ANALYSIS OF A REFORMULATED BLOCK HYBRID LINEAR MULTISTEP METHOD INTO RUNGE –KUTTA TYPE METHOD FOR FIRST ORDER INITIAL VALUE PROBLEM (IVP)

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Abstract

Problems arises from science and technology are expressed in differential equations. These differential equation are sometimes in ordinary differential equations. Reliability with high accuracy and stability are necessary for a numerical method for the solution of differential equations. This research paper presents the analysis of a reformulated block hybrid linear multistep method into Runge-Kutta type method (RKTM) for first order initial value problems (IVPs). In view of this, the block hybrid method derived is of uniform order 6 with error constants of $-\frac{263}{1935360}, -\frac{11}{120960}, -\frac{9}{71680}, -\frac{1}{15120}$ and $-\frac{9}{4480}$ while the Runge-Kutta type method reformulated maintain the order of the derived block hybrid linear multistep method which are of uniform order 6 but with error constants of $-\frac{263}{4232632320}, -\frac{11}{264539520}, -\frac{1}{17418240}, -\frac{1}{33067440}\ and\ -\frac{1}{1088640}$. Testing for convergence of both the derived block hybrid linear multistep method and the Runge-Kutta type method shows that the two methods are consistent and are also zero stable.

Keywords: Block hybrid, Convergency, Linear multistep method, Order and error constants, Runge-Kutta type method.

Introduction

Science and engineering modelled their dynamics system in form of differential equations which require numerical methods for their effective solution (Rice *et al.,* 2023). Traditional explicit and implicit numerical approaches such as Runge-Kutta and linear multistep methods fail to simultaneously achieve accuracy and stability while keeping low computational costs especially for stiff differential equations (Parveen,and Ahmad, 2024).. Block hybrid linear multistep methods has being adopted to address these challenges but the method still have limitations in solving complex problems (Areo *et al.,*2024).

Classical fourth order Runge Kutta type method is of the form:

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$where:$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$$

$$k_4 = f\left(x_n + h, y_n + hk_3\right)$$

The general representation of linear multistep method have the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j}$$

where:

y_{n+j} are the approximate solutions,

$f_{n+j} = f(x_{n+j}, y_{n+j})$,

h is the step size,

α_j and β_j are the method's coefficient

k is the step number which determine how many previous steps are used.

While the block hybrid linear multistep method has the general form:

$$Y = BY + hCF$$

where:

Y is the vector of function values at both step and off step points

B is the coefficient matrix for dependent variables

C is the coefficient matrix for function evaluations

F is the vector of function evaluation f(x, y).

However, conversion of block hybrid linear multistep methods into Runge-Kutta type method (RKTM) will address these challenges (Aliyu *et al.,* 2025).

The field of numerical methods requires an enhanced method which delivers high-order accuracy along with stability preserving capabilities (Sharma *et al.*, 2020). The main challenge involves in developing a scheme is uniform accuracy, minimal error constants, computational efficiency and faster convergence (Sharma *et al.*, 2020). A RKTM transformed from block hybrid linear multistep method will address these challenges through improved error constant, convergence rate and computational efficiency (Muhammad, 2020).

This research adds value to numerical analysis because it develops a new solution technique for first-order IVPs. This research investigates the reformulated method through extensive analysis and improvement to establish it as useful practical instrument applied in engineering practice and applied sciences. The research addresses only first-order IVPs by explaining the derivation process while conducting order assessments and stability examination. The proposed approach should be expanded to solve higher-order problems and complex differential equations in future research.

METHODOLOGY

Derivation of the Method

The derivation of six – stage Runge – Kutta type method reformulated from three step block hybrid linear multistep method with two off-grids points is discussed in this section. First order three step block hybrid linear multistep method (TSBHLMM) with two off-grids point has been reformulated to first order Runge - Kutta Type method.

The general form of first order initial value problem (IVP) in ordinary differential equations (ODEs) is given as:

$$y'(t) = f(t, y), y(t_0) = y_0 \tag{4}$$

We use power series as the basis function given in the form:

$$y(t) = \sum_{i=0}^{n} \varphi_i t^i \tag{5}$$

Differentiating (3) we obtained:

$$y^1(t) = \sum_{i=0}^{n} i\varphi_i t^{i-1}$$

(6)

Using matrix inversion techniques to find the values of φ_i's we obtain a continuous implicit scheme of the form:

$$y(t) = \alpha_0(t) y_n + h\left(\sum_{i=0}^{1} \beta_{n+\frac{2i+1}{2}} f_{n+\frac{2i+1}{2}} + \sum_{i=0}^{3} \beta_i f_{n+i}\right) \tag{7}$$

Interpolating (3) at $t = \{t_n\}$ and collocating (4) at $t = \left\{t_n, t_{n+\frac{1}{2}}, t_{n+1}, t_{n+\frac{3}{2}}, t_{n+2}, t_{n+3}\right\}$.

