DYNAMIC ANALYSIS OF NON-LINEAR CYLINDRICAL SHELL USING THE REGULAR PERTURBATION TECHNIQUE

BY

JIYA, MOHAMMED

Ph.D/SSSE/2004/127

DEPARTMENT OF MATHEMATICS/COMPUTER SCIENCE, FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA

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THESIS SUBMITTED TO THE POST GRADUATE SCHOOL IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF DOCTOR OF PHILOSOPHY (Ph.D) IN APPLIED MATHEMATICS IN THE DEPARTMENT OF MATHEMATICS/COMPUTER SCIENCE, FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA

OCTOBER, 2009

DECLARATION

hereby declare that the thesis titled DYNAMIC ANALYSIS OF NON-LINEAR YLiNDRICAL SHELL USING THE REGULAR PERTURBATION TECHNIQUE is an riginal work carried out by JIYA, Mohammed *(Ph.D/SSSE/2004/127)* under the supervision of Dr. Y. M. Aiyesimi. It has never been presented elsewhere for the award any degree; and that all works related to the field of study, before the present arch have been duly acknowledge and referenced

JIYA, Mohammed ~~~1~ \\'l-0» "'1

Name of Student Signature and Date

CERTIFICATION

fhis thesis titled: DYNAMIC ANALYSIS OF NON-LINEAR CYLINDRICAL SHELL SING THE REGULAR PERTURBATION TECHNIQUE by JIYA, Mohammed (Ph.D/SSSE/2004/127) meets the regulations governing the award of the degree of octor of Philosophy (Ph.D) of the Federal University of Technology, Minna and is approved for its contribution to scientific knowledge and literary presentation.

. Dr. Y. M. Aiyesimi — -----~----~l~[~d{

Major Supervisor — Signature and Date

Dr. S. Sadiku

Co- Supervisor

Dr. N. I. Akinwande — ---- --------------------j]~]~

Co-Supervisor — Signature and Date

Dr. N. I. Akinwande — -- -------- --------------------~[\\

Head of Department
Mathematics/Computer Science — Signature and Date

Prof. M. Galadima

Dean, School of Science and Science Education

Prof. S. L. Lamai — ~ ~J]~]_~~_

Dean, Post Graduate School — Signature and Date

DEDICATION

iThisthesis is dedicated to all those who have contributed in making me what I am.

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ABSTRACT

Ihe research work focuses on the dynamic analysis of the non-linear circular .ylindrical shell. The mathematical model of the problem is based on onnell-Mushtari-Vlasov theory of shallow shells. The ensuing nonlinear partial fifferential equation is approximated by asymptotic series, resulting from regular aerturbation technique and the obtained equations were solved analytically. The ases of free and forced vibration were considered. In the latter, the cases of ulsating pressure and a moving force having constant magnitude were studied. he stability of the cylindrical shell with or without initial imperfection was also tudied. The result indicates exponential decay, away from the edge of the shell, vhich is one of the unique characteristics of a shell. From the numerically imulated results it is observed that higher modes of vibration can be neglected, is is because the contribution to the dynamic displacement is mainly in the first ode and the characteristic shape of the first mode is similar to the force istribution on the system. It is also observed that the increase in the excitation mplitude produces a wrinkling effect on the shell, which results in the shell eformation.

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CHAPTER ONE

1.0 GENERAL INTRODUCTION

1.1 Background of the Study

The word shell is an old one and is commonly used to describe the hard covering of eggs, crustacea, tortoises, etc. The dictionary says that the word shell is derived from the Latin scalus, as in fish scale. But to us now there is a clear difference between the tough but flexible scaly covering of a fish and the tough but rigid shell of, say, a turtle. In this work we shall be concerned with man-made shell structures as used in various branches of Science and Engineering. Shells are classified by their geometry (cylindrical, spherical, elliptic paraboloid, hyperbolic paraboloid, circular cone shell etc.). Many cylindrical shells were analyzed using approximate methods, in that when extended in the long direction they approach beams in behavior, and when shortened in the same direction approach arches in behavior. Hence, they fall between the limiting cases of beams and arches, Richard et.al (2002). There are many interesting aspects of the use of shells in Science and Engineering, but one alone stands out as being of paramount importance namely the structural aspect. The list of applications from a historical point of view, and, to take as a connecting theme, the way in which the introduction of the thin shells have made important contributions to the development of several branches of Science and Engineering is given below.

(i) Architecture and Building: The development of masonry domes and vaults in the middle ages made possible the construction of more spacious buildings. In more recent times the availability of reinforced concrete has simulated interest in the use of shells for roofing purposes

(ii) Power and Chemical Engineering: The development of steam power during the industrial revolution depended to some extent on the construction of suitable boilers. These thin shells were constructed from plates suitably formed and joined by riveting and/or welding. More recently the use of welding in pressure vessel construction has led to more efficient designs. Pressure vessels and associated pipe work are key components in thermal and nuclear power plants, and in all branches of the chemical and petroleum industries.

(iii) Structural Engineering: An important problem in the early development of steel for structural purposes was to design compression members against buckling. A striking advance was the use of tabular members in the construction of the Forth railway bridge in 1889: steel plates were riveted together to form reinforced tubes as large as 12 feet in diameter and having a radius/thickness ratio of between 60 and 180.

(iv) Vehicle Body Structure: The construction of vehicle bodies in the early days of road transportation involved a system of structural ribs and non structural paneling or sheeting. The modern form of vehicle construction, in which the skin plays an important structural part, followed the introduction of sheet-metal components, performed into thin doubly curved shells by large power presses, and firmly connected to each other by welds along the boundaries. The use of curved skin of vehicles as a load bearing member has similarly revolutionalized the construction of railway carriages and aircrafts. In the construction of all kinds of spacecraft, the idea of a thin but strong skin has been used from the beginning.

(v) Composite Construction: The introduction of fiberglass and similar lightweight composite materials has had a great impact on the construction of vehicles ranging from boats, racing cars, fighter and stealth aircraft, and so on. The exterior skin can be used as a strong structural shell.

Miscellaneous Examples: Other examples of the impact of shell structures include water cooling towers for power stations, grain silos, armour, arch dams, tunnels, submarines, and so forth.

1.1.1 Thin Shell

Thin shells are shell structures that have $h \ll R$, where h is the shell thickness and R the mean shell radius, h is small compared with its other dimensions and compared with its principal radius of curvature. Middle surface of thin shells is the surface that bisects the shell. It specifies the form of this surface and the thickness h at every point. Thin shells are considered form resistant structures, as they resist loads by virtue of their shape. It will not function if flat, and carry loads predominantly through in-plane stresses rather than by bending, granted that thin shells bend as well as compress.

1.1.2 Classification in Terms of Thickness Ratio

Shell mathematical models can be classified in terms of the ratio of the thickness to a characteristic dimension:

~Very thick: 3D effects

~Thick: stretching, bending and higher order transverse shear

~Moderately thick: stretching, bending and first order transverse shear

~ Thin shells: stretching and bending energy considered but transverse shear neglected

~Very thin shells: dominated by stretching effects. Also called membranes.

The main difference from flat plates is that the determination of characteristic dimensions is more complex.

1.1.3 Closed and Open Shell

Before describing the main body of the theory it is useful to discuss quantitatively an important practical point that state that closed box is rigid, whereas an open box is easily deformable for example aluminum can easily be squashed far more easily after the ends have been removed . In practice, it is not possible to make completely closed structural boxes. In a ship for example, there will be various cutouts in the deck for things such as hatches and stairways. It is sometimes possible to close such opening with doors, hatch covers that provide continuity. A more extreme example is provided by shell roofs in general. Here the shell is usually very open, being merely a cap of a shell, and the provision of adequate edge rips, together with suitable supports, is of crucial importance. A main objective in the design of shell roofs is to eliminate those aspects of behavior that spring from open nature of the shell. The effect of small cutouts on the overall rigidity of a shell may be trivial; the effect of a large cutout can be serious.

1.1.4 A Simple Geometric Approach

The notion that a closed surface is rigid is well known in the field of pure Euclidean geometry. There is a theorem of Cauchy which states that a convex polyhedron is rigid. The concept of rigidity is, of course, hedged around with suitable restrictions, but will be an obvious one to anybody who has made cardboard cutout models of polyhedral. It is significant that the qualifier *convex* appears in the theorem. Although it is possible to demonstrate by means of simple examples that some non-convex polyhedral (that is, polyhedral with regions of non-convexity) are rigid, it is also possible to demonstrate special cases of non-convex polyhedral which are not rigid, and are capable of undergoing infinitesimal distortions at least. This is a accurate results only for very thin shells, and also we consider the simply supported boundary condition which is most commonly used in practical applications.

1.4 Motivations/Significance of the Study

The motivations for this study come about from growing number of applications involving shell models for which explicit solutions are typically unavailable. The problems of nonlinear vibration of shell have received considerable attention but not sufficient enough as many questions await complete clarifications. The sources of the nonlinearities in the governing equations may be geometric, inertial, material, or any combination. The geometric nonlinearity stems from nonlinear strain-displacement relations, the inertial nonlinearity may be caused by the presence of concentrated or distributed masses, and the material nonlinearity occurs when the stresses are nonlinear functions of the strain. It is largely for this reason that the subject of shell structures generally is a difficult one.

1.5 Methodology

We consider the approximation of the model equation using a regular perturbation technique, where a small perturbed parameter $\varepsilon \ll 1$ which emerges naturally by introducing the dimensional quantities, is used to reduce the nonlinear equation and thereby solving the resulting linear equation analytically. Hence we obtain the analytical approximations both for free and forced vibrations of the cylindrical shell that reveal the essential dependence of the exact solution on the parameter in a more satisfactory way. Stability of shell was also analyzed; were the initial imperfection was taken into consideration.

1.6 Layout of Thesis

The first chapter is an introduction; it introduces the concept of shells, the aims and objective of the study, the methodology adopted are all explained in this chapter.

Chapter two is on literature review about relevant previous works carried out on shell structures. Mathematical review was also included were various methods of solving Partial Differential Equation both analytical and numerical approaches were highlighted and discussed. The semi analytic method Perturbation method which is also the approach used for the study was also discussed.

Chapter three the governing differential equation for the system was stated. The solutions of both free/forced vibration of the cylindrical shell were obtained. The stability of the cylindrical shell with or without initial imperfection was also analyzed.

Chapter four, deals with the numerical simulation of the results obtained in last chapter (chapter three). Graphs were plotted to show the effect of various parameters on the amplitude of vibration.

Finally, Chapter five gives discussion of the results obtained, conclusions on the present study and recommendations for future research.

Additionally, there is an appendix providing results of some terms in the solution.

1.7 Definition of Terms

I. Large Amplitude: in context of shell theory large amplitudes signify amplitudes exceeding the shell thickness, or of several times. The shell thickness, which in other context may still be considered to be small.

II. Middle Surface: The surface that bisects the shell is called the middle surface.

iii. Damping: Retardation of oscillatory motion due to an opposing medium

iv. Resonance: Is a technical term that describes the sudden amplification of amplitude of vibration when the frequency of oscillation of the driving unit approaches the oscillating unit

v. Buckling: Is the phenomenon that accounts for when the inplane loads becomes compressive, upon attaining certain discrete values, these compressive loads do result in producing lateral displacement, thus, there does occur a coupling between inplane loads and lateral displacement

vi. Stress: Field created by a force on a plane of body

vii. Strain: Displacement caused by stress field

viii. Rotary Inertia: The reluctance of a body to start circular motion

ix. Rigidity: The resistance of a body to deform on application of stress

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Previous Related Works

A number of non-linear governing equations have been proposed for the dynamic response of shells. They include the theories of Donnell (1934), Sanders (1963) and Reissner (1955).The main differences among these theories are the approximations used in relating the strains and curvatures to the displacements. Donnell's theory is the most widely used of all these theories. Donnell (1934) established the nonlinear theory of circular cylindrical shells under the simplifying shallow-shell hypothesis. Due to its relative simplicity and practical accuracy, this theory has been widely used. The most frequently used form of Donnell's nonlinear shallow-shell theory (also referred to as Donnell-Mushtari-Vlasov theory) introduces a stress function in order to combine the three equations of equilibrium involving the shell displacements in the radial, circumferential, and axial directions into two equations involving only the radial displacement *Wand* the stress function *F.*

fig. 2.1 showing the shell coordinates and the geometry

Evensen (1963) noted in his experiments that the nonlinearity in closed shells is of the *softening* type and *weak.* Evensen and Fulton (1967) used Donnell's nonlinear shallow-shell theory, but with a different form for the assumed flexural displacement w, involving more modes. Specifically, he included the companion mode in the analysis, as well as an axisymmetric contraction having twice the frequency of the mode excited:

$$w = [A_{mn}(t)\cos n\theta + B_{mn}(t)\sin n\theta]\sin(m\pi x/L) + (n^2/4R)[A_{mn}^2(t) + B_{mn}^2(t)]\sin^2(m\pi x/L) \qquad 2.1$$

where *n* is the circumferential wave number, m is the number of axial half-waves, $A_{mn}(t)$ and $B_{mn}(t)$ are two time functions associated to the driven and companion mode, respectively. Evensen's assumed modes are not moment-free at the ends of the shell, as they should be for classical simply supported shells, and the homogeneous solution for the stress function is neglected; however, the continuity of the circumferential displacement is exactly satisfied. Evensen studied the free vibrations and the response to a modal excitation without considering damping and discussed the stability of the response curves. Mayers and Wrenn (1967) analyzed free vibrations of thin, complete circular cylindrical shells. They used both Donnell's nonlinear shallow-shell theory and the Sanders-Koiter nonlinear theory of shells. Their analysis is based on the energy approach and shows that non-periodic (more specifically, quasiperiodic) motion is obtained for free nonlinear vibrations. In their analysis based on Donnell's theory, the same expansion introduced by Evensen, without the companion mode, was initially applied; the backbone curves (pertaining to free vibrations) of Evensen were confirmed almost exactly. A second expansion with an additional degree of freedom was also applied for shells with many axial waves; finally, an expansion with more axisymmetric terms was introduced, but the corresponding backbone curves were not reported. In their analysis based on the Sanders-Koiter theory, an original expansion for the three shell displacements (i.e., radial, circumferential, and axial) was used, involving seven degrees of freedom. However, only one term was used for the flexural displacement, and the axisymmetric radial contraction was neglected; axisymmetric terms were considered for the in-plane displacements. The authors found that the backbone curves for a mode with two circumferential waves predict a hardening-type nonlinearity which increases with shell thickness. Dowell and Ventres (1968) used a different expansion and approach in order to satisfy exactly the out-of-plane boundary conditions and to satisfy *on the average* the in-plane boundary conditions. They studied shells with restrained inplane displacement at the ends and obtained the particular and the homogeneous solutions for the stress function. Their interesting approach was followed by Atluri (1972) who found that some terms were missing in one of the equations used by Dowell and Ventres (1968); recently, Dowell *et al* (1998) corrected these omissions. The boundary conditions assumed both by Dowell and Ventres and by Atluri constrain the axial displacement at the shell extremities to be zero, so that they are different from the classical constraints of a simply supported shell (zero axial force at both ends). Atluri (1972) found that the nonlinearity is of the hardening type for a closed circular cylindrical shell, in contrast to what was found in experiments. The axisymmetric term used by Dowell and Ventres and Atluri in their mode expansion is a sine in the axial coordinate (specifically, it is the first axisymmetric mode of linear vibrations) and has an independent time variation, i.e.

