OPTIMAL CONTROL STRATEGIES IN DIFFERENTIAL GAMES

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CERTIFICATION

This thesis titled – "OPTIMAL CONTROL STRATEGIES IN DIFFFERENTIAL GAMES"", by Shehu Musa Danjuma, meets the regulations governing the award of the degree of Masters of Technology, in Mathematics, Federal University of Technology, Minna and is approved for its contribution to knowledge and literacy presentation

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DEDICATION

This research work is dedicated to the glory of God and to the memories of my father, grand parents and my brother Danjuma Malik.

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ABSTRACT

The study of game theory has gone on for ages. From most of the researches carried out, it is worth saying that the theory of differential games is still in a state of flux, partly because of the difficulties in formulating the "rules of the game". This research work lays emphasis on the formulation of differential games which are models of two-person zero sum games (game theory) via the optimal control theory. The derivation of differential games from optimal control is analysed.

TABLE OF CONTENT

		PAGES
CERTIFICATION		ii
DEDICATION		iii
ACKNOWLEDGEMENT		iv
ABSTRACT		v
TABLE OF CONTENT		vi
CHAPTER ONE		
FUNDAMENTALS OF OPTIMIZATION THEORY		
1.1	HISTORICAL DEVELOPMENT OF OPTIMIZATION THEORY	2
1.2	MOTIVATION FOR STUDYING OPTIMIZATION	3
1.2.1	OPTIMIZATION AS A BRANCH OF MATHEMATICS	4
1.2.2	THE SCOPE OF OPTIMIZATION	5
1.3	DEFINITION OF OPTIMIZATION PROBLEMS	7
1.4	CLASSIFICATION OF OPTIMIZATION PROBLEMS	8
1.5	AIMS AND OBJECTIVES	10
1.6	AREAS OF APPLICATIONS OF OPTIMIZATION	10
1.6.1	ENGINEERING APPLICATIONS OF OPTIMIZATION	10
1.6.2	OPTIMIZATION AS A TOOL IN DECISION MAKING	11
1.6.3	APPLICATION OF OPTIMIZATION IN THE STUDY OF GAMES	19
CHAPTER TWO		
AN OVER-VIEW OF GAME THEORY		
2.1	INTRODUCTION TO GAME THEORY	21
2.1.1	MOTIVATION FOR STUDYING GAME THEORY	22
2.2	CLASSIFICATION OF GAME PROBLEMS	23
2.2.1	DISCRETE GAME PROBLEM	23
2.2.2	CONTINUOUS GAME PROBLEMS	25
2.3	MINIMAX PROBLEMS	27
2.4	TWO-PERSON ZERO-SUM GAME WITH A SADDLE POINT	32
2.5	TWO-PERSON ZERO-SUM GAME WITH MIXED STRATEGIES	34

2.6 APPLICATIONS OF GAME THEORY.. 46

CHAPTER THREE

APPLICATIONS OF OPTIMAL CONTROL THEORY IN DIFFERENTIAL GAMES

3.1 BASIC CONCEPTS OF CONTROL THEORY.. 50

3.1.2 STATEMENT OF OPTIMAL CONTROL PROBLEM............................. 53

3.1.3 NECESSARY CONDITIONS FOR DIFFERENTIAL GAMES.................... 53

3.2 INTRODUCTION TO DIFFERENTIAL GAMES................................ 57

3.3 STATEMENT OF DIFFERENTIAL GAME PROBLEM....................... 58

3.3.1 SADDLE POINT PROPERTY FOR DIFFERENTIAL GAME PROBLEMS... 59

3.4 SOLUTION STRATEGIES IN DIFFERENTIAL GAMES..................... 59

3.4.1 SOLUTION STRATEGY OF DIFFERENTIAL GAME USING EQUATION OF MOTION. 59

3.4.2 SOLUTION STRATEGY OF DIFFERENTIAL GAME USING LINEAR EQUATION. 61

3.5 APPLICATION OF DIFFERENTIAL GAMES.................................... 66

CHAPTER FOUR

PROGRAM DESIGN

4.1 INTRODUCTION 72

4.2 FLOW CHART.. 73

4.3 COMPUTATIONAL RESULTS... 74

4.4 COMPUTATIONAL RESULTS (GRAPHICAL SOLUTIONS) 82

CHAPTER FIVE

5.1 DISCUSSION.. 88

5.2 RECOMMENDATION... 88

REFERENCES 89

CHAPTER ONE

INTRODUCTION

1.1 Historical Development Of Optimization Theory

The existence of optimization methods can be traced back to the days of Newton, Lagrange and Cauchy. One of the first recorded instances of optimization theory concerns the finding of a geometric curve of given length which will together with a straight line, enclose the largest possible area. Archimedes conjectured correctly that the optimal curve is a semi-circle. Some of the early results are in the form of principles which attempt to describe and explain natural phenomena.

One of the earliest examples was presented approximately hundred years after Archimedes' conjecture. It was formulated by Heron of Alexandria in 100 B.C., who postulated that light always travels by the shortest path. It was not until 1657 that Fermat correctly generalized this postulate by stating that light always travels by the path which incures least time rather than least distance.

The fundamental problem of another branch of optimization is concerned with the choosing of a function that minimizes certain *functionals*. Two problems of this nature were known at the time of Newton.

The first involves finding a curve such that the solid of revolution created by rotating the curve about a line through its endpoints causes the minimum resistance when this solid is moved through the air at constant velocity.

The second problem is called the *brachistochrone*. In this problem two points in space are given. One wishes to find the shape of a curve joining the two points, such that a frictionless bead traveling on the curve from one point to the other will cover the journey in least time. This problem was posed as a competition by John Bernoulli in 1965. The problem was successfully solved by Bernoulli himself, de l' Hôpital, Leibniz and Newton.

Definition 1.1.1

A functional is a special type of function whose domain is a set of real-valued functions.

In 1760 Lagrange invented a method for solving optimization problems that had equality constraints using his Lagrange multipliers. Lagrange transformations are among other uses employed to examine the behaviour of a function in the neighbourhood of a suspected optimum. And Gauss, who made contributions to many fields, developed the method of Least squares curve fitting which is of interest to those working in optimization.

