ONE-STEP SECOND DERIVATIVES METHOD WITH INTRA-STEP POINTS FOR SOLVING INITIAL VALUE PROBLEM IN ORDINARY DIFFERENTIAL EQUATION

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Abstract
An optimized one-step hybrid block method for the numerical solution of first-order initial value problems is constructed. The method takes into consideration two and four intra-step points which are chosen appropriately to optimize the local truncation errors of the main formulas for the block. The method is zero-stable and consistent with fifth and seven algebraic order. Some standard examples such as Riccati equation, system of linear equation and chemical equation are discussed to show the accuracy of the proposed method over the existing method in the literature.

Keywords: Second derivative, Intra-step, initial value problems, Variable step size.

Introduction
The mathematical models in engineering and many spheres of human endeavours often lead to initial value problem of ordinary differential equations of the form:

$$y' = f(x, y)$$
$$y(a) = \alpha \qquad (1)$$
$$x \in [a, b]$$

According to Modebei *et al*. (2019), differential equation is a mathematical equation that relates one or more functions and their derivatives. Functions are used to represent physical quantities in applications, derivatives are used to characterize their rates of change, and differential equations are used to create a relationship between them. The primary goal of studying differential equations is to examine their solutions (the set of functions that satisfy each equation) as well as their attributes. Only the most fundamental differential equations can be solved using explicit formulas; nevertheless, many aspects of differential equation solutions can be known without precisely computing them. When there is not a closed-form equation for the solution, it's often possible to approximate it numerically using computers. While the theory of dynamical systems stresses qualitative analysis of systems described by differential equations, many numerical approaches have been developed to determine solutions with a specified degree of precision.

There are many problems in mathematics for which no analytic solution exists, as well as others for which analytic solutions are repetitious and the answer is in the form of a boundless solution that must be deciphered after considerable computational effort. Hilbert *et al*., (1998). With the advent of fast modern computers, numerical methods have become more appealing for addressing practical issues by eliminating the tiresome dull redundant human mathematical calculations. This is due to the systematic approach numerical method receives in critical thinking, which is similar to computation. These calculations are effectively converted into machinereadable code. As a result, numerical analysis serves as a bridge between mathematics and computers.

Several numerical methods have been designed and proposed in literature for solving second order ordinary differential equations. For example, Areo and Adeniyi (2013) developed a self-starting linear multistep method and applied it to solve second order IVPs of ODEs directly. Two intra step grid points were considered by means of collocation and interpolation approach. Abdelrehim and Omar (2015) proposed a single-step hybrid block method of order five to solve second order ODEs. In the work of Olabode and Momoh (2016), continuous hybrid multistep method with Legendre polynomial as the approximate solutions was investigated to obtain the approximation of stiff second order ODEs. Also, two intra step grid points were considered by means of collocation and interpolation approach. More so, Sunday *et al*. (2014) developed numerical solution of stiff and oscillatory first order differential equations, using the combination of power series and exponential function as basis function. Momoh *et al*. (2014) used the same basis function to produce a new numerical integration for the solution of stiff first order ODEs. Most of the methods proposed for the solution of stiff problems are numerically unstable unless the step size is taken to be extremely small and the adoption of implicit A-stable schemes is better for the solution of stiff or stiff oscillatory problems. Above all, most proposed numerical methods implemented in block modes were problem dependent. In other words, the numbers of interpolation are subject to the order of the problem.

In this research, a class of one-step second derivatives method with intra-step points for the solution of initial value problems of ODEs is proposed. The method is implemented in block mode and problem independent.

Derivation of the method

The proposed one-step second derivative, intra-step block numerical method for the solution of stiff systems of first order ordinary differential equations is given as

$$y_{n+1} = y_n + h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\gamma_k g_{n+k} \tag{2}$$

And the additional formula

$$y_{n+\eta} = y_n + h\sum_{j=o}^{k}\beta_j f_j + h^2\gamma_k g_{n+k} \tag{3}$$

Where β_1 and γ_1 are not equal to zero, β_j and γ_j are constant coefficient to be determine, η is use to denote the intra-step points.

