4-Step Block Hybrid Backward Differentiation Formula For Solving Second Order (BHBDF II) Ordinary Differential Equations

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Abstract

This research work presents the derivation and implementation of a 4-step linear multistep method of block hybrid backward differentiation formula for solving nonlinear second-order initial value problems of ordinary differential equations. Collocation and interpolation methods are adopted in the derivation of the proposed numerical scheme where the legendary polynomial is adopted as a basic function. The 4-step BHBDF has higher order of accuracy $p = 11$ which implies that it is consistent. The proposed numerical block method is further applied to finding direct solution to nonlinear second order ordinary differentiation equations. This implementation strategy is more accurate than some existing methods considered in the literature.

Keywords: linear multi step method, block method, hybrid, collocation and interpolation methods

1. Introduction

The differential equation is a fundamental tool utilized in the modeling of physical problems that arise across virtually all knowledge domains. In the fields of science and engineering, ordinary differential Equations (ODEs) are crucial for stimulating real-world problem. For instance, mathematical models are frequently created to facilitate comprehension of physical phenomena (Pimenov and Hendy 2015). There are various differential equations that we study in calculus to obtain closed form solutions, but not all differential equations have finite answers. Even if they have closed form solutions, obtaining them can be difficult. In such cases, depending on the need, numerical analysis researchers and scholars made significant contributions to the development and derivation of several methods for solving this type of equation (Muhammad and Yahaya, 2012; Kuboye and Omar, 2015; Abdulrahim and Omar, 2016; Muhammad, *et al*., 2020). Because exact solutions to differential equations are difficult to find, numerical methods play an important role in providing approximate solutions to them. The type and category of ordinary differential equations (ODEs) to be solved drive the creation of numerical methods such as the linear multistep approach, the single-step method, finite difference, finite element, and finite volume. The mathematical representation of physical problems leads to the approximation of ordinary differential equations (ODEs) using initial or boundary conditions (Omole *et al*., 2023). Differential equations fall into two categories: first are ordinary differential equations (ODEs) (Zwillinger, Danie and l 2021)
Zwillinger, Danie, and and Vladimir Dobrushkin. l. *Handbook of differential equations. .* 2021.

, where the unknown function is a function of just one independent variable and is expressed as follows:

$$f\left(t, u(t), \frac{du(t)}{dt}, \frac{d^2u(t)}{dt^2}, \frac{d^3u(t)}{dt^3}, \ldots, \frac{d^nu(t)}{dt^n}\right) = 0 \quad (1)$$

And partial differential equations (PDEs) in which the unknown function is a function of two or more independent variables and the equation involves its partial derivatives. In real life, ordinary differential equations are used to explain thermodynamics ideas, compute the movement or flow of electricity, and move an object back and forth like a pendulum.

2. Literature Review

Muhammad *et al.* (2020) reformulated an implicit block hybrid backward differentiation formula for $k = 2$ into a Runge-Kutta Type Method (RKTM of the same step number. The method so formed can be used to solve both first and second order IVPs. Mohammed *et al.* (2022) formulated block hybrid backward differentiation formula for the solution of stiff systems of ordinary differential equation through continuous collocation approach. In the research work, k off - grid points were incorporated at interpolation in order to retain the single function evaluation characteristic, which is peculiar to backward differentiation formula. They are frequently employed in fields like physics and engineering, where it might be challenging to get the precise answer. Akinfenwa *et al.*, (2013), Adeyeye and Omar (2016), Audu *et al.*, (2022), Cardone and D'Ambrosio (2018), Yahaya and Mohammed (2010), Mohammed and Yahaya (2010), Mohammed (2010), Muhammad *et al*. (2015a), Muhammad *et al*. (2015b). Also Akinfenwa *et al.* (2017) derived a self-starting second derivative multistep block method which uses logic behind the Simpson's 3/8 rule for quadrature using collocation and interpolation techniques to obtain approximate solutions to stiff differential equations. Their method is of order eight and A-Stable, effective and reliable for stiff systems of ordinary differential equations. Using a new class of orthogonal polynomials that Adeyefa (2017) created, a one-step hybrid block technique was developed using this new basis function. The others are Yukusak and Owolanke (2018), Abualnaja (2015), Olabode and Momoh (2016), Mohammed *et al*. (2021), and Olanegan *et al.* (2015).

3. Methodology

The derivation process starts from an orthogonal polynomial of the form:

$$U(t) = \sum_{i=0}^{r+s-1} c_j L_i(t) \qquad (2)$$

where r and s are number of interpolations and collocations points respectively, c_i's are coefficients (to be determined) of the Legendre polynomial. In this project a class of 4-step hybrid block method is proposed for solving ODEs in second order.

3.1 4-Step Block Hybrid BDF for Second Order (BHBDF II)

As specified above, the degree of the Legendre polynomial for the proposed method for approximating the exact solution of ODEs is 12 and it is given be

$$\left.\begin{aligned}
U(t) = {} & c_0 + c_1 t + \left(\frac{3}{2}t^2 - \frac{1}{2}\right)c_2 + \left(\frac{5}{2}t^3 - \frac{3}{2}t\right)c_3 + \left(\frac{35}{8}t^4 - \frac{15}{4}t^2 + \frac{3}{8}\right)c_4 + \left(\frac{63}{8}t^5 - \frac{35}{4}t^3 + \frac{15}{8}t\right)c_5 \\
& + \left(\frac{231}{16}t^6 - \frac{315}{16}t^4 + \frac{105}{16}t^2 - \frac{5}{16}\right)c_6 + \left(\frac{429}{16}t^7 - \frac{693}{16}t^5 + \frac{315}{16}t^3 - \frac{35}{16}t\right)c_7 \\
& + \left(\frac{6435}{128}t^8 - \frac{3003}{32}t^6 + \frac{3465}{64}t^4 - \frac{315}{32}t^2 + \frac{35}{128}\right)c_8 + \left(\frac{12155}{128}t^9 - \frac{6435}{32}t^7 + \frac{9009}{64}t^5 - \frac{1155}{32}t^3 + \frac{315}{128}t\right)c_9 \\
& + \left(\frac{46189}{256}t^{10} - \frac{109395}{256}t^8 + \frac{45045}{128}t^6 - \frac{15015}{256}t^4 + \frac{3465}{256}t^2 - \frac{63}{256}\right)c_{10} \\
& + \left(\frac{88179}{256}t^{11} - \frac{230935}{256}t^9 + \frac{109395}{128}t^7 - \frac{45045}{128}t^5 + \frac{15015}{256}t^3 - \frac{693}{256}t\right)c_{11} \\
& + \left(\frac{676039}{512}t^{12} - \frac{969969}{512}t^{10} + \frac{2078505}{1024}t^8 - \frac{225225}{1024}t^6 + \frac{225225}{1024}t^4 - \frac{9009}{512}t^2 + \frac{231}{1024}\right)c_{12}
\end{aligned}\right\}$$

