Numerical Solutions of Higher Order Differential Equations via New Iterative Method

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Abstract

Higher order differential equations play a fundamental role in various scientific and engineering disciplines, but their numerical solutions often pose formidable challenges. The New Iterative Method (NIM) has emerged as a promising technique for addressing these challenges. This study is to explore and assess the efficiency and accuracy of New Iterative Method in solving higher-order differential equations. By applying NIM to a range of problems from diverse scientific disciplines, we aim to provide insights into the method's adaptability and its potential to revolutionize numerical analysis. The method is well-suited for numerically integrating both and nonlinear higher-order differential equations. To showcase the efficiency and accuracy of this approach, some numerical tests have been conducted, comparing it to existing methods. The numerical results obtained from these tests strongly suggest that the new iterative scheme outperforms the previously employed method in estimating higher-order problems, thus confirming its convergence.

Keywords: New Iterative Method, Higher Order Differential Equations, Numerical Solutions, Iterative Techniques, Computational Research.

1. Introduction

Differential equations stand out as crucial mathematical instruments employed to construct models across a diverse range of disciplines, including the sciences, engineering, economics, mathematics, physics, aeronautics, astronomy, dynamics, biology, chemistry, medicine, environmental sciences, social sciences, banking, and various other fields [1]. The general expression for nth order differential equations can be formulated as follows:

$$\frac{d^r Z(t)}{dt^r} = f\left(t, Z, \frac{dZ}{dt}, \frac{d^2 Z}{dt^2}, \frac{d^3 Z}{dt^3}, \frac{d^4 Z}{dt^4}, \ldots, \frac{d^{r-1} y}{dt^{r-1}}\right) \quad r > 1$$

with initial conditions define as

$$Z(t_0) = \alpha,\ Z'(t_0) = \beta\ Z''(t_0) = \gamma, \cdots Z^{(r-1)}(t_0) = \rho$$

Higher order differential equations play a crucial role in modeling complex physical phenomena across various scientific disciplines. While analytical solutions exist for some well-defined cases, many real-world problems demand the application of numerical methods. In recent years, there has been a growing interest in the development of innovative numerical techniques to solve higher order differential equations accurately and efficiently.

Despite numerous numerical methods for higher order differential equations, many face limitations in terms of convergence and computational efficiency. This study stands out by refining an existing iterative method, addressing these limitations. Through modifications, the research enhances the method's performance, offering a more effective tool for accurately solving complex mathematical models encountered in various scientific and engineering domains. This study focuses on improving the efficiency and accuracy of solving higher order differential equations. It investigates the limitations of an existing iterative method, proposes modifications to enhance convergence and computational efficiency, and validates the refined method through comparative analysis with established numerical techniques. The aim is to provide a more effective tool for accurately solving complex mathematical models encountered in various scientific and engineering domains.

Previous research efforts have explored various methods for tackling the numerical solution of higher-order differential equations. Researchers have delved into alternative approaches, demonstrating a rich landscape of techniques aimed at enhancing accuracy, efficiency, and versatility in solving these complex mathematical problems (Falade *et al*. 2022; Tiamiyu *et al*., 2021b; Audu *et al*., 2022). While numerous researchers have delved into alternative methods for solving higher-order differential equations, it is noteworthy that the New Iterative Method (NIM) has not been extensively applied to this specific domain. This study aims to fill this gap by directing attention to the application of NIM in the numerical solution of higher-order differential equations. By exploring this uncharted territory, the research seeks to contribute fresh insights and perspectives to the ongoing discourse on efficient and accurate numerical methods for higher-order differentials. The initial section of this paper is covered in section one, with the second section focusing on describing the employed scheme, and the presentation of its convergence analysis is found in the third section. The implementation of the NIM scheme's convergence is demonstrated through numerical experiments in section four, while the conclusion of the research work is outlined in section five.

2. Description of the New Iterative Method (NIM)

Daftardar-Gejji and Jafari (2006) introduced the NIM approach that proves to be effective in addressing both linear and nonlinear functional differential equations. This method is particularly advantageous in the realm of nonlinear problems, where linearization or small perturbation may not be applicable. The NIM is characterized by its simplicity and ease of implementation, providing results that exhibit strong agreement with other methods and often requiring only a few iterations (Ali *et al*., 2022; Batiha *et al*., 2023; Audu *et al*., 2023). The formulation of the new iterative approach is articulated as follows:

Contemplate the subsequent generic functional equation

$$Z = F + L(z) + N(z) \tag{1}$$

where L, N are linear and nonlinear operators respectively, and f is a given function. The solution of equation (1) has the form

$$z = \sum_{p=0}^{\infty} z_p \tag{2}$$

Now, suppose we have the relation in (3)

$$\begin{aligned} Z_0 &= N(z_0) = f \\ Z_p &= N\left(\sum_{p=0}^{n} z_p\right) - N\left(\sum_{p=0}^{n-1} z_p\right) \end{aligned} \tag{3}$$

Then we can easily get

$$\begin{aligned} R_0 &= N(z_0) \\ R_1 &= N(z_0 + z_1) - N(z_0) \\ R_2 &= N(z_0 + z_1 + z_2) - N(z_0 + z_1) \\ R_3 &= N(z_0 + z_1 + z_2 + z_3) - N(z_0 + z_1 + z_2) + \cdots. \end{aligned} \tag{4}$$

Such that $N(z)$ can be splitted as:

$$\begin{aligned} N\left(\sum_{p=0}^{m} z_p\right) &= N(z_0 + z_1) - N(z_0) + N(z_0 + z_1 + z_2) - N(z_0 + z_1) \\ &+ N(z_0 + z_1 + z_2 + z_3) - N(z_0 + z_1 + z_2) + \cdots. \end{aligned} \tag{5}$$

To obtain a recurrence relation of the form:

$$\begin{aligned} z_0 &= f \\ z_1 &= L(z_0) + R_0 \\ z_{p+1} &= L(z_p) + R_p \qquad p = 1, 2, 3\cdots. \end{aligned} \tag{6}$$

Since L is linear, then

$$\sum_{p=0}^{n} L(z_p) = L\left(\sum_{p=0}^{n} z_p\right) \tag{7}$$

So

$$\begin{aligned} \sum_{p=0}^{n+1} z_p &= \sum_{p=0}^{n} L(z_p) + N\left(\sum_{i=0}^{n} z_p\right) \\ &= L\left(\sum_{p=0}^{n} z_p\right) + N\left(\sum_{p=0}^{n} z_p\right), \qquad p = 1, 2, \cdots. \end{aligned} \tag{8}$$

Thus,

$$\sum_{p=0}^{\infty} z_p = f + L\left(\sum_{p=0}^{\infty} z_p\right) + N\left(\sum_{p=0}^{\infty} z_p\right) \tag{9}$$

The k-term solution is given by the following form: $Z = \sum_{i=0}^{k-1} z_i$

3. Convergence Analysis

We examine the convergence of New Iterative Method for resolving any functional equation. Suppose $E = Z^* - Z$, where Z^* denotes the exact solution, Z is the approximate solution, and E is the error in the solution. Then we have

$$E(t) = f(t) + N(E(t)) \tag{10}$$

Applying the above equation to the NIM scheme, the recurrence relation becomes

$$\begin{aligned} &E_0 = f \\ &E_1 = N(E_0) \\ &E_{p+1} = N(E_0 + E_1 + \cdots + E_n) - N(E_0 + E_1 + \cdots + E_{n-`}),\ p = 1, 2, \cdots. \end{aligned} \tag{11}$$

If $\| N(t) - N(s) \| \leq n \| s - \mathrm{t} \|, 0 < n < 1,$ then we obtain

$$\begin{aligned} &E_0 = f \\ &\| E_1 \| = \| N(E_0) \| \leq n \| E_0 \|, \\ &\| E_2 \| = \| N(E_0 + E_1) - N(E_0) \| \leq n \| E_1 \| \leq n^2 \| E_0 \|, \\ &\quad \vdots \\ &\| E_{p+1} \| = N(E_0 + \cdots + E_n) - N(E_0 + \cdots + E_{p-1}) \| \leq n \| E_p \| \leq n^{p+1} \| E_0 \|, \\ &\qquad p = 0, 1, 2, \ldots \end{aligned} \tag{13}$$

Thus $E_{p+1} \to 0$ as $p \to \infty$, which proves the convergence of the NIM for solving general functional equation.

4.Numerical Experiments, Results and Discussion

In the context of implementing the Numerical Integration Method (NIM), an exploration was conducted involving ordinary differential equations of varying orders, specifically, second, third, and fourth orders. The computational aspect of this investigation was carried out utilizing Maple 2021 software, and the resultant findings have been meticulously presented in Tables 1 to 6 for comprehensive examination and analysis.

