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ORIGINAL PAPER

Fixed Point Results of Interpolative Kannan-Type Contraction in Generalized Metric Space

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Abstract
In this manuscript, a new concept of interpolative contraction, namely interpolative Kannan-type (G-α-μ)-contraction is introduced and some fixed point results in generalized metric space that are not deducible from their akin in metric space are obtained. The preeminence of this class of contractions is that it complements and subsumes a few corresponding notions in the literature. Consequently, substantial examples are constructed to validate the assumptions of our obtained theorems and to show their distinction from corresponding results. As an application, we examine Ulam-type stability and well-posedness for the new contraction proposed herein.

Keywords G-metric · Fixed point · Kannan-type contraction · Interpolation · Ulam-type stability

Mathematics Subject Classification 47H10 · 4H25 · 46L07

Introduction

Following the introduction of the well-known Banach contraction principle, investigation of the existence and uniqueness of fixed points (FP) of contraction mappings in the framework of metric spaces (ms) is one of the centres of interest in linear and nonlinear functional analysis, given its important applications in applied mathematics, engineering and social sciences.

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Several extensions of Banach contraction have been obtained over the years by either generalizing the contractive conditions, introducing additional algebraic structures or altering the metric structures of the underlying space (see e.g [5, 6, 9]). In this connection, Kannan [18] proposed a new generalization of Banach contraction principle in ms, which is characterized by the completeness of underlying ms and does not necessarily require a continuous mapping. Existence and uniqueness of FP of a self-mapping was thereafter, investigated in that space. Some recent fixed point results involving other well-known generalizations of Banach contraction principle can be found in [2, 3, 27, 31] and the reference therein.

By modifying the defining structures of ms, Mustafa [24] pioneered an extension of ms by the name, generalized ms (or more precisely, G-ms) and proved some FP results for Banach-type contraction mappings. This new generalization was brought to spotlight by Mustafa and Sims [25]. Subsequently, Mustafa et al. [25] and several other authors (see, e.g. [1, 4, 8, 23]) obtained some engrossing FP results for Lipschitzian-type mappings on G-ms. However, Jleli and Samet [17], as well as Samet et al. [29] noted that most of the FP results in G-ms are direct consequences of existence results in corresponding ms. Jleli and Samet [17] further observed that if a G-metric is consolidated into a quasi-metric, then the resultant FP results become the known FP results in the setting of quasi-ms. Motivated by the latter observation, many investigators (see, e.g. [7, 13–15, 19]) have established techniques of obtaining FP results in G-ms that are not deducible from their equivalents in ms or quasi-ms.

Recently, Karapınar [20] published a new type of contraction obtained from the definition of Kannan contraction by means of interpolation. This interpolative method has been used by several researchers to obtain generalizations of other forms of contractions (see e.g., [10–12, 28, 30]).

It is noted from the review of existing literature that FP results of interpolative contractions in G-ms in the manner proposed by Karapınar [20] have not been adequately investigated. Hence, motivated by the ideas in [7, 13, 19, 20], we introduce a new concept of interpolative Kannan-type (G-α-μ)-contraction in G-ms and prove some corresponding FP theorems. An example is constructed to demonstrate that our result is valid and the main ideas obtained herein do not reduce to any existence result in ms. Finally, Ulam-type stability and well-posedness of this type of interpolative contraction in G-ms are established.

Preliminaries

In this section, we present some fundamental notations and results that will be deployed subsequently.

Throughout, every set Φ is considered non-empty, $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ represents the set of real numbers and $\mathbb{R}_+$, the set of non-negative real numbers.

Definition 1 [25] Let Φ be a non-empty set and let $G : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be a function satisfying:

(G_1) $G(r, s, t) = 0$ if $r = s = t$;
(G_2) $0 < G(r, r, s)$ for all $r, s \in \Phi$ with $r \neq s$;
(G_3) $G(r, r, s) \leq G(r, s, t)$, for all $r, s, t \in \Phi$ with $t \neq s$;
(G_4) $G(r, s, t) = G(r, t, s) = G(s, r, t) = \cdots$ (symmetry in all three variables);
(G_5) $G(r, s, t) \leq G(r, a, a) + G(a, s, t)$, for all $r, s, t, a \in \Phi$ (rectangle inequality).