(3)

Interpolating (4) at $t=\frac{t_{i+j}}{5}, 0\le j<4, \left(t_{i+j}=t_i+jh\right)$ and collocating its second derivative at $t=t_{i+4}$, lead to the following 13×13 system of equations. h is a given step size.

$$\left.\begin{array}{l} U(t_i)=u_i,\ U\left(t_{i+\frac{1}{5}}\right)=u_{i+\frac{1}{5}},\ U\left(t_{i+\frac{2}{5}}\right)=u_{i+\frac{2}{5}} \\ U(t_{i+1})=u_{i+1},\ U\left(t_{i+\frac{6}{5}}\right)=u_{i+\frac{6}{5}}\ U\left(t_{i+\frac{7}{5}}\right)=u_{i+\frac{7}{5}}, \\ U(t_{i+2})=u_{i+2},\ U\left(t_{i+\frac{11}{5}}\right)=u_{i+\frac{11}{5}},\ U\left(t_{i+\frac{12}{5}}\right)=u_{i+\frac{12}{5}} \\ U(t_{i+3})=u_{i+3},\ U\left(t_{i+\frac{16}{5}}\right)=u_{i+\frac{16}{5}},\ U\left(t_{i+\frac{17}{5}}\right)=u_{i+\frac{17}{5}} \\ U''(t_{i+4})=v_{i+4} \end{array}\right\} \quad (4)$$

Using the matrix inversion technique via the Maple software, (5) is solved for c_j's whose values are the substituted back into (3.4) and then simplified to obtain the continuous collocation method of the form:

$$\begin{aligned} U(t)=&\alpha_0(t)u_i+\alpha_{\frac{1}{5}}(t)u_{i+\frac{1}{5}}+\alpha_{\frac{2}{5}}(t)u_{i+\frac{2}{5}}+\alpha_1(t)u_{i+1}+\alpha_{\frac{6}{5}}(t)u_{i+\frac{6}{5}}+\alpha_{\frac{7}{5}}(t)u_{i+\frac{7}{5}}+\alpha_2(t)u_{i+2} \\ &+\alpha_{\frac{11}{5}}(t)u_{i+\frac{11}{5}}+\alpha_{\frac{12}{5}}(t)u_{i+\frac{12}{5}}+\alpha_3(t)u_{i+3}+\alpha_{\frac{16}{5}}(t)u_{i+\frac{16}{5}}+\alpha_{\frac{17}{5}}(t)u_{i+\frac{17}{5}}+h^2\beta_4(t)v_{i+4} \end{aligned} \quad (5)$$

Equation (6) is the continuous scheme of the proposed method whose by the tradition of BDF has only one evaluation point which is $t=t_i+4h$. And upon implementing this, the following obtainable scheme from (6) is given below.

$$\begin{aligned} u_{i+4}=&-\frac{4899694293}{3719054980}u_i+\frac{993505500}{120322367}u_{i+\frac{1}{5}}-\frac{169955950}{10938397}u_{i+\frac{2}{5}}+\frac{94709296656}{601611835}u_{i+1}-\frac{50567692500}{120322367}u_{i+\frac{6}{5}} \\ &+\frac{4172844600}{10938397}u_{i+\frac{7}{5}}-\frac{34685775774}{54691985}u_{i+2}+\frac{10813166000}{10938397}u_{i+\frac{11}{5}}-\frac{63868346100}{120322367}u_{i+\frac{12}{5}} \\ &+\frac{9906198264}{54691985}u_{i+3}-\frac{77374571625}{481289468}u_{i+\frac{16}{5}}+\frac{96247674900}{2045480239}u_{i+\frac{17}{5}}+\frac{1120392}{54691985}h^2v_{i+4} \end{aligned}$$

(6)

The proposed method is aimed to be implemented to solve ODEs via block method. Hence there is need to generate additional schemes to (7) from the same continuous scheme that would meet the algebraic condition for unique solution. To do this, the second derivative of (6) is evaluated at $t_{i+j};\ j=\frac{1}{5},\frac{2}{5},\frac{6}{5},\frac{7}{5},2,\frac{11}{5},\frac{12}{5},3,\frac{16}{5},\frac{17}{5},4$

$$u_{i+\frac{1}{5}} = \frac{118776743969751}{132317548848290}u_i - \frac{7430962304040}{10118400794281}u_{i+\frac{2}{5}} + \frac{2516108892252}{299360970245}u_{i+1} - \frac{13699667032800}{778338522637}u_{i+\frac{6}{5}} + \frac{762446279980}{59872194049}u_{i+\frac{7}{5}} - \frac{43811811253144}{3891692613185}u_{i+2} + \frac{11085490664136}{778338522637}u_{i+\frac{11}{5}} - \frac{369275089230}{59872194049}u_{i+\frac{12}{5}} + \frac{50435475197052}{50592003971405}u_{i+3} - \frac{999470883023}{1556677045274}u_{+\frac{16}{5}} + \frac{1674488981985}{13231754884829}u_{i+\frac{17}{5}} - \frac{727709675616}{14788431930103}h^2v_{i+\frac{1}{5}} - \frac{2064130992}{73942159650515}h^2v_{i+4} \quad (7)$$

$$u_{i+\frac{2}{5}} = -\frac{162949331766897}{6730172803077020}u_i + \frac{688549885920}{1041822415337}u_{i+\frac{1}{5}} + \frac{8642709843456}{5209112076685}u_{i+1} - \frac{49774082502690}{19794625891403}u_{i+\frac{6}{5}} + \frac{2621794350240}{1799511444673}u_{i+\frac{7}{5}} - \frac{7843200018381}{8997557223365}u_{i+2} + \frac{20104511685440}{19794625891403}u_{i+\frac{11}{5}} - \frac{427817388609}{1041822415337}u_{i+\frac{12}{5}} + \frac{513518289696}{8997557223365}u_{i+3} - \frac{2803874420295}{79178503565612}u_{i+\frac{16}{5}} + \frac{2274177560928}{336508640153851}u_{i+\frac{17}{5}} - \frac{47778918096}{1799511444673}h^2v_{i+\frac{2}{5}} - \frac{11843832}{8997557223365}h^2v_{i+4} \quad (8)$$

