FOURTH ORDER TRIGONOMETRICALLY-FITTED IMPROVED RUNGE-KUTTA METHOD

Abdulrahman Ndanusa[1*] Aliyu Umar Mustapha[2]

[1, 2]Department of Mathematics, Federal University of Technology, Minna, Nigeria

[1]as.ndanusa@futminna.edu.ng

[2]aliyuumar423@gmail.com

[*]Corresponding author

Abstract

A trigonometrically - fitted fourth order Improved Runge – Kutta (IRK) method whose coefficients depend on the frequency and step – size is constructed. The method is applied to solve sample initial value problems (IVPs) with oscillatory solutions. Numerical results obtained are compared with those of the non - fitted IRK method of the same order. The comparison shows that the trigonometrically fitted method is much more effective and efficient than the non – fitted method, with similar cost.

Keywords: improved Runge–kutta method, initial value problem, oscillatory solution, trigonometrically – fitted, Taylor series

1. Introduction

A great many numerical methods for approximating the solution of initial value problems (IVP) of order ordinary differential equations have been in use. Runge – Kutta (RK) method is just one class of such methods while the Improved Runge - Kutta (IRK) method arises from the classical RK method. Most efforts to increase the order of RK methods have been accomplished by increasing the number of Taylors series terms used and thus the number of function evaluations required (Gear 1971; Butcher 1987). Many authors have attempted to increase the efficiency of RK methods by trying to lower the number of function evaluations required (Rabiei *et al*., 2013). The IRK methods are a special class of two-step methods, that is, the approximate solution y_{n+1} is calculated using the value of y_n and y_{n-1}. The method introduces the new terms of k_{-i}, which are calculated using $k_i (i \geq 2)$, from the previous step. The schemes usually have lower number of function evaluations than the Runge-Kutta methods. The general $s - stage$ IRK method has the form

$$y_{n+1} = y_n + h\left(b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{s} b_i \left(k_i - k_{-i}\right)\right) \qquad (1)$$

for $1 \leq n \leq N - 1$, where

$$\left.\begin{aligned} k_1 = f(x_n, y_n), \qquad k_{-1} = f(x_{n-1}, y_{n-1}) \\ k_i = f\left(x_n + c_i h, y_n + h\sum_{j=1}^{i-1} a_{i,j}k_j\right), \qquad 2 \le i \le s \\ k_{-i} = f\left(x_{n-1} + c_i h, y_{n-1} + h\sum_{j=1}^{i-1} a_{i,j}k_{-j}\right), \qquad 2 \le i \le s \end{aligned}\right\} \tag{2}$$

for $c_2 \dots c_s \epsilon [0,1]$ and f depends on both x and y while k_i and k_{-i} depend on the values of k_j and k_{-j} for $j = 1, \dots\dots, i-1$. Here s is the number of function evaluations performed at each step and increases with the order of local accuracy of the IRK method. The IRK method is not self- starting therefore a one-step method must provide the approximate solution of y_1 at the first step. The one-step method must be of appropriate order to ensure that the difference $\left(y_1 - y(x_1)\right)$ is order of p or higher (Rabiei and Ismail, 2012).

Trigonometrically – fitted methods are especially developed to improve the accuracy of numerical solutions of initial value problems. In particular, when the solution exhibits periodic or oscillatory behaviour, trigonometrically fitted methods are much more efficient than non-fitted methods, with the same cost. These methods are generally obtained to exactly solve IVPs whose solutions are linear combinations of the functions $\{1, x, x^2, \dots, \cos(\omega x), \sin(\omega x)\}$. The procedure has been applied to linear multistep methods, Runge_Kutta-type methods, and hybrid methods (Ramos and Vigo-Aguiar, 2010). The pioneering work in the development of trigonometrically – fitted methods is attributed to Gautschi (1961); ever since, authors who have made their contributions in this direction include, Franco (2004), Wang (2006), Fang and Wu (2008), to mention but just a few. This present research is an attempt to design a trigonometrically – fitted IRK method of order four, thereby improving its efficiency and effectiveness.

