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On Fixed Point of Quasi Contraction with Application to Integral Equation

Rhoda Chiroma[1], Mohammed Shehu Shagari[2*] and Jamilu Abubakar Jiddah[3]

Abstract. It is observed from the surveyed literature that there is no sufficient study of quasi-weakly contractive operators in the context of b-metric-like spaces. From this background information, this paper introduces a new unified notion of the quasi-weakly contractive operator in b-metric-like space. It examines the existence and uniqueness of invariant points of such operators. The idea put forward herewith subsumes a few known results in the literature. Non-trivial illustrations are constructed to verify our proposed concepts and to compare them with other corresponding ones. Corollaries which reduce our findings to other famous ideas are presented and discussed. As an application, one of our obtained corollaries is utilised to investigate new existence criteria for solving a class of boundary value problem.

1. Introduction

Fixed point theory plays an important role in many branches of mathematics and applied sciences. The fixed point theorems include results that deal with fixed points and offer a practical way to locate these fixed points. In applications, the existence and uniqueness of solutions of specific differential equations are described as a fixed point problem of an appropriate integral operator, whose fixed point is equivalent to the solutions of the differential equations.

One of the most outstanding results in metric fixed point theory is the Banach fixed point theorem, well known as the contraction mapping principle, published in the PhD thesis of Banach in 1922. Following this result, numerous metric space generalizations have been introduced

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and thoroughly studied by researchers in different directions (see e.g, [10, 15, 19]), one of which is weakening the metric space's defining axioms. Among the earliest generalizations in this direction is the quasi-metric space, established by Wilson [23]. In 1989, Bakhtin [5] (and also Czerwik [9]) introduced some generalizations of the well-known Banach fixed point theorem in b-metric spaces and obtained some new fixed point results in such space. Similarly, Matthews [18] proposed the concept of partial metric space as a part of the study of denotational semantics of data flow networks. The key contribution in [18] is the demonstration that self-distance in the partial space needs not be zero. Putting these two ideas together, Shukla [22] proposed a new generalization called a b-partial-metric space. In order to improve the partial metric space, Amini-Harandi [4] introduced the notion of metric-like space by relaxing the axiom of non-negativity and small self-distances in partial metric space. Recently, Noorwali and Shagari [20] used Geraghty auxiliary mappings to study some fixed point results of weak contractive operators and improved the ideas in [4], among others. Alghamdi et al. [3] introduced the concept of b-metric-like space, which generalizes the ideas of partial b-metric space, metric-like space and b-metric space. They proved the existence and uniqueness of fixed points and applied these results to generate new coupled and fixed point results in partial metric spaces, metric-like spaces, and b-metric spaces. In addition, a few applications to integral equations and several examples were given. In another direction, Alber et al.[1] introduced the notion of weak contraction mappings in the setting of Hilbert space by defining additional algebraic structure on the space. Following this, Cho [7] and Hoa [14] established some fixed point results for weakly contractive mappings in metric space and b-metric space, extending some known results in the literature.

Our review of the existing literature shows that little or no work has been done on quasi-weakly contractive operators in the context of b-metric-like space. Hence, motivated by the ideas in [7, 14], we introduce in this manuscript, a new concept of generalized quasi-weakly contractive operator in b-metric-like space and investigate the existence and uniqueness of fixed points of such operators. The idea proposed in this manuscript generalizes some well-known results in the literature. Substantial examples are presented to verify our proposed idea and to compare them with other corresponding results. A few corollaries which collapse our new concepts to other famous ideas in the literature are presented and analyzed. As an application, one of our obtained corollaries is employed to investigate novel existence conditions to solve a class of boundary value problem. Our proposed ideas extend the results of [6, 7, 14] and some references from complete b-metric space to complete b-metric-like space. It is well-known fact that metric spaces are Hasudorff (see [18]), and so cannot be used to study non-Hausdorff topologies such as those needed in the Tarskian approach to programming language semantics. Hence, this manuscript, being discussed in the framework of metric-like spaces, possibly contributes to how metric fixed point results can be examined in the non-Hausdorff topologies.

2. Preliminaries

In this section, we record specific basic concepts that are needed in the sequel. First, we recall some definitions and basic results in b-metric, partial metric, b-partial metric, and metric-like spaces. For more details, we refer to [16].

Definition 2.1 ([9])**.** Let X be a nonempty set and $s \geq 1$ be a real constant. A function $d_b : X \times X \longrightarrow \mathbb{R}$ is called a b-metric if for all $x, y, z \in X$, it satisfies the following axioms:

(i) $d_b(x,y) = 0 \quad \Leftrightarrow \quad x = y$;
(ii) $d_b(x,y) = d_b(y,x)$;
(iii) $d_b(x,y) \leq s[d_b(x,z) + d_b(z,y)]$

Then the pair (X, d_b) is called a b-metric space with $s \geq 1$ a real constant

Definition 2.2 ([18])**.** Let X be a nonempty set. A function $p : X \times X \longrightarrow \mathbb{R}^+$ is called a partial metric on X if, for all $x, y, z \in X$, the following are satisfied:

(i) $p(x,x) = p(x,y) = p(y,y) \quad \Leftrightarrow \quad x = y$;
(ii) $p(x,x) \leq p(x,y)$;
(iii) $p(x,y) = p(y,x)$;
(iv) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Then the pair (X, p) is called a partial metric space.

Definition 2.3 ([22])**.** Let X be a nonempty set and $s \geq 1$ be a real constant. A function $p_b : X \times X \longrightarrow \mathbb{R}^+$ is called a partial metric on X if, for all $x, y, z \in X$, the following are satisfied:

(i) $p_b(x,x) = p_b(x,y) = p_b(y,y) \quad \Leftrightarrow \quad x = y$;
(ii) $p_b(x,x) \leq p_b(x,y)$;
(iii) $p_b(x,y) = p_b(y,x)$;
(iv) $p_b(x,z) \leq s[p_b(x,y) + p_b(y,z)] - p_b(y,y)$.

Then the pair (X, p_b) is called a partial b-metric space.

Definition 2.4 ([4])**.** A mapping $\sigma : X \times X \longrightarrow \mathbb{R}_+$ is said to be a metric-like on X if for any $x, y, z \in X$, the following four conditions hold:

(*i*) $\sigma(x,y)=0 \quad \Rightarrow \quad x=y$;
(*ii*) $\sigma(x,y)=\sigma(y,x)$;
(*iii*) $\sigma(x,z)\leq\sigma(x,y)+\sigma(y,z)$.

Then the pair (X,σ) is called a metric-like space.

Combining the axioms of the above Definitions 2.1-2.4, Alghamdi et al. [3] introduced another generalization of metric spaces known as b-metric-like space in the following manner:

Definition 2.5 ([3])**.** Let X be a non-empty set and $s\geq 1$ be a real constant. A mapping $\sigma_b(x,y): X\times X\longrightarrow[0,\infty)$ is said to be a b-metric-like if and only if, for all $x,y,z\in X$, the following conditions are satisfied:

(*i*) $\sigma_b(x,y)=0 \quad \Rightarrow$ x=y;
(*ii*) $\sigma_b(x,y)=d_b(y,x)$;
(*iii*) $\sigma_b(x,z)\leq s[d_b(x,y)+d_b(y,z)]$.

Then the pair (X,σ_b) is called a b-metric-like space with constant $s\geq 1$.

Definition 2.6 ([3])**.** Let (X,σ_b) be a b-metric-like space and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points of X. A point $x\in X$ is said to be the limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$ if $\lim\limits_{n\to\infty}\sigma_b(x,x_n)=\sigma_b(x,x)$, and we say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to x and denoted it by $x_n\to x$ as $n\to\infty$.

Definition 2.7 ([3])**.** Let (X,σ_b) be a b-metric-like space.

(*i*) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be a Cauchy sequence if and only if $\lim\limits_{n,m\to\infty}\sigma_b(x_n,x_m)$ exists and is finite.
(*ii*) A b-metric-like space (X,σ_b) is said to be complete if and only if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ $x\in X$in X, so that

$$\begin{aligned}\lim_{n,m\to\infty}\sigma_b(x_n,x_m)&=d_b(x,x)\\&=\lim_{n,\to\infty}\sigma_b(x_n,x).\end{aligned}$$

Remark 2.8 ([3])**.** It is clear that every partial b-metric space is a b-metric-like space with $s\geq 1$ a real constant and every b-metric space is also a b-metric-like space with the same $s\geq 1$ a real constant. However, the converses is not true.

Definition 2.9 ([17])**.** Let $T,S: X\longrightarrow X$ be two mappings. A point $x\in X$ is said to be a common fixed point of T and S, if $Tx=Sx=x$.

Definition 2.10 ([17])**.** Let T and S be two self-mapping on a nonempty set X. If $w=Tx=Sx$ for some $x\in X$,then x is said to be the coincidence point of T and S, where w is called the point of coincidence of T and S. Let $C(T,S)$ denote the set of all coincidence points of T and S.

Definition 2.11 ([17])**.** Let T and S be two self-mappings defined on a nonempty set X. Then , T and S is said to be weakly compatible if they commute at every coincidence point, that is, $Tx = Sx \Rightarrow TSx = STx$ for every $x \in C(T, S)$.

Remark 2.12. In Definition 2.11 , if $T = S$, then $T(orS)$ is said to be weakly self compatible.

Definition 2.13 ([8])**.** Let (X, d) be a metric space. A self-mapping $T : X \longrightarrow X$ is said to be a quasi-contraction if there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \max\left\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right\}.$$

Definition 2.14. Let (X, d) be a b-metric space. A mapping $T : X \longrightarrow X$ is said to be weakly contractive, if for all $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \lambda(d(x, y)),$$

where $\lambda : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous and non-decreasing function such that $\lambda(0) = 0$ and $\lim\limits_{t \to +\infty} \lambda(t) = +\infty$.

Definition 2.15. A function $T : X \longrightarrow [0, \infty]$, where X is a b-metric space, is called lower semi-continuous if, for all $x \in X$ and $\{x_n\}_{n\in\mathbb{N}} \subset X$ are b-convergent with $\lim\limits_{n\to\infty} x_n = x$, we have

$$T(x) \leq \liminf_{n\to\infty} x_n.$$

Let $\Psi = \{\psi : [0, \infty) \to [0, \infty) | \psi$ is continuous and $\psi(t) = 0 \Leftrightarrow t = 0\}$. Also, let $\Phi = \{\phi : [0, \infty) \longrightarrow [0, \infty) | \phi$ is lower semi continuous and $\phi(t) = 0 \Leftrightarrow t = 0\}$.

Hoa [14] obtained the following results in the context of b-metric space.