Which can be expressed in matrix form:

$$\begin{bmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 \\ 0 & 1 & 2t_{n+\frac{1}{2}} & 3\left(t_{n+\frac{1}{2}}\right)^2 & 4\left(t_{n+\frac{1}{2}}\right)^3 & 5\left(t_{n+\frac{1}{2}}\right)^4 & 6\left(t_{n+\frac{1}{2}}\right)^5 & 7\left(t_{n+\frac{1}{2}}\right)^6 \\ 0 & 1 & 2t_{n+1} & 3(t_{n+1})^2 & 4(t_{n+1})^3 & 5(t_{n+1})^4 & 6(t_{n+1})^5 & 7(t_{n+1})^6 \\ 0 & 1 & 2t_{n+\frac{3}{2}} & 3\left(t_{n+\frac{3}{2}}\right)^2 & 4\left(t_{n+\frac{3}{2}}\right)^3 & 5\left(t_{n+\frac{3}{2}}\right)^4 & 6\left(t_{n+\frac{3}{2}}\right)^5 & 7\left(t_{n+\frac{3}{2}}\right)^6 \\ 0 & 1 & 2t_{n+2} & 3(t_{n+2})^2 & 4(t_{n+2})^3 & 5(t_{n+2})^4 & 6(t_{n+2})^5 & 7(t_{n+2})^6 \\ 0 & 1 & 2t_{n+3} & 3(t_{n+3})^2 & 4(t_{n+3})^3 & 5(t_{n+3})^4 & 6(t_{n+3})^5 & 7(t_{n+3})^6 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} = \begin{bmatrix} y_n \\ hf_n \\ hf_{n+\frac{1}{2}} \\ hf_{n+1} \\ hf_{n+\frac{3}{2}} \\ hf_{n+2} \\ hf_{n+3} \end{bmatrix} \tag{8}$$

Making Use of Maple Mathematical software to obtain the values of λ's in (8) that result in the continuous formula:

$$y(t) = \alpha_0 y_n + h\left[\beta_n f_n + \beta_{n+\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{n+1} f_{n+1} + \beta_{n+\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_{n+2} f_{n+2} + \beta_{n+3} f_{n+3}\right] \tag{9}$$

Taking $t_1 = \frac{t - t_{n+2}}{h} \Rightarrow \frac{dt_1}{dt} = \frac{1}{h}$

(10)

We evaluate (7) at $t = t_{n+i}$, $i = \frac{1}{2}, 1, \frac{3}{2}, 2 and 3$, we obtain a block hybrid linear multistep method as

$$\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} y_n \end{bmatrix} + h \begin{bmatrix} \frac{959}{5760} & \frac{35}{72} & -\frac{487}{1920} & \frac{49}{360} & -\frac{211}{5760} & \frac{1}{640} \\ \frac{169}{1080} & \frac{32}{45} & \frac{11}{120} & \frac{8}{135} & -\frac{7}{360} & \frac{1}{1080} \\ \frac{103}{640} & \frac{27}{40} & \frac{243}{640} & \frac{13}{40} & -\frac{27}{640} & \frac{1}{640} \\ \frac{7}{45} & \frac{32}{45} & \frac{4}{15} & \frac{32}{45} & \frac{7}{55} & 0 \\ \frac{11}{40} & 0 & \frac{81}{40} & -\frac{8}{5} & \frac{81}{40} & \frac{11}{40} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \tag{11}$$

Reformulation of Block Hybrid Linear Multistep Method (BHLMM) into Runge – Kutta Type Method (RKTM)

Proposed RKTM for first order differential equation

Aliyu *et al.*, (2025) defined an s-stage implicit Runge - Kutta methods for fist order ODEs in the form:

$$\left.\begin{aligned} & y_{n+1} = y_n + h\sum_{i=0}^{s} a_{ij} k_i \\ & k_i = f\left(x_i + \alpha_i h, y_n + h\sum_{i=0}^{s} a_{ij} k_i\right) \\ & where i = 1, 2, 3, \ldots, n \end{aligned}\right\} \tag{12}$$

