell^{\rho+1}(a)\right]^{\frac{\varsigma}{k}} \int_a^b |y(\gamma)| d\gamma$$

$$< \infty.$$

Hence, $\int_a^b Y(h,\gamma) y(\gamma) d\gamma$ is integrable function on $[a,b]$. So the generalized proportional (k,ρ)-$\mathscr{F}\mathscr{I}\mathscr{E}$ $\left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}\right] y(h)$exists. □

In the next Theorem 4, we will prove that generalized proportional (k,ρ)-$\mathscr{F}\mathscr{I}\mathscr{E}$ satisfies the commutative property.

Theorem 4 *Assume that y be a real-valued continuous function on $[a,b]$, a, k are non-negative and the terms $\varsigma, u > 0$. Then*

$$\left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}\right]\left[\left[{}^k_\rho\mathcal{I}^{u,\theta}_{a,\ell}\, y(h)\right]\right] = \left[{}^k_\rho\mathcal{I}^{\varsigma+u,\theta}_{a,\ell}\, y(h)\right] = \left[{}^k_\rho\mathcal{I}^{u,\theta}_{a,\ell}\right]\left[\left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}\, y(h)\right]\right].$$

Proof Using Dirichlet's formula and the concept of generalized proportional (k,ρ)-$\mathscr{F}\mathscr{I}\mathscr{E}$ (1), we have

$$\left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}\right]\left[\left[{}^k_\rho\mathcal{I}^{u,\theta}_{a,\ell}\, y(h)\right]\right]$$

$$= \frac{(\rho+1)^{1-\frac{\varsigma}{k}}}{\theta^{\frac{\varsigma}{k}} k^{\frac{\varsigma}{k}} \Gamma\left[\frac{\varsigma}{k}\right]} \int_a^h e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]}{\theta}\right]} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1} \ell^{\rho}(\gamma)\ell\prime(\gamma) {}^k_\rho J^{u,\theta}_{a,\ell}\, y(\gamma) d\gamma$$

$$= \frac{(\rho+1)^{1-\frac{\varsigma}{k}}}{\theta^{\frac{\varsigma}{k}} k^{\frac{\varsigma}{k}} \Gamma\left[\frac{\varsigma}{k}\right]} \int_a^h e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]}{\theta}\right]} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1} \ell^{\rho}(\gamma)\ell\prime(\gamma)$$

$$\left[\frac{(\rho+1)^{1-\frac{u}{k}}}{\theta^{\frac{u}{k}} k^{\frac{u}{k}} \Gamma\left[\frac{u}{k}\right]} \int_a^\gamma e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(\gamma)-\ell^{\rho+1}(r)\right]}{\theta}\right]} \left[\ell^{\rho+1}(\gamma) - \ell^{\rho+1}(r)\right]^{\frac{u}{k}-1} \ell^{\rho}(r)\ell\prime(r) dr\right] d\gamma$$

$$= \frac{(\rho+1)^{2-\left[\frac{\varsigma+u}{k}\right]}}{\theta^{\left[\frac{\varsigma+u}{k}\right]} k^{\left[\frac{\varsigma+u}{k}\right]} \Gamma\left[\frac{\varsigma}{k}\right]\Gamma\left[\frac{u}{k}\right]} \int_a^h e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(r)\right]}{\theta}\right]} \ell^{\rho}(r)\ell\prime(r)$$

$$\left[\int_r^h \left[\ell^{\rho+1}(\gamma) - \ell^{\rho+1}(r)\right]^{\frac{\varsigma}{k}-1} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(\gamma)\right]^{\frac{u}{k}-1} \ell^{\rho}(\gamma)\ell\prime(\gamma) d\gamma\right] dr.$$

If we choose, $y = \frac{\left[\ell^{\rho+1}(\gamma)-\ell^{\rho+1}(r)\right]}{\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(r)\right]}$, then we have

$$\int_r^h \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1} \left[\ell^{\rho+1}(\gamma) - \ell^{\rho+1}(r)\right]^{\frac{u}{k}-1} \ell^{\rho}(\gamma)\ell\prime(\gamma)d\gamma$$
$$= \frac{1}{(\rho+1)} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(r)\right]^{\frac{\varsigma+u}{k}-1} \int_0^1 (1-y)^{\frac{\varsigma}{k}-1} y^{\frac{u}{k}-1} dy$$
$$= \frac{1}{(\rho+1)} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(r)\right]^{\frac{\varsigma+u}{k}-1} \; kB_k(\varsigma, u).$$

Hence, we have

$$\left[{}^{k}_{\rho}\mathcal{I}^{\varsigma,\theta}_{a,\ell}\right]\left[\left[{}^{k}_{\rho}\mathcal{I}^{u,\theta}_{a,\ell}\, y(h)\right]\right]$$
$$= \frac{(\rho+1)^{1-\left[\frac{\varsigma+u}{k}\right]}}{\theta^{\left[\frac{\varsigma+u}{k}\right]}{}_k\left[\frac{\varsigma+u}{k}\right]\Gamma\left[\frac{\varsigma+u}{k}\right]} \int_a^h e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(r)\right]}{\theta}\right]} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(r)\right]^{\frac{\varsigma+u}{k}-1} \ell^{\rho}(r)\ell\prime(r)y(r)dr$$
$$= {}^{k}_{\rho}\mathcal{I}^{\varsigma+u,\theta}_{a,\ell}\, y(h).$$

Hence completes the proof. □

4 Generalization of $\mathscr{DFPT}$

We will prove a generalized fixed point result in this section, which serves as an extended version of [22, 23].

Theorem 5 *Let $\mathscr{B}$ be a $\mathscr{NBCC}$ subset of a Banach space $\mathbb{E}$ and $G : \mathscr{B} \to \mathscr{B}$ is a continuous mapping that satisfies,*

$$\Upsilon\left[\int_0^{\mathscr{A}(G\mathscr{B})} \wp(\gamma)d\gamma\right] \leq \mathscr{F}\left[\Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B})} \wp(\gamma)d\gamma\right], \Upsilon\left[\int_0^{\mathscr{A}(G\mathscr{B})} \wp(\gamma)d\gamma\right]\right], \quad (2)$$

where $\mathscr{A}$ is a $\mathscr{MNC}$ on $\mathbb{E}$, $\wp : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous functions, $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing and continuous function and $\mathscr{F} : \mathbb{R}_+^2 \to \mathbb{R}$ is a continuous function with $\mathscr{F}(\mu, \varsigma) = \mu \implies \mu = 0$ or $\varsigma = 0$ for all $\mu, \varsigma \in \mathbb{R}$. Then G has at least one FP.

Proof Let $\mathscr{B}_n$ is a closed and bounded sequence in $\mathscr{B}$ with $\mathscr{B}_0 = \mathscr{B}$ and $\mathscr{B}_{n+1} = conv(G\mathscr{B}_n)$ for all $n \geq 0$.

Also, $G\mathscr{B}_0 = G\mathscr{B} \subseteq \mathscr{B} = \mathscr{B}_0,\ \mathscr{B}_1 = conv(G\mathscr{B}_0) \subseteq \mathscr{B} = \mathscr{B}_0$. In similar manner, we have

$$\mathscr{B}_0 \supset \mathscr{B}_1 \supset, ..., \supset \mathscr{B}_n \supset, ... \quad (3)$$

Equation (3) shows that $\mathscr{B}_n$ is a nested sequence. The fact that $\{\mathscr{A}(\mathscr{B}_n)\}_{n\in\mathbb{N}}$ is a non-negative decreasing and bounded below sequence which implies that $\{\mathscr{A}(\mathscr{B}_n)\}$ is a convergent sequence. Let $\lim_{n\to\infty} \mathscr{A}(\mathscr{B}_n) = r$. We claim that $\left\{ \int_0^{\mathscr{A}(\mathscr{B}_{n+1})} \wp(\gamma)d\gamma \right\}$ is a decreasing sequence.

By Eq. (2), we have

$$\begin{aligned}
\Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_{n+1})} \wp(\gamma)d\gamma\right] &= \Upsilon\left[\int_0^{\mathscr{A}(conv(G\mathscr{B}_n))} \wp(\gamma)d\gamma\right] = \Upsilon\left[\int_0^{\mathscr{A}(G\mathscr{B}_n)} \wp(\gamma)d\gamma\right] \\
&\leq \mathscr{F}\left[\Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right], \Upsilon\left[\int_0^{\mathscr{A}(G\mathscr{B}_n)} \wp(\gamma)d\gamma\right]\right] \\
&\leq \Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right],
\end{aligned}$$

since Υ is nondecreasing function, we get

$$\left[\int_0^{\mathscr{A}(\mathscr{B}_{n+1})} \wp(\gamma)d\gamma\right] \leq \left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right].$$

Hence, $\left\{ \int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma \right\}$ is a decreasing and bounded below sequence, then $\lim_{n\to\infty}\left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right] = l.$

Again, we have

$$\begin{aligned}
\Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_{n+1})} \wp(\gamma)d\gamma\right] &\leq \mathscr{F}\left[\Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right], \Upsilon\left[\int_0^{\mathscr{A}(G\mathscr{B}_n)} \wp(\gamma)d\gamma\right]\right] \\
&\leq \Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right].
\end{aligned}$$

Taking the lim of whole inequality, we have

$$
\begin{aligned}
&\lim_{n\to\infty} \Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_{n+1})} \wp(\gamma)d\gamma\right] \\
&\leq \mathscr{F}\left[\lim_{n\to\infty}\left[\Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right]\right], \lim_{n\to\infty}\left[\Upsilon\left[\int_0^{\mathscr{A}(G\mathscr{B}_n)} \wp(\gamma)d\gamma\right]\right]\right] \\
&\leq \lim_{n\to\infty} \Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right]
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \mathscr{F}&\left[\lim_{n\to\infty}\left[\Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right]\right], \lim_{n\to\infty}\left[\Upsilon\left[\int_0^{\mathscr{A}(G\mathscr{B}_n)} \wp(\gamma)d\gamma\right]\right]\right] \\
&= \lim_{n\to\infty}\left[\Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right]\right].
\end{aligned}
$$

Therefore by the $(\mathscr{F}_2)$ property, we have

$$
\lim_{n\to\infty}\left[\Upsilon\left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right]\right] = 0.
$$

Hence, $\lim_{n\to\infty}\left[\int_0^{\mathscr{A}(\mathscr{B}_n)} \wp(\gamma)d\gamma\right] = 0$, but for every $\epsilon > 0$, $\int_0^{\epsilon} \wp(\gamma)d\gamma > 0$, then $\mathscr{A}(\mathscr{B}_n) \to 0$ *as* $n \to \infty$. Since $\mathscr{B}_n$ is nested sequence, so by the (N_6) property of $\mathscr{M}\mathscr{N}\mathscr{C}$, we conclude that $\mathscr{B}_\infty = \cap_{n=1}^{\infty}\mathscr{B}_n$ is $\mathscr{N}\mathscr{B}\mathscr{C}\mathscr{C}$ subset and belongs to $ker\mathscr{A}$. In light of this, $\mathscr{B}_\infty$ is invariant and compact under G. As a result, according to the $SFPT$, G has a FP. □

Corollary 1 *By replacing* $\wp(\gamma) = 1$, $F(\mu, \varsigma) = \mu - \varsigma$ *and* $\Upsilon(\gamma) = \gamma$ *in Theorem 5, then we get the* $\mathscr{D}\mathscr{F}\mathscr{P}\mathscr{T}$.

5 Existence Solution of Generalized Proportional (k, ρ)-$\mathscr{F}\mathscr{I}\mathscr{E}$ (4)

In this section, the $\mathscr{M}\mathscr{N}\mathscr{C}$ is the space of all continuous functions $C(I)$, we will show the existence solution of the following $\mathscr{F}\mathscr{I}\mathscr{E}$ for the continuous function $y(h)$,

$$
y(h) = V\left[h, y(h), \left[{}^{k}_{\rho}\mathcal{I}^{\varsigma,\theta}_{a,\ell} Gy\right](h)\right], \tag{4}
$$

where $\theta \in (0, 1],\ \rho \in \mathbb{R}^+,\ a,\ \varsigma,\ k > 0,\ h \in I,\ V : I \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $\ell : I \to \mathbb{R}_+$ is continuously differentiable and nondecreasing function. Let $B_{r_0} = \{y \in C(I) : \ \|y\| \le r_0\}$.

The $\mathscr{MNC}$ in the space of all continuous functions $C(I)$ and its norm is defined as

$$\| \ y \ \| = \sup \{|y(h)| : h \in I = [a, b]\}, \ \ y \in C(I).$$

Let $\mathcal{X}$ be a nonempty and bounded subset of a Banach space $C(I)$ and for any function $y \in \mathcal{X}$ and $\epsilon > 0$, the modulus of the continuity of function y is defined as

$$\mu(y, \epsilon) = \sup \{|y(h_1) - y(h_2)| : h_1, h_2 \in I, |h_1 - h_2| \le \epsilon\}.$$

Then the function is defined by

$$\mu_0(\mathcal{X}) = \lim_{\epsilon \to 0} \mu(\mathcal{X}, \epsilon)$$

is a $\mathscr{MNC}$ in $C(I)$, where

$$\mu(\mathcal{X}, \epsilon) = \sup \{\mu(y, \epsilon) : y \in \mathcal{X}\}.$$

The relation between the $\mathscr{MNC}$, $\mu_0(\mathcal{X})$ and $\mathscr{A}(\mathcal{X})$ on $C(I)$ for the nonempty bounded set $\mathcal{X}$ is as $\mathscr{A}(\mathcal{X}) = \frac{1}{2}\mu_0(\mathcal{X})$ (see [24]).