Equations (2) and (3) are derived using the interpolation and collocation techniques of a trial function of the form

$$y(x) = \sum_{j=0}^{r+2s-1} ajx^j \tag{4}$$

where a_j are unknown coefficient to be determined, r is the number of interpolation and s is the number of collocation points. Interpolating (4) at x_n and collocating its first and second derivatives at x_{n+1} and a countable number of intra-point is defined as $x_{n+\eta} = x_n + h\eta$, $\left(\eta \in (0,1)\right)$. This leads to a system of non-linear equations in the form:

$$\left.\begin{aligned} &y(x)=y_n\\ &y'(x_{n+j})=f_{n+j}\\ &y''(x_{n+j)}=g_{n+k}\\ &j=0......k \end{aligned}\right\} \quad (5)$$

This is evaluated using matrix inversion method to have a_j, s and then substituting into equation (4) to obtain to obtain the continuous scheme of the form

$$y(x)=y_n+h(\beta_0(x)f_n+\beta_\eta(x)f_{n+\eta}+\beta_1(x)\,f_{n+1})+h^2\gamma_1 g_{n+1} \quad (6)$$

The equation (6) generates the main and additional algorithm which are merged to generate approximate solution simultaneously. In this paper, two sets of intra-step points are applied: $\left\{\frac{1}{3},\frac{2}{3}\right\}$ and $\left\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\right\}$

One-step second derivative block method with two intra-step points (OSDBM2)

The specification for deriving this method is given as: $k=1,\eta=\left\{\frac{1}{3},\frac{2}{3}\right\}, x\in\left[x_{n,}x_{n+1}\right]$

and following (3) to (5), we obtain one-step second derivative block method:

$$y_{n+1}=y_n+\frac{13}{120}hf_n+\frac{9}{20}hf_{n+\frac{1}{3}}+\frac{9}{40}hf_{n+\frac{2}{3}}+\frac{13}{40}hf_{n+1}-\frac{1}{60}h^2g_{n+1} \quad (7)$$

$$y_{n+\frac{1}{3}}=y_n+\frac{367}{3240}hf_n+\frac{19}{60}hf_{n+\frac{1}{3}}-\frac{7}{40}hf_{n+\frac{2}{3}}+\frac{127}{1620}hf_{n+1}-\frac{19}{1620}h^2g_{n+1} \quad (8)$$

$$y_{n+\frac{2}{3}}=y_n+\frac{43}{405}hf_n+\frac{7}{15}hf_{n+\frac{1}{3}}+\frac{1}{15}hf_{n+\frac{2}{3}}+\frac{11}{405}hf_{n+1}-\frac{2}{405}h^2g_{n+1} \quad (9)$$

One-step second derivative block method with 4 intra-step points (OSDBM4)

The specification for deriving this method is given as: $k=1,\eta=\left\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\right\}, x\in\left\{x_n, x_{n+1}\right\}$

and the block method are obtained as follows:

$$y_{n+1}=y_n+h\left(\frac{61}{1008}f_n+\frac{2375}{8064}f_{n+\frac{1}{5}}+\frac{125}{1512}f_{n+\frac{2}{5}}+\frac{625}{2016}f_{n+\frac{3}{5}}+\frac{125}{1008}f_{n+\frac{4}{5}}+\frac{3108}{24192}f_{n+1}\right)$$
$$+h^2\left(-\frac{11}{2016}g_{n+1}\right) \quad (10)$$

$$y_{n+\frac{1}{5}}=y_n+h\left(\frac{2627}{4200}f_n+\frac{4919}{22400}f_{n+\frac{1}{5}}-\frac{6347}{37800}f_{n+\frac{2}{5}}+\frac{2563}{16800}f_{n+\frac{3}{5}}-\frac{307}{2800}f_{n+\frac{4}{5}}+\frac{129571}{302400}f_{n+1}\right)$$
$$+h^2\left(-\frac{863}{25200}g_{n+1}\right) \quad (11)$$

$$y_{n+\frac{2}{5}}=y_n+h\left(\frac{943}{15750}f_n+\frac{3797}{12600}f_{n+\frac{1}{5}}-\frac{38}{4725}f_{n+\frac{2}{5}}+\frac{283}{3150}f_{n+\frac{3}{5}}-\frac{227}{3150}f_{n+\frac{4}{5}}+\frac{5489}{18900}f_{n+1}\right)$$

