MODELLING MAGNETOHYDRODYNAMIC OSCILLATORY FLOW WITH VISCOUS ENERGY DISSIPATION THROUGH A POROUS CHANNEL SATURATED WITH POROUS MEDIUM

BY

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MTech/SPS/2019/10492

DEPARTMENT OF MATHEMATICS
FEDERAL UNIVERSITY OF TECHNOLOGY
MINNA, NIGERIA.

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL
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IN MATHEMATICS

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DECLARATION

I hereby declare that this thesis titled: "**Modelling Magnetohydrodynamic Oscillatory Flow with Viscous Energy Dissipation through a porous channel saturated with Porous Medium**"is a collection of my original research work and it has not been presented for any other qualification anywhere. Information from other sources (published or unpublished) has been duly acknowledged.

...……………………….

ADEBAYO, Helen Olaife SIGNATURE/DATE

MTech/SPS/2019/10492

FEDERAL UNIVERSITY OF TECHNOLOGY,

MINNA, NIGERIA

CERTIFICATION

The thesis titled: "**Modelling Magnetohydrodynamic Oscillatory Flow with Viscous Energy Dissipation through a porous channel saturated with porous Medium"**by ADEBAYO Helen Olaife, (MTech/SPS/2019/10492) meets the regulations governing the award of the degree of Masters of Technology of the Federal University of Technology, Minna and it is approved for its contribution to scientific knowledge and literary presentation.

Prof. R.O. Olayiwola ----------------------------
SUPERVISOR Signature &Date

Prof. A.I. Enagi ----------------------------
HEAD OF DEPARTMENT Signature &Date

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Engr. Prof. O. K. Abubakre -----------------------------
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DEDICATION

This thesis is dedicated to the glory of God, for His divine grace, protection, favour and inspiration throughout this programme.

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ABSTRACT

This thesis presents mathematical model of unsteady laminar flow of an incompressible viscous fluid through a channel with slip at the cold plate in the presence of viscous energy dissipation. The partial differential equations governing the phenomenon were non-dimensionalized using some dimensionless quantities. The dimensionless coupled non-linear partial differential equations were solved using harmonic solution technique by transforming into ordinary differential equations. The results obtained were presented graphically and discussed. From the results obtained, it was observed that increase in Peclet number, Eckert number, Thermal radiation parameter, Grashof thermal number and term due to thermal radiation increases the temperature profiles and increase in Hartmann number, Sunction/Injection parameter and Reynolds number leads to decrease in the temperature profile. Increase in the Sunction/Ijection parameter and Kinematic viscosity reduces the velocity profile and increase in the Reynold number increases the velocity profile.

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

The study of oscillatory flow of an electrically conducting fluid through a porous channel saturated with porous medium is important in many physiological flows and engineering applications such as Magneto-hydrodynamic (MHD) generators, arterial blood flow, petroleum engineering and many more. Oscillatory flow is a periodic flow that oscillates around a zero value. Oscillatory flow is a single swing or movement in one direction of an oscillating body. They are generally used in the literature to describe the flows in which velocity or pressure or both depend on time. Oscillatory flow is always important for it has many practical applications for example in the aerodynamics of helicopter rotor or in fluttering airfoil and also in a variety of bioengineering problems (Baba *et al*., 2020).

The effect of heat transfer on an oscillatory flow of an electrically conducting fluid in vertical media is encountered in a wide range of engineering areas, science and technology such as MHD power generators, plasma studies, nuclear reactor, geothermal energy extraction, electromagnetic propulsion, the boundary layer control in the field of aerodynamics. It also has numerous industrial applications in molten iron flow, recovery extraction of crude oil, electrostatic precipitation, petroleum industry and polymer technology (Mehta *et al*., 2020).

Flow of conducting fluid in external magnetic field produce a variety of new effects, which are not realized in usual hydrodynamics. MHD analyzes these phenomena. It also studies the arising of a flow of conducting fluid due to the current passing through the fluid (so-called electrically induced vortex-type flows). Eclectromagnetic methods of action on electrically conducting medium are mostly used both in technical devices such as pump, flow meters, generators and industrial processes in metallurgy and material processing. Another common application of MHD in metallurgy is MHD separation that is used for electromagnetic removal of non-metalic inclusions from melts and metal extraction from Oxides i.e. MHD is used for cleaning liquid metals of impurities as well as for the separation of multiphase systems into their components (Herman and Yeshajahu, 1993).

1.2 Statement of the Research Problem

The study of oscillatory flow of an electrically conducting fluid through a porous channel saturated with porous medium is important in many physiological flows and engineering applications such as magneto-hydrodynamic (MHD) generators, arterial blood flow, petroleum engineering and many more. Therefore, investigation of oscillatory flow through porous channel is important for improving the existing industrial processes and for developing new MHD devices.

1.3 Aim and Objectives of the Study

1.3.1 Aim

The aim of this research work is to investigate the effect of viscous energy dissipation on oscillatory flow through a porous medium.

1.3.2 Objectives

The objectives of this study are to:

i. Formulate a model for the oscillatory flow through a porous medium with viscous energy dissipation.

ii. Obtain the analytical solution of the model using harmonic solution technique.

iii. Provide the graphical representation of the system responses.

iv. Analyses the result obtained.

1.4.1 Significance of the Study

Oscillatory flow of an electrically conducting fluid through a porous channel saturated with porous medium is important in many physiological flows and engineering applications such as magneto-hydrodynamic (MHD) generators, arterial blood flow, petroleum engineering e.t.c. Magnetohydrodynamics (MHD) finds its application in meteorology, solar physics, geophysics and motion of the earth core. MHD flow have also significant applications in the field of planetary magnetospheres, aeronautical plasma flows, chemical engineering e.t.c which arises to the use of Partial Differential Equations (PDE) to model these physical phenomena. Based on this, a mathematical model to analyse effect of viscous energy dissipation on oscillatory flow through a porous medium using harmonic technique was presented.

1.5 Scope and Limitation

1.5.1 Scope

The scope of this research work is to study the governing equations by providing analytical solution using harmonic solution method for the analysis of oscillatory flow through a porous medium with viscous energy dissipation.

1.5.2 Limitation

This research work is limited to the mathematical study of oscillatory flow through a porous medium with viscous energy dissipation.

1.6 Definition of Terms

Convection: This is heat transfer by mass motion of fluid such as air or water when the heated fluid is caused to move away from the source of heat, carrying energy on it.

Compressible fluid: This is the one in which the fluid density changes when it is subjected to high pressure-gradients.

Electrodynamics: The branch of mechanics concerned with the interaction of electric currents with magnetic fields or with other electric currents.

Fluid: A substance that has no fixed shape and yield easily to external pressure, either gas or liquid.

Heat transfer: Exchange of thermal energy between physical systems. The rate of heat transfer is dependent on the temperature of the system and the properties of the intervening medium through which the heat is transferred.

Incompressible fluid: Fluid whose volume or density does not change with pressure.

Magnetic Field: Region around a magnetic material or a moving electric charge within which the force of magnetism acts.

Magnetism: A physical process produced by the motion of electric charge, which results in attractive and repulsive forces between objects.

Mathematical Modelling: A representation of a system, process or relationship in a mathematical form in which equations are used to simulate the behavior of the system or process under study.

Order: Order of a differential equation is the order of the highest derivative involved in the equation.

Ordinary Differential Equation (ODE): An equation containing a single independent variable.

Partial Differential Equation (PDE): An equation containing two or more independent variables.

Unsteady flow: Flow in which quantity of liquid flowing per second is not constant, that is, if at any point in the fluid, the conditions change with time. Unsteady flow is a transient phenomenon.

Steady flow: A flow of fluid is steady if its velocity, pressure and all the numerical values relating to its substance (e.g. density and viscosity) are independent of time at every point in the flow field.

Viscosity: Measure of fluids resistance to gradual deformation by shear stress or tensile stress. It is the friction between the molecules of fluids.

Viscous: A thick, sticky consistency between solid and liquid.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Related Literatures

The study of oscillatory fluid flow and heat transfer in porous medium with inclined magnetic field has been carried out by several authors like: Makinde and Aziz (2010) who analyzed MHD mixed convection from a vertical plate embedded in a porous medium with a convective boundary condition.

Several authors have studied the flow and heat transfer in oscillatory fluid problems. To mention just a few, Makinde and Mhone (2005) investigated the forced convective MHD oscillatory fluid flow through a channel filled with porous medium, and analyses were based on the assumption that the plates are impervious. In a related study, Mehmood (2007) investigated the effect of slip on the free convective oscillatory flow through vertical channel with periodic temperature and dissipative heat. In addition, Chauchan and Kumar (2011) studied the steady flow and heat transfer in a composite vertical channel. Palani and Abbas (2009) investigated the combined effects of magneto-hydrodynamics and radiation effect on free convection flow past an impulsively started isothermal vertical plate using the Rosseland approximation. Hussain *et al*. (2010) presented analytical study of oscillatory second grade fluid flow in the presence of a transverse magnetic field and many more. In all the studies above, the channel walls were assumed to be impervious. This assumption was not valid in studying flows such as blood flow in miniature level where digested food particles were diffused into the bloodstream through the wall of the blood capillary.

Umavathi *et al*. (2009) investigated the unsteady flow of viscous fluid through a horizontal composite channel whose half width was filled with porous medium. Ajibade and Jha (2010) presented the effects of suction and injection on hydrodynamics of oscillatory fluid through parallel plates. Ajibade and Jha (2012) extended the problem to heat generating/absorbing fluids in and the effect of viscous dissipation of the free convective flow with time dependent boundary condition was investigated. Adesanya and Makinde (2012) further investigated the effect of radiative heat transfer on the pulsatile couple stress fluid flow with time dependent boundary condition on the heated plate. It is well known that the no-slip condition is not realistic in some flows involving Nanochannel, micro-channel and flows over coated plates with hydrophobic substances. In view of this, Adesanya and Gbadeyan (2010) studied the flow and heat transfer of steady non Newtonian fluid flow noting the fluid slip in the porous channel.

Vieru and Rauf (2011) obtained the exact solutions of Stokes flows for a Maxwell fluid whereas Vieru and Zafar (2013) recently investigated some Couette flows of a Maxwell fluid. Cookey *et al.* (2010) contributed to MHD oscillatory Couette flow of a radiating viscous fluid in a porous medium with periodic wall temperature. Makinde and Chinyoka (2010a) discussed MHD transient flows and heat transfer of dusty fluid in a channel with variable physical properties and Navier slip condition. A numerical investigation of transient heat transfer to hydromagnetic channel flow with radiative heat and convective cooling has been carried out by Makinde and Chinyoka (2010b). Gireesha *et al.* (2010) analyzed unsteady flow and heat transfer of a dusty fluid through a rectangular channel under the influence of pulsatile pressure gradient and uniform magnetic field. Prakash *et al.* (2011) investigated MHD free convective flow of a viscoelastic (Kuvshinski type) dusty gas through a porous medium induced by the motion of a semi-infinite flat plate under the influence of radiative heat transfer.

Zubi, (2018) studied MHD and mass transfer of an oscillatory flow over a vertical permeable plate in a porous medium with chemical reaction. He considered a two-dimensional, unsteady, laminar non-Darcian mixed convection flow of an incompressible, viscous, electrically conducting fluid. He applied a magnetic field of strength vertically to the sufface, neglecting the effect of induced magnetic field.

Mansour *et al*. (2007) studied the free convection flow of micropolar fluid in slip flow regime through porous medium with periodic temperature and concentration. (El-Hakiem 2000) analyzed thermal radiation effects on transient, two-dimensional hydromagnetic free convection along a vertical surface in a highly porous medium using the Roseland diffusion approximation for the radiative heat flux in the energy equation, for the case where free-stream velocity of the fluid vibrates about mean constant value and the surface absorbs the fluid with constant velocity.

Fetecau *et al*. (2021) analysis maxwell fluid flow through a porous plate channel induced by a constant accelerating oscillating wall. They considered an incompressible fluid at rest in a porous medius and used finite fourier sine transform to establish exact expressions for the dimensionless velocity and the shear stress fields corresponding to the two different motions of incompressible fluid.

Hamza *et al*. (2011) investigated unsteady heat transfer to MHD oscillatory flow through a porous medium under slip condition. They investigated the effects of slip condition, transverse magnetic field and radiative heat transfer to unsteady flow of a conducting optically thin fluid through a channel filled with porous medium. Exact solution of the governing equations for fully developed flow was obtained in closed form.

Kulshretha and Puri (1981) had investigated the couette flow of a dusty gas due to an oscillatory motion of the plate. The time dependent plate and transient effects had been included. The dusty gas contained between two parallel plates was disturbed by the motion of the lower plate with an arbitrary velocity $F(t)$. When $F(t)$ contained a factor of the type $\exp\{-(\lambda - i\omega)t\}$., two distinct types of waves were generated, one of which was oscillatory and the other was non-oscillatory which disappears for $\lambda = 0$. Reflections of these waves were studied and graph for the wave speeds were presented. Long time approximations for this type of $F(t)$ were evaluated and steady state solutions were obtained for $F(t)$ of the type $\exp(i\omega)$.

Kodi and Mopuri (2021) studied unsteady MHD oscillatory casson fluid flow past an inclined vertical porous plate in the presence of chemical reaction with heat absorption and soret effects. They perform unsteady hydrodynamic flow over an inclined plate embedded in a porous medium with soret-alligned magnetic field and chemical reaction.

Olayiwola (2016) presented an analytical method for studying chemically reacting flow in a laminar premixed flame of carbon monoxide/oxygen mixture in the region of stagnation point its result show that velocity increased as prandtl number increased, Biot number decreased the fluid velocity and enhanced the species concentration and flame temperature.

Krsihna *et al*. (2018) discussed heat and mass transfer on unsteady MHD oscillatory flow of blood through porous arteriole. They developed a mathematical model for unsteady state situations using slip conditions, analytical expressions were obtained and computationally discussed with respect to the non-dimensional parameters.

Falade *et al*. (2016) studied MHD oscillatory flow through a porous channel saturated with porous medium. They investigated the effect of suction/injection on the unsteady oscillatory flow through a vertical channel with non-uniform wall temperature. The fluid was subjected to a transverse magnetic field and the velocity slip at the lower plate was taken into consideration. Exact solutions of the dimensionless equations governing the fluid flow were obtained and the effects of the flow parameters on temperature, velocity profiles, skin friction and rate of heat transfer are discussed and shown graphically.