Then the function G is called a generalized metric, or more specifically, a G-metric on Φ, and the pair (Φ, G) is called a G-ms.

Example 1 [25] Let (Φ, d) be a usual ms, then (Φ, G_j) and (Φ, G_m) are G-ms, where

$$G_j(r, s, t) = d(r, s) + d(s, t) + d(r, t) \quad \forall r, s, t \in \Phi. \tag{1}$$

$$G_m(r, s, t) = \max\{d(r, s), d(s, t), d(r, t)\} \quad \forall r, s, t \in \Phi. \tag{2}$$

Definition 2 [25] Let (Φ, G) be a G-ms and let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in Φ. Then $\{r_n\}_{n\in\mathbb{N}}$ is G-convergent to r if $\lim_{n,m\to\infty} G(r, r_n, r_m) = 0$; that is, for any $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that $G(r, r_n, r_m) < \epsilon$, $\forall n, m \geq n_0$. We refer to r as the limit of the sequence $\{r_n\}_{n\in\mathbb{N}}$.

Proposition 1 *[25] Let (Φ, G) be a G-ms. Then the following are equivalent:*

(i) $\{r_n\}_{n\in\mathbb{N}}$ *is G-convergent to r.*
(ii) $G(r, r_n, r_m) \longrightarrow 0$, *as* $n, m \to \infty$.
(iii) $G(r_n, r, r) \longrightarrow 0$, *as* $n \to \infty$.
(iv) $G(r_n, r_n, r) \longrightarrow 0$, *as* $n \to \infty$.

Definition 3 [25] Let (Φ, G) be a G-ms. A sequence $\{r_n\}_{n\in\mathbb{N}}$ is G-Cauchy if given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(r_n, r_m, r_l) < \epsilon$, $\forall n, m, l \geq n_0$, that is, $G(r_n, r_m, r_l) \longrightarrow 0$, as $n, m, l \to \infty$.

Proposition 2 *[25] In a G-ms (Φ, G), the following are equivalent:*

(i) The sequence $\{r_n\}_{n\in\mathbb{N}}$ *is G-Cauchy.*
(ii) For every $\epsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $G(r_n, r_m, r_m) < \epsilon$, $\forall n, m \geq n_0$.

Definition 4 [25] Let (Φ, G) and (Φ', G') be two G-ms and let $f : (\Phi, G) \longrightarrow (\Phi', G')$ be a function. Then f is said to be G-continuous at a point $a \in \Phi$ if and only if given $\epsilon > 0$, there exists $\delta > 0$ such that $r, s \in \Phi$ and $G(a, r, s) < \delta$ implies $G'(f(a), f(r), f(s)) < \epsilon$. A function f is G-continuous on Φ if and only if it is G-continuous at all $a \in \Phi$.

Proposition 3 *[25] Let (Φ, G) and (Φ', G') be two G-ms. Then a function $f : (\Phi, G) \longrightarrow (\Phi', G')$ is said to be G-continuous at a point $r \in \Phi$ if and only if it is G-sequentially continuous at r. That is, whenever $\{r_n\}_{n\in\mathbb{N}}$ is G-convergent to r, $\{f r_n\}_{n\in\mathbb{N}}$ is G-convergent to fr.*

Definition 5 [25] A G-ms (Φ, G) is called symmetric G-ms if

$$G(r, r, s) = G(s, r, r) \quad \forall r, s \in \Phi.$$

Proposition 4 *[25] Let (Φ, G) be a G-ms. Then the function $G(r, s, t)$ is jointly continuous in all three of its variables.*

Proposition 5 *[25] Every G-ms (Φ, G) defines a ms (Φ, d_G) by*

$$d_G(r, s) = G(r, s, s) + G(s, r, r), \quad \forall r, s \in \Phi. \tag{3}$$

Note that if (Φ, G) is a symmetric G-ms, then

$$(\Phi, d_G) = 2G(r, s, s), \quad \forall r, s \in \Phi. \tag{4}$$

However, if (Φ, G) is not symmetric, then it holds by the G-metric properties that

$$\frac{3}{2}G(r, s, s) \leq d_G(r, s) \leq 3G(r, s, s), \quad \forall r, s \in \Phi, \tag{5}$$

and that in general, these inequalities are sharp.