$$u_{i+1} = \frac{43265431323857}{303982967085952}u_i - \frac{143367625}{426884812}u_{i+\frac{1}{5}} + \frac{3314955875}{11099005112}u_{i+\frac{2}{5}} + \frac{2649296125}{1280654436}u_{i+\frac{6}{5}} - \frac{2223555125}{1494096842}u_{i+\frac{7}{5}} + \frac{1117406411}{853769624}u_{i+2} - \frac{1942982125}{1173933233}u_{i+\frac{11}{5}} + \frac{1223646125}{1707539248}u_{i+\frac{12}{5}} - \frac{964258361}{8324253834}u_{i+3} + \frac{1019235875}{13660313984}u_{i+\frac{16}{5}} - \frac{1814552875}{123369710668}u_{i+\frac{17}{5}} + \frac{164075955}{14514083608}hu_i' + \frac{693}{213442406}h^2v_{i+4} \quad (9)$$

$$u_{i+\frac{6}{5}} = -\frac{59524779287}{159151539026620}u_i + \frac{1326433964}{468092761843}u_{i+\frac{1}{5}} - \frac{83450292145}{12170411807918}u_{i+\frac{2}{5}} + \frac{1191667384144}{2340463809215}u_{i+1} + \frac{230525811560}{468092761843}u_{i+\frac{7}{5}} + \frac{132388629703}{4680927618430}u_{i+2} - \frac{3012664208}{66870394549}u_{i+\frac{11}{5}} + \frac{9969019315}{468092761843}u_{i+\frac{12}{5}} - \frac{111667102712}{30426029519795}u_{i+3} + \frac{4435631215}{18723710473372}u_{i+\frac{16}{5}} - \frac{3713585660}{7957576951331}u_{i+\frac{17}{5}} - \frac{8663210424}{468092761843}h^2v_{i+\frac{6}{5}} + \frac{236808}{2340463809215}h^2v_{i+4}$$

(10)

$$u_{i+\frac{7}{5}} = -\frac{35729475129}{31239901916740}u_i + \frac{4920894145}{643174451227}u_{i+\frac{1}{5}} - \frac{1683160836}{108587894363}u_{i+\frac{2}{5}} + \frac{539977310436}{3215872256135}u_{i+1} + \frac{42463381920}{91882064461}u_{i+\frac{6}{5}} + \frac{307950693356}{292352023285}u_{i+2} - \frac{93728957670}{91882064461}u_{i+\frac{11}{5}} + \frac{236475241011}{643174451227}u_{i+\frac{12}{5}} - \frac{163964431473}{3800576302705}u_{i+3} + \frac{9558245185}{367528257844}u_{i+\frac{16}{5}} - \frac{53101972005}{10933965670859}u_{i+\frac{17}{5}} - \frac{4725387504}{108587894363}h^2 v_{i+\frac{7}{5}} + \frac{484488}{542939471815}h^2 v_{i+4}$$

(11)

$$u_{i+2} = -\frac{220572654003}{153425883810892}u_i + \frac{281300270000}{29331418963847}u_{i+\frac{1}{5}} - \frac{673791569625}{34664404230001}u_{i+\frac{2}{5}} + \frac{1165672110144}{4190202709121}u_{i+1} - \frac{3771743943750}{4190202709121}u_{i+\frac{6}{5}} + \frac{220713488000}{205114817929}u_{i+\frac{7}{5}} + \frac{439481520000}{4190202709121}u_{i+\frac{11}{5}} + \frac{2032984418625}{4190202709121}u_{i+\frac{12}{5}} - \frac{2223584219904}{34664404230001}u_{i+3} + \frac{4503232956875}{117325675855388}u_{i+\frac{16}{5}} - \frac{38984754000}{5479495850389}u_{i+\frac{17}{5}} - \frac{23626937520}{380927519011}h^2 u_{i+2} + \frac{484488}{380927519011}h^2 v_{i+4}$$

(12)

$$u_{i+\frac{11}{5}} = -\frac{1610931429}{55387820088340}u_i + \frac{199983510}{1140337472407}u_{i+\frac{1}{5}} - \frac{4610182940}{14824387141291}u_{i+\frac{2}{5}} + \frac{9496656336}{5701687362035}u_{i+1} - \frac{155117232}{162905353201}u_{i+\frac{6}{5}} - \frac{1732516335}{325810706402}u_{i+\frac{7}{5}} + \frac{410605748388}{814526766005}u_{i+2} + \frac{577473159735}{1140337472407}u_{i+\frac{12}{5}} - \frac{1448598058281}{148243871412910}u_{i+3} + \frac{21929922459}{4561349889628}u_{i+\frac{16}{5}} - \frac{2149394970}{2769391004417}u_{i+\frac{17}{5}} - \frac{2887736808}{162905353201}h^2 v_{i+\frac{11}{5}} + \frac{78936}{814526766005}h^2 v_{i+4}$$

(13)

$$u_{i+\frac{12}{5}} = -\frac{737000782381}{1880754589636660}u_i + \frac{139727744540}{553163114599}u_{i+\frac{1}{5}} - \frac{353004129610}{7191120489787}u_{i+\frac{2}{5}} + \frac{1571408160272}{2765815572995}u_{i+1} - \frac{893315875620}{553163114599}u_{i+\frac{6}{5}} + \frac{875764873192}{553163114599}u_{i+\frac{7}{5}} - \frac{10327761506018}{2765815572995}u_{i+2} + \frac{2546494175120}{553163114599}u_{i+\frac{11}{5}} - \frac{23121588004856}{35955602448935}u_{i+3} + \frac{35955602448935}{2212652458396}u_{i+\frac{16}{5}} - \frac{486856004716}{9403772948183}u_{i+\frac{17}{5}} + \frac{60642472968}{553163114599}h^2 v_{i+\frac{12}{5}} + \frac{19215504}{2765815572995}h^2 v_{i+4} \quad (14)$$

$$u_{i+3} = \frac{21843227531489}{19311753828207236}u_i - \frac{2070677464375}{283996379826577}u_{i+\frac{1}{5}} + \frac{365735910500}{25817852711507}u_{i+\frac{2}{5}} - \frac{46377146659948}{283996379826577}u_{i+1} + \frac{131376152180000}{283996379826577}u_{i+\frac{6}{5}} - \frac{11647928985875}{25817852711507}u_{i+\frac{7}{5}} + \frac{27445244184332}{25817852711507}u_{i+2} - \frac{576419714438750}{283996379826577}u_{i+\frac{11}{5}} + \frac{417083651983375}{283996379826577}u_{i+\frac{12}{5}} + \frac{751171463501875}{1135985519306308}u_{i+\frac{16}{5}} - \frac{93560142294125}{4827938457051809}u_{i+\frac{17}{5}} - \frac{716683771440}{25817852711507}h^2 v_{i+3} - \frac{35531496}{25817852711507}h^2 v_{i+4} \quad (15)$$