2. Materials and Methods

An alternative form of the general IRK method (1) and (2) is

$$\begin{aligned} y_{n+1} = y_n + hb_1 f(x_n, y_n) - hb_{-1} f(x_{n-1}, y_{n-1}) \\ + h\sum_{i=2}^{s} b_i\left(f(x_n + c_i h, Y_i) - f(x_{n-1} + c_i h, Y_{-i})\right) \end{aligned} \tag{3}$$

where,

$$Y_i = y_n + h\sum_{j=1}^{i-1} a_{i,j} f(x_n + c_j h, Y_j) \tag{4}$$

$$Y_{-i} = y_{n-1} + h\sum_{j=1}^{i-1} a_{i,j} f\left(x_{n-1} + c_j h, Y_{-j}\right) \quad (5)$$

where y_{n+1} and y_n are approximations to $y(x_{n+1})$ and $y(x_n)$ respectively.

If a function $y(x)$ is integrated exactly by TFIRK method for all problems whose solution is $y(x)$ then,

$$y_n = y(x_n) = e^{i\omega x_n} \quad (6)$$

$$y_{n-1} = y(x_{n-1}) = y(x_n - h) = e^{i\omega(x_n - h)} \quad (7)$$

Thus

$$y_n^{'} = i\omega e^{i\omega x_n} = f\left(x_n, y_n\right) \quad (8)$$

$$y_{n-1}^{'} = i\omega e^{i\omega(x_n - h)} = f\left(x_{n-1}, y_{n-1}\right) \quad (9)$$

$$Y_i = e^{i\omega(x_n + c_i h)} \quad (10)$$

$$Y_{-i} = e^{i\omega(x_{n-1} + c_i h)} \quad (11)$$

Consequently, we obtain the recursive relations

$$cos(c_i z) = 1 - z\sum_{j=1}^{i-1} a_{ij} \sin\left(c_j z\right), \qquad i = 2,3, \dots, s \quad (12)$$

$$sin(c_i z) = z\sum_{j=1}^{i-1} a_{ij}\, cos\left(c_j z\right), i = 2,3, \dots, s \quad (13)$$

$$cos(z) = 1 - zb_{-1}\, sin(z) - z\sum_{i=2}^{s} b_i\, sin(c_i z) + z\sum_{i=2}^{s} b_i\, sin\left(z(c_i - 1)\right) \quad (14)$$

$$sin(z) = zb_1 - zb_{-1}\, cos(z) + z\sum_{i=2}^{s} b_i\, cos(c_i z) - z\sum_{i=2}^{s} b_i\, cos\left(z(c_i - 1)\right) \quad (15)$$

The relations (12), (13), (14) and (15) are the relations of order conditions of the trigonometrically-fitted method. They replace the equations of order conditions of two-step Improved Runge-Kutta (IRK) method, which can be solved to give the coefficients of a particular method based on existing coefficients.

For the proposed fourth order three stage trigonometrically – fitted IRK (TFIRK43) method ($s = 3, p = 4$), the order conditions up to order four for classical IRK methods, according to Rabiei *et al*. (2013), is

$$\left.\begin{aligned} &\text{First order:} \qquad b_1 - b_{-1} = 1 \\ &\text{Sercond order:}\; b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2} \\ &\text{Third order 3} \quad : \sum_{i=2}^{s} b_i\, c_i = \frac{5}{12} \\ &\qquad\qquad\qquad \sum_{i=2}^{s} b_i\, c_i^2 = \frac{1}{3} \\ &\text{Fourth order 4:} \sum_{i=2, j=1}^{s} b_i\, a_{ij} c_j = \frac{1}{6} \\ &\qquad\qquad\qquad \sum_{i=1}^{s} b_i\, c_i^3 = \frac{31}{120} \end{aligned}\right\} \tag{16}$$

And the classical fourth order three stage IRK method (IRK43) has the butcher tableau as

Table 1 Butcher Tableau for IRK43

0			
$\frac{31}{60}$	$\frac{31}{60}$		
$\frac{62}{85}$	$\frac{7502}{24565}$	$\frac{10416}{24565}$	
$\frac{157}{23064}$	$\frac{23221}{23064}$	$-\frac{1800}{6727}$	$\frac{122825}{161448}$