Theorem 6 *If generalized proportional* (k, ρ)-$\mathscr{FIE}$ *(4) satisfies the following assumptions:*

(I) *Let* $V : I \times \mathbb{R}^2 \to \mathbb{R}$ *be a continuous functions with* $|V(h, 0, 0)| = 0,\ 0 \le \alpha_1 < 1$, *and* $\alpha_2 > 0$ *satisfying*

$$|V[h, y_1(h), I_1] - V[h, y_2(h), I_2]| \le \alpha_1 |y_1(h) - y_2(h)| + \alpha_2 |I_1 - I_2|,\ h,\ I_1,\ I_2 \in \mathbb{R}.$$

(II) $\sup\{|V(h, y(h), I_1)| : h \in I,\ \ y(h) \in [r_0, r_0],\ \ I_1 \in [-J, J]\} \le r_0$, *where* $J = \sup\{| \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell} Gy\right](h)| : h \in I, y(h) \in [-r_0, r_0]\}$.

(III) $\alpha_1 + \dfrac{\alpha_2 (\rho+1)^{-\frac{\varsigma}{k}} e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a) - \ell^{\rho+1}(b)\right]}{\theta}\right]}}{\varsigma \theta^{\frac{\varsigma}{k}} k^{\left[\frac{\varsigma}{k} - 1\right]} \Gamma\left[\frac{\varsigma}{k}\right]} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(a)\right]^{\frac{\varsigma}{k}} \le 1.$

Then, Eq. (4) has a solution in $C(I)$, *where* $a > 0$.

Proof Let us consider the operator $G : C(I) \to C(I)$ is defined as

$$(Gy)(h) = V\left[h, y(h), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell} Gy\right](h)\right],$$

We will complete the proof by **step I–step III**.

Step I: G **is a self-mapping on** B_{r_0}.

Let $y \in B_{r_0}$, then by using the Eq. (4) and considered assumptions, we have

$$\begin{aligned}|(Gy)(h)| &\le \left|V\left[h, y(h), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h)\right] - V(h,0,0)\right| + |V(h,0,0)| \\ &\le \alpha_1|y(h)| + \alpha_2\left|\left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h)\right|,\end{aligned}$$

and

$$\begin{aligned}&\left|\left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h)\right| \\ &= \left|\frac{(\rho+1)^{1-\frac{\varsigma}{k}}}{\theta^{\frac{\varsigma}{k}}k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\int_a^h e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]}{\theta}\right]}\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1}\ell^{\rho}(\gamma)\ell\prime(\gamma)y(\gamma)d\gamma\right| \\ &\le \frac{(\rho+1)^{1-\frac{\varsigma}{k}}}{\theta^{\frac{\varsigma}{k}}k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\int_a^h e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]}{\theta}\right]}\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1}\ell^{\rho}(\gamma)\ell\prime(\gamma)|y(\gamma)|d\gamma \\ &\le \frac{r_0(\rho+1)^{1-\frac{\varsigma}{k}}e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(b)\right]}{\theta}\right]}}{\theta^{\frac{\varsigma}{k}}k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\int_a^h\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1}\ell^{\rho}(\gamma)\ell\prime(\gamma)d\gamma \\ &= \frac{r_0(\rho+1)^{-\frac{\varsigma}{k}}e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(b)\right]}{\theta}\right]}}{\varsigma\theta^{\frac{\varsigma}{k}}k^{\frac{\varsigma}{k}-1}\Gamma\left[\frac{\varsigma}{k}\right]}\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(a)\right]^{\frac{\varsigma}{k}}.\end{aligned}$$

If $\|y\| \le r_0$, then $\|Gy\| \le r_0$, by the assumption (III). Hence, G is self-mapping on B_{r_0}.

Step II: G **is continuous mapping on** B_{r_0}.

Let $\epsilon > 0$ and $y_1,\ y_2 \in B_{r_0}$ with $\|y_1 - y_2\| \le \epsilon$, then we have

$$\begin{aligned}|(Gy_1)(h) - (Gy_2)(h)| &= \left|V\left[h, y_1(h), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy_1\right](h)\right] - V\left[h, y_2(h), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy_2\right](h)\right]\right| \\ &\le \alpha_1|y_1(h) - y_2(h)| + \alpha_2\left|\left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy_1\right](h) - \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy_2\right](h)\right|,\end{aligned}$$

$$\begin{aligned}&\left|\left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy_1\right](h) - \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy_2\right](h)\right| \\ &\le \frac{(\rho+1)^{1-\frac{\varsigma}{k}}}{\theta^{\frac{\varsigma}{k}}k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\int_a^h e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]}{\theta}\right]}\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1}\ell^{\rho}(\gamma)\ell\prime(\gamma)|y_1(\gamma)-y_2(\gamma)|d\gamma \\ &\le \frac{(\rho+1)^{1-\frac{\varsigma}{k}}e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(b)\right]}{\theta}\right]}}{\theta^{\frac{\varsigma}{k}}k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\int_a^h\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1}\ell^{\rho}(\gamma)\ell\prime(\gamma)\|y_1-y_2\|d\gamma \\ &\le \frac{\epsilon(\rho+1)^{1-\frac{\varsigma}{k}}e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(b)\right]}{\theta}\right]}}{\theta^{\frac{\varsigma}{k}}k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\int_a^h\left[\ell^{\rho+1}(h)-\ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1}\ell^{\rho}(\gamma)\ell\prime(\gamma)d\gamma\end{aligned}$$

$$= \frac{\epsilon(\rho+1)^{-\frac{\varsigma}{k}} e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(b)\right]}{\theta}\right]}}{\varsigma\theta^{\frac{\varsigma}{k}} k^{\frac{\varsigma}{k}-1}\Gamma\left[\frac{\varsigma}{k}\right]} \left[\ell^{\rho+1}(h)-\ell^{\rho+1}(a)\right]^{\frac{\varsigma}{k}}.$$

Therefore,

$$|(Gy_1)(h)-(Gy_2)(h)| \leq \epsilon\alpha_1 + \frac{\epsilon\alpha_2(\rho+1)^{-\frac{\varsigma}{k}} e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(b)\right]}{\theta}\right]}}{\varsigma\theta^{\frac{\varsigma}{k}} k^{\frac{\varsigma}{k}-1}\Gamma\left[\frac{\varsigma}{k}\right]} \left[\ell^{\rho+1}(h)-\ell^{\rho+1}(a)\right]^{\frac{\varsigma}{k}}.$$

Hence, $|(Gy_1)(h)-(Gy_2)(h)| \to 0$ as $\epsilon \to 0$. Hence, G is continuous on B_{r_0}.

Step III: **An estimate of G with respect to $\mathscr{MNCA}$.**
Let $h_1,\ h_2 \in I$, then for any $\epsilon > 0$, we have $|h_2-h_1| \leq \epsilon$, whenever $h_2 \geq h_1$. Then we have

$$\begin{aligned}
&|(Gy)(h_2)-(Gy)(h_1)| \\
&= \left|V\left[h_2, y(h_2), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_2)\right] - V\left[h_1, y(h_1), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_1)\right]\right| \\
&\leq \left|V\left[h_2, y(h_2), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_2)\right] - V\left[h_2, y(h_2), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_1)\right]\right| \\
&+ \left|V\left[h_2, y(h_2), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_1)\right] - V\left[h_2, y(h_1), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_1)\right]\right| \\
&+ \left|V\left[h_2, y(h_1), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_1)\right] - V\left[h_1, y(h_1), \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_1)\right]\right| \\
&\leq \alpha_2\left|\left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_2) -, \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_1)\right| + \alpha_1|y(h_2)-y(h_1)| + \mu_V(I,\epsilon),
\end{aligned}$$

$$\begin{aligned}
&where \\
&\mu_V(I,\epsilon) \\
&= \sup\left\{|V[h_2, y(h_1), I_1] - V[h_1, y(h_1), I_1]| : h_1, h_2 \in I, I_1 \in [-J, J], y(h_1) \in [-r_0, r_0]\right\}, \\
&and \\
&\mu(y,\epsilon) = \sup\{|y(h_2)-y(h_1)| : h_1, h_2 \in I, y(h_2), y(h_1) \in [-r_0, r_0]\}.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left|\left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_2) - \left[{}^k_\rho\mathcal{I}^{\varsigma,\theta}_{a,\ell}Gy\right](h_1)\right| \\
&= \frac{r_0(\rho+1)^{1-\frac{\varsigma}{k}}}{\theta^{\frac{\varsigma}{k}} k^{\frac{\varsigma}{k}}\Gamma\left[\frac{\varsigma}{k}\right]}\left|\int_a^{h_2} e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h_2)-\ell^{\rho+1}(\gamma)\right]}{\theta}\right]}\left[\ell^{\rho+1}(h_2)-\ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1}\ell^{\rho}(\gamma)\ell\prime(\gamma)d\gamma\right. \\
&\left. - \int_a^{h_1} e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(h_1)-\ell^{\rho+1}(\gamma)\right]}{\theta}\right]}\left[\ell^{\rho+1}(h_1)-\ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1}\ell^{\rho}(\gamma)\ell\prime(\gamma)d\gamma\right|
\end{aligned}$$

$$\leq \frac{r_0(\rho+1)^{1-\frac{\varsigma}{k}} e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(b)\right]}{\theta}\right]}}{\theta^{\frac{\varsigma}{k}} k^{\frac{\varsigma}{k}} \Gamma\left[\frac{\varsigma}{k}\right]} \Bigg[\int_a^{h_2} \left[\ell^{\rho+1}(h_2) - \ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1} \ell^{\rho}(\gamma)\ell\prime(\gamma)y(\gamma)d\gamma$$
$$- \int_a^{h_1} \left[\ell^{\rho+1}(h_1) - \ell^{\rho+1}(\gamma)\right]^{\frac{\varsigma}{k}-1} \ell^{\rho}(\gamma)\ell\prime(\gamma)d\gamma\Bigg]$$

$$= \frac{r_0(\rho+1)^{-\frac{\varsigma}{k}} e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(b)\right]}{\theta}\right]}}{\varsigma\theta^{\frac{\varsigma}{k}} k^{\frac{\varsigma}{k}-1} \Gamma\left[\frac{\varsigma}{k}\right]} \left(\left[\ell^{\rho}(h_2) - \ell^{\rho}(a)\right]^{\frac{\varsigma}{k}} - \left[\ell^{\rho}(h_1) - \ell^{\rho}(a)\right]^{\frac{\varsigma}{k}}\right).$$

Hence, $|\left[{}^{k}_{\rho}\mathcal{I}^{\varsigma,\theta}_{a,\ell} Gy\right](h_2) -, \left[{}^{k}_{\rho}\mathcal{I}^{\varsigma,\theta}_{a,\ell} Gy\right](h_1)| \to 0$, since $\ell(h)$ is a continuous function.

Therefore, we have

$$|(Gy)(h_2) - (Gy)(h_1)| \leq \alpha_1\mu(y,\epsilon) + \mu_V(I,\epsilon),$$

this implies

$$\mu(Gy,\epsilon) \leq \alpha_1\mu(y,\epsilon) + \mu_V(I,\epsilon).$$

Since V is uniformly continuous on $I \times [-r_0, r_0] \times [-J, J]$, so we get $\mu_V(I,\epsilon) \to 0$.

Further, if we take $\sup_{y\in V}$ and $\epsilon \to 0$, we have

$$\mu_0(Gy) \leq \alpha_1\mu_0(y), \quad \Longrightarrow \quad \mathscr{A}(Gy) \leq \alpha_1\mathscr{A}(y).$$

As a direct consequence of $\mathscr{DFPT}$, G has a FP in B_{r_0}, and the Eq. (4) possesses solution in $C(I)$. □

Example 1 Take a look at the following $\mathscr{FIE}$

$$y(h) = \frac{h^2 y(h)}{1+h^2} + \frac{\left[{}^{\frac{1}{2}}_{\frac{1}{2}}J^{2,\frac{1}{2}}_{1,h}\, y\right](h)}{m}, \tag{5}$$

for $\ell(h) = h,\ m \geq 30$, where $h \in [1, 2] = I$,

and

$$\left[{}^{\frac{1}{2}}_{\frac{1}{2}}\mathcal{I}^{2,\frac{1}{2}}_{1,h}\, y\right](h) = \frac{2^{11}}{3^3\Gamma(4)} \int_1^h e^{-(h^{\frac{3}{2}}-\gamma^{\frac{3}{2}})}(h^{\frac{3}{2}} - \gamma^{\frac{3}{2}})^3\gamma^{\frac{1}{2}} y(\gamma)d\gamma.$$

Also, $V(h, y(h), J) = \frac{h^2 y(h)}{1+h^2} + \frac{J}{m}$. It is obvious that V is continuous function satisfying

$$|V(h, y_1(h), J_1) - V(h, y_2(h), J_2)| \leq \frac{h^2}{1+h^2}|y_1(h) - y_2(h)| + \frac{1}{m}|J_1 - J_2|.$$

Therefore, $\alpha_1 = \frac{1}{3},\ \alpha_2 = \frac{1}{m}$. Further,

$$|V(h, y(h), J_1)| \leq \frac{r_0}{3} + \frac{r_0 2^{10} e^{-\left[1-2^{\left(\frac{3}{2}\right)}\right]}}{m 3^4 \Gamma(4)} \left[2^{\left(\frac{3}{2}\right)} - 1\right]^4 \leq r_0.$$

If we choose $r_0 \geq 0$, then we have

$$\alpha_1 + \frac{\alpha_2 (\rho+1)^{-\frac{\varsigma}{k}} e^{\left[\frac{(\theta-1)\left[\ell^{\rho+1}(a)-\ell^{\rho+1}(b)\right]}{\theta}\right]}}{\varsigma \theta^{\frac{\varsigma}{k}} k^{\frac{\varsigma}{k}-1} \Gamma\left[\frac{\varsigma}{k}\right]} \left[\ell^{\rho+1}(h) - \ell^{\rho+1}(a)\right]^{\frac{\varsigma}{k}}$$

$$= \frac{1}{3} + \frac{2^{10} e^{-\left[1-2^{\left(\frac{3}{2}\right)}\right]}}{30(3^4)\Gamma(4)} \left[2^{\left(\frac{3}{2}\right)} - 1\right]^4 \leq 1,\ where\ \alpha_2 = \frac{1}{30}.$$

Therefore, all the assumptions from $(I)-(III)$ of the Theorem 6 are satisfied. Hence, the Eq. (4) has a solution in $C[1, 2]$.