Definition 6 [25] A G-ms (Φ, G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (Φ, G) is G-convergent in (Φ, G).

Proposition 6 *[25] A G-ms (Φ, G) is G-complete if and only if (Φ, d_G) is a complete ms.*

Karapınar [20] obtained the following result for interpolative Kannan-type contraction in ms.

Definition 7 [20] Let (Φ, d) be a ms. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called an interpolative Kannan-type contraction if there exist $\mu \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(\Gamma r, \Gamma s) \leq \mu d(r, \Gamma r)^{\alpha} \cdot d(s, \Gamma s)^{1-\alpha} \tag{6}$$

for all $r, s \in \Phi \backslash Fix(\Gamma)$, where $Fix(\Gamma) = \{r \in \Phi : \Gamma r = r\}$.

Theorem 1 *[20]. Let (Φ, d) be a complete ms and let $\Gamma : \Phi \longrightarrow \Phi$ be an interpolative Kannan-type contraction. Then T has a unique fixed point in Φ.*

However, Karapınar et al. [21] observed that the fixed point obtained in the above Theorem 1 is not necessarily unique. Hence, a robust version of the results in [20] is provided therein. For some extensions of the idea of interpolative contractions in FP theory, we refer to [10–12, 26, 30] and the reference therein.

Main Results

We begin this section by defining the notion of interpolative Kannan-type $(G$-α-$\mu)$-contraction in G-ms.

Definition 8 Let (Φ, G) be a G-ms. A self-mapping $\Gamma : \Phi \longrightarrow \Phi$ is called an interpolative Kannan-type $(G$-α-$\mu)$-contraction if there exist $\mu \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$G(\Gamma r, \Gamma s, \Gamma^2 s) \leq \mu G(r, \Gamma r, \Gamma^2 r)^{\alpha} \cdot G(s, \Gamma s, \Gamma^2 s)^{1-\alpha} \tag{7}$$

for all $r, s \in \Phi \backslash Fix(\Gamma)$, where $Fix(\Gamma) = \{r \in \Phi : \Gamma r = r\}$.

The following is our main result.

Theorem 2 *Let (Φ, G) be a complete G-ms and let $\Gamma : \Phi \longrightarrow \Phi$ be an interpolative Kannan-type $(G$-α-$\mu)$-contraction on (Φ, G). Then Γ has a unique FP in Φ.*

Proof Let $r_0 \in \Phi$ be an arbitrary point and define a sequence $\{r_n\}_{n\in\mathbb{N}}$ in Φ by $r_n = \Gamma^n r_0$. If there exists some $m \in \mathbb{N}$ such that $\Gamma r_m = r_{m+1} = r_m$, then r_m is a FP of Γ, and so the proof is complete. Assume now that $r_n \neq r_{n+1}$ for any $n \in \mathbb{N}$. Since Γ is an interpolative Kannan-type $(G$-α-$\mu)$-contraction, then we have from (7) that

$$\begin{aligned} G(r_n, r_{n+1}, r_{n+2}) &= G(\Gamma r_{n-1}, \Gamma r_n, \Gamma^2 r_n) \\ &\leq \mu G(r_{n-1}, \Gamma r_{n-1}, \Gamma^2 r_{n-1})^{\alpha} \cdot G(r_n, \Gamma r_n, \Gamma^2 r_n)^{1-\alpha} \\ &= \mu G(r_{n-1}, r_n, r_{n+1})^{\alpha} \cdot G(r_n, r_{n+1}, r_{n+2})^{1-\alpha}, \end{aligned}$$

implying that

$$G(r_n, r_{n+1}, r_{n+2})^{\alpha} \leq \mu G(r_{n-1}, r_n, r_{n+1})^{\alpha}, \text{ or}$$

$$G(r_n, r_{n+1}, r_{n+2}) \leq \mu^{\frac{1}{\alpha}} G(r_{n-1}, r_n, r_{n+1}) \leq \mu G(r_{n-1}, r_n, r_{n+1}).$$