Theorem 2.16 ([14])**.** *Let (X, σ_b) be a complete b-metric space with $s \geq 1$ a real constant, and let $T, S : X \longrightarrow X$ be given self-mappings satisfying S as injective and $T(X) \subset S(X)$, where $S(X)$ is closed. Suppose $\varphi : X \longrightarrow [0, \infty)$ is a lower semi-continuous function and $p \geq 2$ is a constant. If there are functions $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(s^p(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty))) \leq \psi(M(x, y, \varphi)) - \phi(L(x, y, \varphi)),$$

for all $x, y \in X$. where

$$M(x, y, \varphi) = \max\left\{\begin{array}{c} d(Sx, Sy) + \varphi(Sx) + \varphi(Sy), \\ \frac{1}{2}\left[d(Tx, Sx) + \varphi(Tx) + \varphi(Sx) + d(Ty, Sy) + \varphi(Ty) + \varphi(Sy)\right], \\ \frac{1}{2s}\left[d(Tx, Sy) + \varphi(Tx) + \varphi(Sy) + d(Ty, Sx) + \varphi(Ty) + \varphi(Sx)\right] \end{array}\right\}$$

and

$$L(x, y, \varphi) = \max\left\{d(Sx, Sy) + \varphi(Sx) + \varphi(Sy), d(Ty, Sy) + \varphi(Ty) + \varphi(Sy)\right\}$$

Then T and S have a unique coincidence point in X. Moreover, T And S have a unique common fixed point provided T and S are weakly compatible.

Theorem 2.17 ([14]). *Let (X, d_b) be a complete b-metric space with $s \geq 1$ a real constant, and let $T, S : X \longrightarrow X$ be given self-mappings, and one of T and S is continuous. Suppose $\varphi : X \longrightarrow [0, \infty)$ is a lower semi-continuous function and $p \geq 3, 0 < \lambda \leq \frac{1}{4}$ are two constants. If there are functions $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$n(x, y, \varphi) = \lambda \max \left\{ \begin{array}{c} d(x, y) + \varphi(x) + \varphi(y), \\ d(Tx, x) + \varphi(Tx) + \varphi(x) + d(y, Sy) + \varphi(y) + \varphi(Sy), \\ \frac{1}{s}[d(Tx, y) + \varphi(Tx) + \varphi(y) + d(x, Sy) + \varphi(x) + \varphi(Sy)] \end{array} \right\}$$

and

$$\eth(x, y, \varphi) = \max \{d(x, y) + \varphi(x) + \varphi(y), d(y, Sy) + \varphi(y) + \varphi(Sy)\}$$

Then T and S have a unique common fixed point in X.

Lemma 2.18 ([16]). *Let (X, d_σ) be a b-metric-like space with $s \geq 1$ a real constant and suppose that $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ are b-convergent to x, y respectively. Then we have*

$$\begin{aligned} \frac{1}{s} d_\sigma(x, y) - \frac{1}{s} d_\sigma(x, x) - d_\sigma(y, y) &\leq \liminf_{n \to +\infty} d_\sigma(x_n, y_n) \\ &\leq \limsup_{n \to +\infty} d_\sigma(x_n, y_n) \\ &\leq s^2 d_\sigma(x, x) \\ &\leq s^2 d_\sigma(y, y) \\ &\leq s^2 d_\sigma(x, y). \end{aligned}$$

In particular, if $d_\sigma(x, y) = 0$, then we have $\lim_{n \to +\infty} d_\sigma(x_n, y_n) = 0$. Moreover for each $z \in X$ we have

$$\begin{aligned} \frac{1}{s} d_\sigma(x, z) - \frac{1}{s} d_\sigma(x, x) &\leq \liminf_{n \to +\infty} d_\sigma(x_n, z) \\ &\leq \limsup_{n \to +\infty} d_\sigma(x, z) \\ &\leq s d_\sigma(x, z) + s d_\sigma(x, z) \end{aligned}$$

In particular if $d_\sigma(x, x) = 0$ then,

$$\begin{aligned} \frac{1}{s} d_\sigma(x, z) &\leq \liminf_{n \to +\infty} d_\sigma(x_n, z) \\ &\leq \limsup_{n \to +\infty} d_\sigma(x_n, z) \\ &\leq s d_\sigma(x, z). \end{aligned}$$

3. Main Results

We start this section by introducing the following definition.

Definition 3.1. Let (X, d_σ) be a b-metric-like space with constants $s \geq 1$ and $p \geq 2$. The pair (T, S) of self-mappings of X is said to form a generalized quasi-weakly contractive operator, if it satisfies the following condition,

$$\begin{aligned}(3.1) \qquad &\psi(s^p(d(Tx,Tx)+d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)))\\ &\leq \psi(\mathcal{M}(x,y,\varphi)) - \phi(\mathcal{L}(x,y,\varphi)),\end{aligned}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\phi \in \Phi$ and

$$\mathcal{M}(x,y,\varphi) = \max\left\{\begin{array}{c} d(Sx,Sy)+\varphi(Sx)+\varphi(Sy),\\ \frac{1}{2}[d(Tx,Sx)+\varphi(Tx)+\varphi(Sx)\\ +d(Ty,Sy)+\varphi(Ty)+\varphi(Sy)+d(Tx,Tx)],\\ \frac{1}{2s}\left[d(Tx,Sy)+\varphi(Tx)+\varphi(Sy)+d(Ty,Sx)+\varphi(Ty)+\varphi(Sx)\right]\end{array}\right\}$$

and

$$\begin{aligned}\mathcal{L}(x,y,\varphi) = \max\{&d(Sx,Sy)+\varphi(Sx)+\varphi(Sy), d(Ty,Sy)+\varphi(Ty)\\ &+\varphi(Sy)+d(Tx,Tx)\}.\end{aligned}$$

Theorem 3.2. *Let (X, d_σ) be a complete b-metric-like space with constants $s \geq 1$ and $p \geq 2$. If the pair (T, S) of self-mappings of X forms a generalized quasi-weakly contraction such that S is a one-one mapping and $T(X) \subset S(X)$, where $S(X)$ is closed in X. Then, T and S have a unique coincidence point in X. Furthermore, T and S have a unique common fixed point provided T and S are weakly compatible.*

Proof. Let $x_0 \in X$. As $T(X) \subset S(X)$, there exists $x_1 \in X$ with $Tx_0 = Sx_1$. Now, we define the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X by $yn = Tx_n = Sx_{n+1}$ for all $n \in \mathbb{N}$. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$; then we have $y_n = y_{n+1} = Tx_{n+1} = Sx_{n+1}$ and T and S have a coincidence point. Without loss of generality, we assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Applying (3.1) with $x = x_n$ and $y = x_{n+1}$, we obtain:

$$\begin{aligned}(3.2)\\ &\psi(d(y_n,y_n)+d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1}))\\ &\quad\leq \psi\left(s^p(d(y_n,y_n)+d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1}))\right)\\ &\quad= \psi\left(s^p(d(Tx_n,Tx_n)+d(Tx_n,Tx_{n+1})+\varphi(Tx_n)+\varphi(Tx_{n+1}))\right)\\ &\quad\leq \psi\left(\mathcal{M}(x_n,x_{n+1},\varphi)\right) - \phi\left(\mathcal{L}(x_n,x_{n+1},\varphi)\right),\end{aligned}$$

where,

(3.3)

$$\begin{aligned}
&\mathcal{M}(x_n, x_{n+1}, \varphi)\\
&\quad= \max\left\{\begin{array}{c} d(Sx_n, Sx_{n+1}) + \varphi(Sx_n) + \varphi(Sx_{n+1}),\\ \frac{1}{2}[d(Tx_n, Sx_n) + \varphi(Tx_n) + \varphi(Sx_n)\\ +d(Tx_{n+1}, Sx_{n+1}) + \varphi(Tx_{n+1}) + \varphi(Sx_{n+1}) + d(Tx_n, Tx_n)],\\ \frac{1}{2s}[d(Tx_n, Sx_{n+1}) + \varphi(Tx_n) + \varphi(Sx_{n+1})\\ +d(Tx_{n+1}, Sx_n) + \varphi(Tx_{n+1}) + \varphi(Sx_n)] \end{array}\right\}\\
&\quad= \max\left\{\begin{array}{c} d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n),\\ \frac{1}{2}[d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1}) + d(y_{n+1}, y_n)\\ +\varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n)],\\ \frac{1}{2s}[d(y_n, y_n) + \varphi(y_n) + \varphi(y_n)\\ +d(y_{n+1}, y_{n-1}) + \varphi(y_{n+1}) + \varphi(y_{n-1})] \end{array}\right\}\\
&\quad\leq \max\left\{\begin{array}{c} d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n),\\ \frac{1}{2}[d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1}) + d(y_{n+1}, y_n)\\ +\varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n)],\\ \frac{1}{2s}[d(y_n, y_n) + \varphi(y_n) + \varphi(y_n) + sd(y_{n+1}, y_n) + sd(y_{n+1}, y_n)\\ +\varphi(y_{n+1}) + \varphi(y_{n-1})] \end{array}\right\}\\
&\quad\leq \max\left\{\begin{array}{c} d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n),\\ \frac{1}{2}[d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1}) + d(y_{n+1}, y_n)\\ +\varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n)] \end{array}\right\}\\
&\quad\leq \max\left\{\begin{array}{c} d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n),\\ \max\{d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1}),\\ d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n)\} \end{array}\right\}\\
&\quad\leq \max\left\{\begin{array}{c} d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n),\\ d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n) \end{array}\right\}.
\end{aligned}$$

Hence, (3.3) becomes

(3.4)

$$\mathcal{M}(x_n, x_{n+1}, \varphi) \leq \max\left\{\begin{array}{c} d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n),\\ d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n) \end{array}\right\}.$$

Similarly,

(3.5)

$$\begin{aligned}
&\mathcal{L}(x_n, x_{n+1}, \varphi)\\
&\quad\leq \max\left\{\begin{array}{c} d(Sx_n, Sx_{n+1}) + \varphi(Sx_n) + \varphi(Sx_{n+1}),\\ d(Tx_{n+1}, Sx_{n+1}) + \varphi(Tx_{n+1}) + \varphi(Sx_{n+1}) + d(Tx_n, Tx_n) \end{array}\right\}\\
&\quad= \max\left\{\begin{array}{c} d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n),\\ d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n) \end{array}\right\}.
\end{aligned}$$

If $d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1}) < d(y_n, y_n) + d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n)$ for some positive integer n, then it follows from (3.2), (3.4) and (3.5) that

$$\begin{aligned}&\psi(d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n))\\ &\quad\le \psi(d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n))\\ &\qquad - \phi(d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n)),\end{aligned}$$

which implies that

$$\phi(d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n)) = 0,$$

and hence,

$$d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n) = 0,$$

from which we notice that

$$y_n = y_{n+1}, \qquad \varphi(y_n) = \varphi(y_{n+1}) = 0$$

is a contradiction. Therefore,

(3.6)
$$d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n) + d(y_n, y_n) < d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1})$$

for all $n = 1, 2, 3, \ldots$

Hence,

$$\mathcal{M}(x_{n-1}, x_n, \varphi) = d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n)$$

and

$$\mathcal{L}(x_{n-1}, x_n, \varphi) = d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n).$$

From (3.2), we have

$$\begin{aligned}&\psi(d(y_n, y_n) + d(y_n, y_{n+1}) + \varphi(y_n) + \varphi(y_{n+1}))\\ &\quad\le \psi(d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n))\\ &\qquad - \phi(d(y_{n-1}, y_n) + \varphi(y_{n-1}) + \varphi(y_n)).\end{aligned} \tag{3.7}$$

It follows from (3.6) that the sequence

$$\{d(y_n, y_n) + d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n)\}$$

is non-decreasing. Therefore,

$$d(y_n, y_n) + d(y_n, y_{n+1}) + \varphi(y_{n+1}) + \varphi(y_n) \longrightarrow r \quad \text{as} \quad n \to +\infty,$$

for some $r \ge 0$.