$$u_{i+\frac{16}{5}} = \frac{15521609114019}{4030212808091735}u_i - \frac{1152542307152}{47414268330491}u_{i+\frac{1}{5}} + \frac{28416106407480}{616385488296383}u_{i+\frac{2}{5}} - \frac{114114431032512}{237071341652455}u_{i+1} + \frac{61723730000688}{47414268330491}u_{i+\frac{6}{5}} - \frac{56873025990880}{47414268330491}u_{i+\frac{7}{5}} + \frac{502758613885528}{237071341652455}u_{i+2} - \frac{161062432009920}{47414268330491}u_{i+\frac{11}{5}} + \frac{88992550681380}{47414268330491}u_{i+\frac{12}{5}} + \frac{362584687682976}{3081927441481915}u_{i+3} + \frac{511877886577680}{806042561618347}u_{i+\frac{17}{5}} - \frac{1455419351232}{47414268330491}h^2 v_{i+\frac{16}{5}} - \frac{4128261984}{237071341652455}h^2 v_{i+4} \quad (16)$$

$$u_{i+\frac{17}{5}} = \frac{370330170269}{25192113745610}u_i - \frac{700687613545}{7557634123683}u_{i+\frac{1}{5}} + \frac{3983479568552}{22672902371049}u_{i+\frac{2}{5}} - \frac{9876629706908}{5398310088345}u_{i+1} + \frac{5337402515680}{1079662017669}u_{i+\frac{6}{5}} - \frac{34398062616676}{7557634123683}u_{i+\frac{7}{5}} + \frac{43482341208152}{5398310088345}u_{i+2} - \frac{41985676126040}{3238986053007}u_{i+\frac{11}{5}} + \frac{2611462130050}{359887339223}u_{i+\frac{12}{5}} - \frac{120657940271924}{37788170618415}u_{i+3} + \frac{48130212038465}{15115268247366}u_{i+\frac{16}{5}} + \frac{3850315744}{359887339223}h^2 v_{i+\frac{17}{5}} - \frac{136668752}{1799436696115}h^2 v_{i+4} \quad (17)$$

Also, the first derivative of (3.6) is evaluated at the same points as above in order to cater for the first derivative term of a general second order ODE in the implementation process of the method.

$$\begin{aligned} u_{i+\frac{1}{5}} = &-\frac{8675155969326}{60196975329685}u_i + \frac{14021918174880}{9206596226893}u_{i+\frac{2}{5}} - \frac{11285871779952}{3540998548805}u_{i+1} + \frac{4482715805568}{708199709761}u_{i+\frac{6}{5}} \\ &- \frac{3135340103280}{708199709761}u_{i+\frac{7}{5}} + \frac{13130988065888}{3540998548805}u_{i+2} - \frac{3288004229664}{708199709761}u_{i+\frac{11}{5}} + \frac{1411908344280}{708199709761}u_{i+\frac{12}{5}} \\ &- \frac{14573246657904}{46032981134465}u_{i+3} + \frac{143792861982}{708199709761}u_{i+\frac{16}{5}} - \frac{480094679295}{12039395065937}u_{i+\frac{17}{5}} - \frac{121284945936}{708199709761}hu'_{i+\frac{1}{5}} \\ &+ \frac{30735936}{3540998548805}h^2 v_{i+4} \end{aligned} \tag{18}$$

$$\begin{aligned} u_{i+\frac{2}{5}} = &-\frac{31702636997577}{76743011328170}u_i + \frac{2216148348960}{451429478401}u_{i+\frac{1}{5}} - \frac{54994118804352}{2257147392005}u_{i+1} + \frac{20478615049620}{451429478401}u_{i+\frac{6}{5}} \\ &- \frac{1250059329792}{41039043491}u_{i+\frac{7}{5}} + \frac{4908496858083}{205195217455}u_{i+2} - \frac{13353572261120}{451429478401}u_{i+\frac{11}{5}} + \frac{5677108568586}{451429478401}u_{i+\frac{12}{5}} \\ &- \frac{401274736128}{205195217455}u_{i+3} + \frac{1126773690705}{902858956802}u_{i+\frac{16}{5}} - \frac{1873115296416}{7674301132817}u_{i+\frac{17}{5}} + \frac{71668377144}{41039043491}hu'_{i+\frac{2}{5}} \\ &+ \frac{10732176}{205195217455}h^2 v_{i+4} \end{aligned} \tag{19}$$

$$\begin{aligned} u_{i+3} = &-\frac{1422747654237}{209457042650042}u_i + \frac{24226868250}{560045568583}u_{i+\frac{1}{5}} - \frac{46424385000}{560045568583}u_{i+\frac{2}{5}} + \frac{5565659882328}{6160501254413}u_{i+1} \\ &- \frac{15362049976000}{6160501254413}u_{i+\frac{6}{5}} + \frac{1319675686875}{560045568583}u_{i+\frac{7}{5}} - \frac{2661475477752}{560045568583}u_{i+2} + \frac{50987263918500}{6160501254413}u_{i+\frac{11}{5}} \\ &- \frac{2924930531250}{560045568583}u_{i+\frac{12}{5}} + \frac{27262360841625}{12321002508826}u_{i+\frac{16}{5}} - \frac{1289764271250}{5512027438159}u_{i+\frac{17}{5}} - \frac{358341885720}{560045568583}hu'_{i+3} \\ &+ \frac{10732176}{560045568583}h^2 v_{i+4} \end{aligned} \tag{20}$$

$$\begin{aligned}u_{i+\frac{16}{5}} &= -\frac{147308208201}{48908643201535}u_i + \frac{10988954592}{575395802371}u_{i+\frac{1}{5}} - \frac{272379024720}{7480145430823}u_{i+\frac{2}{5}} + \frac{1118270418048}{2876979011855}u_{i+1} \\ &- \frac{611251923744}{575395802371}u_{i+\frac{6}{5}} + \frac{570588800320}{575395802371}u_{i+\frac{7}{5}} - \frac{5398656452112}{2876979011855}u_{i+2} + \frac{1805963499648}{575395802371}u_{i+\frac{11}{5}} \\ &- \frac{1068920095320}{575395802371}u_{i+\frac{12}{5}} + \frac{56898239340096}{37400727154115}u_{i+3} - \frac{2185664231520}{9781728640307}u_{i+\frac{17}{5}} + \frac{121284945936}{575395802371}hu'_{i+\frac{16}{5}} \\ &+ \frac{30735936}{2876979011855}h^2 v_{i+4}\end{aligned} \tag{21}$$

$$\begin{aligned}u_{i+\frac{17}{5}} &= \frac{215079595038}{39009452634895}u_i - \frac{271730159085}{7801890526979}u_{i+\frac{1}{5}} + \frac{20119102506272}{304273730552181}u_{i+\frac{2}{5}} - \frac{27057554793808}{39009452634895}u_{i+1} \\ &+ \frac{14671221962880}{7801890526979}u_{i+\frac{6}{5}} - \frac{13562180574096}{7801890526979}u_{i+\frac{7}{5}} + \frac{122356216558752}{39009452634895}u_{i+2} - \frac{119450162548640}{23405671580937}u_{i+\frac{11}{5}} \\ &+ \frac{22640428824552}{7801890526979}u_{i+\frac{12}{5}} - \frac{756052232336016}{507122884253635}u_{i+3} + \frac{16153821432210}{7801890526979}u_{i+\frac{16}{5}} + \frac{687281360304}{7801890526979}hu'_{i+\frac{17}{5}} \\ &- \frac{986965056}{39009452634895}h^2 v_{i+4}\end{aligned} \tag{22}$$