The parameters α_j, k_j and a_{ij} defined the method. In Butcher array form, it can be written as

$$\begin{array}{c|c} \alpha & \beta \\ \hline & W^T \end{array} \tag{13}$$

Presenting (11) in the Butcher table we have

C	A						
0	0	0	0	0	0	0	0
$\frac{1}{2}$	$\frac{959}{5760}$	$\frac{35}{72}$	$-\frac{487}{1920}$	$\frac{49}{360}$	$-\frac{211}{5760}$	$\frac{1}{640}$	
1	$\frac{169}{1080}$	$\frac{32}{45}$	$\frac{11}{120}$	$\frac{8}{135}$	$-\frac{7}{360}$	$\frac{1}{1080}$	
$\frac{3}{2}$	$\frac{103}{640}$	$\frac{27}{40}$	$\frac{243}{640}$	$\frac{13}{40}$	$-\frac{27}{640}$	$\frac{1}{640}$	
2	$\frac{7}{45}$	$\frac{32}{45}$	$\frac{4}{15}$	$\frac{32}{45}$	$\frac{7}{55}$	0	
3	$\frac{11}{40}$	0	$\frac{81}{40}$	$-\frac{8}{5}$	$\frac{81}{40}$	$\frac{11}{40}$	
b=3	$\frac{11}{40}$	0	$\frac{81}{40}$	$-\frac{8}{5}$	$\frac{81}{40}$	$\frac{11}{40}$	

The Butcher array conditions for first order differential equation is stated as:

$$\left.\begin{aligned} &(\text{i}).\sum_{j=1}^{s} a_{ij} = c_i \\ &(ii).\sum_{j=1}^{s} b_j = 1 \end{aligned}\right\} \quad (14)$$

Source:(Muhammad, 2020)

Since the butcher table (13) does not satisfied the conditions hence it was modified to satisfied the conditions (14).

Therefore six-stage implicit Ruge-Kutta type method is obtained as

$$\left.\begin{aligned} &y_{n+1} = y_n + h\left(\frac{11}{120}k_1 + \frac{27}{40}k_3 - \frac{8}{15}k_4 + \frac{27}{40}k_5 + \frac{11}{120}k_6\right) \\ &where \\ &k_1 = f(x_n, y_n) \\ &k_2 = f\left(x_n + \frac{h}{6}, y_n + h\left(\frac{959}{17280}k_1 + \frac{35}{216}k_2 - \frac{487}{5760}k_3 + \frac{49}{1080}k_4 - \frac{211}{17280}k_5 + \frac{1}{1920}k_6\right)\right) \end{aligned}\right\} \quad (
$$

$$k_3 = f\left(x_n + \frac{h}{3}, y_n + h\left(\frac{169}{3240}k_1 + \frac{32}{135}k_2 + \frac{11}{360}k_3 + \frac{8}{405}k_4 - \frac{7}{1080}k_5 + \frac{1}{3240}k_6\right)\right)$$

$$k_4 = f\left(x_n + \frac{h}{2}, y_n + h\left(\frac{103}{1920}k_1 + \frac{9}{40}k_2 + \frac{81}{640}k_3 + \frac{13}{120}k_4 - \frac{9}{640}k_5 + \frac{1}{1920}k_6\right)\right)$$

$$k_5 = f\left(x_n + \frac{2h}{3}, y_n + h\left(\frac{7}{135}k_1 + \frac{32}{135}k_2 + \frac{4}{45}k_3 + \frac{32}{135}k_4 + \frac{7}{135}k_5\right)\right)$$

$$k_6 = f\left(x_n + h, y_n + h\left(\frac{11}{120}k_1 + 0k_2 + \frac{27}{40}k_3 - \frac{8}{15}k_4 + \frac{27}{40}k_5 + \frac{11}{120}k_6\right)\right)$$

RESULT AND DISCUSSION

Order and Error Constant of the TSBHLMM

We analyze the approximation of the order and error constant of the derived block method using the difference equation:

$$L[y(t),h]=\sum_{i=0}^{k}\left[\alpha_i y\left(t+ih\right)-h\beta y'\left(t+ih\right)\right]$$

(16)