$$w = [A_{mn}(t)\cos n\theta + B_{mn}(t)\sin n\theta]\sin(m\pi x/L) + [A_{mo}(t)]\sin(m\pi x/L) \qquad 2.2$$

Their approach was criticized by Evensen (2000) because it gives a hardening-type result; he pointed out that satisfaction of continuity of the circumferential displacement *on the average* is not a good enough approximation, in view of its importance in nonlinear vibrations. Amabili *et al* (2000) showed that at least the first and third axisymmetric modes (axisymmetric modes with an even number of longitudinal half-waves do not give any contribution) must be included in the mode expansion (for modes with a single longitudinal half-wave), as well as using both the driven and companion modes, to correctly predict the trend of nonlinearity with sufficiently good accuracy: Leissa and Kadi (1971) studied linear and nonlinear free vibrations of shallow shell panels, simply supported at the four edges without in-plane restraints. Panels with two different curvatures in orthogonal directions were studied. This is the first study on the effect of curvature of the generating lines on large-amplitude vibrations of shallow shells. Donnell's nonlinear shallow-shell theory was used in a slightly modified version to take into account the meridional curvature. A single mode expansion of the transverse displacement was used. The Galerkin method was applied; the compatibility equation was exactly satisfied, and in-plane boundary conditions were satisfied on the average. Results were obtained by numerical integration and non-simple harmonic oscillations were found. For cylindrical and spherical panels, phase plots show that, during vibrations, inward deflections are larger than outward deflections, as previously found by Reissner (1955).Chen and Babcock (1975) used the perturbation method to solve the nonlinear equations obtained by Donnell's nonlinear shallow-shell theory, without selecting a particular deflection solution. They solved the classical simply supported case and studied the driven mode response, the companion mode participation, and the appearance of a travelling wave. A damped response to an external excitation was found. The solution involved a sophisticated mode expansion, including boundary layer terms in order to satisfy the boundary conditions. They also presented experimental results in good agreement with their theory, showing a softening nonlinearity. Regions with amplitude-modulated response were also experimentally detected. Radwan and Genin (1976) derived nonlinear modal equations by using the Sanders-Koiter nonlinear theory of shells and the Lagrange equations of motion, taking into account imperfections. However, the equations of motion were derived only for perfect, closed shells, simply supported at the ends. The nonlinear coupling between the linear modes, that are the basis for the expansion of the shell displacements, was neglected. As previously observed, this single-mode approach gives the wrong trend of nonlinearity for a closed circular shell. The numerical results give only the coefficients of the Duffing equation, obtained while solving the problem Harari (1976) used a perturbation method to study the nonlinear equations of motion of shallow shells and plates. Applications are given only for plates and no numerical results are presented Nayfeh and Raouf (1987) studied vibrations of closed shells by using plane-strain theory of shells and a perturbation analysis; thus, their study is suitable for rings but not for supported shells of finite length. They investigated the response when the frequency of the axisymmetric mode is approximately twice that of the asymmetric mode (two-to-one internal resonance). The phenomenon of saturation of the response of the directly excited mode was observed. Raouf and Nayfeh (1990) studied the response of the shell by using the same shell theory previously used by Nayfeh and Raouf (1987), retaining both the driven and companion modes in the expansion, finding amplitude-modulated and chaotic solutions. Only two degrees of freedom were used in this study; an axisymmetric term and terms with twice the number of circumferential waves of the driven and companion modes were obtained by perturbation analysis and added to the solution without independent degrees of freedom. The method of multiple scales was applied to obtain a perturbation solution from the equations of motion. By using a similar approach, Nayfeh *et al* (1991) investigated the behavior of shells, considering the presence of a two-to-one internal resonance between the axisymmetric and asymmetric modes for the problem previously studied by Nayfeh and Raouf (1990).Chiba (1993) studied experimentally large-amplitude vibrations of two cantilevered circular cylindrical shells made of polyester sheet. He found that responses of almost all modes display a softening nonlinearity. He observed that for modes with the same axial wave number, the weakest degree of softening nonlinearity can be attributed to the mode having the minimum natural frequency. He also found that shorter shells have a larger softening nonlinearity than longer ones. Travelling wave modes were also observed. Ganapathi and Varadan (1996) used the finite element method to study large-amplitude vibrations of doubly-curved composite shells. Numerical results were given for isotropic circular cylindrical shells. They showed the effect of including the axisymmetric contraction mode with the asymmetric linear modes, confirming the effectiveness of the mode expansions used by many authors, as discussed in the foregoing. Only free vibrations were investigated in the paper, using Novozhilov's theory of shells. A four-node finite element was developed with five degrees of freedom for each node. Ganapathi and Varadan also pointed out problems in the finite element analysis of closed shells that are not present in open shells. The same approach was used to study numerically laminated composite circular cylindrical shells. Selmane and Lakis (1997) applied the finite element method to study free vibrations of open and closed orthotropic cylindrical shells. Their method is a hybrid of the classical finite element method and shell theory. They used the refined Sanders-Koiter nonlinear theory of shells. The formulation was initially general but in the end, to simplify the solution, only a single linear mode was retained. As previously discussed, this approximation gives erroneous results for a complete circular shell. In fact, numerical results for free vibrations of the same closed circular cylindrical shell, simply supported at the ends. Amabili *et al (1998)* investigated the nonlinear free and forced vibrations of a simply supported, complete circular cylindrical shell, empty or fluid-filled. Donnell's nonlinear shallow-shell theory was used. The boundary conditions on the radial displacement and the continuity of circumferential displacement were exactly satisfied, while the in-plane constraints were satisfied on the average. Galerkin projection was used and the mode shape was expanded by using three degrees of freedom; specifically, two asymmetric modes (driven and companion modes), plus an axisymmetric term involving the first and third axisymmetric modes (reduced to a single term by an artificial constraint), were employed. The time dependence of each term of the expansion was general. Different axial constraints were imposed at the shell ends. Coupling with an in viscid fluid was considered. Solutions were obtained both numerically and by the method of normal forms. Numerical results were obtained for both free and forced vibrations of empty and water-filled shells. Vol'mir and Ponomarev (1973) studied the nonlinear response of orthotropic circular cylindrical shells subjected to *(i)* a time-varying axial load (constant load plus harmonic component) plus a static external pressure and *(ii)* a harmonic external pressure and a static axial load. They used Donnell's nonlinear shallow-shell theory and a mode expansion involving three terms: one asymmetric and two axisymmetric. In particular, the first axisymmetric term is analogous of that used by Evensen and the second one is just a constant radial displacement of the shell. Continuity of the circumferential displacement was satisfied. The system was reduced to one of a single degree of freedom and the equation of motion shows nonlinearity, a nonlinear damping coupled to the radial displacement *w,* and a quadratic term coupled to the radial accelerations. Results show both hardening and softening nonlinearity, depending on the system parameters. Linear stability of the Mathieu equation was also investigated. Nagai and Yamaki (1978) studied circular cylindrical shells subjected to compressive axial periodic forces (constant force plus harmonic load) by means of Donnell's nonlinear shallow-shell theory. The superposition of an unperturbed axisymmetric vibration of the shell and an asymmetric incremental deformation was assumed. Results show that the shell has generally bounded solutions for the radial displacement and the unperturbed axisymmetric vibration is stable. However, under specific axial forces, having specific constant and variable amplitude and a particular frequency, the shell can become unstable. The stability was studied for different shell characteristics and boundary conditions. Nonlinear vibrations of shells of revolution under constant plus harmonic loads were studied by Gotsulyak *et al* (1982) using a perturbation approach. The method was applied to shallow spherical shells under uniformly distributed constant plus harmonic loads. Results show that a critical load value gives a bifurcation point, from which a branch, corresponding to a non-axisymmetric solution, emerges. Popov *et al* (1998) investigated large-amplitude vibrations of complete, infinitely long circular cylindrical shells with axially periodic deformations, excited by an axial force (constant plus harmonic load). Donnell's nonlinear shallow-shell theory was utilized. The equations of motion were obtained by using an energy approach, including dissipation. Discretization was obtained by using a mode expansion with four degrees of freedom, capable of describing the well-known diamond pattern of buckled shells, but neglecting the companion mode. Moreover, some results were obtained with a reduced expansion. The bifurcation analysis and the study of stability under a harmonic axial load were performed, with models having one or two degrees of freedom. Gonc.alves and Del Prado (2000) investigated the nonlinear response of infinitely long circular cylindrical shells with axially periodic deformations subjected to axial excitations (constant force plus harmonic load). They found that the constant axial force necessary to reach instability must be lower than the static critical load for the softening character of the post-buckling response. Donnell's nonlinear shallow-shell theory was used and the solution was obtained by Galerkin projection. Both a simple two-mode expansion of the flexural displacement and a refined 18-mode expansion were used; both of them neglected the companion mode. Jumps to the bifurcated equilibrium position under excitation were found, and the convergence of the solution with the number of modes employed in the expansion was numerically verified. They also studied the effect of axial pre-stress on the free vibration (backbone curve) of circular cylindrical shells. It is seen that the axial load largely increases the softening-type nonlinearity of the shell, especially when the critical load P_{cr} is approached. Vol'mir et. al. (1973) studied nonlinear oscillations of simply supported, circular cylindrical panels and plates subject to an initial deviation from the equilibrium position (response of the panel to initial conditions) by using Donnell's nonlinear shallow theory. Results were calculated by numerical integration of the equations of motion obtained by Gerlerkin projection, retaining three or five modes in the expansion.

Mikhlin (2000) studied vibrations of circular cylindrical shells under a radial excitation and an axial static load, using Donnell's nonlinear shallow-shell theory with Gerlerkin projection and two different mode expansions. Amabili (2005) studied Large amplitude (geometrically non-linear) vibrations of doubly curved shallow shells with rectangular base, simply supported at the four edges and subjected to harmonic excitation normal to the surface in the spectral neighbourhood of the fundamental mode are investigated. Two different non-linear strain-displacement relationships, from the Donnell's and Novozhilov's shell theories, are used to calculate the elastic strain energy. In-plane inertia and geometric imperfections are taken into account. The solution is obtained by Lagrangian approach. The non-linear equations of motion are studied by using (i) a code based on arc length continuation method that allows bifurcation analysis and (ii) direct time integration. Amabili et. al. (2003) experimentally studied large amplitude vibrations of a stainless-steel circular cylindrical panel supported at four edges. The nonlinear response to harmonic excitation of different magnitudes in the neighborhood of three resonances was investigated. Experiments showed that the curved panel exhibited a relatively strong geometric nonlinearly of softening type. Nayfeh (1983) used a perturbation technique to reduce the eight-order vibration problem for presstred, clamped cylindrical shells to an equivalent sixth-order membrane problem. In the transformation to a membrane problem composite expansion are utilized, uniformly over the length of the shell, to formed modified boundary conditions that account for the effects of bending near the shells.

From the above reviews it can be noticed that the solutions to the nonlinear partial differential equation governing shell motion can be broadly classified into three approaches: purely numerical methods, perturbation methods, and a combination of the Galerkin procedure with either perturbation or numerical methods. Where perturbation techniques were used in the literature, linear models were mostly employed for the study where asymptotically closed form of solution were obtained, also in most of these works the lateral distributed loading were not considered. We considered in this project nonlinear cylindrical shell subjected to pulsating force, moving force and the axial-in plane stress function as that of an equivalent lateral distributed load, which involves solving the two partial differential equations simultaneously.

2.2 Geometric Analysis of shell

Consider the position vector

where *.t;(a.j3),f2(a.j3)andf3(a.j3)are* continuous, singled valued functions. The surface is determined by *a* and *j3* uniquely. *a* and *j3* are called curvilinear coordinates. *u,* v and ware unit vectors in the Cartesian coordinate system.

Orthogonality is addressed as

$\frac{ar}{aa} \cdot \frac{ar.}{aj3} = 0$ Inner product İSzero

The distance between *(a.j3)and(a+da.j3+dj3)is*

$$ds = \frac{ar}{aa} da + \frac{ar}{ajJ} djJ = 0$$

The scalar product of ds with itself is

Where A and B are called Lame's parameters or measure numbers. The formula shown above is the first quadratic form of the theory of surfaces.

~ Gaussian Curvature

where $r,$ and r_y are the principal radii of curvature

- $k > 0$: Synclastic shells, i.e., spherical domes and elliptic paraboloids.

- $k = 0$: (either -, or r_y is zero): Single-curvature shells, i.e., cylinders and cones.

- $k < 0$: Anticlastic shells, i.e., hyperbolic paraboloids and hyperbolas of revolution.

~ Stress Resultants and Stress Couples of Cylindrical Shells

aN
~ +_!!_d¢
O ç

O +----:-:"=:"dx
_x Ox

I'

x

fig 2.2 showing stress resultant of a cylindrical shell

statically indeterminate.

In most cylindrical shells. u, will be small and Q_x will be small also M_{xo}, M_{ex} are

~ Classifications of Shells

Oral (2004) classifies shells as follows:

i. Long shells $\frac{L}{r} \sim 2.5$

Line loads produce significant magnitudes of $M()$ and $Q()$ membrane forces become insignificant. Stresses can be estimated using the beam theory.

ii. Intermediate shells $0.5 \sim \frac{L}{r} < 2.5$

iii. Short shells $\frac{L}{r} < 0.5$

The line loads produce internal forces generally in the region near the longitudinal edge. Greater part of the shell behaves with membrane value.

Beam Theory for long cylindrical shells

- ~ For long shells the stresses can be estimated closely by beam theory. The shell is considered as a beam of a curved cross section between end supports.
- ~ Assumption: relative displacements within each transverse cross section are negligible.

Membrane theory

- ~ For a certain class of shells which the stress couples are an order of magnitude smaller than the extensional and in-plane shear stress resultants, the transverse shear stress resultants are similarly small and may be neglected in the force equilibrium.
- ~ The assumption is valid only if at least one radius of curvature is finite. (Flat plates are excluded from resisting transverse loading in this manner.)

- The shell may achieve force equilibrium through the action of inplane forces alone. Hence, the state of state of stress in the shell is completely determined by equations of equilibrium i.e., to shell is statically determinate.
- The boundary conditions must provide for those shell edge forces which are computed from the equations of equilibrium. The boundary conditions must also [permit those shell edge displacement (translation and rotations) which are computed from the forces found by the membrane theory.

2.3 Partial Differential Equation

A partial differential equation is an equation relating an unknown function (the dependent variable) of two or more variables with one or more of its partial derivatives with respect to those variables. The order of the highest derivative is called the order of the equation.

The equation

$$A(x,y)u_{xx}+ B(x,y)u_{xy}+ C(x,y)u_{yy}+ O(x,y)u_{x}+ E(x,y)u_{y}+ F(x,y)u + G(x,y) = /(x,y) \qquad 2.3$$

eqn. (2.3) is the general second order linear Partial differential equation in two independent variable. If $/(x, y) = 0$ then (2.3) is homogeneous else is non-homogeneous.

