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Implementation of New Iterative Method for Solving Nonlinear Partial Differential Problems

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ABSTRACT
Nonlinear partial differential equations (PDEs) are prevalent in various scientific and engineering fields, demanding efficient solution methods. This study focuses on the practical application and evaluation of a well-established iterative method; New Iterative Method (NIM) for solving nonlinear PDEs. The primary aim is to assess the method's performance and applicability in solving nonlinear PDEs. We present the chosen iterative method, discuss its mathematical basis, and analyze its convergence properties, accuracy, and computational efficiency. We also provide insights into practical implementations and conduct numerical experiments on diverse nonlinear PDEs. Numerical experiments across various nonlinear PDEs confirm the method's accuracy and computational efficiency, positioning it favorably compared to existing approaches. The NIM's versatility and computational efficiency makes it a valuable tool for tackling complex problems. This innovation has the potential to greatly benefit scientific and engineering communities dealing with nonlinear PDEs, offering a promising solution for challenging real-world problems.
Keywords: Nonlinear Partial Differential Problems, Iterative Method, Computational Efficiency, Practical Implementation, Numerical Experiments

INTRODUCTION
Nonlinear partial differential equations (PDEs) are ubiquitous in various scientific and engineering disciplines, yet their efficient and accurate solutions remain a challenging endeavor. This study is focused on applying the New Iterative Method (NIM) to numerically solve nonlinear Partial Differential Equations that models physical complex processes. The NIM scheme have proven to be an effective mathematical tool in dealing with various scenarios and addressing linear and nonlinear differential equations. The method offers promise as numerical techniques that amalgamate analytical and iterative strategies to approximate solutions for differential equations. The NIM, initially introduced by Daftardar and Jafari in 2006 and subsequently referred to as the Daftardar-Jafari method by Batiha and Ghanim (2022), is a well-established technique. It is a straightforward and effective semi-analytical approach utilized for solving differential equations with applications spanning various fields. The NIM employs an iterative framework to linearize the problem and enhance the solution. Its successful applications encompass differential equations featuring variable coefficients (as shown by Falade *et al*., 2020), fractional differential equations (as demonstrated by Batiha *et al*., 2023), and cancer model (as illustrated by Falade and Tiamiyu, 2021).
Many natural phenomena, encompassing chemical, physical, and biological processes, often find representation through nonlinear differential equations. In addition to seeking exact solutions, there's a practical necessity for approximating these solutions to make them applicable. Consequently, a multitude of both numerical and analytical approximate methods have been devised and put into practice for addressing nonlinear models. As an illustration of this, the authors in Zada *et al.,* (2021) employed a numerical approach to solve fractional order inhomogeneous PDE. The numerical solutions for system of coupled Fractional-order drinfeld–sokolov–wilson and Fractional shallow water equations using NIM was studied by Ali *et al*., (2023). Zeleke and Regassa (2017) applied the reduced differential transform method to provide solutions concerning some PDEs such as beam and airy equations. Recently, Shihab *et al*., (2023) conducted an analysis and application of a Variational Iteration based scheme for approximating solutions of some PDEs such as Korteweg-De-Vries, Benjamin, and Airy equations. Their study indicates that the approach allows for rapid implementation without the need to deconstruct the nonlinear variables.
The implementation of the New Iterative Method (NIM) for tackling nonlinear partial differential problems presents a novel and highly motivated approach in the realm of computational mathematics. The novelty lies in NIM's ability to efficiently address complex nonlinear PDEs, offering a unique perspective on solving these intricate equations. The motivation for this topic stems from the growing importance of accurately simulating real-world phenomena in fields such as physics, engineering, and biology, where nonlinear PDEs play a pivotal role. The justification for this research lies in the demand for improved numerical methods that can provide more accurate and stable solutions to nonlinear PDEs, ultimately advancing our understanding and problem-solving capabilities in various scientific and engineering applications.