6 Conclusion

The central claim of this article is to demonstrate a new approach to fractional integral referred to as generalized proportional (k, ρ)-$\mathscr{FIE}$ and a generalization of $\mathscr{DFPT}$. First, we defined a generalized proportional (k, ρ)-$\mathscr{FIE}$ and then show the existence of generalized proportional (k, ρ)-$\mathscr{FIE}$ with some of its properties. After that, we generalized $\mathscr{DFPT}$ and established the existence of a solution of generalized proportional (k, ρ)-$\mathscr{FIE}$ via $\mathscr{DFPT}$. In order to support the current findings, we have now provided an example.

Competing Interest. The authors declare that they do not have any competing interests.

Availability of data and materials. Not applicable.

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New Topologies on Partial Metric Spaces and M-Metric Spaces

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1 New Weak Topology for Partial M-Metric Spaces

2 Introduction

After introducing partial metric spaces by Matthews in [34] many papers are written specially in fixed point theory which all of them turn on $p(a, a)$ is not zero. In this work we introduce a weaker topology of the PMS, and we remove the condition $p(x, x) \leq p(x, y)$ in the main definition of PMS.

Definition 1 ([34]) Let X be a nonempty set and $p : X \times X \to \mathbb{R}^+$ be a self mapping of X such that for all $x, y, z \in X$ the followings are satisfied:

p1 $x = y \iff p(x, x) = p(x, y) = p(y, y)$,
p2 $p(x, x) \leq p(x, y)$,
p3 $p(x, y) = p(y, x)$,
p4 $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then p is called partial metric on X and the pair (X, p) is called partial metric space (in short PMS).

At first, we show that the condition p2 is redundant in Definition 1 of partial metric. By $p4$ if we put $y = x$, then

$$p(x, x) \leq p(x, z) + p(z, x) - p(z, z).$$

$$p(x, x) + p(z, z) \leq 2p(x, z).$$

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Therefore we have

$$2p(x, x) \leq p(x, x) + p(z, z) \leq 2p(x, z)$$

or

$$2p(z, z) \leq p(x, x) + p(z, z) \leq 2p(x, z).$$

So $p(x, x) \leq p(x, z)$, for every $x, z \in X$.

Note also that each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls

$$\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$$

where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$.

It's time to introduce new definition of partial metric.

Definition 2 ([7]) Let X be a nonempty set and $p : X \times X \to \mathbb{R}^+$ be a self mapping of X such that for all $x, y, z \in X$ the followings are satisfied:

$p1$ $p(x, x) = p(x, y) = p(y, y) \iff x = y$,
$p3$ $p(x, y) = p(y, x)$,
$p4$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then p is called partial metric on X and the pair (X, p) is called partial metric space.

Put

$$d(x, y) := p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)|. \tag{1}$$

Proposition 1 *d is a metric on X.*

Proof (1) If $x = y$, then

$$d(x, x) = p(x, x) - \min\{p(x, x), p(x, x)\} + k|p(x, x) - p(x, x)| = 0.$$

(2) And if $d(x, y) = 0$, then

$$p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)| = 0.$$

So

$$p(x, y) \leq p(x, y) + k|p(x, x) - p(y, y)| = \min\{p(x, x), p(y, y)\} \leq p(x, y).$$

Thus $p(x, y) = p(x, x)$ or $p(x, y) = p(y, y)$. Hence

$$p(x, y) + k|p(x, x) - p(y, y)| = p(x, y) \Rightarrow p(x, x) = p(y, y).$$

Therefore $p(x, y) = p(x, x) = p(y, y)$ which means $x = y$.

(3) Symmetry is obvious.

(4) For triangle inequality by the following inequality

$$\min\{a, c\} + \min\{c, b\} \leq \min\{a, b\} + c \quad \forall a, b, c \in \mathbb{R}^+, \tag{2}$$

we have

$$\begin{aligned} d(x, y) &= p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)| \\ &\leq p(x, z) + p(z, y) - p(z, z) \\ &- \min\{p(x, x), p(z, z)\} - \min\{p(z, z), p(y, y)\} + p(z, z) \\ &+ k|p(x, x) - p(z, z)| + k|p(z, z) - p(y, y)| \\ &= d(x, z) + d(x, z). \end{aligned}$$

□

3 Main Results

We define weak topology τ_d by the balls

$$B_d^k(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\},$$

for every $k \in (0, 1)$, makes τ_d is T_0.

Theorem 1 *Balls $B_d^k(x, \varepsilon)$ for every $x \in X$ and $\varepsilon > 0$ makes a base for topology τ_d.*

Theorem 2 *Topology τ_d is weaker than topology τ_p.*

Proof Put $y \in B_d^k(x, \varepsilon)$. Hence

$$p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)| < \varepsilon$$

thus

$$p(x, y) - p(x, x) \leq \rho(x, y) - \min\{\rho(x, x), \rho(y, y)\} + k|\rho(x, x) - \rho(y, y)| < \varepsilon$$

$$p(x, y) - p(x, x) < \varepsilon \Rightarrow y \in B_p(x, \varepsilon)$$

which means $B_d^k(x, \varepsilon) \subseteq B_p(x, \varepsilon)$. □

4 Second Weak Topology

If we put

$$D(x, y) := p(x, y) - \min\{p(x, x), p(y, y)\} \tag{3}$$

and

$$B_D(x, \varepsilon) = \{y \in X : D(x, y) < \varepsilon\},$$

then

$$\bigcap_{k \in (0,1)} B_d^k(x, \varepsilon) = B_D(x, \varepsilon).$$

Also, we know that

$$p(x, y) - p(x, x) \leq D(x, y) := p(x, y) - \min\{p(x, x), p(y, y)\}$$

We define weak topology τ_D which is T_0, by the balls

$$B_D(x, \varepsilon) = \{y \in X : D(x, y) < \varepsilon\}.$$

Remark 1 D isn't a metric. Put $X := \{1, 2\}$ and define p as follows:

$$p(1, 1) = 1, p(2, 2) = 2, p(1, 2) = p(2, 1) = 3,$$

So p is a partial metric and $D(2, 2) = p(2, 2) - \min\{p(1, 1), p(2, 2)\} = 2 - 1 = 1.$

Theorem 3 *Balls $B_D(x, \varepsilon)$ for every $x \in X$ and $\varepsilon > 0$ makes a base for topology τ_D.*

Theorem 4 *Topology τ_d is weaker than topology τ_D and topology τ_D is weaker than topology τ_p.*

Proof Put $y \in B_d^k(x, \varepsilon)$. Hence

$$p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)| < \varepsilon$$

thus

$$p(x, y) - p(x, x) \leq p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)| < \varepsilon$$

$$p(x, y) - p(x, x) \leq D(x, y) \leq d(x, y) < \varepsilon \Rightarrow y \in B_D(x, \varepsilon) \subseteq B_p(x, \varepsilon).$$

which means $B_d^k(x, \varepsilon) \subseteq B_D(x, \varepsilon) \subseteq B_p(x, \varepsilon)$. $\square$

Definition 3 ([7]) Let (X, p) be a partial metric space. Then

- a sequence $\{a_n\}$ in (X, p) is said to be convergent to a point $a \in X$ if and only if

$$\lim_{n\to\infty} d(a_n, a) = 0 \iff a_n \xrightarrow{\tau_d} a.$$

$$(\lim_{n\to\infty} D(a_n, a) = 0 \iff a_n \xrightarrow{\tau_D} a).$$

- a sequence $\{a_n\}$ is called a Cauchy sequence if

$$\lim_{m,n\to\infty} d(a_m, a_n) \quad (\lim_{m,n\to\infty} D(a_m, a_n))$$

  exists and finite;
- (X, p) is said to be complete if every Cauchy sequence $\{a_n\}$ in X converges to a point $a \in X$ with respect to τ_d. Furthermore,

$$\lim_{m,n\to\infty} d(a_m, a_n) = \lim_{n\to\infty} d(a, a_n) = 0$$

- A mapping $f : X \to X$ is said to be continuous at $a_0 \in X$ if for

$$\forall \varepsilon > 0\, \exists \delta > 0 \quad f(B_d^k(a_0, \delta)) \subseteq B_d^k(f(a_0), \varepsilon).$$

$$(\forall \varepsilon > 0\, \exists \delta > 0 \quad f(B_D(a_0, \delta)) \subseteq B_D(f(a_0), \varepsilon)).$$

Example 1 Let $X := \{1, 2, 3\}$, $x_n := 1$, $x = 3$ and $p(x, y) = \max\{x, y\}$. Hence $x_n \to x$ in τ_p but $x_n \not\to x$ in τ_d.

Example 2 Let $X := \{\frac{n+1}{n} : n \in \mathbb{N}\} \cup \{1\}$, $x_n := \frac{n+1}{n}$, $x = 1$ and $p(x, y) = \max\{x, y\}$. Hence $x_n \to x$ in τ_d, so $x_n \to x$ in τ_p.

Lemma 1 *Let (X, p) be a partial metric space. If $\{a_n\}$ be a sequence in (X, p) such that $p(a_n, a_{n+1}) \to 0$ as $n \to \infty$. Then $d(a_n, a_{n+1}) \to 0$ as $n \to \infty$.*

Proof By $p(a_n, a_n) \leq p(a_n, a_{n+1})$ so $p(a_n, a_n) \to 0$ as $n \to \infty$ with respect τ_p. Therefore $d(a_n, a_{n+1}) \to 0$ as $n \to \infty$. □

The next lemma states that converse convergent condition in τ_d and τ_p topologies.

Lemma 2 *Let (X, p) be a partial metric space. If $a_n \xrightarrow{\tau_p} a$ and $\lim_{n\to\infty} p(a_n, a_n)$ exists. Then*

$$\lim_{n\to\infty} d(a_n, a) = \lim_{n\to\infty} D(a_n, a) = (k+1)(p(a, a) - \lim_{n\to\infty} p(a_n, a_n)).$$

Furthermore $\lim_{n\to\infty} p(a_n, a_n) = p(a, a)$, then

$$\lim_{n\to\infty} d(a_n, a) = 0, \quad \text{and} \quad \lim_{n\to\infty} D(a_n, a) = 0,$$

or

$$a_n \xrightarrow{\tau_d} a, \quad \text{and} \quad a_n \xrightarrow{\tau_D} a.$$

Proof According to $d(a_n, a) = p(a_n, a) - \min\{p(a, a), p(a_n, a_n)\} + k|p(a, a) - p(a_n, a_n)|$ and $p(a_n, a_n) \leq p(a_n, a) + p(a, a_n) - p(a, a)$ assertion is clear. □

About the condition $\lim_{n\to\infty} p(a_n, a_n) = p(a, a)$, in Lemma 2, look at the Examples 1 and 2.

Next theorem is an application in fixed point theory as base on Banach's theorem.

Theorem 5 *Let (X, p) be a complete partial metric space. T a self mapping on X and*

$$\begin{aligned} &p(Tx, Ty) - \min\{p(Tx, Tx), p(Ty, Ty)\} + k|p(Tx, Tx) - p(Ty, Ty)| \\ &\leq l(p(x, y) - \min\{p(x, x), p(y, y)\} + k|p(x, x) - p(y, y)|), \end{aligned}$$

for some $l \in [0, 1)$ and for every $x, y \in X$. Then T has a unique fixed point on X.

Proof By Proposition 1, d is a metric and $d(Tx, Ty) \leq ld(x, y)$. □

By the new topology and metric d, many complicated contractions could be verified in the same way.

Corollary 1 *Let (X, p) be a complete partial metric space. T a self mapping on X and*

$$p(Tx, Ty) - \min\{p(Tx, Tx), p(Ty, Ty)\} \leq l(p(x, y) - \min\{p(x, x), p(y, y)\}),$$

for some $l \in [0, 1)$ and for every $x, y \in X$. Then T has a unique fixed point on X.

