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Analytical Study of Leakage of Viscous Flow in a Cylindrical Pipe

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Abstract

This research work presents the transient flow analysis of viscous fluid within a pipe. The model equations evolved were considered for leak and no leak conditions. The equations were further solved analytically using eigen vector expansion method. The results obtained were presented graphically and analyzed. The analyses were undertaken using flow velocity, pressure, density, measured inlet mass flow, measured outlet mass flow, elevation, leak rate, leak velocity and Reynolds' number. Based on the results obtained, these fundamental tools of analysis proved effective in detecting, locating and describing the type and behaviour of leakage in a pipe.

Keywords: Leakage, Viscous Flow, Mass Flow Rate and Reynolds number.

1.0 Introduction

Pipelines are media needed for the movement of crude oil from reservoir wellbore and other stations to be delivered to destination point such as separator, storage tanks and the likes (Oyedeko and Balogun, 2015). The use of pipelines is considered as a major medium of transporting petroleum products like gases, fossil fuels, chemicals and other important hydrocarbons (Rehman and Nawaz, 2017).

Until crude oil is converted into useful products, there are intermediary processes that need one or more-unit operations which will involve leakages with one another with the help of pipelines (Chinwuko *et al.*,2016). It has been proven that gas and oil pipeline system are the safest and most economical media of transporting crude oil and they fulfill a high demand for reliability and efficiency (Boaz *et al.,* 2014; Xiao *et al.,* 2018). Transportation of crude oil in pipelines need serious monitoring to detect pipeline failure or malfunctioning like leaks (Hauge *et al.,* 2007). Overtime, these pipelines due to design faults, stress corrosion and fatigue cracks, operation outside design limit or intentional damage in act of vandalism, ageing and their likes result to leaks (Oyedeko and Balogun, 2015). Sudden pipeline burst result in rapid change in pressure, causing economic loss and environmental problem without locating the leakages position and repairing in time (Yang *et al.*, 2011).

The failure of pipelines is either deliberate vandalism or device failure or corrosion damage (Ajao *et al*., 2018; White *et al*., 2019). This leads to pipeline failure and thus causing an irreversible damage such as financial losses and extreme environmental pollution specifically when the leakage is not located timely (Arifin *et al*., 2018; Mokhatab *et al*., 2012; Mutiu *et al*., 2019). Crude oil is in some sense hazardous. Hence, it is required to install leak detection and localization systems (LDS). Leak detection systems that have the capability of detecting the particular spot. Both the transporters and producers of these hydrocarbons experience the problems of pipeline leaks from time to time and failure to locate it can lead to human casualties, direct cause of loss of product and lie downtime, environmental cleanup cost and possible fines and legal suits from habitants (Oyedeko and Balogun, 2015; Chinwuko *et al*., 2016). Hauge *et al*., (2007) suggested a set of two coupled one dimensional first order nonlinear hyperbolic partial differential equations governing the flow dynamics based on the assumption that measurements are only available at the inlet and outlet of the pipe, and output is applied in the form of boundary conditions. The deficiency of the model is that leak is only accurately located and quantified successfully when the pipeline is shut-down. In this work, we solved and extended the model for leakage in pipeline. The model was modified to include: no leak and leak situations for viscous fluid. The two were solved analytically by eigen-function expansion technique.

2.0 Mathematical Formulation

In formulating the models, relevant assumptions were made in line with Chinwuko *et al.,* (2016). The model equation was also modified to depict two different situations. They are:

1) Case 1: No leak situation for a viscous fluid.
2) Case 2: Leak situation for a viscous fluid.

The following assumptions were made:

(i) Pipe cross-sectional area remains constant;
(ii) Isothermal and adiabatic flow;
(iii) One-dimensional flow (unidirectional)
(iv) No chemical reaction between the transporting fluid and internal wall of the pipe;
(v) Constant density throughout the pipeline segments

(vi) Homogenous fluid (either oil or gas) being transported in the pipeline.

(vii) Energy conservation equation is neglected, since the leak considered in the transient model affects only the downstream temperature at the fluid flowing velocity.

Based on the above assumptions, the continuity equation also known as the mass balance equation which is based on the law of conservation of mass for a one-dimensional flow can be expressed as;

$$\frac{\partial(\rho A)}{\partial t}+\frac{\partial(\rho A U)}{\partial x}=0 \tag{1}$$

In the occurrence of a leak, the continuity equation becomes

$$\frac{\partial(\rho A)}{\partial t}+\frac{\partial(\rho A U)}{\partial x}+\dot{M}_{leak}=0 \tag{2}$$

where the $\dot{M}_{leak}$ is the leak rate defined as:

$$\dot{M}_{leak}=M_I-\dot{M}_I-(M_0-\dot{M}_0) \qquad \text{(Tetzner, 2003)} \tag{3}$$

The leak position is also given by

$$x_{leak}=-\frac{(M_0-\dot{M}_0)}{\dot{M}_{leak}}L \tag{4}$$

The momentum equation describes the force balance on the fluid within a segment of the pipeline. From the Navier-Stokes equations, the conservation of momentum in one – dimensional flow is:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial \rho}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2} \tag{5}$$

Substituting the viscosity μ from the expression for Reynolds' number and later substituting with the relation of frictional factor f gives:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2} \tag{6}$$

Substituting $p=P+\rho g H$ into (6) and introducing the leak term gives

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial P}{\partial x}-\rho g\frac{\partial H}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2}+\rho u_{leak} \tag{7}$$

where: u is the one-dimensional velocity of fluid; t is time; x is the spatial space; p is the static pressure; μ is the viscosity of the fluid; H is the elevation; ρ is the density of the fluid; A is the cross sectional area of the pipe; $\dot{M}_{Leak}$ is the leak rate; x_{Leak} is the leak position; L is the length of the pipe; g is the acceleration due to gravity; u_{Leak} is the leak velocity; U_0 is the reference flow velocity; M_0 is the estimated outlet mass flow rate; M_I is the estimated inlet mass flow rate; $\dot{M}_0$ is the measured outlet mass flow rate and $\dot{M}_I$ is the measured inlet mass flow rate.