Continuing inductively, we obtain

$$G(r_n, r_{n+1}, r_{n+2}) \leq \mu^n G(r_0, r_1, r_2) \quad \forall n \in \mathbb{N}.$$

Now, since

$$G(r_n, r_n, r_{n+1}) \leq G(r_n, r_{n+1}, r_{n+2}) \leq \mu^n G(r_0, r_1, r_2)$$

for all $n \in \mathbb{N}$ with $r_n \neq r_{n+1}$, then for any $n, m \in \mathbb{N}$ with $n < m$ and by rectangle inequality, we have

$$\begin{aligned} G(r_n, r_n, r_m) &\leq G(r_n, r_n, r_{n+1}) + G(r_{n+1}, r_{n+1}, r_{n+2}) + \cdots + G(r_{m-1}, r_{m-1}, r_m) \\ &\leq \mu^n (1 + \mu + \mu^2 + \cdots + \mu^{m-n-1}) G(r_0, r_1, r_2) \\ &= \mu^n \sum_{i=0}^{m-n-1} \mu^i G(r_0, r_1, r_2) \leq \mu^n \sum_{i=0}^{\infty} \mu^i G(r_0, r_1, r_2) \\ &= \frac{\mu^n}{1-\mu} G(r_0, r_1, r_2). \end{aligned}$$

Since $\mu \in [0, 1)$, then taking $m, n \to \infty$, we have

$$G(r_n, r_n, r_m) \longrightarrow 0.$$

Therefore, $\{r_n\}_{n \in \mathbb{N}}$ is a G-Cauchy sequence in (Φ, G) and so by the completeness of (Φ, G), there exists a point $t \in \Phi$ such that $\{r_n\}_{n \in \mathbb{N}}$ is G-convergent to t, that is,

$$\lim_{n \to \infty} G(r_n, r_n, t) = 0.$$

We will now show that t is a FP of Γ. Notice that

$$G(\Gamma t, \Gamma r_n, \Gamma^2 r_n) \leq \mu G(t, \Gamma t, \Gamma^2 t)^{\alpha} \cdot G(r_n, \Gamma r_n, \Gamma^2 r_n)^{1-\alpha},$$

implying that

$$G(\Gamma t, r_{n+1}, r_{n+2}) \leq \mu G(t, \Gamma t, \Gamma^2 t)^{\alpha} \cdot G(r_n, r_{n+1}, r_{n+2})^{1-\alpha}. \tag{8}$$

Since G is continuous, then taking limit as $n \to \infty$ in (8), we have

$$\lim_{n \to \infty} G(\Gamma t, r_{n+1}, r_{n+2}) \leq \mu G(t, \Gamma t, \Gamma^2 t)^{\alpha} \cdot \lim_{n \to \infty} G(r_n, r_{n+1}, r_{n+2})^{1-\alpha},$$

from which we obtain

$$G(\Gamma t, t, t) \leq \mu G(t, \Gamma t, \Gamma^2 t)^{\alpha} \cdot G(t, t, t)^{1-\alpha},$$

which resolves to

$$G(\Gamma t, t, t) = 0.$$

Hence, $\Gamma t = t$, that is, t is a FP of Γ.

To see uniqueness of t, assume contrary that there exists a point $u \in \Phi$ such that $\Gamma u = u$ and $u \neq t$. Then we have by (7) that

$$\begin{aligned} G(\Gamma u, \Gamma t, \Gamma^2 t) &\leq \mu G(u, \Gamma u, \Gamma^2 u)^{\alpha} \cdot G(t, \Gamma t, \Gamma^2 t)^{1-\alpha} \\ &= \mu G(u, u, u)^{\alpha} \cdot G(t, t, t)^{1-\alpha} = 0, \end{aligned}$$

implying that $\Gamma u = \Gamma t$, and so $u = t$, a contradiction. Hence, the FP of Γ is unique. □