Substituting $s = 3, c_1 = 0$ in the recursive relations (12) and (13)

when $i = 2$

$$\cos(c_2 z) - 1 = 0 \tag{17}$$

$$\sin(c_2 z) - z a_{2,1} = 0 \tag{18}$$

when $i = 3$

$$\cos(c_3 z) - 1 + z a_{3,2} \sin(c_2 z) = 0 \tag{19}$$

$$\sin(c_3 z) - z\left[a_{3,1} + a_{3,2} \cos(c_2 z)\right] = 0 \tag{20}$$

Substituting $s = 3, c_1 = 0$ into (14) and (15)

$$\cos(z) - 1 + z b_{-1} \sin(z) + z[b_2 \sin(c_2 z) + b_3 \sin(c_3 z)] - z\left[b_2 \sin\left((c_2 - 1)z\right) + b_3 \sin\left((c_3 - 1)z\right)\right] = 0 \tag{21}$$

$$\sin(z) - z b_1 + z b_{-1} \cos(z) - z[b_2 \cos(c_2 z) + b_3 \cos c_3 z] + z\left[b_2 \cos\left((c_2 - 1)z\right) + b_3 \cos\left((c_3 - 1)z\right)\right] = 0 \tag{22}$$

Equations $(19) - (22)$ are now the equations of order conditions for the fourth order three stage trigonometrically-fitted IRK method which replaces the order conditions of the original method.

Equations (21) and (22), together with two additional equations from the order conditions (16), that is,

$$b_1 - b_{-1} = 1 \tag{23}$$

$$b_{-1} + b_2 + b_3 = \frac{1}{2} \tag{24}$$

are combined to constitute a system a four equations in six unknowns $(b_{-1}, b_1, b_2, b_3, c_2 \text{and} c_3)$. The resulting system is solved in terms of two free parameters $\left(c_2 = \frac{31}{60}, c_3 = \frac{62}{85}\right)$ whose values are obtained from Table 1. Using Maple software, the following values are obtained for the unknowns.

$$\left.\begin{aligned} b_{-1} &= -\frac{1}{2}\frac{M_1}{M_2} \\ b_1 &= \frac{1}{2}\frac{\mathrm{M}_3}{\mathrm{M}_4} \\ b_2 &= -\frac{1}{2}\frac{\mathrm{M}_5}{\mathrm{M}_6} \\ b_3 &= \frac{1}{2}\frac{M_7}{M_8} \end{aligned}\right\} \tag{25}$$

where,

$$M_1 = -\frac{1}{2}[-2z \sin\left(\frac{31}{60}z\right) - 2z \sin\left(\frac{29}{60}z\right) - 2\cos\left(\frac{31}{60}z\right) + 2\cos\left(\frac{62}{85}z\right) + 2\cos\left(\frac{29}{60}z\right)$$

$$-2\cos\left(\frac{23}{85}z\right) + 2\cos\left(\frac{31}{60}z\right)\cos(z) - 2\cos\left(\frac{29}{60}z\right)\cos(z) + 2\sin(z)\sin\left(\frac{31}{60}z\right)$$

$$+2\sin(z)\sin\left(\frac{29}{60}z\right) - z\cos\left(\frac{29}{60}z\right)\sin\left(\frac{62}{85}z\right) - z\cos\left(\frac{29}{60}z\right)\sin\left(\frac{23}{85}z\right) - 2\sin\left(\frac{62}{85}z\right)\sin(z)$$

$$+2sin\left(\frac{62}{85}z\right)-2sin\left(\frac{23}{85}z\right)sin(z)+2zsin\left(\frac{23}{85}z\right)-2cos(z)cos\left(\frac{62}{85}z\right)\sin(z)$$

$$+2cos(z)cos\left(\frac{23}{85}z\right)+zcos\left(\frac{23}{85}z\right)sin\left(\frac{29}{60}z\right)+zcos\left(\frac{23}{85}z\right)sin\left(\frac{31}{60}z\right)$$