2.3.1 Linear and Nonlinear Partial Differential Equations

A POE is linear if the dependent variable and its derivatives appear in linear combination. When the POE is not linear, a distinction is made between so-called quasilinear POEs and Fully Nonlinear POEs. The former is defined as an equation, in which the highest derivative is still linear, but not necessarily the lower derivatives or the dependent variable itself. For example:

Quasilinear: $u_{xx} = u_{yy} + u_t^2$ 2.4

Fully nonlinear: $u_{tt} = u_{xx} + u_{yy}^2$

TABLE 2.1: Elliptic, Parabolic, and Hyperbolic Second Order PDEs

Criteria	Type of POE	Example	Properties
$B^2 - AC < 0$	Elliptic	Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0$	Boundary Value Problem
$B^2 - AC = 0$	Parabolic	Fourier's equation $a\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$	Mixed BV and IV problem
$B^2 - AC > 0$	Hyperbolic	Wave equation $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$	Mixed BV and IV problem or IV problem

and the set criteria are as follows:

TABLE 2.2: Set Criteria in vector-matrix form

The Set is	If the eigenvalues of $\det\lvert A - \lambda B\rvert = 0$
Elliptic	Imaginary
Parabolic	Real and identical
Hyperbolic	Real and distinct

2.3.2 Methods of Solving Partial Differential Equations

There are various methods of solving partial differential equation which can be classified as analytic or numerical

(a) Analytical Methods

These methods produce exact and closed form solution of the partial differential equation the following are among the most important;

(i) Separation of Variables/Superposition Methods

Suppose a solution of the form $U\{x, y, z, t)$is sought for some partial differential equation (expressed in Cartesian co-ordinate). If the product of the form can be obtained.

$$U(x,y,z,t)= X(x)Y(y)Z(z)T(t) \neq 0 \qquad 2.5$$

A solution of the form (2.5) is said to be separable in $x, y,$ z and t seeking solution of this form is called the method of separation of variables.

(ii) Laplace Transformation

If $f\{t$)is a given function that is defined for all $t > 0$. Then we multiply $f\{t)$ by

e^{-st} and integrate with respect to *tfrom* zero to infinity. Then, if the resulting integral exists, it is a function of s, say F(s)(sbeing the Laplace parameter).

$$F(s) = \int_0^{\infty} e^{-st} J(t)\,dt$$

The function $F(s)$ of the variable s is called the Laplace transform of the original function $J(t)$ and will be denoted by $L(J)$ thus

$$F(S) = L(J) = \int_0^{\infty} e^{-st} J(t)\,dt \qquad 2.6$$

The operator described, which yields $F(s)$ from given $J(t)$, is called the Laplace transformation. The original function $f(t)$ in eqn. (2.5) is called the inverse transform or inverse of $J(t)$ and it is denoted by $L^{-'}(F)$; that is $J(t) = L^{-'}(F)$

(iii) Application of Laplace Transformation to Partial Differential Equations

Laplace transformation is a very useful tool in the solution of certain class of initial value problems and it is carried out as follows:

~ The Laplace transformation with respect to one of the two or more variables in the time domain, usually t is taken. This gives rise to an equation for the transform of the unknown function. This is so since the derivatives of this function with respect to other variable slip into the transformed equation. The latter also incorporate the given boundary and initial condition.

~ The ordinary differential equation is solved to obtain the transform of the unknown function.

- Applying the inverse transformation to the result, obtain the solution of the given problem.

If the coefficient of the given equation does not depend on t, the transformation will simplify the problem. This method is usually applied for IVPs.

(iv) Fourier Transformation

Given the equation

$$\tilde{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

Then $\tilde{f}_c(w)$ is called the Fourier Cosine transform of $f(x)$.

Then $f(x)$ is called the inverse Fourier Cosine transform of $\tilde{f}_c(w)$. The process of obtaining the transform $\tilde{f}_c(w)$ from $f(x)$ is called the Fourier Cosine Transformation.

Also given the equation

$$\tilde{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

Then $\tilde{f}_s(w)$ is called the Fourier sine transform of $f(x)$.

If $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{f}_s(w) \sin wx \, dx$

Then $[(x)$ is called the inverse Fourier Sine transform of $!sew)$. The process of obtaining the transform $!sew)$ from $[(x)$ is called the Fourier Sine Transformation.

Similarly given the equation,

$$few) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [(x)e^{-iwx} \quad dw$$

Then [(x)is called the inverse Fourier transform of $few)$ and it is denoted by

$i = F^{-1}c!(w))$. The process of obtaining the Fourier transform $F([) = !$ from a given [is called Fourier transformation.

(v) Application of Fourier Transformation to Partial Differential Equations

Fourier series are applied in the solution of partial differential equation in a way that if the initial boundary data of problem is given on the positive half-plane (Right or Left).The Fourier Cosine or Sine transformation may be appropriate but in case where the initial boundary data are not given the Fourier transformation is used.

(b) Numerical Methods

The process of solving partial differential equation with its auxiliary conditions can be reduce to a finite number of arithmetical calculation that can be carried out by computer in an iterative procedure. Hence numerical methods are appropriate schemes of solving partial differential equation. These include the following:

(i) Finite Difference

Finite difference consists of replacing each derivatives by a quotient.

~ Forward Difference Approximation

$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \quad + S,$

truncating the series after the first derivative yields

we then have

$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + o(h)$

$f(x_{i+1}) - f(x_i)$ is called the first forward difference and h is called the step size, that is, the length of the interval (mesh) over which the approximation is made is termed the forward difference because it is utilizes data at i and $i + 1$ to estimate the derivative

~ Backward Difference Approximation

The Taylors series can be expanded backward to calculate a previous value on the basis of a present value as in.

$f(x_{i-1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \quad + R_n$ 2.7

truncating this equation after the first derivative and rearranging yields

$f'(x_i) = f(X_i)\sim f(X_{i-1}) \quad + o(h) = \nabla f'$

where the error is O(h)and Vi; is referred to as the first backward difference.

~ Central Difference Approximation

A third way to approximate the first derivative is to subtract equation (2.6) from the forward Taylor's series expansion

$f(x_{i+1}) = f(x_i) + hf'(x_i) + \sim: r(x_i) +$.

to yield

$f(x_{i+1}) = f(x_{i-1}) + 2hf'(x_i) + \sim: r(x_i) +$.

which can be solved for

$f'(x_i) = f(x'\cdot')2\sim f(x_{i-1}) + h: r(x_i) +$.

Hence

2.8

eqn. (2.7) is centered difference representation of the first derivative. The Taylor's series analysis yields the practical information at the centered difference is a more accurate representation of the derivatives. For example, if we have a step size using a forward or backward difference, we would approximately halve truncation error whereas for centered difference the error would be quartered.

~ Relationship Between the Difference Operators

from the Taylors series expansion

$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \cdots + R_n$

but in general $f^{(n)}(x_i) = D^n f(x_i)$

hence

$f(x_{i+1}) = \left[1 + hD + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \cdots\right] f(x_i) + R_n$

$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \cdots + R_n$

where $e^{hD} = \left[1 + hD + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \cdots\right] + R_n$

but from shift operator $f(x_{i+1}) = Ef(x_i)$

we have $[E - e^{hD}] f(x_i) = 0$

since $f(x_i) \neq 0$ then $E = e^{hD}$

also from the Taylors series expansion

$f(x_{i-1}) = f(x_i) - hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \cdots + R_n$

we also have $f(x_{-}) = \left[1 - hD + \frac{(hD)^2}{2!} - \frac{(hD)^3}{3!} + \quad \right] f(x) + R_n$

$f(x_{-}) = e^{-hD} f(x)$

By definition forward difference operator is given as

rearranging we have $\Delta f(x_i) = [e^{hD} - 1] f(x_i)$

The backward difference operator is given as

$\nabla f(x_i) = f(x_i) - f(x_{i-1})$

$\nabla f(x_i) = f(x_i) - e^{-hD} f(x_i)$

$\nabla = 1 - e^{-hD}$

The central operator is given as

similarly $f\left(x_i - \frac{h}{2}\right) = e^{-\frac{hD}{2}} f(x_i)$

therefore $If(xJ = (e^{\frac{hD}{--:;}} - e^{\frac{hD}{---:;}}) j(xJ$

$(j= (e^{\frac{hD}{2}} - e^{\frac{hD}{2}})$

hence $0 = 2Sinh(h\sim)$

Accuracy of Finite Difference Methods

The Finite Difference formulas and their subsequent use in boundary problems must assure accuracy in portraying the physical aspect of the problem that has been modeled. The accuracy depends on consistency, stability and convergence as defined below:

a. Consistency: That the finite difference procedure may in fact approximate the solution of the Partial Differential Equation under consideration and not the solution of some other POE. For example:

   $O(\sim t)+O(\&y = T.E$

   Since the $T.E \sim 0$ $as\sim t,(\&)\sim 0$

   The forward difference scheme is consistent with the original POE

b. Stability: A numerical scheme used for the approximation of a partial differential equation is stable if the error remains bounded. Certain criteria must be satisfied in other to achieve stability.

c. Convergence: The departure of the forward difference approximation from the solution of the POE at any grid point is known as the local discretization error, e.

i.e. if u is the exact solution and v the forward approximation then,

$e=u-v$

The forward difference method is said to converge if $e \sim 0$ as the grid step lengths tend to zero

(ii) Finite Element Method

In finite element methods difference equations are generated using approximate methods with the piecewise polynomial solutions. The finite element methods include weighted residual, least squares, partitions, Galerkin, moment and collocation methods. The domain R is divided into finite number of non-overlapping sub-domains called finite elements. Any regular or irregular network may be used as finite elements. Generally the straight line segments are used for one dimensional case, triangles, rectangles or elements with algebraic curves as boundaries in the plane, and tetrahedron or hexahedron in three space dimensions Okedayo (2008).

2.3.3 Choice of Method of Solution

The analytical method of solving a Partial Differential Equation requires that the problem must be sufficiently idealized (linear) for techniques to be effective. For nore practical problems, either the boundary geometry or the governing equations lre simpler, and one must often be content with approximate solutions.

mong the methods of approximation, two are most important. Perturbation nethods are powerful tool for getting analytical results. If however, the problem is 'ar from anything that can be solved exactly, strictly numerical methods via discretization or subdivision into elements must be employed. In general, analytical perturbation methods are much more effective in gaining a qualitative insight, while numerical methods are good in producing quantitative information. Sometimes the two can be mixed for studying small departures from a basic state that must itself be solved numerically. The method employed in this work is the regular perturbation technique

2.3.4 Perturbation Analysis

Perturbation methods are used when a small parameter (or a larger parameter) is introduced in a given equation or data for the problem. Then the (assumed) solution is expanded in a series of powers (or inverse powers) of the parameter and this expansion is inserted into the equation and data for the problem. By equating like powers of the parameter, a collection of problems results whose solution is expected to be simpler than that of the given problem. The series expansion of the solution converges or is expected to converge. Since only the first few terms in the series are determined and the distinction between convergent and divergent asymptotic series becomes irrelevant.

Ne shall give a brief account of the analytical approach of perturbation method. Perturbatlon methods are one of the most powerful tools for getting analytical 'esults .

.::>erturbation analysis can be classified into two as follows:-

(a) Regular perturbation

(b) Singular perturbation

2.3.5 Regular Perturbation Analysis

W consider a linear or nonlinear differential equation

$$L(u,c)=O \qquad 2.9$$

that depends (smoothly) on the small positive parameter e and problem for (2.9) given over a bounded or unbounded spatial region G. If (2.9) is of elliptic type, appropriate boundary condition are assigned on 8G or at infinity. If (2.9) is of the hyperbolic or parabolic type, in addition to the boundary conditions assigned on 8G or at infinity for all $t >- 0$, initial data are given in G at the time $t = 0$. The boundary or initial data may depend on e, but the boundary 8G for the present is assumed to be specified independently of c.

The reduced or unperturbed problem associated with the problem for (2.9) is obtained on formally setting $c = 0$ and its data. That is, we consider the equation

$$L(u,O) = 0$$

With the reduced data obtained from the data for the given problem for (2.9). If the reduced problem has a unique solution, then the given problem is called a regular ierturbation. If this is not the case, we have a singular perturbation problem Erich 1983). Generally speaking, if the reduced equation is of a different type or order han the given equation, we have a singular perturbation problem. It may happen, rowever, that the reduced problem can be solved even if the order or type of the jiven equation is changed.

An introductory outline of the typical ideas and procedure of regular perturbations given below by Okedayo (2001).

(i) Identify a small parameter. This is very important first step, which must be taken by recognizing the physical scale relevant to the problem.

(ii) One then normalizes all variable with respect to this characteristic scale. In the normalized form, the governing equations will display certain dimensionless parameters of certain physical relevance to the problem. If one of the parameters say e, is much less than unity, then & can be chosen as the perturbation parameter.

(iii) Expand the solution as an ascending power of the small parameter. As an example a power series s

2.10

where Un' ($n = 1,2,3,$) is called the nth order term. Collect terms of the same order in all governing equations and auxiliary conditions and get perturbation equations at each order.

(iv) Starting from the lowest order, solve the problem at each order successively up to a certain order say O(e''';

(v) Substitute the results for Un' (n = 1,2,3,) back into (2.10) to get the final results which is accurate up to some described order 0(&m)Example:

Let us examine the quadratic equation

$$u^2 + e\,U - 1 = 0 \qquad 2.11$$

here e is much less than unity let us propose to find the solution as perturbation ries

d substitute this into equation (2.11)

Expanding and collecting terms of equal powers, we get

$$(U\sim -1)+e(2U_0U_1 \quad +U_0)+e^2(2U_oU_2 \qquad +U_{12} \;+UJ+ \;\ldots = 0$$

With the coefficient of each power of e equated to zero, a sequence of perturbation equation is obtained of various orders.

0(&0):

0(&):

$0(e^{2})$:

the lowest order solution is

$$U_o = +1$$

th this result higher order problems are solved successivety.

$U_1 = -Yz$ and

n this case the efficacy of the approximate result can be judged by comparing with he exact solution

Clearly, this result confirms the perturbation series solution to the accuracy calculated.

2.3.6 Singular Perturbation Analysis

In singular perturbation theory we are concerned with the study of partial differential equation which contains a small parameter that multiplies one or more of the highest derivative terms in the equations. Thus when that parameter is equated to zero either the order or the type (or both the order and type) of the given equation is changed. Generally, this means that a regular perturbation series solution proves inadequate to handle the initial and/or boundary data for the given problem. It thus becomes necessary to introduce boundary or initial layers where the solution of the given problem undergoes a rapid transition from a form that satisfies all the data iven for the problem to a form represented by the perturbation series. Singular rturbation theory is sometimes taken to encompass any problem where regular erturbation theory is inadequate for any reason (see Aiyesimi(2007)) . This may ot involve the presence of a small parameter multiplying the highest derivative, but nay be due to the presence of secular terms that result in a non-uniformity of the solution over an infinite region or the occurrence of a small parameter in the data or the problem. Singular perturbation analysis requires the following steps.

(i) Diagnose the failure of the regular expansion check which of the original assumptions is violated when failure occurs. Examine the quantitative nature of the breakdown

(ii) Choose new terms that should be important near breakdown and start a new perturbation analysis.

Example: Consider the following cubic equation

which can also be solved exactly. For a small e let us try the straightforward expansion

Substituting this series into the equation we have

Equating equal powers of e yields the perturbation equations

$$U_0 = 1$$

$$U_1 = U_0^3$$

$$U_2 = 3U_0^2 U_1$$

Ihe situations are obviously

$$U_0 = 1, U_1 = 1, U_2 = 3$$

ence the final solution is

$$U = 1 + \varepsilon + 3\varepsilon^2 \qquad + O(\varepsilon^3)$$

e notice that the other two solutions of the original cubic equation disappear. Hence we seek a better expansion, which leads to a singular perturbation.