Their model equations are:

$$\frac{\partial u'}{\partial t'} - V_0 \frac{\partial u'}{\partial y'} = \frac{1}{\rho}\frac{dP'}{dx'} + \upsilon \frac{\partial^2 u'}{\partial y'^2} - \frac{\upsilon}{k}u' - \frac{\sigma_e B_0^2}{\rho}u' + g\beta\left(T' - T_0\right) \tag{2.1}$$

$$\frac{\partial T'}{\partial t'} - V_0 \frac{\partial T'}{\partial y'} = \frac{k_f}{\rho C_p}\frac{\partial^2 T'}{\partial y'^2} + \frac{4\alpha^2}{\rho C_p}\left(T' - T_0\right) \tag{2.2}$$

where

$u(y,t)$ is the axial velocity

V_0 is the constant horizontal Velocity

ρ is the fluid density

P' is the fluid pressure

υ is the Kinematic viscosity

K is the porous permeability

σ_e is the electrical conductivity

B_0 is the magnetic field intensity

g is the gravitational acceleration

β is the volumetric expansion

C_p is the specific heat at constant pressure

α is the term due to thermal radiation

k is the thermal conductivity

T is the fluid temperature

T_0 is the referenced fluid temperature

2.2 Summary of Review and Gap to fill

In reviewing literature of this study, it has been discussed that several works had been carried out on oscillatory fluid flow and heat transfer in porous medium with inclined magnetic field without considering viscous energy dissipation. In view of the above, this research work seeks to consider oscillatory fluid flow and heat transfer in porous medium with inclined magnetic field in the presence of viscous energy dissipation. Here, in formulating our model, the research work of Falade *et al*. (2016) was extended by incorporating viscous energy dissipation to the energy equation.

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Mathematical Formulation

Consider the unsteady laminar flow of an incompressible viscous electrically conducting fluid through a channel with slip at the cold plate. An external magnetic field is placed across the normal to the channel. It is assumed that the fluid has small electrical conductivity and the electro-magnetic force produced is also very small. The flow is subjected to suction at the cold wall and injection at the heated wall. We choose a Cartesian coordinate system (x', y') where x' lies along the centre of the channel, and y' is the distance measured in the normal section such that $y' = a$ is the channel's half width.

Under these assumptions, the equations governing the flow are as follows:

$$\frac{\partial u'}{\partial t'} - V_0 \frac{\partial u'}{\partial y'} = \frac{1}{\rho}\frac{dP'}{dx'} + \upsilon \frac{\partial^2 u'}{\partial y'^2} - \frac{\upsilon}{k}u' - \frac{\sigma_e B_0^2}{\rho}u' + g\beta\left(T' - T_0\right) \tag{3.1}$$

$$\frac{\partial T'}{\partial t'} - V_0 \frac{\partial T'}{\partial y'} = \frac{k_f}{\rho C_p}\frac{\partial^2 T'}{\partial y'^2} + \frac{4\alpha^2}{\rho C_p}\left(T' - T_0\right) + \frac{\upsilon}{C_p}\left(\frac{\partial u'}{\partial y'}\right)^2 \tag{3.2}$$

with the boundary conditions,

$$\left.\begin{array}{llll} u'(y,0) = 0, & u' = \dfrac{\sqrt{k}}{\alpha_s}\dfrac{du'}{dy'}, & T' = T_0 & on \quad y' = 0 \\ T'(y,0) = T_0 & u' = 0, & T' = T_0 & on \quad y' = h \end{array}\right\} \tag{3.3}$$

where

$u(y,t)$ is the axial velocity

V_0 is the constant horizontal Velocity

ρ is the fluid density

P' is the fluid pressure

υ is the Kinematic viscosity

K is the porous permeability

σ_e is the electrical conductivity

B_0 is the magnetic field intensity

g is the gravitational acceleration

β is the volumetric expansion

C_p is the specific heat at constant pressure

α is the term due to thermal radiation

k is the thermal conductivity

T is the fluid temperature

T_0 is the referenced fluid temperature

3.2 Non-dimensionalisation

Equations (3.1), (3.2) and (3.3) were non-dimensionalised using the following dimensionless variables:

$$\left.\begin{aligned} &x=\frac{x'}{h}, \quad y=\frac{y'}{h}, \quad \theta=\frac{T'-T_0}{T_1-T_0}, \quad u=\frac{u'}{U}, \quad v=\frac{v_0}{U}, \quad t=\frac{Ut'}{h}, \\ &P=\frac{P'}{\rho U^2}, \quad \frac{dP'}{dx'}=\frac{\rho U^2}{h}\frac{dP}{dx}, \quad u'=Uu, \quad t'=\frac{h}{U}t, \quad y'=hy, \end{aligned}\right\} \tag{3.4}$$

From equation (3.4) equation (3.5) was obtained

$$\left.\frac{\partial}{\partial t'}=\frac{U}{h}\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial y'}=\frac{1}{h}\frac{\partial}{\partial y}, \quad \frac{\partial^2}{\partial y'^2}=\frac{1}{h^2}\frac{\partial^2}{\partial y^2}\right\} \tag{3.5}$$

Equation (3.1) becomes,

$$\frac{U^2}{h}\frac{\partial u}{\partial t}-\frac{vU^2}{h}\frac{\partial u}{\partial y}=-\frac{U^2}{h}\frac{dP}{dx}+\frac{\upsilon U}{h^2}\frac{\partial^2 u}{\partial y^2}-\frac{\upsilon U}{k}u-\frac{\sigma_e B_0^2 U}{\rho}u+g\beta\left(T_1-T_0\right)\theta \tag{3.6}$$

Multiply equation (3.6) by $\frac{h}{U^2}$ gives,

$$\frac{\partial u}{\partial t}-v\frac{\partial u}{\partial y}=-\frac{dP}{dx}+\frac{\upsilon}{hU}\frac{\partial^2 u}{\partial y^2}-\frac{hv}{Uk}u-\frac{\sigma_e B_0^2 h}{\rho U}u+\frac{g\beta h\left(T_1-T_0\right)}{U^2}\theta \tag{3.7}$$

$$\frac{\partial u}{\partial t}-v\frac{\partial u}{\partial y}=-\frac{dP}{dx}+\frac{1}{R_e}\frac{\partial^2 u}{\partial y^2}-Su-H_a^2u+G_{r\theta}\theta \tag{3.8}$$

So,

$$\frac{\partial u}{\partial t}-v\frac{\partial u}{\partial y}=-\frac{dP}{dx}+\frac{1}{R_e}\frac{\partial^2 u}{\partial y^2}-\left(S+H_a^2\right)u+G_{r\theta}\theta \tag{3.9}$$

Equation (3.2) becomes,

$$\frac{U\left(T_1-T_0\right)}{h}\frac{\partial\theta}{\partial t}-\frac{U\left(T_1-T_0\right)v}{h}\frac{\partial\theta}{\partial y}=\frac{k_f\left(T_1-T_0\right)}{\rho C_p h^2}\frac{\partial^2\theta}{\partial y^2}+\frac{4\alpha^2\left(T_1-T_0\right)}{\rho C_p}\theta+\frac{\upsilon U^2}{C_p h^2}\left(\frac{\partial u}{\partial y}\right)^2 \tag{3.10}$$

Multiply equation (3.10) by $\frac{h}{U\left(T_1-T_0\right)}$ gives,

$$\frac{\partial\theta}{\partial t}-v\frac{\partial\theta}{\partial y}=\frac{k_f}{\rho C_p hU}\frac{\partial^2\theta}{\partial y^2}+\frac{4\alpha^2 h}{\rho C_p U}\theta+\frac{\upsilon U}{C_p h\left(T_1-T_0\right)}\left(\frac{\partial u}{\partial y}\right)^2 \tag{3.11}$$

So,

$$\frac{\partial\theta}{\partial t}-v\frac{\partial\theta}{\partial y}=\frac{1}{P_e}\frac{\partial^2\theta}{\partial y^2}+\delta\theta+\frac{E_c}{R_e}\left(\frac{\partial u}{\partial y}\right)^2 \tag{3.12}$$

Equation (3.3) also becomes,

$$
\left.
\begin{array}{l}
u = \dfrac{u'}{U}, \quad u\left(y,0\right) = \dfrac{0}{U} = 0, \\
\dfrac{\sqrt{k}}{\alpha_s}\dfrac{du'}{dy'}\bigg|_{y'=0} - u'\big|_{y'=0} = 0, \quad \dfrac{\sqrt{k}U}{h\alpha_s}\dfrac{du}{dy}\bigg|_{y=0} - U\,u\big|_{y=0} = 0, \\
\dfrac{du}{dy}\bigg|_{y=0} - \dfrac{h\alpha_s}{\sqrt{k}}u\big|_{y=0} = 0, \quad \dfrac{du}{dy}\bigg|_{y=0} - \alpha\,u\big|_{y=0} = 0, \quad \alpha = \dfrac{h\alpha_s}{\sqrt{k}}, \\
u\left(1,t\right) = \dfrac{0}{U} = 0 \\
u\left(y,0\right) = 0, \quad \dfrac{du}{dy}\bigg|_{y=0} - \alpha\,u\big|_{y=0} = 0, \quad u\left(1,t\right) = 0 \\
\theta = \dfrac{T' - T_0}{T_1 - T_0} \\
\theta\left(y,0\right) = \dfrac{T_0 - T_0}{T_1 - T_0} = 0, \quad \theta\left(0,t\right) = \dfrac{T_0 - T_0}{T_1 - T_0} = 0, \quad \theta\left(1,t\right) = \dfrac{T_1 - T_0}{T_1 - T_0} = 1 \\
\theta\left(y,0\right) = 0, \quad \theta\left(0,t\right) = 0, \quad \theta\left(1,t\right) = 1
\end{array}
\right\} \tag{3.13}
$$

Therefore the dimensionless equations become,

$$
\frac{\partial u}{\partial t} - v\frac{\partial u}{\partial y} = -\frac{dP}{dx} + \frac{1}{R_e}\frac{\partial^2 u}{\partial y^2} - \left(S + H_a^2\right)u + G_{r\theta}\theta \tag{3.14}
$$

$$
\frac{\partial \theta}{\partial t} - v\frac{\partial \theta}{\partial y} = \frac{1}{P_e}\frac{\partial^2 \theta}{\partial y^2} + \delta\theta + \frac{E_c}{R_e}\left(\frac{\partial u}{\partial y}\right)^2 \tag{3.15}
$$

with corresponding dimensionless initial and boundary conditions as.

$$
\left.
\begin{array}{l}
u\left(y,0\right) = 0, \quad \dfrac{du}{dy}\bigg|_{y=0} - \alpha\,u\big|_{y=0} = 0, \quad u\left(1,t\right) = 0 \\
\theta\left(y,0\right) = 0, \quad \theta\left(0,t\right) = 0, \quad \theta\left(1,t\right) = 1
\end{array}
\right\} \tag{3.16}
$$

where,

$\dfrac{dU}{\upsilon} = R_e$ = Reynolds number

$$\frac{hv}{Uk} = S$$ = Suction/Injection parameter

$$\frac{g\beta h\left(T_1 - T_0\right)}{U^2} = G_{r\theta}$$ = Thermal Grashof number

$$\frac{4\alpha^2 h}{\rho C_p U} = \delta$$ = Thermal radiation parameter

$$\frac{\rho C_p dU}{k} = P_e$$ = Peclet Energy number

$$\frac{U^2}{C_p\left(T_w{}' - T_d{}'\right)} = E_c$$ = Eckert number

$$\frac{\sigma_e B_0^2 h}{\rho U} = H_a^2$$ = Hartman number

v = Kinematic viscosity

α = Term due to thermal radiation

3.3 Method of solution

For a purely oscillating flow,

$$\left.-\frac{dP}{dx} = \lambda e^{i\omega t}, \quad u\left(y,t\right) = u\left(y\right)e^{i\omega t}, \quad \theta\left(y,t\right) = \theta\left(y\right)e^{2i\omega t}\right\} \tag{3.19}$$

So,

$$\left.\begin{aligned}&\frac{\partial u(y,t)}{\partial t}=i\omega e^{i\omega t}u(y),\quad \frac{\partial u(y,t)}{\partial y}=e^{i\omega t}\frac{du(y)}{dy},\quad \frac{\partial^2 u(y,t)}{\partial y^2}=e^{i\omega t}\frac{d^2u(y)}{dy^2},\\&\frac{\partial \theta(y,t)}{\partial t}=2i\omega e^{2i\omega t}\theta(y),\quad \frac{\partial \theta(y,t)}{\partial y}=e^{2i\omega t}\frac{d\theta(y)}{dy},\quad \frac{\partial^2 \theta(y,t)}{\partial y^2}=e^{2i\omega t}\frac{d^2\theta(y)}{dy^2},\\&\left(\frac{\partial u(y,t)}{\partial y}\right)^2=e^{2i\omega t}\left(\frac{du(y)}{dy}\right)^2\end{aligned}\right\}\tag{3.20}$$

Put equation (3.20) in equation (3.14) gives,

$$i\omega e^{i\omega t}u-ve^{i\omega t}\frac{du}{dy}=\lambda e^{i\omega t}+\frac{1}{R_e}e^{i\omega t}\frac{d^2u}{dy^2}-\left(S+H_a^2\right)e^{i\omega t}u+G_{r\theta}e^{2i\omega t}\theta \tag{3.21}$$

Multiply equation (3.21) by $\frac{R_e}{e^{i\omega t}}$ gives,

$$i\omega R_e u-vR_e\frac{du}{dy}=\lambda R_e+\frac{d^2u}{dy^2}-\left(S+H_a^2\right)R_e u+G_{r\theta}e^{i\omega t}R_e\theta \tag{3.22}$$

$$\frac{d^2u}{dy^2}+vR_e\frac{du}{dy}-\left(S+H_a^2+i\omega\right)R_e u=-\lambda R_e-G_{r\theta}e^{i\omega t}R_e\theta \tag{3.23}$$

That is,

$$\frac{d^2u}{dy^2}+b_1\frac{du}{dy}-c_1^2u=-\lambda R_e-G_{r\theta}e^{i\omega t}R_e\theta \tag{3.24}$$

where

$$\left.\begin{aligned}&b_1=vR_e\\&c_1=\sqrt{\left(S+H_a^2+i\omega\right)R_e}\end{aligned}\right\}\tag{3.25}$$

Put equation (3.20) in equation (3.15) gives,

$$2i\omega e^{2i\omega t}\theta - ve^{2i\omega t}\frac{d\theta}{dy} = \frac{1}{P_e}e^{2i\omega t}\frac{d^2\theta}{dy^2} + \delta e^{2i\omega t}\theta + \frac{E_c}{R_e}e^{2i\omega t}\left(\frac{du}{dy}\right)^2 \tag{3.26}$$

Multiply equation (3.26) by $\frac{P_e}{e^{2i\omega t}}$ gives,

$$2i\omega P_e\theta - vP_e\frac{d\theta}{dy} = \frac{d^2\theta}{dy^2} + \delta P_e\theta + \frac{E_cP_e}{R_e}\left(\frac{du}{dy}\right)^2 \tag{3.27}$$

$$\frac{d^2\theta}{dy^2} + vP_e\frac{d\theta}{dy} + (\delta - 2i\omega)P_e\theta = -\frac{E_cP_e}{R_e}\left(\frac{du}{dy}\right)^2 \tag{3.28}$$