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Description of New Iterative Method (NIM)

To elucidate the concept of the New Iterative Method, contemplate the subsequent generic functional equation;

$$Z = F + L(z) + N(z) \tag{1}$$

where L , N are linear and nonlinear operators respectively, and f is a given function. The solution of equation (1) has the form

$$z = \sum_{p=0}^{\infty} z_p \tag{2}$$

Now, suppose we have the relation in (3)

$$\begin{aligned} Z_0 &= N(z_0) = f \\ Z_p &= N\left(\sum_{p=0}^{n} z_p\right) - N\left(\sum_{p=0}^{n-1} z_p\right) \end{aligned} \tag{3}$$

Then we can easily get

$$\begin{aligned} R_0 &= N(z_0) \\ R_1 &= N(z_0 + z_1) - N(z_0) \\ R_2 &= N(z_0 + z_1 + z_2) - N(z_0 + z_1) \\ R_3 &= N(z_0 + z_1 + z_2 + z_3) - N(z_0 + z_1 + z_2) + \cdots. \end{aligned} \tag{4}$$

Such that $N(z)$ can be splitted as:

$$\begin{aligned} N\left(\sum_{p=0}^{m} z_p\right) &= N(z_0 + z_1) - N(z_0) + N(z_0 + z_1 + z_2) - N(z_0 + z_1) \\ &+ N(z_0 + z_1 + z_2 + z_3) - N(z_0 + z_1 + z_2) + \cdots. \end{aligned} \tag{5}$$

To obtain a recurrence relation of the form:

$$\begin{aligned} z_0 &= f \\ z_1 &= L(z_0) + R_0 \\ z_{p+1} &= L(z_p) + R_p \quad p = 1, 2, 3\cdots. \end{aligned} \tag{6}$$

Since L is linear, then

$$\sum_{p=0}^{n} L(z_p) = L\left(\sum_{p=0}^{n} z_p\right) \tag{7}$$

So

$$\begin{aligned} \sum_{p=0}^{n+1} z_p &= \sum_{p=0}^{n} L(z_p) + N\left(\sum_{i=0}^{n} z_p\right) \\ &= L\left(\sum_{p=0}^{n} z_p\right) + N\left(\sum_{p=0}^{n} z_p\right), \quad p = 1, 2, \cdots. \end{aligned} \tag{8}$$

Thus,

$$\sum_{p=0}^{\infty} z_p = f + L\left(\sum_{p=0}^{\infty} z_p\right) + N\left(\sum_{p=0}^{\infty} z_p\right) \tag{9}$$

The k-term solution is given by the following form: $Z = \sum_{i=0}^{k-1} z_i$

Convergence Analysis

We examine the convergence of NIM for resolving any functional equation. Suppose $E = Z^* - Z$, where Z^* denotes the exact solution, Z is the approximate solution, and E is the error in the solution. Then we have

$$E(x) = f(x) + N(E(x)) \tag{10}$$

Applying the above equation to the NIM scheme, the recurrence relation becomes

$$E_0 = f$$
$$E_1 = N(E_0) \quad (11)$$
$$E_{p+1} = N(E_0 + E_1 + \cdots + E_n) - N(E_0 + E_1 + \cdots + E_{n-}), \; p = 1, 2, \cdots.$$

If $\| N(x) - N(t) \| \leq n \| x - t \|, \; 0 < n < 1,$ then we obtain

$$E_0 = f$$
$$\| E_1 \| = \| N(E_0) \| \leq n \| E_0 \|,$$
$$\| E_2 \| = \| N(E_0 + E_1) - N(E_0) \| \leq n \| E_1 \| \leq n^2 \| E_0 \|, \quad (12)$$
$$\vdots$$
$$\| E_{p+1} \| = N(E_0 + \cdots + E_n) - N(E_0 + \cdots + E_{p-1}) \| \leq n \| E_p \| \leq n^{p+1} \| E_0 \|,$$
$$p = 0, 1, 2, \ldots$$

Thus $E_{p+1} \to 0$ as $p \to \infty$, which proves the convergence of the NIM for solving general functional equation.

Implementations and Results

We implemented the NIM method on some PDEs, by applying it to a set of benchmark problems with known solutions. This is to assess its performance in terms of accuracy, convergence behavior, and computational efficiency. Furthermore, the results obtained through Maple 2021 software were compared with other established methods to determine its advantages and limitations. The computed results are tabulated in Tables 1-3.