$$-zcos\left(\frac{62}{85}z\right)sin\left(\frac{29}{60}z\right)-zcos\left(\frac{62}{85}z\right)sin\left(\frac{31}{60}z\right)+zcos\left(\frac{31}{60}z\right)sin\left(\frac{62}{85}z\right)$$

$$+zcos\left(\frac{31}{60}z\right)sin\left(\frac{23}{85}z\right)] \qquad (26)$$

$$M_2=z[\cos(z)\,sin\left(\frac{31}{60}z\right)+cos(z)sin\left(\frac{29}{60}z\right)-sin\left(\frac{31}{60}z\right)-cos\left(\frac{62}{85}z\right)\sin(z)$$

$$+cos\left(\frac{62}{85}z\right)sin\left(\frac{31}{60}z\right)+cos\left(\frac{62}{85}z\right)sin\left(\frac{29}{60}z\right)+cos\left(\frac{23}{85}z\right)sin(z)$$

$$-cos\left(\frac{23}{85}z\right)sin\left(\frac{31}{60}z\right)-cos\left(\frac{23}{85}z\right)sin\left(\frac{29}{60}z\right)-cos(z)sin\left(\frac{62}{85}z\right)-cos(z)sin\left(\frac{23}{85}z\right)$$

$$+sin\left(\frac{62}{85}z\right)+sin\left(\frac{23}{85}z\right)+cos\left(\frac{31}{60}z\right)sin(z)-cos\left(\frac{31}{60}z\right)sin\left(\frac{31}{60}z\right)$$

$$-cos\left(\frac{31}{60}z\right)sin\left(\frac{23}{85}z\right)-cos\left(\frac{29}{60}z\right)sin(z)+cos\left(\frac{29}{60}z\right)+cos\left(\frac{29}{60}z\right)sin\left(\frac{62}{85}z\right)$$

$$+cos\left(\frac{29}{60}z\right)sin\left(\frac{23}{85}z\right)] \qquad (27)$$

$$M_3=\frac{1}{2}[2\,sin\left(\frac{23}{85}z\right)+2\,sin\left(\frac{62}{85}z\right)sin(z)+2cos(z)cos\left(\frac{62}{85}z\right)-2cos(z)cos\left(\frac{23}{85}z\right)$$

$$+2cos\left(\frac{31}{60}z\right)-2cos\left(\frac{62}{85}z\right)-2cos\left(\frac{29}{60}z\right)+2cos\left(\frac{23}{85}z\right)+2cos\left(\frac{31}{60}z\right)\sin(z)$$

$$-2cos\left(\frac{31}{60}z\right)cos(z)+2cos\left(\frac{29}{60}z\right)cos(z)-2sin(z)sin\left(\frac{31}{60}z\right)-2sin(z)sin\left(\frac{29}{60}z\right)$$

$$+2zcos(z)sin\left(\frac{31}{60}z\right)+2zcos(z)sin\left(\frac{29}{60}z\right)-cos\left(\frac{29}{60}z\right)zsin(z)$$

$$+3zcos\left(\frac{29}{60}z\right)sin\left(\frac{62}{85}z\right)+3zcos\left(\frac{29}{60}z\right)sin\left(\frac{23}{85}z\right)-3zcos\left(\frac{23}{85}z\right)sin\left(\frac{29}{60}z\right)$$

$$-3zcos\left(\frac{23}{85}z\right)sin\left(\frac{31}{60}z\right)+3zcos\left(\frac{62}{85}z\right)sin\left(\frac{29}{60}z\right)+3zcos\left(\frac{62}{85}z\right)sin\left(\frac{31}{60}z\right)$$

$$-3zcos\left(\frac{31}{60}z\right)sin\left(\frac{62}{85}z\right)-3zcos\left(\frac{31}{60}z\right)sin\left(\frac{23}{85}z\right)-2zsin(z)cos\left(\frac{62}{85}z\right)]$$

$$-2zsin(z)cos\left(\frac{23}{85}z\right)-2sin\left(\frac{62}{85}z\right)zcos(z)sin\left(\frac{23}{85}z\right)-2sin\left(\frac{23}{85}z\right)zcos(z)] \qquad (28)$$