That is,

$$\frac{d^2\theta}{dy^2} + b_2\frac{d\theta}{dy} + c_2^2\theta = -\frac{E_cP_e}{R_e}\left(\frac{du}{dy}\right)^2 \tag{3.29}$$

where,

$$\left.\begin{aligned} b_2 &= vP_e, \\ c_2 &= \sqrt{(\delta - 2i\omega)P_e} \end{aligned}\right\} \tag{3.30}$$

The corresponding boundary conditions are

$$\left.\begin{aligned}
&e^{i\omega t}\left.\frac{du}{dy}\right|_{y=0}-\alpha e^{i\omega t}\left.u\right|_{y=0}=0, \quad \Rightarrow \left.\frac{du}{dy}\right|_{y=0}-\alpha\left.u\right|_{y=0}=0\\
&u(1)=\frac{0}{e^{i\omega t}}=0,\\
&\left.\frac{du}{dy}\right|_{y=0}-\alpha\left.u\right|_{y=0}=0, \qquad u(1)=0\\
&\theta(y)=\frac{\theta(y,t)}{e^{2i\omega t}}, \quad \theta(0)=\frac{0}{e^{2i\omega t}}=0, \quad \theta(1)=\frac{1}{e^{2i\omega t}}=e^{-2i\omega t}\\
&\theta(0)=0, \quad \theta(1)=e^{-2i\omega t}
\end{aligned}\right\} \tag{3.31}$$

That is,

$$\left.\begin{aligned}
&\frac{d^2u}{dy^2}+b_1\frac{du}{dy}-c_1^2u=-\lambda R_e-G_{r\theta}e^{i\omega t}R_e\theta\\
&\left.\frac{du}{dy}\right|_{y=0}-\alpha\left.u\right|_{y=0}=0, \qquad u(1)=0
\end{aligned}\right\} \tag{3.32}$$

and,

$$\left.\begin{aligned}
&\frac{d^2\theta}{dy^2}+b_2\frac{d\theta}{dy}-c_2^2\theta=-\frac{E_cP_e}{R_e}\left(\frac{du}{dy}\right)^2\\
&\theta(0)=0, \quad \theta(1)=e^{-2i\omega t}
\end{aligned}\right\} \tag{3.33}$$

Let

$0<G_{r\theta}<<1$ such that,

$$\left.\begin{aligned}
&u(y)=u_0(y)+G_{r\theta}u_1(y)+\ldots\\
&\theta(y)=\theta_0(y)+G_{r\theta}\theta_1(y)+\ldots
\end{aligned}\right\} \tag{3.34}$$

Put equation (3.34) in equations (3.32) and (3.33) gives,

$$\left.\begin{aligned}&\frac{d^2u_0}{dy^2}+G_{r\theta}\frac{d^2u_1}{dy^2}+\ldots+b_1\left(\frac{du_0}{dy}+G_{r\theta}\frac{du_1}{dy}+\ldots\right)-c_1^2\left(u_0+G_{r\theta}u_1+\ldots\right)=-\lambda R_e-\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad G_{r\theta}e^{i\omega t}R_e\left(\theta_0+G_{r\theta}\theta_1+\ldots\right)\\&\left.\frac{du_0}{dy}\right|_{y=0}+G_{r\theta}\left.\frac{du_1}{dy}\right|_{y=0}+\ldots-\alpha G_{r\theta}\left.\left(u_0+G_{r\theta}u_1+\ldots\right)\right|_{y=0}=0\end{aligned}\right\}\tag{3.35}$$

$$\left.\begin{aligned}&\frac{d^2\theta_0}{dy^2}+G_{r\theta}\frac{d^2\theta_1}{dy^2}+\ldots+b_2\left(\frac{d\theta_0}{dy}+G_{r\theta}\frac{d\theta_1}{dy}+\ldots\right)+c_2^2\left(\theta_0+G_{r\theta}\theta_1+\ldots\right)=\\&\qquad\qquad\qquad\qquad\qquad\qquad-\frac{E_cP_e}{R_e}\left(\frac{du_0}{dy}+G_{r\theta}\frac{du_1}{dy}+\ldots\right)^2\end{aligned}\right\}\tag{3.36}$$

For Order 0, that is $O\left(G_{r\theta}^0\right):1$ gives,

$$\left.\begin{aligned}&\frac{d^2u_0}{dy^2}+b_1\frac{du_0}{dy}-c_1^2u_0=-\lambda R_e\\&\left.\frac{du}{dy}\right|_{y=0}-\alpha\left.u\right|_{y=0}=0,\quad u_0(1)=0\end{aligned}\right\}\tag{3.37}$$

$$\left.\begin{aligned}&\frac{d^2\theta_0}{dy^2}+b_2\frac{d\theta_0}{dy}+c_2^2\theta_0=-\frac{E_cP_e}{R_e}\left(\frac{du_0}{dy}\right)^2\\&\theta_0(0)=0,\quad \theta_0(1)=e^{-2i\omega t}\end{aligned}\right\}\tag{3.38}$$

also for Order 1, That is $O\left(G_{r\theta}^1\right):G_{r\theta}$ gives,

$$\left.\begin{aligned}&\frac{d^2u_1}{dy^2}+b_1\frac{du_1}{dy}-c_1^2u_1=-e^{i\omega t}R_e\theta_0\\&\left.\frac{du_1}{dy}\right|_{y=0}-\alpha\left.u_0\right|_{y=0}=0,\quad u_1(1)=0\end{aligned}\right\}\tag{3.39}$$

$$\left.\begin{aligned}&\frac{d^2\theta_1}{dy^2}+b_2\frac{d\theta_1}{dy}+c_2^2\theta_1=-\frac{2E_cP_e}{R_e}\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\\&\theta_1(0)=0,\quad \theta_1(1)=0\end{aligned}\right\} \tag{3.40}$$

Solving equation (3.37)

$$\left.\begin{aligned}&\frac{d^2u_0}{dy^2}+b_1\frac{du_0}{dy}-c_1^2u_0=-\lambda R_e\\&\left.\frac{du_0}{dy}\right|_{y=0}-\alpha\, u_0\big|_{y=0}=0,\quad u_0(1)=0\end{aligned}\right\} \tag{3.41}$$

Solving the homogeneous part of equation (3.41), that is

$$\frac{d^2u_0}{dy^2}+b_1\frac{du_0}{dy}-c_1^2u_0=0 \tag{3.42}$$

seek,

$$u_0=e^{my} \tag{3.43}$$

Differentiating equation (3.43) gives,

$$u_0'=me^{my} \tag{3.44}$$

Differentiating equation (3.44) gives,

$$u_0''=m^2e^{my} \tag{3.45}$$

Put equations (3.45), (3.44) and (3.43) into equation (3.42)

$$m^2e^{my}+bme^{my}-c_1^2e^{my}=0 \tag{3.46}$$

$$\left(m^2 + bm - c_1^2\right)e^{my} = 0 \tag{3.47}$$

$$m^2 + bm - c_1^2 = 0 \text{ and } e^{my} \neq 0$$

Using formular method, we have

$$m_{1,2} = \frac{-b_1 \pm \sqrt{b_1^2 + 4c_1^2}}{2} \tag{3.48}$$

$$m_1 = \frac{-b_1 + \sqrt{b_1^2 + 4c_1^2}}{2} \tag{3.49}$$

$$m_2 = \frac{-b_1 - \sqrt{b_1^2 + 4c_1^2}}{2} \tag{3.50}$$

That is,

$$u_{0c} = A_1 e^{m_1 y} + A_2 e^{m_2 y} \tag{3.51}$$

Assume a particular solution of the form,

$$u_{0p} = A_3 \tag{3.52}$$

$$u'_{0p} = 0 \tag{3.53}$$

$$u'''_{0p} = 0 \tag{3.54}$$

Put equations (3.54), (3.53) and (3.52) into equation (3.41) gives,

$$-c_1^2 A_3 = -\lambda R_e \tag{3.55}$$

$$A_3 = \frac{\lambda R_e}{c_1^2} \tag{3.56}$$

and,

$$u_0(y) = u_{0c}(y) + u_{0p}(y) \tag{3.57}$$

That is,

$$u_0(y) = A_1 e^{m_1 y} + A_2 e^{m_2 y} + A_3 \tag{3.58}$$

Applying the boundary condition,

$$\left.\frac{du_0}{dy}\right|_{y=0} - \alpha \left.u_0\right|_{y=0} = 0 \text{ implies,}$$

$$A_1 m_1 + A_2 m_2 - \alpha A_1 - \alpha A_2 - \alpha A_3 = 0 \tag{3.59}$$

$$A_1(m_1 - \alpha) + A_2(m_2 - \alpha) - \alpha A_3 = 0 \tag{3.60}$$

$$A_1 = \frac{\alpha A_3 - A_2(m_2 - \alpha)}{(m_1 - \alpha)} \tag{3.61}$$

Using the boundary condition,

$u_0(1) = 0$ implies,

$$e^{m_1}\left(\frac{\alpha A_3 - A_2(m_2 - \alpha)}{(m_1 - \alpha)}\right) + A_2 e^{m_2} + A_3 = 0 \tag{3.62}$$

$$\alpha A_3 e^{m_1} - A_2 e^{m_1}(m_2 - \alpha) + (m_2 - \alpha) A_2 e^{m_2} + (m_2 - \alpha) A_3 = 0 \tag{3.63}$$

$$\left(\left(m_2-\alpha\right)e^{m_2}-e^{m_1}\left(m_2-\alpha\right)\right)A_2=-\left(\alpha e^{m_1}+\left(m_2-\alpha\right)\right)A_3 \tag{3.64}$$

$$A_2=\frac{-\left(\alpha e^{m_1}+\left(m_2-\alpha\right)\right)A_3}{\left(\left(m_2-\alpha\right)e^{m_2}-e^{m_1}\left(m_2-\alpha\right)\right)} \tag{3.65}$$

So, the solution to equation (3.37) is,

$$u_0\left(y\right)=A_1e^{m_1y}+A_2e^{m_2y}+A_3 \tag{3.66}$$

Where,

$$\left.\begin{aligned}
A_1&=\frac{\alpha A_3-A_2\left(m_2-\alpha\right)}{\left(m_1-\alpha\right)}\\
A_2&=\frac{-\left(\alpha e^{m_1}+\left(m_2-\alpha\right)\right)A_3}{\left(\left(m_2-\alpha\right)e^{m_2}-e^{m_1}\left(m_2-\alpha\right)\right)}\\
A_3&=\frac{\lambda R_e}{c_1^2}
\end{aligned}\right\} \tag{3.67}$$

Solving equation (3.39) that is,

$$\left.\begin{aligned}
&\frac{d^2\theta_0}{dy^2}+b_2\frac{d\theta_0}{dy}+c_2^2\theta_0=-\frac{E_cP_e}{R_e}\left(\frac{du_0}{dy}\right)^2\\
&\theta_0\left(0\right)=0,\quad \theta_0\left(1\right)=e^{-2i\omega t}
\end{aligned}\right\} \tag{3.68}$$

Differentiating equation (3.66) gives,

$$\frac{du_0}{dy}=A_1m_1e^{m_1y}+A_2m_2e^{m_2y} \tag{3.69}$$

$$\left(\frac{du_0}{dy}\right)^2=\left(A_1m_1e^{m_1y}+A_2m_2e^{m_2y}\right)^2 \tag{3.70}$$

$$\left(\frac{du_0}{dy}\right)^2 = A_1^2 m_1^2 e^{2m_1 y} + 2A_1 A_2 m_1 m_2 e^{(m_1+m_2)y} + A_2^2 m_2^2 e^{2m_2 y} \tag{3.71}$$

Put equation (3.71) into equation (3.68) gives,

$$\frac{d^2\theta_0}{dy^2} + b_2\frac{d\theta_0}{dy} + c_2^2\theta_0 = -\frac{E_c P_e}{R_e}\left(A_1^2 m_1^2 e^{2m_1 y} + 2A_1 A_2 m_1 m_2 e^{(m_1+m_2)y} + A_2^2 m_2^2 e^{2m_2 y}\right) \tag{3.72}$$

Solving the homogeneous part,

$$\frac{d^2\theta_0}{dy^2} + b_2\frac{d\theta_0}{dy} + c_2^2\theta_0 = 0 \tag{3.73}$$

seek,

$$\theta_0(y) = e^{my} \tag{3.74}$$

Differentiating equation (3.74) gives,

$$\theta_0'(y) = me^{my} \tag{3.75}$$

Differentiating equation (3.75) gives,

$$\theta_0''(y) = m^2 e^{my} \tag{3.76}$$

Put equations (3.76), (3.75) and (3.74) into equation (3.73) gives,

$$m^2 e^{my} + b_2 m e^{my} + c_2^2 e^{my} = 0$$

$$\left(m^2 + b_2 m + c_2^2\right)e^{my} = 0 \tag{3.77}$$

$$m^2 + b_2 m + c_2^2 = 0 \text{ and } e^{my} \neq 0 \tag{3.78}$$

Applying formular method gives,

$$m_{3,4} = \frac{-b_2 \pm \sqrt{b_2^2 - 4c_2^2}}{2} \tag{3.79}$$

$$m_3 = \frac{-b_2 + \sqrt{b_2^2 - 4c_2^2}}{2} \tag{3.80}$$

$$m_4 = \frac{-b_2 - \sqrt{b_2^2 - 4c_2^2}}{2} \tag{3.81}$$

That is,

$$\theta_{0c}\left(y\right) = A_4 e^{m_3 y} + A_5 e^{m_4 y} \tag{3.82}$$

We assume a particular solution of,

$$\theta_{0p}\left(y\right) = A_6 e^{2m_1 y} + A_7 e^{(m_1+m_2)y} + A_8 e^{2m_2 y} \tag{3.83}$$

Differentiating equation (3.83) gives,

$$\theta'_{0p}\left(y\right) = 2A_6 m_1 e^{2m_1 y} + A_7\left(m_1 + m_2\right) e^{(m_1+m_2)y} + 2A_8 m_2 e^{2m_2 y} \tag{3.84}$$

Differentiating equation (3.84) equation,

$$\theta''_{0p}\left(y\right) = 4A_6 m_1^2 e^{2m_1 y} + A_7\left(m_1 + m_2\right)^2 e^{(m_1+m_2)y} + 4A_8 m_2^2 e^{2m_2 y} \tag{3.85}$$