3.1 Convergence Properties of the Methods

In this section, we examine the order and error constant for consistency and also zero stability. According Lambert (1973), zero stability and consistency of a linear multistep method are necessary and sufficient conditions for its convergence.

3.3 Order and error constants

The linear differential operator L associated with (6) is defined by

$$L\left[u(t);h\right] = \sum_{j=0}^{k}\left[\alpha_j u(t+jh) - h^n \beta_j u^{(n)}(t+jh)\right] \tag{23}$$

where n is the order of the differential equation considered. Expanding (8) in Taylor series, we have

$$L\left[u(t);h\right] = c_0 u(t) + c_1 hu'(t) + c_2 hu''(t) + \ldots + c_q h^q u^q(t) \tag{24}$$

where

$$\begin{aligned}c_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_k \\ c_1 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + k\alpha_k \\ c_2 &= \frac{1}{2!}\left(\alpha_1 + 2^2\alpha_2 + 3^2\alpha_3 + \ldots + k^2\alpha_k\right) - \left(\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_k\right) \\ &\vdots \\ c_p &= \frac{1}{p!}\left(\alpha_1 + 2^p\alpha_2 + 3^p\alpha_3 + \ldots + k^p\alpha_k\right) - \frac{1}{(p-n)!}\left(\beta_1 + 2^{p-2}\beta_2 + 3^{p-2}\beta_3 + \ldots + k^{p-2}\beta_k\right) \\ p &\geq n\end{aligned}$$

Definition 3.1: The LMM (8) is said to be of order *p* if $c_0 = c_1 = c_2 = ... = c_p = c_{p+1} = 0$ and $c_{p+n} \neq 0$ is the error constant.

Definition 3.2: The LMM (8) is said to be consistent if it has order of accuracy *p*>1

3.2 Order and error constants of BHBDF II

Considering the discrete scheme (7), the coefficients are given as follows:

$$\alpha_0 = -\frac{4899694293}{3719054980},\ \alpha_{\frac{1}{5}} = \frac{993505500}{120322367},\ \alpha_{\frac{2}{5}} = -\frac{169955950}{10938397},\ \alpha_1 = \frac{94709296656}{601611835}$$
$$\alpha_{\frac{6}{5}} = -\frac{50567692500}{120322367},\ \alpha_{\frac{7}{5}} = \frac{4172844600}{10938397},\ \alpha_2 = -\frac{34685775774}{54691985},$$
$$\alpha_{\frac{11}{5}} = \frac{10813166000}{10938397},\ \alpha_{\frac{12}{5}} = -\frac{63868346100}{120322367},\ \alpha_3 = \frac{9906198264}{54691985},$$
$$\alpha_{\frac{16}{5}} = -\frac{77374571625}{481289468},\ \alpha_{\frac{17}{5}} = \frac{96247674900}{2045480239},\ \alpha_4 = -1,\quad \beta_4 = -\frac{1120392}{54691985}$$

and applying (9),

$$c_0 = \alpha_0 + \alpha_{\frac{1}{5}} + \alpha_{\frac{2}{5}} + \alpha_1 + \alpha_{\frac{6}{5}} + \alpha_{\frac{7}{5}} + \alpha_2 + \alpha_{\frac{11}{5}} + \alpha_{\frac{12}{5}} + \alpha_3 + \alpha_{\frac{16}{5}} + \alpha_{\frac{17}{5}} + \alpha_4 = 0$$

$$c_1 = \left(\frac{1}{5}\right)\alpha_{\frac{1}{5}} + \left(\frac{2}{5}\right)\alpha_{\frac{2}{5}} + \alpha_1 + \left(\frac{6}{5}\right)\alpha_{\frac{6}{5}} + \left(\frac{7}{5}\right)\alpha_{\frac{7}{5}} + (2)\alpha_2 + \left(\frac{11}{5}\right)\alpha_{\frac{11}{5}} + \left(\frac{12}{5}\right)\alpha_{\frac{12}{5}} + (3)\alpha_3 + \left(\frac{16}{5}\right)\alpha_{\frac{16}{5}} + \left(\frac{17}{5}\right)\alpha_{\frac{17}{5}} + (4)\alpha_4 = 0$$

$$c_2 = \frac{1}{2!}\left(\left(\frac{1}{5}\right)^2\alpha_{\frac{1}{5}} + \left(\frac{2}{5}\right)^2\alpha_{\frac{2}{5}} + \alpha_1 + \left(\frac{6}{5}\right)^2\alpha_{\frac{6}{5}} + \left(\frac{7}{5}\right)^2\alpha_{\frac{7}{5}} + (2)^2\alpha_2 + \left(\frac{11}{5}\right)^2\alpha_{\frac{11}{5}} + \left(\frac{12}{5}\right)^2\alpha_{\frac{12}{5}} + (3)^2\alpha_3 + \left(\frac{16}{5}\right)^2\alpha_{\frac{16}{5}} + \left(\frac{17}{5}\right)^2\alpha_{\frac{17}{5}} + (4)^2\alpha_4\right) - \beta_4 = 0$$

$$c_3 = \frac{1}{3!}\left(\left(\frac{1}{5}\right)^3\alpha_{\frac{1}{5}} + \left(\frac{2}{5}\right)^3\alpha_{\frac{2}{5}} + \alpha_1 + \left(\frac{6}{5}\right)^3\alpha_{\frac{6}{5}} + \left(\frac{7}{5}\right)^3\alpha_{\frac{7}{5}} + (2)^3\alpha_2 + \left(\frac{11}{5}\right)^3\alpha_{\frac{11}{5}} + \left(\frac{12}{5}\right)^3\alpha_{\frac{12}{5}} + (3)^3\alpha_3 + \left(\frac{16}{5}\right)^3\alpha_{\frac{16}{5}} + \left(\frac{17}{5}\right)^3\alpha_{\frac{17}{5}} + (4)^3\alpha_4\right) - (4\beta_4) = 0$$