We may assume

$U = X\varepsilon^{-1/2}$ so that

The original cubic equation becomes

$$X\varepsilon^{-1/2} = 1 + X^3 \varepsilon^{-1/2} \qquad 2.12$$

Substituting the new expansion

$X = X_0 + \varepsilon^{1/2}X_1 + \varepsilon X_2 + \varepsilon^{3/2}X_3 + \ldots$ into (2.12) and collecting powers of e we get the perturbation equations

$O(\varepsilon^0)$

$O(\varepsilon^{1/2})$

$O(\varepsilon)$

The solutions at successive order give

$x_0 = (0, -1)$

$$x_1 = \frac{-1}{3x_0^2 - 1}$$

..pecific value can be obtained depending on the value of x_0

e.g. When $x_0 = -1$

$$x = -1 - \frac{\&1/2}{2} + \frac{3\&}{8} \ ...$$

o that

$$U = -\& \ 1/2 - \frac{1}{2} + \frac{3}{8} \& \ 1/2 + \ldots$$

CHAPTER THREE

.0 GOVERNING EQUATION AND SOLUTIONS

.1 Non-linear Shell Model Equation

he Donnell's non - linear shallow shell theory, gives the equation for transverse ibration of a very, thin, circular cylindrical shell as

ere $D = \frac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity, E is the Young's Modulus, ν is the oisson's ratio, h the shell thickness, R the mean shell radius, p the mass density the shell, c the damping coefficient and q the radial pressures applied to the urface of the shell as a consequence of external forces. The radial deflection w is sitive inward $W = \frac{\partial w}{\partial t}$, $W = \frac{\partial^2 w}{\partial t^2}$ and F is the in plane stress function; F is defined

3.1.2

biharmonic operator is defined as

lso using Donnell's non-linear shallow shell theory, the middle surface strain lisplacement relationships are obtained.

$$_{xB} = \frac{a\sim}{RoB} + \frac{fJv}{ax} + \frac{Ow}{ax}\frac{Ow}{RoB}$$

n-plane displacements are assumed to be infinitesimal, i.e. $|c; 1 \ll h, 1\ v\ 1 \ll h$

hereas w is of the same order of the shell thickness.

Solution of the Nonlinear Shell Model Equation

he biharmonic Operator

3.2.1

ubstituting eqn. (3.2.1) into (3.1.1) gives

3.2.2

introduce the following quantities length, *1*- unit length, T_0 - Time and N_0 - prestress, which has the unit of force and length. Hence

$$= ul \Rightarrow u = \frac{w}{l}$$

$$r = \frac{R}{l}$$

Where u, ξ, τ and r are all dimensionless quantities

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial \xi}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{l}\frac{\partial^2 u}{\partial \xi^2}, \quad \frac{\partial^4 w}{\partial x^4} = \frac{1}{l^3}\frac{\partial^4 u}{\partial \xi^4}$$

$$\frac{\partial w}{\partial t} = \frac{l}{T_0}\frac{\partial u}{\partial \tau}, \quad \frac{\partial^2 w}{\partial t^2} = \frac{l}{T_0^2}\frac{\partial^2 u}{\partial \tau^2}$$

In this thesis we will consider two cases: case 1 we neglect the radial pressure i.e. $q = 0$ throughout due to free vibration and case 2 where $q \neq 0$ for forced vibration. Hence for the former, substituting the dimensionless quantities into eqn. (3.2.2) we

$$\frac{1}{R^2}\left[\frac{N_0}{t}\frac{\partial^2 f}{\partial\theta^2}\frac{\partial^2 u}{\partial\xi^2} + \frac{2N_0}{t}\frac{\partial^2 f}{\partial\xi\partial\theta}\frac{\partial^2 u}{\partial\xi\partial\theta} + \frac{N_0}{t}\frac{\partial^2 f}{\partial\xi^2}\frac{\partial^2 u}{\partial\theta^2}\right] \qquad 3.2.3$$

earranging eqn. (3.2.3) we have

3.2.4

eqn. (3.2.4) is arranged to obtain

$$\frac{\partial^4 u}{\partial\xi^4} + \eta\frac{\partial^4 u}{\partial\xi^2\partial\theta^2}$$

here $\eta = \frac{l^2}{R^2}$, $\alpha = \frac{l^4}{R^4}$,

$\varepsilon = \frac{N_0 l^2}{DR^2}$ this emerges naturally as the perturbation parameter. Physically, this means that the terms arising from the nonlinear behavior are now a perturbation of the systems linear characterization. On formally setting ε equal zero, we recover

of the linear system. This implies that regular perturbation can be

We now assume solution of the form

$$u = U_m(\xi, T)\sin m\theta$$
$$f = f_m(\xi)\sin m\theta \qquad 3.2.6$$

Substituting eqn. (3.2.6) into (3.2.5) gives

$$\frac{4U}{ae;} \; Sin\, m O + 7] \left(\frac{a^2 u}{ae;} f - \left(m^2 Sin\, m O \right) \right) + am^4 u m Sin m B + c^* \frac{au}{aT} - m\, Szn m B + f3 \frac{a^2 u}{aT} f - Szh m B$$

$$-\& \left\{ \frac{R}{l} - Sin m B - \frac{d? f}{d\sim 2} - m_2 Sin_2 m B . + \frac{a_2 u}{a\sim 2} - 2m Cos m B - \frac{df}{d\sim} \left(m Cos m B - \frac{au}{a\sim} \right) + Sin m B - \frac{d? f}{d\sim 2} \left(2 - m \;\; Sin\, mil \right) u \right\}$$

- 3.2.7

sqn. (3.2.7) was arranged to obtain

3.2.8

eqn. (3.2.8) gives

Seeking asymptotic expansion of the form

-3.2.10

Substituting (3.2.10) into (3.2.9) gives

$$\frac{\partial^4}{\partial \xi^4}\left[U_{mo} + \varepsilon U_{m1} + \varepsilon^2 U_{m2} + \dots\dots\dots\right] - 17 m^2 \frac{\partial^2}{\partial \xi^2}\left[U_{mo} + \varepsilon U_{m1} + \varepsilon^2 U_{m2} + \dots\right]$$

$$|- \tilde{C}\frac{\partial}{\partial r}\left[U_{mo} + \varepsilon U_{m1} + \varepsilon^2 U_{m2} + \dots\right] + \beta\frac{\partial^2}{\partial r^2}\left[U_{mo} + \varepsilon U_{m1} + \varepsilon^2 U_{m2} + \dots\right] + a\, m^4\left[U_{mo} + \varepsilon U_{m1} + \varepsilon^2 U_{m2} + \dots\right]$$

$$e\left\{\frac{R}{l}\frac{d^2}{d\xi^2}\left[f_{mo} + \varepsilon f_{m1} + \varepsilon^2 f_{m2} + \dots\right] - m^2 \mathrm{Sin}\, m\Pi\left(\left[f_{mo} + \varepsilon f_{m1} + \varepsilon^2 f_{m2} + \dots\right]\right.\right.$$

$$8', [U_{mo} + \varepsilon U_{m1} + \varepsilon^2 U_{m2} + \dots\dots\dots][U_{mo} + \varepsilon U_{m1} + \varepsilon^2 U_{m2} + \dots\dots\dots], \quad [f_{mo} + \varepsilon f_{m1} + \varepsilon^2 f_{m2} + \dots]$$

$$\frac{2m^2 \mathrm{Cos}^2 m\Pi}{\mathrm{Sin}\, m\Pi}\left(\frac{d}{d\xi}\left[f_{mo} + \varepsilon f_{m1} + \varepsilon^2 f_{m2} + \dots\right]\frac{d}{d\xi}\left[U_{mo} + \varepsilon U_{m1} + \varepsilon^2 U_{m2} + \dots\right]\right)\Bigg\} \quad - 3.2.12$$

ubstituting eqn. (3.2.1) into the equation of the inplane stress function eqn. (3.1.2)

-3.2.13

substituting the dimensionless quantities we obtain

-3.2.14

-3.2.15

earranging eqn. (3.2.15) we have

-3.2.16

Vhere $7] = \frac{2/2}{-2 R}$ *and* $a = \frac{14}{-4 R}$

$\frac{Ehl^4}{"I_o R2} = \frac{Nol^2}{DR2} \left(\frac{!2\_}{N_o}\right)^{12} 12(1\text{-}v2)$

,qn. (3.2.16) now becomes

-3.2.17

here $p, \sim (:$ r 12(1- $v')$

n formally setting e equal zero in eqn. (3.2.17), the order of the linear equation is ot lost. This implies that regular perturbation can also be applied.

assuming the solution of the form

$U = Urn(\sim, \tau) SinmB\}$
$f = frn(\sim) SinmB$ 3.2.6

Substituting eqn. (3.2.6) into eqn. (3.2.17) we have

3.2.18

Rearranging 3.2.18 gives

- 3.2.19

Seeking asymptotic expansion of the form

3.2.10

Substituting eqn. (3.2.10) and eqn. (3.2.19) we have

equating coefficient of equal powers of e in eqn. (3.2.12) and eqn. (3.2.20) we have

.Order $(\&0)$

3.2.21

3.2.22

Order $(\&)$

3.2.23

-3.2.24

3.2.1 Order $(\&0)$ Solution for Radial Deflection $u_{mo}(\sim,t)$ Free Vibration

3.2.21

eqn. (3.2.21) is the order $(\&0)$ for the Radial Displacement

3.2.25

eqn. (3.2.25) is chosen to satisfy the simply supported end boundary conditions

$$\frac{\partial^2 u_{mo}}{\partial \sim^2}(0,r) = 0$$

Substituting eqn. (3.2.25) into eqn. (3.2.21) we obtain

(kn t U; (r) Sin kn~ + 217m2 (kn? Uo (r) Sinkn~ + c· dUo/dt (r) Sinkrd; + fJ d2U02(r)/dt Sin kn~

3.2.26

Rearranging eqn. (3.2.26) we have

3.2.27

eqn. (3.2.27) is the free motion of a mass

solving eqn. (3.2.27) we let $U_a(r) = Ae'IT$

$f3A^2 + c^* A + Z = 0$

The solution depends on the nature of the roots. For vibration to occur then

the general solution is

$U;(r) = e^{-\frac{c}{2P}r} (AkCos\ \phi r + BkSin\phi r)$

Where $\phi = \sim \frac{4/3z - (c^* t}{2/3}$

ence

$$u_{mo}(\varsigma, \tau) = e^{-\frac{c}{2\beta}\tau}(A_k Cos\ \phi\tau + B_k Sin\ \phi\tau) Sin\ k\pi\varsigma \quad -3.2.28$$

A_k and B_k are constants to be obtained by the given initial conditions.

$u_{mo}(\varsigma, 0) = 0$

$$\frac{\partial U_{mo}(\varsigma, 0)}{\partial \tau} = \tau_0$$

where τ_0 is the velocity at initial equilibrium position

There is no initial displacement hence at $u_{mo}(\varsigma, 0) = 0$ it implies that

then,

$$u_{mo}(\varsigma, \tau) = e^{-\frac{c}{2\beta}\tau}(B_k Sin\phi\tau) Sin k\pi\varsigma$$

From the second condition we have

$$B_k = \frac{2\tau_0}{\phi k\pi}[1 - Cos k\pi]$$

We finally obtain

$$u_{mo}(\varsigma, \tau) = \sum_{k=1}^{\infty} \frac{2\tau}{\phi k\pi}[1 - Cos k\pi] e^{-\frac{c}{2\beta}\tau} Sin\ \phi t\ Sin k\pi\varsigma; \quad -3.2.29$$

3.2.2 Order (50) Solution for the Inplane - Stress Function $f_{mo}(\xi)$

3.2.22

Eqn. (3.2.22) is the order (5°) problem for inplane stress function

Eqn. (3.2.22) can be rewritten as

$$\frac{d^4 f_{mo}}{d\xi^4} - 2\left(\frac{lm}{R}\right)^2 \frac{d^2 f_{mo}}{d\xi^2} + \left(\frac{lm}{R}\right)^4 f_{mo} = 0 \qquad 3.2.30$$

We let $f_{mo} = Ae^{r\xi}$ - -3.2.31

Substituting (3.2.31) into (3.2.30)

$$r = \frac{lm}{R}, \frac{lm}{R}, -\frac{lm}{R}, -\frac{lm}{R}$$

The general solution is given as

-3.2.32

The constants are to be obtained using the boundary condition rom the given boundary condition the solution can generally be represented by the series expansion

$$f_{rno}(r) = \sum_{p=1}^{\infty} B_p \sin p\pi r \qquad 3.2.33$$

For all values of p

Substituting eqn. (3.2.33) into eqn. (3.2.30) we have

since $\sin p\pi t$ is arbitrary chosen for $0 \leq t \leq 1$

and $B_p \neq 0$ then

3.2.3 Order (c) Solution for the Radial Deflection $U_{rn1}(\tau, r)$ Free Vibration

-3.2.23

Eqn. (3.2.23) is the order (c) problem substituting the values of

Where

$$H(\sim)= - R \, I \sum_{p=l} Bp(p7rY \quad Sin\, p\, rd;$$

To solve eqn. (3.2.34), the function $U_{m1}(\sim, r)$ can be expressed as a series of Eigen function

$$Uml(\sim,r)= \sum_{k=l}^{oo} bk(r)(h(\sim) \qquad 3.2.35$$

$v_k(\sim)$ is chosen so as to satisfy the simply supported boundary conditions

$v_k(\sim)|\sim=O,l = 0$ and

Hence we let

The Eigen function is now given by

where c_k remain arbitrary constant chosen to normal'

THEOREM 1.1

Let $\{\phi_n(\sim)\}$ be a countably infinite orthonormal set in a Hilbert space H. Then the series $\sum_{n=1}^{\infty} a_n\phi_n(\sim)$, where a_n are scalars, converges if and only if the series $\sum_{n=1}^{\infty} |a_n|^2$ converges

Proof:

if $\sum_{n=1}^{\infty} a_n\phi_n(\sim)$ converges,

set $u = \sum_{n=1}^{\infty} a_n\phi_n(\sim)$

Then from Bessel's inequality

$n=1$

But every bounded series of positive numbers converges, so necessity is proved.

Next, let $\sum_{n=1}^{\infty} |a_n|^2$ converge.

Put $a_n = \sum_{k=1}^{\infty} a_k\phi_k(\sim)$

then $\|a_n - a_m\|_H^2 = \sum_{k=m+1}^{n} |a_k|^2$

i.e. $\{\phi_n(\sim)\}$ is Cauchy sequence in H.