Put equations (3.85), (3.84) and (3.83) into equation (3.72) gives,

$$4A_6m_1^2e^{2m_1y}+A_7\left(m_1+m_2\right)^2e^{(m_1+m_2)y}+4A_8m_2^2e^{2m_2y}+b_2\begin{pmatrix}2A_6m_1e^{2m_1y}+A_7\left(m_1+m_2\right)e^{(m_1+m_2)y}+\\2A_8m_2e^{2m_2y}\end{pmatrix}$$
$$+c_2^2\left(A_6e^{2m_1y}+A_7e^{(m_1+m_2)y}+A_8e^{2m_2y}\right)=-\frac{E_cP_e}{R_e}\left(A_1^2m_1^2e^{2m_1y}+2A_1A_2m_1m_2e^{(m_1+m_2)y}+A_2^2m_2^2e^{2m_2y}\right) \quad (3.86)$$

Comparing the variables,

$$4A_6m_1^2+2A_6b_2m_1+c_2^2A_6=-\frac{E_cP_eA_1^2m_1^2}{R_e} \quad (3.87)$$

$$\Rightarrow A_6=-\frac{E_cP_eA_1^2m_1^2}{R_e\left(4m_1^2+2b_2m_1+c_2^2\right)} \quad (3.88)$$

$$A_7\left(m_1+m_2\right)^2+b_2A_7\left(m_1+m_2\right)+c_2^2A_7=-\frac{2E_cP_eA_1A_2m_1m_2}{R_e} \quad (3.89)$$

$$\Rightarrow A_7=\frac{-2E_cP_eA_1A_2m_1m_2}{R_e\left(\left(m_1+m_2\right)^2+b_2\left(m_1+m_2\right)+c_2^2\right)} \quad (3.90)$$

$$4A_8m_2^2+2b_2A_8m_2+c_2^2A_8=-\frac{E_cP_eA_2^2m_2^2}{R_e} \quad (3.91)$$

$$\Rightarrow A_8=\frac{-E_cP_eA_2^2m_2^2}{R_e\left(4m_2^2+2b_2m_2+c_2^2\right)} \quad (3.92)$$

$$\theta_0\left(y\right)=\theta_{0c}\left(y\right)+\theta_{0p}\left(y\right) \quad (3.93)$$

That is,

$$\theta_0\left(y\right)=A_4e^{m_3y}+A_5e^{m_4y}+A_6e^{2m_1y}+A_7e^{(m_1+m_2)y}+A_8e^{2m_2y} \quad (3.94)$$

Applying the boundary condition,

$\theta_0(0)=0$ implies,

$$A_4+A_5+A_6+A_7+A_8=0 \tag{3.95}$$

$$A_4=-(A_6+A_7+A_8)-A_5 \tag{3.96}$$

$$A_4=-A_9-A_5 \tag{3.97}$$

where,

$$A_9=A_6+A_7+A_8 \tag{3.98}$$

Also,

$\theta_0(1)=e^{-2i\omega t}$ implies,

$$A_4e^{m_3}+A_5e^{m_4}+A_6e^{2m_1}+A_7e^{(m_1+m_2)}+A_8e^{2m_2}=e^{-2i\omega t} \tag{3.99}$$

Put equation (3.97) into equation (3.99)

$$-A_9e^{m_3}-A_5e^{m_3}+A_5e^{m_4}+A_6e^{2m_1}+A_7e^{(m_1+m_2)}+A_8e^{2m_2}=e^{-2i\omega t} \tag{3.100}$$

$$A_5=\frac{e^{-2i\omega t}+A_9e^{m_3}-A_6e^{2m_1}-A_7e^{(m_1+m_2)}-A_8e^{2m_2}}{e^{m_4}-e^{m_3}} \tag{3.101}$$

and

$$A_4=-A_9-A_5 \tag{3.102}$$

$$A_4=-A_9-\frac{e^{-2i\omega t}+A_9e^{m_3}-A_6e^{2m_1}-A_7e^{(m_1+m_2)}-A_8e^{2m_2}}{e^{m_4}-e^{m_3}} \tag{3.103}$$

$$A_4 = \frac{-A_9\left(e^{m_4}-e^{m_3}\right)-A_9e^{m_3}+A_6e^{2m_1}+A_7e^{(m_1+m_2)}+A_8e^{2m_2}-e^{-2i\omega t}}{e^{m_4}-e^{m_3}} \tag{3.104}$$

$$A_4 = \frac{-A_9e^{m_4}+A_6e^{2m_1}+A_7e^{(m_1+m_2)}+A_8e^{2m_2}-e^{-2i\omega t}}{e^{m_4}-e^{m_3}} \tag{3.105}$$

Therefore,

$$\theta_0\left(y\right) = A_4e^{m_3y}+A_5e^{m_4y}+A_6e^{2m_1y}+A_7e^{(m_1+m_2)y}+A_8e^{2m_2y} \tag{3.106}$$

where,

$$\left.\begin{aligned}
A_4 &= \frac{-A_9e^{m_4}+A_6e^{2m_1}+A_7e^{(m_1+m_2)}+A_8e^{2m_2}-e^{-2i\omega t}}{e^{m_4}-e^{m_3}}\\
A_5 &= \frac{e^{-2i\omega t}+A_9e^{m_3}-A_6e^{2m_1}-A_7e^{(m_1+m_2)}-A_8e^{2m_2}}{e^{m_4}-e^{m_3}}\\
A_6 &= -\frac{E_cP_eA_1^2m_1^2}{R_e\left(4m_1^2+2b_2m_1+c_2^2\right)}\\
A_7 &= \frac{-2E_cP_eA_1A_2m_1m_2}{R_e\left(\left(m_1+m_2\right)^2+b_2\left(m_1+m_2\right)+c_2^2\right)}\\
A_8 &= \frac{-E_cP_eA_2^2m_2^2}{R_e\left(4m_2^2+2b_2m_2+c_2^2\right)}
\end{aligned}\right\} \tag{3.107}$$

Solving equation (3.8)

$$\left.\begin{aligned}
&\frac{d^2u_1}{dy^2}+b_1\frac{du_1}{dy}-c_1^2u_1 = -e^{i\omega t}R_e\theta_0\\
&\left.\frac{du_1}{dy}\right|_{y=0}-\alpha\left.u_1\right|_{y=0} = 0, \quad u_1\left(1\right)=0
\end{aligned}\right\} \tag{3.108}$$

Put equation (3.106) into equation (3.108) gives,

$$\frac{d^2u_1}{dy^2}+b_1\frac{du_1}{dy}-c_1^2u_1=-e^{i\omega t}R_e\left(A_4e^{m_3y}+A_5e^{m_4y}+A_6e^{2m_1y}+A_7e^{(m_1+m_2)y}+A_8e^{2m_2y}\right) \tag{3.109}$$

Solving the homogeneous part that is,

$$\frac{d^2u_1}{dy^2}+b_1\frac{du_1}{dy}-c_1^2u_1=0 \tag{3.110}$$

seek,

$$u_1(y)=e^{my} \tag{3.111}$$

Differentiating equation (3.111) gives,

$$u_1'(y)=me^{my} \tag{3.112}$$

Differentiating equation (3.112) gives,

$$u_1''(y)=m^2e^{my} \tag{3.113}$$

Put equations (3.113), (3.112) and (3.111) into equation (3.110) gives,

$$m^2e^{my}+b_1me^{my}-c_1^2e^{my}=0 \tag{3.114}$$

$$\left(m^2+b_1m-c_1^2\right)e^{my}=0 \tag{3.115}$$

$$e^{my}\neq 0 \tag{3.116}$$

$$m^2+b_1m-c_1^2=0 \tag{3.117}$$

Using formular method gives,

$$m_{1,2} = \frac{-b_1 \pm \sqrt{b_1^2 + 4c_1^2}}{2} \tag{3.118}$$

$$m_1 = \frac{-b_1 + \sqrt{b_1^2 + 4c_1^2}}{2} \tag{3.119}$$

$$m_2 = \frac{-b_1 - \sqrt{b_1^2 + 4c_1^2}}{2} \tag{3.120}$$

The complimentary solution becomes,

$$u_{1c}(y) = A_{10}e^{m_1 y} + A_{11}e^{m_2 y} \tag{3.121}$$

Assume a particular solution of,

$$u_{1p}(y) = A_{12}e^{m_1 y} + A_{13}e^{m_2 y} + A_{14}e^{2m_1 y} + A_{15}e^{(m_1+m_2)y} + A_{16}e^{2m_2 y} \tag{3.122}$$

Differentiating equation (3.122) gives

$$u'_{1p}(y) = A_{12}m_1e^{m_1 y} + A_{13}m_2e^{m_2 y} + 2A_{14}m_1e^{2m_1 y} + A_{15}(m_1 + m_2)e^{(m_1+m_2)y} + 2A_{16}m_2e^{2m_2 y} \tag{3.123}$$

Differentiating equation (3.123) gives

$$u''_{1p}(y) = A_{12}m_1^2e^{m_1 y} + A_{13}m_2^2e^{m_2 y} + 4A_{14}m_1^2e^{2m_1 y} + A_{15}(m_1 + m_2)^2 e^{(m_1+m_2)y} + 4A_{16}m_2^2e^{2m_2 y} \tag{3.124}$$

Put equations (3.124), (3.123) and (3.121) into equation (3.109) gives,

$$\left.\begin{array}{l} A_{12}m_1^2e^{m_1 y}+A_{13}m_2^2e^{m_2 y}+4A_{14}m_1^2e^{2m_1 y}+A_{15}\left(m_1+m_2\right)^2e^{(m_1+m_2)y}+4A_{16}m_2^2e^{2m_2 y}+ \\ b_1\left(A_{12}m_1e^{m_1 y}+A_{13}m_2e^{m_2 y}+2A_{14}m_1e^{2m_1 y}+A_{15}\left(m_1+m_2\right)e^{(m_1+m_2)y}+2A_{16}m_2e^{2m_2 y}\right)- \\ c_1^2\left(A_{12}e^{m_1 y}+A_{13}e^{m_2 y}+A_{14}e^{2m_1 y}+A_{15}e^{(m_1+m_2)y}+A_{16}e^{2m_2 y}\right)=-e^{i\omega t}R_e\begin{pmatrix} A_4e^{m_3 y}+A_5e^{m_4 y}+ \\ A_6e^{2m_1 y}+A_7e^{(m_1+m_2)y} \\ +A_8e^{2m_2 y} \end{pmatrix} \end{array}\right\} \quad (3.125)$$

Comparing the variables gives,

$$A_{12}m_1^2+b_1A_{12}m_1-c_1^2A_{12}=-e^{i\omega t}R_eA_4 \quad (3.126)$$

$$\Rightarrow A_{12}=\frac{-e^{i\omega t}R_eA_4}{m_1^2+b_1m_1-c_1^2} \quad (3.127)$$

$$A_{13}m_2^2+b_1A_{13}m_2-c_1^2A_{13}=-e^{i\omega t}R_eA_5 \quad (3.128)$$

$$\Rightarrow A_{13}=\frac{-e^{i\omega t}R_eA_5}{m_2^2+b_1m_2-c_1^2} \quad (3.129)$$

$$4A_{14}m_1^2+2b_1A_{14}m_1-c_1^2A_{14}=-e^{i\omega t}R_eA_6 \quad (3.130)$$

$$\Rightarrow A_{14}=\frac{-e^{i\omega t}R_eA_6}{4m_1^2+2b_1m_1-c_1^2} \quad (3.131)$$

$$A_{15}\left(m_1+m_2\right)^2+b_1A_{15}\left(m_1+m_2\right)-c_1^2A_{15}=-e^{i\omega t}R_eA_7 \quad (3.132)$$

$$\Rightarrow A_{15}=\frac{-e^{i\omega t}R_eA_7}{\left(m_1+m_2\right)^2+b_1\left(m_1+m_2\right)-c_1^2} \quad (3.133)$$

$$4A_{16}m_2^2+2b_1A_{16}m_2-c_1^2A_{16}=-e^{i\omega t}R_eA_8 \quad (3.134)$$

$$\Rightarrow A_{16} = \frac{-e^{i\omega t} R_e A_8}{4m_2^2 + 2b_1 m_2 - c_1^2} \tag{3.135}$$

and

$$u_1(y) = u_{1c}(y) + u_{1p}(y) \tag{3.136}$$

That is,

$$u_1(y) = A_{10}e^{m_1 y} + A_{11}e^{m_2 y} + A_{12}e^{m_1 y} + A_{13}e^{m_2 y} + A_{14}e^{2m_1 y} + A_{15}e^{(m_1+m_2)y} + A_{16}e^{2m_2 y} \tag{3.137}$$

Applying the boundary condition

$$\left.\frac{du_1}{dy}\right|_{y=0} - \alpha \left. u_1\right|_{y=0} = 0,$$

Differentiating equation (3.137) gives,

$$\frac{du_1}{dy} = A_{10}m_1e^{m_1 y} + A_{11}m_2e^{m_2 y} + A_{12}m_1e^{m_1 y} + A_{13}m_2e^{m_2 y} + 2A_{14}m_1e^{2m_1 y} + A_{15}(m_1+m_2)e^{(m_1+m_2)y} + 2A_{16}m_2e^{2m_2 y} \tag{3.138}$$

This implies,

$$\left.\begin{array}{l} A_{10}m_1 + A_{11}m_2 + A_{12}m_1 + A_{13}m_2 + 2A_{14}m_1 + A_{15}(m_1+m_2) + 2A_{16}m_2 - \\ \alpha(A_{10} + A_{11} + A_{12} + A_{13} + A_{14} + A_{15} + A_{16}) = 0 \end{array}\right\} \tag{3.139}$$

$$\left.\begin{array}{l} (m_1-\alpha)A_{10} + (m_2-\alpha)A_{11} + (m_1-\alpha)A_{12} + (m_2-\alpha)A_{13} + (2m_1-\alpha)A_{14} + \\ (m_1+m_2-\alpha)A_{15} + (2m_2-\alpha)A_{16} = 0 \end{array}\right\} \tag{3.140}$$

$$(m_1-\alpha)A_{10} + (m_2-\alpha)A_{11} + B_4 = 0 \tag{3.141}$$

Where,

$$B_4 = (m_1 - \alpha) A_{12} + (m_2 - \alpha) A_{13} + (2m_1 - \alpha) A_{14} + (m_1 + m_2 - \alpha) A_{15} + (2m_2 - \alpha) A_{16} \quad (3.142)$$

So,

$$(m_1 - \alpha) A_{10} + (m_2 - \alpha) A_{11} = -B_4 \quad (3.143)$$

$$A_{10} = \frac{-\left(B_4 + (m_2 - \alpha) A_{11}\right)}{(m_1 - \alpha)} \quad (3.144)$$

Also,

$u_1(1) = 0$ implies,

$$\Rightarrow A_{10} e^{m_1} + A_{11} e^{m_2} + A_{12} e^{m_1} + A_{13} e^{m_2} + A_{14} e^{2m_1} + A_{15} e^{(m_1 + m_2)} + A_{16} e^{2m_2} = 0 \quad (3.145)$$