$$+h^{2}\left(-\frac{37}{15750}g_{n+1}\right) \qquad (12)$$

$$y_{n+\frac{3}{5}}=y_{n}+h\left(\frac{849}{14000}f_{n}+\frac{6567}{22400}f_{n+\frac{1}{5}}+\frac{127}{1400}f_{n+\frac{2}{5}}+\frac{1233}{5600}f_{n+\frac{3}{5}}+\frac{291}{2800}f_{n+\frac{4}{5}}+\frac{4393}{11200}f_{n+1}\right)$$

$$+h^{2}\left(-\frac{87}{2800}g_{n+1}\right) \qquad (13)$$

$$y_{n+\frac{4}{5}}=y_{n}+h\left(\frac{158}{2625}f_{n}+\frac{52}{175}f_{n+\frac{1}{5}}+\frac{344}{4725}f_{n+\frac{2}{5}}+\frac{176}{525}f_{n+\frac{3}{5}}+\frac{2}{175}f_{n+\frac{4}{5}}+\frac{548}{23625}f_{n+1}\right)$$

$$+h^{2}\left(-\frac{16}{7875}g_{n+1}\right) \qquad (14)$$

Analysis of Basic Properties of the Methods.

3.1 Order and error constants

The approach of Garba and Mohammed (2020) for determining the order of a numerical scheme is adopted in the analysis of order and error constants of the new block methods. Hence the new methods have uniform orders and their respective error constants are presented below.

Table 1: Order and Error Constants for (OSDBM2)

Equation	order p	error constants C_{P+1}
(7)	5	$\frac{1}{64800}$
(8)	5	$\frac{-283}{2624400}$
(9)	5	$\frac{2}{164025}$

Table 2: Order and Error Constants for (OSDBM4)

Equation	**order p**	**error constants C_{P+1}**
(10)	7	$\frac{11}{529200000}$
(11)	7	$\frac{2633}{110250000000}$
(12)	7	$\frac{187}{10335937500}$
(13)	7	$\frac{257}{12250000000}$

(14)	7	$\frac{16}{861328125}$

3.2 Consistency: Also, as stated in Mohammed *et al.* (2021), a linear multistep method is said to be consistent if it has order $p \geq 1$. As seen from the tables above, the two new methods satisfy the condition for consistency.

3.3. Zero stability

Definition: A block linear multistep method is said to be zero stable if the roots of the first characteristic polynomial $p(\lambda)$ satisfied $|\lambda_j| \leq 1, j = 1, 2.......$ and for those roots with $|\lambda_j| = 1,$ the multiplicity must not exceed 1. To establish the zero-stability of the new methods, we write the proposed one-step second derivative block methods as a matrix differences equation as follow

$$A^{(1)} = A^{(0)}Y_{w-1} + h\left[B^{(0)}F_{W-1} + B^{(1)}F_W\right] + h^2\left[C^{(0)}G_{W-1} + C^{(0)}G_W\right] \qquad (15)$$

And the matrices $A^{(1)}, A^{(0)}, \mathrm{B}^{(1)}, \mathrm{B}^{(0)}, C^{(1)}$ and $C^{(0)}$ are matrices whose first entries are given by the coefficients of the methods whose first characteristic polynomials is given as

$$P(\lambda) = \left|\lambda A^{(1)} - A^{(0)}\right| \qquad (16)$$

Equations (6) to (8) are written in the form of equation (14) for **(OSDBM2)** and we have

$$A^{(0)} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \qquad A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^{(0)} = \begin{bmatrix} 0 & 0 & \frac{367}{3240} \\ 0 & 0 & \frac{43}{405} \\ 0 & 0 & \frac{13}{120} \end{bmatrix} \qquad B^{(1)} = \begin{bmatrix} \frac{19}{60} & \frac{-7}{40} & \frac{127}{1620} \\ \frac{7}{15} & \frac{1}{15} & \frac{11}{405} \\ \frac{9}{20} & \frac{9}{40} & \frac{13}{60} \end{bmatrix} \qquad C^{(1)} = \begin{bmatrix} 0 & 0 & \frac{-19}{1620} \\ 0 & 0 & \frac{-2}{405} \\ 0 & 0 & \frac{-1}{60} \end{bmatrix}$$