The major developments in the area of numerical methods of unconstrained optimization have been made in the United Kingdom in the 1960s. The development of *simplex* method by Dantzig in 1947 or linear programming problems and the principle of optimality in 1957 by Bellman for dynamic programming problems paved the way for development of the methods of constrained optimization.

Geometric programming was developed in the 1960s by Duffin, Zenar and Peterson. Gomory did pioneering work in integer programming areas of optimization. The reason for this is that most of the real-world applications fall under this category of problems.

1.2 Motivation for studying Optimization

The efficient operation of systems activity (e.g. chemical processing plants) and theoretical entities (e.g. economic models) requires an attempt at the optimization of various indices which measure the performance of the system.

Sometimes these indices are quantified and represented as algebraic variables. Then values for these variables must be found which maximize the gain or profit of the system and minimize the waste or loss. The variables are assumed to be dependent upon a number of factors, some of these factors often under the control, or partial control of the analyst responsible for the performance of the system.

The process of attempting to manage the limited resources of a system can usually be divided into six phases:-

i. Mathematical analysis of the system

ii. Construction of a mathematical model which reflects the important aspects of the system.

iii. Validation of the model

iv. Manipulation of the model to produce a satisfactory, if not optimal solution to the model.

v. Implementation of the solution selected and

vi. The introduction of a strategy which monitors the performance of the system after implementation.

The theory of optimization is concerned with the fourth phase which is the manipulation of the model to produce optimal solution to the model.

Based on the fact that the theory of optimization provides this link in the chain of systems management is an important body of mathematical knowledge.

1.2.1 Optimization as a Branch of Mathematics

Typically the theory of optimization is mathematical in nature, it involves the maximization or minimization of a function (sometimes unknown) which represents the performance of the system. This is carried out by finding values for those variables (which are both quantifiable and controllable) which cause the function to yield an optimal value.

A knowledge of linear algebra and differential multivariable calculus is required in order to understand how the algorithms operate. Secondly, a knowledge of analysis is necessary for an understanding of the theory.

Some of the problems of optimization theory can be solved by the classical techniques of advanced calculus: Such as *Jacobian* methods and the use of *Lagrange multipliers*.

However most optimization problems do not satisfy the conditions necessary for solution in this manner. Of the remaining problems, many although amenable to the classical techniques are solved more efficiently by methods designed for the purpose. Throughout recorded mathematical history a collection of such techniques has been built up. Some have been forgotten and reinvented, others received little attention until modern-day computers made them feasible.

The bulk of the material of the subject is of recent origin because many of the problems, such as traffic flow, are only now a concern and also because of the large numbers of people now available to analyse such problems. When the material is catalogued into a meaningful whole, the result is a new branch of applied mathematics.

1.2.2 <u>**The Scope of Optimization**</u>

One of the most important tools of optimization is linear programming. A linear programming problem is specified by a linear, multivariable function which is to be optimized (maximized or minimized) subject to a number of linear constraints. Problems from a wide variety of fields of human endeavor can be formulated and solved by means of linear programming. Resource allocation problems in government planning, network analysis for urban and regional planning, production planning problems in industry, and the management of transportation distribution are just a few. Thus linear programming is one of the successes of modern optimization.

Integer programming is concerned with the solution of optimization problems in which at least some of the variables must assume only integer values. Many problems of a combinatorial nature can be formulated in terms of integer programming. Some practical examples include facility location, job sequencing in production lines, assembly line balancing, matching problems, inventory control and machine replacement.

Another class of problems involves the management of a network. Problems in traffic flow, communications, the distribution of goods and project scheduling are often of this type. Many of these problems can be solved by linear or integer programming. However because these problems usually have a special structure, more efficient specialised techniques have been developed for their solution. The ideas of labelling method for maximizing the flow of a commodity through a network and the out-of-kilter method for minimizing the cost of transporting a given quantity of a commodity through a network can be combined with those of integer programming to analyze a whole host of practical network problems.

Some problems can be decomposed into parts, the decision processes of which are then optimized. This decomposition process is very powerful, as it allows one to solve a series of smaller, easier problems rather than one large; intractable problem. Systems for which this approach will yield a valid optimum are called serial multistage systems. Dynamic programming is one of the best known techniques to attack such problems.

In the formulation of many optimization problems the assumption of linearity cannot be made, as it was in the case of linear programming. There do not exist general procedures for nonlinear problems. A large number of specialized algorithms have been developed to treat special cases. Many of these procedures are based on the mathematical theory concerned with analysing the structure of such problems. This theory is usually termed classical optimization.

1.3 **Definition of Optimization Problems**

An optimization problem begins with a set of independent variables or parameters, and often includes conditions or restrictions that define acceptable values of the variables. Such restrictions are termed the *Constraints* of the problem. The other essential component of an optimization problem is a single measure of "goodness", termed the *Objective* function, which depends in some way on the variables.

The solution of an optimization problem is a set of allowed values of the variables for which the objective function assumes an "optimal" value. In mathematical terms, optimization usually involves maximizing or minimizing for example, an engineer may wish in the process of his design to maximize profit or minimize weight.

Problems in all areas of mathematics, applied sciences, engineering, economics, medicine and statistics can be posed in terms of optimization. In particular, mathematical models are often developed in order to analyze and understand complex phenomena. Optimization is used in this context to determine the form and characteristics of the model that corresponds most closely to reality.

Furthermore, most decision making procedures involve explicit solution of an optimization problem to make the "best" choice. In addition to this role, optimization problems often arise as critical sub-problems within other numeral processes. This situation is so common that the existence of the optimization problem may pass unmarked. For example when an optimization problem must be solved to find points where a function reaches a certain critical value.