Put equation (3.144) into equation (3.145) gives,

$$\left(\frac{-\left(B_4 + (m_2 - \alpha) A_{11}\right)}{(m_1 - \alpha)}\right) e^{m_1} + A_{11} e^{m_2} + A_{12} e^{m_1} + A_{13} e^{m_2} + A_{14} e^{2m_1} + A_{15} e^{(m_1 + m_2)} + A_{16} e^{2m_2} = 0 \quad (3.146)$$

$$\frac{-B_4 e^{m_1}}{(m_1 - \alpha)} - \frac{(m_2 - \alpha) A_{11} e^{m_1}}{(m_1 - \alpha)} + A_{11} e^{m_2} + A_{12} e^{m_1} + A_{13} e^{m_2} + A_{14} e^{2m_1} + A_{15} e^{(m_1 + m_2)} + A_{16} e^{2m_2} = 0 \quad (3.147)$$

$$\frac{A_{11} (m_2 - \alpha) e^{m_2} - (m_2 - \alpha) A_{11} e^{m_1}}{(m_1 - \alpha)} = \frac{B_4 e^{m_1}}{(m_1 - \alpha)} - A_{12} e^{m_1} - A_{13} e^{m_2} - A_{14} e^{2m_1} - A_{15} e^{(m_1 + m_2)} - A_{16} e^{2m_2} \quad (3.148)$$

$$A_{11} = \frac{B_4 e^{m_1} - A_{12} (m_1 - \alpha) e^{m_1} - A_{13} (m_1 - \alpha) e^{m_2} - A_{14} (m_1 - \alpha) e^{2m_1} - A_{15} (m_1 - \alpha) e^{(m_1 + m_2)} - A_{16} (m_1 - \alpha) e^{2m_2}}{(m_2 - \alpha) e^{m_2} - (m_2 - \alpha) e^{m_1}} \quad (3.149)$$

$$A_{10}=\frac{-B_4}{(m_1-\alpha)}-\frac{(m_2-\alpha)}{(m_1-\alpha)}\left(\frac{\begin{array}{l}B_4e^{m_1}-A_{12}(m_1-\alpha)e^{m_1}-A_{13}(m_1-\alpha)e^{m_2}-A_{14}(m_1-\alpha)e^{2m_1}-\\A_{15}(m_1-\alpha)e^{(m_1+m_2)}-A_{16}(m_1-\alpha)e^{2m_2}\end{array}}{(m_2-\alpha)e^{m_2}-(m_2-\alpha)e^{m_1}}\right) \tag{3.150}$$

Therefore,

$$u_1(y)=A_{10}e^{m_1y}+A_{11}e^{m_2y}+A_{12}e^{m_1y}+A_{13}e^{m_2y}+A_{14}e^{2m_1y}+A_{15}e^{(m_1+m_2)y}+A_{16}e^{2m_2y} \tag{3.151}$$

where,

$$\left.\begin{array}{l}A_{10}=\dfrac{-B_4}{(m_1-\alpha)}-\dfrac{(m_2-\alpha)}{(m_1-\alpha)}\left(\dfrac{\begin{array}{l}B_4e^{m_1}-A_{12}(m_1-\alpha)e^{m_1}-A_{13}(m_1-\alpha)e^{m_2}-\\A_{14}(m_1-\alpha)e^{2m_1}-A_{15}(m_1-\alpha)e^{(m_1+m_2)}-\\A_{16}(m_1-\alpha)e^{2m_2}\end{array}}{(m_2-\alpha)e^{m_2}-(m_2-\alpha)e^{m_1}}\right)\\
A_{11}=\dfrac{\begin{array}{l}B_4e^{m_1}-A_{12}(m_1-\alpha)e^{m_1}-A_{13}(m_1-\alpha)e^{m_2}-A_{14}(m_1-\alpha)e^{2m_1}-\\A_{15}(m_1-\alpha)e^{(m_1+m_2)}-A_{16}(m_1-\alpha)e^{2m_2}\end{array}}{(m_2-\alpha)e^{m_2}-(m_2-\alpha)e^{m_1}}\\
A_{12}=\dfrac{-e^{i\omega t}R_eA_4}{m_1^2+b_1m_1-c_1^2}\\
A_{13}=\dfrac{-e^{i\omega t}R_eA_5}{m_2^2+b_1m_2-c_1^2}\\
A_{14}=\dfrac{-e^{i\omega t}R_eA_6}{4m_1^2+2b_1m_1-c_1^2}\\
A_{15}=\dfrac{-e^{i\omega t}R_eA_7}{(m_1+m_2)^2+b_1(m_1+m_2)-c_1^2}\\
A_{16}=\dfrac{-e^{i\omega t}R_eA_8}{4m_2^2+2b_1m_2-c_1^2}\end{array}\right\} \tag{3.152}$$

Solving equation (3.40) that is,

$$\left.\begin{aligned}&\frac{d^2\theta_1}{dy^2}+b_2\frac{d\theta_1}{dy}+c_2^2\theta_1=-\frac{2E_cP_e}{R_e}\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\\&\theta_1(0)=0,\quad \theta_1(1)=0\end{aligned}\right\}\tag{3.153}$$

Differentiating equation (3.58) gives

$$\frac{du_0}{dy}=A_1m_1e^{m_1y}+A_2m_2e^{m_2y}\tag{3.154}$$

Differentiating equation (3.1531) gives,

$$\frac{du_1}{dy}=A_{10}m_1e^{m_1y}+A_{11}m_2e^{m_2y}+A_{12}m_1e^{m_1y}+A_{13}m_2e^{m_2y}+2A_{14}m_1e^{2m_1y}+A_{15}\left(m_1+m_2\right)e^{(m_1+m_2)y}+2A_{16}m_2e^{2m_2y}\tag{3.155}$$

That is,

$$\left.\begin{aligned}\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)=&A_1A_{10}m_1^2e^{2m_1y}+A_1A_{11}m_1m_2e^{(m_1+m_2)y}+A_1A_{12}m_1^2e^{2m_1y}+A_1A_{13}m_1m_2e^{(m_1+m_2)y}+\\&2A_1A_{14}m_1^2e^{3m_1y}+A_1A_{15}m_1\left(m_1+m_2\right)e^{(2m_1+m_2)y}+2A_1A_{16}m_1m_2e^{(m_1+2m_2)y}+\\&A_2A_{10}m_1m_2e^{(m_1+m_2)y}+A_2A_{11}m_2^2e^{2m_2y}+A_2A_{12}m_1m_2e^{(m_1+m_2)y}+A_2A_{13}m_2^2e^{2m_2y}+\\&2A_2A_{14}m_1m_2e^{(2m_1+m_2)y}+A_2A_{15}m_2\left(m_1+m_2\right)e^{(m_1+2m_2)y}+2A_2A_{16}m_2^2e^{3m_2y}\end{aligned}\right\}\tag{3.156}$$

$$\left.\begin{aligned}\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)=&\left(A_1A_{10}+A_1A_{12}\right)m_1^2e^{2m_1y}+\left(A_1A_{11}+A_1A_{13}+A_2A_{10}+A_2A_{12}\right)m_1m_2e^{(m_1+m_2)y}+\\&2A_1A_{14}m_1^2e^{3m_1y}+\left(A_1A_{15}\left(m_1+m_2\right)+2A_2A_{14}m_2\right)m_1e^{(2m_1+m_2)y}+\\&\begin{pmatrix}2A_1A_{16}m_1+\\A_2A_{15}\left(m_1+m_2\right)\end{pmatrix}m_2e^{(m_1+2m_2)y}+\left(A_2A_{11}+A_2A_{13}\right)m_2^2e^{2m_2y}+2A_2A_{16}m_2^2e^{3m_2y}\end{aligned}\right\}\tag{3.157}$$

Put equation (3.157) into equation (3.153) gives,

$$\frac{d^2\theta_1}{dy^2}+b_2\frac{d\theta_1}{dy}+c_2^2\theta_1=-\frac{2E_cP_e}{R_e}\begin{pmatrix}\left(A_1A_{10}+A_1A_{12}\right)m_1^2e^{2m_1y}+\begin{pmatrix}A_1A_{11}+A_1A_{13}+\\A_2A_{10}+A_2A_{12}\end{pmatrix}m_1m_2e^{(m_1+m_2)y}+\\2A_1A_{14}m_1^2e^{3m_1y}+\left(A_1A_{15}\left(m_1+m_2\right)+2A_2A_{14}m_2\right)m_1e^{(2m_1+m_2)y}+\\\left(2A_1A_{16}m_1+A_2A_{15}\left(m_1+m_2\right)\right)m_2e^{(m_1+2m_2)y}+\\\left(A_2A_{11}+A_2A_{13}\right)m_2^2e^{2m_2y}+2A_2A_{16}m_2^2e^{3m_2y}\end{pmatrix} \quad (3.158)$$

Solving the homogeneous part, that is

$$\frac{d^2\theta_1}{dy^2}+b_2\frac{d\theta_1}{dy}+c_2^2\theta_1=0 \quad (3.159)$$

seek

$$\theta_1(y)=e^{my} \quad (3.160)$$

Differentiating equation (3.160) gives,

$$\theta_1'(y)=me^{my} \quad (3.161)$$

Differentiating equation (3.161) gives,

$$\theta_1''(y)=m^2e^{my} \quad (3.162)$$

Put equations (3.162), (3.161) and (3.160) into equation (3.159) gives,

$$m^2e^{my}+b_2me^{my}+c_2^2e^{my}=0 \quad (3.163)$$

$$\left(m^2+b_2m+c_2^2\right)e^{my}=0 \quad (3.164)$$

$$e^{my}\neq 0 \text{ and } m^2+b_2m+c_2^2=0 \quad (3.165)$$

Using the formular method we have,

$$m_{3,4} = \frac{-b_2 \pm \sqrt{b_2^2 - 4c_2^2}}{2} \tag{3.166}$$

$$m_3 = \frac{-b_2 + \sqrt{b_2^2 - 4c_2^2}}{2} \tag{3.167}$$

$$m_4 = \frac{-b_2 - \sqrt{b_2^2 - 4c_2^2}}{2} \tag{3.168}$$

That is,

$$\theta_{1c}\left(y\right) = A_{17}e^{m_3 y} + A_{18}e^{m_4 y} \tag{3.169}$$

Assume a particular solution of,

$$\theta_{1p}\left(y\right) = A_{19}e^{2m_1 y} + A_{20}e^{\left(m_1+m_2\right)y} + A_{21}e^{3m_1 y} + A_{22}e^{\left(2m_1+m_2\right)y} + A_{23}e^{\left(m_1+2m_2\right)y} + A_{24}e^{2m_2 y} + A_{25}e^{3m_2 y} \tag{3.170}$$

Differentiating equation (3.170) gives,

$$\left.\begin{aligned} \frac{d\theta_{1p}\left(y\right)}{dy} = {} & 2A_{19}m_1e^{2m_1 y} + A_{20}\left(m_1+m_2\right)e^{\left(m_1+m_2\right)y} + 3A_{21}m_1e^{3m_1 y} + A_{22}\left(2m_1+m_2\right)e^{\left(2m_1+m_2\right)y} + \\ & A_{23}\left(m_1+2m_2\right)e^{\left(m_1+2m_2\right)y} + 2A_{24}m_2e^{2m_2 y} + 3A_{25}m_2e^{3m_2 y} \end{aligned}\right\} \tag{3.171}$$

Differentiating equation (3.171) gives,

$$\left.\begin{aligned} \frac{d^2\theta_{1p}\left(y\right)}{dy^2} = {} & 4A_{19}m_1^2e^{2m_1 y} + A_{20}\left(m_1+m_2\right)^2e^{\left(m_1+m_2\right)y} + 9A_{21}m_1^2e^{3m_1 y} + A_{22}\left(2m_1+m_2\right)^2e^{\left(2m_1+m_2\right)y} \\ & + A_{23}\left(m_1+2m_2\right)^2e^{\left(m_1+2m_2\right)y} + 4A_{24}m_2^2e^{2m_2 y} + 9A_{25}m_2^2e^{3m_2 y} \end{aligned}\right\} \tag{3.172}$$

Put equations (3.172), (3.171) and (3.170) into equation (3.158) gives,

$$
\left.\begin{array}{l}
4A_{19}m_1^2e^{2m_1y}+A_{20}\left(m_1+m_2\right)^2e^{\left(m_1+m_2\right)y}+9A_{21}m_1^2e^{3m_1y}+A_{22}\left(2m_1+m_2\right)^2e^{\left(2m_1+m_2\right)y}+\\
A_{23}\left(m_1+2m_2\right)^2e^{\left(m_1+2m_2\right)y}+4A_{24}m_2^2e^{2m_2y}+9A_{25}m_2^2e^{3m_2y}+\\
b_2\begin{pmatrix}2A_{19}m_1e^{2m_1y}+A_{20}\left(m_1+m_2\right)e^{\left(m_1+m_2\right)y}\\
+3A_{21}m_1e^{3m_1y}+A_{22}\left(2m_1+m_2\right)e^{\left(2m_1+m_2\right)y}\\
+A_{23}\left(m_1+2m_2\right)e^{\left(m_1+2m_2\right)y}+2A_{24}m_2e^{2m_2y}\\
+3A_{25}m_2e^{3m_2y}\end{pmatrix}+\\
c_2^2\left(A_{19}e^{2m_1y}+A_{20}e^{\left(m_1+m_2\right)y}+A_{21}e^{3m_1y}+A_{22}e^{\left(2m_1+m_2\right)y}+A_{23}e^{\left(m_1+2m_2\right)y}+A_{24}e^{2m_2y}+A_{25}e^{3m_2y}\right)=\\
-\dfrac{2E_cP_e}{R_e}\begin{pmatrix}\left(A_1A_{10}+A_1A_{12}\right)m_1^2e^{2m_1y}+\begin{pmatrix}A_1A_{11}+A_1A_{13}+\\A_2A_{10}+A_2A_{12}\end{pmatrix}m_1m_2e^{\left(m_1+m_2\right)y}+\\
2A_1A_{14}m_1^2e^{3m_1y}+\left(A_1A_{15}\left(m_1+m_2\right)+2A_2A_{14}m_2\right)m_1e^{\left(2m_1+m_2\right)y}+\\
\left(2A_1A_{16}m_1+A_2A_{15}\left(m_1+m_2\right)\right)m_2e^{\left(m_1+2m_2\right)y}+\\
\left(A_2A_{11}+A_2A_{13}\right)m_2^2e^{2m_2y}+2A_2A_{16}m_2^2e^{3m_2y}\end{pmatrix}
\end{array}\right\} \tag{3.173}
$$