Example 2 Let $\Phi = [-1, 1]$ and let $\Gamma : \Phi \longrightarrow \Phi$ be a self-mapping on Φ defined by

$$\Gamma r = \begin{cases} \frac{r}{5}, & \text{if} \quad r \in \{-1, 1\}; \\ \frac{1}{5}, & \text{if} \quad r \in (-1, 1) \end{cases}$$

for all $r \in \Phi$. Define $G : \Phi \times \Phi \times \Phi \longrightarrow \mathbb{R}_+$ by

$$G(r, s, \Gamma s) = |r - s| + |r - \Gamma s| + |s - \Gamma s| \quad \forall\, r, s \in \Phi.$$

Then (Φ, G) is a complete G-ms.
To see that Γ is an interpolative Kannan-type $(G$-α-$\mu)$-contraction, notice that $G(\Gamma r, \Gamma s, \Gamma^2 s) = 0$ for all $r, s \in (-1, 1)$. Hence, inequality (7) holds for all $r, s \in (-1, 1)$. Now, for $r, s \in \{-1, 1\}$, if $r = s = 1$, then $G(\Gamma r, \Gamma s, \Gamma^2 s) = 0$, thereby satisfying inequality (7). If $r = s = -1$, then letting $\mu = \frac{1}{2}$ and $\alpha = \frac{3}{5}$, we obtain

$$\begin{aligned} G(\Gamma r, \Gamma s, \Gamma^2 s) &= G\left(\frac{-1}{5}, \frac{-1}{5}, \frac{1}{5}\right) \\ &= \frac{4}{5} < \frac{6}{5} = \frac{1}{2}\left(\frac{12}{5}\right) \\ &= \frac{1}{2}\left(G\left(-1, \frac{-1}{5}, \frac{1}{5}\right)^{\frac{3}{5}} \cdot G\left(-1, \frac{-1}{5}, \frac{1}{5}\right)^{\frac{2}{5}}\right) \\ &= \mu G(r, \Gamma r, \Gamma^2 r)^{\alpha} \cdot G(s, \Gamma s, \Gamma^2 s)^{1-\alpha}. \end{aligned}$$

If $r \neq s$, then letting $\mu = \frac{12}{25}$ and $\alpha = \frac{1}{2}$, we obtain

$$\begin{aligned} G(\Gamma r, \Gamma s, \Gamma^2 s) &= G\left(\frac{-1}{5}, \frac{1}{5}, \frac{1}{5}\right) = G\left(\frac{1}{5}, \frac{-1}{5}, \frac{1}{5}\right) \\ &= \frac{4}{5} < \frac{47}{50} = \frac{12}{25}\left(\frac{47}{24}\right) \\ &= \frac{12}{25}\left(G\left(-1, \frac{-1}{5}, \frac{1}{5}\right)^{\frac{1}{2}} \cdot G\left(1, \frac{1}{5}, \frac{1}{5}\right)^{\frac{1}{2}}\right) \\ &= \frac{12}{25}\left(G\left(1, \frac{1}{5}, \frac{1}{5}\right)^{\frac{1}{2}} \cdot G\left(-1, \frac{-1}{5}, \frac{1}{5}\right)^{\frac{1}{2}}\right) \\ &= \mu G(r, \Gamma r, \Gamma^2 r)^{\alpha} \cdot G(s, \Gamma s, \Gamma^2 s)^{1-\alpha}. \end{aligned}$$

Hence, inequality (7) is satisfied for all $r, s \in \Phi$. Therefore, Γ is an interpolative Kannan-type $(G$-α-$\mu)$-contraction which satisfies all the assumptions of Theorem 2 and $r = \frac{1}{5}$ is the FP of Γ.

We now demonstrate that our result is independent of the result of Karapınar [20]. Let $d : \Phi \times \Phi \longrightarrow \mathbb{R}_+$ be defined by

$$d(r, s) = |r - s| \quad \forall\, r, s \in \Phi.$$

Consider $r, s \in \{-1, 1\}$ and take $r \neq s$, $\mu = \frac{11}{25}$ and $\alpha = \frac{1}{2}$. Then inequality (7) becomes

$$\begin{aligned} G(\Gamma r, \Gamma s, \Gamma^2 s) &= G\left(\frac{-1}{5}, \frac{1}{5}, \frac{1}{5}\right) = G\left(\frac{1}{5}, \frac{-1}{5}, \frac{1}{5}\right) \\ &= \frac{4}{5} < \frac{43}{50} = \frac{11}{25}\left(\frac{43}{22}\right) \end{aligned}$$