$$M_4=z[\cos(z)\,sin\left(\frac{31}{60}z\right)+cos(z)sin\left(\frac{29}{60}z\right)-sin\left(\frac{31}{60}z\right)-sin\left(\frac{29}{60}z\right)$$

$$-cos\left(\frac{62}{85}z\right)sin(z)+cos\left(\frac{62}{85}z\right)sin\left(\frac{31}{60}z\right)+cos\left(\frac{62}{85}z\right)sin\left(\frac{29}{60}z\right)+\cos\left(\frac{23}{85}z\right)sin(z)$$

$$-cos\left(\frac{23}{85}z\right)sin\left(\frac{31}{60}z\right)-cos\left(\frac{23}{85}z\right)sin\left(\frac{29}{60}z\right)-cos(z)sin\left(\frac{62}{85}z\right)-cos(z)sin\left(\frac{23}{85}z\right)$$

$$+sin\left(\frac{62}{85}z\right)sin\left(\frac{23}{85}z\right)+cos\left(\frac{31}{60}z\right)sin(z)-cos\left(\frac{31}{60}z\right)sin\left(\frac{62}{85}z\right)$$

$$-cos\left(\frac{31}{60}z\right)sin\left(\frac{23}{85}z\right)-cos\left(\frac{29}{60}z\right)sin(z)+cos\left(\frac{29}{60}z\right)sin\left(\frac{62}{85}z\right)$$

$$+cos\left(\frac{29}{60}z\right)sin\left(\frac{23}{85}z\right)] \qquad (29)$$

$$M_5=[-2sin(z^2)-4\,cos(z)+2zsin(z)+2sin\left(\frac{23}{85}z\right)sin(z)+zsin(z)cos\left(\frac{62}{85}z\right)$$

$$-zsin(z)cos\left(\frac{23}{85}z\right)+2sin\left(\frac{62}{85}z\right)sin(z)+2-4cos\lfloor z\rfloor-2cos\left(\frac{62}{85}z\right)+2cos\left(\frac{23}{85}z\right)$$

$$+2cos(z^2)+sin\left(\frac{62}{85}z\right)cos(z)+sin\left(\frac{23}{85}z\right)zcos(z)-3zsin\left(\frac{62}{85}z\right)-3zsin\left(\frac{23}{85}z\right)$$

$$+2cos(z)cos\left(\frac{62}{85}z\right)-2cos(z)cos\left(\frac{23}{85}z\right) \qquad (30)$$

$$M_6=[z[\cos(z)\,sin\left(\frac{31}{60}z\right)+cos(z)sin\left(\frac{29}{60}z\right)-sin\left(\frac{31}{60}z\right)-sin\left(\frac{26}{60}z\right)$$

$$-cos\left(\frac{62}{85}z\right)sin\left(\frac{31}{60}z\right)+cos\left(\frac{62}{85}z\right)sin\left(\frac{31}{60}z\right)+cos\left(\frac{62}{85}z\right)sin\left(\frac{29}{60}z\right)$$

$$+cos\left(\frac{23}{85}z\right)sin(z)-cos\left(\frac{23}{85}z\right)sin\left(\frac{31}{60}z\right)-cos\left(\frac{23}{85}z\right)sin\left(\frac{29}{60}z\right)$$

$$-cos(z)sin\left(\frac{62}{85}z\right)-cos(z)sin\left(\frac{23}{85}z\right)+sin\left(\frac{62}{85}z\right)+sin\left(\frac{23}{85}z\right)]+cos\left(\frac{31}{60}z\right)sin(z)$$

$$-cos\left(\frac{31}{60}z\right)sin\left(\frac{62}{85}z\right)-cos\left(\frac{31}{60}z\right)sin\left(\frac{23}{85}z\right)-cos\left(\frac{29}{60}z\right)sin(z)+cos\left(\frac{29}{60}z\right)sin\left(\frac{62}{85}z\right)$$

$$+cos\left(\frac{29}{60}z\right)sin\left(\frac{23}{85}z\right)]] \qquad (31)$$

$$M_7 = [2cos(z^2) - 4\,cos(z) + 2zsin(z) + 2 + cos\left(\frac{31}{60}z\right)zsin(z)$$