The general form of optimization problem may be expressed in mathematical terms as:

$$\underset{x \in \mathbb{R}^n}{\text{Minimize}}\ F(x) \qquad (1.3.1)$$

$$\text{Subject to} \quad \left.\begin{aligned} c_i(x) &= 0 \quad i=1,2,\ldots..m' ; \\ c_i(x) &\geq 0 \quad i=m'+1,\ldots.,m. \end{aligned}\right\} \qquad (1.3.2)$$

The objective function F and constraint function $\{c_i\}$ (which taken together, are termed the problem functions) are real - valued scalar functions.

1.4 Classification of Optimization Problems

Most Optimization problems can be expressed in the form:

$$\begin{aligned}&\text{Minimize } F(x)\\&\quad x \in \mathbb{R}^n\\&\text{Subject to } c_i(x) = 0 \quad i=1,2,\ldots..m';\\&\qquad\qquad c_i(x) \geq 0 \quad i=m'+1,\ldots.,m.\end{aligned}$$

where F is the objective function,

c_I is the constraint function.

The most extreme form of classification would be to assign every problem to a separate category. However, this approach is based on the false premise that every difference is significant with respect to solving the problem. For example, with such a scheme it would be necessary to change methods if the number of variables in a problem changed from 3 to 4.

The most obvious distinctions in problems involve variations in the mathematical characteristics of the objective and constraint functions. For example, the objective function may be very smooth in some cases, and discontinuous in others, the problem functions may be of a simple form with well understood properties or their computation may require the solution of several complicated sub-problems.

The following table gives a typical classification scheme based on the nature of the problem functions, where significant algorithmic advantage can be taken of each characteristic:

Properties of f(x)	Properties of $\{c_i(x)\}$
Function of single variable	No constraints
Linear function	Simple bound
Sum of squares of linear functions	Linear functions
Quadratic function	Sparse linear functions
Sum of squares on nonlinear functions	Smooth nonlinear functions
Smooth nonlinear function	Sparse nonlinear functions
Sparse nonlinear functions	Non-smooth nonlinear functions

For example, a particular problem might be categorised as the minimization of a smooth nonlinear function subject to upper and lower bounds on the variables.

Other features may also be used to distinguish amongst optimization problems. The size of the problem affects both the storage and the amount of computational effort required to obtain the solution, and hence techniques that are effective for a problem with a few variables are usually unsuitable when there are hundreds of thousands of variables.

Another way in which optimization problems vary involves the computable information that may be available to an algorithm during the solution process. For example, it may be possible to compute analytic first and second derivatives of the objective function in one instance, while in another case only the function values may be provided. A further refinement of such a classification would reflect the amount of computation associated with obtaining the information. The best strategy for a problem clearly depends on the relative effort required to compute the function value compared to operations associated with the solution method.

Applications of optimization may include special needs that reflect the source of the problem and the frame work within which it is to be solved. Such "external" factors often dictate requirements in solving the problem that are not contained in the mathematical statement of the problem. For example, in some problems it may be highly desirable for certain constraints to be satisfied exactly at every iteration. The required accuracy also differs with the application; for example, it might be wasteful to solve an optimization problem with maximum accuracy when the results are used only in a minor way within some outer iteration.

1.5 AIMS AND OBJECTIVES

The aims and objectives of the study are

1. Formulation of differential games via optimal control theory
2. Consideration of control systems whose dynamics is described by a system of Ordinary Differential Equation, under the influence of two players.
3. Construction of strategies for the players in order to determine the optimal strategy a player can use to counteract the effect as a result of perturbation generated by the opponent.
4. To apply the solution strategies to real life problems.

1.6 Areas of Applications of Optimization

1.6.1 Engineering Application of Optimization

Optimization in its broadest sense can be applied to solve any engineering problem. Below are some typical applications from different engineering disciplines.

a. Design of water resources systems for maximum benefit.
b. Design of material handling equipment like conveyors, trucks for minimum cost.
c. Optimum design of electrical networks.
d. Optimum production planning, controlling and scheduling.
e. Optimal plastic design of structures.
f. Design of aircraft and aerospace structures for minimum weight.
g. Design of civil engineering structures like frames, foundations, bridges, towers and dams for minimum cost.

h. Design of pumps, turbines and heat transfer equipment for maximum efficiency.

i. Planning of maintenance and replacement of equipment to reduce operating cost.

j. Controlling the waiting and idle times and queuing in production lines to reduce the costs.

k. Optimum design of control system.

The works of Johnson, Kirsh, Haug and Arora in the eighteenth century deal with the optimum design of machines and structure system. The application of optimization methods in the design of thermal system was presented by Stocker.

1.6.2 Optimization as a tool in Decision making

One of the basic tools of solving optimization problems is the linear programming method and this technique can be used to find the best uses of an organisation's resources. It is used in Operational Research (OR) to solve particular types of problem known as allocation, transportation or assignment problems.

All organisations have to make decisions about how to allocate their resources, and there is no organisation that operates permanently with unlimited resources.

Consequently, management must continually allocate scarce resources to achieve an organisation's goals.

Below are some computational examples.

Example 1.6.21

A manufacturer who manufactures chairs and tables, which must be processed through assembly and finishing departments. Assembly has 60 hours available, finishing can handle up to 48 hours of work. A table requires 4 hours in assembly and 2 hours in finishing. Similarly a chair requires 2 hours in assembly and 4 hours in finishing. If profit is ₦70 per table and ₦30 per chair, determine the best possible combination of tables and chairs to produce in order to realise the maximum profit.

Solution

The information supplied above can be summarised as follows:

		Hours required for a unit product		Total hours available
		Tables	Chairs	
Departments	Assembly	4	2	60
	Finishing	2	4	40
Contribution per unit		₦70	₦30	

Step one

Identify and formulate the objective function

Let ₦70T = Total profit from sales of tables

₦30C = Total profit from sales of chairs

Thus the objective function is

Z = ₦70T + ₦30C

which shows the relationship of output to profit.

Step two

Identify the constraints:

Assembly departments:

$4T + 2C \leq 60$

Finishing department:

$2T + 4C \leq 48$

The problem can now be summarised in a mathematical form

Maximize: Profit = ₦70T + ₦30C

Subject to

$4T + 2C \leq 60$
$2T + 4C \leq 48$
} Structural constraints

$T \geq 0$
$C \geq 0$
} Non negative constraints

Step three

We now plot a graph to locate the feasible region in order to get the optimal solution for the problem.