Suppose that $r > 0$, taking the upper limit in (3.7) as $n \to +\infty$, using continuity of ψ and the lower semi-continuity of ϕ, we have

$$\begin{aligned}&\limsup_{n\to+\infty} \psi(d(y_n, y_n) + d(y_{n+1}, y_n) + \varphi(y_{n+1}) + \varphi(y_n))\\ &\quad\le \limsup_{n\to+\infty}\ \psi(d(y_n, y_{n-1}) + \varphi(y_n) + \varphi(y_{n-1}))\end{aligned}$$

$$-\liminf_{n\to+\infty}\ \phi(d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1})),$$

which implies that

$$\begin{aligned}\psi(r)&\leq\psi(r)-\liminf_{n\to+\infty}\ \psi(d(y_n,y_{n-1})+\varphi(y_n)+\varphi(y_{n-1}))\\&\leq\psi(r)-\phi(r)\leq\psi(r),\end{aligned}$$

a contradiction. Thus,

$$\lim_{n\to+\infty}d(y_n,y_n)+d(y_n,y_{n+1})+\varphi(y_n)+\varphi(y_{n+1})=0,$$

from which we have

$$\lim_{n\to+\infty}d(y_n,y_n)=\lim_{n\to+\infty}d(y_n,y_{n+1})=0 \tag{3.8}$$

and

$$\lim_{n\to+\infty}\varphi(y_n)=\lim_{n\to+\infty}+\varphi(y_{n+1})=0. \tag{3.9}$$

Now, we shall prove that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Suppose on the contrary that $\{y_n\}_{n\in\mathbb{N}}$ is not Cauchy. It follows that there exists $\epsilon>0$ for which one can find sequences $\{y_{m(k)}\}_{k\in\mathbb{N}}$ and $\{y_{n(k)}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ satisfying $n(k)$ is the smallest index for which $n(k)>m(k)>k$,

$$\epsilon\leq d\left(y_{m(k)},y_{n(k)}\right), \tag{3.10}$$

$$d\left(y_{m(k)},y_{n(k-1)}\right)<\epsilon. \tag{3.11}$$

By the triangle inequality in b-metric-like space and (3.10) and (3.11), we have

$$\begin{aligned}\epsilon&\leq d\left(y_{m(k)},y_{n(k)}\right)\\&\leq sd\left(y_{m(k)},y_{n(k-1)}\right)+sd\left(y_{n(k-1)},y_{n(k)}\right)\\&<s\epsilon+sd\left(y_{n(k-1)},y_{n(k)}\right).\end{aligned}$$

Taking the upper limit as $k\to+\infty$ in the above inequality, we have

$$\epsilon\leq\limsup_{k\to+\infty}\ d\left(y_{m(k)},y_{n(k)}\right)<s\epsilon. \tag{3.12}$$

Also,

$$d\left(y_{m(k)},y_{n(k)}\right)\leq sd\left(y_{m(k)},y_{n(k-1)}\right)+sd\left(y_{n(k-1)},y_{n(k)}\right), \tag{3.13}$$

$$d\left(y_{m(k)},y_{n(k)}\right)\leq sd\left(y_{m(k)},y_{m(k-1)}\right)+sd\left(y_{m(k-1)},y_{n(k)}\right), \tag{3.14}$$

$$d\left(y_{m(k-1)},y_{n(k)}\right)\leq sd\left(y_{m(k-1)},y_{m(k)}\right)+sd\left(y_{m(k)},y_{n(k)}\right). \tag{3.15}$$

From (3.10), (3.11) and (3.13), we obtain

$$\frac{\epsilon}{s}\leq\limsup_{k\to+\infty}\ d\left(y_{m(k)},y_{n(k-1)}\right)\leq\epsilon. \tag{3.16}$$

Using (3.10), (3.14) and (3.15), we get:

$$\frac{\epsilon}{s} \leq \limsup_{k\to+\infty} d\left(y_{m(k-1)}, y_{n(k)}\right) \leq s^2\epsilon. \tag{3.17}$$

Similarly,

$$d\left(y_{m(k-1)}, y_{n(k-1)}\right) \leq sd\left(y_{m(k-1)}, y_{m(k)}\right) + sd\left(y_{m(k)}, y_{n(k-1)}\right)$$

and

$$\begin{aligned} d\left(y_{m(k)}, y_{n(k)}\right) \leq sd\left(y_{m(k)}, y_{m(k-1)}\right) + s^2 d\left(y_{m(k-1)}, y_{n(k-1)}\right) \\ + s^2 d\left(y_{n(k-1)}, y_{n(k)}\right), \end{aligned}$$

so there is

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to+\infty} d\left(y_{m(k-1)}, y_{n(k-1)}\right) \leq s\epsilon. \tag{3.18}$$

Using the same method, one can obtain that

$$\begin{aligned} \epsilon &\leq \liminf_{k\to+\infty} d\left(y_{m(k)}, y_{n(k)}\right) \leq s\epsilon, \\ \frac{\epsilon}{s} &\leq \liminf_{k\to+\infty} d\left(y_{m(k)}, y_{n(k-1)}\right) \leq \epsilon, \\ \frac{\epsilon}{s} &\leq \liminf_{k\to+\infty} d\left(y_{m(k-1)}, y_{n(k)}\right) \leq s^2\epsilon, \\ \frac{\epsilon}{s^2} &\leq \liminf_{k\to+\infty} d\left(y_{m(k-1)}, y_{n(k-1)}\right) \leq s\epsilon. \end{aligned}$$

In view of the definition of $\mathcal{M}(x, y, \varphi)$, we deduce

$$\begin{aligned} &\mathcal{M}\left(x_{m(k)}, x_{n(k)}, \varphi\right) \\ &= \max\left\{\begin{array}{c} d(Sx_{m(k)}, Sx_{n(k)}) + \varphi(Sx_{m(k)}) + \varphi(Sx_{n(k)}), \\ \frac{1}{2}[d(Tx_{m(k)}, Sx_{m(k)}) + \varphi(Tx_{m(k)}) + \varphi(Sx_{m(k)}) \\ +d(Tx_{n(k)}, Sx_{n(k)}) + \varphi(Tx_{n(k)}) + \varphi(Sx_{n(k)}) \\ +d(Tx_{n(k)}, Tx_{n(k)})], \\ \frac{1}{2s}[d(Tx_{m(k)}, Sx_{n(k)}) + \varphi(Tx_{m(k)}) + \varphi(Sx_{n(k)}) \\ +d(Tx_{n(k)}, Sx_{m(k)}) + \varphi(Tx_{n(k)}) + \varphi(Sx_{m(k)})] \end{array}\right\} \\ &= \max\left\{\begin{array}{c} d\left(y_{m(k-1)}, y_{n(k-1)}\right) + \varphi(y_{m(k-1)}) + \varphi(y_{n(k-1)}), \\ \frac{1}{2}[d\left(y_{m(k)}, y_{m(k-1)}\right) + \varphi(y_{m(k)}) + \varphi(y_{m(k-1)}) \\ +d(y_{n(k)}, y_{n(k-1)}) + \varphi(y_{n(k)}) + \varphi(y_{n(k-1)}) \\ +d(y_{m(k)}, y_{m(k)})], \\ \frac{1}{2s}[d\left(y_{m(k)}, y_{n(k-1)}\right) + \varphi(y_{m(k)}) + \varphi(y_{n(k-1)}) \\ +d(y_{n(k)}, y_{m(k-1)}) + \varphi(y_{n(k)}) + \varphi(y_{m(k-1)})] \end{array}\right\} \end{aligned} \tag{3.19}$$

Taking the upper limit as $k \to +\infty$ in (3.19), we obtain

$$\begin{aligned} \limsup_{k\to+\infty} \mathcal{M}\left(x_{m(k)}, x_{n(k)}, \varphi\right) &\leq \max\left\{s\epsilon, 0, \frac{\epsilon + s^2\epsilon}{2s}\right\} \\ &= s\epsilon \end{aligned}$$

Also, we have

$$\begin{aligned}
&\mathcal{L}\left(x_{m(k)}, x_{n(k)}, \varphi\right)\\
&\quad = \max\left\{\begin{array}{c} d(Sx_{m(k)}, Sx_{n(k)}) + \varphi(Sx_{m(k)}) + \varphi(Sx_{n(k)}),\\ d(Tx_{n(k)}, Sx_{n(k)}) + \varphi(Tx_{n(k)}) + \varphi(Sx_{n(k)}\\ +d(Tx_{m(k)}, Tx_{m(k)}) \end{array}\right\}\\
&\quad = \max\left\{\begin{array}{c} d\left(y_{m(k-1)}, y_{n(k-1)}\right) + \varphi(y_{m(k-1)}) + \varphi(y_{n(k-1)}),\\ d(y_{n(k)}, y_{n(k-1)}) + \varphi(y_{n(k)}) + \varphi(_{n(k-1)})\\ +d(y_{m(k)}, y_{m(k)}) \end{array}\right\}.
\end{aligned}$$

It follows that,

$$\frac{\epsilon}{s^2} \leq \liminf_{k\to+\infty} \ \mathcal{L}\left(x_{m(k)}, x_{n(k)}, \varphi\right) \leq s\epsilon. \tag{3.20}$$

Applying (3.1) with $x = x_{m(k)}$ and $y = x_{n(k)}$, one can get

$$\begin{aligned}
\psi(s\epsilon) \leq \psi(s^p\epsilon) &\leq \psi\Big(s^p\Big(\limsup_{k\to+\infty}\ d(y_{m(k)}, y_{m(k)})\\
&\qquad + d\left(y_{m(k)}, y_{n(k)}\right) + \varphi(y_{m(k)}) + \varphi\left(_{n(k)}\right)\Big)\Big)\\
&\leq \psi\left(\limsup_{k\to+\infty}\ \mathcal{M}(x_{m(k)}, x_{n(k)}\varphi)\right)\\
&\qquad - \phi\left(\liminf_{k\to+\infty}\ \mathcal{L}(x_{m(k)}, x_{n(k)}\varphi)\right)\\
&\leq \psi(s\epsilon) - \phi\left(\liminf_{k\to+\infty}\ \mathcal{L}\left(x_{m(k)}, x_{n(k)}, \varphi\right)\right),
\end{aligned} \tag{3.21}$$

which implies that $\liminf\limits_{k\to+\infty}(\mathcal{L}(x_{m(k)}, x_n, \varphi) = 0$, a contradiction to (3.20).