The following initial and boundary conditions are later applied:

$$\left.\begin{aligned} u(x,0)&=0\\ u(0,\mathrm{t})&=U_0\\ u(L,t)&=0 \end{aligned}\right\} \tag{8}$$

2.1 Method of Solution

If we assume $\frac{\partial H}{\partial x}$ to be parabolic, i. e. $\frac{\partial H}{\partial x}=\frac{H_0}{L}x\left(1-\frac{x}{L}\right)$ (9)

Then, equations (2) and (7) satisfying (8) become (10) and (11) respectively

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\dot{M}_{leak}=0 \tag{10}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial P}{\partial x}-\frac{\rho g H_0}{L}x\left(1-\frac{x}{L}\right)+\mu\frac{\partial^2 u}{\partial x^2}+\rho u_{leak} \tag{11}$$

2.2 Non-Dimensionalization

Here, we non-dimensionalize equations (10) and (11) satisfying (8) using the following dimensionless variables: $u'=\frac{u}{U_0}, t'=\frac{U_0 t}{L}, x'=\frac{x}{L}; \mathrm{p}'=\frac{p}{\rho_0 U_0^2}, \rho'=\frac{\rho}{\rho_0}$ (12)

Substituting (12) into (10), (11) and using (8), we have

$$\frac{\rho_0 U_0}{L}\frac{\partial \rho'}{\partial t'}+\frac{\rho_0 U_0}{L}\frac{\partial\left(\rho' u'\right)}{\partial x'}+\dot{M}_{leak}=0 \tag{13}$$

$$\frac{\rho_0 \rho' U_0^2}{L}\left(\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}\right)=-\frac{\rho_0 U_0^2}{L}\frac{\partial \rho}{\partial x'}-\frac{\rho_0 \rho' H_0 g L}{L}x'\left(1-\frac{Lx'}{L}\right)+\frac{\mu U_0}{L^2}\frac{\partial^2 u'}{\partial x'^2}+\rho_0 \rho' u_{leak} \tag{14}$$

and
$$\left.\begin{aligned} u(x,0) &= U_0 u'(x',0) = 0, \\ u(0,t) &= U_0 u'(0,t') = u_0, \\ \mathrm{u}(L,t) &= U_0 u'(1,t') = 0 \end{aligned}\right\} \quad (15)$$

i.e.
$$\frac{\partial \rho'}{\partial t'} + \frac{\partial(\rho' u')}{\partial x'} + \frac{L\dot{M}_{leak}}{\rho_0 U_0} = 0 \quad (16)$$

and
$$\rho'\left(\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'}\right) = -\frac{\partial P}{\partial x'} - \frac{g\rho' H_0 L}{U_0^2} x'(1-x') + \frac{\mu}{\rho_0 L U_0}\frac{\partial^2 u'}{\partial x'^2} + \frac{L\rho'}{U_0^2} u_{leak} \quad (17)$$

$$\left.\begin{aligned} u'(x',0) &= 0 \\ u'(0,t') &= 1 \\ u'(1,t') &= 0 \end{aligned}\right\} \quad (18)$$

Dropping prime we have the following dimensionless equations and initial and boundary conditions.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + R_{leak} = 0 \quad (19)$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial P}{\partial x} + \beta x(1-x) + \frac{1}{R_\ell}\frac{\partial^2 u}{\partial x^2} + V_{leak} \quad (20)$$

$$\left.\begin{aligned} u(x,0) &= 0 \\ u(0,t) &= 1 \\ u(1,t) &= 0 \end{aligned}\right\} \quad (21)$$

where R_{Leak} : is the leak rate, V_{Leak} : is the leak velocity, β : is the elevation and

R_e : is the Reynolds' Number for the flow

The above equations (19) – (21) will now be considered under two (2) cases. They are;

1. Non-viscous flow without leakage
2. Non-viscous flow with leakage

Note that;

1. If there is no leak, the transient pressure drop per unit length along the pipeline is constant i.e., no leak implied

$$\frac{\partial \rho}{\partial x} = f\left(\dot{M}_I\right) = \dot{M}_I = \text{constant} \tag{22}$$

2. If a leak occurs, the flow rate upstream of the leak will be greater than the flow rate downstream of the leak, therefore the pressure drop per unit length upstream of the leak will also be greater than pressure downstream of the leak i.e.,

$$\left.\frac{\partial \rho}{\partial x}\right|_{0\le x\le x_{leak}} - \left.\frac{\partial \rho}{\partial x}\right|_{0<x\le L} = f\left(\dot{M}_I\right) - f\left(\dot{M}_0\right) = \dot{M}_I + \left(\dot{M}_0 - \dot{M}_I\right)x > 0 \tag{23}$$

2.3 Viscous flow without leakage

In this case, equation (19) and (21) reduce to

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{24}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\dot{M}_I - \beta x(1-x) + \frac{1}{R_e}\frac{\partial^2 u}{\partial x^2} \tag{25}$$

$$\left.\begin{aligned} u(x,0) &= 0 \\ u(0,t) &= 1 \\ u(1,t) &= 0 \end{aligned}\right\} \tag{26}$$

2.4 Analytical solution of Viscous Flow without Leakage

It is simple to eliminate the continuity equation (24) by means of streamlines function

$$\eta(x,t) = \left(\rho^2\right)^{-\frac{1}{2}} \int_0^x \rho(s,t)\,ds \tag{27}$$

Let $u = u(x,t)$ such that $\eta = \eta(x,t)$ and $x = x(t)$ then by chain rule, we have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial t} \tag{28}$$

and

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x} \tag{29}$$

$$u=-\frac{1}{\rho}\int_0^x \frac{\partial \rho}{\partial t}ds \tag{30}$$

then $$\frac{\partial \eta}{\partial t}=-u \tag{31}$$

and $$\frac{\partial \eta}{\partial x}=1 \tag{32}$$

The coordinate transformations become

$$\frac{\partial}{\partial x}\to\frac{\partial}{\partial \eta}.\frac{\partial \eta}{\partial x}=\frac{\partial}{\partial \eta} \tag{33}$$

$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial \eta}.\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial t}=\frac{\partial}{\partial t}-u\frac{\partial}{\partial \eta} \tag{34}$$

Using (27) - (34) then equations (24) – (26) can be simplified as:

$$\rho\frac{\partial u}{\partial t}=\frac{1}{R_e}\frac{\partial^2 u}{\partial \eta^2}-\dot{M}_I-\beta\eta(1-\eta) \tag{35}$$

$$\left.\begin{array}{l} u(\eta,0)=0 \\ u(0,t)=1 \\ u(1,t)=0 \end{array}\right\} \tag{36}$$

Equation (35) – (36) is a non-homogenous boundary value problem. Hence, there is need to transform (35) – (36) to homogenous boundary value problem. To do this, let

$$\mu(\eta,t)=\alpha(t)+\frac{\eta}{L}\left(\beta(t)-\alpha(t)\right)=(1-\eta)t^0 \tag{37}$$

and $$u(\eta,t)=A(\eta,t)+\mu(\eta,t) \tag{38}$$

Then, $$\mu(\eta,0)=A(\eta,0)+\mu(\eta,0)=A(\eta,0)+(1-\eta)=0 \tag{39}$$

$$\Rightarrow A(\eta,0)=(\eta-1) \tag{40}$$

$$u(0,t) = A(0,t) + \mu(0,t) = A(0,t) + 1 = 1 \Rightarrow A(0,t) = 0 \tag{41}$$

$$u(1,t) = A(1,t) + \mu(1,t) = A(1,t) + 0 = 0 \Rightarrow A(1,t) = 0 \tag{42}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial A}{\partial t} + \frac{\partial \mu}{\partial t} = \frac{\partial A}{\partial t} + 0 = \frac{\partial A}{\partial t} \tag{43}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial A}{\partial \eta} + \frac{\partial \mu}{\partial \eta} = \frac{\partial A}{\partial \eta} - 1 \tag{44}$$

$$\frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 A}{\partial \eta^2} + \frac{\partial^2 \mu}{\partial \eta^2} = \frac{\partial^2 A}{\partial \eta^2} + 0 = \frac{\partial^2 A}{\partial \eta^2} \tag{45}$$