Since H is complete

$\{\phi_n(\sim)\}$ is convergent, as asserted

Hence the sufficiency and thus end of proof.

from eqn. (3.2.34) we can take

$$G(\xi) = \sum_{k=1}^{\infty} d_k \psi_k(\xi)$$

$$H(\xi) = \sum_{k=1}^{\infty} a_k \psi_k(\xi)$$

substituting eqn. (3.2.37) into (3.2.34)

and since $\psi_k(\xi) \neq 0$ then

-3.2.38

equation (3.2.38) is the forced motion of a mass

where

solving homogeneous part of eqn. (3.2.38) we have

3.3.39

The solution depends on the nature of the roots. For vibration to occur then

$(c')^2 - 4/3z < 0$

$$A = \left[\frac{=i}{2/3} \pm i \frac{\sqrt{4/3z - (c')^2}}{2/3} \right]$$

the general solution is obtained as

$$bk(r) = e^{-\frac{c}{2P}r} (AkCos\phi r + BkSin\phi r) \quad 3.2.40$$

Where $\phi = \frac{\sqrt{4/3z - (c')^2}}{2/3}$

We seek for the solution of the non homogeneous part (particular integral) by the way of variation of parameter. We assume particular integral of the form

3.2.41

ere $\omega = -\frac{c}{2/3}$

$$C_1' e^{\omega t} Cos\phi t + C_2' e^{\omega t} Sin\phi t = 0 \quad 3.2.42$$

3.2.43

$$C_1' = \frac{-C_2' Sin \phi t}{Cos \phi t}$$ 3.2.44

ubstituting (3.2.44) into (3.2.43) and rearranging we obtain

$$C_2' = \frac{d_k}{2\phi} Sin' Id n:$$

3.2.45

imilarly $C_2' = -\frac{C_1' Cos \phi r}{Sin \phi t}$ 3.2.46

Substituting (3.2.46) into (3.2.43) and rearranging we have

$$C_1 = \frac{d_k}{4\phi}(Sin 2\phi r - 2r)$$ 3.2.47

3.2.48

Hence we have

$$b_k(r) = \frac{1}{4} e^{\frac{1}{2}P}\left(A\ Cos\ \phi r + B\ \ Sin\ \phi r - \frac{d_k}{\phi} 2r Cos\ \phi r\right) + \frac{a_k}{z}$$ - 3.2.49

e will obtain the constants A and B subject to the initial condition rom the first condition

$A = \frac{-4a_k}{z}$

For the second condition we obtain

We now have

$bk(r)=-e \frac{1}{4} -2// c ((\frac{4}{¢} ro - \frac{c}{2f3} \frac{ak}{z} + \frac{dk.}{2¢} \rfloor Szn ¢r -- \frac{4ak}{z} Cos ¢r -- \frac{d}{¢} 2rCos ¢r \rfloor + \frac{ak}{z}$ -3.2.50

Finally

$ull11(\sim,r)= \sum_{k=l} C_k \{ \frac{1}{4} e^{--T} ((t((\frac{4}{¢} ro \frac{cad}{2f3} \frac{k}{z} + \frac{k}{2¢} \rfloor Sin ¢r \frac{4a}{z} k Cos ¢r -- \frac{d}{¢} k Zt Cos ¢r \rfloor \} \frac{a}{z} Sinkrd;$

3.2.51

To compute C_k from orthonormal sets

$\int_0^1 ¢k(\sim)¢m(\sim)d\sim = \{\sim\sim:: 0 \sim \sim \sim 1$

ence

$\int_0^1 ¢2k(\sim)d\sim = \int_0^1 c:v:(\sim)d\sim$

Zr: k

kst =Coskx Sinkn - 3.2.52

.2.4 Order (s) Solution for the Inplane - Stress Function $fml (C;)$

- 3.2.24

qn. (3.2.24) is the order (s) problem substituting the values of

$U_{Ina'} \frac{OU_{lNI!}}{Oc;}, \frac{02U_{rna}}{Oc;2}$ Into the equation gives

$fJ_I m^2 COS^2 me \; fJ \; kn^{2}_{(2)} \; Sin \; mil \left[e^{-\frac{c}{T}} 2(1 \; Sin \; qrr \right]^2 Cos2 \; krcc; - fJl(krc?fJ2m2Sinme \left['e^{-\frac{c}{T}} 2(1 \; Sin \; \phi r \right] Sin \; krd;$

3.2.53

$= \sum_{k=l}^{0} \frac{2r}{\phi k7r} [l-Coskrc]$

sing the Finite Fourier Sine transformation (FFST) of eqn. (3.2.53) subject to the

nditions

$IIIl0) = !'nlO) = 0$

obtained

3.2.54

Finally, on taking the inverse Fourier transform of *L, (~)* we have;

r: (~)= 2I-1 {fJlRfJ2 (kn.y e-;> Sin ¢n (fJ4)+ fJlm2 Cos.2 m()(fJ2k1r r [e-;> Sin ¢n]2 (fJs)
\1'=fJ3 I Sin m ()

- 13,(k7r)2 132m2Sin m B (e-;> Sin ¢r }p6) }Sin \'f,,-~ -3.2.55

here

fJ - _ k Cos kn Sin lj/n -lj/ Sin kst Cos lj/n
4 - n(k2 _1j/2)

fJ - _ 1j/2 - 2k21j/2 Cos lj/nCos(kn Y - 2lj/kSinlj/n Sin kn Cos kn + 2k2 Coslj/n
5 - 1j/.7Z{4k21j/2)

fJ - lj/ Sin kn Cos lj/n - k Cos kn Sin lj/n
6- n(k2_1j/2)

3.2.5 Order (&0) Solution for Radial Deflection u_{m0} (~, r) Forced Vibration

(a) We will first consider a case of Pulsating Pressure acting on the surface of the shell

fig 3.1 force acting on a shell surface

From eqn. (3.2.2) when q "* 0, and expanding in power series of E we have

3.2.56

3.2.57

Substituting eqn.(3.2.57) into (3.2.56) we have

-3.2.58

Jsing the Fourier Sine transform subject to the simply supported boundary ondition

'8 2 umo (0, r) = 0
ae

mO(~,r)= fUmoSinIf/Jr~d~ (lower limit 0) -3.2.59

Evaluating the integrals we obtain

-3.2.60

rearranging eqn. (3.2.60) we obtain

3.2.61

Where 'II = 2r+l, r E Z

rearranging eqn. (3.2.61) we have

-3.2.62

where fJ8 = ~* / fJ, fJ9 = a_o / fJ and fJo = 13 / DfJ Sinmtl

~aking the Laplace transform of eqn. (3.2.62) subject to the initial condition we have

-3.2.63

Evaluating eqn. (3.2.63) we have

-3.2.64

Rearranging eqn. (3.2.64) we have

fimo(n,r)= 2fJoPo / 'lin [Q / s^2 + ~] (fJ / s^2 + s fJ8 + fJ9) - 3.2.65

3.2.66

esolving eqn. (3.2.66) into partial fractions we have for

$$\frac{1}{(s-a_2)(s-a_3)} = \frac{a}{(s-a_2)} + \frac{b}{(s-a_3)}$$

'0. $+ b = 0 \sim \quad a = -b$

-3.2.67

-3.2.68

n other to obtain the Laplace inversion of equation (3.2.68) we shall adopt the allowing representations:

3.2.69

3.2.70

Since $\sim mo(n;r) is$ expressed in a fairly simple way in terms of functions whose Laplace transform are easily recognizable. Then the Laplace inverse of eqn. (3.2.70) is the convolution denoted by the integral Omolafe et. al. (2008)

$$\int_0^T f_i(\tau-u)g(u)du \quad , \quad i=1,2$$

eqn. (3.2.70) now becomes

- 3.2.71

Evaluating eqn. (3.2.71) we have

3.2.72

ffhe inverse Fourier sine transform is given by

$$mOen,r) = 2\sum_{'1'=1} UmO(n, r)Sin \quad II/1r\sim$$ 3.2.73

Substitutingeqn. (3.2.73) into (3.2.72) we have

(b) We will also consider a case of moving force acting on the surface of the shell.

From eqn. (3.2.2) when q "* 0, and expanding in power series of E we have

3.2.56

3.2.75

Where qj (~, r) - is continuous moving force acting on the surface of the shell

Vi - the velocity and

8(~-vir) - Direc delta function which is defined as

("* v(r

(= v(r

Substituting eqn. (3.2.75) into (3.2.56) we have

3.2.76

sing the Fourier sine transform subject to the simply supported boundary condition

mO (~, r) = fUmoSinlflJr~ d~ -3.2.59

:valuating the integrals we obtain

()4 IflJr + 217m 2 (IflJr)2 + m 4 a Jl mo + c * au / aT + t

.arranging eqn. (3.78) we obtain

rearranging eqn. (3.79) we have

-3.2.79

Where fJs = ~* / f'J   fJ9 = a_o / fJ   and fJo = 1_3 / DfJ Sin mil

Taking the Laplace transform of eqn. (3.2.79) subject to initial condition we have

Evaluating eqn. (3.2.80) we have

3.2.81

3.2.82

earranqinq eqn.(3.2.82) we have

3.2.83

where $a_2 = \frac{-\beta_8 + \sqrt{\beta_8^2 - 4\beta_9}}{2}$ *and* $a_3 = \frac{-\beta_8 - \sqrt{\beta_8^2 - 4\beta_9}}{2}$

resolving eqn. (3.2.83) into partial fractions we have for

-3.2.84

3.2.85

In other to obtain the Laplace inversion of equation (3.2.85) we shall adopt the following representations:

3.2.86

3.2.87

,ince $V_i^*(n, r)$ is expressed in a fairly simple way in terms of functions whose .aplace transform are easily recognizable. Then the Laplace inverse of eqn. .2.87) is the convolution denoted by the integral

;;(r-u)g(u)du , i=1,2

n. (3.2.87) becomes

- 3.2.88

Evaluating eqn. (3.2.88) we have

3.2.89

The inverse Fourier sine transform is given by

$$U_{m0}(n,\mathrm{r}) = 2L_{'1'=1}\, u_{m0}(n,r)Sin\ 1f/1C\sim \qquad 3.2.90$$

Substituting eqn. (3.2.89) into (3.2.90) we have

-inally eqn. (3.2.91) becomes

3.3 Stability of the Circular Cylindrical Shell

p

L

rig 3.2 axially-loaded cylindrical shell

('Ie tend to find under what loading condition, lateral deflection might occur due to ixial load P. The buckling of the circular cylindrical shell under axially symmetric)ading can be obtained by modifying equation (3.2.21) to have

$$\frac{{}^{4}»;}{3\text{-}\sim\text{-}4} + a\ m^{4} »; = \frac{P\ 7]\ m^{2} I}{D} \qquad 3.3.1$$

3.3.2

is chosen as a mode shape (buckled shape) which satisfies the simply supported undary conditions of circular cylindrical shell.

bstituting eqn.(3.3.2) into eqn.(3.3.1) we obtained

3.3.3

Since $Sinkrd;$ is arbitrarily chosen for $0 < \sim < 1$

and Ak "* 0 then

$$[k^4 1r^4 + am^4 + \frac{P}{D} 17 m2Z k^2 1r^2] = 0 \quad 3.3.4$$

$$P = -D [k^4 1r^4 + am^4] \frac{1}{17 m^2 Zk^2 1r^2} \quad 3.3.5$$

earranging (3.3.5) we have

3.3.6

or each value of k there is a unique buckling mode shape and a unique buckling ad. The least load that buckles is where $k = 1$ hence we obtained

3.3.7

iJ.1 Buckling of the Shell with an Initial Imperfection

n. 3.3.1 can be modified to give

3.3.8

3.3.9

Substituting (3.3.9) into (3.3.8) we have

3.3.10

Since *Sinkiti;* is arbitrarily chosen for $0 < f, < 1$

3.3.11

3.3.12

Hence

$$»;^{(f,)} = -\frac{D}{P}\left[\frac{rc2}{17m2/}^{BJ} + \frac{am'}{17rc2/}\right] +- \frac{D}{P}\left[\frac{4rc2}{17m2/}^{B2} + \frac{am2]}{174rc2/}\right] + \quad . \qquad 3.3.13$$

The least load that buckles the structure occurs at $k = 1$ i.e.

3.3.14

The shell will either inelastically deform or strain harden or it will fracture. When $k < 1$, it has no physical significance to the dynamics of the system.

CHAPTER FOUR

4.0 NUMERICAL SIMULATION

4.1 Numerical Simulation

Considering the solution of the order (£0) and order (s) and for different values of *e* we employ the MAPLE 11 Mathematical computer software for the purpose of this simulation. The displacement profiles of the shell are displayed graphically in what follows demonstrating the effect of the, initial velocity at initial equilibrium stage, mode number, damping parameter and external force (pulsating force and a moving force) on the amplitude of vibration.

The material employ is the circular cylindrical shell, simply supported at the ends, having the following dimensions and properties: $1= O.2m$, $R=O.1$ m, $h=O.247x10^{-3}m$, $E=71.02x10^{9}Pa$, $p=2796kg/m^{3}$ and $v=O.31$