Proof By Definition 3, $D(Tx, Ty) \leq lD(x, y)$. □

5 Introduction and Preliminaries to M-Metric space

In 1994 Matthew in the article [34] introduced the partial metric space notation and proved Banach's fixed point contraction theorem. After that, many mathematicians in the world proved fixed point spaces with various well-known structures for partial metric. In this regard, in this article we try to introduce a new definition for M-metric space Let us introduce (introduced in 2014 by Asadi and his colleagues in [9]) that the extension is partial metric and give an example that this is not necessarily a new Hausdorff space. By presenting the topology of this space, while examining some of its properties, we define the new topology and state that it is weaker than the previously defined topology. For more detail, see [1–70].

The following views are necessary in the rest of the article:

(1) $m_{xy} := \min\{m(x,x), m(y,y)\}$,
(2) $M_{xy} := \max\{m(x,x), m(y,y)\}$.
(3) $0 \leq M_{xy} - m_{xy} = |m(x,x) - m(y,y)|$

Now we define the new version of a M-metric space as follows:

Definition 4 ([10]) Let X be an infinite set. Function $m : X \times X \to \mathbb{R}^+$ An m is called a metric whenever

$(m1)$ $m(x,x) = m(y,y) = m(x,y) \iff x = y$
$(m2)$ $m(x,y) = m(y,x)$
$(m3)$ $\left(m(x,y) - m_{xy}\right) \leq (m(x,z) - m_{xz}) + \left(m(z,y) - m_{zy}\right)$.

In this case, (X, m) is called M-metric space.

We note that the condition $m_{xy} \leq m(x,y)$ in [9] is obtained from the condition $(m3)$ assuming $x = y$ and $z = y$. It is enough to note that

$$M_{x,x} = m_{x,x} = m(x,x).$$

Example 3 Suppose $X := [0, \infty)$ and $m(x,y) := x + y$. In this case, m is an m-metric on X.

There are examples where m is a metric but p is not a metric.

Example 4 Suppose $X = \{1, 2, 3\}$ and put:

$$\begin{aligned}
m(1,1) &= 1 \\
m(2,2) &= 9 \\
m(3,3) &= 5 \\
m(1,2) &= m(2,1) = 9 \\
m(1,3) &= m(3,1) = 7 \\
m(3,2) &= m(2,3) = 7.
\end{aligned}$$

m is a metric that is not p metric.

Example 5 Suppose m is a m metric in this case

$$D(x,y) = m(x,y) - m_{x,y}.$$

D is not ordinary metric. It is enough to assume $X := \{1, 2\}$ and

$$m(1,2) = m(2,1) = m(1,1) = 1 \quad \text{and} \quad m(2,2) = 2.$$

In this case

$$D(1,2) = m(1,2) - m_{1,2} = 1 - 1 = 0$$

But $1 \neq 2$.

Example 6 Suppose (X, d) is a metric space and $\phi : [0, \infty) \to [\phi(0), \infty)$ is a one-to-one ascending or strictly ascending mapping such that $\phi(0) \geq 0$ be defined and

$$\phi(x + y) \leq \phi(x) + \phi(y) - \phi(0), \quad \forall x, y \geq 0.$$

in this case $m(x, y) = \phi(d(x, y))$ and m is an m-metric.

Example 7 For metric space (X, d) Assuming $a, b > 0$ in this case $m(x, y) = ad(x, y) + b$ and m is a metric because it is enough to assume according to the previous example $\phi(t) = at + b$.

Example 8 Suppose m is a m metric, then m^w and m^s are normal metrics:

(1) $m^w(x, y) = m(x, y) - 2m_{xy} + M_{xy}$,
(2) $m^s(x, y) = m(x, y) - m_{xy}$ when $x \neq y$ and $m^s(x, y) = 0$ if $x = y$.

Remark 2 For each $x, y \in X$ we have

(1) $M_{x,y} - m_{x,y} = |m(x, x) - m(y, y)|$,
(2) $m(x, y) - M_{xy} \leq m^w(x, y) \leq m(x, y) + M_{xy}$,
(3) $(m(x, y) - M_{xy}) \leq m^s(x, y) \leq m(x, y)$.

In other words

$$|m^w(x, y) - m(x, y)| \leq M_{xy} \quad \text{and} \quad |m^s(x, y) - m(x, y)| \leq M_{xy}.$$

Lemma 3 *Each p is an m metric.*

6 Initial Topology of M-Metric space

In [9] authors verified that every m-metric m on X produces a T_0 for the topology τ_m. Collection

$$\{B_m(x, \varepsilon) : x \in X, \varepsilon > 0\},$$

where in

$$B_m(x, \varepsilon) = \{y \in X : m(x, y) < m_{x,y} + \varepsilon\}.$$

makes a base to τ_m.

Theorem 6 *Topology τ_m is not Hausdorff.*

Example 9 Let $X = \{1, 2, 3\}$ and

$$m(1,1) = m(1,3) = m(3,1) = m(1,2) = m(2,1) = 1,$$
$$m(2,2) = 3,$$
$$m(3,2) = m(2,3) = m(3,3) = 2.$$

This is an m-metric but it is not a p-metric. Put

$$x_n := 1, \quad x = 2, 3$$

in M-metric space $x_n \to 2$ and $x_n \to 3$ in τ_m.

$$\lim_{n\to\infty} (m(x_n, 2) - m_{x_n,2}) = 0,$$

$$\lim_{n\to\infty} (m(x_n, 3) - m_{x_n,3}) = 0,$$

which means a sequence has two different limits.

7 A New Topology τ^m for M-Metric Space

Now we define a weaker topology than the last topology called τ^m, which is generated by open balls as follows:

$$\{B_k^m(x, \varepsilon) : x \in X, \varepsilon > 0\},$$

where $0 < k < 1$ and

$$B_k^m(x, \varepsilon) = \{y \in X : m(x, y) - m_{x,y} + k(M_{x,y} - m_{x,y}) < \varepsilon\}.$$

τ_m is T_0. Since, if $x \neq y$ and $\varepsilon := m(x, y) - m_{x,y} + M_{x,y} - m_{x,y}$, then $y \notin B(x, \varepsilon)$.

$B_k^m(x, \varepsilon)$ generates a base for topology τ^m. Fixed $0 < k < 1$. For every $x \in X$ and $\varepsilon > 0$

$$y \in B_k^m(x, \varepsilon) \Rightarrow m(x, y) - m_{x,y} + k(M_{x,y} - m_{x,y}) < \varepsilon.$$

We claim

$$\exists \delta > 0 \quad B_k^m(y, \delta) \subseteq B_k^m(x, \varepsilon).$$

Let

$$\delta := \varepsilon - (m(x, y) - m_{x,y} + k(M_{x,y} - m_{x,y})).$$

$$z \in B(y, \delta) \Rightarrow m(y, z) - m_{y,z} + k(M_{y,z} - m_{y,z}) < \delta$$

$$m(y,z) - m_{y,z} + k(M_{y,z} - m_{y,z}) < \varepsilon - (m(x,y) - m_{x,y} + k(M_{x,y} - m_{x,y}))$$

by the following property

$$M_{x,z} - m_{x,z} = |m(x,x) - m(z,z)| \leq |m(x,x) - m(y,y)| + |m(y,y) - m(z,z)|$$

or

$$M_{x,z} - m_{x,z} \leq M_{x,y} - m_{x,y} + M_{y,z} - m_{y,z}$$

by $(m3)$ and $0 < k < 1$

$$\begin{aligned} m(z,x) - m_{z,x} + k(M_{x,z} - m_{x,z}) &\leq m(y,z) - m_{y,z} + k(M_{y,z} - m_{y,z}) \\ &+ m(x,y) - m_{x,y} + k(M_{x,y} - m_{x,y}) < \varepsilon \end{aligned}$$

so

$$z \in B_k^m(x,\varepsilon).$$

Theorem 7 *Topology τ^m is weaker than topology τ_m.*

Proof It's clear that, if $y \in B_k^m(x,\varepsilon) \subset B_m(x,\varepsilon)$, then

$$m(x,y) - m_{x,y} \leq m(x,y) - m_{x,y} + k(M_{x,y} - m_{x,y}) < \varepsilon.$$

□

We also note that

$$\bigcap_{k\in(0,1)} B_k^m(x,\varepsilon) = B_m(x,\varepsilon).$$

Fixed $0 < k < 1$.

Definition 5 Let (X,m) be a M-metric space. Then:

(1) A sequence $\{x_n\}$ in a M-metric space (X,m) converges to a point $x \in X$ if and only if

$$\lim_{n\to\infty} \big(m(x_n,x) - m_{x_n,x} + k(M_{x_n,x} - m_{x_n,x})\big) = 0. \tag{4}$$

(2) A sequence $\{x_n\}$ in a M-metric space (X,m) is called a m-Cauchy sequence if

$$\lim_{n\to\infty} \big(m(x_n,x_m) - m_{x_n,x_m} + k(M_{x_n,x_m} - m_{x_n,x_m})\big) \tag{5}$$

there exist (and are finite).

(3) A M-metric space (X,m) is said to be complete if every m-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ^m, to a point $x \in X$ such that

$$\lim_{n\to\infty} \big(m(x_n,x) - m_{x_n,x} + k(M_{x_n,x} - m_{x_n,x})\big) = 0.$$

Lemma 4 *Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an M-metric space (X, m). Then*

$$\lim_{n\to\infty} (m(x_n, y_n) - m_{x_n,y_n}) + k(M_{x_n,x} - m_{x_n,x}) = m(x, y) - m_{xy} + k(M_{x,y} - m_{x,y}).$$

Proof We note that

$$\left|\left(m(x_n, y_n) - m_{x_n,y_n}\right) - \left(m(x, y) - m_{x,y}\right)\right| \leq \left(m(x_n, x) - m_{x_n,x}\right) + \left(m(y, y_n) - m_{y,y_n}\right).$$

So

$$|\left(m(x_n, y_n) - m_{x_n,y_n} + k(M_{x_n,y_n} - m_{x_n,y_n})\right) - \left(m(x, y) - m_{x,y} + k(M_{x,y} - m_{x,y})\right)| \leq \left(m(x_n, x) - m_{x_n,x} + k(M_{x_n,x} - m_{x_n,x})\right) + \left(m(y_n, y) - m_{y_n,y} + k(M_{y_n,y} - m_{y_n,y})\right)$$

□

From Lemma 4 we can deduce the following lemma.

Lemma 5 *Assume that $x_n \to x$ as $n \to \infty$ in a M-metric space (X, m). Then*

$$\lim_{n\to\infty} (m(x_n, y) - m_{x_n,y}) + k(M_{x_n,x} - m_{x_n,x}) = m(x, y) - m_{xy} + k(M_{x,y} - m_{x,y}).$$

for all $y \in X$.

About uniqueness of a limit a sequence, we have the following lemma.

Lemma 6 *Assume that $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in a M-metric space (X, m). Then $m(x, y) - m_{xy} + k(M_{x,y} - m_{x,y}) = 0$. Further if $m(x, x) = m(y, y)$, then $x = y$.*

Proof By Lemma 4 if we put $y_n = x_n$, then we have

$$0 = \lim_{n\to\infty} (m(x_n, y_n) - m_{x_n,y_n}) + k(M_{x_n,x} - m_{x_n,x}) = m(x, y) - m_{xy} + k(M_{x,y} - m_{x,y}).$$

□

Example 10 Let $X = \mathbb{R}$ and $m(x, y) = \max\{x, y\}$. So m is a m-metric on X. Put $x_n = 1 + \frac{1}{n}$. So $x_n \to 1$ in τ^m and τ_m.

Example 11 Let $X = \{1, 2, 3\}$ and

$$m(1, 1) = m(1, 3) = m(3, 1) = m(1, 2) = m(2, 1) = 1,$$
$$m(2, 2) = 3,$$
$$m(3, 2) = m(2, 3) = m(3, 3) = 2.$$

This is an m-metric but it is not a p-metric. Put

$$x_n := 1, \quad x = 1, 3$$

in M-metric space $x_n \to 1$ in τ^m and τ_m and $x_n \to 3$ in τ_m but $x_n \not\to 3$ in τ^m.

$$\lim_{n\to\infty} \big((m(x_n, 1) - m_{x_n,1}) + k(M_{x_n,1} - m_{x_n,1})\big) = 0.$$

$$\lim_{n\to\infty} \big(m(x_n, 1) - m_{x_n,1}\big) = 0$$

$$\lim_{n\to\infty} (m(x_n, 3) - m_{x_n,3}) = 0,$$

while

$$\lim_{n\to\infty} \big((m(x_n, 3) - m_{x_n,3}) + k(M_{x_n,3} - m_{x_n,3})\big) \neq 0.$$

We know

$$0 \leq M_{x_n,1} - m_{x_n,1} = |m(x_n, x_n) - m(1, 1)| = 1 - 1 = 0$$
$$0 \leq M_{x_n,3} - m_{x_n,3} = |m(x_n, x_n) - m(3, 3)| = |1 - 2| = 1.$$

Proposition 2 *If*

$$D_m(x, x) = m(x, x) - \min\{m(x, x), m(x, x)\} + k|m(x, x) - m(x, x)|.$$

Then d is a metric on X.