Then equation (35) – (36) reduce to

$$\frac{\partial A}{\partial t} = \frac{1}{\rho R_e}\frac{\partial^2 A}{\partial \eta^2} - \frac{\dot{M}_I}{\rho} - \frac{\beta}{\rho}(1-\eta) \tag{46}$$

$$\left.\begin{aligned} &A(\eta,0) = \eta - 1 \\ &A(0,t) = 0 \\ &A(1,t) = 0 \end{aligned}\right\} \tag{47}$$

Consider equation 48 to 53 (Myint-U and Debnath, 1987)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \alpha u + F(x,t) \tag{48}$$

$$\left.\begin{aligned} &u(x,0) = F(x) \\ &u(0,t) = 0 \\ &u(\mathrm{L},t) = 0 \end{aligned}\right\} \tag{49}$$

Assuming $\quad u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi x}{L}\right),$ (50)

where $\quad u_n(t) = \int_0^t \ell^{\left(\alpha - k\left(\frac{n\pi}{L}\right)^2\right)(t-\tau)} F_n(\tau)\,d\tau + b_n \ell^{\left(\alpha - k\left(\frac{n\pi}{L}\right)^2\right)t}$ (51)

$$F_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin\frac{n\pi x}{L}dx \tag{52}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{53}$$

Compare (46) and (47) with (48) and (49) respectively, we have

$$u = A, K = \frac{1}{\rho R_e}, \alpha = 0, F(\eta,t) = -\frac{1}{\rho}\left(\dot{M}_I - \beta\eta(1-\eta)\right), x = \eta, f(\eta) = \eta - 1, L = 1 \tag{54}$$

$$f(\eta) = \eta - 1 \Rightarrow b_n = 2\int_0^L \eta - 1\sin n\pi\eta d\eta \tag{55}$$

Then, $f(\eta) = 2\left(\int_0^1 \eta\sin n\pi\eta d\eta - \int_0^1 \sin n\pi\eta d\eta\right)$ (56)

Let $b_n = 2\left(\frac{(1-(-1)^n)}{n\pi} - \frac{(-1)^n}{n\pi}\right) = 2\left(\frac{(1-2(-1)^n)}{n\pi}\right) = q$ (57)

and $F_n(t) = -\frac{2}{\rho}\left(\int_0^1 \dot{M}_I \sin n\pi\eta d\eta - \beta\int_0^1 \eta\sin n\pi\eta d\eta + \beta\int_0^1 \eta^2\sin n\pi\eta d\eta\right)$ (58)

$$F_n(t) = -\frac{2}{\rho}\left(\frac{\left(\dot{M}_I(1-(-1)^n)\right)}{n\pi} - \frac{\beta(-1)^n}{n\pi} + \frac{\left((2-(n\pi)^2)(-1)^n - 2\right)}{(n\pi)^3}\right) \tag{59}$$

Let $F_n(t) = q_1$ (60)

Also, $A_n(t) = q_1\ell^{-\frac{1}{\rho R_e}(n\pi)^2 t}\int_0^t \ell^{-\frac{1}{\rho R_e}(n\pi)^2\tau}d\tau + q\ell^{-\frac{1}{\rho R_e}(n\pi)^2 t}$ (61)

Let $q_2 = \frac{1}{\rho R_e}(n\pi)^2$ (62)

Then $A_n(t) = \frac{q_1}{q_2}\ell^{-q_2 t}.\ell^{q_2\tau}\Big|_0^t + q\ell^{-q_2 t}$ (63)

$$A_n(t)=\frac{q_1}{q_2}\left(1-\ell^{-q_2t}\right)+q\ell^{-q_2t}=\left(\frac{q_1}{q_2}+\left(q-\frac{q_1}{q_2}\right)\ell^{-q_2t}\right) \tag{64}$$

Therefore, $$A(\eta,t)=\sum_{n=1}^{\infty}\left(\frac{q_1}{q_2}+\left(q-\frac{q_1}{q_2}\right)\ell^{-q_2t}\right)\sin n\pi\eta \tag{65}$$

Thus $$u(\eta,t)=(1-\eta)+\sum_{n=1}^{\infty}\left(\frac{q_1}{q_2}+\left(q-\frac{q_1}{q_2}\right)\ell^{-q_2t}\right)\sin n\pi\eta \tag{66}$$

2.5 Case 2: Viscous flow with leakage

In this case, equations (19)- (21) reduce to

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+R_{Leak}=0 \tag{67}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial(\rho u)}{\partial x}\right)=-\left(\dot{M}_I+\left(\dot{M}_0-\dot{M}_I\right)x\right)-\beta x(1-x)+\frac{1}{R_e}\frac{\partial^2 u}{\partial x^2}+V_{Leak} \tag{68}$$

$$\left.\begin{aligned}u(x,0)&=0\\ u(0,t)&=1\\ u(1,t)&=0\end{aligned}\right\} \tag{69}$$

2.6 Analytical solution of Viscous Flow with Leakage

Using (27) – (29) and from continuity equation (67), we obtain

$$u=-\left(\frac{1}{\rho}\int_0^x\frac{\partial\rho}{\partial t}ds+\frac{R_{Leak}}{\rho}\eta\right) \tag{70}$$

i.e. $$-\left(u+\frac{R_{Leak}}{\rho}\eta\right)=\frac{1}{\rho}\int_0^x\frac{\partial\rho}{\partial t}ds \tag{71}$$

Then $$\frac{\partial\eta}{\partial t}=-\left(u+\frac{R_{Leak}}{\rho}\eta\right) \tag{72}$$

$$\frac{\partial\eta}{\partial x}=1 \tag{73}$$

Then, the coordinate transformation become

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \eta} \tag{74}$$

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t}-\left(u+\frac{R_{Leak}}{\rho}\eta\right)\frac{\partial}{\partial \eta} \tag{75}$$