$$+2cos\left(\frac{31}{60}z\right)cos(z) - 2cos\left(\frac{31}{60}z\right) - cos\left(\frac{29}{60}z\right)zsin(z) - 2cos\left(\frac{29}{60}z\right)cos(z)$$

$$+2cos\left(\frac{29}{60}z\right) - 2sin(z^2) + 2\sin(z)\,sin\left(\frac{31}{60}z\right) + 2sin(z)sin\left(\frac{29}{60}z\right)$$

$$+cos(z)sin\left(\frac{31}{60}z\right) + zcos(z)cos\left(\frac{29}{60}z\right) - 3zsin\left(\frac{31}{60}z\right) - 3zsin\left(\frac{29}{60}z\right) \quad (32)$$

$$M_8 = [z[\cos(z)\,sin\left(\frac{31}{60}z\right) + cos(z)sin\left(\frac{29}{60}z\right) - sin\left(\frac{31}{60}z\right) - sin\left(\frac{26}{60}z\right)$$

$$-cos\left(\frac{62}{85}z\right)sin(z) + cos\left(\frac{62}{85}z\right)sin\left(\frac{31}{60}z\right) + cos\left(\frac{62}{85}z\right)sin\left(\frac{29}{60}z\right)$$

$$+cos\left(\frac{23}{85}z\right)sin(z) - cos\left(\frac{23}{85}z\right)sin\left(\frac{32}{60}z\right) - cos\left(\frac{23}{85}z\right)sin\left(\frac{23}{85}z\right)$$

$$+sin\left(\frac{62}{85}z\right) + sin\left(\frac{23}{85}z\right) + cos\left(\frac{31}{60}z\right)sin(z) - cos\left(\frac{31}{60}z\right)sin\left(\frac{62}{85}z\right)$$

$$-cos\left(\frac{31}{60}z\right)sin\left(\frac{23}{85}z\right) - cos\left(\frac{29}{60}z\right)sin(z) + cos\left(\frac{29}{60}z\right)sin\left(\frac{62}{85}z\right)$$

$$+cos\left(\frac{29}{60}z\right)sin\left(\frac{23}{85}z\right)] \quad (33)$$

The Taylor series expansions of the coefficients $b_{-1}, b_1, b_2, \text{and} b_3$ in (25) are obtained as

$$\left.\begin{aligned} b_{-1} &= \frac{157}{23064} - \frac{455}{553536}z^2 + o(z^4) \\ b_1 &= \frac{23221}{23064} - \frac{445}{553536}z^2 + o(z^4) \\ b_2 &= -\frac{1800}{6727} + \frac{65}{23064}z^2 + o(z^4) \\ b_3 &= \frac{122825}{161448} - \frac{1105}{553536}z^2 + o(z^4) \end{aligned}\right\} \quad (34)$$

From (34), it is evident that the classical IRK43 method, shown in Table 1, is recovered as z approaches zero. And in order to re-establish the order of the method as four, the Taylor expansions of the order conditions from the order conditions (16) is obtained after substituting the coefficients (34).

$$
\left.\begin{aligned}
&\text{First order :} && b_1 - b_{-1} = 1 + o(z^4) \\
&\text{Second order :} && b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2} + o(z^4) \\
&\text{Third order :} && \sum_{i=2}^{s} b_i\, c_i = \frac{5}{12} + o(z^4) \\
& && \sum_{i=2}^{s} b_i\, c_i^2 = \frac{1}{3} - \frac{91}{293760} z^2 + o(z^4) \\
&\text{Fourth order :} && \sum_{i=2,j=1}^{s} b_i a_{ij}\, c_j = \frac{1}{6} - \frac{91}{208080} z^2 + o(z^4)
\end{aligned}\right\} \qquad (35)
$$

From (35) it observed that as z approaches zero the order conditions (16) of the IRK methodsare recovered up to order four, implying that the coefficients of the TFIRK43 method satisfies the IRK order four conditions.