Proof (1) If $x = y$, then

$$D_m(x, x) = m(x, x) - \min\{m(x, x), m(x, x)\} + k|M_{x,x} - m_{x,x}| = 0.$$

(2) And if $D_m(x, y) = 0$, then

$$m(x, y) - \min\{m(x, x), m(y, y)\} + k|M_{x,y} - m_{x,y}| = 0.$$

So

$$m(x, y) \leq m(x, y) + k|M_{x,y} - m_{x,y}| = \min\{m(x, x), m(y, y)\} \leq m(x, y).$$

Thus $m(x, y) = m(x, x)$ or $m(x, y) = m(y, y)$. Hence

$$m(x, y) + k|m(x, x) - m(y, y)| = m(x, y) \Rightarrow m(x, x) = m(y, y).$$

Therefore $m(x, y) = m(x, x) = m(y, y)$ which means $x = y$.

(3) Symmetry is obvious.

(4) For triangle inequality by the following inequality

$$|M_{x,y} - m_{x,y}| \leq |M_{x,z} - m_{x,z}| + |M_{z,y} - m_{z,y}| \quad (6)$$

we have

$$\begin{aligned} D_m(x, y) &= m(x, y) - m_{x,y} + k|M_{x,y} - m_{x,y}| \\ &\leq m(x, z) - m_{x,z} + k|M_{x,z} - m_{x,z}| \\ &\quad +(z, y) - m_{z,y} + k|M_{z,y} - m_{z,y}| \\ &\leq D_m(x, z) + D_m(x, z). \end{aligned}$$

□

Theorem 8 *Topology τ^m is Hausdorff.*

8 Conclusion

We verified two topology τ^m and τ_m. Topology τ^m is Hausdorff while topology τ_m isn't. We present, new definitions for partial metric space and M-metric space. By presenting the topology of those spaces, we obtain some of its properties. Also, a new topology that is weaker than the previously defined topology.

After introducing of partial metric spaces by Matthews, many papers are written specially in fixed point theory. All of them turn on $p(a, a) \neq 0$. In this article, we made a weaker than its topology and we remove the condition $p(x, x) \leq p(x, y)$ from partial metric space and $m_{x,y} \leq m(x, y)$ from M-metric space.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors have read and approved the final manuscript.

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Some Recent Fixed Point Results in S_b−Metric Spaces and Applications

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1 Introduction

A branch of mathematics that is still developing is fixed point theory, which is connected to functional analysis and topology. The rapidly expanding fields of nonlinear operators and nonlinear analysis heavily rely on fixed point theory. This branch of science is still rather new, yet it is growing quickly. Fixed point theorems and their fixed points have historically been fundamental theoretical tools in fields as numerous as topology, differential equations, economy, game theory, optimal control, dynamics, and functional analysis. Fixed point methods have become a crucial tool in the arsenal of the applied mathematician as a result of effective methods for computing fixed points and the development of accuracy, which has greatly increased the concept's importance for applications.

Major areas of mathematics such as general topology, set theory, functional analysis, and algebraic topology are just a few that provide ideal contexts for fixed point theorems. Fixed point theorems are used in a variety of disciplines, including potential theory, approximation theory, mathematical economics, game theory, theory of differential equations, and others, to answer problems in these fields. When someone is passionate about a system of integral, differential, or functional equations, it is possible to analyze a variety of problems from engineering and science utilizing fixed

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point methodologies. When dealing with control systems and elasticity concerns, this approach is quite helpful. Younis et al. in [1] solved a functional equation representing a dynamic programming problem related to a multistage process in 2022 and in [2] solved a nonlinear model indicating a rocket's ascending motion. In [3], authors used their graphical contraction to deal with a model of elastic beam deformation and in [4], authors focus on contemporary applications, mainly pertaining to the presence of solutions for diverse models associated with engineering challenges.

In order to demonstrate the solutions to various mathematical models (variational inequalities, integral, partial, and differential equations, etc.), which represent phenomena arising in a variety of fields like chemical reactions, steady-state temperature distributions, epidemics, economic theories, Neutron transport theories, and fluid flow, fixed point theorems are one of the most crucial tools available. They are also used to examine how challenging it is to choose the appropriate central for these systems.

Following Poincare [5], metric fixed point theory, initially established by Banach [6], has expanded significantly, encompassing both linear and nonlinear expressions in metric spaces (MS) (have a look at, [7, 8] and so on). In 1968, Kannan [9] used the terms $d(\chi, \mathcal{T}\chi)$, $d(y, \mathcal{T}y)$ instead of single term $d(\chi, y)$ used by Banach [6], whence Chatterjea [10] used the terms $d(\chi, \mathcal{T}\chi)$, $d(y, \mathcal{T}y)$ to extend the theory. Again, in [11] Riech combined all the three terms used by Banach and Kannan to define the contraction. Later in 2023, Alam et al. [12] defined rational contractions utilizing auxiliary functions to generalize the theory.

As a further extension of metric space, Sedghi et al. [13] presented $S-$metric space. The assertion that $S-$metric space is an extension of $G-$metric space [14] was also made by them. The claim, according to several researchers, is false. Additionally, it is asserted that the classes of $S-$metrics and $G-$metrics are both distinct. A new concept of $b-$metric space is firstly developed by Bakhtin [15]. Czerwick makes considerable use of the Bakhtin notion in [16, 17]. By combining the ideas of $S-$metric and $b-$metric, Souayah and Mlaiki [18] established the idea of S_b-metric space. A broader definition of S_b-metric space was provided by Rohen et al. [19]. Research publications in [20] provide additional findings on S_b-metric space.

Of course, it is impossible to address all these ideas in a brief overview. In this study, we limit ourselves to summarizing the generalizations of fixed point theory in S_b-metric structure, which is one of the most interesting generalizations of the metric concept. Before we explain the definition of S_b-metric, for self-control let us recall the terms $S-$metric and $b-$metric.

2 Definitions and Examples

First, we go over the basic definitions and characteristics to know S_b−metric space.

Definition 1 ([13]) An S−metric space is a pair $(\mathcal{Y}, S_b)$, where $\mathcal{Y}$ is any set and $S : \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$, so that for $u, v, w, \chi \in \mathcal{Y}$ if

(Sb1) $S(u, v, w) = 0 \Leftrightarrow u = v = w$,
(Sb2) $S(u, v, w) \leq S(u, u, s) + S(v, v, \chi) + S(w, w, \chi)$.

In this case, S is known as Smetric on $\mathcal{Y}$.

Remark 1 An S−metric is automatically symmetric with respect to the variables.

Definition 2 ([15]) A b−metric space is a pair $(\mathcal{Y}, b)$ with some $s \geq 1$, where $\mathcal{Y}$ is any set and $b : \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$, so that for $u, v, w \in \mathcal{Y}$ if

(Sb1) $b(u, w) = 0 \Leftrightarrow u = w$,
(Sb2) $b(u, w) = b(w, u)$,
(Sb3) $b(u, w) \leq s[b(u, v) + b(v, w,)]$.

In this case, b is known as b−metric on $\mathcal{Y}$.

Definition 3 ([18]) An S_b−metric space is a pair $(\mathcal{Y}, S_b)$ with some $s \geq 1$, where $\mathcal{Y}$ is any set and $S_b : \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$, so that for $u, v, w, \chi \in \mathcal{Y}$ if

(Sb1) $S_b(u, v, w) = 0 \Leftrightarrow u = v = w$,
(Sb2) $S_b(u, u, w) = S_b(w, w, u)$,
(Sb3) $S_b(u, v, w) \leq s[S_b(u, u, s) + S_b(v, v, \chi) + S_b(w, w, \chi)]$.

In this case, S_b is known as S_b−metric on $\mathcal{Y}$.

Example 1 ([18]) Consider a pair $(\mathcal{Y}, S_b)$ for $s \geq 1$, where $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$ with $card(\mathcal{Y}_1) \geq 4$, $card(\mathcal{Y}) \geq 5$, $card(\mathcal{Y}_1 \cap \mathcal{Y}_2) = 0$ and S_b: $\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as

$$S_b(u, v, w) = \begin{cases} 0, & \text{if } u = v = w = 0, \\ 3s, & \text{if } (u, v, w) \in \mathcal{Y}_1^3, \\ 1, & \text{if } (u, v, w) \notin \mathcal{Y}_1^3. \end{cases}$$

Then the pair $(\mathcal{Y}, S_b)$ is a S_b−metric space for $s \geq 1$.

The following definition is about the convergence of the sequences in the S_b−metric structure.

Definition 4 ([19]) In an S_b−metric space $(\mathcal{Y}, S_b)$, a sequence $\{w_m\}$ is

(a) convergent to $w \in \mathcal{Y}$ if and only if $\lim_{m \to \infty} S_b(w_m, w_m, w) = 0$.
(b) Cauchy if and only if $\lim_{m,p \to \infty} S_b(w_m, w_m, w_p) = 0$.

Again an S_b−metric space $(\mathcal{Y}, S_b)$ is complete if all Cauchy sequences in $(\mathcal{Y}, S_b)$ is convergent in $(\mathcal{Y}, S_b)$.

Now we give some definitions which are used by the researchers to generalize the theory in the context of S_b-metric structure.

Definition 5 Let $(\mathcal{Y}, S_b)$ be an S_b-metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{Y}$, a point $w \in \mathcal{Y}$ is a fixed point of $\mathcal{T}$ if $\mathcal{T}w = w$.

Definition 6 Let $(\mathcal{Y}, S_b)$ be an S_b-metric space for $s \geq 1$ and $\mathcal{T}_1, \mathcal{T}_2, \ldots : \mathcal{Y} \longrightarrow \mathcal{Y}$, a point $w \in \mathcal{Y}$ is a common fixed point of $\mathcal{T}_1, \mathcal{T}_2, \ldots$ if $\mathcal{T}_1 w = \mathcal{T}_2 w = \cdots = w$.

3 Results and Examples

Now we give fixed point results and their generalizations in S_b-metric structure.
Souayah et al. [18]

Souayah and Mlaiki [18] presented their first contraction theorem by extending Banach [6] in S_b-metric space utilizing an auxiliary function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, which is an increasing function so that $\lim_{m \to \infty} \psi^m(r) = 0, \ \forall\, r > 0$. Let us consider Ψ as the collection of such auxiliary functions.

Theorem 17.1 ([18]) *Let $(\mathcal{Y}, S_b)$ be a complete S_b-metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{Y}$ be a continuous function, so that*

$$S_b(\mathcal{T}u, \mathcal{T}v, \mathcal{T}w) \leq \psi(S_b(u, v, w)), \forall u, v, w \in \mathcal{Y}, \tag{1}$$

where $\psi \in \Psi$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{Y}$.

Sedghi et al. [21]
In that same year, Sedghi et al. [21], utilizing the compatible condition of mappings, presented their common fixed point result for four self mappings in S_b-metric structure.

The concept of compatible mappings is presented in a manner similar to how $S-$metric spaces were.

Definition 7 ([21]) In an S_b-metric space $(\mathcal{Y}, S_b)$, two mappings $\mathcal{T}_1, \mathcal{T}_2$ are called compatible if for any sequence $\{w_m\}$ in $\mathcal{Y}$ with $\lim_{m \to \infty} \mathcal{T}_1 w_m = \lim_{m \to \infty} \mathcal{T}_2 w_m = w$, for $w \in \mathcal{Y}$, implies $\lim_{m \to \infty} S_b(\mathcal{T}_1\mathcal{T}_2 w_m, \mathcal{T}_1\mathcal{T}_2 w_m, \mathcal{T}_2\mathcal{T}_1 w_m) = 0$.

Theorem 17.2 ([21]) *Let $(\mathcal{Y}, S_b)$ be a complete S_b-metric space for $s \geq 1$ and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4 : \mathcal{Y} \longrightarrow \mathcal{Y}$ are self mappings having compatible pairs $\{\mathcal{T}_1, \mathcal{T}_4\}, \{\mathcal{T}_2, \mathcal{T}_3\}$ with $\mathcal{T}_1(\mathcal{Y}) \subseteq \mathcal{T}_4(\mathcal{Y}), \mathcal{T}_2(\mathcal{Y}) \subseteq \mathcal{T}_3(\mathcal{Y})$, so that*

$$S_b(\mathcal{T}_1 u, \mathcal{T}_1 u, \mathcal{T}_2 w) \leq \frac{\xi}{s^4} M(u, u, w), \forall u, w \in \mathcal{Y}, \ 0 < \xi < 1, \ s \geq \frac{3}{2}, \tag{2}$$

where

$$M(u, u, w) = \sup \Big\{ S_b(\mathcal{T}_3 u, \mathcal{T}_3 u, \mathcal{T}_4 w), S_b(\mathcal{T}_1 u, \mathcal{T}_1 u, \mathcal{T}_3 u), S_b(\mathcal{T}_2 w, \mathcal{T}_2, \mathcal{T}_4 w), \frac{1}{2}[S_b(\mathcal{T}_3 u, \mathcal{T}_3 u, \mathcal{T}_2 w) + S_b(\mathcal{T}_1 u, \mathcal{T}_1 u, \mathcal{T}_4 w)] \Big\}.$$

Then the mappings $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ have a unique common fixed point in $\mathcal{Y}$.

Example 2 ([21]) Consider the S_b-metric space $(\mathcal{Y}, S_b)$ for $s = 4$, where $\mathcal{Y} = [0, 1]$ and $S_b \colon \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as

$$S_b(u, v, w) = (|v + w - 2u| + |v - w|)^2, \ \forall\, u, v, w \in \mathcal{Y}.$$

Then the mappings $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ defined as $\mathcal{T}_1 w = \left(\frac{w}{4}\right)^8$, $\mathcal{T}_2 w = \left(\frac{w}{8}\right)^4$, $\mathcal{T}_3 w = \left(\frac{w}{4}\right)^4$, $\mathcal{T}_4 w = \left(\frac{w}{8}\right)^2$ $\forall\, w \in \mathcal{Y}$ satisfy Theorem 17.2 for $\xi = \frac{25}{4^4} < 1$ having a unique common fixed point 0.