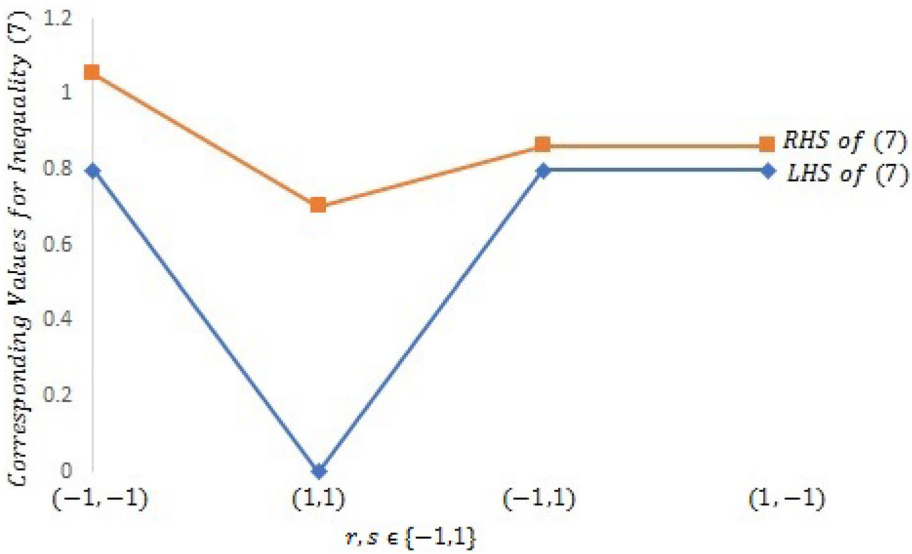

Fig. 1 Illustration of contractive inequality (7) for all $r, s \in \{-1, 1\}$

$$
\begin{aligned}
&= \frac{11}{25}\left(G\left(-1, \frac{-1}{5}, \frac{1}{5}\right)^{\frac{1}{2}} \cdot G\left(1, \frac{1}{5}, \frac{1}{5}\right)^{\frac{1}{2}}\right) \\
&= \frac{11}{25}\left(G\left(1, \frac{1}{5}, \frac{1}{5}\right)^{\frac{1}{2}} \cdot G\left(-1, \frac{-1}{5}, \frac{1}{5}\right)^{\frac{1}{2}}\right) \\
&= \mu G(r, \Gamma r, \Gamma^2 r)^{\alpha} \cdot G(s, \Gamma s, \Gamma^2 s)^{1-\alpha},
\end{aligned}
$$

while inequality (6) due to Karapınar [20] yields

$$
\begin{aligned}
d(\Gamma r, \Gamma s) &= d\left(\frac{-1}{5}, \frac{1}{5}\right) = d\left(\frac{1}{5}, \frac{-1}{5}\right) \\
&= \frac{2}{5} > \frac{9}{25} = \frac{11}{25}\left(\frac{9}{11}\right) \\
&= \frac{11}{25}\left(d\left(-1, \frac{-1}{5}\right)^{\frac{1}{2}} \cdot d\left(1, \frac{1}{5}\right)^{\frac{1}{2}}\right) \\
&= \frac{11}{25}\left(d\left(1, \frac{1}{5}\right)^{\frac{1}{2}} \cdot d\left(-1, \frac{-1}{5}\right)^{\frac{1}{2}}\right) \\
&= \mu d(r, \Gamma r)^{\alpha} \cdot d(s, \Gamma s)^{1-\alpha}.
\end{aligned}
$$

The above comparison is illustrated graphically for all $r, s \in \{-1, 1\}$, using the following Figs. 1 and 2.