Using (71) - (75), then equations (67) – (69) can be simplified as

$$\rho\left(\frac{\partial u}{\partial t}-\frac{R_{Leak}}{\rho}\eta\frac{\partial u}{\partial \eta}\right)=\frac{1}{R_e}\frac{\partial^2 u}{\partial \eta^2}-\left(\dot{M}_I+\left(\dot{M}_0-\dot{M}_I\right)\eta\right)-\beta\eta\left(1-\eta\right)+V_{Leak} \tag{76}$$

Suppose the solution $u(\eta,t)$ can be expressed as:

$$u(\eta,t)=u_0(\eta,t)+R_{Leak}u_1(\eta,t) \tag{77}$$

Then, equation (76) becomes

$$\rho\left(\frac{\partial}{\partial t}(u_0+R_{Leak}u_1)-\frac{R_{Leak}}{\rho}\eta\frac{\partial}{\partial \eta}(u_0+R_{Leak}u_1)\right)=\frac{1}{R_e}\frac{\partial^2}{\partial \eta^2}(u_0+R_{Leak}u_1)- \left(\dot{M}_I+\left(\dot{M}_0-\dot{M}_I\right)\eta\right)-\beta\eta\left(1-\eta\right)+V_{Leak} \tag{78}$$

Collecting the like powers of R_{Leak} , we have for

${R_{Leak}}^0$:

$$\rho\frac{\partial u_0}{\partial t}=\frac{1}{R_e}\frac{\partial^2 u_0}{\partial \eta^2}-\left(\dot{M}_I+\left(\dot{M}_0-\dot{M}_I\right)\eta\right)-\beta\eta\left(1-\eta\right)+V_{Leak} \tag{79}$$

$$\left.\begin{array}{l} u_0(\eta,0)=0 \\ u_0(0,t)=1 \\ u_0(1,t)=0 \end{array}\right\} \tag{80}$$

${R_{Leak}}^1$:

$$\rho\left(\frac{\partial u_1}{\partial t}-\frac{1}{\rho}\eta\frac{\partial u_0}{\partial \eta}\right)=\frac{1}{R_e}\frac{\partial^2 u_1}{\partial \eta^2} \tag{81}$$

$$\left.\begin{aligned} u_1(\eta,0)&=0\\ u_1(0,t)&=0\\ u_1(1,t)&=0 \end{aligned}\right\} \tag{82}$$

Equations (79) – (80) is a non-homogeneous boundary value problem. So, we let

$$\mu_1(\eta,t)=\alpha(t)+\frac{\eta}{L}\left(\beta(t)-\alpha(t)\right)=(1-\eta)t^0 \tag{83}$$

and $\quad u_0(\eta,t)=B(\eta,t)+\mu_1(\eta,t)$ (84)

Then, equations (79) and (80) become

$$\frac{\partial B}{\partial t}=\frac{1}{\rho R_e}\frac{\partial^2 B}{\partial \eta^2}-\frac{1}{\rho}\left(\dot{M}_I+\left(\dot{M}_0-\dot{M}_I\right)\eta\right)-\frac{\beta}{\rho}\eta(1-\eta)+\frac{V_{Leak}}{\rho} \tag{85}$$

$$\left.\begin{aligned} B(\eta,0)&=\eta-1\\ B(0,t)&=0\\ B(1,t)&=0 \end{aligned}\right\} \tag{86}$$

Compare (85) and (86) with (48) and (49) respectively we have

$$\left.\begin{aligned} &u=B, k=\frac{1}{\rho R_e}, \alpha=0, F(\eta)=\eta-1, \mathrm{L}=1, x=\eta\\ &F(\eta,t)=\frac{1}{\rho}\left(V_{Leak}-\left(\dot{M}_I+\left(\dot{M}_0-\dot{M}_I\right)\eta\right)-\beta\eta(1-\eta)\right) \end{aligned}\right\} \tag{87}$$

Then $F(\eta)=n-1 \Rightarrow b_n=q$ (88)

and

$$F_n(t)=\frac{2}{\rho}\left(\begin{aligned} &\left(V_{Leak}-\dot{M}_I\right)\int_0^1 \sin n\pi\eta d\eta-\left(\dot{M}_0-\dot{M}_I+\beta\right)\int_0^1 \eta\sin n\pi\eta d\eta\\ &+\beta\int_0^1 \eta^2 \sin n\pi\eta d\eta \end{aligned}\right) \tag{89}$$

$$F_n(t)=\frac{2}{\rho}\left(\frac{\left(V_{Leak}-\dot{M}_I\right)\left(1-(-1)^n\right)}{n\pi}+\frac{\left(\dot{M}_0-\dot{M}_I+\beta\right)(-1)^n}{n\pi}+\frac{\left(\left(2-(n\pi)^2\right)(-1)^n-2\right)}{(n\pi)^3}\right) \tag{90}$$

Let $F_n(t) = q_3$ (91)

$$B_n(t) = q_3\ell^{-q_2 t}\int_0^t \ell^{q_2\tau}d\tau + q\ell^{-q_2 t} \quad (92)$$

$$B_n(t) = \frac{q_3}{q_2}\left(1-\ell^{-q_2 t}\right) + q\ell^{-q_2 t} = \left(\frac{q_3}{q_2} + \left(q - \frac{q_3}{q_2}\right)\ell^{-q_2 t}\right) \quad (93)$$

Therefore, $B(\eta,t) = \sum_{n=1}^{\infty}\left(\frac{q_3}{q_2} + \left(q - \frac{q_3}{q_2}\right)\ell^{-q_2 t}\right)\sin n\pi\eta$ (94)

Thus, $u_0(\eta,t) = (1-\eta) + \sum_{n=1}^{\infty}\left(\frac{q_3}{q_2} + \left(q - \frac{q_3}{q_2}\right)\ell^{-q_2 t}\right)\sin n\pi\eta$ (95)

But $\frac{\partial u_0}{\partial \eta} = -1 + \sum_{n=1}^{\infty} n\pi\left(\frac{q_3}{q_2} + \left(q - \frac{q_3}{q_2}\right)\ell^{-q_2 t}\right)\cos n\pi\eta$ (96)

Then, equations (81) and (82) become

$$\frac{\partial u_1}{\partial t} = \frac{1}{\rho R_e}\frac{\partial^2 u_1}{\partial \eta^2} + \sum_{n=1}^{\infty}(n\pi)\left(\frac{q_3}{q_2} + \left(q - \frac{q_3}{q_2}\right)\ell^{-q_2 t}\right)\eta\cos n\pi\eta - \eta \quad (97)$$

Compare (97) with (48) and (49), we have;

$$\left.\begin{aligned} &u = u_1, k = \frac{1}{\rho R_e}, \alpha = 0, f(\eta) = 0, \mathrm{L} = 1, x = \eta \\ &F(\eta,t) = \sum_{n=1}^{\infty}(n\pi)\left(\frac{q_3}{q_2} + \left(q - \frac{q_3}{q_2}\right)\ell^{-q_2 t}\right)\eta\cos n\pi\eta - \eta \end{aligned}\right\} \quad (98)$$