Comparing the variables,

$$
4A_{19}m_1^2+2b_2A_{19}m_1+c_2^2A_{19}=-\frac{2E_cP_e\left(A_1A_{10}+A_1A_{12}\right)m_1^2}{R_e} \tag{3.174}
$$

$$
\Rightarrow A_{19}=-\frac{2E_cP_e\left(A_1A_{10}+A_1A_{12}\right)m_1^2}{R_e\left(4m_1^2+2b_2m_1+c_2^2\right)}
$$

$$
A_{20}\left(m_1+m_2\right)^2+b_2A_{20}\left(m_1+m_2\right)+c_2^2A_{20}=-\frac{2E_cP_e\left(A_1A_{11}+A_1A_{13}+A_2A_{10}+A_2A_{12}\right)m_1m_2}{R_e} \tag{3.175}
$$

$$
\Rightarrow A_{20}=-\frac{2E_cP_e\left(A_1A_{11}+A_1A_{13}+A_2A_{10}+A_2A_{12}\right)m_1m_2}{R_e\left(\left(m_1+m_2\right)^2+b_2\left(m_1+m_2\right)+c_2^2\right)} \tag{3.176}
$$

$$9A_{21}m_1^2 + 3b_2A_{21}m_1 + c_2^2A_{21} = -\frac{4E_cP_eA_1A_{14}m_1^2}{R_e} \tag{3.177}$$

$$\Rightarrow A_{21} = -\frac{4E_cP_eA_1A_{14}m_1^2}{R_e\left(9m_1^2 + 3b_2m_1 + c_2^2\right)}$$

$$A_{22}\left(2m_1 + m_2\right)^2 + b_2A_{22}\left(2m_1 + m_2\right) + c_2^2A_{22} = -\frac{2E_cP_e\left(A_1A_{15}\left(m_1 + m_2\right) + 2A_2A_{14}m_2\right)m_1}{R_e} \tag{3.178}$$

$$\Rightarrow A_{22} = -\frac{2E_cP_e\left(A_1A_{15}\left(m_1 + m_2\right) + 2A_2A_{14}m_2\right)m_1}{R_e\left(\left(2m_1 + m_2\right)^2 + b_2\left(2m_1 + m_2\right) + c_2^2\right)} \tag{3.179}$$

$$A_{23}\left(m_1 + 2m_2\right)^2 + b_2A_{23}\left(m_1 + 2m_2\right) + c_2^2A_{23} = -\frac{2E_cP_e\left(2A_1A_{16}m_1 + A_2A_{15}\left(m_1 + m_2\right)\right)m_2}{R_e} \tag{3.180}$$

$$\Rightarrow A_{23} = -\frac{2E_cP_e\left(2A_1A_{16}m_1 + A_2A_{15}\left(m_1 + m_2\right)\right)m_2}{R_e\left(\left(m_1 + 2m_2\right)^2 + b_2\left(m_1 + 2m_2\right) + c_2^2\right)} \tag{3.181}$$

$$4A_{24}m_2^2 + 2b_2A_{24}m_2 + c_2^2A_{24} = -\frac{2E_cP_e\left(A_2A_{11} + A_2A_{13}\right)m_2^2}{R_e} \tag{3.182}$$

$$\Rightarrow A_{24} = -\frac{2E_cP_e\left(A_2A_{11} + A_2A_{13}\right)m_2^2}{R_e\left(4m_2^2 + 2b_2m_2 + c_2^2\right)} \tag{3.183}$$

$$9A_{25}m_2^2 + 3b_2A_{25}m_2 + c_2^2A_{25} = -\frac{4E_cP_eA_2A_{16}m_2^2}{R_e} \tag{3.184}$$

$$\Rightarrow A_{25} = -\frac{4E_cP_eA_2A_{16}m_2^2}{R_e\left(9m_2^2 + 3b_2m_2 + c_2^2\right)} \tag{3.185}$$

and,

$$\theta_1(y)=\theta_{1c}(y)+\theta_{1p}(y) \tag{3.186}$$

$$\left.\begin{aligned}\theta_1(y) &= A_{17}e^{m_3 y}+A_{18}e^{m_4 y}+A_{19}e^{2m_1 y}+A_{20}e^{(m_1+m_2)y}+A_{21}e^{3m_1 y}+A_{22}e^{(2m_1+m_2)y}+\\ &\quad A_{23}e^{(m_1+2m_2)y}+A_{24}e^{2m_2 y}+A_{25}e^{3m_2 y}\end{aligned}\right\} \tag{3.187}$$

Applying the boundary condition

$\theta_1(0)=0$ implies,

$$A_{17}+A_{18}+A_{19}+A_{20}+A_{21}+A_{22}+A_{23}+A_{24}+A_{25}=0 \tag{3.188}$$

$$A_{17}=-(A_{19}+A_{20}+A_{21}+A_{22}+A_{23}+A_{24}+A_{25})-A_{18} \tag{3.189}$$

$$A_{17}=-A_{26}-A_{18} \tag{3.190}$$

Where,

$$A_{26}=A_{19}+A_{20}+A_{21}+A_{22}+A_{23}+A_{24}+A_{25} \tag{3.191}$$

Also, $\theta_1(1)=0$ implies,

$$\left.\begin{aligned}&A_{17}e^{m_3}+A_{18}e^{m_4}+A_{19}e^{2m_1}+A_{20}e^{(m_1+m_2)}+A_{21}e^{3m_1}+A_{22}e^{(2m_1+m_2)}+A_{23}e^{(m_1+2m_2)}+\\ &A_{24}e^{2m_2}+A_{25}e^{3m_2}=0\end{aligned}\right\} \tag{3.192}$$

$$\left.\begin{aligned}&(-A_{26}-A_{18})e^{m_3}+A_{18}e^{m_4}+A_{19}e^{2m_1}+A_{20}e^{(m_1+m_2)}+A_{21}e^{3m_1}+A_{22}e^{(2m_1+m_2)}+A_{23}e^{(m_1+2m_2)}+\\ &A_{24}e^{2m_2}+A_{25}e^{3m_2}=0\end{aligned}\right\} \tag{3.193}$$

$$\left.\begin{aligned}&-A_{26}e^{m_3}-A_{18}e^{m_3}+A_{18}e^{m_4}+A_{19}e^{2m_1}+A_{20}e^{(m_1+m_2)}+A_{21}e^{3m_1}+A_{22}e^{(2m_1+m_2)}+A_{23}e^{(m_1+2m_2)}+\\ &A_{24}e^{2m_2}+A_{25}e^{3m_2}=0\end{aligned}\right\} \tag{3.194}$$

$$A_{18}=\frac{A_{26}e^{m_3}-A_{19}e^{2m_1}-A_{20}e^{(m_1+m_2)}-A_{21}e^{3m_1}-A_{22}e^{(2m_1+m_2)}-A_{23}e^{(m_1+2m_2)}-A_{24}e^{2m_2}-A_{25}e^{3m_2}}{e^{m_4}-e^{m_3}} \quad (3.195)$$

Put equation (3.195) into equation (3.190) gives,

$$A_{17}=-A_{26}-\frac{\begin{array}{l}A_{26}e^{m_3}-A_{19}e^{2m_1}-A_{20}e^{(m_1+m_2)}-A_{21}e^{3m_1}-A_{22}e^{(2m_1+m_2)}-A_{23}e^{(m_1+2m_2)}-\\A_{24}e^{2m_2}-A_{25}e^{3m_2}\end{array}}{e^{m_4}-e^{m_3}} \quad (3.196)$$

$$A_{17}=\frac{\begin{array}{l}-A_{26}\left(e^{m_4}-e^{m_3}\right)-A_{26}e^{m_3}+A_{19}e^{2m_1}+A_{20}e^{(m_1+m_2)}+A_{21}e^{3m_1}+A_{22}e^{(2m_1+m_2)}+A_{23}e^{(m_1+2m_2)}\\+A_{24}e^{2m_2}+A_{25}e^{3m_2}\end{array}}{e^{m_4}-e^{m_3}} \quad (3.197)$$

$$A_{17}=\frac{-A_{26}e^{m_4}+A_{19}e^{2m_1}+A_{20}e^{(m_1+m_2)}+A_{21}e^{3m_1}+A_{22}e^{(2m_1+m_2)}+A_{23}e^{(m_1+2m_2)}+A_{24}e^{2m_2}+A_{25}e^{3m_2}}{e^{m_4}-e^{m_3}} \quad (3.198)$$

$$\left.\begin{array}{l}\theta_1(y)=A_{17}e^{m_3y}+A_{18}e^{m_4y}+A_{19}e^{2m_1y}+A_{20}e^{(m_1+m_2)y}+A_{21}e^{3m_1y}+A_{22}e^{(2m_1+m_2)y}+\\ \qquad A_{23}e^{(m_1+2m_2)y}+A_{24}e^{2m_2y}+A_{25}e^{3m_2y}\end{array}\right\} \quad (3.199)$$

where,

$$
\left.\begin{aligned}
A_{17} &= \frac{-A_{26}e^{m_4} + A_{19}e^{2m_1} + A_{20}e^{(m_1+m_2)} + A_{21}e^{3m_1} + A_{22}e^{(2m_1+m_2)} + A_{23}e^{(m_1+2m_2)} + A_{24}e^{2m_2} + A_{25}e^{3m_2}}{e^{m_4} - e^{m_3}} \\
A_{18} &= \frac{A_{26}e^{m_3} - A_{19}e^{2m_1} - A_{20}e^{(m_1+m_2)} - A_{21}e^{3m_1} - A_{22}e^{(2m_1+m_2)} - A_{23}e^{(m_1+2m_2)} - A_{24}e^{2m_2} - A_{25}e^{3m_2}}{e^{m_4} - e^{m_3}} \\
A_{19} &= -\frac{2E_cP_e\left(A_1A_{10} + A_1A_{12}\right)m_1^2}{R_e\left(4m_1^2 + 2b_2m_1 + c_2^2\right)} \\
A_{20} &= -\frac{2E_cP_e\left(A_1A_{11} + A_1A_{13} + A_2A_{10} + A_2A_{12}\right)m_1m_2}{R_e\left(\left(m_1 + m_2\right)^2 + b_2\left(m_1 + m_2\right) + c_2^2\right)} \\
A_{21} &= -\frac{4E_cP_eA_1A_{14}m_1^2}{R_e\left(9m_1^2 + 3b_2m_1 + c_2^2\right)} \\
A_{22} &= -\frac{2E_cP_e\left(A_1A_{15}\left(m_1 + m_2\right) + 2A_2A_{14}m_2\right)m_1}{R_e\left(\left(2m_1 + m_2\right)^2 + b_2\left(2m_1 + m_2\right) + c_2^2\right)} \\
A_{23} &= -\frac{2E_cP_e\left(2A_1A_{16}m_1 + A_2A_{15}\left(m_1 + m_2\right)\right)m_2}{R_e\left(\left(m_1 + 2m_2\right)^2 + b_2\left(m_1 + 2m_2\right) + c_2^2\right)} \\
A_{24} &= -\frac{2E_cP_e\left(A_2A_{11} + A_2A_{13}\right)m_2^2}{R_e\left(4m_2^2 + 2b_2m_2 + c_2^2\right)} \\
A_{25} &= -\frac{4E_cP_eA_2A_{16}m_2^2}{R_e\left(9m_2^2 + 3b_2m_2 + c_2^2\right)}
\end{aligned}\right\} \tag{3.200}
$$

The general solution to the momentum and energy equation respectively is,

$$u\left(y\right) = u_0\left(y\right) + G_{r\theta}u_1\left(y\right) \tag{3.201}$$

$$\theta\left(y\right) = \theta_0\left(y\right) + G_{r\theta}\theta_1\left(y\right) \tag{3.202}$$

and,

$$u(y) = \left(A_1 e^{m_1 y} + A_2 e^{m_2 y} + A_3\right) + G_{r\theta}\begin{pmatrix} A_{10}e^{m_1 y} + A_{11}e^{m_2 y} + A_{12}e^{m_1 y} + A_{13}e^{m_2 y} + \\ A_{14}e^{2m_1 y} + A_{15}e^{(m_1+m_2)y} + A_{16}e^{2m_2 y} \end{pmatrix} \tag{3.203}$$

$$\theta(y) = \begin{pmatrix} A_4 e^{m_3 y} + A_5 e^{m_4 y} + A_6 e^{2m_1 y} \\ +A_7 e^{(m_1+m_2)y} + A_8 e^{2m_2 y} \end{pmatrix} + G_{r\theta}\begin{pmatrix} A_{17}e^{m_3 y} + A_{18}e^{m_4 y} + A_{19}e^{2m_1 y} + A_{20}e^{(m_1+m_2)y} \\ +A_{21}e^{3m_1 y} + A_{22}e^{(2m_1+m_2)y} + A_{23}e^{(m_1+2m_2)y} + \\ A_{24}e^{2m_2 y} + A_{25}e^{3m_2 y} \end{pmatrix} \tag{3.204}$$

3.4: Skin-friction of the Fluid Velocity $u(y,t)$.

The dimensionless stress tensor in terms of the skin-friction coefficient at the plate $y = 0$ is given by

$$\begin{aligned} CF_0 &= \left(\frac{\partial u(y,t)}{\partial y}\right)_{y=0} = \left(\frac{du}{dy}\right)_{y=0}\left(e^{i\omega t}\right) \\ CF_0 &= \left(A_1 m_1 + A_2 m_2\right) + G_{r\theta}\begin{pmatrix} \left(A_{10} + A_{12} + 2A_{14}\right)m_1 + \\ \left(A_{11} + A_{13} + 2A_{16}\right)m_2 + A_{15}\left(m_1 + m_2\right) \end{pmatrix} e^{i\omega t} \end{aligned} \tag{3.205}$$

3.5: Nusselt-number of the Temperature of the Fluid $\theta(y,t)$.

The dimensionless rate of heat transfer in terms of the Nusselt number at the plate $y = 0$ is given by

$$\begin{aligned} Nu_0 &= -\left(\frac{\partial \theta(y,t)}{\partial y}\right)_{y=0} = \left(\frac{d\theta}{dy}\right)_{y=0}\left(e^{2i\omega t}\right) \\ Nu_0 &= -\begin{pmatrix} A_4 m_3 + A_5 m_4 + 2A_6 m_1 \\ +A_7\left(m_1 + m_2\right) + 2A_8 m_2 \end{pmatrix} + G_{r\theta}\begin{pmatrix} A_{17}m_3 + A_{18}m_4 + \left(2A_{19} + 3A_{21}\right)m_1 \\ +A_{20}\left(m_1 + m_2\right) + A_{22}\left(2m_1 + m_2\right) \\ +A_{23}\left(m_1 + 2m_2\right) + \left(2A_{24} + 3A_{25}\right)m_2 \end{pmatrix} e^{2i\omega t} \end{aligned} \tag{3.206}$$

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Results

From the graph in chapter three, the effects of Peclet number (P_e), Hartman number (H_a), Suction/Injection parameter (S), Eckert number (E_c), Reynolds number (R_e), kinematic viscosity (v), Grashof thermal number $(G_{r\theta})$, term due to thermal radiation (α), Thermal radiation parameter (R_a) and time (t) on the velocity $u(y,t)$ and temperature $\theta(y,t)$ of the fluid. Skin friction of the fluid velocity and Nusset number of the temperature of the fluid given by equations (3.206) and (3.207) respectively were computed using computer symbolic algebraic package (MAPLE 17).