Using equation (15) we obtain

$$P(\lambda) = \lambda^2(\lambda - 1) = 0$$

$$\lambda = \{0, 0, 1\} \qquad (17)$$

Hence the method is zero stable since it satisfies $\left|\lambda_j\right| \leq 1$

Similarly, equations (9) to (13) for **(OSDBM4)** are also written in the form of equation (15) as follows:

$$A^{(0)}=\begin{bmatrix}0&0&0&0&-1\\0&0&0&0&-1\\0&0&0&0&-1\\0&0&0&0&-1\\0&0&0&0&-1\end{bmatrix}\quad A^{(1)}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\quad B^{(0)}=\begin{bmatrix}0&0&0&0&\frac{2627}{42000}\\0&0&0&0&\frac{943}{15750}\\0&0&0&0&\frac{849}{14000}\\0&0&0&0&\frac{158}{2625}\\0&0&0&0&\frac{61}{1008}\end{bmatrix}$$

$$B^{(1)}=\begin{bmatrix}\frac{4919}{22400}&\frac{-6347}{37800}&\frac{2563}{16800}&\frac{-307}{2800}&\frac{129571}{3024000}\\\frac{3797}{12600}&\frac{-38}{4725}&\frac{283}{3150}&\frac{-227}{3150}&\frac{5489}{189000}\\\frac{6567}{22400}&\frac{127}{1400}&\frac{1233}{5600}&\frac{-291}{2800}&\frac{4393}{112000}\\\frac{52}{175}&\frac{344}{4725}&\frac{176}{525}&\frac{2}{175}&\frac{548}{23625}\\\frac{2375}{8064}&\frac{125}{1512}&\frac{625}{2016}&\frac{125}{1008}&\frac{3103}{24192}\end{bmatrix}\quad C^{(1)}=\begin{bmatrix}0&0&0&0&\frac{-863}{252000}\\0&0&0&0&\frac{-37}{15750}\\0&0&0&0&\frac{-87}{28000}\\0&0&0&0&\frac{-16}{7875}\\0&0&0&0&\frac{-11}{2016}\end{bmatrix}$$

Then the first characteristic polynomial is given as

$$P(\lambda)=\lambda^4(\lambda-1)=0$$

$$\lambda=\{0,0,0,0,1\} \tag{18}$$

Hence the method is zero stable since it satisfies $|\lambda_j|\le 1$.

Convergence of the method

Consistency and zero stability are both required and sufficient for a linear multistep technique to reach convergence in the spirit of Lambert (1973) and Kuboye and Adeyefa (2021). As a result, we infer that our technique is convergent since it has an order of accuracy greater than 1 (which implies consistency) and zero-stable.

Numerical Experiment

In this section, the first-order dynamical systems with applications in population dynamics, chemical equations, and vibration theory are implemented. The resultant iterative techniques are discretized into the following form:

$$y'_{n+j}=f\left(x_{n+j},y_{n+j}\right),\quad y''_{n+j}=g\left(x'_{n+j},f_{n+j}\right),\quad j\in[0,1] \tag{19}$$

and treated as a block which requires no starting values and predictors. Using the known initial condition, $y(x_n)$ for $n=0,1,\ldots,N-1$, the first order IVPs are solved in the N non-overlapping block points $[x_0,x_1],\ldots,[x_{N-1},x_N]$, with the step size defined in the usual way as $h=x_{n+1}-x_n$.

Problem 1: Consider the Riccati equation

$y' = -y^2 + 2y + 1, \quad y(0) = 0$,

with the exact solution: $y(x) = 1 + \sqrt{2}\tanh\left[\sqrt{2}x + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$

Problem 2: Consider the system of linear equation

$$\begin{aligned} y_1' &= -y_1 - 15y_2 + 15e^{-x} \qquad & y_1(0) &= 1 \\ y_2' &= 15y_1 - y_2 - 15e^{-x} \qquad & y_2(0) &= 1 \end{aligned},$$

with the exact solution: $y_1(x) = e^{-x}; \qquad y_2(x) = e^{-x}$

Problem 3: Consider the following system of chemical equation

$$\begin{aligned} y_1' &= -500000.5y_1 + 499999.5y_2 \qquad & y_1(0) &= 0 \\ y_2' &= 499999.5y_1 - 500000.5y_2 \qquad & y_2(0) &= 2 \end{aligned},$$

with the exact solution: $y_1(x) = e^{-x} - e^{-10^6 x}; \qquad y_2(x) = e^{-x} + e^{-10^6 x}$