Then $f(\eta) = 0 \Rightarrow b_n = 0$ (99)

and $F_n(t) = 2\left(\sum_{n=1}^{\infty}(n\pi)\left(\frac{q_3}{q_2} + \left(q - \frac{q_3}{q_2}\right)\ell^{-q_2 t}\right)\int_0^1 \eta\cos n\pi\eta\sin n\pi\eta d\eta - \int_0^1 \eta\sin n\pi\eta d\eta\right)$ (100)

$$F_n(t) = 2\left(\sum_{n=1}^{\infty}\frac{\left(1-2(-1)^{2n}\right)}{4}\left(\frac{q_3}{q_2} + \left(q - \frac{q_3}{q_2}\right)\ell^{-q_2 t}\right) + \frac{(-1)^n}{n\pi}\right) \quad (101)$$

Let $F_n(t) = q_4$ (102)

Then $u_{1n}(t) = q_4 \ell^{-q_2 t} \int_0^t \ell^{q_2 \tau} d\tau + 0$ (103)

$$u_{1n}(t) = \frac{q_4}{q_2}\left(1 - \ell^{-q_2 t}\right) \quad (104)$$

Therefore; $u_1(\eta, t) = \sum_{n=1}^{\infty} \frac{q_4}{q_2}\left(1 - \ell^{-q_2 t}\right) \sin n\pi\eta$ (105)

Thus, $u(\eta, t) = u_0(\eta, t) + R_{Leak} u_1(\eta, t)$ (106)

$$u(\eta, t) = (1-\eta) + \sum_{n=1}^{\infty}\left(\frac{q_3}{q_2} + \left(q - \frac{q_3}{q_2}\right)\ell^{-q_2 t}\right)\sin n\pi\eta + R_{Leak}\sum_{n=1}^{\infty}\frac{q_4}{q_2}\left(1 - \ell^{-q_2 t}\right)\sin n\pi\eta \quad (107)$$

3.0 Results and Discussion

In analyzing the solution, we examine the effect of the measured inlet mass flow rate (m_i), measured outlet mass flow rate (m_0), elevation, pressure, the leak rate (R_{leak}), leak velocity (V_{leak}), the Reynold number (R_e) on the flow distribution.

3.1 Analysis of Results

The graphs for different cases are given below:

Case 1: Viscous flow without leakage

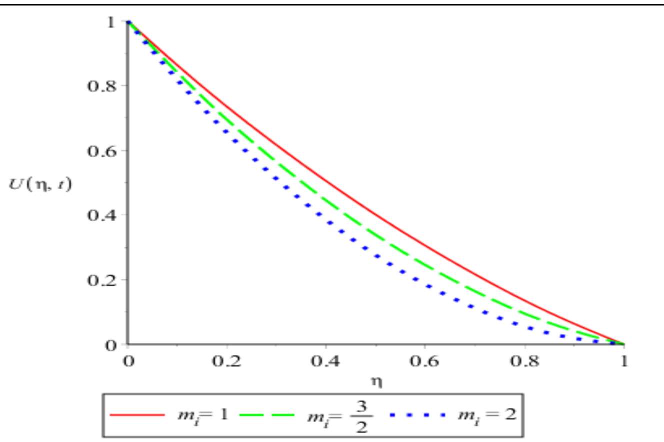

Figure 4.1: Flow distribution against distance at various values of m_i

Figure 4.1 depicts the graph of flow velocity against distance for various values of measured inlet mass flow rate m_i. It is observed that the flow velocity decreases along distance and decreases as measured inlet mass flow rate m_i value increases.

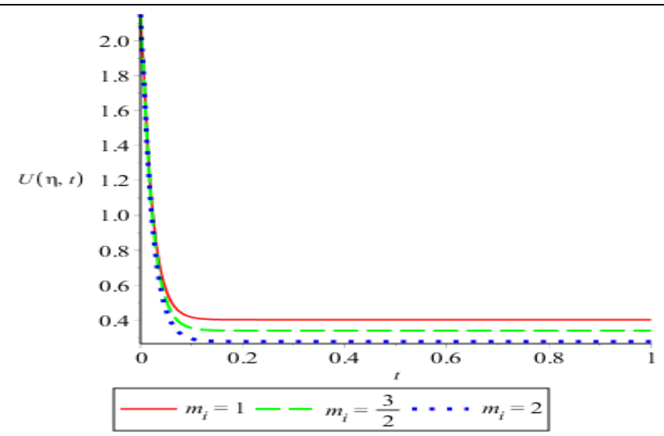

Figure 4.2: Flow distribution against time at various values of m_i

Figure 4.2 depicts the graph of flow velocity against time for various values of measured inlet mass flow rate m_i. It is observed that the flow velocity decreases and later becomes steady with time and decreases as measured inlet mass flow rate m_i value increases.

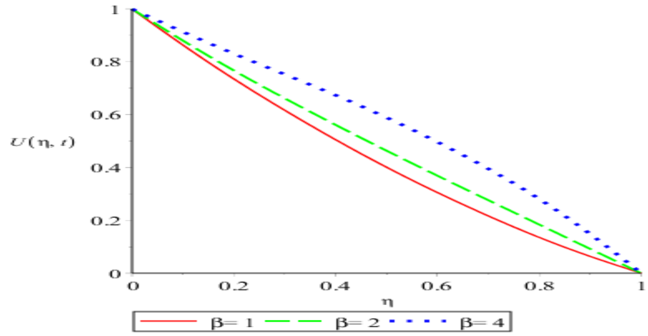

Figure 4.3: Flow distribution against distance at various values of β

Figure 4.3 depicts the graph of flow velocity against distance for various values of elevation β . It is observed that the flow velocity decreases along distance and increases as elevation value increases.

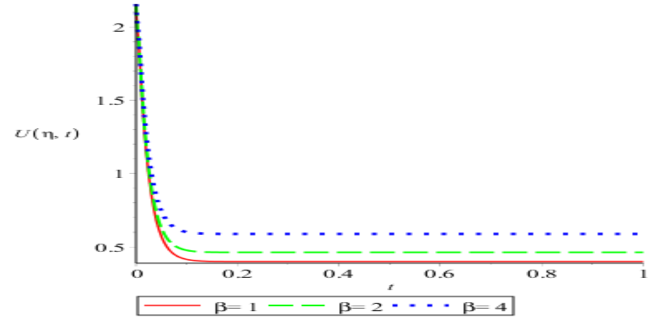

Figure 4.4: Flow distribution against time at various values of β

Figure 4.4 depicts the graph of flow velocity against time for various values of elevation β . It is observed that the flow velocity decreases and later becomes steady with time and increases as elevation value increases.