From the constraint

$4T + 2C \leq 60$

we have $4T + 2C = 60$

for $T = 0$; $C = 30$

for $C = 0$; $T = 15$

Therefore the two points needed are:

(0,30) and (15,0)

Similarly from the constraint

$2T + 4C \leq 48$

we have $2T + 4C = 48$

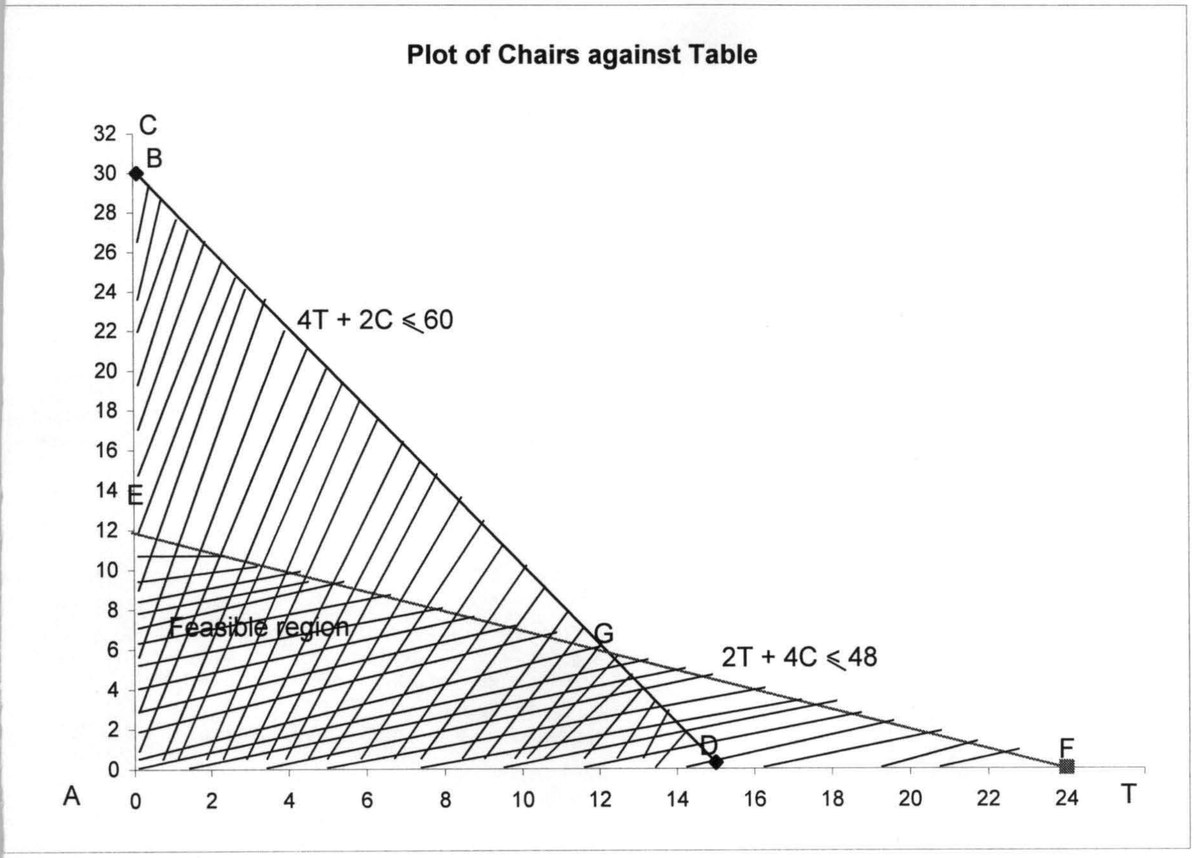
for $T = 0$, $C = 12$;

for $C = 0$, $T = 24$

Therefore the two points needed are:

(0,12) and (24,0)

We now proceed to plotting the graph.



Graphical solution to problem 1.5.21

The feasible region is bounded by points:

A(0,0): where Z= ₦70(0) + ₦30(0) = ₦0

E(0,12): where Z= ₦70(0) + ₦30(12) = ₦360

D(15,0): where Z= ₦70(15) + ₦30(0) = ₦1,050

G(12,6): where Z= ₦70(12) + ₦30(6) = ₦1,020

Therefore the point which yields the highest profit is point G=(12,6), which is the optimum point for the problem.

Example 1.6.22

A farm management wishes to minimize the cost of feeding the cattle of a client on a mixture. The minimum nutritional requirement is that the mixture must contain at least 10,000 units of nutrient A, 20,000 units of nutrient B and 15,000 units of nutrient C. There are two commercial feeds available to the consultant; each kilogram of the first feed costs ₦150 and contains 100 units of nutrient A, 400 units of nutrient B and 200 units of nutrient C. Each kilogram of the second feed costs ₦200 and contains 200 units of nutrient A , 250 units of nutrient B and 200 units of nutrient C.

A decision is to be taken on how to minimize total daily feed cost while at the same time meeting the nutrient requirements.

Solution

The information above can be summarized as follows:

Step 1

	Feed 1	Feed 2	Minimum units
A	100	200	10,000
B	400	250	20,000
C	200	200	15,000
Cost	₦150	₦ 200	

Let x_1 be the number of kilograms of feed 1 produced

x_2 be the number of kilograms of feed 2 produced

Then:

We have to;

Minimize: $z = 150x_1 + 200x_2$;

where z is the objective function, subject to the following constraints

$100x_1 + 200x_2 \geq 10000$ - (i)

$400x_1 + 250x_2 \geq 20000$ - (ii)

$200x_1 + 200x_2 \geq 15000$ - (iii)

$x_1 \geq 0$

$x_2 \geq 0$

Step 2

We now plot a graph to locate the feasible region in order to get the optimal solution for the problem.