Problem 4: Consider the following system of nonlinear equation

$$y_1' = \mu y_1 + y_2^2, \qquad y_1(0) = -\frac{1}{(\mu+2)}$$

$$y_2' = -y_2, \qquad y_2(0) = 1$$

Where $\mu = 10000$, The exact solution is $y_1(x) = -\frac{e^{-2x}}{(\mu+2)}, \; y_2(x) = e^{-x}$

Table 1: Comparison of Absolute Errors for Problem 1 with variable step size

x	Error in Kashkari and Syam (2019) *h* = 0.05	Abolarin *et. al*, (2020) *h* = 0.1	**Error in (OSDBM2) *h* = 0.05**	**Error in (OSDBM4) *h* = 0.1**
1.0	1.418×10^{-11}	9.104×10^{-15}	1.280×10^{-10}	3.952×10^{-12}
2.0	7.234×10^{-13}	7.105×10^{-15}	8.254×10^{-11}	1.123×10^{-13}
3.0	1.163×10^{-13}	8.882×10^{-15}	1.664×10^{-13}	1.360×10^{-15}
4.0	2.132×10^{-14}	2.121×10^{-14}	4.447×10^{-13}	1.271×10^{-15}
5.0	2.664×10^{-15}	1.368×10^{-13}	5.380×10^{-14}	1.634×10^{-16}
6.0	4.441×10^{-16}	7.983×10^{-13}	4.804×10^{-15}	1.491×10^{-17}
7.0	4.441×10^{-16}	3.699×10^{-12}	3.808×10^{-16}	1.240×10^{-18}
8.0	4.441×10^{-16}	-	2.817×10^{-17}	8.988×10^{-20}
9.0	4.441×10^{-16}	-	2.001×10^{-18}	2.972×10^{-20}
10	4.441×10^{-16}	-	1.382×10^{-19}	5.885×10^{-20}

Table 2: Comparison of Absolute Errors for Problem 2

x	**Error in Akinfenwa *et. al,* (2015)** y_1 y_2	**Error in Ezzeddine and Hojjati (2012)** y_1 y_2	**Error in (OSDBM2)** y_1 y_2	**Error in (OSDBM4)** y_1 y_2
5.0	9.730×10^{-17}	2.230×10^{-15}	3.071×10^{-19}	4.123×10^{-26}
	9.540×10^{-18}	3.040×10^{-15}	2.340×10^{-19}	3.171×10^{-26}
10	4.680×10^{-19}	2.270×10^{-17}	4.588×10^{-21}	6.167×10^{-28}
	2.710×10^{-19}	2.830×10^{-17}	2.228×10^{-21}	3.028×10^{-28}
15	3.920×10^{-21}	6.560×10^{-19}	4.826×10^{-23}	6.493×10^{-30}
	2.440×10^{-21}	1.500×10^{-19}	1.248×10^{-23}	1.709×10^{-30}
20	4.300×10^{-23}	2.760×10^{-21}	4.262×10^{-25}	5.740×10^{-32}
	4.140×10^{-24}	1.520×10^{-22}	2.300×10^{-26}	3.348×10^{-33}