$$c_4 = \frac{1}{4!}\left(\left(\frac{1}{5}\right)^4\alpha_{\frac{1}{5}} + \left(\frac{2}{5}\right)^4\alpha_{\frac{2}{5}} + \alpha_1 + \left(\frac{6}{5}\right)^4\alpha_{\frac{6}{5}} + \left(\frac{7}{5}\right)^4\alpha_{\frac{7}{5}} + (2)^4\alpha_2 + \left(\frac{11}{5}\right)^4\alpha_{\frac{11}{5}} + \left(\frac{12}{5}\right)^4\alpha_{\frac{12}{5}} + (3)^4\alpha_3 + \left(\frac{16}{5}\right)^4\alpha_{\frac{16}{5}} + \left(\frac{17}{5}\right)^4\alpha_{\frac{17}{5}} + (4)^4\alpha_4\right) - \frac{1}{2!}(4^2\beta_4) = 0$$

$$c_5 = \frac{1}{5!}\left(\left(\frac{1}{5}\right)^5\alpha_{\frac{1}{5}} + \left(\frac{2}{5}\right)^5\alpha_{\frac{2}{5}} + \alpha_1 + \left(\frac{6}{5}\right)^5\alpha_{\frac{6}{5}} + \left(\frac{7}{5}\right)^5\alpha_{\frac{7}{5}} + (2)^5\alpha_2 + \left(\frac{11}{5}\right)^5\alpha_{\frac{11}{5}} + \left(\frac{12}{5}\right)^5\alpha_{\frac{12}{5}} + (3)^5\alpha_3 + \left(\frac{16}{5}\right)^5\alpha_{\frac{16}{5}} + \left(\frac{17}{5}\right)^5\alpha_{\frac{17}{5}} + (4)^5\alpha_4\right) - \frac{1}{3!}(4^3\beta_4) = 0$$

$$c_5 = \frac{1}{5!}\left(\left(\frac{1}{5}\right)^5\alpha_{\frac{1}{5}} + \left(\frac{2}{5}\right)^5\alpha_{\frac{2}{5}} + \alpha_1 + \left(\frac{6}{5}\right)^5\alpha_{\frac{6}{5}} + \left(\frac{7}{5}\right)^5\alpha_{\frac{7}{5}} + (2)^5\alpha_2 + \left(\frac{11}{5}\right)^5\alpha_{\frac{11}{5}} + \left(\frac{12}{5}\right)^5\alpha_{\frac{12}{5}} + (3)^5\alpha_3 + \left(\frac{16}{5}\right)^5\alpha_{\frac{16}{5}} + \left(\frac{17}{5}\right)^5\alpha_{\frac{17}{5}} + (4)^5\alpha_4\right) - \frac{1}{3!}(4^3\beta_4) = 0$$

$$c_7 = \frac{1}{7!}\begin{pmatrix} \left(\frac{1}{5}\right)^7 \alpha_{\frac{1}{5}} + \left(\frac{2}{5}\right)^7 \alpha_{\frac{2}{5}} + \alpha_1 + \left(\frac{6}{5}\right)^7 \alpha_{\frac{6}{5}} + \left(\frac{7}{5}\right)^7 \alpha_{\frac{7}{5}} + (2)^7 \alpha_2 + \left(\frac{11}{5}\right)^7 \alpha_{\frac{11}{5}} + \left(\frac{12}{5}\right)^7 \alpha_{\frac{12}{5}} + (3)^7 \alpha_3 \\ + \left(\frac{16}{5}\right)^7 \alpha_{\frac{16}{5}} + \left(\frac{17}{5}\right)^7 \alpha_{\frac{17}{5}} + (4)^7 \alpha_4 \end{pmatrix} - \frac{1}{5!}\left(4^5 \beta_4\right) = 0$$

$$c_8 = \frac{1}{8!}\begin{pmatrix} \left(\frac{1}{5}\right)^8 \alpha_{\frac{1}{5}} + \left(\frac{2}{5}\right)^8 \alpha_{\frac{2}{5}} + \alpha_1 + \left(\frac{6}{5}\right)^8 \alpha_{\frac{6}{5}} + \left(\frac{7}{5}\right)^8 \alpha_{\frac{7}{5}} + (2)^8 \alpha_2 + \left(\frac{11}{5}\right)^8 \alpha_{\frac{11}{5}} + \left(\frac{12}{5}\right)^8 \alpha_{\frac{12}{5}} + (3)^8 \alpha_3 \\ + \left(\frac{16}{5}\right)^8 \alpha_{\frac{16}{5}} + \left(\frac{17}{5}\right)^8 \alpha_{\frac{17}{5}} + (4)^8 \alpha_4 \end{pmatrix} - \frac{1}{6!}\left(4^6 \beta_4\right) = 0$$

$$c_9 = \frac{1}{9!}\begin{pmatrix} \left(\frac{1}{5}\right)^9 \alpha_{\frac{1}{5}} + \left(\frac{2}{5}\right)^9 \alpha_{\frac{2}{5}} + \alpha_1 + \left(\frac{6}{5}\right)^9 \alpha_{\frac{6}{5}} + \left(\frac{7}{5}\right)^9 \alpha_{\frac{7}{5}} + (2)^9 \alpha_2 + \left(\frac{11}{5}\right)^9 \alpha_{\frac{11}{5}} + \left(\frac{12}{5}\right)^9 \alpha_{\frac{12}{5}} + (3)^9 \alpha_3 \\ + \left(\frac{16}{5}\right)^9 \alpha_{\frac{16}{5}} + \left(\frac{17}{5}\right)^9 \alpha_{\frac{17}{5}} + (4)^9 \alpha_4 \end{pmatrix} - \frac{1}{7!}\left(4^7 \beta_4\right) = 0$$

$\vdots$

$$c_{13} = \frac{1}{13!}\begin{pmatrix} \left(\frac{1}{5}\right)^{13} \alpha_{\frac{1}{5}} + \left(\frac{2}{5}\right)^{13} \alpha_{\frac{2}{5}} + \alpha_1 + \left(\frac{6}{5}\right)^{13} \alpha_{\frac{6}{5}} + \left(\frac{7}{5}\right)^{13} \alpha_{\frac{7}{5}} + (2)^{13} \alpha_2 + \left(\frac{11}{5}\right)^{13} \alpha_{\frac{11}{5}} \\ + \left(\frac{12}{5}\right)^{13} \alpha_{\frac{12}{5}} + (3)^{13} \alpha_3 + \left(\frac{16}{5}\right)^{13} \alpha_{\frac{16}{5}} + \left(\frac{17}{5}\right)^{13} \alpha_{\frac{17}{5}} + (4)^{13} \alpha_4 \end{pmatrix} - \frac{1}{11!}\left(4^{11} \beta_4\right) = \frac{190648242}{1175023115234375}$$