Remark 2 ([21]) 1) If the mappings $\mathcal{T}_1, \mathcal{T}_2$ (respectively $\mathcal{T}_3, \mathcal{T}_4$) considered to be identity maps in Theorem 17.2, then the mappings $\mathcal{T}_3, \mathcal{T}_4$ (respectively $\mathcal{T}_1, \mathcal{T}_2$) has a unique common fixed point.
2) If the mappings $\mathcal{T}_3, \mathcal{T}_4$ considered to be identity maps in Theorem 17.2 and $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$, then the mapping $\mathcal{T}$ has a unique fixed point.

Remark 3 Note that, form the Remark 2, Theorem 17.2 is a clear generalization of Theorem 17.1, as the continuity condition of the self mapping in Theorem 17.1 of Souayah et al. [18] is released in Theorem 17.2 of Sedghi et al. [21].

Rohen et al. [19]
In S_b-metric structure, Rohen et al. [19] introduced coupled coincidence point results for rational contractions containing a sum of seven constant multiples of rational terms.

Remark 4 In 2017, Rohen et al. [19] released the condition (Sb2) of Definition 3 to redefine S_b-metric space including only the rest two conditions (Sb1) and (Sb3). If the condition (Sb2) is added to the conditions of new redefine S_b-metric space, in that case the S_b-metric space will be called symmetric S_b-metric space.

The following example illustrates the above remark.

Example 3 ([19]) Consider a pair $(\mathcal{Y}, S_b)$ for $s \geq 2$, where $\mathcal{Y} = \mathbb{R}$ and $S_b \colon \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as

$$S_b(u, v, w) = \begin{cases} 0, \text{ if } & u = v = w, \\ 2, \text{ if } & u = v = 0, w = 1, \\ 4, \text{ if } & u = v = 1, w = 0, \\ 1, \text{ if } & \text{otherwise}. \end{cases}$$

Then the pair $(\mathcal{Y}, S_b)$ is a non-symmetric S_b-metric space for $s \geq 2$.

Definition 8 ([19]) Let $(\mathcal{Y}, S_b)$ be an S_b-metric space for $s \geq 1$ and $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathcal{Y}$, a point $(u, w) \in \mathcal{Y} \times \mathcal{Y}$ is a coupled coincidence point of $\mathcal{T}_1, \mathcal{T}_2$ if $\mathcal{T}_1(u, w) = \mathcal{T}_2(u, w)$ and $\mathcal{T}_1(w, u) = \mathcal{T}_2(w, u)$. In case of $\mathcal{T}_1(u, w) = \mathcal{T}_2(u, w) = u$ and $\mathcal{T}_1(w, u) = \mathcal{T}_2(w, u) = w$, the point (u, w) is called a common coupled fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Theorem 17.3 ([19]) *Let $(\mathcal{Y}, S_b)$ be a symmetric complete S_b-metric space for $s \geq 1$ and $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathcal{Y}$, so that*

$$S_b(\mathcal{T}_1(u, v), \mathcal{T}_1(u, v), \mathcal{T}_2(w, \chi)) \leq M(u, v, w, \chi), \forall u, v, w, \chi \in \mathcal{Y}, \tag{3}$$

where

$$\begin{aligned} M(u, v, w, \chi) = {} & \xi_1 \frac{S_b(u, u, w) + S_b(v, v, \chi)}{2} + \xi_2 \frac{S_b(\mathcal{T}_1(u, v), \mathcal{T}_1(u, v), \mathcal{T}_2(w, \chi)) S_b(u, u, w)}{1 + S_b(u, u, w) + S_b(v, v, \chi)} \\ & + \xi_3 \frac{S_b(\mathcal{T}_1(u, v), \mathcal{T}_1(u, v), \mathcal{T}_2(w, \chi)) S_b(v, v, \chi)}{1 + S_b(u, u, w) + S_b(v, v, \chi)} \\ & + \xi_4 \frac{S_b(u, u, \mathcal{T}_1(u, v)) S_b(u, u, w)}{1 + S_b(u, u, w) + S_b(v, v, \chi)} + \xi_5 \frac{S_b(u, u, \mathcal{T}_1(u, v)) S_b(v, v, \chi)}{1 + S_b(u, u, w) + S_b(v, v, \chi)} \\ & + \xi_6 \frac{S_b(w, w, \mathcal{T}_2(w, \chi)) S_b(u, u, w)}{1 + S_b(u, u, w) + S_b(v, v, \chi)} + \xi_7 \frac{S_b(w, w, \mathcal{T}_2(w, \chi)) S_b(v, v, \chi)}{1 + S_b(u, u, w) + S_b(v, v, \chi)}, \end{aligned}$$

with $\xi_i \geq 0, \sum_{i=1}^{7} \xi_i < 1$ and $s < \frac{1-\xi_2-\xi_3-\xi_6-\xi_7}{\xi_1+\xi_4+\xi_5}$. Then the mappings $\mathcal{T}_1, \mathcal{T}_2$ have a unique common coupled fixed point in $\mathcal{Y}$.

Remark 5 If the mappings $\mathcal{T}_1, \mathcal{T}_2$ considered to be one, that is, if $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$ in Theorem 17.3, then the mapping $\mathcal{T}$ has a unique coupled fixed point.

The following is a consequent corollary of Theorem 17.3 in [19].

Corollary 1 ([19]) *Let $(\mathcal{Y}, S_b)$ be a symmetric complete S_b-metric space for $s \geq 1$ and $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathcal{Y}$, so that for all $u, v, w, \chi \in \mathcal{Y}$,*

$$\begin{aligned} & S_b(\mathcal{T}_1(u, v), \mathcal{T}_1(u, v), \mathcal{T}_2(w, \chi)) \\ & \leq \xi_1 \frac{S_b(u, u, w) + S_b(v, v, \chi)}{2} \\ & + \xi_2 \frac{S_b(u, u, \mathcal{T}_1(u, v)) S_b(w, w, \mathcal{T}_2(w, \chi))}{1 + s[S_b(u, u, w) + S_b(v, v, \chi) + S_b(u, u, \mathcal{T}_2(u, v)) + S_b(w, w, \mathcal{T}_1(w, \chi))]}, \end{aligned} \tag{4}$$

where with $\xi_i \geq 0, \sum_{i=1}^{2} \xi_i < 1$ and $s < \frac{1-\xi_2}{\xi_1}$. Then the mappings $\mathcal{T}_1, \mathcal{T}_2$ have a unique common coupled fixed point in $\mathcal{Y}$.

Remark 6 If $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$ in Corollary 1, then the mapping $\mathcal{T}$ has a unique coupled fixed point.

The following example illustrates Theorem 17.3.

Example 4 ([19]) Consider the S_b−metric space $(\mathcal{Y}, S_b)$ for $s = 2$, where $\mathcal{Y} = \mathbb{R}$ and $S_b \colon \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as

$$S_b(u, v, w) = |u - w| + |v - w|, \ \forall\ u, v, w \in \mathcal{Y}.$$

Then the mappings $\mathcal{T}_1, \mathcal{T}_2$ defined as $\mathcal{T}_1(u, w) = \mathcal{T}_2(u, w) = \frac{2u-w+11}{12}$, $\forall\ u, w \in \mathbb{R}$ satisfy Theorem 17.3 for $\xi_1 = \frac{1}{3}, \xi_i = 0,\ i = 2, 3, 4, 5, 6, 7$ having a unique coupled fixed point $(1, 1)$.

Remark 7 Note that, Example 4 also satisfy Corollary 1 for $\xi_1 = \frac{1}{3}, \xi_2 = 0$ concluding the result.

Sedghi et al. [22]
Utilizing the $R-$ weakly commuting condition for a pair of self mappings with the help of an increasing continuous functions $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying $\phi(r) < r,\ r > 0$ and $\phi(0) = 0$, Sedghi et al. [22] proposed the following common fixed point results. Let the collection of such functions be denoted by Φ.

Definition 9 ([22]) In an S_b−metric space $(\mathcal{Y}, S_b)$, two mappings $\mathcal{T}_1, \mathcal{T}_2$ are called R−weakly commuting if there is some $R \geq 0$, so that

$$S_b(\mathcal{T}_1\mathcal{T}_2 w, \mathcal{T}_1\mathcal{T}_2 w, \mathcal{T}_2\mathcal{T}_1 w) \leq R S_b(\mathcal{T}_1 w, \mathcal{T}_1 w, \mathcal{T}_2 w), \ \forall\ w \in \mathcal{Y}.$$

In case of $R = 1$, we simply call $\mathcal{T}_1, \mathcal{T}_2$ are weakly commuting maps.

Example 5 ([22]) In an S_b−metric space $(\mathcal{Y}, S_b)$ for $s \geq 1$, where $\mathcal{Y} = \mathbb{R}$ and $S_b \colon \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be given by

$$S_b(u, v, w) = (|v + w - 2u| + |v - w|)^2, \ \forall\ u, v, w \in \mathcal{Y},$$

the mappings $\mathcal{T}_1, \mathcal{T}_2$ defined as $\mathcal{T}_1 w = 2w - 1,\ \mathcal{T}_2 w = w^2,\ \forall\ , w \in \mathbb{R}$ are $R(= 4)$−weakly commuting, but not weakly commuting.

Theorem 17.4 ([22]) *Let $(\mathcal{Y}, S_b)$ be a symmetric complete S_b−metric space for $s \geq 1$ and $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{Y} \longrightarrow \mathcal{Y}$ be two R−weakly commuting functions with $\mathcal{T}_1(\mathcal{Y}) \subseteq \mathcal{T}_2(\mathcal{Y})$, so that*

$$S_b(\mathcal{T}_1 u, \mathcal{T}_1 u, \mathcal{T}_1 w) \leq \frac{1}{4\,s^6}\phi(S_b(\mathcal{T}_2 u, \mathcal{T}_2 u, \mathcal{T}_2 w)), \forall u, w \in \mathcal{Y}, \tag{5}$$

where $\phi \in \Phi$. If in addition, anyone of the mappings $\mathcal{T}_1, \mathcal{T}_2$ is continuous, then $\mathcal{T}_1, \mathcal{T}_2$ have a unique common fixed point in $\mathcal{Y}$.

We now have an example that satisfies Theorem 17.4.

Example 6 ([22]) Consider the S_b-metric space $(\mathcal{Y}, S_b)$ for $s = 4$, where $\mathcal{Y} = \mathbb{R}$ and S_b: $\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as

$$S_b(u, v, w) = (|v + w - 2u| + |v - w|)^2, \ \forall\ u, v, w \in \mathcal{Y}.$$

Then the $R-$weakly commuting mappings $\mathcal{T}_1, \mathcal{T}_2$ defined as $\mathcal{T}_1 w = 1,\ \mathcal{T}_2 w = 2w - 1,\ \forall\ , w \in \mathbb{R}$ satisfy Theorem 17.4 for $\phi(r) = \frac{3r}{4}$ having a unique common fixed point 1.

Remark 8 If the mapping $\mathcal{T}_2$ considered to be identity in Theorem 17.4, then the mapping $\mathcal{T}_1$ has a unique fixed point.

Mlaiki et al. [23]
To generalize $\psi-$contraction, the concept of Souayah et al. [18], Mlaiki et al. [23] in 2017 presented $\alpha - \psi-$contraction results utilizing $\alpha-$admissibility condition of the self mapping.

Definition 10 ([23]) In an S_b-metric space $(\mathcal{Y}, S_b)$, a self mapping $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{Y}$ is called $\alpha-$admissible if

$$\alpha(u, v, w \geq 1) \Longrightarrow S_b(\mathcal{T}u, \mathcal{T}v, \mathcal{T}w) \geq 1,$$

where $\alpha : \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$.

Theorem 17.5 ([23]) *Let $(\mathcal{Y}, S_b)$ be a symmetric complete S_b-metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{Y}$ be an $\alpha-$admissible continuous function, so that $\exists\ w_0 \in \mathcal{Y}$ with $\alpha(w_0, w_0, \mathcal{T}w_0) \geq 1$ and for some $\psi \in \Psi$,*

$$\alpha(u, v, w) S_b(\mathcal{T}u, \mathcal{T}v, \mathcal{T}w) \leq \psi(S_b(u, v, w)), \forall u, v, w \in \mathcal{Y}. \tag{6}$$

Then $\mathcal{T}$ has a fixed point in $\mathcal{Y}$.

Remark 9 Notice that, the Theorem 17.5 includes the continuity restriction on the mapping $\mathcal{T}$. To release the continuity restriction Mlaiki et al. [23] defined the following corollary by adding a sequential condition, known as $\alpha-$regularity condition.