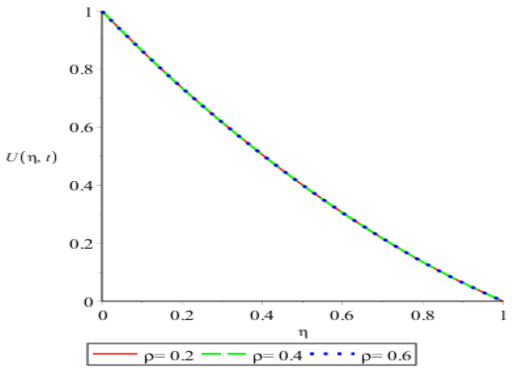

Figure 4.5: Flow distribution against distance at various values of ρ

Figure 4.5 depicts the graph of flow velocity against distance for various values of density ρ . It is observed that the flow velocity decreases along distance and decreases as density ρ value increases.

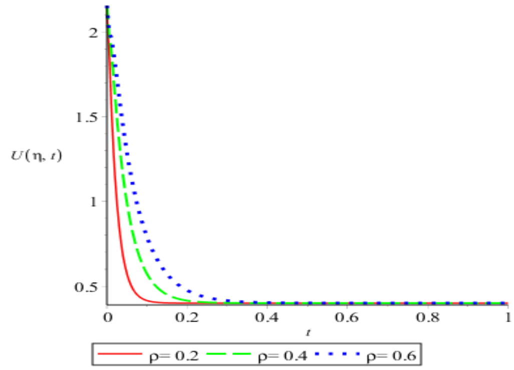

Figure 4.6: Flow distribution against time at various values of ρ

Figure 4.6 depicts the graph of flow velocity against time for various values of density ρ . It is observed that the flow velocity decreases and later becomes steady with time and increases as density ρ value increases.

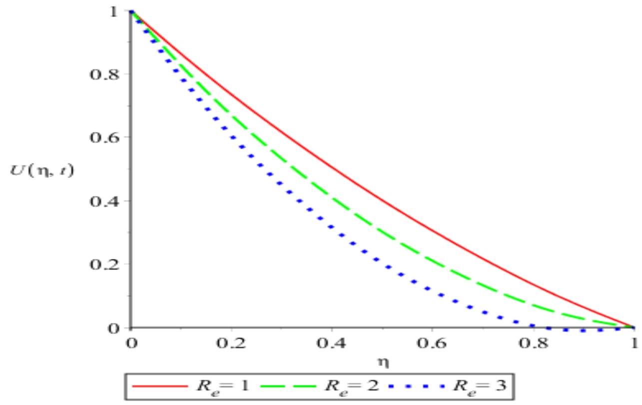

Figure 4.7: Flow distribution against distance at various values of R_e

Figure 4.7 depicts the graph of flow velocity against distance for various values of Reynolds' number R_e . It is observed that the flow velocity decreases along distance and decreases as Reynolds' number R_e value increases.

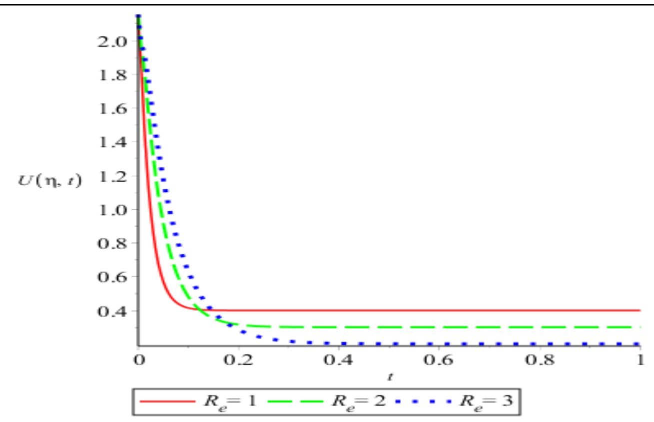

Figure 4.8: Flow distribution against time at various values of R_e

Figure 4.8 depicts the graph of flow velocity against time for various values of Reynolds' number R_e . It is observed that the flow velocity decreases and later becomes steady with time and increases as Reynolds' number R_e value increases.

Case 2: Viscous flow with leakage

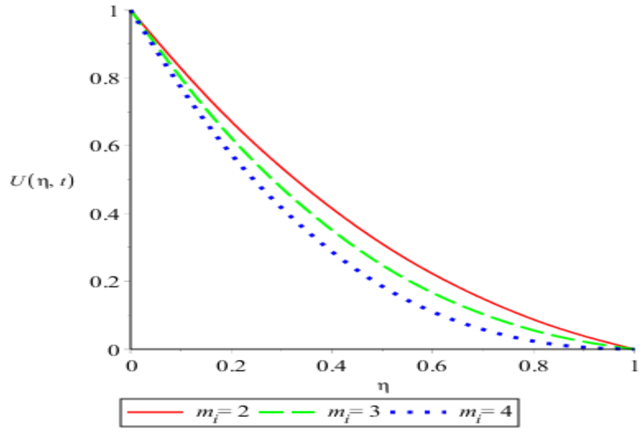

Figure 4.9: Flow distribution against distance at various values of m_i

Figure 4.9 depicts the graph of flow velocity against distance for various values of measured inlet mass flow rate m_i. It is observed that the flow velocity decreases along distance and decreases as measured inlet mass flow rate m_i value increases.

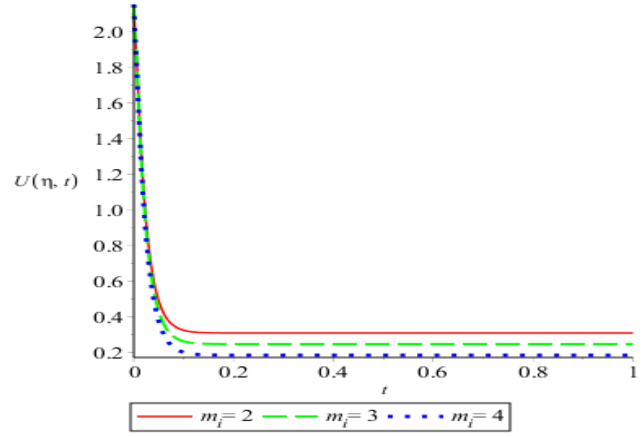

Figure 4.10: Flow distribution against time at various values of m_i

Figure 4.10 depicts the graph of flow velocity against time for various values of measured inlet mass flow rate m_i. It is observed that the flow velocity decreases and later becomes steady with time and decreases as measured inlet mass flow rate m_i value increases.