Table 3: Comparison of Absolute Errors for Problem 3

x	**Tahmasbi (2008),** y_1 y_2 ***h*** = 0.00001	**Error in Akinfenwa *et. al,* (2017)** y_1 y_2 ***h*** = 0.0001	**Error in (OSDBM2)** y_1 y_2 ***h*** = 0.0001	**Error in (OSDBM4)** y_1 y_2 ***h*** = 0.001
0.2	6.200×10^{-14}	3.930×10^{-25}	2.527×10^{-26}	1.245×10^{-29}
	6.200×10^{-14}	3.930×10^{-25}	2.527×10^{-26}	1.245×10^{-29}
0.4	1.020×10^{-13}	6.570×10^{-25}	4.138×10^{-26}	2.038×10^{-29}
	1.020×10^{-13}	6.570×10^{-25}	4.138×10^{-26}	2.038×10^{-29}
0.6	6.050×10^{-14}	8.000×10^{-25}	5.082×10^{-26}	2.504×10^{-29}
	6.050×10^{-14}	8.000×10^{-25}	5.082×10^{-26}	2.504×10^{-29}
0.8	4.480×10^{-14}	8.720×10^{-25}	5.547×10^{-26}	2.733×10^{-29}
	4.480×10^{-14}	8.720×10^{-25}	5.547×10^{-26}	2.733×10^{-29}
1.0	4.410×10^{-14}	8.900×10^{-25}	5.677×10^{-26}	2.797×10^{-29}
	4.410×10^{-14}	8.900×10^{-25}	5.677×10^{-26}	2.797×10^{-29}

Table 4: Comparative Analysis of Result of Problem 4

x	Error in Mehdizadeh *et.al,* (2012) h = 0.0001 y_1 y_2	Error in Akinfenwa *et. al.* (2017) h = 0.1 y_1 y_2	**Error in (OSDBM4)** h=0.1 y_1 y_2
3	2.48×10^{-11}	2.03×10^{-19}	2.95×10^{-21}
	2.47×10^{-06}	1.44×10^{-14}	2.96×10^{-16}
5	3.45×10^{-14}	1.20×10^{-20}	9.00×10^{-23}
	2.30×10^{-08}	3.21×10^{-15}	6.68×10^{-17}
10	3.46×10^{-18}	1.11×10^{-20}	8.17×10^{-27}
	3.15×10^{-11}	4.38×10^{-17}	9.00×10^{-19}

Discussion of Results

Problem is a quadratic Riccati differential equation. The derived methods are implemented on this nonlinear equation. This problem has been solved by Kashkari and Syam (2019) and Abolarin *et. al,* (2020), with different step sizes. In Table 1 the newly derived methods outperform the methods in Kashkari and Syam (2019), and Abolarin *et. al,* (2020) even with higher step sizes.

Problem 2 is a stiff system of ordinary differential equation; this system has eigenvalues of large modulus lying close to the imaginary axis $-115i$. This problem is solved using the newly derived methods and the absolute errors are compared in table 2 with the methods of Ezzeddine and Hojjati (2012) and Akinfenwa *et. al,* (2015). It is seen that the single step method with 2 intra-step points is superior to the method of Ezzeddine and Hojjati (2012) and that of Akinfenwa *et. al,* (2015) (also a 2 – step method).

Problem 3 is a strongly stiff system with stiffness ration $1:10^6$. This problem is solved using the proposed methods in this paper on the range $[0,1]$. This problem is also solved in the methods of Tahmasbi (2008), and Akinfenwa *et. al,* (2017). The results in these methods are presented in table 3 and compared the one of the proposed methods. It is seen that the method **(OSDBM2)** using h = 0.0001 outperforms that of Akinfenwa *et. al,* (2017) using the same step size, and also performs better than that of Tahmasbi (2008) whose step size is smaller (h = 0.00001).

The last problem considered in this work is the nonlinear system of ordinary differential equation in problem 4. This problem is solved using the new methods in this paper on the range $[0,1]$. This problem is also solved in the methods of Mehdizadeh *et.al,* (2012) and Akinfenwa *et. al,* (2017). The results in these methods are presented in table 4 and compared with the one of the **(OSDBM4).** It is seen that the method **(OSDBM4)** using $h = 0.1$ outperforms that of Akinfenwa *et. al,* (2017) using the same step size, and also performs better than that of Mehdizadeh *et.al,* (2012) whose step size is smaller (h = 0.0001).

Conclusion

In this paper, we construct an optimized one-step hybrid block method for the numerical solution of first-order initial value problems. The continuous formulation of the method takes into consideration two and four intra-step points which are chosen appropriately to optimize the local truncation errors of the main formulas for the block. The methods are zero-stable and consistent with fifth and seven algebraic order respectively. Some standard examples such as the nonlinear Riccati equation, system of nonlinear equation and chemical equation are discussed to demonstrate the effectiveness of the methods over the existing method in the literature. As a result, these methods are recommended for solving real-life problems especially the ones without any known analytical solution.

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