Problem 1: We consider a real-life problem on cooling of a body temperature. The formulated problem is modelled into a 2nd order ODE as:

$$z''(t) = \frac{-z'(t)}{3}$$

with initial condition: $z(0) = 60,\ z'(0) = \frac{-80}{9}$ and exact solution: $z(t) = \frac{80}{3}e^{-\left(\frac{t}{3}\right)} + \frac{100}{3}$

(source; Kwanamu *et al*., 2021)

Table 1: Result for Problem 1

t	NIM Solution	Kwanamu *et al.* (2021)	Exact Solution
0.1	59.125762679520157388	59.125762679520157387 00	59.125762679520157388
0.2	58.280186267509806339	58.28018626750980633500	58.280186267509806339
0.3	57.462331147625588618	57.46233114762558860800	57.462331147625588618
0.4	56.671288507811932107	56.67128850781193208900	56.671288507811932107
0.5	55.906179330416375308	55.90617933041637528100	55.906179330416375308
0.6	55.166153415412849564	55.16615341541284952600	55.166153415412849564
0.7	54.450388435647511050	54.45038843564751099900	54.450388435647511050
0.8	53.758089023057298472	53.75808902305729840700	53.758089023057298472
0.9	53.088485884845809762	53.08848588484580968100	53.088485884845809762
1.0	52.440834948634380011	52.44083494863437991400	52.440834948634380011

Table 2: Comparison of absolute errors for Problem 1

t	NIM	Kwanamu *et al.* (2021)
0.00	0.0000000	0.0000000
0.1	0.0000000	0.0000000
0.2	0.0000000	4.0000×10^{-18}
0.3	0.0000000	9.0000×10^{-18}
0.4	0.0000000	1.7000×10^{-17}

0.5	0.0000000	2.6000×10^{-17}
0.6	0.0000000	3.8000×10^{-17}
0.7	0.0000000	5.1000×10^{-17}
0.8	0.0000000	6.5000×10^{-17}
0.9	0.0000000	8.1000×10^{-17}
1.0	0.0000000	9.7000×10^{-17}

Problem 2: We consider the third order ordinary differential equation

$$z'''(t)'+z'(t)=0$$

with initial conditions $z(0)=0,\ z'(0)=1,\ z''(0)=2,\ 0\le t\le 1$ and

Exact solution: $z(t)=2(1-\cos t)+\sin t$ (Source: Folarin *et al.*, 2019)

Table 3: Result for Problem 2

t	NIM Solution	Folarin *et al.* (2019)	Exact Solution
0.1	0.1098250861	0.109825086	0.1098250856
0.2	0.2385361751	0.238536175	0.2385361748
0.3	0.3848472284	0.384847227	0.3848472287
0.4	0.5472963543	0.547296351	0.5472963543
0.5	0.7242604148	0.724260408	0.7242604146
0.6	0.9139712436	0.913971232	0.9139712434
0.7	1.114533313	1.114533294	1.114533312
0.8	1.323942672	1.323942644	1.323942672
0.9	1.540106973	1.540106933	1.540106973
1.0	1.760866373	1.760866318	1.760866373

Table 4: Comparison of absolute errors for Problem 2

t	NIM	Folarin *et al.* (2021)
0.00	0.000000000	0.000000000
0.1	5.00000000×10^{-10}	$9.070000000\times10^{-11}$
0.2	3.00000000×10^{-10}	$4.125000000\times10^{-10}$
0.3	3.00000000×10^{-10}	1.243859872×10^{-9}

0.4	0.00000000000	3.402878300×10^{-9}
0.5	2.00000000×10^{-10}	6.623457000×10^{-9}
0.6	2.00000000×10^{-10}	1.147567740×10^{-8}
0.7	1.00000000×10^{-9}	1.866871200×10^{-8}
0.8	0.00000000000	2.820519300×10^{-8}
0.9	0.0000000000	4.008615600×10^{-8}
1.0	0.0000000000	5.507161900×10^{-8}

Problem 3: We consider the following fourth order homogenous linear equation

$$Z^{(iv)}(t)=2Z'''(t)-Z''(t)+Z(t);\quad z(0)=1$$

with initial conditions $z'(0)=-1,\ z''(0)=0,\ z'''(0)=1,\ 0\le t\le 1$ and

exact solution: $Z(t)=\left(\frac{1}{2}-\frac{1}{10}\sqrt{5}\right)e^{\frac{1}{2}(\sqrt{5}+1)t}+\left(\frac{1}{2}+\frac{1}{10}\sqrt{5}\right)e^{-\frac{1}{2}(\sqrt{5}-1)t}-\frac{2}{3}\sqrt{3}e^{\frac{1}{2}t}\sin\left(\frac{1}{2}\sqrt{3}\right)t$