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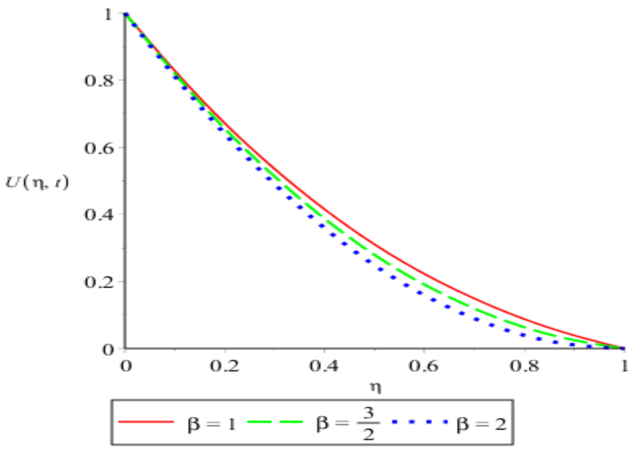


Figure 4.11: Flow distribution against distance at various values of β

Figure 4.11 depicts the graph of flow velocity against distance for various values of elevation β . It is observed that the flow velocity decreases along distance and decreases as elevation value increases.

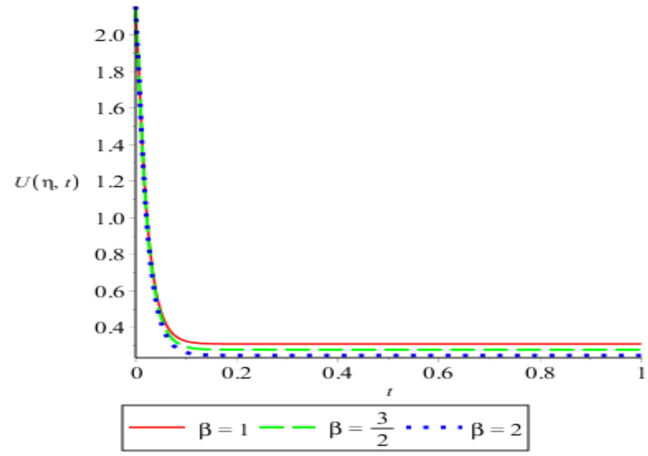


Figure 4.12: Flow distribution against time at various values of β

Figure 4.12 depicts the graph of flow velocity against time for various values of elevation β . It is observed that the flow velocity decreases and later becomes steady with time and decreases as elevation value increases.

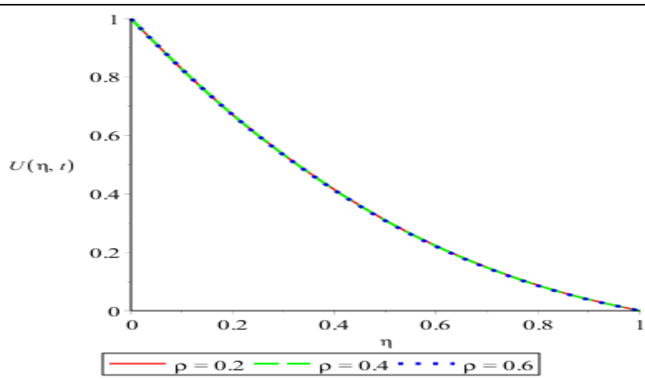


Figure 4.13: Flow distribution against distance at various values of ρ

Figure 4.13 depicts the graph of flow velocity against distance for various values of density ρ . It is observed that the flow velocity decreases along distance and decreases as density ρ value increases.

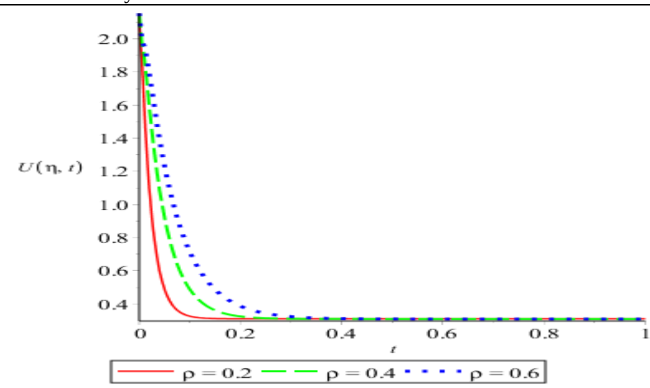


Figure 4.14: Flow distribution against time at various values of ρ

Figure 4.14 depicts the graph of flow velocity against time for various values of density ρ . It is observed that the flow velocity decreases and later becomes steady with time and increases as density ρ value increases.

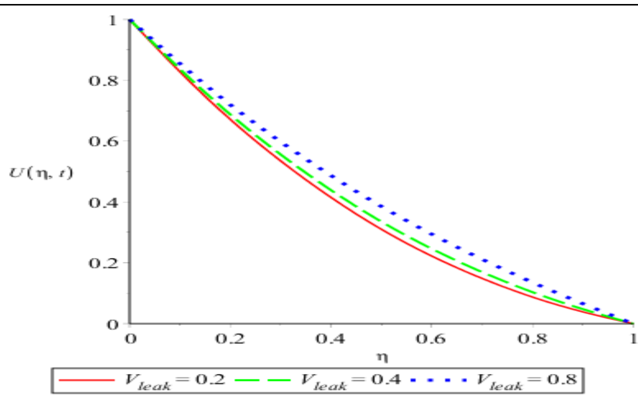


Figure 4.15: Flow distribution against distance at various values of V_{leak}

Figure 4.15 depicts the graph of flow velocity against distance for various

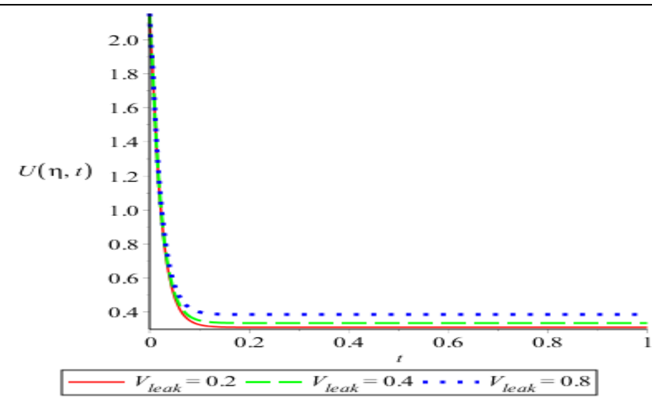


Figure 4.16: Flow distribution against time at various values of V_{leak}

Figure 4.16 depicts the graph of flow velocity against time for various values

values of leak velocity V_{leak} . It is observed that the flow velocity decreases along distance and increases as leak velocity V_{leak} value increases.

of leak velocity V_{leak} . It is observed that the flow velocity decreases and later becomes steady with time and increases as leak velocity V_{leak} value increases.