4.1.1 Computations for Free Vibration

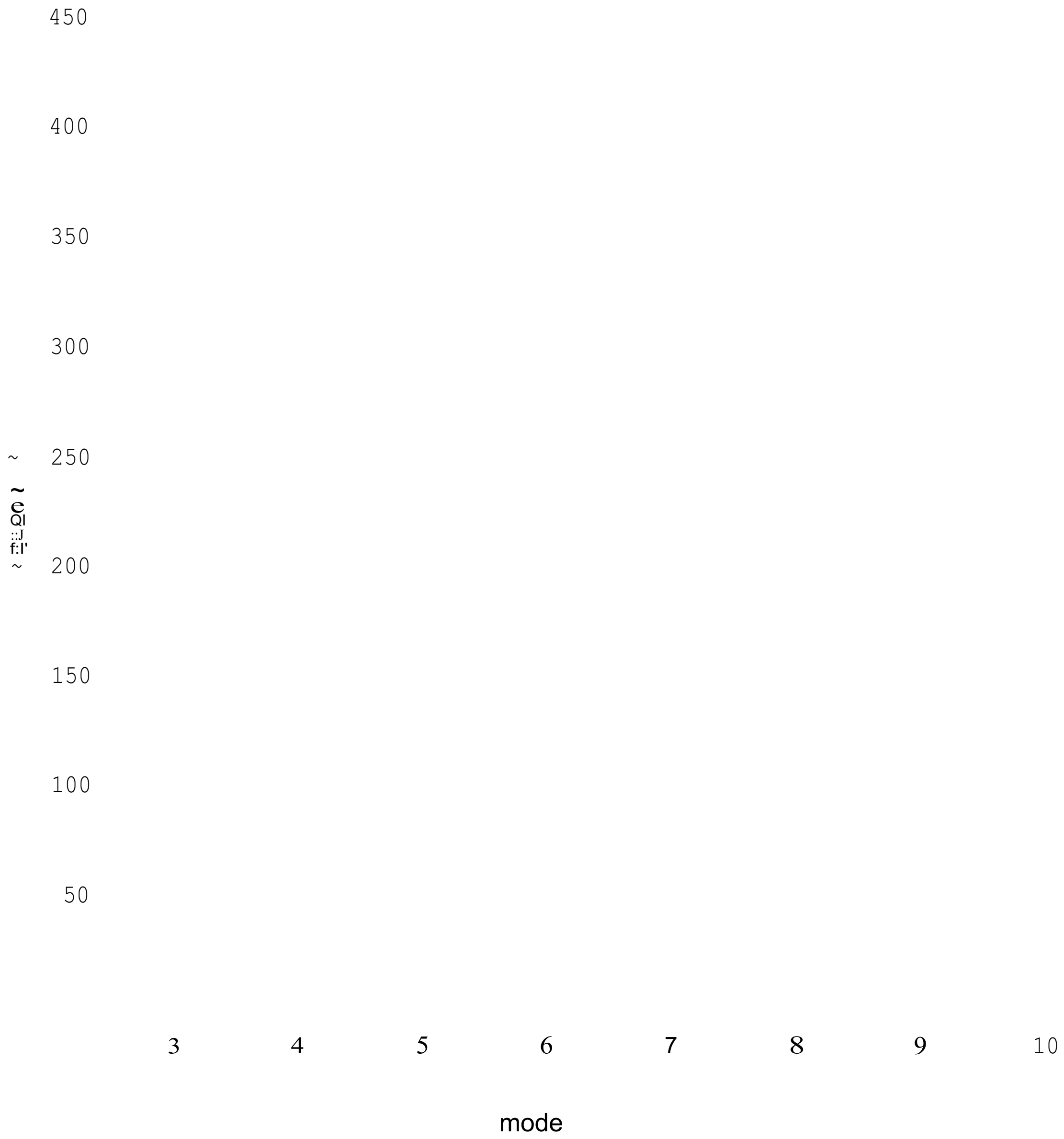


Fig 4.1 Frequency of the structure over mode number m

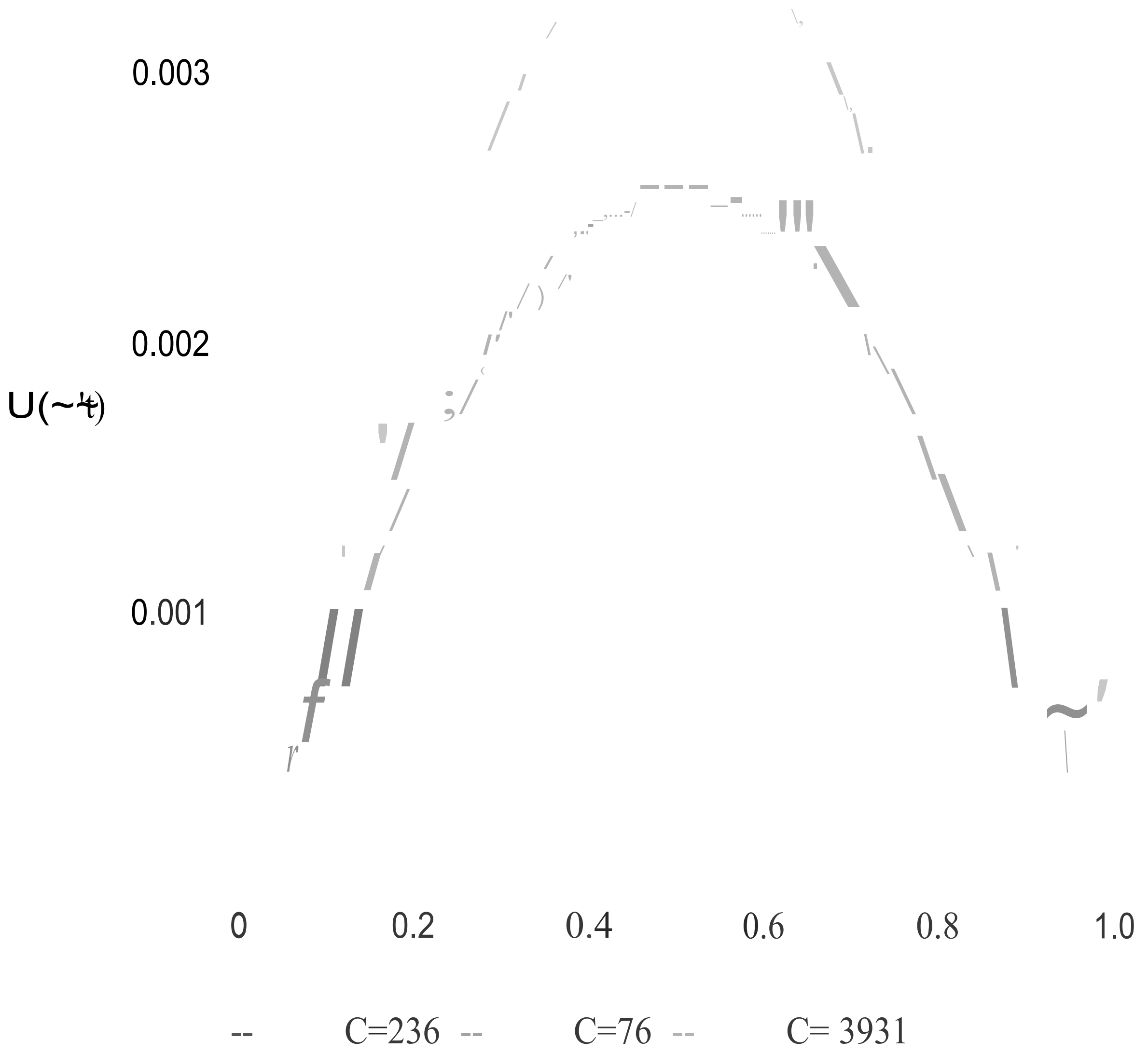


fig. 4.2 Response of the cylindrical shell to the effect of damping coefficient

0.02
0.0

0.02
0.0

(a) (b)

0.02
0.0

(c)

fig. 4.3 Response of the cylindrical shell to the effect of damping coefficients in three dimension (a) c=76 (b) c=236 (c) c=393

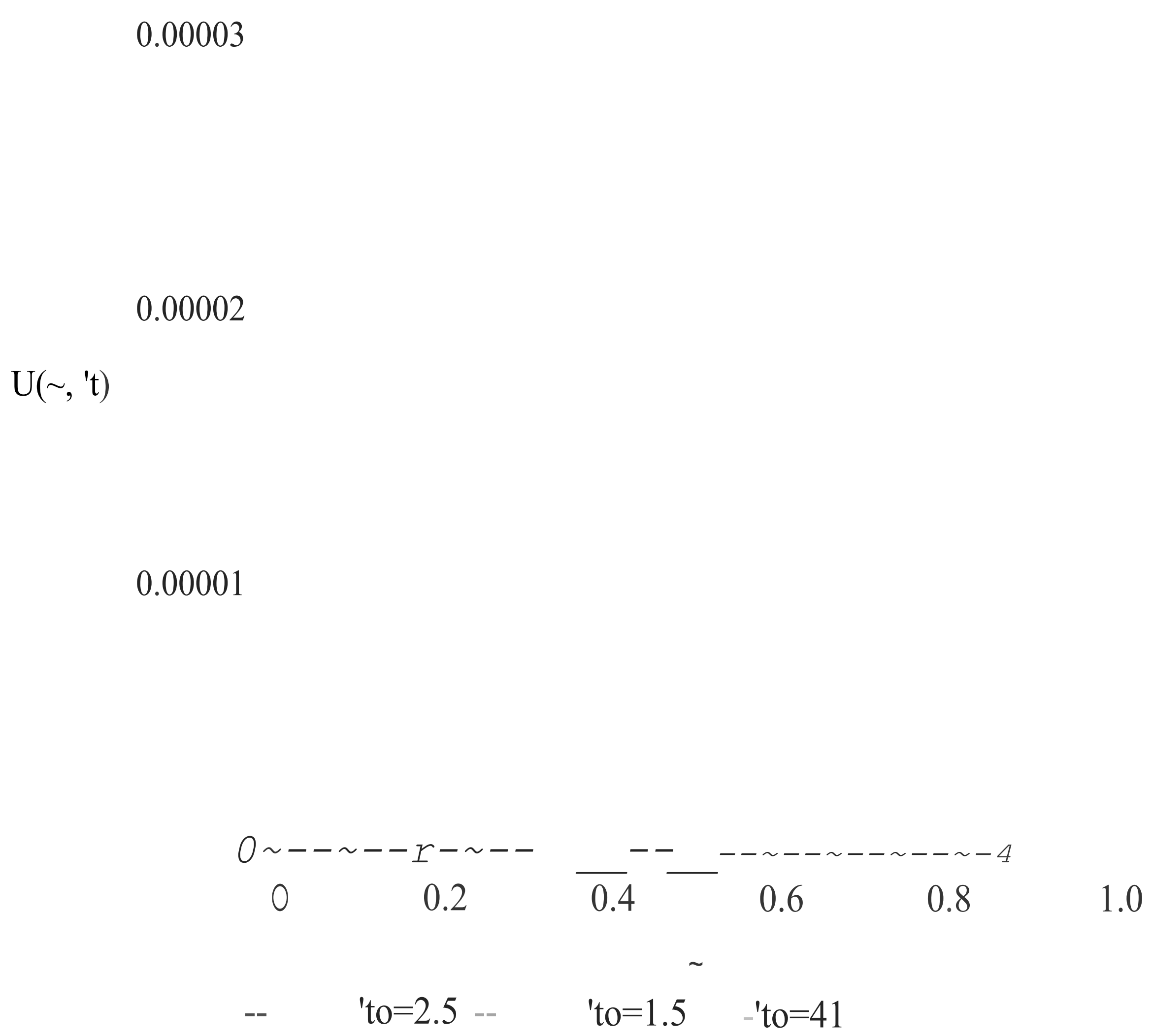


fig. 4.4Response of the cylindrical shell to the effect of initial velocity at initial equilibrium stage

0.1
0.0

(a) (b)

(c)

fig 4.5 Response of the cylindrical shell to the effect of velocity at initial equilibrium stage
(a) To=1.5 (b) To=2.5 (c) TO=4

-1

-2

-3

-4

-.5

-6

-- m=4 -- m=.5 --- m=6 -- m=71

fig. 4.6 displacement profile for the cylindrical shell with different wave numbers

0.1
0.0

(a) (b)

0.1
0.0

(b) (d)

fig. 4.7 displacement profile for the cylindrical shell with different wave numbers (a) m=4, (b) m=5 (c) rn=6 and (d) m=7

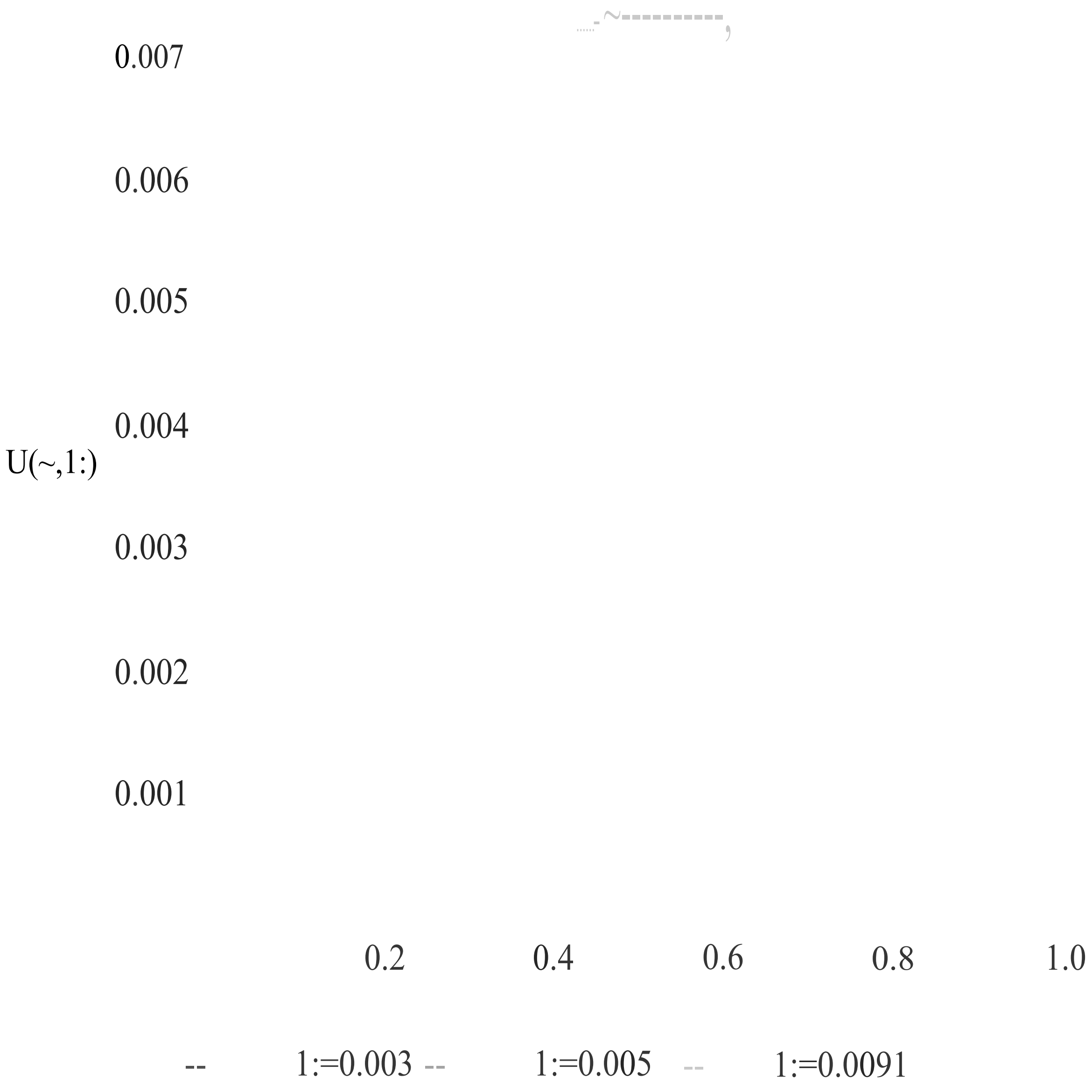


fig 4.8 time response of the free vibrating cylindrical shell

5

4

f(~ 3

2

1

fig. 4.9 mode shape profile of Inplane stress function

4.1.2 Computation for Forced Vibration

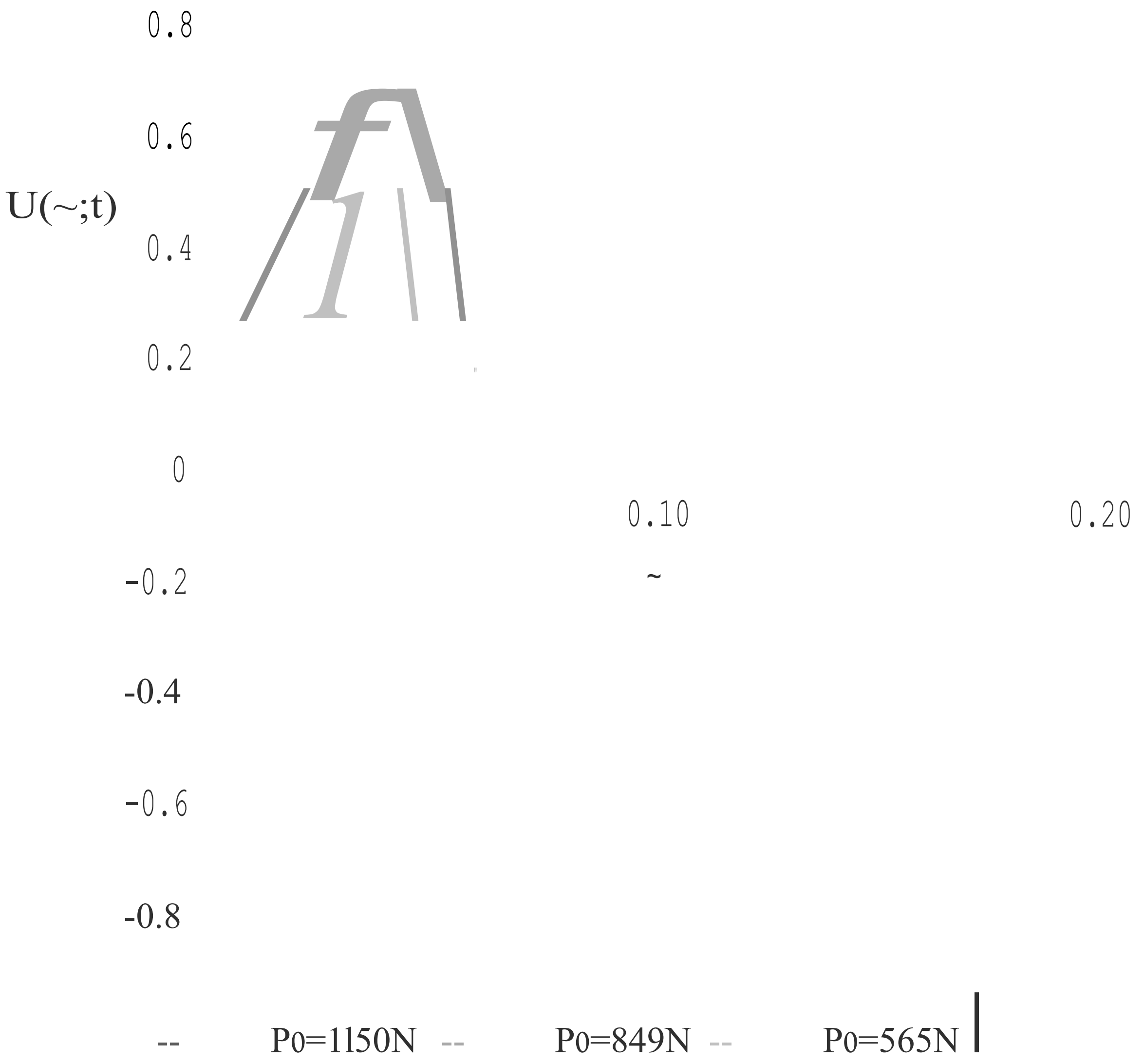


fig. 4.10 response of forced vibrating cylindrical shell to different values of pulsating force