The results obtained from the solutions are shown in Figure 4.1 through 4.16. The effect of Peclet number (P_e) on velocity $u(y,t)$ against distance is depicted in figure 4.1. The effect of Peclet number (P_e) on temperature $\theta(y,t)$ against distance is depicted in figure 4.2. The effect of Hartman number (H_a) on temperature $\theta(y,t)$ against distance is depicted in figure 4.3. The effect of Suction/Injection parameter (S) on velocity $u(y,t)$ against distance is depicted in figure 4.4. The effect of Suction/Injection parameter (S) on temperature $\theta(y,t)$ against distance is depicted in figure 4.5. The effect of Eckert number (E_c) on velocity $u(y,t)$ against distance is depicted in figure 4.6. The effect of Eckert number (E_c) on temperature $\theta(y,t)$ against distance is depicted in figure 4.7. The effect of

Reynolds number (R_e) on temperature $\theta(y,t)$ against distance is depicted in figure 4.8. The effect of kinematic viscosity (v) on velocity $u(y,t)$ against distance is depicted in figure 4.9. The effect of Grashof thermal number $(G_{r\theta})$ on velocity $u(y,t)$ against distance is depicted in figure 4.10. The effect of Grashof thermal number $(G_{r\theta})$ on velocity $u(y,t)$ against distance is depicted in figure 4.11. The effect of term due to thermal radiation (α) on temperature $\theta(y,t)$ against distance is depicted in figure 4.12. The effect of Thermal radiation parameter (R_a) on temperature $\theta(y,t)$ against distance is depicted in figure 4.13. The effect of time (t) on velocity $u(y,t)$ against distance is depicted in figure 4.14. The effect of time (t) on temperature $\theta(y,t)$ against distance is depicted in figure 4.15.

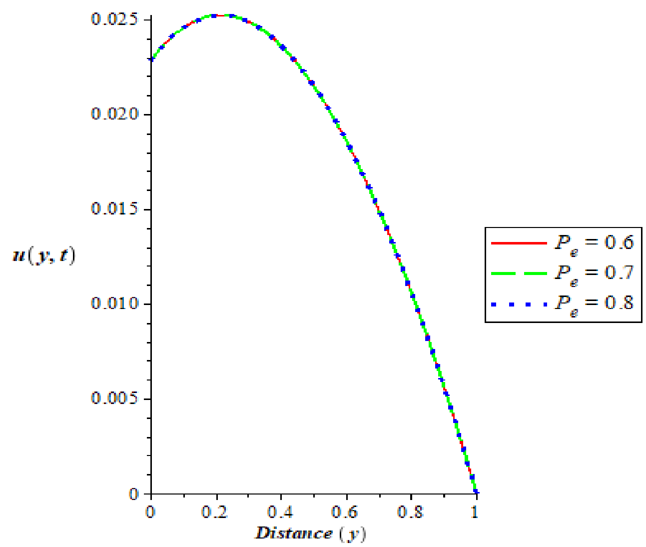

Figure 4.1: Effect of Peclet number (P_e) on Velocity $u(y,t)$ against Distance

It is observed that velocity of the fluid does not change much with an increase in the Peclet number (P_e) at steady time.

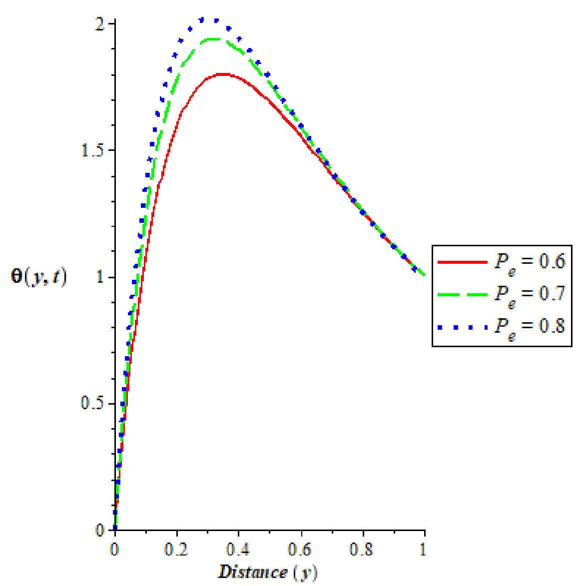

Figure 4.2: Effect of Peclet number (P_e) on Temperature $\theta(y,t)$ against Distance

It is observed that temperature of the fluid increases with an increase in the Peclet number (P_e) at steady time.

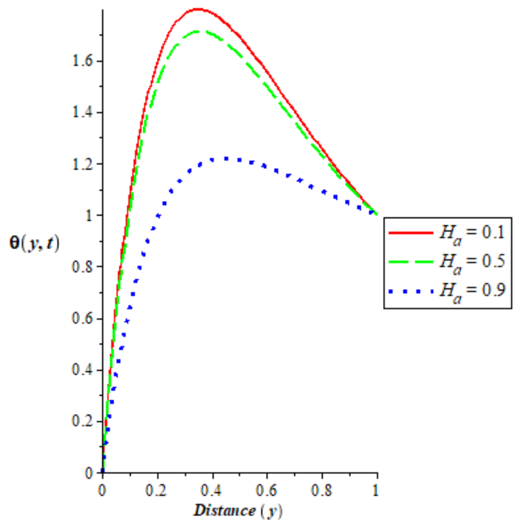

Figure 4.3: Effect of Hartman number (H_a) on Temperature $\theta(y,t)$ against Distance

It is observed that temperature of the fluid reduces with an increase in the Hartman number (H_a) at steady time.

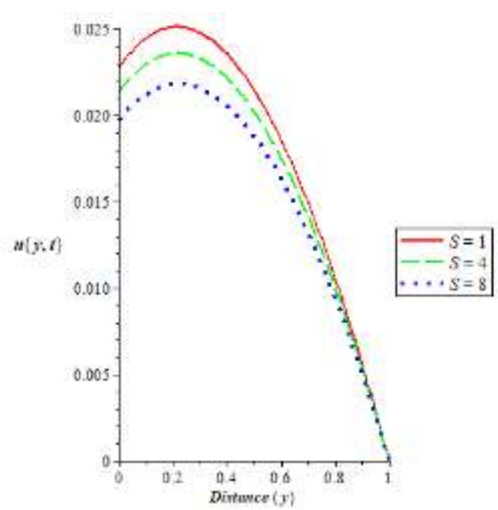

Figure 4.4: Effect of Suction/Injection parameter (S) on Velocity $u(y,t)$ against Distance

It is observed that velocity of the fluid reduces with an increase in the Suction/Injection parameter (S) at steady time.

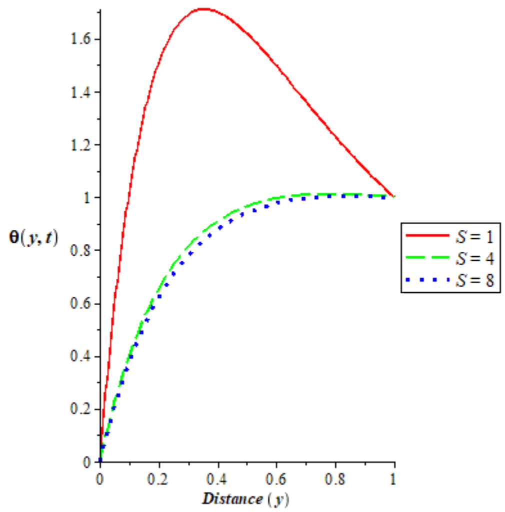

Figure 4.5: Effect of Suction/Injection parameter (S) on Temperature $\theta(y,t)$ against Distance.

It is observed that temperature of the fluid reduces with an increase in the Suction/Injection parameter (S) at steady time.

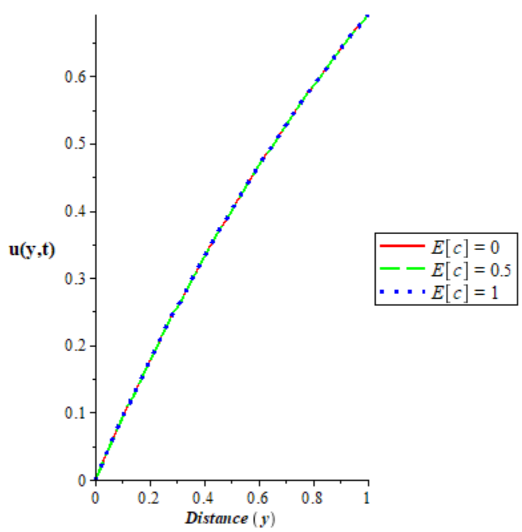

Figure 4.6: Effect of Eckert number (E_c) on Velocity $u(y,t)$ against Distance

It is observed that velocity of the fluid does not change much with an increase in the Suction/Injection parameter (S) at steady time.

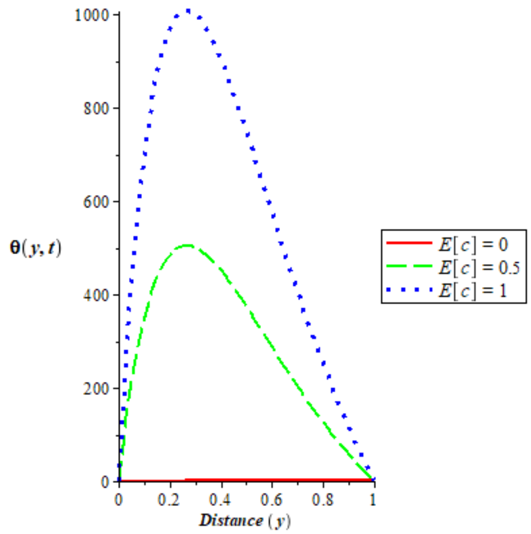

Figure 4.7: Effect of Eckert number (E_c) on Temperature $\theta(y,t)$ against Distance

It is observed that temperature of the fluid increases with an increase in the Eckert number (E_c) at steady time.

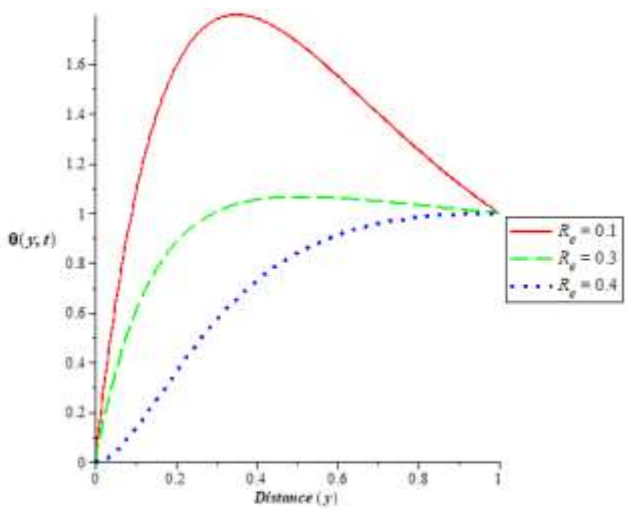

Figure 4.8: Effect of Reynolds number (R_e) on Temperature $\theta(y,t)$ against Distance

It is observed that temperature of the fluid reduces with an increase in the Reynolds number (R_e) at steady time.

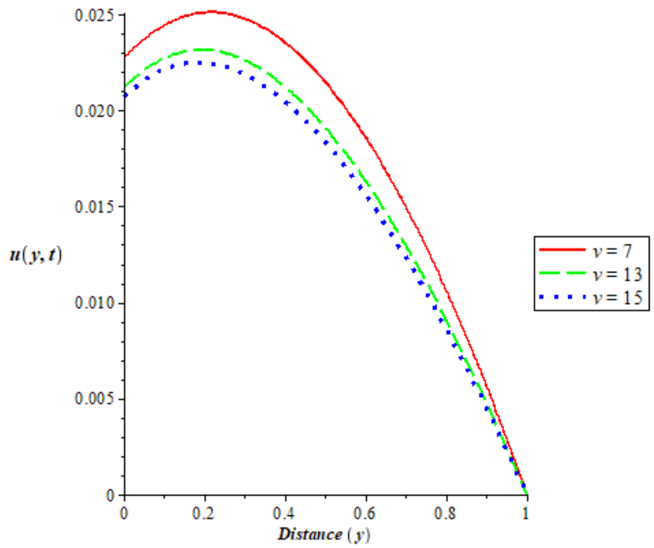

Figure 4.9: Effect of Kinematic Viscosity (v) on Velocity $u(y,t)$ against Distance

It is observed that velocity of the fluid reduces with an increase in the kinematic viscosity (v) at steady time.

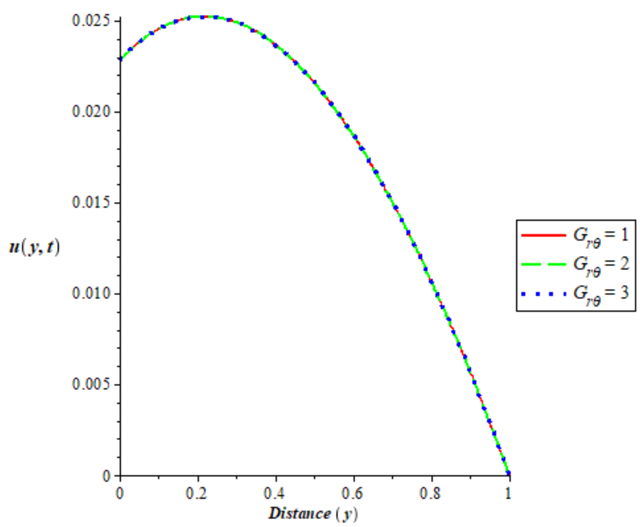

Figure 4.10: Effect of Grashof Thermal number $(G_{r\theta})$ on Velocity $u(y,t)$ against Distance

It is observed that the velocity of the fluid does not change much with an increase in the Grashof thermal number $(G_{r\theta})$ at steady time.

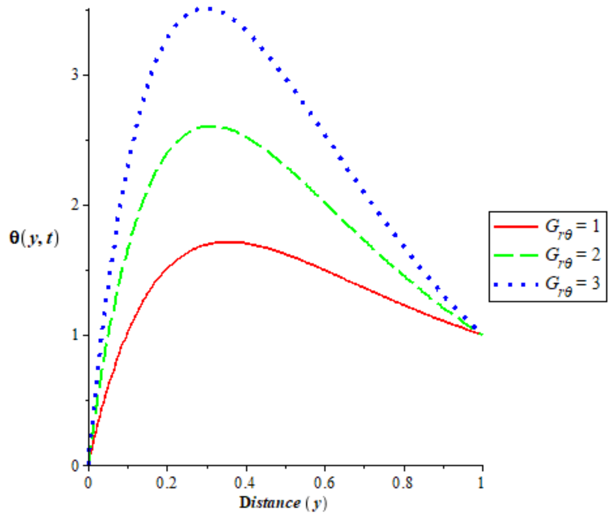

Figure 4.11: Effect of Grashof Thermal number $(G_{r\theta})$ on Velocity $u(y,t)$ against Distance

It is observed that the temperature of the fluid increases with an increase in the Grashof thermal number $(G_{r\theta})$ at steady time.