We assume $y(t)$ to have as many higher derivatives we require, expanding the terms in (16) as a Taylor series about the point t we obtain the expansion

$$\left.\begin{array}{l} L[y(t),h]=C_0 y(t)+C_1 h y^{(1)}(t)+\ldots+C_q h^q y^{(q)}(t)+\ldots \\ \text{where} \\ C_0=\alpha_0+\alpha_1+\ldots+\alpha_k \\ C_1=(\alpha_1+\alpha_2+\ldots+k\alpha_k)-(\beta_0+\beta_1+\ldots.+k^{(q-1)}\beta_k where q=2,3,\ldots,k \end{array}\right\}$$

(17)

The order and error constant is obtained by applying (13) and (14) in (11)

Table 3.1: Order and Error Constants of the TSBHLMM

Scheme	Order	Error Constant
$y_{n+\frac{1}{2}}$	6	$-\frac{263}{1935360}$
y_{n+1}	6	$-\frac{11}{120960}$
$y_{n+\frac{3}{2}}$	6	$-\frac{9}{71680}$
y_{n+2}	6	$-\frac{1}{15120}$
y_{n+3}	6	$-\frac{9}{4480}$

Table 3.1 shows that the block method (11) is of uniform order 6 with the respective error constants as shown in the table.

Order and Error Constant of the RKTM

Using the idea of Muhammad, (2020), we choose $k_i=f_{c_i}$, which implies that $k_1=f_{c1},k_2=f_{c2},k_3=f_{c3},k_4=f_{c4},k_5=f_{c_5}, and k_6=f_{c6}$

$$\Rightarrow c_1=0,c_2=\frac{1}{6},c_3=\frac{1}{3},c_4=\frac{1}{2},c_5=\frac{2}{3},c_6=1$$

$$k_1=f_0=f_n,k_2=f_{\frac{1}{6}}=f_{n+\frac{1}{6}},k_3=f_{\frac{1}{3}}=f_{n+\frac{1}{3}},k_4=f_{\frac{1}{2}}=f_{n+\frac{1}{2}},k_5=f_{\frac{2}{3}}=f_{n+\frac{2}{3}}, and k_6=f_1=f_{n+1}$$

Hence (15) becomes

$$\left.\begin{aligned}
y_{n+1} &= y_n + h\left(\frac{11}{120}f_n + \frac{27}{40}f_{n+\frac{1}{3}} + \frac{8}{15}f_{n+\frac{1}{2}} + \frac{27}{40}f_{n+\frac{2}{3}} + \frac{11}{120}f_{n+1}\right)\\
y_{n+\frac{2}{3}} &= y_n + h\left(\frac{7}{135}f_n + \frac{32}{135}f_{n+\frac{1}{6}} + \frac{4}{45}f_{n+\frac{1}{3}} + \frac{32}{135}f_{n+\frac{1}{2}} + \frac{7}{135}f_{n+\frac{2}{3}}\right)
\end{aligned}\right\},$$

$$y_{n+\frac{1}{2}} = y_n + h\left(\frac{103}{1920}f_n + \frac{9}{40}f_{n+\frac{1}{6}} + \frac{81}{640}f_{n+\frac{1}{3}} + \frac{13}{120}f_{n+\frac{1}{2}} - \frac{9}{640}f_{n+\frac{2}{3}} + \frac{1}{1920}f_{n+1}\right)$$

$$y_{n+\frac{1}{3}} = y_n + h\left(\frac{169}{3240}f_n + \frac{32}{135}f_{n+\frac{1}{6}} + \frac{11}{360}f_{n+\frac{1}{3}} + \frac{8}{405}f_{n+\frac{1}{2}} - \frac{7}{1080}f_{n+\frac{2}{3}} + \frac{1}{3240}f_{n+1}\right)$$

$$y_{n+\frac{1}{6}} = y_n + h\left(\frac{959}{17280}f_n + \frac{35}{216}f_{n+\frac{1}{6}} - \frac{487}{5760}f_{n+\frac{1}{3}} + \frac{49}{1080}f_{n+\frac{1}{2}} - \frac{211}{17280}f_{n+\frac{2}{3}} + \frac{1}{1920}f_{n+1}\right)$$