3. Numerical Examples

In the following numerical examples, the approximate solutions are sought on the partition$[x_0,\ x_N]$, and the errors are calculated at the endpoints as $|y_n - y(x_n)|$, where y_n and $y(x_n)$ are the approximate and exact solutions at the point x_n respectively. The sample problems are solved with the aid of Maple software and the results presented in Table 2 - 5

Example 1

$y'(x) = x\cos(x) + \sin(x), \quad y(0) = 0,$ Exact solution: $y(x) = x\sin(x), \qquad \omega = 1$

Example 2

$y'(x) = -\pi\sin(\pi x), \quad y(0) = 0;$ Exact solution: $y(x) = \cos(\pi x), \qquad \omega = \pi$

Table 2: Results of Example 1 on [0, 1], $h = 0.05$, $\omega = 1$

x	Exact	TFIRK43	Error	IRK43	Error
0.05	0.0024989585	4.9958341150E-03	2.4968756150E-03	4.9958341150E-03	2.4968756150E-03
0.10	0.0099833417	9.9833416599E-03	4.8246343683E-12	9.9833416502E-03	1.4478106272E-11
0.15	0.0224157199	2.2415719861E-02	9.6220972565E-12	2.2415719842E-02	2.8867686620E-11

0.20	0.0397338662	3.9733866145E-02	1.4380397506E-11	3.9733866116E-02	4.3120760966E-11
0.25	0.0618509898	6.1850989795E-02	1.9087641844E-11	6.1850989756E-02	5.7189775782E-11
0.30	0.0886560620	8.8656061975E-02	2.3732064611E-11	8.8656061927E-02	7.1027752760E-11
0.35	0.1200142326	1.2001423258E-01	2.8302057169E-11	1.2001423252E-01	8.4588435762E-11
0.40	0.1557673369	1.5576733689E-01	3.2786196916E-11	1.5576733683E-01	9.7826435600E-11
0.45	0.1957344904	1.9557344903E-01	3.7173275840E-11	1.9573449024E-01	1.1069737222E-11
0.50	0.2397127693	2.3971276926E-01	4.1452328526E-11	2.3971276918E-01	1.2315801384E-11
0.55	0.2874779759	2.8747797587E-01	4.5612659572E-11	2.8747797577E-01	1.3516641267E-11
0.60	0.3387854840	3.3887854840E-01	4.9643870317E-11	3.3878548389E-01	1.4668203676E-10
0.65	0.3933711637	3.9337116367E-01	5.3535884834E-11	3.9337116357E-01	1.5766589752E-10
0.70	0.4509523811	4.5095238101E-01	5.7278975112E-11	4.5095238090E-01	1.6808067100E-10
0.75	0.5112290700	5.1122906996E-01	6.0863785376E-11	5.1122906984E-01	1.7789082521E-10
0.80	0.5738848727	5.7388487266E-01	6.4281355468E011	5.7388487253E-01	1.8706271336E-10
0.85	0.6385883444	6.3858834430E-01	6.7523143240E-11	6.3858834417E-01	1.9556470038E-10
0.90	0.7049942187	7.0449942186E-01	7.0581045914E-11	7.0499421846E-01	2.0336725388E-10
0.95	0.7727447295	7.7274472948E-01	7.3447420323E-11	7.7274472934E-01	2.1044304064E-10
1.00	0.8414709848	8.4147098473E-01	7.6115102026E-11	8.4147098459E-01	2.1676701450E-10

Table 3: Results of Example 2 on [0, 1], $h = 0.05$, $\omega = \pi$

x	Exact	TFIRK43	Error	IRK43	Error
0.05	0.9876883400	-0.0245726683	1.0122610080+000	-0.0245726683	1.0122610080+000
0.10	0.9510565163	0.9510565163	4.5570800000E-25	0.9510565186	2.2762336676E-09