0.1
0.0

0.1
0.0

(a)

(b)

0.1
0.0

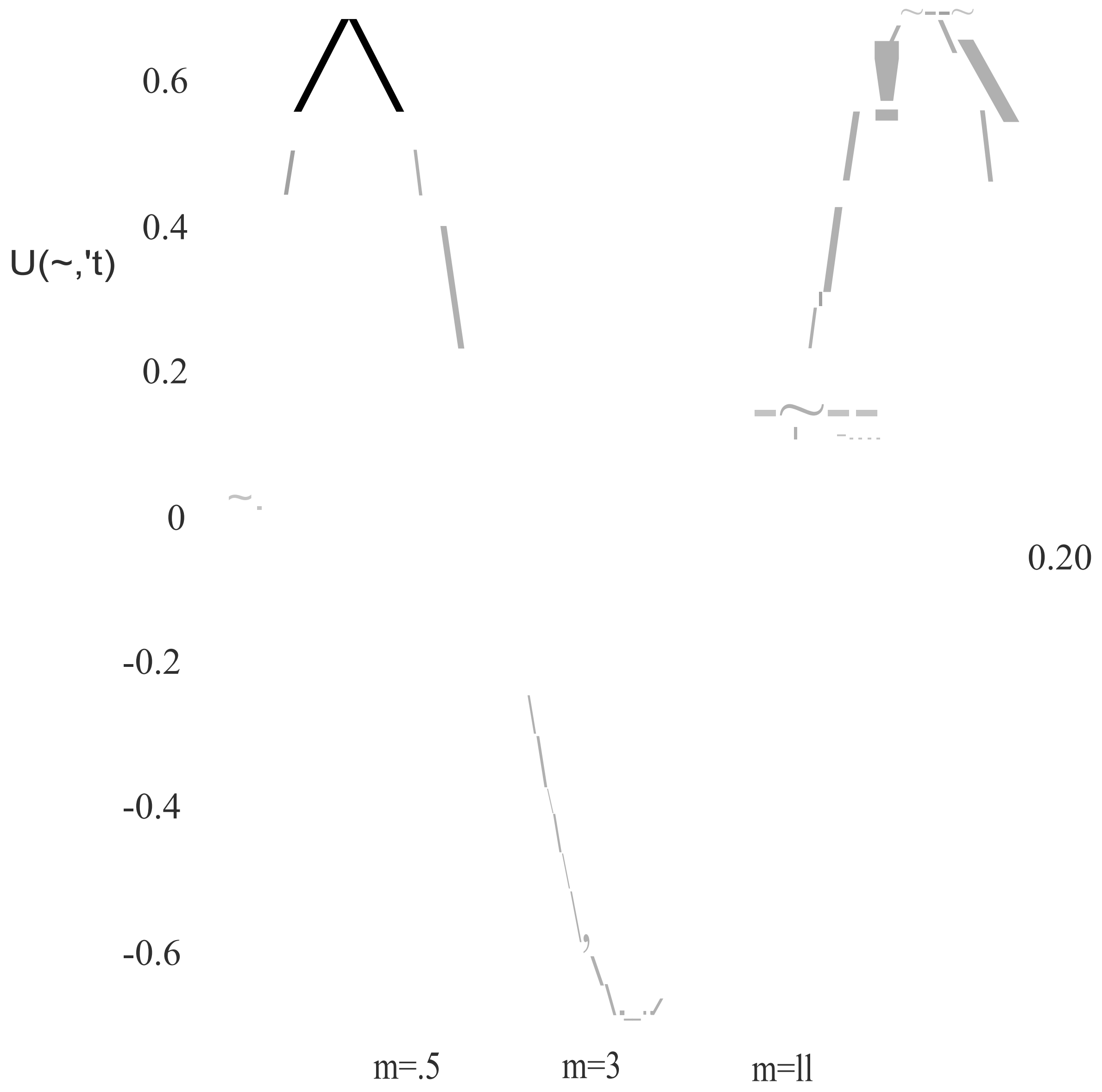
(c)

fig. 4.11 response of forced vibrating cylindrical shell to different values of pulsating force (a) po=565N/m2(b) po=849N/m2and (c) po=1150Nlm2

U(~;t)

-- '1;=0.8 -- '1;=0.31

fig. 4.12 time response of a forced vibrating cylindrical shell



19 4.13 displacement profile with different mode index

60

U(~,'t) 40

20

0.20

-20

-40

-60

-- ~=1.5 x103 -- ~=2 x 103 -- ~=1 x 1031

4.14 response of forced vibrating cylindrical shell to different values of moving force

0.1
0.0

(a)

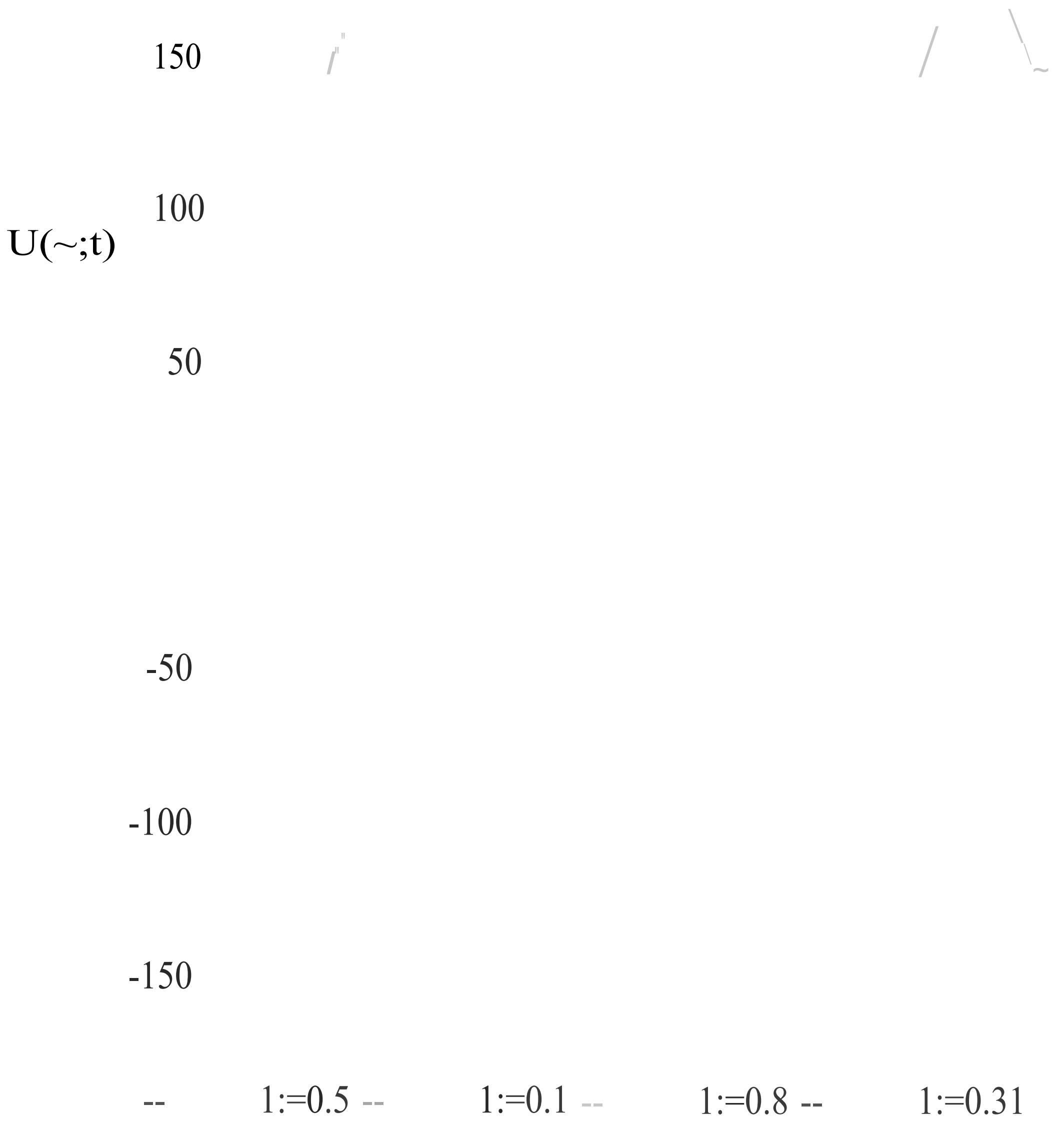
0.1
0.0

(b)

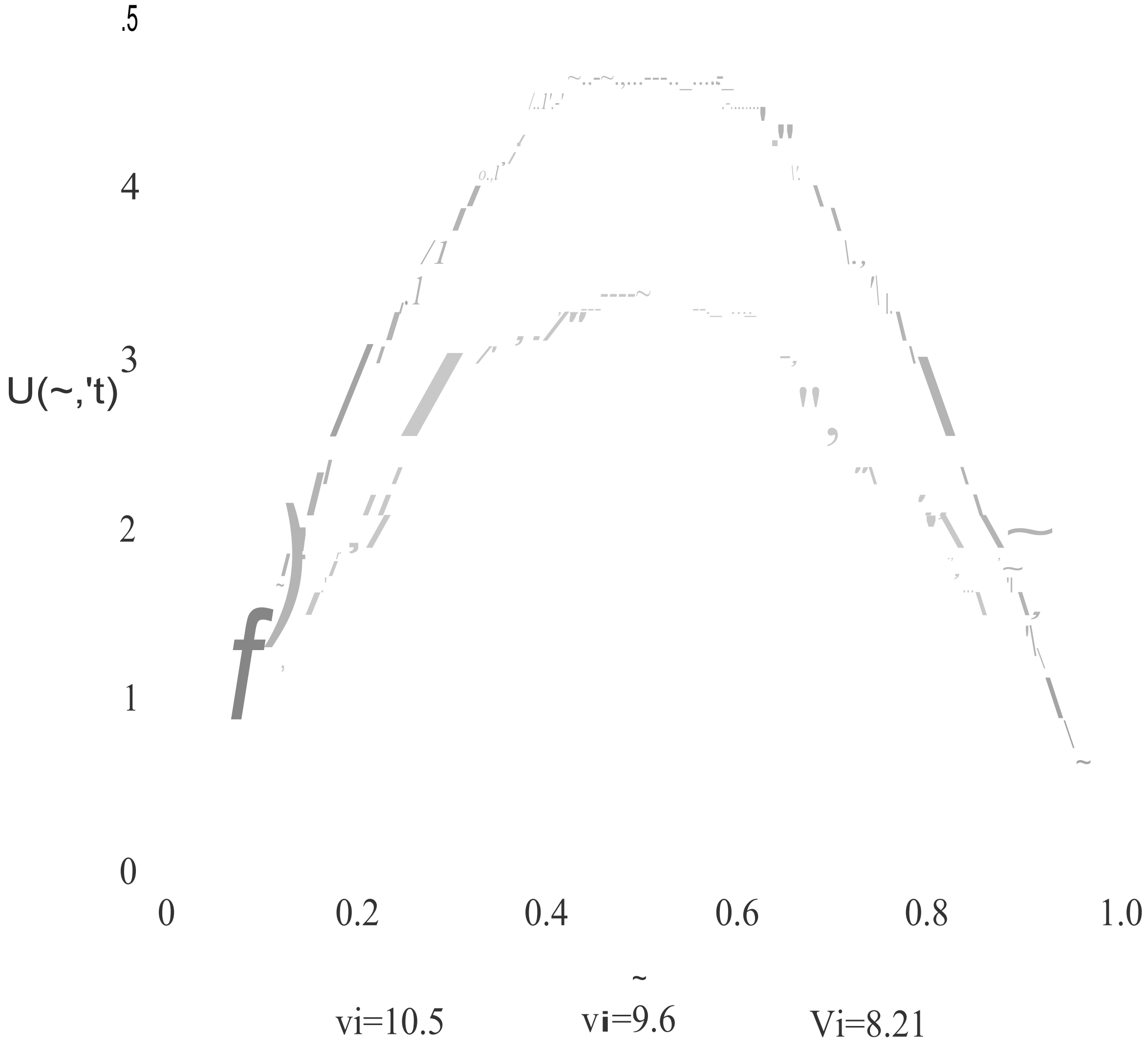
0.1
0.0

(c)

g. 4.15 response of forced vibrating cylindrical shell to different values of moving force (a) $= 1 \times 10^3 N/m^2$ (b) $q_f = 1.5 \times 10^3 N/m^2$ and (c) $q_f = 2 \times 10^3 N/m^2$



19.4.16 time response of a forced vibrating cylindrical shell subjected to a moving force



g. 4.17 response of forced vibrating cylindrical shell subjected to a moving force with arious values of velocity