Taylor series expansion of

$$y_{n+1} = y(n+h) = y(n) + hy'(n) + \frac{(h)^2}{2!}y''(n) + \frac{(h)^3}{3!}y'''(n) + \frac{(h)^4}{4!}y^{iv}(n) + \ldots + \frac{(h)^s}{s!}y^s(n)$$

$$y_{n+\frac{2}{3}} = y(n+\frac{2}{3}h) = y(n) + \frac{2}{3}hy'(n) + \frac{(\frac{2}{3}h)^2}{2!}y''(n) + \frac{(\frac{2}{3}h)^3}{3!}y'''(n) + \frac{(\frac{2}{3}h)^4}{4!}y^{iv}(n) + \ldots + \frac{(\frac{2}{3}h)^s}{s!}y^s(n)$$

$$y_{n+\frac{1}{2}} = y(n+\frac{1}{2}h) = y(n) + \frac{1}{2}hy'(n) + \frac{(\frac{1}{2}h)^2}{2!}y''(n) + \frac{(\frac{1}{2}h)^3}{3!}y'''(n) + \frac{(\frac{1}{2}h)^4}{4!}y^{iv}(n) + \ldots + \frac{(\frac{1}{2}h)^s}{s!}y^s(n)$$

$$y_{n+\frac{1}{3}} = y(n+\frac{1}{3}h) = y(n) + \frac{1}{3}hy'(n) + \frac{(\frac{1}{3}h)^2}{2!}y''(n) + \frac{(\frac{1}{3}h)^3}{3!}y'''(n) + \frac{(\frac{1}{3}h)^4}{4!}y^{iv}(n) + \ldots + \frac{(\frac{1}{3}h)^s}{s!}y^s(n)$$

$$y_{n+\frac{1}{6}} = y(n+\frac{1}{6}h) = y(n) + \frac{1}{6}hy'(n) + \frac{(\frac{1}{6}h)^2}{2!}y''(n) + \frac{(\frac{1}{6}h)^3}{3!}y'''(n) + \frac{(\frac{1}{6}h)^4}{4!}y^{iv}(n) + \ldots + \frac{(\frac{1}{6}h)^s}{s!}y^s(n)$$

$$f_1 = f(n+h) = y'(n) + hy''(n) + \frac{(h)^2}{2!}y'''(n) + \frac{(h)^3}{3!}y^{iv}(n) + \frac{(h)^4}{4!}y^{v}(n) + \ldots + \frac{(h)^{s-1}}{(s-1)!}y^{s-1}(n)$$

$$f_{\frac{2}{3}} = f(n+\frac{2}{3}h) = y'(n) + \frac{2}{3}hy''(n) + \frac{(\frac{2}{3}h)^2}{2!}y'''(n) + \frac{(\frac{2}{3}h)^4}{3!}y^{iv}(n) + \frac{(\frac{2}{3}h)^5}{4!}y^{v}(n) + \ldots + \frac{(\frac{2}{3}h)^{s-1}}{(s-1)!}y^{s-1}(n)$$

$$f_{\frac{1}{2}} = f(n+\frac{1}{2}h) = y'(n) + \frac{1}{2}hy''(n) + \frac{(\frac{1}{2}h)^2}{2!}y'''(n) + \frac{(\frac{1}{2}h)^4}{3!}y^{iv}(n) + \frac{(\frac{1}{2}h)^5}{4!}y^{v}(n) + \ldots + \frac{(\frac{1}{2}h)^{s-1}}{(s-1)!}y^{s-1}(n)$$

$$f_{\frac{1}{3}} = f(n+\frac{1}{3}h) = y'(n) + \frac{1}{3}hy''(n) + \frac{(\frac{1}{3}h)^2}{2!}y'''(n) + \frac{(\frac{1}{3}h)^3}{3!}y^{iv}(n) + \frac{(\frac{1}{3}h)^4}{4!}y^{v}(n) + \ldots + \frac{(\frac{1}{3}h)^{s-1}}{(s-1)!}y^{s-1}(n)$$

$$f_{\frac{1}{6}} = f(n+\frac{1}{6}h) = y'(n) + \frac{1}{6}hy''(n) + \frac{(\frac{1}{6}h)^2}{2!}y'''(n) + \frac{(\frac{1}{6}h)^3}{3!}y^{iv}(n) + \frac{(\frac{1}{6}h)^4}{4!}y^{v}(n) + \ldots + \frac{(\frac{1}{6}h)^{s-1}}{(s-1)!}y^{s}(n)$$

Applying (16) and (17) to the new reformulated BHLMM from the RKTM gives

Table 3.2: The Order and Error Constant for the Proposed RKTM

Scheme	Order	Error Constants
y_{n+1}	6	$-\frac{263}{4232632320}$
$y_{n+\frac{2}{3}}$	6	$-\frac{11}{264539520}$
$y_{n+\frac{1}{2}}$	6	$-\frac{1}{17418240}$
$y_{n+\frac{1}{3}}$	6	$-\frac{1}{33067440}$
$y_{n+\frac{1}{6}}$	6	$-\frac{1}{1088640}$