It follows that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. The completeness of X ensures that there exists a $u \in X$ such that

$$\begin{aligned}
\lim_{n\to+\infty} d(y_n, u) &= \lim_{n\to+\infty} d(Tx_n, u)\\
&= \lim_{n\to+\infty} d(Sx_{n+1}, u)\\
&= \lim_{n\to+\infty} d(y_n, y_m)\\
&= 0
\end{aligned} \tag{3.22}$$

Further more, we have $u \in S(X)$, since $S(X)$ is closed. It follows that we can choose a $z \in X$ such that $u = Sz$, and one can write (3.21) as

$$\begin{aligned}
\lim_{n\to+\infty} d(y_n, Sz) &= \lim_{n\to+\infty} d(Tx_n, Sz)\\
&= \lim_{n\to+\infty} d(Sx_{n+1}, Sz)\\
&= 0.
\end{aligned} \tag{3.23}$$

Following from the definition of φ, we get

$$\begin{aligned} \varphi(Sz) = \varphi(u) &\leq \liminf_{n\to+\infty} \ \varphi(y_n) \\ &= 0. \end{aligned} \tag{3.24}$$

That is, $\varphi(Sz) = \varphi(u) = 0$.

If $Tz \neq Sz$, taking $x = x_{n(k)}$ and $y = z$ in contractive condition (3.1), we deduce that

(3.25)

$$\begin{aligned} &\psi(d(Tx_{n(k)}, Tx_{n(k)}) + d(Tx_{n(k)}, Tz) + \varphi(Tx_{n(k)}) + \varphi(Tz) \\ &\quad \leq \psi(s^p(d(Tx_{n(k)}, Tx_{n(k)}) + d(Tx_{n(k)}, Tz) + \varphi(Tx_{n(k)}) + \varphi(Tz)) \\ &\quad \leq \psi(\mathcal{M}(x_{n(k)}, z, \varphi) - \phi(\mathcal{L}(x_{n(k)}, z, \varphi). \end{aligned}$$

Now, letting $k \to +\infty$ in (3.25) and using (3.23) and (3.24) we get,

$$\begin{aligned} &\mathcal{M}(x_{n(k)}, z, \varphi) \\ &= \max \left\{ \begin{array}{c} d(Sx_{n(k)}, Sz) + \varphi(Sx_{n(k)}) + \varphi(Sz), \\ \frac{1}{2s}[d(Tx_{n(k)}, Sx_{n(k)}) + \varphi(Tx_{n(k)}) + \varphi(Sx_{n(k)}) \\ +d(Tz, Sz) + \varphi(Tz) + \varphi(gz) + d(Tx_{n(k)}, Tx_{n(k)})], \\ \frac{1}{2s}[d(Tx_{n(k)}, Sz) + \varphi(Tx_{n(k)}) + \varphi(Sz) \\ +d(Tz, Sx_{n(k)}) + \varphi(Tz) + \varphi(Sx_{n(k)}) \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} d(y_{n(k-1)}, Sz) + \varphi(y_{n(k-1)}) + \varphi(Sz), \\ \frac{1}{2}[d(y_{n(k)}, y_{n(k-1)}) + \varphi(y_{n(k)}) + \varphi(y_{n(k-1)}) \\ +d(Tz, Sz) + \varphi(Tz) + \varphi(Sz) + d(y_{n(k)}, y_{n(k)}), \\ \frac{1}{2s}[d(y_{n(k)}, Sz) + \varphi(y_{n(k)}) + \varphi(Sz) \\ +d(Tz, y_{n(k-1)}) + \varphi(Tz) + \varphi(y_{n(k-1)}) \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} d(gz, gz), \frac{1}{2}[Sz, Sz) + d(Tz, Sz) + \varphi(Tz) + d(Sz, Sz)], \\ \frac{1}{2s}[Sz, Sz) + d(Tz, Sz) + \varphi(Tz)] \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} d(u, u), \frac{1}{2}[d(u, u) + d(Tz, Sz) + \varphi(Tz) + d(u, u)], \\ \frac{1}{2s}[d(u, u) + d(Tz, Sz) + \varphi(Tz)] \end{array} \right\} \\ &= \max \left\{ d(u, u), \frac{1}{2}[d(u, u) + d(Tz, Sz) + \varphi(Tz) + d(u, u)] \right\} \\ &\leq \max \{d(u, u), \max\{d(u, u), d(Tz, Sz) + \varphi(Tz) + d(u, u)\}]\} \\ &= \max \{d(u, u), d(Tz, Sz) + \varphi(Tz) + d(u, u)\} \\ &= \max \{d(Tz, Sz) + \varphi(Tz) + d(u, u)\}, \end{aligned}$$

which implies,

$$\mathcal{M}(x_{n(k)}, z, \varphi) = d(Tz, Sz) + \varphi(Tz) + d(u, u). \tag{3.26}$$

Similarly,

$$\mathcal{L}(x_{n(k)}, z, \varphi)$$

$$
\begin{aligned}
&= \max\left\{\begin{array}{c} d(Sx_{n(k)}, Sz) + \varphi(Sx_{n(k)}) + \varphi(Sz), \\ d(Tz, Sz) + \varphi(Tz) + \varphi(Sz) + d(Tx_{n(k)}, Tx_{n(k)})\} \end{array}\right\} \\
&= \max\left\{\begin{array}{c} d(y_{n(k-1)}, Sz) + \varphi(y_{n(k-1)}) + \varphi(Sz), \\ d(Tz, Sz) + \varphi(Tz) + \varphi(Sz) + d(y_{n(k)}, y_{n(k)})\} \end{array}\right\} \\
&= \max\left\{d(Sz, Sz), d(Tz, Sz) + \varphi(Tz) + d(Sz, Sz)\right\} \\
&= \max\left\{d(u, u), d(Tz, Sz) + \varphi(Tz) + d(u, u)\right\},
\end{aligned}
$$

which implies,

$$
d(Tz, Sz) + \varphi(Tz) + d(u, u) \leq \mathcal{L}(x_{n(k)}, z, \varphi). \tag{3.27}
$$

Hence, (3.25) reduces to

$$
d(u, u) + d(Tz, Sz) + \varphi(Tz) = 0,
$$

which implies that

$$
Tz = Sz, \qquad d(u, u) = 0, \qquad \varphi(Tz) = 0,
$$

and z is the unique coincidence point of T and S.

For uniqueness, suppose z' is another coincidence of T and S such that $z \neq z'$. Now (3.1) with $x = z$ and $y = z'$, we obtain that

$$
\begin{aligned}
&\psi(d(u, u) + d(Tz, Tz') + \varphi(Tz) + \varphi(Tz^{'})) \\
&\quad \leq \psi(s^p(d(u, u) + d(Tz, Tz') + \varphi(Tz) + \varphi(Tz^{'}))) \\
&\quad \leq \psi(\mathcal{M}(z, z', \varphi)) - \phi(\mathcal{L}(z, z', \varphi)),
\end{aligned} \tag{3.28}
$$

where

$$
\begin{aligned}
\mathcal{M}(z, z', \varphi) &= \max\left\{\begin{array}{c} d(Sz, Sz') + \varphi(Sz) + \varphi(Sz'), \\ \frac{1}{2s}[d(Tz, Sz) + \varphi(Tz) + \varphi(Sz) \\ +d(Tz', Sz') + \varphi(Tz') + \varphi(Sz') + d(Tz, Tz)], \\ \frac{1}{2s}[d(Tz, Sz') + \varphi(Tz) + \varphi(Sz') \\ +d(Tz', Sz) + \varphi(Tz') + \varphi(Sz)] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} d(Sz, Sz') + \varphi(Sz) + \varphi(Sz'), \\ \frac{1}{2}[d(u, u) + \varphi(Sz) + \varphi(Sz) \\ +d(v, v) + \varphi(Sz') + \varphi(Sz') + d(u, u)], \\ \frac{1}{2s}[d(Sz, Sz') + \varphi(Sz) + \varphi(Sz') \\ +d(Sz', Sz) + \varphi(Sz') + \varphi(Sz)] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} d(Sz, Sz') + \varphi(Sz'), \\ \frac{1}{2}[2d(u, u) + d(v, v) + 2\varphi(Sz') + \varphi(Sz')], \\ \frac{1}{2s}[2d(Sz, Sz') + 2\varphi(Sz')] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} d(Sz, Sz') + \varphi(Sz'), \varphi(Sz'), \\ \frac{1}{s}[d(gz, gz') + \varphi(gz')] \end{array}\right\} \\
&= \max\left\{d(Sz, Sz') + \varphi(Sz')\right\}
\end{aligned}
$$

$$\mathcal{L}(z,z',\varphi)=\max\left\{\begin{array}{c} d(Sz,Sz')+\varphi(Sz)+\varphi(Sz'),\\ d(Tz',Sz')+\varphi(Tz')+\varphi(Sz')+d(Tz',Tz') \end{array}\right\}$$

$$\mathcal{L}(z,z',\varphi)=\max\left\{\begin{array}{c} d(Sz,Sz')+\varphi(Sz'),\\ d(v,v)+\varphi(Tz')+\varphi(Sz')+d(v,v) \end{array}\right\}$$

$$\begin{aligned}\mathcal{L}(z,z',\varphi)&=\max\left\{d(Sz,Sz')+\varphi(Sz'),2\varphi(Sz')\right\}\\ &\geq d(Sz,Sz')+\varphi(Sz').\end{aligned}$$

It follows from (3.28) that

$$\begin{aligned}&\psi(d(Sz,Sz')+\varphi(Sz'))\\ &\quad\leq \psi(d(Sz,Sz')+\varphi(Sz'))-\phi(d(Sz,Sz')+\varphi(Sz')).\end{aligned}\tag{3.29}$$

Hence, we get that

$$d(Sz,Sz')+\varphi(Sz')=0,$$

which implies that,

$$Sz=Sz',\qquad \varphi(Sz')=0.$$

Since S is an injective mapping, then $z=z'$, that is, z is a unique coincidence point of T and S. Furthermore, if T and S are weakly compatible, then it is easy to show that z is a unique common fixed point of T and S. □

We construct the following example to verify the hypotheses of Theorem 3.2.