From the constraint

$100x_1 + 200x_2 \geq 10000$

we have that

$100x_1 + 200x_2 = 10000$

when $x_1 = 0$; $x_2 = 50$

when $x_2 = 0$; $x_1 = 100$

Therefore the two points needed are (0,50) and (100,0)

From the constraint

$400x_1 + 250x_2 \geq 20000$

we have that

$400x_1 + 250x_2 = 20000$

when $x_1 = 0$; $x_2 = 80$

when $x_2 = 0$; $x_1 = 50$

Therefore the two points needed are (0,80) and (50,0)

From the constraint

$200x_1 + 200x_2 \geq 15000$

we have that

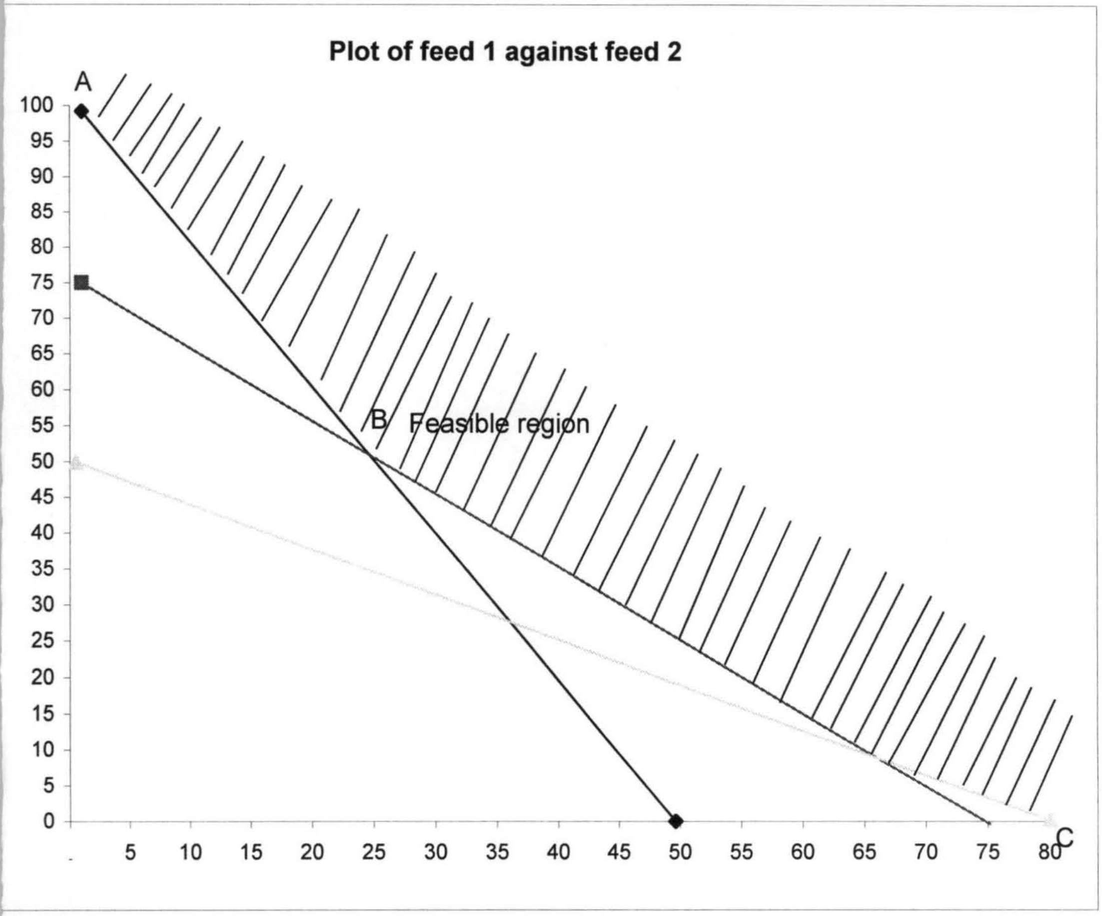
$200x_1 + 200x_2 = 15000$

when $x_1 = 0$; $x_2 = 75$

when $x_2 = 0$; $x_1 = 75$

Therefore the two points needed are (0,75) and (75,0)

The graphical solution is as shown below:



Graphical solution to problem 1.5.22

The feasible region is bounded by points:

A(100,0); where z = 150(100) + 200(0) = ₦15,000

B(50,25): where z = 150(50) + 200(25) = ₦12,500

C(0,80) ; where z = 150(0) + 200(80) = ₦16,000

The optimum point that will minimise the total daily feed cost is (50,25).

1.6.3 <u>Applications of Optimization in the study of Games</u>

Game theory has a close relation with linear programming. In a linear programming problem, the decision-maker is concerned with the allocation of the available limited resources to optimize his objective function. The various alternatives available for allocation of the resources are competitive in the sense that out of the various choices, only one will correspond to the optimum result.

However in a game theory problem, a decision maker has to select such strategies which will enable him to gain as much as possible or to lose as little as possible by taking into consideration the possible strategies of his competitors. In any game problem, each player is interested in determining his own optimal strategy.

In the next chapter we are going to discuss various techniques on how optimization could be applied to the study of game theory and also see its application to real life problems.

CHAPTER TWO

AN

OVER-VIEW OF

GAME THEORY

2.1 Introduction to Game Theory

The existence of game theory can be traced back to the days of John Von Neumann in 1928, who is the original developer of the theory. The mathematical relationship between game theory and linear programming was also initially recognised by Von Neumann. However, George B. Dantzig was the first person to apply the simplex method successfully to solve a game theory problem.

Whenever there is a situation of conflict and competition between two opponents, we refer to the situation as a "game". The opponents of a game may be individuals, groups of individuals or organisations. The opponents in this situation are usually called "players". Each player has a number of choices called "*Strategy*". These strategies can either be finite or infinite. A player is supposed to choose his strategies without any knowledge of the strategies selected by the other players. The net outcome of all the strategies chosen by all the players may represent a gain or loss or a draw to any particular player.

Game theory is concerned with discrete optimization problems involving two players with conflicting interests. A game can be formulated in the matrix below.