Hence the method is of order p = 11 and the error constant is $c_{p+2} = \dfrac{190648242}{1175023115234375}$

Similar procedure is applied to the discrete schemes that constitute the block members of BHBDF II and the summary of the order and error constants is given in the tables below:

Table 3.1: Order and Error Constants of BHBDF II

Equation	Order p	Error constants c_{p+2}
3.7	11	1.622506311 x 10^{-9}
3.8	11	3.097970596 x 10^{-9}
3.9	11	-9.197700413 x 10^{-11}
3.10	11	3.621094306 x 10^{-10}
3.11	11	9.875049942 x 10^{-12}
3.12	11	6.196932974 x 10^{-11}
3.13	11	8.33642171 x 10^{-11}
3.14	11	3.623624551 x 10^{-12}
3.15	11	3.18967290 x 10^{-10}
3.16	11	-8.96155326 x 10^{-11}

3.17	11	-4.181635836 x 10^{-10}
3.18	11	-1.622744053 x 10^{-10}

Table 1 displays the order of accuracy of each member of the derived block method (**BHBDF II**) with its corresponding error constant. It is revealed from analysis that the block method is of uniform order *p* = 11 which is greater than 1. Hence the method is said to be consistent.

3.3 Zero stability

In order to characterize the method for stability, we rewrite the derived schemes as a matrix difference equation as follows:

$$P^{(1)}U_w = P^{(0)}U_{w-1} + h^n Q^{(1)} V_w \quad (25)$$

where

$$\left.\begin{aligned} U_w &= \left(u_{n+\frac{1}{5}}, u_{n+\frac{2}{5}}, u_{n+1}, u_{n+\frac{6}{5}}, u_{n+\frac{7}{5}}, u_{n+2}, u_{n+\frac{11}{5}}, u_{n+\frac{12}{5}}, u_{n+3}, u_{n+\frac{16}{5}}, u_{n+\frac{17}{5}}, u_{n+4}\right)^T \\ U_{w-1} &= \left(u_{n-\frac{1}{5}}, u_{n-\frac{2}{5}}, u_{n-1}, u_{n-\frac{6}{5}}, u_{n-\frac{7}{5}}, u_{n-2}, u_{n-\frac{11}{5}}, u_{n-\frac{12}{5}}, u_{n-3}, u_{n-\frac{16}{5}}, u_{n-\frac{17}{5}}, u_{n-4}\right)^T \\ V_{w+1} &= \left(v_{n+\frac{1}{5}}, v_{n+\frac{2}{5}}, v_{n+1}, v_{n+\frac{6}{5}}, v_{n+\frac{7}{5}}, v_{n+2}, v_{n+\frac{11}{5}}, v_{n+\frac{12}{5}}, v_{n+3}, v_{n+\frac{16}{5}}, v_{n+\frac{17}{5}}, v_{n+4}\right)^T \end{aligned}\right\}$$

and $P^{(1)}, P^{(0)}, Q^{(1)}$ are matrices whose entries are given by the coefficients of the block method, whose first characteristic polynomial is given as

$$\rho(\lambda) = \left|\lambda P^{(1)} - P^{(0)}\right| \quad (26)$$

Definition (Zero Stability) 3.3 **:** A block linear multistep method is said to be zero stable if the roots (λ_k) of the difference equation in (3.46) as $h \to 0$ is $|\lambda_k| \le 1,\ k = 1, \dots, n$ and the multiplicity of the roots $|\lambda_k| = 1$ is not greater than the order of the ODE.

Zero Stability of BHBDF II, The first characteristics polynomial is given as:

$$\rho(\lambda) = \left|\lambda P^{(1)} - P^{(0)}\right| = 0.000013519\lambda^{12}(\lambda + 1) = 0 \quad (27)$$

$$\Rightarrow \ \lambda = \{0,0,0,0,0,0,0,0,0,0,0,1\}$$

Hence the proposed method BHBDF II is zero stable.

4. Results and Discussion

Problem 2:

Consider the nonlinear problem in Guler *et al.* (2019): $u''+(u')^2-u^2=1-\sin t, u(0)=0, u'(0)=1$ and h = 0.01 Exact Solution: $U(t)=\sin t$

Table 4.2: Comparison of the Exact and Numerical Results for Problem 2

T	Exact solution	Numerical solution
0.1	0.09983341664682815230 7	0.09983341664682814101 2
0.2	0.19866933079506121546	0.19866933079506114447
0.3	0.29552020666133957511	0.29552020666133937661
0.4	0.38941834230865049167	0.38941834230865008339
0.5	0.47942553860420300027	0.47942553860420229076
0.6	0.56464247339503535720	0.56464247339503424468
0.7	0.64421768723769105367	0.64421768723768942938
0.8	0.71735609089952276163	0.71735609089952051408
0.9	0.78332690962748338846	0.78332690962748041151
1.0	0.84147098480789650665	0.84147098480789267764

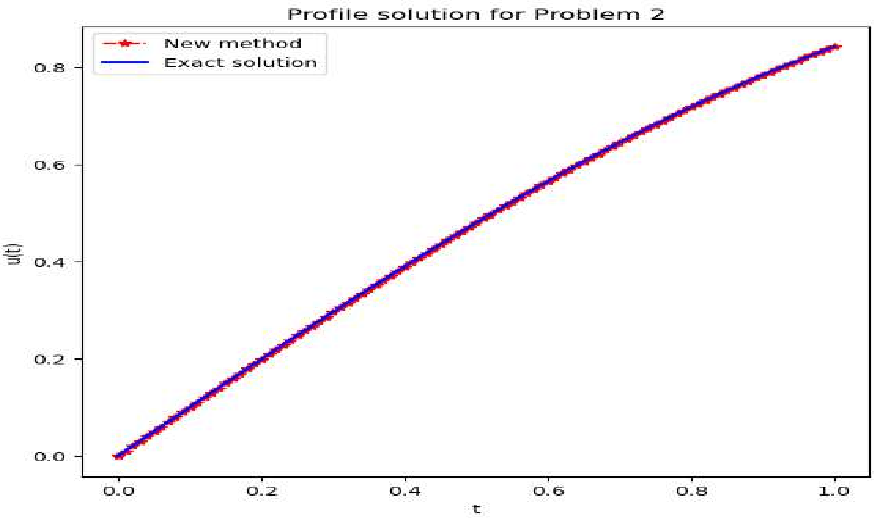

Figure 4.2: Graph of Exact and Numerical Solutions for Problem 2

Problem 3:

Consider the nonlinear problem: $u''+u^3+u=(\cos t+\varepsilon\sin(10t))^3-99\varepsilon\sin t$, with the following initial conditions $u(0)=1, u'(0)=10\varepsilon$ and h = 0.005

Exact Solution: $U(t)=(\cos t+\varepsilon\sin(10t))$, where $\varepsilon=10^{-10}$

Table 4.3: Comparison of the Exact and Numerical Results Problem 3

T	Exact solution	Numerical solution
0.005	0.99998750003103956190	0.99998750003104148000

0.010	0.99995000042664861944	0.99995000042666361119
0.015	0.99988750212430299300	0.99988750212435328633
0.020	0.99980000668644471149	0.99980000668656354220
0.025	0.99968751630044298218	0.99968751630067456136
0.030	0.99955003377853953694	0.99955003377893889990
0.035	0.99938756255777835656	0.99938756255841116595
0.040	0.99920010669991977454	0.99920010670086213063
0.045	0.99898767089133896263	0.99898767089267724763
0.050	0.99875026044290880042	0.99875026044473940375

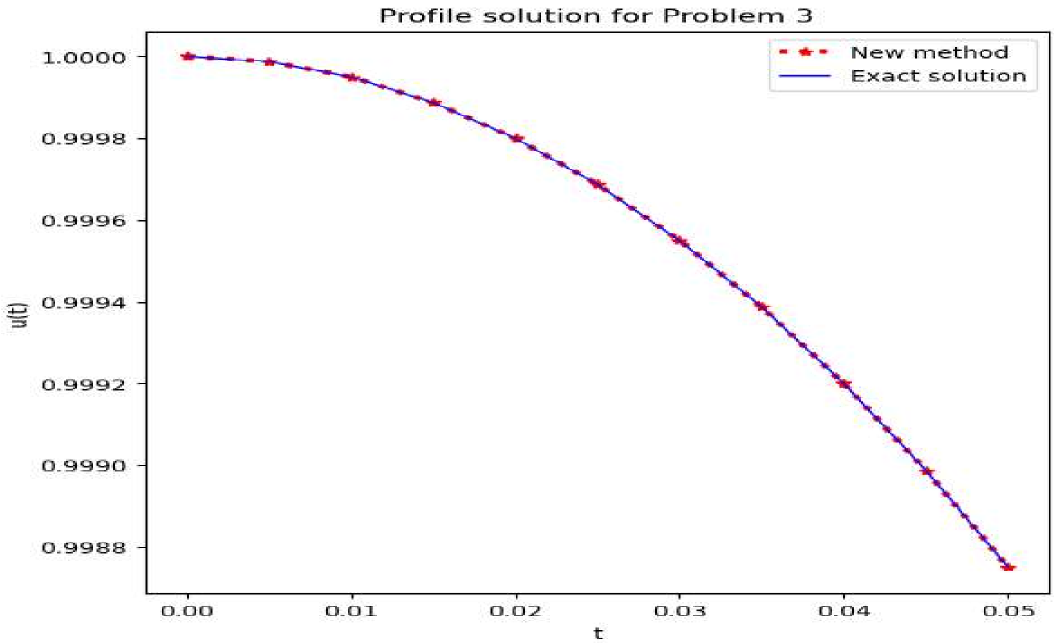

Figure 4.3: Graph of Exact and Numerical Solutions for Problem 3

Table 4.5: Comparison of Accuracy for Problem 2

T	Exact solution	Error in Guler *et al.* (2019)	Error in Ogunlaran and Kehinde (2022)	Error in BHBDF II
0.1	0.09983341664682815230	3.490×10^{-08}	9.300×10^{-10}	1.130×10^{-17}
0.2	0.19866933079506121546	1.160×10^{-07}	1.990×10^{-09}	7.099×10^{-17}
0.3	0.29552020666133957511	-	-	1.985×10^{-16}

0.4	0.38941834230865049167	1.130×10^{-07}	3.180×10^{-09}	4.083×10^{-16}
0.5	0.47942553860420300027	4.610×10^{-07}	3.230×10^{-09}	7.095×10^{-16}
0.6	0.56464247339503535720	7.800×10^{-07}	3.510×10^{-09}	1.113×10^{-15}
0.7	0.64421768723769105367	1.400×10^{-06}	3.740×10^{-09}	1.624×10^{-15}
0.8	0.71735609089952276163	4.1600×10^{-06}	3.530×10^{-09}	2.248×10^{-15}
0.9	0.78332690962748338846	1.400×10^{-05}	3.030×10^{-09}	2.977×10^{-15}
1.0	0.84147098480789650665	4.100×10^{-05}	2.750×10^{-09}	3.829×10^{-15}

Table 4.6: Comparison of Accuracy for Problem 3

T	Exact solution	Numerical solution	Error in BHBDF II
0.005	0.99998750003103956190	0.99998750003104148000	1.918×10^{-15}
0.010	0.99995000042664861944	0.99995000042666361119	1.500×10^{-14}
0.015	0.99988750212430299300	0.99988750212435328633	5.029×10^{-14}
0.020	0.99980000668644471149	0.99980000668656354220	1.188×10^{-13}
0.025	0.99968751630044298218	0.99968751630067456136	2.316×10^{-13}
0.030	0.99955003377853953694	0.99955003377893889990	3.994×10^{-13}
0.035	0.99938756255777835656	0.99938756255841116595	6.328×10^{-13}
0.040	0.99920010669991977454	0.99920010670086213063	9.424×10^{-13}
0.045	0.99898767089133896263	0.99898767089267724763	1.338×10^{-12}
0.050	0.99875026044290880042	0.99875026044473940375	1.831×10^{-12}

5. Conclusion

In this project, an effective numerical approach for solving higher-order ordinary differential equations is sought. To achieve this milestone, the research focused on developing a class of block hybrid backward differentiation formula (BDF) that simultaneously generates approximate solutions to an equation on the entire interval of integration. The techniques of interpolation and collocation are employed for the derivation of the method. In the derivation process of the method, which is four-step, that is, k = 4, a number of off-grid points were carefully selected at the interpolation points over the interval [0, 4] for the second ODE, BHBDF II. Convergence analysis of the methods reveals that BHBDF II has an order of accuracy (p + 2) of eleven. The methods are zero stable and consistent, which implies their convergence. Furthermore, numerical experiments were carried out where BHBDF II was implemented on several nonlinear second problems. The graphical illustrations displayed therein show that the method conforms to the exact solution. Further comparative analyses of BHBDF II in tables 2–4 reveal that the method has the advantage of producing smaller global errors over several existing methods in the literature, including some of the most recent ones. Error analysis in comparison with some existing methods further testifies the efficacy of the method.

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