0.15	0.8910065242	0.8910065242	1.1772170000E-24	0.8910065286	4.4488188493E-09
0.20	0.8090169944	0.8090169944	2.1467630000E-24	0.8090170008	6.4642592875E-09
0.25	0.7071067812	0.7071067812	3.3404710000E-24	0.7071067895	8.2729281496E-09
0.30	0.5877852523	0.5877852523	4.7289480000E-24	0.5877852600	9.8302900058E-09
0.35	0.4539904997	0.4539904997	6.2780070000E-24	0.4539905108	1.1097997439E-08
0.40	0.3090169944	0.3090169944	7.9494970000E-24	0.3090170064	1.2044835283E-08
0.45	0.1564344650	0.1564344650	9.7022770000E-24	0.1564344777	1.2647489251E-08
0.50	0.0000000000	0.0000000000	1.1493168000E-23	1.2891119999	1.2891119999E-08
0.55	-0.1564344650	-0.1564344650	1.3278081000E-23	-0.1564344523	1.2769728531E-08
0.60	-0.3090169944	-0.3090169944	1.5013077000E-23	-0.3090169821	1.2286303908E-08
0.65	-0.4539904997	-0.4539904997	1.6655413000E-23	-0.4539904883	1.1452749648E-08
0.70	-0.5877852523	-0.5877852523	1.8164670000E-23	-0.5877852420	1.0289590624E-08
0.75	-0.7071067812	-0.7071067812	1.9503666000E-23	-0.7071067724	8.8254676701E-09
0.80	-0.8090169944	-0.8090169944	1.0639442000E-23	-0.8090169873	7.0964323538E-09
0.85	-0.8910065242	-0.8910065242	2.1544032000E-23	-0.8910065190	5.1450592626E-09
0.90	-0.9510565163	-0.9510565163	2.2195157000E-23	-0.9510565133	3.0193976782E-09
0.95	-0.9876883406	-0.9876883406	2.2576787000E-23	-0.9876883398	7.7178844349E-10
1.00	-1.0000000000	-0.9999999999	2.2679523000E-23	-1.0000000015	1.5424248428E-09

Table 4: Maximum errors for Example 1 on [0, 100], $\omega = 1$

h	TFIRK43	IRK43	NFEs
$\frac{1}{20}$	1.0291918182E-10	4.7352725236E-09	6000
$\frac{1}{40}$	3.1201619496E-12	1.4777985444E-10	12000
$\frac{1}{80}$	9.6134575308E-14	4.6151065805E-12	24000
$\frac{1}{160}$	1.2950265575E-14	1.4417554518E-13	48000
$\frac{1}{320}$	3.7783169258E-12	4.5047630316E-15	96000
$\frac{1}{640}$	5.1468240093E-11	1.4076266174E-16	192000

Table 5: Maximum errors for Example 2 on [0, 100], $\omega = \pi$

h	TFIRK43	IRK43	NFEs
$\frac{1}{20}$	1.6696492000E-23	1.6757378568E-08	6000
$\frac{1}{40}$	1.0045783210E-21	4.9322096769E-10	12000
$\frac{1}{80}$	2.0575750783E-20	1.4938496673E-11	24000
$\frac{1}{160}$	8.3151927728E-19	4.5942435517E-13	48000
$\frac{1}{320}$	2.0753909807E-17	1.4241446777E-14	96000
$\frac{1}{640}$	4.8402497964E-15	4.4324050858E-16	192000

4. Discussion of Results

Table 2 displays the results of comparison of performance between TFIRK43 and IRK43 when applied to solve Example 1 on the interval $[0, 1]$. It is observed that TFIRK43 performs better than the classical IRK43 by exhibiting lesser errors. Similarly, Table 3 shows the results of comparing the performance of TFIRK43 and IRK43 applied to the solution of the problem of Example 2. There is an observable display of better accuracy by TFIRK43, considering its minimal display of errors relative to the errors of IRK43. Table 4 gives the results of the maximum errors obtained on the interval $[0, 100]$ by TFIRK43 and IRK43 applied to Problem 1. There is a general observation of better performance by TFIRK even though this reduces as the step size tends towards zero. And, in Table 5, similar observation to the results of Table 4 is noted.

5. Conclusion

We have derived a fourth order three stage trigonometrically – fitted Improved Runge – Kutta method. The methods were applied to solve highly oscillatory initial value problems of first order ordinary differential equations. The results established that the trigonometrically – fitted method is more accurate than the non – fitted method. Also, the TFIRK43 requires only three function evaluations at each integration step and in general requires $\left(3 \cdot \left(\frac{T}{h}\right)\right)$ NFEs on the entire interval of integration.

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