Example 3.3. Let $X=[0,\infty)$ and $d_\sigma(x,y)=(x+y)^2$ for all $x,y\in X$. Then obviously, d_b is a b-metric-like on X with constant $s=2$ and (X,d_σ) is complete. Also, d_b is not a metric-like or b-metric (and not a metric) on X. Define the self-mappings $T,S:X\longrightarrow X$ by $Tx=\frac{x}{2}$ and $Sx=\frac{3x}{2}$ for all $x\in X$. Clearly, $T(X)\subset S(X)$, S is one-to-one and $S(X)$ is closed in X. To see that the pair $\{T,S\}$ is a generalized quasi-weakly contraction; let $\psi(t)=t$, $\phi(t)=\frac{t}{9}$, $\varphi(t)=t^2$ and the constant $p=2$. Without loss of generality, let $x\leq y$ for all $x,y\in X$. If $x=y$, then

$$\begin{aligned}&\psi(s^p(d(Tx,Tx)+d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)))\\ &\quad=2^2\left(\left(\frac{x}{2}+\frac{x}{2}\right)^2+\left(\frac{x}{2}+\frac{y}{2}\right)^2+\left(\frac{x}{2}\right)^2+\left(\frac{y}{2}\right)^2\right)\\ &\quad=4\left(x^2+x^2+\frac{x^2}{2}\right)\\ &\quad=10x^2\end{aligned}$$

$$\begin{aligned}
&\leq 12x^2 = \frac{27x^2}{2} - \frac{3x^2}{2}\\
&= \frac{27x^2}{2} - \frac{1}{9}\left(\frac{27x^2}{2}\right)\\
&= \psi\left(\max\left\{\frac{27x^2}{2}, 7x^2, \frac{13x^2}{4}\right\}\right) - \phi\left(\max\left\{\frac{27x^2}{2}, \frac{15x^2}{2}\right\}\right)\\
&= \psi(\mathcal{M}(x,y,\varphi)) - \phi(\mathcal{L}(x,y,\varphi)).
\end{aligned}$$

If $x < y$, then

$$\begin{aligned}
&\psi(s^p(d(Tx,Tx) + d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)))\\
&\quad = 4x^2 + (x+y)^2 + x^2 + y^2\\
&\quad = 6x^2 + 2y^2 + 2xy\\
&\quad < 4x^2 + 4y^2 + 4xy\\
&\quad = \frac{1}{2}\left(9x^2 + 9y^2 + 9xy - \left(x^2 + y^2 + xy\right)\right)\\
&\quad = \frac{1}{2}\left(9x^2 + 9y^2 + 9xy\right) - \frac{1}{2}\left(x^2 + y^2 + xy\right)\\
&\quad = 2\left(\left(\frac{3x}{2}\right)^2 + \left(\frac{3y}{2}\right)^2 + \frac{9xy}{4}\right)\\
&\qquad - \frac{1}{9}\left(2\left(\left(\frac{3x}{2}\right)^2 + \left(\frac{3y}{2}\right)^2 + \frac{9xy}{4}\right)\right)\\
&\quad = \left(\left(\frac{3x}{2} + \frac{3y}{2}\right)^2 + \left(\frac{3x}{2}\right)^2 + \left(\frac{3y}{2}\right)^2\right)\\
&\qquad - \frac{1}{9}\left(\left(\frac{3x}{2} + \frac{3y}{2}\right)^2 + \left(\frac{3x}{2}\right)^2 + \left(\frac{3y}{2}\right)^2\right)\\
&\quad = \psi(\mathcal{M}(x,y,\varphi)) - \phi(\mathcal{L}(x,y,\varphi)).
\end{aligned}$$

Therefore, all the hypotheses of Theorem 3.2 are verified. It follows that $x = 0$ is the unique common fixed point of T and S.

The following Figure 1, demonstrates the behaviour of contractive inequality (3.1) using Example 3.3 for some selected values of $x, y \in X$. Specifically, it shows that the right-hand side (RHS) of (3.1) dominates its left-hand side (LHS).

Since d_σ as we defined in this example is not a metric or b-metric, the result of Hao [14] is not applicable to this example.

The following corollaries are some consequence of Theorem 3.2.

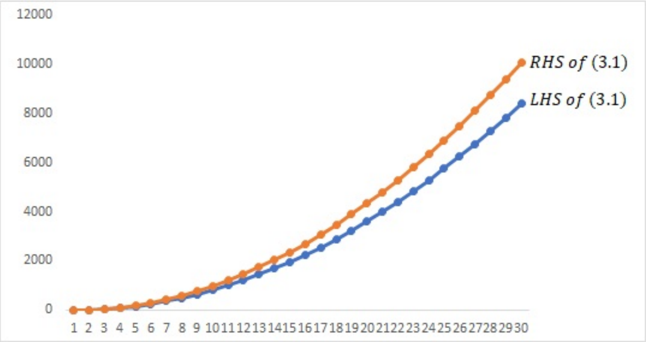

FIGURE 1. Illustration of inequality (3.1) using Example 3.3 for $x, y \in X$ with $x = y$

Corollary 3.4. *Let* (X, d_σ) *be a complete b-metric-like space,* $T : X \longrightarrow X$ *be a given self-mapping with constants* $s \geq 1$ *and* $p \geq 1$ *and let* $\varphi : X \longrightarrow [0, \infty)$ *be a lower semi-continuous function. Suppose further that the following conditions are satisfied:*

(*i*) $\psi(s^p(d(Tx, Tx) + d(Tx, Ty) + \varphi(Tx) + \varphi(Ty))) \leq \psi(\mathcal{M}_o(x, y, \varphi)) - \phi(\mathcal{L}_o(x, y, \varphi))$, *where* $\psi \in \Psi$, $\phi \in \Phi$ *and*

$$\mathcal{M}_o(x, y, \varphi) = \max \left\{ \begin{array}{c} d(Tx, Tx) + d(Tx, Ty) + \varphi(Tx) + \varphi(Ty), \\ \frac{1}{2}[d(Tx, Tx) + \varphi(Tx) + d(Ty, Ty) + \varphi(Ty)], \\ \frac{1}{2s}[d(Tx, Ty) + \varphi(x) + \varphi(Ty)] \end{array} \right\}$$

and

$$\mathcal{L}_o(x, y, \varphi) = \max \left\{ \begin{array}{c} d(Tx, Tx) + d(Tx, Ty) + \varphi(Tx) + \varphi(Ty), \\ d(Ty, Ty) + 2\varphi(Ty) \end{array} \right\}$$

(*ii*) *T is weakly self-compatible.*

Then T has a fixed point in X.

Proof. Take $T = S$ in Theorem 3.2. □

Corollary 3.5. *Let* (X, d_σ) *be a b-complete metric-like space with constant* $s \geq 1$ *and let* $T, S : X \longrightarrow X$ *be a given self-mappings satisfying S as injective and* $T(X) \subset S(X)$ *where* $S(X)$ *is closed. Suppose* $p \geq 2$ *is a constant, such that*

$$\psi(s^p(d(Tx, Tx) + d(Tx, Ty))) \leq \psi(\mathcal{M}_1(x, y)) - \phi(\mathcal{L}_1(x, y)), \tag{3.30}$$

for all $x, y \in X$, *where* $\psi \in \Psi$, $\phi \in \Phi$ *and*

$$\mathcal{M}_1(x, y) = \max \left\{ \begin{array}{c} d(Sx, Sx) + d(Sx, Sy), \\ \frac{1}{2}[d(Tx, Sx) + d(Ty, Sy) + d(Ty, Ty)], \frac{1}{2s}[d(Tx, Sy) + d(Ty, Sx)] \end{array} \right\}$$

$\mathcal{L}_1(x,y) = \max\{d(Sx,Sx)+d(Sx,Sy), d(Ty,Sy)+d(Ty,Ty)\}$.

Then T and S have a unique coincidence point in X. Moreover, T and S have a unique common fixed point provided that T and S are weakly compatible.

Proof. Put $\varphi(t)=0$, for all $t \in \mathbb{R}^+$ in Theorem 3.2. □

Corollary 3.6. *Let (X, d_σ) be a complete b-metric-like space with constant $s \geq 1$, $T, S : X \longrightarrow X$ be a given self mappings satisfying S as injective and $T(X) \subset S(X)$ where $S(X)$ is closed and let $\varphi : X \longrightarrow [0,\infty)$ be a lower semi-continuous function. Suppose $p \geq 2$ is a constant, such that*

$$\psi(s^p(d(Tx,Tx)+d(Tx,Ty)+\varphi(Tx)+\varphi(Ty))) \leq \psi(\mathcal{M}(x,y,\varphi)),$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\mathcal{M}(x,y,\varphi)$ is the same as Theorem 3.2 . Then T and S have a unique coincidence point in X. Moreover, T and S have a unique common fixed point provided that T and S are weakly compatible.

Proof. Take $\phi(t)=0$, for all $t \in \mathbb{R}^+$ in Theorem 3.2. □

The following is a version of Theorem 3.2 in the setting of metric-like space with $s=1$ in Theorem 3.2.

Corollary 3.7. *Let (X, σ) be a complete metric-like space, $T, S : X \longrightarrow X$ be a given self mappings satisfying S as injective and $T(X) \subset S(X)$ where, $S(X)$ is closed. Suppose $\varphi : X \longrightarrow [0,\infty)$ is a lower semi-continuous function, such that*

$$\begin{aligned}&\psi(d(Tx,Tx)+d(Tx,Ty)+\varphi(Tx)+\varphi(Ty))\\ &\quad\leq \psi(\mathcal{M}(x,y,\varphi)) - \phi(\mathcal{L}(x,y,\varphi)),\end{aligned}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\phi \in \Phi$ and $\mathcal{L}(x,y,\varphi)$ and $\mathcal{L}(x,y,\varphi)$ are the same as Theorem 3.2 .

Then T and S have a unique coincidence point in X. Moreover, T and S have a unique common fixed point provided that T and S are weakly compatible.

Theorem 3.8. *Let (X, d_σ) be a complete b-metric-like space with constants $s \geq 1$, $p \geq 3$ and $T, S : X \longrightarrow X$ be given self-mappings and one of T and S is continuous. Suppose $\varphi : X \longrightarrow [0,+\infty)$ is a lower semi-continuous function, such that*

$$\begin{aligned}&\psi(s^p(d(Tx,Tx)+d(Tx,Sy)+\varphi(Tx)+\varphi(Sy)))\\ &\quad\leq \psi(q(x,y,\varphi)) - \phi(\Im(x,y,\varphi)),\end{aligned} \tag{3.31}$$

for all $x, y \in X$*, where* $\psi \in \Psi$, $\phi \in \Phi$ *and*

$$q(x,y,\varphi)$$
$$= \max\left\{\begin{array}{c} d(x,y)+\varphi(x)+\varphi(y), \\ \frac{1}{2}[d(Tx,x)+\varphi(Tx)+\varphi(x)+d(y,y)+d(y,Sy)+\varphi(y)+\varphi(Sy)], \\ \frac{1}{2s}[d(Tx,y)+\varphi(Tx)+\varphi(y)+d(x,Sy)+\varphi(x)+\varphi(Sy)] \end{array}\right\}$$

and

$$r(x,y,\varphi)$$
$$= \max\left\{d(x,y)+\varphi(x)+\varphi(y), d(y,y)+d(y,Sy)+\varphi(y)+\varphi(Sy)\right\}$$