Problem 1: We compute the nonlinear PDE (Shihab *et al*., 2023);

$z_t + a(z_x)^2 + bz_{xxx} = 0,$ with initial condition: $z(x,0) = A\tanh(Bx)$,

where $A = \frac{6bB}{a}$, $B = \frac{1}{2}\sqrt{\frac{v}{b}}$ True solution: $z(x,t) = A\tanh[B(x - vt)]$

As a result, the PDE problem described above can be expressed as the subsequent set of integral equations:

$$Z(x,t) = -I_t\left(aZ_x^2 + bZ_{xxx}\right)$$

Taking

$$N(Z) = -I_t\left(aZ_x^2 + bZ_{xxx}\right)$$

Hence, by examining equations (3), (4), (5)and (6), we can readily deduce the initial components of the New iterative solution for problem 1 as

$$z_0 := A\tanh(Bx)$$
$$z_1 := \left(2AB^3\left(1 - \tanh(Bx)^2\right)^2 - 4AB^3\tanh(Bx)^2\left(1 - \tanh(Bx)^2\right) - A^2B^2\left(1 - \tanh(Bx)^2\right)^2\right)t$$

The remaining components of the iterative formula (9) are derived using Maple software 2023 version and the computed solutions are tabulated in Table 1.

Problem 2: Considering the nonlinear PDE (Shihab *et al*., 2023)

$z_{tt} + a(zz_x)_x + \beta z_{xxxx} = 0$, with initial conditions $z(x,0) = A\,\mathrm{sec}h^2(Bx)$

$z_t(x,0) = 2ABv\,\mathrm{sech}^2(Bx)\tanh(Bx)$

where $A = \frac{12\beta B^2}{a}$, $B = \frac{1}{2}\frac{v}{\sqrt{-\beta}}$ True solution: $z(x,t) = A\,\mathrm{sech}^2[B(x - vt)]$

Consequently, the previously described PDE problem can be reformulated as the following collection of integral equations:

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$Z(x,t) = -I_t^2\left(a\left(ZZ_x\right)_x + \beta Z_{xxxx}\right)$

taking

$N(Z) = -I_t^2\left(a\left(ZZ_x\right)_x + \beta Z_{xxxx}\right)$

Therefore, by analyzing equations (3), (4), (5), and (6), we can easily infer the initial components of the New Iterative solution for problem 2, such as:

$$z_0 := 2\,t\,A\,B\,\mathrm{sech}(B\,x)^2\,\tanh(B\,x) + A\,\mathrm{sech}(B\,x)^2$$

$$\begin{aligned}
z_1 := {} & 48\,t^2 A\,B^5\,\mathrm{sech}(B\,x)^2\tanh(B\,x)^5 - 624\,t^2 A\,B^5\,\mathrm{sech}(B\,x)^2\tanh(B\,x)^3\left(1-\tanh(B\,x)^2\right) \\
& + 408\,t^2 A\,B^5\,\mathrm{sech}(B\,x)^2\tanh(B\,x)\left(1-\tanh(B\,x)^2\right)^2 + 48\,t\,A\,B^4\,\mathrm{sech}(B\,x)^2\tanh(B\,x)^4 \\
& - 264\,t\,A\,B^4\,\mathrm{sech}(B\,x)^2\tanh(B\,x)^2\left(1-\tanh(B\,x)^2\right) + 48\,t\,A\,B^4\,\mathrm{sech}(B\,x)^2\big(1 \\
& - \tanh(B\,x)^2\big)^2 + \frac{1}{3}\Big(\Big(-4\,A\,B^2\,\mathrm{sech}(B\,x)^2\tanh(B\,x)^2 + 2\,A\,B^2\,\mathrm{sech}(B\,x)^2\big(1 \\
& - \tanh(B\,x)^2\big)\Big)\Big(8\,A\,B^3\,\mathrm{sech}(B\,x)^2\tanh(B\,x)^3 - 16\,A\,B^3\,\mathrm{sech}(B\,x)^2\tanh(B\,x)\big(1 \\
& - \tanh(B\,x)^2\big)\Big)\,t^3\Big) + \frac{1}{2}\Big(\Big(-2\,A\,B\,\mathrm{sech}(B\,x)^2\tanh(B\,x)\Big(8\,A\,B^3\,\mathrm{sech}(B\,x)^2\tanh(B\,x)^3 \\
& - 16\,A\,B^3\,\mathrm{sech}(B\,x)^2\tanh(B\,x)\left(1-\tanh(B\,x)^2\right)\Big) + \Big(-4\,A\,B^2\,\mathrm{sech}(B\,x)^2\tanh(B\,x)^2 \\
& + 2\,A\,B^2\,\mathrm{sech}(B\,x)^2\left(1-\tanh(B\,x)^2\right)\Big)\Big(4\,A\,B^2\,\mathrm{sech}(B\,x)^2\tanh(B\,x)^2 \\
& - 2\,A\,B^2\,\mathrm{sech}(B\,x)^2\left(1-\tanh(B\,x)^2\right)\Big)\Big)\,t^2\Big) \\
& - 2\,A\,B\,\mathrm{sech}(B\,x)^2\tanh(B\,x)\Big(4\,A\,B^2\,\mathrm{sech}(B\,x)^2\tanh(B\,x)^2 - 2\,A\,B^2\,\mathrm{sech}(B\,x)^2\big(1 \\
& - \tanh(B\,x)^2\big)\Big)\,t
\end{aligned}$$