Let us consider a game formulated in the matrix below:

Player C chooses

		C_1	C_2	C_3 ....C_n
Player	R_1	a_{11}	a_{12}	a_{13}...............a_{1n}
	R_2	a_{21}	a_{22}	a_{23}...............a_{2n}
R				
	R_3	a_{31}	a_{32}	a_{33}...............a_{3n}
Chooses	:	:		
	:	:		
	R_m	a_{m1}	a_{m2}	a_{m3}...............a_{mn}

Figure 2.1.1 (A typical matrix game)

From the above figure we have two players referred to as R and C. The player R controls the rows(which represent his strategies) and the player C controls the columns (which represent his strategies).

If player R chooses the ith strategy (ith row) and player C chooses the jth strategy (jth column), then the element aij is assumed to represent the payoff(payment) from player C to player R. Thus if aij is a positive number, it represents payment of C to R and if it is a negative number, it represents payment of R to C.

In a game theory problem, a decision maker has to select such strategies which will enable him to gain as much as possible, or to lose as little as possible by taking into consideration the possible strategies of his competitors.

2.1.1 **Motivation for studying Game Theory**

Game theory has a wide range of applications, it has been for long applied to solve several mathematical, economics and military problems.

Whenever there is a situation of conflict and competition between two or more opponents, then this situation is referred to as a "game". The opponents of a game may be individuals, groups of individuals or organisations.

The formulation of game problems in the form of a game matrix has been used to solve many practical problems in decision making environments.

All organisations have to make decision about how to locate their branches in different locations for optimum profit, the application of game theory in this regard is being widely used in deciding the optimal strategies in terms of location.

2.2 Classification of Game problems

Game problems can basically be classified into Discrete and Continuous.

2.2.1 Discrete Game problems:

Discrete game problems are often represented in matrices form, which take the form of either n x n or m x n matrix and it involves two players with selection of strategies.

Example 2.2.1

Let us consider a 2 x 2 matrix where there are two players R and C and a selection of strategies Ri, i=1,2,....m and C_j, j=1,2,....n for each player. For each pair of strategies there is a corresponding pay-off P = G_{ij}. Player R attempts to minimize the pay-off, while C attempts to maximize it. We assume that each player has all the information above and that each player knows the other player's choice of strategies.

If C(the maximizer) plays first, he should pick the column with the largest minimum, since he knows R will subsequently pick the row with the minimum. Similarly if R(the minimizer) should play first, he should pick the row with the smallest maximum, since he knows C will subsequently pick the column with the maximum. This can be represented in 2 x 2 matrix game shown below, with values given to G_{ij} as in the matrix.

		Player C	
		C_1	C_2
Player R	R_1	$G_{11} = 1$	$G_{12} = 6$
	R_2	$G_{21} = 4$	$G_{22} = 8$

Figure 2.2.1 A (2 x 2 matrix) discrete game

From the above matrix, player C is maximizing while R is minimizing.

R_1 is the row with the smaller maximum, while C_2 is the column with the larger minimum.

The optimal choices for the game of figure 2.2.1 are R1 and C2 with the pay-off **6** regardless of who plays first.

Example 2.2.2

We further consider a 2 x 2 matrix game with a different set of pay-off and where the order of play makes no difference.

		Player C	
		C_1	C_2
Player R	R_1	$G_{11} = 3$	$G_{12} = 9$
	R_2	$G_{21} = 2$	$G_{22} = 5$

Figure 2.2.2 A (2 x 2 matrix) discrete game

Similarly from the above matrix, player C is maximizing, while R is minimizing.

R_2 is the row with smaller maximum, while C_2 is the column with the larger minimum.

The optimal choices for the game of figure 2.2.2 are R_2 and C_2 with the pay-off **5** regardless of who plays first.

Example 2.2.3

We shall further consider a discrete game where order of play makes no difference.

		Player C	
		C_1	C_2
Player R	R_1	$G_{11} = 2$	$G_{12} = 7$
	R_2	$G_{21} = 5$	$G_{22} = 9$

Figure 2.2.3 A (2 x 2 matrix) discrete game

From the above matrix, player C is maximizing, while R is minimizing. If player C (the maximizer) plays first, he should obviously pick the column with the larger minimum i.e column C_2, since he knows player R will subsequently pick the row with the minimum. Similarly if player R(the minimizer) plays first he should pick the row with smallest maximum i.e R_1, since he knows player C will pick the column with the maximum.

The optimal choices for the game of figure 2.2.3 are R_1 and C_2 with the pay-off **7**, regardless of who plays first.

2.2.2 Continuous Game

In a continuous game the choices of R and C are continuous instead of discrete. Therefore there must be a continuous pay-off function G(R,C) instead of a pay-off matrix Gij as explained in discrete games.

We look for a pair of choices

$$G(R^o,C) \leq G(R^o,C^o) \leq G(R,C^o) \text{ for all } R, C \quad - \quad (2.2.1)$$

The necessary and sufficient conditions for R^o, C^o are

$\partial G/\partial R = 0, \quad \partial G/\partial C = 0$ - (2.2.2)

$\partial^2 G/\partial R^2 \geq 0, \partial^2 G/\partial C^2 \leq 0$ - (2.2.3)

Any R^o, C^o satisfying the sufficient condition, is called a game – theoretic saddle point.

Example 2.2.4
Let us consider the case where

$J = G(R,C) = \frac{1}{2}(R^2 - C^2); \; -1 \leq R \leq 1 \; ; \; -1 \leq C \leq 1$ - (2.2.4)

Solution
$\partial G/\partial R = R$
$\Rightarrow \partial G/\partial R = 0$ for $-1 < R < 1$
$\Rightarrow R^o = 0$

Similarly
$\partial G/\partial C = - C$

$\Rightarrow \partial G/\partial C = 0$ for $-1 \leq C \leq 1$
$\Rightarrow C^o = 0$

Then
$\partial^2 G/\partial R^2 = 1$ and

$\partial^2 G/\partial C^2 = -1$

We now have that
$\partial G/\partial R = 0 \quad \Rightarrow R^o = 0,$

$\partial G/\partial C = 0 \quad \Rightarrow C^o = 0$ and

$\partial^2 G/\partial R^2 = 1 \quad \Rightarrow \partial^2 G/\partial R^2 > 0$

$\partial^2 G/\partial C^2 = -1 \quad \Rightarrow \partial^2 G/\partial C^2 < 0$

which shows that the function

$J = G(R,C) = \frac{1}{2}(R^2 - C^2); \; -1 \leq R \leq 1 \; ; \; -1 \leq C \leq 1$

is a game theoretic saddle point.