Corollary 2 ([23]) *Let $(\mathcal{Y}, S_b)$ be a symmetric complete S_b-metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{Y}$ be an $\alpha-$admissible function, so that $\exists\ w_0 \in \mathcal{Y}$ with $\alpha(w_0, w_0, \mathcal{T}w_0) \geq 1$ and for some $\psi \in \Psi$,*

$$\alpha(u, v, w) S_b(\mathcal{T}u, \mathcal{T}v, \mathcal{T}w) \leq \psi(S_b(u, v, w)), \forall u, v, w \in \mathcal{Y}. \tag{7}$$

If for any convergent sequence $\{w_m\}$ in $\mathcal{Y}$, converging to $w \in \mathcal{Y}$ we have α $(w_m, w_m, w_{m+1}) \geq 1 \Rightarrow \alpha(w_m, w_m, w) \geq 1,\ \forall\ m \in \mathbb{N}$, then $\mathcal{T}$ has a fixed point in $\mathcal{Y}$.

Remark 10 We see in Theorem 17.5 and Corollary 2 of Mlaiki et al. [23], the fixed point may not be unique. For the uniqueness of fixed point, in addition to the conditions of Theorem 17.5 we have to add the following condition:

$\exists\, v \in \mathcal{Y}$ so that $\alpha(u, u, v) \geq 1, \alpha(w, w, v) \geq 1$ for any two fixed points $u, w \in \mathcal{Y}$.

The example below illustrates Remark 10 with Theorem 17.5.

Example 7 ([23]) Consider the S_b−metric space $(\mathcal{Y}, S_b)$ for $s \geq 1$, where $\mathcal{Y} = [0, 3] \setminus (1, 2)$ and S_b: $\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as

$$S_b(u, v, w) = \begin{cases} |u - w| + |v - w|, & \text{if } u, v, w \in [0, 1] \\ \sup\{u, v, w\}, & \text{otherwise.} \end{cases}$$

Then the mapping $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ defined as

$$\mathcal{T}w = \begin{cases} \frac{1+w}{2}, & \text{if } 0 \leq w \leq 1 \\ \frac{3}{2}, & \text{if } w = 2 \\ \frac{2+w}{2}, & \text{if } 2 < w \leq 3, \end{cases}$$

satisfy Theorem 17.5, for $\psi(r) = \frac{1}{2}r$ and

$$\alpha(u, v, w) = \begin{cases} e^{\sup\{u,v\}-w}, & \text{if } \sup\{u, v\} - w \geq 0 \\ 0, & \text{if } \sup\{u, v\} - w \leq 0 \end{cases}$$

having a unique fixed point 1.

Saleem et al. [24]
Combining the concept of compatibility of pair of mappings and monotonicity of the triple (ψ, ϕ, F) defined in Definition 11, Saleem et al. [24] generalized the results of Souayah and Mlaiki [18] and Sedghi et al. [21] as follows.

Definition 11 ([24]) A three tuple (ψ, ϕ, F) is known as monotone if $u \leq w \Rightarrow F(\psi(u), \phi(u)) \leq F(\psi(w), \phi(w))$, $\forall\, u, w \in \mathbb{R}^+$, where $F : [0, \infty)^2 \longrightarrow (-\infty, \infty)$ and satisfies $F(r_1, r_2) \leq r_1$, and $F(r_1, r_2) = r_1 \Rightarrow$ either $r_1 = 0$ or $r_2 = 0$; $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is increasing continuous and satisfies $\psi(r) = 0 \Leftrightarrow r = 0$; $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and satisfies $\phi(r) > 0$, $\forall\, r > 0$ and $\phi(0) \geq 0$.

Example 8 ([24]) Let $\psi(r) = \sqrt{r}, 0 \leq r \leq 1, \psi(r) = r^2, r > 1; \phi(r) = \sqrt{r}$; $F(r_1, r_2) = r_1 - r_2$. Then the triple (ψ, ϕ, F) is monotone.

Example 9 ([24]) Let $\psi(r) = \sqrt{r}, 0 \leq r \leq 1, \psi(r) = r^2, r > 1; \phi(r) = r^2$; $F(r_1, r_2) = r_1 - r_2$. Then the triple (ψ, ϕ, F) is monotone.

Theorem 17.6 ([24]) *Let* $(\mathcal{Y}, S_b)$ *be a symmetric complete* S_b*−metric space for* $s \geq 1$ *and* $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4 : \mathcal{Y} \longrightarrow \mathcal{Y}$ *are self mappings having compatible pairs* $\{\mathcal{T}_1, \mathcal{T}_3\}, \{\mathcal{T}_2, \mathcal{T}_4\}$ *with* $\mathcal{T}_1(\mathcal{Y}) \subseteq \mathcal{T}_4(\mathcal{Y}), \mathcal{T}_2(\mathcal{Y}) \subseteq \mathcal{T}_3(\mathcal{Y})$*, so that*

$$\psi(s^4 S_b(\mathcal{T}_1 u, \mathcal{T}_1 u, \mathcal{T}_2 w)) \leq F[\psi(M(u, w)), \phi(M(u, w))],\ \forall\ u, w \in \mathcal{Y}, \quad (8)$$

where

$$M(u, w) = \sup \Big\{ S_b(\mathcal{T}_3 u, \mathcal{T}_3 u, \mathcal{T}_4 w), S_b(\mathcal{T}_1 u, \mathcal{T}_1 u, \mathcal{T}_3 u), S_b(\mathcal{T}_2 w, \mathcal{T}_2 w, \mathcal{T}_4 w), \frac{1}{2}[S_b(\mathcal{T}_3 u, \mathcal{T}_3 u, \mathcal{T}_2 w) + S_b(\mathcal{T}_1 u, \mathcal{T}_1 u, \mathcal{T}_4 w)] \Big\}$$

and (ψ, ϕ, F) *is monotone defined in Definition 11. If* $s > \frac{\sqrt{3}+1}{2}$ *and* $\mathcal{T}_3, \mathcal{T}_4$ *are continuous, then the mappings* $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ *have a unique common fixed point in* $\mathcal{Y}$.

Example 10 ([24]) Consider the S_b-metric space $(\mathcal{Y}, S_b)$ for $s = 2$, where $\mathcal{Y} = [0, 1]$ and $S_b\colon \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as

$$S_b(u, v, w) = (|v + w - 2u| + |v - w|)^{\frac{3}{2}},\ \forall\ u, v, w \in \mathcal{Y}.$$

Then the mappings $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ defined as $\mathcal{T}_1 w = \left(\frac{w}{3}\right)^4, \mathcal{T}_2 w = \left(\frac{w}{9}\right)^2, \mathcal{T}_3 w = \left(\frac{w}{3}\right)^2$, $\mathcal{T}_4 w = \frac{w}{3}\ \forall\ w \in \mathcal{Y}$ satisfy Theorem 17.6 for $\psi(r) = r, \phi(r) = \frac{729}{160 \times \sqrt{10}} - 1, F(r_1, r_2) = \frac{r_1}{r_2 + 1}$, having a unique common fixed point 0.

Remark 11 ([24]) (1) If the mappings $\mathcal{T}_1, \mathcal{T}_2$ (respectively $\mathcal{T}_3, \mathcal{T}_4$) considered to be identity maps in Theorem 17.6, then the mappings $\mathcal{T}_3, \mathcal{T}_4$ (respectively $\mathcal{T}_1, \mathcal{T}_2$) has a unique common fixed point.
(2) If the mappings $\mathcal{T}_3, \mathcal{T}_4$ considered to be identity maps in Theorem 17.6 and $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$, then the mapping $\mathcal{T}$ has a unique fixed point.

Thounaojam et al. [25]
In the year 2021, Thounaojam et al. [25] carried out and concluded $\alpha-$Meir-Keeler contraction results, concerning $\alpha-$admissibility of mappings, of Gulyaz et al. [26] in $b-$metric structure, to S_b-metric structure.

Definition 12 ([25]) In an S_b-metric space $(\mathcal{Y}, S_b)$, a self mapping $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{Y}$ is called $\alpha-$Meir-Keeler contraction of type AI, if $\forall\ \varepsilon > 0, \exists\ \delta > 0$, so that

$$\varepsilon \leq M(u, v, w) < \varepsilon + \delta \Longrightarrow \alpha(u, v, w) S_b(\mathcal{T}u, \mathcal{T}v, \mathcal{T}w) < \frac{\varepsilon}{s}$$

where $\alpha : \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ and

$$M(u, v, w) = \sup\{S_b(u, v, w), S_b(u, u, \mathcal{T}u), S_b(v, v, \mathcal{T}v), S_b(w, w, \mathcal{T}w)\}.$$

- If in AI type $\alpha-$Meir-Keeler contraction $u = v$, then the contraction will be called $\alpha-$Meir-Keeler contraction of type AII.

- If in AI type $\alpha-$Meir-Keeler contraction $M(u, v, w) = \sup\{S_b(u, v, w), S_b(u, u, \mathcal{T}u), S_b(v, v, \mathcal{T}v), S_b(w, w, \mathcal{T}w), \frac{1}{4}[S_b(u, u, \mathcal{T}v) + S_b(v, v, \mathcal{T}w) + S_b(w, w, \mathcal{T}u)]\}$, then the contraction will be called $\alpha-$Meir-Keeler contraction of type BI.
- If in BI type $\alpha-$Meir-Keeler contraction $u = v$, then the contraction will be called $\alpha-$Meir-Keeler contraction of type BII.

Theorem 17.7 ([25]) *Let* $(\mathcal{Y}, S_b)$ *be a symmetric complete* S_b-*metric space for* $s \geq 1$ *and* $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{Y}$ *be an* $\alpha-$*admissible* $\alpha-$*Meir-Keeler continuous contraction of type AI. Then* $\mathcal{T}$ *has a fixed point in* $\mathcal{Y}$.

Remark 12 ([25]) (1) We see in Theorem 17.7, the uniqueness of fixed point is not mentioned. In addition to the conditions of Theorem 17.7, if we add the following condition:

$\exists\, v \in \mathcal{Y}$ so that $\alpha(u, u, v) \geq 1, \alpha(w, w, v) \geq 1$ for any two fixed points $u, w \in \mathcal{Y}$.

Then fixed point of $\mathcal{T}$ becomes unique.
(2) To release the continuity condition of $\mathcal{T}$ in Theorem 17.7, we can add $\alpha-$regularity condition defined in Corollary 2.

Remark 13 ([25]) The Theorem 17.7 and Remark 11 is true for all other $\alpha-$Meir-Keeler contraction of types AII, BI and BII.

Example 11 ([25]) Consider the S_b-metric space $(\mathcal{Y}, S_b)$ for $s = 2$, where $\mathcal{Y} = [0, \infty)$ and S_b: $\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as $S_b(u, v, w) = |v + w - 2u|$. Then the mapping $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ defined as

$$\mathcal{T}w = \begin{cases} \frac{w^2}{8}, & \text{if } 0 \leq w \leq 1 \\ \frac{1}{8} + \log(w), & \text{if } w \in (1, \infty), \end{cases}$$

satisfy Theorem 17.7, for

$$\alpha(u, v, w) = \begin{cases} 1, & \text{if } u, v, w \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

having a unique fixed point 0.

Aytimur et al. [27]
Later in 2022, Aytimur [27] presented geometric interpretation concerning Jleli-Samet type contractions to analyze fixed figure (circle, ellipse, hyperbola, Cassini curve, Apollonius circle) problems in the S_b-metric structure.

Definition 13 ([27]) In an S_b-metric space $(\mathcal{Y}, S_b)$ for $s \geq 1$, let $w_0, w_1, w_2 \in \mathcal{Y}$, a circle having radius ρ and centered at w_0 is given by $\mathcal{C}^{S_b}_{w_0,\rho} = \{w \in \mathcal{Y} : S_b(w, w, w_0) = \rho\}$; a disc having radius ρ and centered at w_0 is given by

$\mathcal{D}^{S_b}_{w_0,\rho} = \{w \in \mathcal{Y} : S_b(w, w, w_0) \leq \rho\}$; an ellipse is given by $\mathcal{E}^{S_b}_{\rho}(w_1, w_2) = \{w \in \mathcal{Y} : S_b(w, w, w_1) + S_b(w, w, w_2) = \rho\}$; a hyperbola is given by $\mathcal{H}^{S_b}_{\rho}(w_1, w_2) = \{w \in \mathcal{Y} : |S_b(w, w, w_1) - S_b(w, w, w_2)| = \rho\}$; a Cassini curve is given by $\mathcal{C}^{S_b}_{\rho}(w_1, w_2) = \{w \in \mathcal{Y} : S_b(w, w, w_1)S_b(w, w, w_2) = \rho\}$; an Apollonius circle is given by $\mathcal{A}^{S_b}_{\rho}(w_1, w_2) = \left\{w \in \mathcal{Y} \setminus \{w_2\} : \frac{S_b(w,w,w_1)}{S_b(w,w,w_2)} = \rho\right\}$.

A figure $\mathcal{Fig}$ is called fixed figure of $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{Y}$ if $\mathcal{T}w = w, \forall\, w \in \mathcal{Fig}$.