The remaining elements of the iterative formula (9) are computed through the utilization of Maple software, version 2023, and the resulting solutions are presented in Table 2.

Problem 3: We apply the NIM to solve the following nonlinear PDE;

$Z_{tt} + Z_{xx} + Z_{x^2} = 2x + t^4$

$Z(x,0) = 0; Z(0,t) = at; Z_t(x,0) = a; Z_x(0,t) = t^2$

Exact solution: $Z(x,t) = at + xt^2$

The problem mentioned above is analogous to the subsequent integral equation.

$Z = at + \dfrac{t^6}{30} + xt^2 - I_t^2\left(Z_{xx} + Z_x^2\right)$

Let $N(Z) = -I_t^2\left(Z_{xx} + Z_x^2\right)$, in view of the procedure in equations (3-6), we obtain the following relations

$Z_0 = at + \dfrac{t^6}{30} + xt^2,$

$Z_1 = N(Z_0) = -\dfrac{t^6}{30}$

$Z_2 = 0,\ Z_3 = 0, \cdots.$

The remaining components of the iterative formula (9) are calculated using Maple software, and the outcomes are showcased in Table 3. And consequently, further iterations leads to $Z(x,t) = at + xt^2$, which is the exact solution.

Table 1: Analysis of NIM Result for Problem One

x/t	**NIM Solutions**	**VIM Solutions**	**True Solutions**	**Error of NIM** $\lvert Z_{NIM} - Z\rvert$	**Error of VIM** $\lvert Z_{VIM} - Z\rvert$
0.10	0.03749609425	0.03749606573	0.03749609425	5.0×10^{-11}	2.852×10^{-10}
0.25	0.09369151252	0.09368693050	0.09368901252	2.0×10^{-11}	2.082035141×10^{-8}
0.50	0.1866728295	0.1870085540	0.1870132399	0.00000	4.685844×10^{-6}
0.75	0.2702871420	0.2800088572	0.2796135561	2.40×10^{-8}	3.953011×10^{-6}
1.00	0.3711410670	0.3734682012	0.3711419684	9.014×10^{-7}	2.3262328×10^{-6}
2.00	0.7202371522	0.6889158600	0.7202372565	2.40×10^{-8}	3.13213965×10^{-4}

3.00	1.030174037	1.0547476570	1.030183947	9.910×10^{-6}	2.456277×10^{-4}
4.00	1.29157557	1.4121069	1.291585757	1.1087×10^{-5}	1.205169×10^{-3}
5.00	1.502646418	1.37114764	1.502657418	1.1×10^{-4}	1.316×10^{-2}

Table 1 displays the results calculated using the New Iterative Method, providing approximate solutions in comparison to the exact solutions for problem 1. The selection of parameters $V = 0.5$ & $a = b = 1$ is made to address the PDE in problem 1. After the fourth iteration, it is evident that the computed errors of the NIM approximate solutions are lower than those of the Variational Iterative method (VIM). This observation indicates that the NIM technique is more efficient in computing the solutions of the PDE compared to the VIM technique.