Figure 1 above demonstrates that the right-hand side (RHS) of contractive inequality (7) dominates the left-hand side (LHS), that is, $G(\Gamma r, \Gamma s, \Gamma^2 s) \leq \mu G(r, \Gamma r, \Gamma^2 r)^{\alpha} \cdot G(s, \Gamma s, \Gamma^2 s)^{1-\alpha}$ for all $r, s \in \{-1, 1\}$ as defined in Example 2. On the other hand, Fig. 2 above demonstrates that the right-hand side (RHS) of contractive inequality (6) has been

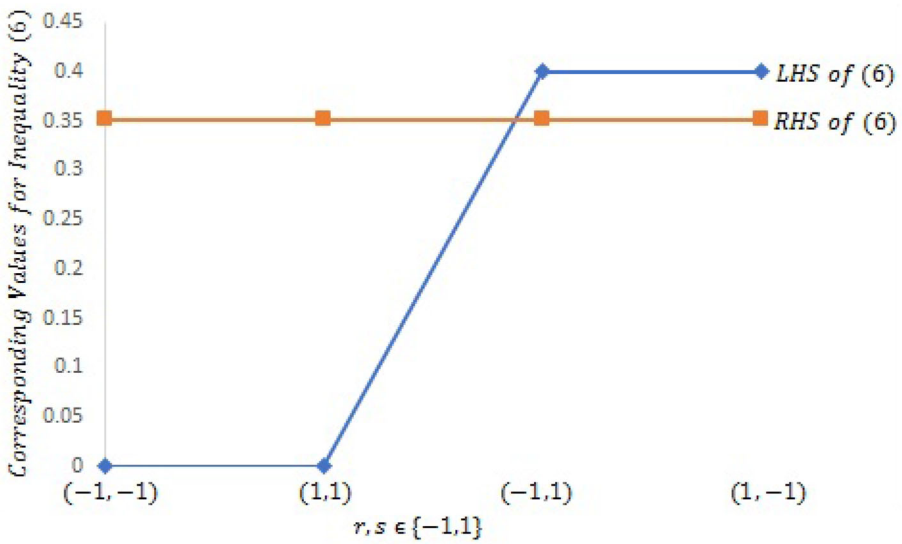

Fig. 2 Illustration of contractive inequality (6) for all $r, s \in \{-1, 1\}$

dominated by the left-hand side (LHS), that is, $d(\Gamma r, \Gamma s) > \mu d(r, \Gamma r)^{\alpha} \cdot d(s, \Gamma s)^{1-\alpha}$ for all $r, s \in \{-1, 1\}$ if $r \neq s$.

Therefore, interpolative Kannan-type (G-α-μ)-contraction is not interpolative Kannan-type contraction defined by Karapınar [20], and so Theorem 2.2 due to Karapınar [20] is not applicable to this example.

Ulam-Type Stability

Ulam introduced Ulam stability, which is thought to be a form of data dependence. Hyers and other scholars developed this idea further. The universal Ulam type stability in the sense of the FP problem in ms was studied by Karapınar and Fulga [22]. In the context of G-ms, Jiddah et al. [16] examine the general Ulam type stability as a FP problem. Here, we study the results of [16] with respect to interpolative Kannan-type (G-α-μ)-contraction.
Given a self-mapping $\Gamma : \Phi \longrightarrow \Phi$ on a G-ms (Φ, G), the FP problem

$$\Gamma r = r \tag{9}$$

has the general Ulam type stability if and only if there exists an increasing function $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, continuous at 0, $\mu(0) = 0$ such that for every $\epsilon > 0$ and for each $s' \in \Phi$ which satisfies the inequality

$$G(s', \Gamma s', \Gamma^2 s') \leq \epsilon, \tag{10}$$

there exists a solution $t \in \Phi$ of (9) such that

$$G(t, s', \Gamma s') \leq \mu(\epsilon). \tag{11}$$

Let $C > 0$ and take $\mu(\lambda) = C\lambda$ for all $\lambda \geq 0$. Then the FP of (9) is said to be Ulam type stable.

On a G-ms (Φ, G), the FP problem (9) is said to be well-posed if the following assumptions are satisfied:

(i) Γ has a unique FP $t \in \Phi$;
(ii) $G(r_n, t, t) = 0$ for each sequence $\{r_n\}_{n\in\mathbb{N}}$ in Φ such that $\lim_{n\to\infty} G(r_n, \Gamma r_n, \Gamma^2 r_n) = 0$.