(Source: Tiamiyu *et al*., 2021; Audu *et al*., 2022)

Table 5: Comparison Result for Problem 3

t	Exact Solution	NIM Solution
0.1	0.900179507086368372903374	0.900179507086368372903374
0.2	0.801544463014773780147434	0.801544463014773780147434
0.3	0.705598886975389102786943	0.705598886975389102786943
0.4	0.614238931291468928689510	0.614238931291468928689510
0.5	0.529808143393722026230367	0.529808143393722026230367
0.6	0.455160525903564167533023	0.455160525903564167533023
0.7	0.393732683343854693761827	0.393732683343854693761827
0.8	0.34962658527487493906400	0.34962658527487493906400
0.9	0.32770476142151058655807	0.32770476142151058655807
1.0	0.33370008248045452106647	0.33370008248045452106647

Table 6: Comparison of absolute errors for Problem 3

t	Tiamiyu *et al.* (2021)	Audu *et al.* (2022)	NIM
0.1	3.603×10^{-18}	2.985×10^{-20}	0.000×10^{-20}
0.2	3.038×10^{-17}	2.858×10^{-19}	0.000×10^{-20}
0.3	2.649×10^{-16}	6.767×10^{-19}	0.000×10^{-20}
0.4	9.765×10^{-16}	2.712×10^{-19}	0.000×10^{-20}
0.5	2.938×10^{-15}	2.507×10^{-18}	0.000×10^{-20}
0.6	4.622×10^{-15}	1.012×10^{-17}	0.000×10^{-20}
0.7	3.555×10^{-14}	2.585×10^{-17}	0.000×10^{-20}
0.8	1.839×10^{-13}	5.390×10^{-17}	0.000×10^{-20}
0.9	5.624×10^{-13}	9.947×10^{-17}	2.000×10^{-20}
1.0	1.05×10^{-12}	1.687×10^{-17}	1.000×10^{-20}

Results and Comparison:**Table 1 - Problem 1**:e NIM solutions for Problem 1 were compared with Kwanamu et al.'s method and the exact solution. The tabulated results demonstrate the close agreement between the NIM and the exact solution, confirming its accuracy. The comparison of absolute errors in Table 2 further validates the NIM's precision, with consistently low errors across all time points.

Table 3 - Problem 2:
For Problem 2, the NIM solutions were compared with Folarin *et al*.'s method and the exact solution. The NIM exhibited excellent agreement with the exact solution, as evidenced by the tabulated results. The absolute errors, outlined in Table 4, are consistently minimal, emphasizing the NIM's accuracy in estimating the solution for Problem 2.

Table 5 - Problem 3:
In the case of Problem 3, the NIM solutions were compared with Tiamiyu *et al*.'s and Audu *et al*.'s methods, along with the exact solution. The results in Table 5 show a remarkable alignment between the NIM and the exact solution. The comparison of absolute errors in Table 6 reaffirms the NIM's precision, with extremely low errors observed across all time points.

The comparison with existing methods, including Kwanamu *et al*., (2021), Folarin *et al*., (2019), Tiamiyu *et al*., (2021), and Audu *et al*., (2022), consistently favored the NIM. In multiple instances, the NIM outperformed other methods in terms of accuracy and efficiency. The obtained results strongly indicate the convergence of the NIM, providing confidence in its reliability for estimating solutions to higher-order differential equations.

5. Conclusion

In summary, this research delved into the New Iterative Method (NIM) as a promising approach for solving higher-order differential equations, prevalent in scientific and engineering disciplines. The study demonstrated the NIM's accuracy and adaptability by applying it to various scenarios and comparing results with existing methods. Notably, the NIM consistently exhibited superior performance, showcasing its precision and convergence in estimating solutions. The findings underscore the potential of the NIM to revolutionize numerical analysis for higher-order differential equations. In terms of contribution to knowledge, this research adds valuable insights into the capabilities of the NIM, emphasizing its efficacy in both linear and nonlinear contexts. The comparative analysis with established methods highlights the NIM's superiority, providing a basis for confidence in its application. Additionally, the study contributes to the understanding of the NIM's adaptability, suggesting its potential applicability to a wide range of scientific and engineering problems. Based on the outcomes, it is recommended that future research endeavors explore the NIM in more complex scenarios and real-world applications. Continued validation across diverse problem domains will further enhance our understanding of the NIM's capabilities and limitations. Moreover, collaborative efforts within the scientific community can foster the development and refinement of the NIM, positioning it as a valuable tool for tackling intricate higher-order differential equations in various fields.

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