Example 2.2.5

We consider the case

$G=R^2-3RC+2C^2 \qquad -1\leq R\leq 1 \; ; \; -1\leq C\leq 1 \qquad -(2.2.5)$

We wish to verify the fact that

$\max_C \min_R G < \min_R \max_C G$

Solution

$\partial G/\partial R = 2R - 3C$

We have that

$2R - 3C = 0$

$=> 2R = 3C$

$R = 3/2\ C$; Substituting the value of R in (2.2.5)

We have

$G = 9/4\ C^2 - 9/2\ C^2 + 2C^2$

$\quad = -1/4\ C^2$

Therefore

$\max[\min G(R,C) = \max[-1/4C^2 \text{ with } R = 3/2\ C] = 0 \text{ with } C = 0$

Similarly

$\min[\max G(R,C)] = \min[R^2 + 3R + 2 \text{ with } C=-R] = 2 \text{ with } R=0$

Therefore we have that

$\max_C \min_R G < \min_R \max_C G$

2.3 MINI MAX (MAXMIN) Principle

In any game problem, each player is interested in determining his own "optimal" strategy. However because of the conflicting nature of the problem, and because of the lack of information regarding the specific strategies selected by the other player(s), optimality has to be based on a conservative principle.

Definition 2.3.1 (Minimax(Maxmin) Principle

A situation where each player selects his strategy which guarantees a pay-off that can never be worsened by the selection of his opponents is referred to as the minimax(maxmin) principle i.e.

maxmin value $\leq$ value of the game $\leq$ minimax value- (2.3.1)

Example 2.3.1

Let us consider a 2x2 matrix game with pay-off matrix as shown in figure 2.3.1

		Player C	
		C1	C2
Player R	R1	$G_{11}=1$	$G_{12}=6$
	R2	$G_{21}=3$	$G_{22}=7$

Figure 2.3.1 A (2x2 matrix) discrete game where the order of play makes no difference.

From the above matrix, player C is maximizing, while player R is minimizing. For each pair of strategies let the corresponding pay-off be P=Gij. Player C attempts to maximize the pay-off while player R attempts to minimize it. Player C will select the column with the largest minimum while R will select the row with the smallest maximum.

The optimal choices for the game of figure 2.3.1 are R_1 and C_2 with the pay-off 6 regardless of who plays first.

i.e. we have

$$\max_{C_j} \min_{R_i} G_{ij} = 6 = \min_{R_i} \max_{C_j} G_{ij}$$

(C plays first) (R plays first)

Example 2.3.2
Let us consider a matrix game with pay-off matrix as shown in figure 2.3.2

Player C

		C1	C2
Player R	R1	$G_{11}=5$	$G_{12}=7$
	R2	$G_{21}=4$	$G_{22}=8$

Figure 2.3.2 A (2x2 matrix) discrete game where the order of play makes no difference.

If player C is maximizing, while player R is minimizing, The optimal choices for the game of figure 2.3.2 are R_2 and C_2 with the pay-off 8 regardless of who plays first.

i.e we have

$$\max_{C_j}\min_{R_i} G_{ij} = 8 = \min_{R_i}\max_{C_j} G_{ij}$$
(C plays first) (R plays first)

Under the minimax(maxmin) principle we shall now consider m x n pay-off matrix game problems.

Example 2.3.3
Find the maxmin(minmax) solution (the optimal strategies for the pay-off matrix of two person game shown in figure 2.3.3 if player R is maximizing and C is minimizing.

Player C

		C1	C2	C3	C4
	R1	$G_{11}=6$	$G_{12}=1$	$G_{13}=7$	$G_{14}=3$
Player R	R2	$G_{21}=4$	$G_{22}=3$	$G_{23}=5$	$G_{24}=6$
	R3	$G_{31}=5$	$G_{32}=1$	$G_{33}=2$	$G_{34}=5$

Figure 2.3.3 A (3 x 4 matrix) discrete game where the order of play makes no difference.

Solution

If player R, (the Maximizer) selects his first strategy (R_1) he may gain; 6, 1, 7 or 3 depending on the strategy selected by player C.

Thus player R is guaranteed a gain of at least
1 = min(6,1,7,3) if he selects strategy R_1 irrespective of the strategy selected by player C.

Similarly R is guaranteed a gain of at least
3 = min(4,3,5,6) for strategy R_2 selection and
-2 = min(5,1,-2,5) for strategy R_3 selection.

Thus for player R to maximize his gain irrespective of the strategies of C, he has to maximize his minimum gain i.e.
3 = max(1,3,-2)

Similarly if player C chooses strategy C_1 he loses 6, 4 or 5 depending on the strategy selected by the player R.

Thus player C loses no more than

6 = max(6,4,5) for C_1 strategy
3 = max(1,3,1) for C_2 strategy
7 = max(7,5,-2) for C_3 strategy
6 = max(3,6,5) for C_4 strategy

Therefore for player C to minimize his loss, irrespective of player R, he has to minimize his maximum losses by selecting

3 = min(6,3,7,6) from strategy C_4

and this is called the minimax value of the game for player C.

Therefore:

$$\underset{R\ \ C}{\text{Maxmin}}\ G_{ij} = 3 = \underset{C\ \ R}{\text{minmax}}\ G_{ij}$$

(R plays first) (C plays first)

Example 2.3.4

Find the optimal strategies for the pay-off matrix of two person game as shown below if player R is maximizing and C is minimizing.