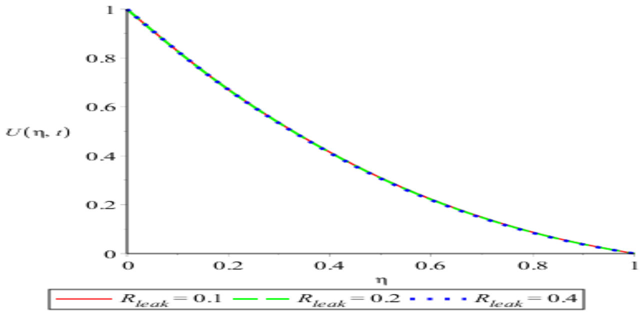


Figure 4.17: Flow distribution against distance at various values of R_{leak}

Figure 4.17 depicts the graph of flow velocity against distance for various values of leak rate R_{leak} . It is observed that the flow velocity decreases along distance and decreases as leak rate R_{leak} value increases.

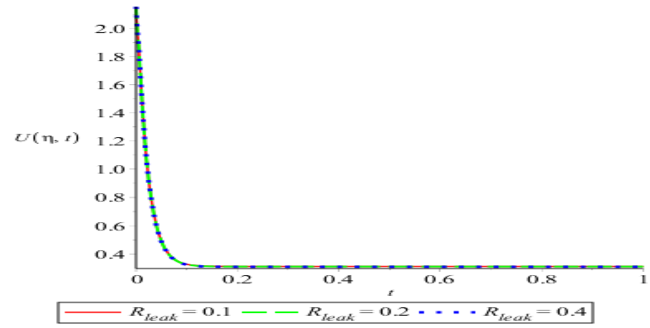


Figure 4.18: Flow distribution against time at various values of R_{leak}

Figure 4.18 depicts the graph of flow velocity against time for various values of leak rate R_{leak} . It is observed that the flow velocity decreases and later becomes steady with time and decreases as leak rate R_{leak} value increases.

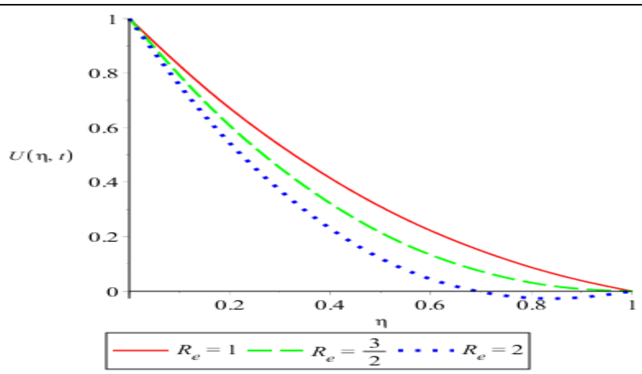


Figure 4.19: Flow distribution against distance at various values of R_e

Figure 4.19 depicts the graph of flow velocity against distance for various values of Reynolds' number R_e. It is observed that the flow velocity decreases along distance and decreases as Reynolds' number R_e value increases.

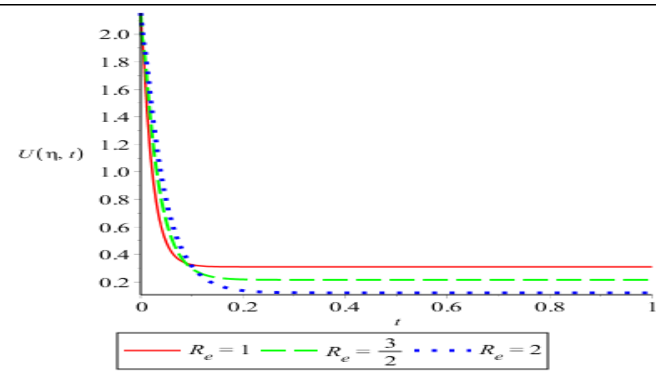
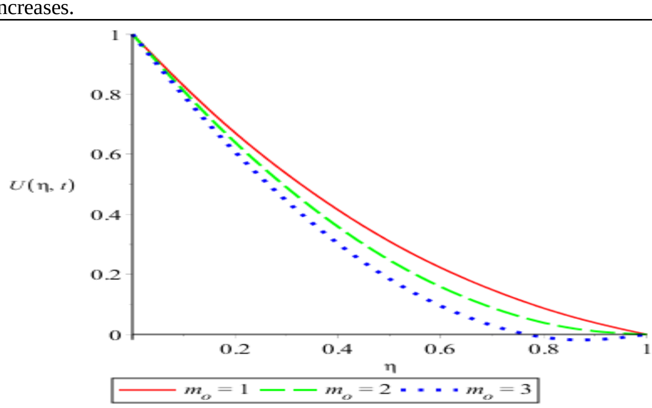


Figure 4.20: Flow distribution against time at various values of R_e

Figure 4.20 depicts the graph of flow velocity against time for various values of Reynolds' number R_e. It is observed that the flow velocity decreases and later becomes steady with time and increases as Reynolds' number R_e value increases.



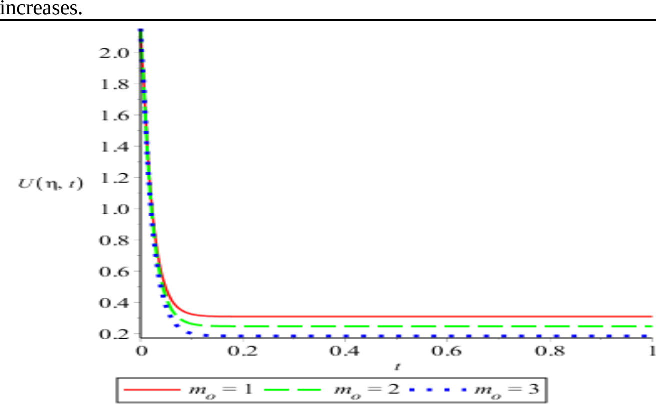


Figure 4.21: Flow distribution against distance at various values of m_0 Figure 4.21 depicts the graph of flow velocity against distance for various values of measured outlet mass flow rate m_0. It is observed that the flow velocity decreases along distance and decreases as measured outlet mass flow rate m_0 value increases.	**Figure 4.22: Flow distribution against time at various values of m_0** Figure 4.22 depicts the graph of flow velocity against time for various values of measured outlet mass flow rate m_0. It is observed that the flow velocity decreases and later becomes steady with time and decreases as measured outlet mass flow rate m_0 value increases.

5.0 Conclusion

In this research work, a mathematical model for detecting leakage of non-viscous and viscous fluid flow in a pipeline was formulated. The model equations evolved were considered in two cases:

i. Viscous flow without leakage ii. Viscous flow with leakage

The two cases were solved analytically and we observed that: the flow velocity, pressure, density measure inlet mass flow rate, measured outlet mass flow rate, elevation, leak velocity and Reynolds' number for the flow in the equations are not only fundamental but useful tools in detecting, localizing and analyzing leakage of viscous and non-viscous fluid in a pipe.

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