Then T *and* S *have a unique common fixed point in* X*.*

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ by $x_{n+1} = Tx_n$, $x_{n+2} = Sx_{n+1}$ for all $n \in \mathbb{N}$. Suppose that $x_{n+1} \neq x_{n+2}$ for all $n \in \mathbb{N}$. Then, taking $x = x_n$ and $y = x_{n+1}$ in (3.31), we obtain:

$$\begin{aligned} (3.32)\quad & \psi(d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})) \\ & \leq \psi(s^p(d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})) \\ & = \psi(s^p(d(Tx,Tx)+d(Tx,Sy)+\varphi(Tx)+\varphi(Sy))) \\ & \leq \psi(q(x_n,x_{n+1},\varphi))-\phi(\Im(x_n,x_{n+1},\varphi)), \end{aligned}$$

where,

(3.33)

$$q(x_n,x_{n+1},\varphi)$$
$$= \max\left\{\begin{array}{c} d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}), \\ \frac{1}{2}[d(Tx_n,x_n)+\varphi(Tx_n)+\varphi(x_n) \\ +d(x_{n+1},x_{n+1})+d(x_{n+1},Sx_{n+1})+\varphi(x_{n+1})+\varphi(Sx_{n+1})], \\ \frac{1}{2s}[d(Tx_n,x_{n+1})+\varphi(Tx_n)+\varphi(x_{n+1}) \\ +d(x_n,Sx_{n+1})+\varphi(x_n)+\varphi(Sx_{n+1})] \end{array}\right\}$$
$$= \max\left\{\begin{array}{c} d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}), \\ \frac{1}{2}[d(x_{n+1},x_n)+\varphi(x_{n+1})+\varphi(x_n) \\ +d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})], \\ \frac{1}{2s}[d(x_{n+1},x_{n+1})+\varphi(x_{n+1})+\varphi(x_{n+1}) \\ +d(x_n,x_{n+2})+\varphi(x_n)+\varphi(x_{n+2})] \end{array}\right\}$$
$$\leq \max\left\{\begin{array}{c} d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}), \\ \frac{1}{2}[d(x_{n+1},x_n)+\varphi(x_{n+1})+\varphi(x_n)+d(x_{n+1},x_{n+1}) \\ +d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})], \\ \frac{1}{2s}[d(x_{n+1},x_{n+1})+\varphi(x_{n+1})+\varphi(x_{n+1})+sd(x_n,x_{n+1}) \\ +sd(x_{n+1},x_{n+2})+\varphi(x_n)+\varphi(x_{n+2})] \end{array}\right\}$$
$$\leq \max\left\{\begin{array}{c} d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}), \\ \frac{1}{2}[d(x_{n+1},x_n)+\varphi(x_{n+1})+\varphi(x_n)+d(x_{n+1},x_{n+1}) \\ +d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})] \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}),\\ \max\{d(x_{n+1},x_n)+\varphi(x_{n+1})+\varphi(x_n),\\ d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})\}\end{array}\right\}$$
$$\leq \max\left\{\begin{array}{c} d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}),\\ d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})\end{array}\right\}.$$

Hence, (3.33) becomes

(3.34)

$$q(x_n,x_{n+1},\varphi) \leq \max\left\{\begin{array}{c} d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}),\\ d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})\end{array}\right\}.$$

Similarly,

(3.35)

$$\Im(x_n,x_{n+1},\varphi) = \max\left\{\begin{array}{c} d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}),\\ d(x_{n+1},x_{n+1})+d(x_{n+1},Sx_{n+1})+\varphi(x_{n+1})+\varphi(Sx_{n+1})\end{array}\right\}$$
$$= \max\left\{\begin{array}{c} d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}),\\ d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})\end{array}\right\}.$$

If $d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}) < d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})$ for some positive integer n, then it follows from (3.32), (3.34) and (3.35) that

$$\begin{aligned}\psi(d(x_{n+1},x_{n+1})&+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))\\ &\leq \psi(d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))\\ &\quad -\phi(d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})),\end{aligned}$$

which implies that

$$\phi(d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))=0,$$

and hence,

$$d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})=0,$$

from which we notice that

$$x_{n+1}=x_{n+2}, \qquad \varphi(x_{n+1})=\varphi(x_{n+2})=0$$

is a contradiction. Therefore,

$$\begin{aligned}(3.36)\qquad d(x_{n+1},x_{n+1})&+d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2})\\ &< d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}),\end{aligned}$$

for all $n=1,2,3,\ldots$

Hence,

$$q(x_n, x_{n+1}, \varphi) = d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})$$

and

$$\Im(x_n, x_{n+1}, \varphi) = d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}).$$

From (3.32), we have

$$\begin{aligned}
&\psi(d(x_{n+1}, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) \\
&\quad \leq \psi(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})) - \phi(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})).
\end{aligned} \tag{3.37}$$

It follows from (3.36) that the sequence

$$\{d(x_{n+1}, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\}$$

is non-decreasing. Therefore,

$$d(x_{n+1}, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}) \longrightarrow r \quad \text{as} \quad n \to +\infty,$$

for some $r \geq 0$.

Suppose that $r > 0$, taking the upper limit in (3.37) as $n \to +\infty$, using continuity of ψ and the lower semi-continuity of ϕ, we have

$$\begin{aligned}
&\limsup_{n\to+\infty} \psi(d(x_{n+1}, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) \\
&\quad \leq \limsup_{n\to+\infty} \ \psi(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})) \\
&\qquad - \liminf_{n\to+\infty} \ \phi(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})),
\end{aligned}$$

which implies that

$$\begin{aligned}
\psi(r) &\leq \psi(r) - \liminf_{n\to+\infty} \ \psi(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})) \\
&\leq \psi(r) - \phi(r) \\
&\leq \psi(r),
\end{aligned}$$

a contradiction. Thus

$$\lim_{n\to+\infty} d(x_{n+1}, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}) = 0,$$

from which we have

$$\begin{aligned}
\lim_{n\to+\infty} d(x_{n+1}, x_{n+1}) &= \lim_{n\to+\infty} d(x_{n+1}, x_{n+2}) \\
&= 0
\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
\lim_{n\to+\infty} \varphi(x_{n+1}) &= \lim_{n\to+\infty} \varphi(x_{n+2}) \\
&= 0.
\end{aligned} \tag{3.39}$$

We shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Suppose on the contrary that $\{x_n\}_{n\in\mathbb{N}}$ is not Cauchy. It follows that there exists $\epsilon > 0$ for which one can find sequences $\{x_{m(k)}\}_{k\in\mathbb{N}}$ and $\{x_{n(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ satisfying $n(k)$ as the smallest index for which $n(k) > m(k) > k$,

$$\epsilon \leq d(x_{m(k+1)}, x_{n(k+1)}), \tag{3.40}$$

$$d(x_{m(k)}, x_{n(k-1)}) < \epsilon. \tag{3.41}$$

Using the same technique in the proof of Theorem 3.2, we can deduce that

$$\epsilon \leq \liminf_{k\to+\infty} \ d(x_{m(k)}, x_{n(k)}) \leq \limsup_{k\to+\infty} \ d(x_{m(k)}, x_{n(k)}) \leq s\epsilon, \tag{3.42}$$

$$\frac{\epsilon}{s} \leq \liminf_{k\to+\infty} \ d(x_{m(k-1)}, x_{n(k)}) \leq \limsup_{k\to+\infty} \ d(x_{m(k-1)}, x_{n(k)}) \leq s^2\epsilon, \tag{3.43}$$

$$\frac{\epsilon}{s^2} \leq \liminf_{k\to+\infty} \ d(x_{m(k-1)}, x_{n(k+1)}) \leq \limsup_{k\to+\infty} \ d(x_{m(k-1)}, x_{n(k+1)}) \leq s^3\epsilon, \tag{3.44}$$

$$\frac{\epsilon}{s} \leq \liminf_{k\to+\infty} \ d(x_{m(k)}, x_{n(k+1)}) \leq \limsup_{k\to+\infty} \ d(x_{m(k)}, x_{n(k+1)}). \tag{3.45}$$

Letting $x = x_{n(k)}$ and $y = x_{m(k-1)}$ in (3.31), we obtain

$$\begin{aligned}
&\psi\left(d(x_{n(k+1)}, x_{n(k+1)}) + d(x_{n(k+1)}, x_{m(k)}) + \varphi(x_{n(k+1)}) + \varphi(x_{m(k)})\right)\\
&\psi\left(s^p(d(Tx_{n(k)}, Tx_{n(k)}) + d(Tx_{n(k)}, Sx_{m(k-1)}) + \varphi(Tx_{n(k)}) + \varphi(Sx_{m(k-1)}))\right)\\
&\psi\left(q(x_{n(k)}, x_{m(k-1)}, \varphi)\right) - \phi(\Im(x_{n(k)}, x_{m(k-1)}, \varphi)).
\end{aligned} \tag{3.46}$$

In view of the definition of $q(x, y, \varphi)$, we deduce

$$\begin{aligned}
&q(x_{n(k)}, x_{m(k-1)}, \varphi)\\
&= \max\left\{\begin{array}{c}
d(x_{n(k)}, x_{m(k-1)}) + \varphi(x_{n(k)}) + \varphi(x_{m(k-1)}),\\
\frac{1}{2}[d(Tx_{n(k)}, x_{n(k)}) + \varphi(Tx_{n(k)})\\
+\varphi(x_{n(k)}) + d(x_{m(k-1)}, x_{m(k-1)})\\
+d(x_{m(k-1)}, Sx_{m(k-1)}) + \varphi(x_{m(k-1)}) + \varphi(Sx_{m(k-1)})],\\
\frac{1}{2s}[d(Tx_{n(k)}, x_{m(k-1)}) + \varphi(Tx_{n(k)}) + \varphi(x_{m(k-1)})\\
+d(x_{n(k)}, Sx_{m(k-1)}) + \varphi(x_{n(k)}) + \varphi(Sx_{m(k-1)})]
\end{array}\right\}\\
&= \max\left\{\begin{array}{c}
d(x_{n(k)}, x_{m(k-1)}) + \varphi(x_{n(k)}) + \varphi(x_{m(k-1)}),\\
\frac{1}{2}[d(x_{n(k+1)}, x_{n(k)}) + \varphi(x_{n(k+1)}) + \varphi(x_{n(k)})\\
+d(x_{m(k-1)}, x_{m(k-1)}) + d(x_{m(k-1)}, x_{m(k)})\\
+\varphi(x_{m(k-1)}) + \varphi(x_{m(k)})],\\
\frac{1}{2s}[d(x_{n(k+1)}, x_{m(k-1)}) + \varphi(x_{n(k+1)}) + \varphi(x_{m(k-1)})\\
+d(x_{n(k)}, x_{m(k)}) + \varphi(x_{n(k)}) + \varphi(x_{m(k)})]
\end{array}\right\}
\end{aligned}$$

It follows from (3.38) – (3.45) that

$$\limsup_{k\to\infty} q(x_{n(k)}, x_{m(k-1)}, \varphi) \leq \max\left\{s^2\epsilon, 0, \frac{s^3\epsilon + s\epsilon}{s}\right\} \tag{3.47}$$

$$\leq s^2\epsilon. \tag{3.48}$$

Also, we have,

$$\begin{aligned}
&\Im(x_{n(k)}, x_{m(k-1)}, \varphi)\\
&\quad = \max\left\{\begin{array}{c} d(x_{n(k)}, x_{m(k-1)}) + \varphi(x_{n(k)}) + \varphi(x_{m(k-1)}),\\ d(x_{m(k-1)}, x_{m(k-1)}) + d(x_{m(k-1)}, Sx_{m(k-1)})\\ +\varphi(x_{m(k-1)}) + \varphi(Sx_{m(k-1)}) \end{array}\right\}\\
&\quad = \max\left\{\begin{array}{c} d(x_{n(k)}, x_{m(k-1)}) + \varphi(x_{n(k)}) + \varphi(x_{m(k-1)}),\\ d(x_{m(k-1)}, x_{m(k-1)}) + d(x_{m(k-1)}, x_{m(k)})\\ +\varphi(x_{m(k-1)}) + \varphi(x_{m(k)}) \end{array}\right\}.
\end{aligned}$$

It follow from (3.38), (3.39) and (3.43) that

$$\frac{\epsilon}{s} \leq \liminf_{k\to+\infty} d(x_{n(k)}, x_{m(k-1)}) \leq s^2\epsilon. \tag{3.49}$$

By virtue of (3.46), (3.47) and (3.49)

$$\begin{aligned}
\psi(s^2\epsilon) &= \psi\left(s^3\frac{\epsilon}{s}\right)\\
&\leq \psi(s^p(\limsup_{k\to+\infty} d(x_{n(k+1)}, x_{n(k+1)})\\
&\quad + d(x_{n(k+1)}, x_{m(k)}) + \varphi(x_{n(k+1)}) + \varphi(x_{m(k)})))\\
&\leq \psi(\limsup_{k\to+\infty} q(x_{n(k)}, x_{m(k-1)}, \varphi)) - \liminf_{k\to+\infty} \phi(\Im(x_{n(k)}, x_{m(k-1)}, \varphi))\\
&\leq \psi(s^2\epsilon) - \liminf_{k\to+\infty} \phi(\Im(x_{n(k)}, x_{m(k-1)}, \varphi)),
\end{aligned} \tag{3.50}$$

which implies that

$$\liminf_{k\to+\infty} \Im(x_{n(k)}, x_{m(k-1)}, \varphi) = 0 \tag{3.51}$$

a contradiction to (3.49). Hence, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. The completeness of X ensures that there exists a $x^* \in X$ such that

$$\begin{aligned}\lim_{n\to+\infty} Tx_n &= \lim_{n\to+\infty} Sx_{(n+1)}\\ &= x^*.\end{aligned} \tag{3.52}$$

By the definition of φ , we deduce that

$$\varphi(x^*) \leq \liminf_{n\to\infty} \varphi(x_n) = 0. \tag{3.53}$$

Now, we will show that if one of the mappings T and S is continuous, then $T(x^*) = S(x^*) = x^*$.