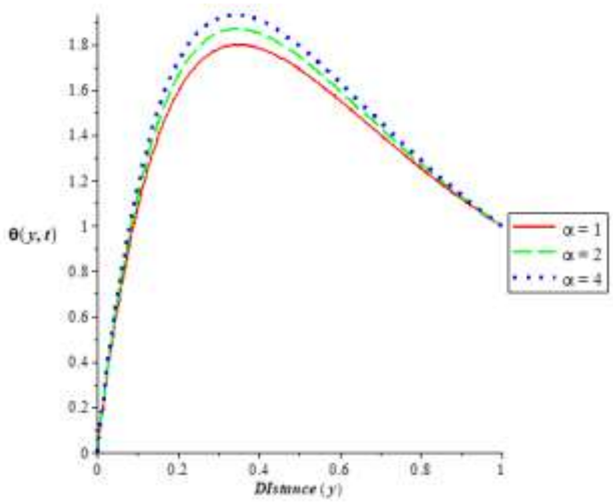

Figure 4.12: Effect of term due to Thermal radiation (α) on Temperature $\theta(y,t)$ against Distance

It is observed that the temperature of the fluid increases with an increase in the term due to thermal radiation (α) at steady time.

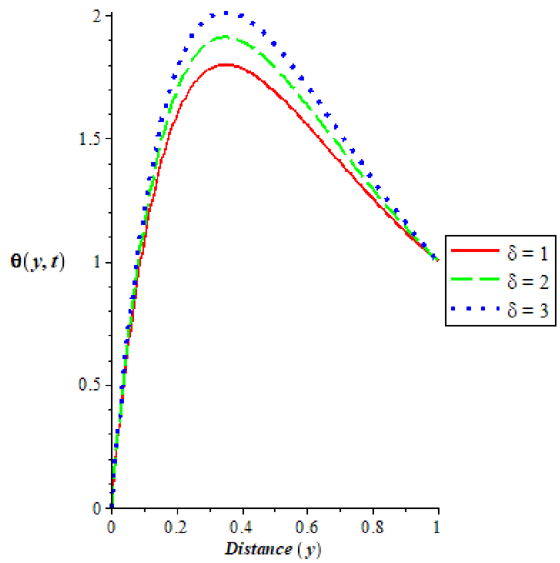

Figure 4.13: Effect of Thermal Radiation parameter (R_a) on Temperature $\theta(y,t)$ against Distance.

It is observed that the temperature of the fluid increases with an increase in the Thermal radiation parameter (δ) at steady time.

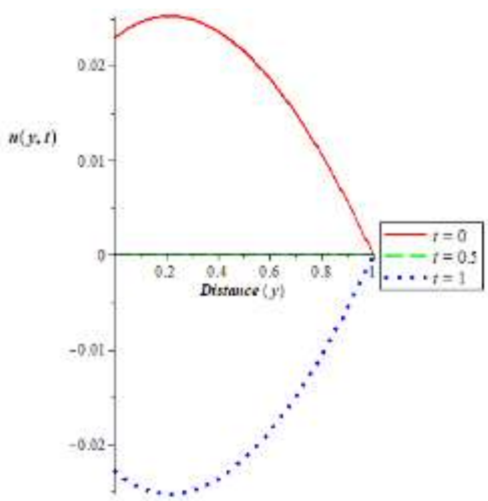

Figure 4.14: Effect of Time (t) on Velocity $u(y,t)$ against Distance

It is observed that the velocity of the fluid reduces with an increase in time (t).

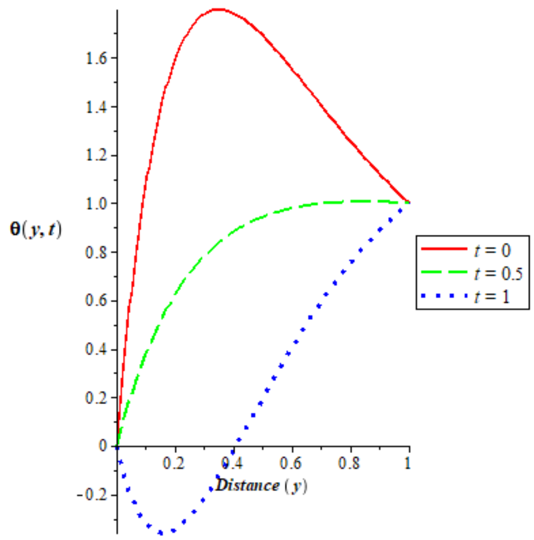

Figure 4.15: Effect of Time (t) on Temperature $\theta(y,t)$ against Distance

It is observed that the temperature of the fluid reduces with an increase in time (t).

Table 4.1.. Numerical values of skin-friction coefficient at the plate $y=0$ for various values of physical parameters

P_e	H_a	E_c	R_e	ν	$G_{r\theta}$	δ	t	CF_0
0.6	0.1	0.0000001	0.1	7	1	1	0	0.02282159761
0.7	0.1	0.0000001	0.1	7	1	1	0	0.2771784025
0.8	0.1	0.0000001	0.1	7	1	1	0	1.222821597
0.6	0.5	0.0000001	0.1	7	1	1	0	0.02270243200
0.6	0.9	0.0000001	0.1	7	1	1	0	-0.5775709957
0.6	0.1	0.0000002	0.1	7	1	1	0	-0.1771783959
0.6	0.1	0.0000003	0.1	7	1	1	0	-0.3771783893
0.6	0.1	0.0000001	0.3	7	1	1	0	0.6557569325
0.6	0.1	0.0000001	0.4	7	1	1	0	0.3670472404
0.6	0.1	0.0000001	0.1	13	1	1	0	0.5172536900
0.6	0.1	0.0000001	0.1	15	1	1	0	0.02076954027
0.6	0.1	0.0000001	0.1	7	2	1	0	0.02282160413
0.6	0.1	0.0000001	0.1	7	3	1	0	0.02282161064
0.6	0.1	0.0000001	0.1	7	1	2	0	1.022821599
0.6	0.1	0.0000001	0.1	7	1	3	0	3.022821607
0.6	0.1	0.0000001	0.1	7	1	1	0.4	0.02609456154
0.6	0.1	0.0000001	0.1	7	1	1	0.7	0.01290161830

Table 4.2: Numerical values of Nusselt number at the plate $y = 0$ for various values of physical parameters.

P_e	H_a	E_c	R_e	v	$G_{r\theta}$	δ	t	Nu_0
0.6	0.1	0.0000001	0.1	7	1	1	0	-14.56136822
0.7	0.1	0.0000001	0.1	7	1	1	0	-17.25372613
0.8	0.1	0.0000001	0.1	7	1	1	0	-19.49438297
0.6	0.5	0.0000001	0.1	7	1	1	0	-13.75969507
0.6	0.9	0.0000001	0.1	7	1	1	0	-8.417759937
0.6	0.1	0.0000002	0.1	7	1	1	0	-24.50266599
0.6	0.1	0.0000003	0.1	7	1	1	0	-34.44396367
0.6	0.1	0.0000001	0.3	7	1	1	0	-8.493814836
0.6	0.1	0.0000001	0.4	7	1	1	0	0.068118293
0.6	0.1	0.0000001	0.1	13	1	1	0	-9.856397265
0.6	0.1	0.0000001	0.1	15	1	1	0	-9.856397265
0.6	0.1	0.0000001	0.1	7	2	1	0	-24.50266594
0.6	0.1	0.0000001	0.1	7	3	1	0	-0.06553324430
0.6	0.1	0.0000001	0.1	7	1	2	0	-15.35806125
0.6	0.1	0.0000001	0.1	7	1	3	0	-16.04140800
0.6	0.1	0.0000001	0.1	7	1	1	0.4	-5.034545097
0.6	0.1	0.0000001	0.1	7	1	1	0.7	0.1168126243

4.2 Discussion of Results

Figure 4.1 displays the effects of Peclet number (P_e) on the velocity of the fluid. It is observed that the velocity of the fluid $u(y,t)$ does not change much with increase in Peclet number (P_e) at a steady time.

Figure 4.2 show the effect of Peclet number (P_e) on the temperature of the fluid $\theta(y,t)$. It is observed that the temperature increases with an increase in the Peclet number at steady time.

Figure 4.3 show the effect of Hartman number (H_a) on the temperature of the fluid $\theta(y,t)$. It is observed that temperature reduces with an increase in the Hartman number at steady time

Figure 4.4 shows the effects of Suction/Injection parameter (S) on the velocity of the fluid. It is observed that the velocity of the fluid $u(y,t)$ reduces with increase in Suction/Injection parameter at a steady time.

Figure 4.5 show the effect of Suction/Injection parameter (S) on the temperature of the fluid $\theta(y,t)$. It is observed that the temperature reduces with an increase in the Suction/Injection parameter at steady time.

Figure 4.6 shows the effects of Eckert number (E_c) on the velocity of the fluid. It is observed that the velocity of the fluid $u(y,t)$ does not change much with increase in the Eckert number at a steady time.

Figure 4.7 displays the graph of Eckert number $\left(E_c\right)$ on the temperature of the fluid $\theta\left(y,t\right)$. It shows that an increase in Eckert number from 0 (no viscous heating) through 0.5 to 1 (high viscous heating) clearly boost temperature in the porous regime. Eckert number signifies the quantity of mechanical energy converted via internal friction to thermal energy.

Figure 4.8 show the effect of Reynolds number $\left(R_e\right)$ on the temperature of the fluid $\theta\left(y,t\right)$. It is observed that temperature decreases with an increase in the Reynolds number at steady time.

Figure 4.9 displays the graph of kinematic viscosity $\left(v\right)$ on the velocity of the fluid $u\left(y,t\right)$. It is observed that the velocity of the fluid reduces with increase in kinematic viscosity at a steady time.

Figure 4.10 show the effect of Grashof thermal number $\left(G_{r\theta}\right)$ on the velocity of the fluid $u\left(y,t\right)$. It is observed that the velocity of the fluid $u\left(y,t\right)$ does not change much with increase in the Grashof thermal number at steady time.

Figure 4.11 shows the effects of Grashof thermal number $\left(G_{r\theta}\right)$ on the velocity of the fluid. It is observed that the temperature increases with an increase in the Grashof thermal number at a steady time.

Figure 4.12 show the effect of the term due to thermal radiation $\left(\alpha\right)$ on the temperature of the fluid $\theta\left(y,t\right)$. It is observed that the temperature of the fluid increases with increase in the term due to thermal radiation at steady time.

Figure 4.13 shows the effects of Thermal radiation parameter (δ) on the temperature of the fluid $\theta(y,t)$. It is observed that the temperature of the fluid increases with increase in Thermal radiation parameter (δ) at a steady time.

Figure 4.14 show the effect of time (t) on the velocity of the fluid $u(y,t)$. It is observed that the velocity of the fluid reduces with increase in time.

Figure 4.15 shows the effect of time (t) on the temperature of the fluid $\theta(y,t)$. It is observed that the temperature of the fluid reduces with increase in time.

Table 4.1 shows that at the plate $(y=0)$ when the Eckert number (E_c), Reynold number (R_e), Hatmann number (H_a), kinematic viscosity (v) and Thermal radiation parameter (δ) increase the skin friction (CF_0) decreases. The rate of skin friction (CF_0) increases for increasing values of Peclet number (P_e).

Table 4.2 shows that the rate of heat transfer at the plate $(y=0)$ increases for increasing values of Reynold number (R_e), Hatmann number (H_a), and Grashof thermal $(G_{r\theta})$, but a reverse trend is observed for increasing values of Peclet number (P_e), Eckert number (E_c) and Thermal radiation parameter (δ).

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATION

5.1 Conclusion

A mathematical analysis has been carried out to study the magnetohydrodynamic oscillatory flow with viscous energy dissipation through a porous channel saturated with porous medium. The dimensionless governing coupled non-linear partial differential equations for this investigation were solved analytically using harmonic solution technique. The effects of the dimensionless parameters as shown on the graph were analyzed. It is concluded that:

(i) Peclet energy number, Eckert number, Thermal radiation parameter, Grashof Thermal and the term due to thermal radiation increase the transient temperature of the fluid.

(ii) Suction/Injection, kinematic viscosity and time reduce the velocity of the fluid.

(iii) Hartman number, Sunction/Injection parameter, Reynolds number and time reduce the transient temperature of the fluid

(iv) Reynolds number increases the velocity of the fluid.

(v) Eckert number (E_c), Reynold number (R_e), Hatmann number (H_a), kinematic viscosity (v) and Thermal radiation parameter (δ) decrease the skin friction (CF_0) at the plate $(y=0)$.

(vi) Peclet number (P_e) increase the rate of skin friction (CF_0) at the plate $(y=0)$

(vii) Reynold number $\left(R_e\right)$, Hatmann number $\left(H_a\right)$, and Grashof thermal $\left(G_{r\theta}\right)$ increases the rate of heat transfer at the plate $\left(y=0\right)$.

(viii) Peclet number $\left(P_e\right)$, Eckert number $\left(E_c\right)$ and Thermal radiation parameter $\left(\delta\right)$ reduces the rate of heat transfer at the plate $\left(y=0\right)$.

5.2 Recommendation

Further work can be carried out on magnetohydrodynamic oscillatory flow with viscous energy dissipation through a porous channel saturated with porous medium using other analytical methods (Parameter expansion, Method of lines, Homotopy perturbation and so on) to ascertain how best the result can be obtained as it is important in many physiological flows and engineering applications such as magneto-hydrodynamic (MHD) generators, arterial blood flow, petroleum engineering, meteorology, solar physics, geophysics and motion of the earth core.

Magnetohydrodynamic oscillatory flows in channels and pipes possess large amounts of mechanical applications which may incorporate cooling frameworks, petrochemical transport (oil and petroleum gas) and biotechnology. Regularly such flows are going with heat transfer, example is the removal of thermal energy from hydronic space heating framework by means of circling water in the heater, after which it is transported to the individual areas through pipes.

5.3 Contributions to knowledge

In this study, the following contributions were made to knowledge:

(i) This present work extends the work of Falade *et al*. (2016) by incorporating viscous energy dissipation term in the heat process

(ii) Magnetohydrodynamic oscillatory flow with viscous energy dissipation through a porous channel saturated with porous medium was solved using harmonic solution technique.

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