4.1.3 Computation for Stability

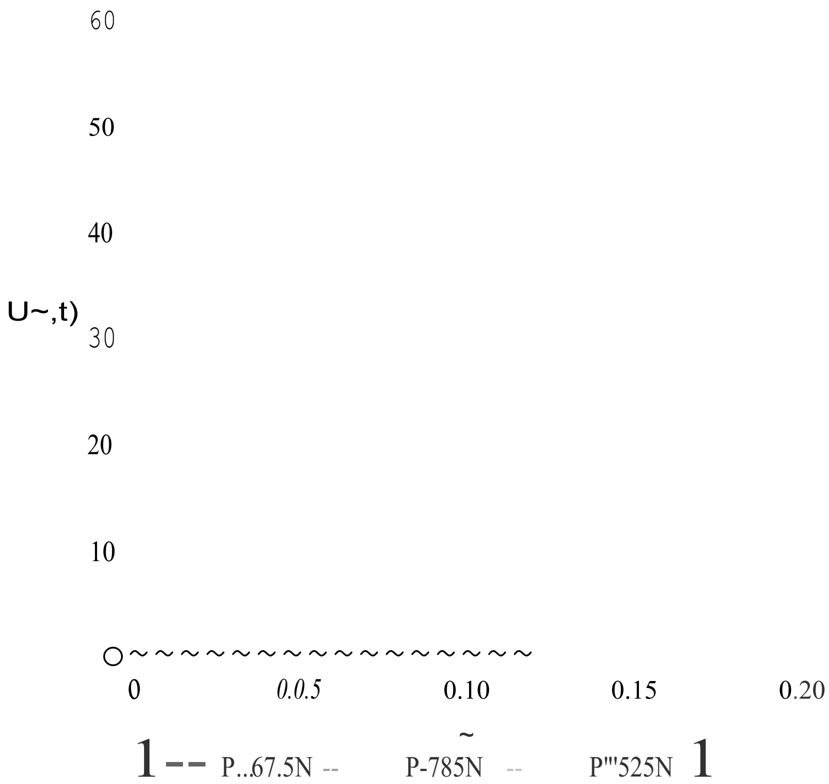


fig. 4.18 Deformed shell shape for different values load at first mode

0.0

. 4.19 Deformed shell shape for increase of load from zero to 525N

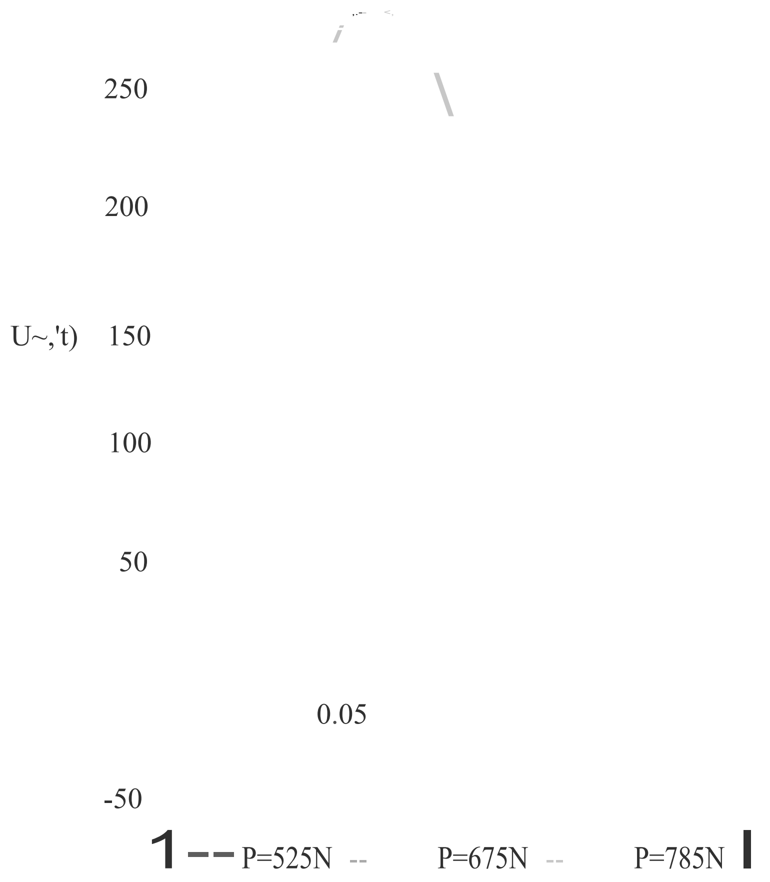


fig. 4.20 Deformed shell shape for different values load at second mode

0.0

4.21 Deformed shell shape for increase of load from zero to 525N

CHAPTER FIVE

5.0 DISCUSSION OF RESULTS, CONCLUSION AND RECOMMENDATION

5.1 Discussion of Results

From the solutions i.e. eqn. (3.30) and eqn. (3.52) the axial-Inplane stress function is that of an equivalent lateral distributed load as far as the lateral displacement u is concerned. The solutions have some constant terms an oscillating (harmonic) factor and a factor which exhibits an exponential decay away from the edge of the shell. This decay in the lateral deflection due to edge stress couple or an edge transverse shear resultants. Since the shape bending number is proportional to the derivatives of the lateral deflections, each of these also decays away outwardly from the edge of the shell.

igA.1 shows frequency plotted over the wave number it is observed that for each mode number m there is a corresponding frequency of the system. In fig. 4.2 the ~ffectof different values of damping coefficients on the amplitude of vibration with =1, T=0.01, and T_o = 1.5, as the value of the damping coefficients increases the mplitude of vibration decreases. The displacements of damping coefficient in z rection are reported in Figs. 4.3(a)-(c). It is observed that with the damping efficient increasing, the risk of resonance will be sufficiently reduced. Fig *4A* ows the shell response to the effect of initial velocity at equilibrium stage with 0.01 and c=76 as the value of the velocity increases the amplitude of vibration increases. The effect in z direction and the shell deformations are reported in figs. 4.5(a)-(c) it is observed that the nonlinearity is of the weak, softening type. The response that is attained physically depends on the initial conditions.

Fig. 4.6 and figA.7 (a)-(d) shows displacement profile for the cylindrical shell with different values of wave numbers with 1:=0.1, p=1 and k=1 it is observed that the model is more accurate for values of m ~ 5 which conforms with the result of Amabili (2005).

Fig 4.8 shows time response of the cylindrical shell, it is observed that amplitude increases with time but dies out quickly due to the effect of damping.

Fig. 4.9 mode shape of inplane stress function is displayed, it is observed that if the axcitation amplitude increases this produces wrinkling of the shell. However, if the 'hell is disturbed the shell may respond with the nonlinear terms.

ig. 4.10 shows the response of forced vibrating cylindrical shell to different values pulsating force with m=5, ljJ=3, and 1:=0.1 it is observed that the largest force has e highest amplitude value and ljJ can take only odd value. The deformations of e shell in z direction are reported in figs. 4.11 (a)-(c) it is observed that higher des of vibration can be neglected, this is because the contribution to the amic displacement is mainly done by the first mode and the characteristic shape of the first mode is similar to the force distribution. Fig. 4.12 shows time response of a forced vibrating cylindrical shell with m=5, I/J =3 and po=565N/m2 it is observed that as the time increases there is corresponding increase of the amplitude. Fig 4.13 shows displacement for various mode indexes.

Fig. 4.14 shows response of forced vibrating cylindrical shell to different values of moving force with m=5, I/J=3,T=0.1and v_i = 8.2 it is observed that I/Jcan assume an integer value and the largest force has the highest amplitude. The deformations of the shell in z direction for different values of moving force with constant magnitude are also reported in figs 4.15(a)-(c). Fig. 4.16 shows time response of a forced vibrating cylindrical shell with m=5, I/J=3 and qf= 1 x 1*03N/m2,* it is observed that as the time increases there is corresponding increase of the excitation amplitude. Cig. 4.17 shows response of the shell to different values of velocity. It is observed hat if the velocity of the moving force acting on the shell increases the critical peed of the vibrating system involving shell under the action of moving forces creases but with increase in the damping coefficient, the risk of resonance will be fficiently reduced.

g. 4.18 and fig. 4.20 shows deformed shell shape for different values load at first d second mode respectively. Fig. 4.19 and fig. 4.21 shows deformed shell shape increase of load from 0 to 525N in first and second mode respectively. Simple iodic motion and chaotic responses have been detected. This indicates the nonlinear dynamics of the cylindrical shell subject to external excitation. It is visible that in some regions the shell presents the same behavior when the load is increased or decreased. This indicates that different stable solutions coexist for the same set of system parameters, so that the solution is largely affected by initial conditions.

These elastic stability considerations are very important in the analysis and design of structure in which compressive stresses result from the loading, because in addition to ensuring that the structure is not merely over stressed or over deflected, in this case a new failure mode has been added, i.e. bucklinq. Initial imperfections in shells can result in their buckling at loads far below their theoretical capacity. Once a shell buckles, its collapse tends to be complete .

.2 Conclusion

Ihe nonlinearshell model equation (in the sense of Donnell - Mushtari - Vlasov eory) was reduced to linear equations using the regular perturbation technique nd the obtained linear equations were solved analytically. Free and forced ibrations were considered. In the case of the latter, pulsating pressure and moving rce having constant magnitude were studied. Finally the stability of the cylindrical ~ellwith or without initial imperfection was studied. The following conclusions were awn for the study carried out:

(a) It is observed that higher modes of vibration can be neglected; this is because the contribution to the dynamic displacement is mainly done by the first mode and the characteristic shape of the first mode is similar to the force distribution on the system.

(b) When the load is increased or decreased on the shell surface some regions in the shell presents the same behavior. This indicates that different stable solutions coexist for the same set of system parameters, so that the solution is largely affected by initial conditions.

(c) The method of regular perturbation presented in this thesis has proved to be successful in reducing the nonlinear partial differential equation into a linear form using the perturbation parameter.

(d) If the shell is subjected to external forces and the excitation amplitude increases this produces wrinkling of the shell, which will result in the shell deformation.

(e) The study has shown that nonlinear theories are very necessary in predicting the stability of the cylindrical shell

(f) When damping coefficient increases the amplitude of she" vibration decreases, hence the risk of resonance is sufficiently reduced.

5.3 Applications

(a) This work will assist the practicing engineers to evaluate the dynamic response of a free and forced vibrating cylindrical she" subjected pulsating pressure, moving force and other forms of external loading with ease.

(b) It can be applied to calculations involving shells often encountered in structural design, and other areas of she" application in engineering.

(c) Also, it can assist the engineers in analyzing the stability of she" with or without initial imperfection

.4 Recommendations

Detailed analysis of she" structures should be carried out before it is used in esigns to prevent collapse which can lead to destruction of life and properties

b) Due to the importance of she" structures in Science and Engineering, we commend that interested researchers should consider the case of a thick she" by nploying other she" models.

Interested researchers can also employ other forms of boundary conditions it is done in science and engineering

(d) Though we consider the case of buckling of the shell, interested researchers can extend the work to include stiffeners

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APPENDIX A

restart;

H(~) :=– ~ 'B[p].(p.n)2·sin(p'n.~);

$$H(\sim) := - \quad R\ B_p\ p2n2\ \sin(p\ \ n\sim)\ l \tag{1}$$

c[k] := 2kn / *k ti - cos(kn) sin(kn)*

ck:=.[2)r----k-n:---- / *k t: - cos(kn:) sin(kn:)* (2)

<|>(~) := *c[k]'sin(k'n'~);*

<|>(~) :=.[2 ------- *kn:* / *k t: - cos(kn:) sin(kn)* *sin(kn~)* (3)

a[k] := $\int_0^1$ *(H(~)* '<|>(~))d~;

ak:= ▌▌ k= lp= 1 *I (_p2* + *k2)* 1 *[R B p p2* n:3/2 .[2 *j* *k t: - cos(kn) sin(kn)* *k* (

-pcos(P1t) sin(k1t) *+ksin(p1t)* cos(h)) 1

APPENDIXB

restart;

Ai := 2 0't [0 11t 0m2 0sin (moe) 0 / <l>

Ai := 2 'to 1t m2 sin (rn e) / <l> (1)

A2 := *(koBo* (1 - *cos(k°1t))* *osin(po1to~)* *osin(k°1t°~));*

A2 := k B (1 - cos(k 1t)) sin (p 1t~) sin (k 1t~) (2)

A3 := (i~B (1 - cos(k°1t)) osin(po1to~) -sin] k°1t°~));

A3:= *p2 B* (1 *-cos(k1t))* *;in(p1t~)* *sin(k1t~)* (3)

A4 :=- 40 r [0] 01t 0rn2 0cos2(rn 0e) / <l>°sm(moe) *(Bopo* (1 - *cos(ko1t))* *ocos(po1to~)* *ocos(k°1to~));*

A4 := - 4't01tm2cos(me)2 *Bp* *(l-cos(k1t))* *cos(p1t~)* *cos(k1t~)* / <l> sin *(rn* e) (4)

G(~) *:=Ai (A2+A3)* *+A4;*

m :~~(2 'o'lOm' sin(me) (k B (1 - cos(h)) *sin(p,,~)* sin(kn:~) (5)

+ *p2 B* (1 - cos(k 1t)) ;in (p 1t~) sin (k 1t ~)))

4't01trn2cos(me)2 *Bp* *(l-cos(k1t))* *cos(p1t~)* *cos(k1t~)*

<l> sin(rn e)

c[k] := *2k1t* / k t: - *cos(k1t)* *sin(k1t)*

ck:=.J2 *k1t* / k t: - *cos(k1t)* *sin(k1t)* (6)

<l>(~) := *c[k]osin(ko1to~);*

<l>(~) :=[2 *k1t* / k t: - *cos(k1t)* *sin(k1t)* *sin(k1t~)* (7)

d[k] := I 0 to 1 (G(~) <l> (~)) d~;

- ~ksin(m8) 1/(_p2+4k2)p (2 t m2 B 12Fj 0 k1t-cos(k1t) k sin(k1t) (4 k2p2

cos(m 8)2 - 2 k4 cosl z rr] cos(m 8)2 + cos(p 1t) p2 k2 - 4 k2p2 cos(k1t) cos(m 8)2

+ 2 k4 cos(m 8) 2 + 5 cos] k 1t) 3p2 k2 COS(p 1t) cos(m 8) 2 - 2 p2 k2 + 2 cos] k 1t) p2 k2

- 2 k4 + 2 cos(k 1t) k4 - cos(k1t) 3p4 COS(p 1t) - 2 cos] k rr) cos(p 1t) k4 - cos(p 1t) p4

+ 2 cos(p 1t) k4 + cos(k1t) cos(p 1t) p4 - cos(k 1t) cos(p 1t) p2 k2

+ 2 cos(k1t) k4 cos(p 1t) cos(m 8)2 + p2 k2 COS(k1t)2 cos(p 1t)

+ p2 k2 COS(p 1t) cos(m 8)2 - p4 COS(k1t)2 cos(p 1t) cos(m 8) 2

- cos(k 1t) 3p2 k2 cos(P 1t) + cos(k n) 3P4 cos(P 1t) cos(m 8) 2

- cos(k1t) p4 cos(p 1t) cos(m 8) 2 + p4 cos(k1t) 2 COS(p 1t) - 2 k4 cos(p 1t) cos(m 8) 2

- 4 p3 k cos(k 1t) sin (k rt] sin (p 1t) cos(m 8) 2 + 2 p3 k cos(k rt] sin (k 1t) sin (p n)

- 5 p2 k2 cos(k 1t) 2 cos(P 1t) cos(m 8) 2 + P4 cos(P 1t) cos(m 8) 2

-cos(k1t) p2 k2cos(p1t) cos(m8)2 +4 cos(k1t)2 / ksin(k1t) sin(p1t) cos(m8)2

- 2 cos(k n) 2/ k sin (k 1t) sin (p rt] - 2 P k3 cos(k n.) sin (k 1t) sin (p 1t) cos(m 8) 2

+2pk3cos(k1t) sin(k1t) sin(p1t) +2cos(k1t)2pk3sin(k1t) sin(p1t) cos(m8)2

- 2 cos(k1t)2 P k3 sin(k1t) sin(p 1t)))

Where

To solve eqn. (3.2.34), the function $U_{m1}(\xi, \tau)$ can be expressed as a series of Eigen function

$$u_{m1}(\xi,\tau)=\sum_{k=1}^{\infty} b_k(\tau)\phi_k(\xi) \quad 3.2.35$$

$\psi_k(\xi)$ is chosen so as to satisfy the simply supported boundary conditions

-ience we let

3.2.36

'he Eigen function is now given by

3.2.37

Iere C_k remain arbitrary constant chosen to normalize the Eigen function