Example 12 ([27]) In an S_b-metric space $(\mathcal{Y}, S_b)$ for $s = 1$, where $\mathcal{Y} = \mathbb{R}^{\nVdash}$ and S_b: $\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be given by

$$S_b(u, v, w) = (|u - v| + |v - w| + |w - u|)^3,\ \forall\, u, v, w \in \mathcal{Y},$$

let $w_0 = (1, 1, 1) = w_1$, $w_2 = (-1, -1, -1)$ and $\rho = 40$. Then

- $\mathcal{C}^{S_b}_{w_0,\rho} = \{(u, v, w) \in \mathcal{Y} : |u - 1|^3 + |v - 1|^3 + |w - 1|^3 = 5\}$
- $\mathcal{D}^{S_b}_{w_0,\rho} = \{(u, v, w) \in \mathcal{Y} : |u - 1|^3 + |v - 1|^3 + |w - 1|^3 \leq 5\}$
- $\mathcal{E}^{S_b}_{\rho}(w_1, w_2) = \big\{(u, v, w) \in \mathcal{Y} : (|u - 1| + |u + 1|)^2 + (|v - 1| + |v + 1|)^2$ $+(|w - 1| + |w + 1|)^2 \leq 50\big\}$
- $\mathcal{H}^{S_b}_{\rho}(w_1, w_2) = \big\{(u, v, w) \in \mathcal{Y} : ||u - 1| - |u + 1||^3 + ||v - 1| - |v + 1||^3$ $+||w - 1| - |w + 1||^3 \leq 5\big\}$
- $\mathcal{C}^{S_b}_{\rho}(w_1, w_2) = \big\{(u, v, w) \in \mathcal{Y} : (|u - 1||u + 1|)^3 + (|v - 1||v + 1|)^3$ $+(|w - 1||w + 1|)^3 \leq 5\big\}$
- $\mathcal{A}^{S_b}_{\rho}(w_1, w_2) = \left\{(u, v, w) \in \mathcal{Y} : \left(\frac{|u-1|}{|u+1|}\right)^3 + \left(\frac{|v-1|}{|v+1|}\right)^3 + \left(\frac{|w-1|}{|w+1|}\right)^3 \leq 5\right\}$

Theorem 17.8 ([27]) *Let $(\mathcal{Y},S_b)$ be a symmetric complete S_b-metric space for $s \geq 1$ and $\mathcal{T} : \mathcal{Y} \longrightarrow \mathcal{Y}$ be a function, so that for $w_0, w_1, w_2 \in \mathcal{Y}$*

$$S_b(w, w, w_0) > 0 \Rightarrow \phi(S_b(w, w, \mathcal{T}w)) \leq [\phi(M(w, w_0, w_1, w_2))]^{\xi}, \tag{9}$$

where $0 < \xi < 1$, $\rho = \min\{S_b(w, w, \mathcal{T}w) : w \neq \mathcal{T}w, w \in \mathcal{Y}\}$, $\phi : (0, \infty) \longrightarrow 91, \infty)$ is increasing and

- *when $M(w, w_0, w_1, w_2) = S_b(w, w, w_0), \forall\, w \in \mathcal{Y}$, the inequality (9) called Jleli-Samet type $\mathcal{D}_{w_0} - S_b-$ contraction, $\mathcal{T}$ fixes the disc $\mathcal{D}^{S_b}_{w_0,\rho}$.*
- *when $M(w, w_0, w_1, w_2) = S_b(w, w, w_1) + S_b(w, w, w_2), \forall w \in \mathcal{Y} \setminus \{w_1, w_2\}$ and $\mathcal{T}w_1 = w_1$, $\mathcal{T}w_2 = w_2$, the inequality (9) called Jleli-Samet type $\mathcal{E}_{w_1,w_2} - S_b-$ contraction, $\mathcal{T}$ fixes the ellipse $\mathcal{E}^{S_b}_{\rho}(w_1, w_2)$.*
- *when $M(w, w_0, w_1, w_2) = |S_b(w, w, w_1) - S_b(w, w, w_2)|, \forall w \in \mathcal{Y} \setminus \{w_1, w_2\}$, $\rho > 0$ and $\mathcal{T}w_1 = w_1$, $\mathcal{T}w_2 = w_2$, the inequality (9) called Jleli-Samet type $\mathcal{H}_{w_1,w_2} - S_b-$ contraction, $\mathcal{T}$ fixes the hyperbola $\mathcal{H}^{S_b}_{\rho}(w_1, w_2)$.*
- *when $M(w, w_0, w_1, w_2) = S_b(w, w, w_1)S_b(w, w, w_2), \forall w \in \mathcal{Y} \setminus \{w_1, w_2\}$ and $\mathcal{T}w_1 = w_1$, $\mathcal{T}w_2 = w_2$, the inequality (9) called Jleli-Samet type $\mathcal{C}_{w_1,w_2} - S_b-$ contraction, $\mathcal{T}$ fixes the Cassini curve $\mathcal{C}^{S_b}_{\rho}(w_1, w_2)$.*

- *when* $M(w, w_0, w_1, w_2) = \frac{S_b(w,w,w_1)}{S_b(w,w,w_2)}, \forall w \in \mathcal{Y} \setminus \{w_1, w_2\}$ *and* $Tw_1 = w_1$, $Tw_2 = w_2$*, the inequality (9) called Jleli-Samet type* $\mathcal{A}_{w_1,w_2} - S_b-$ *contraction,* $\mathcal{T}$ *fixes the Apollonius circle* $\mathcal{A}_\rho^{S_b}(w_1, w_2)$.

The example below satisfies Theorem 17.8 having fixed figures.

Example 13 ([27]) Consider the S_b-metric space $(\mathcal{Y}, S_b)$ for $s = 1$, where $\mathcal{Y} = \{-7, -\sqrt{2}\} \cup [-1, 1] \cup \{\sqrt{2}, \frac{7}{3}, 7, 8, 21\}$ and $S_b: \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as $S_b(u, v, w) = |u + w - 2v| + |u - w|$. Then the mapping $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ defined as

$$\mathcal{T}w = \begin{cases} w, & \text{if } w \in \mathcal{Y} \setminus \{8\} \\ 7, & \text{if } w = 8, \end{cases}$$

satisfy Theorem 17.8, for $\phi(r) = 1 + r,\ r > 0$ and $\rho = 2$ as follows:

- for $\alpha = \frac{1}{2}$, $w_0 = 0$, Jleli-Samet type $\mathcal{D}_0 - S_b-$ contraction and $\mathcal{T}$ fixes the disc $\mathcal{D}_{0,2}^{S_b} = [-1, 1]$.
- for $\alpha = \frac{1}{2}$, $w_1 = -\frac{1}{2}$, $w_2 = \frac{1}{2}$, Jleli-Samet type $\mathcal{E}_{w_1,w_2} - S_b-$ contraction, $\mathcal{T}$ fixes the ellipse $\mathcal{E}_2^{S_b}(w_1, w_2) = [-\frac{1}{2}, \frac{1}{2}]$.
- for $\alpha = \frac{9}{10}$, $w_1 = -1$, $w_2 = 1$, Jleli-Samet type $\mathcal{H}_{w_1,w_2} - S_b-$ contraction, $\mathcal{T}$ fixes the hyperbola $\mathcal{H}_2^{S_b}(-1, 1) = \{-\frac{1}{2}, \frac{1}{2}\}$.
- for $\alpha = \frac{1}{5}$, $w_1 = -1$, $w_2 = 1$, Jleli-Samet type $\mathcal{C}_{w_1,w_2} - S_b-$ contraction, $\mathcal{T}$ fixes the Cassini curve $\mathcal{C}_2^{S_b}(-1, 1) = \{-\sqrt{2}, 0, \sqrt{2}\}$.
- for $\alpha = \frac{1}{5}$, $w_1 = -7$, $w_2 = 7$, Jleli-Samet type $\mathcal{A}_{w_1,w_2} - S_b-$ contraction, $\mathcal{T}$ fixes the Apollonius circle $\mathcal{A}_2^{S_b}(-7, 7) = \{\frac{7}{3}, 21\}$.

Thounaojam et al. [28]
In the context of two S_b-metric spaces Thounaojam et al. [28] investigated coupled coincidence point and coupled fixed point results to enlarge the theory in 2022.

Definition 14 ([28]) In an S_b-metric space $(\mathcal{Y}, S_b)$, two mappings $\mathcal{T}_1 : \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathcal{Y}$, $\mathcal{T}_2 : \mathcal{Y} \longrightarrow \mathcal{Y}$ are called $w-$compatible if

$$\mathcal{T}_1(u, w) = \mathcal{T}_2 u,\ \mathcal{T}_1(w, u) = \mathcal{T}_2 w \Rightarrow \mathcal{T}_2\mathcal{T}_1(u, w) = \mathcal{T}_1(\mathcal{T}_2 u, \mathcal{T}_2 w),\ \forall\ u, w \in \mathcal{Y}.$$

Theorem 17.9 ([28]) *Let* $(\mathcal{Y}_1, S_{b_1})$, $(\mathcal{Y}_2, S_{b_2})$ *be two symmetric* S_b-*metric spaces for* $s \geq 1$ *in an universal set* Y *with* $S_{b_2}(u, u, w) \leq S_{b_1}(u, u, w), \forall\ u, w \in Y$ *and* $\mathcal{T}_1 : \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathcal{Y}$, $\mathcal{T}_2 : \mathcal{Y} \longrightarrow \mathcal{Y}$ *are two mappings having* $w-$*compatibility with* $\mathcal{T}_1(\mathcal{Y} \times \mathcal{Y}) \subseteq \mathcal{T}_2(\mathcal{Y})$*, so that* $\forall\ u, v, w, \chi \in \mathcal{Y}$

$$S_{b_1}(\mathcal{T}_1(u, w), \mathcal{T}_1(u, w), \mathcal{T}_1(v, \chi)) + S_{b_2}(\mathcal{T}_1(w, u), \mathcal{T}_1(w, u), \mathcal{T}_1(\chi, v)) \leq M(u, v, w, \chi)), \tag{10}$$

where

$$
\begin{aligned}
M(u, v, w, \chi) = \xi_1[S_{b_2}(\mathcal{T}_2u, \mathcal{T}_2u, \mathcal{T}_2v) + S_{b_2}(\mathcal{T}_2w, \mathcal{T}_2w, \mathcal{T}_2\chi)] + \xi_2[S_{b_2}(\mathcal{T}_2u, \mathcal{T}_2u, \mathcal{T}_1(u, w)) \\
+S_{b_2}(\mathcal{T}_2w, \mathcal{T}_2w, \mathcal{T}_1(w, u))] + \xi_3[S_{b_2}(\mathcal{T}_2u, \mathcal{T}_2u, \mathcal{T}_1(v, \chi)) \\
+S_{b_2}(\mathcal{T}_1w, \mathcal{T}_1w, \mathcal{T}_1(\chi, v))]
\end{aligned}
$$

and $\xi \in [0, 1],\ 0 \le \xi_1 + \xi_2 + \xi_3 \le \frac{1}{s^2}$. *If* $\mathcal{T}_2(Y)$ *is* $S_{b_1}-$*complete, then the mappings* $\mathcal{T}_1, \mathcal{T}_2$ *have a unique common coupled fixed point in* $\mathcal{Y}$.

Remark 14 (1)If we remove the $w-$compatibility condition from Theorem 17.9, then instead of having a unique common coupled fixed point, $\mathcal{T}_1, \mathcal{T}_2$ have a coupled coincidence point in $\mathcal{Y}$.
(2) Instead of taking two symmetric S_b-metric spaces, if we take one symmetric complete S_b-metric space, then the results can also be concluded.
(3) If the mapping $\mathcal{T}_2$ is considered to be identity, then we will get the corresponding coupled fixed point results.

We have an example that is concluded using Theorem 17.9.

Example 14 ([28]) Consider the two S_b-metric spaces $(\mathcal{Y}_1, S_{b_1}), (\mathcal{Y}_2, S_{b_2})$ for $s = 1$, where $\mathcal{Y} = \mathcal{Y}_1 = \mathcal{Y}_2 = \mathbb{R}$ and $S_{b_1}, S_{b_2}\colon \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \longrightarrow \mathbb{R}^+$ be defined as

$$S_{b_1}(u, v, w) = (u + v - 2w)^2,\ \forall\ u, v, w \in \mathcal{Y}_1,$$

$$S_{b_2}(u, v, w) = \left(\frac{u + v - 2w}{2}\right)^2,\ \forall\ u, v, w \in \mathcal{Y}_2.$$

Then the mappings $\mathcal{T}_1, \mathcal{T}_2$ defined as $\mathcal{T}_1(u, w) = \frac{u-w}{3}$, $\mathcal{T}_2w = 2w,\ \forall\ u, w \in \mathbb{Y}$ satisfy Theorem 17.9 for $\xi_1 = \frac{1}{9}$, $\xi_i = 0,\ i = 2, 3$ having a unique coupled fixed point $(0, 0)$.

There is another generalization by Rao et al. [20], who represented common coupled fixed point results and solved an initial valued problem by utilizing the corresponding integral equation. Recently, Tas et al. [29] investigated some geometric properties by proving common fixed point results in the context of S_b-metric space. Besides generalizing the theory, they proposed that parametric rectified linear unit activation functions also fixes some figures. Consequently, Researchers Investigated the S_b-metric in different ways and generalized the theory in many directions.

4 Conclusion

Starting with the fixed point result for contractions involving only one self mapping, we discussed, with examples and remarkable comments, coincidence fixed point results, common coupled fixed point results for contractions involving up to four mappings satisfying $w-$compatible conditions in the context of S_b-metric structure.

There are also some fixed figure results collected that are interpreting geometric properties of figures like circle, ellipse, hyperbola, Cassini curve, Apollonius circle, etc. in S_b−metric spaces. By demonstrating these results, we want to encourage young researchers that there is still an opportunity to explore this intriguing area with its vast potential for applications.

Declarations

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