Without loss of generality, we suppose that T is continuous. It follows from (3.52) that

$$x^* = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = T(x^*). \tag{3.54}$$

That is x^* is a fixed point of T. From the contractive conditions (3.31), we get

$$\begin{aligned} &\psi(d(x^*,x^*) + d(x^*,Sx^*) + \varphi(x^*) + \varphi(Sx^*)) \\ &\quad \leq \psi(s^p(d(Tx^*,fx^*) + d(Tx^*,Sx^*) + \varphi(x^*) + \varphi(Sx^*))) \\ &\quad \leq \psi(q(x^*,x^*,\varphi)) - \phi(\Im(x^*,x^*,\varphi)), \end{aligned} \tag{3.55}$$

where,

$$\begin{aligned} &q(x^*,x^*,\varphi) \\ &= \max\left\{\begin{array}{c} d(x^*,x^*) + \varphi(x^*) + \varphi(x^*), \\ \frac{1}{2}[d(x^*,x^*) + \varphi(x^*) + \varphi(x^*) \\ +d(x^*,Sx^*) + \varphi(x^*) + \varphi(Sx^*) + d(x^*,x^*)], \\ \frac{1}{2s}[d(x^*,x^*) + \varphi(x^*) + \varphi(x^*) + d(x^*,Sx^*) + \varphi(x^*) + \varphi(Sx^*)] \end{array}\right\} \\ &= \max\left\{\begin{array}{c} d(x^*,x^*), \frac{1}{2}[d(x^*,x^*) + d(x^*,Sx^*) + \varphi(x^*) \\ +\varphi(Sx^*) + d(x^*,x^*)] \frac{1}{2s}[d(x^*,x^*) + d(x^*,Sx^*) + \varphi(x^*)] \end{array}\right\} \\ &= \max\left\{d(x^*,x^*), \frac{1}{2}[d(x^*,x^*) + d(x^*,Sx^*) + \varphi(x^*) + \varphi(Sx^*) + d(x^*,x^*)]\right\} \\ &\leq \max\{d(x^*,x^*), \max\{d(x^*,x^*), d(x^*,Sx^*) + \varphi(x^*) + \varphi(Sx^*) + d(x^*,x^*)\}\} \\ &= \max\{d(x^*,x^*) + d(x^*,Sx^*) + \varphi(Sx^*)\} \end{aligned}$$

and

$$\begin{aligned} \Im(x^*,x^*,\varphi) &= \max\{d(x^*,x^*), d(x^*,x^*) + d(x^*,Sx^*) + \varphi(x^*) + \varphi(Sx^*)\} \\ &= \max\{d(x^*,x^*) + d(x^*,Sx^*) + \varphi(Sx^*)\}. \end{aligned}$$

It follows from (3.55) that,

$$\begin{aligned} &\psi(d(x^*,x^*) + d(x^*,Sx^*) + \varphi(Sx^*)) \\ &\quad \leq \psi(d(x^*,x^*) + d(x^*,Sx^*) + \varphi(Sx^*)) \\ &\qquad - \phi(d(x^*,x^*) + d(x^*,Sx^*) + \varphi(Sx^*)). \end{aligned}$$

Hence,

$$d(x^*,x^*) + d(x^*,Sx^*) + \varphi(Sx^*) = 0$$

This implies that, $d(x^*,x^*) = \varphi(Sx^*) = 0$ and $x^* = Sx^*$. Therefore, x^* is a common fixed point of T and S.

Next, we want to show that T and S have a unique common fixed point. Suppose that v is a fixed point of T, w is a fixed point of S and $v \neq w$. Then $T(v) = v \neq w = S(w)$. It follows that $d(T(v),S(w)) = d(v,w) > 0$. Applying (3.31) with $x = v$ and $y = w$, we obtain

$$\psi(d(v,v) + d(v,w) + \varphi(v) + \varphi(w) \tag{3.56}$$

$$
\begin{aligned}
&\leq \psi(s^p(d(Tv,Tv)+d(Tv,Sw)+\varphi(Tv)+\varphi(Sw)))\\
&\leq \psi(q(v,w,\varphi))-\phi(\Im(v,w,\varphi)),
\end{aligned}
$$

where,

$$
\begin{aligned}
&q(v,w,\varphi)\\
&= \max\left\{\begin{array}{c} d(v,w)+\varphi(v)+\varphi(w),\\ \frac{1}{2}[d(Tv,v)+\varphi(Tv)+\varphi(v)+d(w,Sw)\\ +\varphi(w)+\varphi(Sw)+d(w,w)],\\ \frac{1}{2s}[d(Tv,w)+\varphi(Tv)+\varphi(w)+d(v,Sw)+\varphi(w)+\varphi(Sw)] \end{array}\right\}\\
&= \max\left\{\begin{array}{c} d(v,w)+\varphi(v)+\varphi(w),\\ \frac{1}{2}[d(v,v)+\varphi(v)+\varphi(v)+d(w,w)+\varphi(w)+\varphi(w)+d(w,w)\\ \frac{1}{2s}[d(v,w)+\varphi(v)+\varphi(w)+d(v,w)+\varphi(v)+\varphi(w)] \end{array}\right\}\\
&= \max\left\{\begin{array}{c} d(v,w)+\varphi(v)+\varphi(w),\\ \frac{1}{2}[d(v,v)+2\varphi(v)+2d(w,w), \frac{1}{2s}[2d(v,w)+2\varphi(v)+2\varphi(w)] \end{array}\right\}\\
&= \max\left\{\begin{array}{c} d(v,w)+\varphi(v)+\varphi(w), \frac{1}{2}d(v,v)+\varphi(v)+d(w,w)+\varphi(w),\\ \frac{1}{s}[d(v,w)+\varphi(v)+\varphi(w)] \end{array}\right\}\\
&= \max\left\{d(v,w)+\varphi(v)+\varphi(w), \frac{1}{2}[d(v,v)+\varphi(v)+d(w,w)+\varphi(w)]\right\}\\
&\leq \max\left\{\begin{array}{c} s(d(w,w)+d(w,v))+\varphi(v)+\varphi(w), sd(v,w)\\ +\varphi(v)+d(w,w)+\varphi(w) \end{array}\right\}\\
&\leq \max\left\{s(d(w,w)+d(w,v))+\varphi(v)+\varphi(w)\right\}.
\end{aligned}
$$

and

$$
\begin{aligned}
\Im(v,w,\varphi) &= \max\left\{\begin{array}{c} d(v,w)+\varphi(v)+\varphi(w),\\ d(w,Sw)+\varphi(w)+\varphi(Sw)+d(w,w), \end{array}\right\}\\
&= \max\left\{\begin{array}{c} d(v,w)+\varphi(v)+\varphi(w),\\ d(w,w)+\varphi(w)+\varphi(w)+d(w,w) \end{array}\right\}\\
&= \max\left\{\ d(v,w)+\varphi(v)+\varphi(w), 2\varphi(w)\ \right\}\\
&\geq \max\left\{\ d(v,w)+\varphi(v)+\varphi(w)\ \right\}.
\end{aligned}
$$

It follows from (3.56) that

$$
\begin{aligned}
&\phi(d(v,w)+\varphi(v)+\varphi(w))\\
&\quad\leq \psi(s(d(w,w)+d(v,w))+\varphi(v)+\varphi(w))\\
&\qquad-\psi(d(v,v)+d(v,w)+\varphi(v)+\varphi(w))\\
&\quad\leq \psi(s(d(w,w)+d(v,w))+\varphi(v)+\varphi(w))\\
&\qquad-\psi((d(v,v)+s(d(w,w)+d(w,v))+\varphi(v)+\varphi(w))\\
&\quad\leq \psi(d(v,v)+s(d(w,w)+d(v,w))+\varphi(v)+\varphi(w))\\
&\qquad-\psi((d(v,v)+s(d(w,w)+d(w,v))+\varphi(v)+\varphi(w)),
\end{aligned}
$$

which implies that $d(v,w)+\varphi(v)=0$. That is, $v=w$ and $\varphi(v)=0$. Hence the pair (T,S) has a unique common fixed point. This completes the proof. $\square$

Corollary 3.9. *Let (X,d_σ) be a complete b-metric-like space, $T,S : X \longrightarrow X$ be a given self-mappings, and one of T and S is continuous with constant $s \geq 1$. Suppose $p \geq 3$ is a constant such that*

$$\psi(s^p(d(Tx,Tx)+d(Tx,Sy))) \leq \psi(q(x,y)) - \phi(r(x,y)), \tag{3.57}$$

for all $x,y \in X$, where $\psi \in \Psi$, $\phi \in \Phi$ and

$$q_1(x,y) = \max\left\{ \begin{array}{c} d(x,y), \frac{1}{2}[d(Tx,x)+d(y,y)+d(y,Sy)], \\ \frac{1}{2}[d(Tx,y)+d(x,Sy)] \end{array} \right\}$$

and

$$\Im_1(x,y) = \max\{d(x,y), d(y,y)+d(y,Sy)\}.$$

Then T and S have a unique common fixed point in X.

Proof. Put $\varphi(t)=0$ for all $t \in \mathbb{R}_+$ in Theorem 3.8. $\square$

The following is a version of Theorem 3.8 in the setting of metric-like space with $s=1$ in Theorem 3.8.