Table 2: Analysis of NIM Result for Problem Two

x/t	NIM Solutions	VIM Solutions	True Solutions	ror of NIM $\lvert Z_{NIM} - Z\rvert$	Error of VIM $\lvert Z_{VIM} - Z\rvert$
0.10	0.1874945069	0.1874945069	0.1874945069	0.000000	1.10×10^{-10}
0.25	0.1874656719	0.1874656719	0.1874656719	0.000000	$5.645801890\times10^{-11}$
0.50	0.1873627379	0.1873627380	0.1873627379	0.000000	$8.318579018\times10^{-11}$
0.75	0.1871913487	0.1871913486	0.1871913487	0.000000	2.72945×10^{-11}
1.00	0.1869517547	0.1869517548	0.1869517547	0.000000	4.62×10^{-11}
2.00	0.1853197872	0.1853197871	0.1853197872	0.000000	$4.036710999\times10^{-11}$
3.00	0.1826417751	0.1826417747	0.1826417751	0.000000	$3.194813670\times10^{-10}$
4.00	0.1789784742	0.1789784710	0.1789784742	0.000000	3.2×10^{-9}
5.00	0.1744108363	0.1744108342	0.1744108363	0.000000	2.000290866×10^{-9}

Table 2 exhibits the numerical solutions obtained through the New Iterative Method, the Variational Iterative Method, as well as the exact solutions for problem 2. We have configured the parameters $v = 0.25,\ \beta = -3,\ and\ a = -1$ for problem 2, to assess the accuracy and reliability of the NIM's approximate solution for the considered PDE. A comparison of the results after the fourth iteration of the NIM with the exact solution and the VIM reveals a notable trend. It is evident that the NIM's solution converges more effectively to the exact solution in comparison to the VIM. This demonstrates the NIM's superior efficiency in handling nonlinear PDEs.

Table 3: Analysis of NIM Result for Problem three

x/t	NIM Solutions	VIM Solutions	True Solutions	Error of NIM $\lvert Z_{NIM} - Z\rvert$	Error of VIM $\lvert Z_{VIM} - Z\rvert$
0.10	0.1010000000	0.1010000000	0.10100000000	0.00000	0.0000
0.25	0.2080000000	0.2080000000	0.20800000000	0.00000	0.00000
0.50	0.3270000000	0.3270000000	0.32700000000	0.00000	0.00000
0.75	0.4640000000	0.4640000000	0.46400000000	0.00000	0.00000
1.00	0.625000000	0.6250000000	0.62500000000	0.00000	0.00000
2.00	0.8160000000	0.8160000000	0.8160000000	0.00000	0.00000
3.00	1.0430000000	1.0430000000	1.04300000000	0.00000	0.00000
4.00	1.3120000000	1.3120000000	1.31200000000	0.00000	0.00000
5.00	1.6290000000	1.6290000000	1.62900000000	0.00000	0.00000
0.10	2.0000000000	2.0000000000	2.0000000000	0.00000	0.00000

Within the table provided, you can find the approximate solutions derived using the NIM, the Variational Iterative Method (VIM), and the exact solutions for problem 3. Notably, what's interesting is that, by the 10th iteration, both the NIM and VIM approximate solutions converge remarkably close to the exact solutions. An intriguing observation in the results is that the errors for both methods have reached zero. This outcome strongly indicates the high efficiency and effectiveness of these two methods when it comes to solving problems involving partial differential equations (PDEs).

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CONCLUSION

In this study, we introduced and implemented a numerical approach for tackling nonlinear partial differential equations. We carried out an extensive analysis and practical application of the New Iterative method to approximate solutions for a set of chosen PDEs. This method consistently generates a sequence of solutions that progressively approaches the exact solution. The New Iterative technique marks a substantial leap forward in the domain of solving nonlinear PDEs, and its potential impact extends across a wide range of scientific and engineering domains. Its adaptability, precision, and computational efficiency position it as a promising option for addressing intricate nonlinear issues in the future. The outcomes of this study hold the potential to guide scientists and engineers, enhancing their comprehension of complex physical problems. This, in turn, may pave the way for the formulation of strategies and plans aimed at addressing such issues more effectively. We propose that future research should explore the application of the New Iterative Method to address various other categories of differential equations.

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