Table 3.2 shows that the uniform 6 from the block hybrid linear multistep method was maintained when it was reformulated back to Runge – Kutta type method. The respective error constants are shown in the table

Zero stability of the BHLMM and the RKTM

If (11) is written in the normalized block form we have

$$A_0Y_s = A_1Y_{s-3} + h(B_0F_{s-3} + B_1F_s) \tag{19}$$

Where s represent the block number

$$Y_s = \left(y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+3}\right)^T \tag{20}$$

$$Y_{s-3} = \left(y_{n-\frac{5}{2}}, y_{n-2}, y_{n-\frac{3}{2}}, y_{n-1}, y_n\right)^T \tag{21}$$

$$F_s = \left(f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, f_{n+3}\right)^T \tag{22}$$

$$F_{s-3} = \left(f_{n-\frac{5}{2}}, f_{n-2}, f_{n-\frac{3}{2}}, f_{n-1}, y_n\right)^T \quad (23)$$

Also, A_0, A_1, B_0 *and* B_1 are matrices in which their entries are given by the coefficient of the block method (11) and reformulated (18).

The zero stability of equation (11) and (18) can be obtained as

$$\rho(\lambda) = \left|\lambda A^{(1)} - A^{(0)}\right| \quad (24)$$

where

$$A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad A^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho(\lambda) = \lambda^5(\lambda - 1) = 0\,. \quad \lambda_j = (0,0,0,0,0,1),\ j = 1,2,3,4,5\ and\ 6$$

Clearly the method (11) and (18) are zero stable which satisfies $|\lambda_j| \le 1$

Consistency of TSBHLMM and RKTM

The methods (8) and (27) – (31) are said to be consistent if they have order greater than one with the first and second characteristics polynomial defined respectively by:

$$(i).\rho(r) = \sum_{j=0}^{k} \alpha_j z^j$$

$$(ii).\sigma(r) = \sum_{j=0}^{k} \beta_j z^j$$

where r is the principal root, which satisfy these conditions

$$\sum_{j=0}^{k} \alpha_j = 0$$

and

$$\rho'(1) = \sigma(1)$$

Table 3.3: Parameters for Obtaining Consistency of the BHLMM (8)

Scheme	Order	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
y_{n+3}	**6**	**0**	**3**	**3**
y_{n+2}	**6**	**0**	**2**	**2**
$y_{n+\frac{3}{2}}$	**6**	**0**	$\frac{3}{2}$	$\frac{3}{2}$
y_{n+1}	**6**	**0**	**1**	**1**
$y_{n+\frac{1}{2}}$	**6**	**0**	$\frac{1}{2}$	$\frac{1}{2}$

The first order Runge-Kutta type method in ordinary differential equation (ODE) are said to be consistent if

$\phi(x, y(x), 0) \equiv f(x, y(x))$ holds.

Note: Consistency required that

$$\sum_{j=1}^{s} b_j = 1$$

Sources: (2022; Muhammad, 2020)

From the Butcher table resulted in (15) it shows that the conditions are satisfied for RKTM, derived. Hence the RKTM are consistent.

CONCLUSION

Three step block hybrid linear multistep method was successfully transformed into a Runge-Kutta type method and the uniform order was maintained which enhanced the numerical performance. The research analysis demonstrates that RKTM solution method present consistency and zero-stability properties with convergence characteristics which allows their application to first-order IVPs. The research adds to numerical method development by presenting strategies which enhance hybrid techniques used for differential equation solutions.

RECOMMENDATIONS

Recommendations for Further Studies

i. Researchers should study how to extend the new method to handle both higher-order differential equations alongside systems of IVPs because it could determine its full usefulness.
ii. Insights about RKTM effectiveness on complex differential problems can be acquired by studying its execution on stiff equations during analysis.
iii. The proposed method requires testing through its application to physical problems and engineering applications and financial challenges to prove practical usage.
iv. Evaluation of the reformulated approach requires additional comparison tests with Runge-Kutta methods as well as hybrid methods to demonstrate its relative benefits and drawbacks.
v. The optimization of computational algorithms together with parallel computing research would boost its performance speed for solving significant problems.

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