Corollary 3.10. *Let (X,d_b) be a complete metric-like space, $T,S : X \longrightarrow X$ be a given self-mappings, and one of T and S is continuous with constant $s \geq 1$. Suppose $\varphi : X \longrightarrow [0,\infty)$ is a lower semi-continuous function such that*

$$\begin{aligned} &\psi((d(Tx,Tx)+d(Tx,Sy)+\varphi(Tx)+\varphi(Sx)) \\ &\quad \leq \psi(q(x,y,\varphi)) - \phi(\mathcal{L}(x,y,\varphi)), \end{aligned}$$

for all $x,y \in X$, where $\psi \in \Psi$, $\phi \in \Phi$ and $q(x,y,\varphi)$, $\mathcal{L}(x,y,\varphi)$ are the same as in Theorem 3.8 and Theorem 3.2 respectively. Then T and S have a unique common fixed point in X.

Corollary 3.11. *Let (X,d_σ) be a b-complete metric-like space, $T : X \longrightarrow X$ be a given self-mappings with constants $s \geq 1$ and let $\varphi : X \longrightarrow [0,\infty)$ be a lower semi-continuous function with $\varphi(t)=0$ if and only if $t=0$. Suppose $p \geq 2$ such that*

$$\begin{aligned} &\psi\left(s^p(d(Tx,Tx)+d(Tx,Ty)+\varphi(Tx)+\varphi(Ty))\right) \\ &\quad \leq \psi(j(x,y,\varphi)) - \phi(k(x,y,\varphi)), \end{aligned}$$

for all $x,y \in X$, where $\psi \in \Psi$, $\phi \in \Phi$ and

$$j(x,y,\varphi) = \max\left\{ \begin{array}{c} d(x,y)+\varphi(x)+\varphi(y), \\ \frac{1}{2}[d(Tx,x)+\varphi(Tx)+\varphi(x) \\ +d(y,y)+d(Ty,Sy)+d(y,Ty)+\varphi(y)+\varphi(Ty)], \\ \frac{1}{2s}[d(Tx,y)+\varphi(Tx)+\varphi(y)+d(x,Ty)+\varphi(y)+\varphi(Ty)] \end{array} \right\}$$

and

$$k(x,y,\varphi) = \max\{d(x,y)+\varphi(x)+\varphi(y), d(y,y)+d(y,Ty)+\varphi(y)+\varphi(Ty)]\}$$

Then T have a unique common fixed point in X.

Proof. Take $T = S$ in Theorem 3.8 ☐

Corollary 3.12. *Let (X, d_σ) be a complete b- metric-like space with $s \geq 1$, $T : X \longrightarrow X$ be a given self-mappings and let $\varphi : X \longrightarrow [0,\infty)$ be a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Suppose $p \geq 2$ is a constant such that*

$$\begin{aligned}&\psi\left(s^p(d(Tx,Tx)+d(Tx,Ty)+\varphi(Tx)+\varphi(Ty))\right)\\&\quad\leq \psi(j(x,y,\varphi)) - \phi(j(x,y,\varphi)),\end{aligned}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\phi \in \Phi$ and $j(x,y,\varphi)$ is the same as in Corollary 3.11. Then T has a unique fixed point in X.

4. Application to the Existence of Solutions of Integral Equations

Lipschitzian-type, fixed point results play a vital role in the existence theory of various class of equations. Not long ago, Karapinar et al. [21] applied some fixed point results to study new conditions for the existence of a solution to an ordinary differential equation. Meanwhile, many authors (e.g., see [11–13]) have proposed different techniques for analyzing solvability conditions of either integer or non-integer order differential equations. Motivated by the work in [3, 14], we study the existence of solutions to a nonlinear integral equation using Corollary 3.5. To this effect, consider the integral equation:

$$x(t) = \int_0^\zeta \gamma(t,r)\chi(r,x(r))dr, \quad t \in [0,\zeta]. \tag{4.1}$$

The integral equation (4.1) is equivalent to the following integral equation:

$$x(t) = \int_0^\zeta K(t,r,x(r))dr. \tag{4.2}$$

Let $X = C[0,\zeta]$ be the set of continuous real-valved functions defined on $[0,\zeta]$. Define a function $d_\sigma : X \times X \longrightarrow \mathbb{R}_+$ by

$$d_\sigma(x,y) = \sup_{t\in[0,\zeta]} |x(t)+y(t)|^p, \tag{4.3}$$

for all $x, y \in X$. Then clearly, (X, d_σ) is a complete b-metric-like space with $p \geq 1$ and $s = 2^{p-1}$.

Consider the self-mapping $T : X \longrightarrow X$ defined by

$$Tx(t) = \int_0^\zeta K(t, r, x(r))dr. \tag{4.4}$$

The following result discusses the existence of solution to the integral equation (4.2):

Theorem 4.1. *Suppose in the integral equation* (4.2) *that the following conditions are satisfied*

- *(i)* $K : [0, \zeta] \times [0, \zeta] \times \mathbb{R} \longrightarrow \mathbb{R}_+$ *is continuous;*
- *(ii) there exists a continuous function* $\gamma : [0, \zeta] \times [0, \zeta] \longrightarrow \mathbb{R}^+$ *such that*

$$\sup_{t\in[0,\zeta]} \int_0^\zeta K(t, r, x(r))dr \tag{4.5}$$

- *(iii) there exists a constant* $L \in (0, 1)$ *such that for any* $t, r \in [0, T]$,

$$|K(t, r, x(r)) + K(t, r, y(r))| \le \sqrt[p]{\frac{1 - L}{s^p}}\gamma(t, r)|x(r) + y(r)|. \tag{4.6}$$

Then the integral equation (4.2) *has a unique solution* u^* *in* X.

Proof. For any $x, y \in X$, given assumptions $(i) - (iii)$, we have from (4.3) that

$$\begin{aligned}
d(Tx, Ty) &= \sup_{t\in[0,\zeta]} |Tx(t) + Ty(t)|^p \\
&= \sup_{t\in[0,\zeta]} \left|\int_0^\zeta K(t, r, x(r)) + \int_0^\zeta K(t, r, x(r))dr\right|^p \\
&\le \sup_{t\in[0,\zeta]} \left(\int_0^\zeta |K(t, r, x(r)) + K(t, r, x(r))|\, dr\right)^p \\
&\le \sup_{t\in[0,\zeta]} \left(\int_0^\zeta \sqrt[p]{\frac{1 - L}{s^p}}\gamma(t, r)|x(r) + y(r)|dr\right)^p \\
&\le \frac{1 - L}{s^p} d_b(x, y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi(s^p(d(Tx, Tx) + d(Tx, Ty))) &\le s^p \left(\frac{1 - L}{s^p} d(x, x) + \frac{1 - L}{s^p} d(x, y)\right) \\
&= (1 - L)[d(x, x) + d(x, y)] \\
&= (d(x, x) + d(x, y)) - L(d(x, x) + d(x, y)) \\
&\le \psi(\mathcal{M}_2(x, y)) - \phi(\mathcal{L}_2(x, y)),
\end{aligned}$$

where, $\psi(t) = t$ and $\phi(t) = Lt$. Hence, all the hypotheses of Corollary 3.5 are satisfied with $S = I_X$, the identity mapping on X and $\varphi(t) = 0$, for all $t \in \mathbb{R}^+$. This implies that there exists a unique fixed point u^* in X such that $Tu^* = u^*$ and u^* is a solution of the integral equation (4.2). ☐

Conclusion

This manuscript introduced a new concept of generalized quasi-weakly contractive operator in b-metric-like space, and the existence and uniqueness of fixed points of such operators were investigated. The proposed ideas herein were supported with non-trivial comparative example, demonstrating genuine generalizations of a few concepts in the literature. As an application, one of our obtained results was applied to study new criteria for the existence of solutions to a particular class of boundary value problems.

5. Competing Interests

The authors declare that they have no competing interests.

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References

1. Y. Aber and D.S. Guerre, *Principles of weakly contractive maps in Hilbert spaces*, I Lyubich, Yu (eds), Oper. Theory Adv. Appl., 98, (2011) pp. 7-12.
2. A. Aghaiani, M. Abbas and J.R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces*, Math. Slovaca, 2014 (2014), pp. 941-960.
3. M.A. Alghamdi, N. Hussain and P. Salimi, *Fixed point and coupled fixed point theorem on b-metric-like spaces*, J. Inequal. Appl., 2013 (2013).
4. A. Amini-Harandi, *Metric-like spaces, partial metric spaces and fixed points*, Fixed Point Theory Appl., 2012 (2012).
5. I. Bakhtin, *The contraction mapping principle in quasi metric spaces*, Funct. Anal. Appl., 30 (1989), pp. 26-37.
6. K.P. Chi, E. Karapinar and T.D. Thanh, *On the fixed point theorems for generalized weakly contractive mappings on partial metric spaces*, Bull. Iranian Math. Soc., 39 (2) (2013), pp. 369-381.
7. S. Cho, *Fixed point theorems for generalized weakly contractive mappings in metric spaces with application*, Fixed Point Theory Appl., 2018 (3) (2012).

8. L.B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., 45 (2) (1994), pp. 267-273.
9. S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Univ. Ostrav., 1 (1) (1993), pp. 5-11.
10. S. Gähler, *2-metrische Räume und ihre topologische Struktur*, Math. Nachr., 26 (1-4) (1963), pp. 115-148.
11. R. George, M. Houas, M. Ghaderi, S. Rezapour and S.K. Elagan, *On a coupled system of pantograph problem with three sequential fractional derivatives by using positive contraction-type inequalities*, Results in Phys., 39 (2022), pp. 105-687.
12. R. George, F. Al-shammari, M. Ghaderi and S. Rezapour, *On the boundedness of the solution set for the ψ-Caputo fractional pantograph equation with a measure of non-compactness via simulation analysis*, AIMS Math., 8 (9) (2023), pp. 20125-20142.
13. Z. Heydarpour, M.N. Parizi, R. Ghorbnian, M. Ghaderi, S. Rezapour and A. Mosavi, *A study on a special case of the Sturm-Liouville equation using the Mittag-Leffler function and a new type of contraction*, AIMS Math., 7 (10) (2022), pp. 18253-18279.
14. Y. Hoa and H. Guan, *On some common fixed point results for weakly contraction mappings with application*, J. Funct. Spaces, 2021 (2021), pp. 14.
15. L.G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., 332 (2) (2007), pp. 1468-1476.
16. N. Hussain, J.R. Roshan, V. Parvaneh and Z. Kadelburg, *Fixed points of contractive mappings in b-metric-like spaces*, Sci. World J., (2014), pp. 15.
17. G. Jungck, *Compartible mappings and common fixed points*, Int. J. Math. Sci., 9, (1986), pp. 771-779.
18. S.G. Matthews, *Partial metric topology*, Annals New York Acad. Sci., 728 (1994), pp. 183-197.
19. Z. Mustaf and B. Sims, *A New Approach to Generalized Metric Spaces*, J. Nonlinear Convex Anal., 7 (2) (2006), pp. 289-297.
20. M. Noorwali and M.S. Shagari, *On two-point boundary value problems and fractional differential equations via new quasi-contractions*, Math., 11 (1) (2023), pp. 2477.
21. E. Karapınar, T. Abdeljawad and F. Jarad, *Applying new fixed point theorems on fractional and ordinary differential equations*, Adv.Differ Equ., 2019 (1) (2019), pp. 1-25.
22. S. Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterr. J. Math., (2013).

23. W. A. Wilson, *On quasi-metric spaces*, Amer. J. Math., 53 (1931), pp. 675-684.

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