Theorem 3 *Let (Φ, G) be a complete G-ms which verifies the assumptions of Theorem 2. Then the following hold:*

(i) the FP equation (9) is Ulam-Hyers stable;
(ii) the FP equation (9) is well-posed if for any $\{r_n\}_{n\in\mathbb{N}}$ in Φ $\lim_{n\to\infty} G(r_n, \Gamma r_n, \Gamma^2 r_n) = 0$ and $Fix(\Gamma) = \{t\}$.

Proof (i) In Theorem 2, we have shown that there exists a unique $t \in \Phi$ such that $\Gamma t = t$. Let $s' \in \Phi$ such that for any $\epsilon > 0$, we have

$$G(s', \Gamma s', \Gamma^2 s') \le \epsilon$$

Then obviously, t satisfies (10) and so by rectangular inequality,

$$\begin{aligned} G(t, s', \Gamma s') &\le G(t, \Gamma s', \Gamma^2 s') + G(\Gamma^2 s', s', \Gamma s') \\ &= G(\Gamma t, \Gamma s', \Gamma^2 s') + G(s', \Gamma s', \Gamma^2 s') \\ &\le \mu G(t, \Gamma t, \Gamma^2 t)^{\alpha} \cdot G(s', \Gamma s', \Gamma^2 s')^{1-\alpha} + G(s', \Gamma s', \Gamma^2 s') \\ &= \mu G(t, t, t)^{\alpha} \cdot G(s', \Gamma s', \Gamma^2 s')^{1-\alpha} + G(s', \Gamma s', \Gamma^2 s') \\ &= G(s', \Gamma s', \Gamma^2 s') \le \epsilon, \end{aligned}$$

which implies that

$$G(t, s', \Gamma s') \le C\epsilon, \ \text{ where } C = 1.$$

(ii) Given the additional conditions, we have

$$\begin{aligned} G(t, r_n, \Gamma r_n) &\le G(t, \Gamma r_n, \Gamma^2 r_n) + G(\Gamma^2 r_n, r_n, \Gamma r_n) \\ &= G(\Gamma t, \Gamma r_n, \Gamma^2 r_n) + G(r_n, \Gamma r_n, \Gamma^2 r_n) \\ &\le \mu G(t, \Gamma t, \Gamma^2 t)^{\alpha} \cdot G(r_n, \Gamma r_n, \Gamma^2 r_n)^{1-\alpha} + G(r_n, \Gamma r_n, \Gamma^2 r_n) \\ &= \mu G(t, t, t)^{\alpha} \cdot G(r_n, \Gamma r_n, \Gamma^2 r_n)^{1-\alpha} + G(r_n, \Gamma r_n, \Gamma^2 r_n). \end{aligned}$$

Letting $n \to \infty$ and keeping in mind Proposition 1 and

$$\lim_{n\to\infty} G(r_n, \Gamma r_n, \Gamma^2 r_n) = 0,$$

we obtain

$$\lim_{n\to\infty} G(r_n, t, t) = \lim_{n\to\infty} G(t, r_n, \Gamma r_n) \le \lim_{n\to\infty} G(r_n, \Gamma r_n, \Gamma^2 r_n) = 0.$$

That is, the FP equation (9) is well-posed. □

Remark 1 The results presented in this work cannot be expressed in the form $G(r, s, s)$ or $G(r, r, s)$. Hence, they cannot be obtained from their corresponding versions in ms.

Conclusion

A generalization of ms was introduced by Mustafa and Sims [25], namely G-ms and several FP results were studied in that space. However, Jleli and Samet [17] as well as Samet et al. [29] established that most FP theorems obtained in G-ms are direct consequences of their analogues in ms. Contrary to the above observation, a new notion of contraction, called interpolative Kannan-type (G-α-μ)-contraction is introduced in this manuscript and some FP results that cannot be deduced from their corresponding ones in ms are proved. The main distinction of this class of contractions is that it complements and encompasses a few corresponding notions in the literature. Consequently, nontrivial comparative examples are constructed to validate the assumptions of our obtained theorems. In addition, we looked into well-posedness and Ulam-type stability for the novel contraction put out here. The general ideas in this paper are motivated by [7, 13, 16, 20, 25].

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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