		Player C			
		C1	C2	C3	C4
	R1	$G_{11}=7$	$G_{12}=1$	$G_{13}=8$	$G_{14}=4$
Player R	R2	$G_{21}=5$	$G_{22}=4$	$G_{23}=6$	$G_{24}=7$
	R3	$G_{31}=6$	$G_{32}=2$	$G_{33}=-3$	$G_{34}=6$

Figure 2.3.4 A (4 x 3 matrix) discrete game where the order of play makes no difference.

Solution

Player R may gain; 7, 1, 8 or 4 if he selects his first strategy (R_1) depending on the strategy selected by player C.

Thus player R is guaranteed a gain of at least

1 = min(7,1,8,4) irrespective of the strategy selected by player C.

Similarly for strategy R_2 selection R is guaranteed a gain of at least

4 = min(5,4,6,7)

for strategy R_3 selection

-3 = min(6,2,-3,6)

Therefore for player R to maximize his gain irrespective of the strategies of player C, he has to maximize his minimum gain by selecting

4 = max(1,4,-3)

Similarly player C may lose 7,5 or 6 if he selects his first strategy (C_1) depending on the strategy selected by the player R.

Thus player C loses no more than

7 = max(7,5,6) for C_1 strategy

4 = max(1,4,2) for C_2 strategy

8 = max(8,6,-3) for C_3 strategy and

7 = max(4,7,6) for C_4 strategy regardless of the strategy sleeted by player R.

Therefore for player C to minimize his loss, irrespective of the strategies of player R, he has to minimize his maximum losses by selecting

4 = min(7,4,8,7) from strategy C_2

and this is called the minimax value of the game for player C.

Therefore:

$$\max_R \min_C G_{ij} = 4 = \min_C \max_R G_{ij}$$

(R plays first) (C plays first)

2.4 <u>Two-Person Zero-Sum Game with a Saddle point</u>

Definition 2.4.1(zero-sum game)

A *two-person zero-sum game* is a game played by two persons or groups where the gain of one person will be exactly equal to the loss of the other so that the sum total of the gains and losses will be equal to zero.

A two-person zero-sum game can be formulated conveniently in the form of a matrix known as the pay-off matrix as shown in figure 2.4.1.

	C_1	C_2	C_3	..........C_n
R_1	a_{11}	a_{12}	a_{13}	.... a_{1n}
R_2	a_{21}	a_{22}	a_{23}	.... a_{2n}
R_3	a_{31}	a_{32}	a_{33}	.... a_{3n}
:				
Rm	a_{41}	a_{42}	a_{43}	.... a_{mn}

Figure 2.4.1 The pay-off matrix

From the above pay-off matirx the two players are referred to as R and C. The player R controls the rows (which represent his strategies) and the player C controls the columns (which represent his strategies). If player R chooses the ith strategy (ith row) and player C chooses the jth strategy (jth column), then the element aij is assumed to represent the pay-off from player C to player R. Thus if aij is a positive number it represents the payment from player C to R and if it is a negative number, it denotes the payment of R to C.

Definition 2.4.2(Saddle point)

The element aij of the pay-off matirx is called a *saddle point* if it is the minimum among the ith row and maximum among the jth column elements.

Example 2.4.1

The saddle point could be illustrated as below:

	C_1	C_2	C_3	C_4	Row Minimum
R_1	7	1	8	4	1
R_2	5	4	6	7	4
R_3	6	2	-3	6	4
Column maximum	7	4	8	7	

Figure 2.4.2 The pay-off matrix

From figure 2.4.2 The saddle point element = 4

The value of the game is equal to the saddle point, and the optimal strategies for the two players are given by the row that contains the saddle point for player R, and the column that contains the saddle point for player C.

2.5 **Two-Person Zero-Sum Game with Mixed Strategies**

There are certain two-person zero-sum games which do not have saddle point i.e the optimal pure strategy of the game can not be found readily.

Let us consider a game for which the pay-off matrix is as shown in figure 2.5.1

		Strategies for player B		
		I	II	
Strategies of Player A	I	a	b	a maxmin solution
	II	c	d	[c]
		[c] minimax solution	d	

Figure 2.5.1 (A matrix game with mixed strategies)

If in figure 2.5.1 if the minimax solution is not equal to the maxmin solution then we say the game does not have a saddle point.

Generalised method of solution for matrix game problems with mixed strategies

If the probability of player A playing I is x

Then the probability of playing II is 1-x

Similarly, if the probability of player B playing I is y

Then the probability of B playing II is 1-y

The expected value of A if B plays I through out is

$ax + c(1-x) = bx + d(1-x)$

Then

$$ax - cx - bx + dx = d - c$$

$$x = \frac{d - c}{a + d - (b + c)}$$

$$1 - x = \frac{a - b}{a + d - (b + c)}$$

Similarly,

$$ay + b(1 - y) = cy + d(1 - y)$$

$$ay - by - cy + dy = d - b$$

$$y = \frac{d - b}{a + d - (b + c)}$$

$$1 - y = \frac{a - c}{a + d - (b + c)}$$

The value of the game is

$$ax + c(1 - x)$$

$$= \frac{ad - ac}{a + d - (b + c)} + \frac{ac - cb}{a + d - (b + c)}$$

$$= \frac{ad - bc}{a + d - (b + c)}$$

we now solve some examples using the generalised formula

Example 2.5.1

We consider a matrix game with a pay-off matrix as shown in figure 2.5.2

		Player B: I	Player B: II
Player A	I	11	7
	II	5	9

Figure 2.5.2 A (2x2 matrix) discrete game with mixed strategies.

where a = 11, b = 7
c = 5, d = 9

The value of the game is

$$V = \frac{11 \times 9 - 7 \times 5}{11 + 9 - (7 + 5)}$$

$= 64/8$

Value of the game $= 8$

The solution of two-person zero sum game with mixed strategy could also be obtained graphically.

Let us consider a matrix game with pay-off as shown below;

		Player C	
		C_1	C_2
Player R	R1	$G_{11}=11$	$G_{12}=7$
	R2	$G_{21}=5$	$G_{22}=9$

Figure 2.5.3 A (2x2 matrix) discrete game with mixed strategies.

Thus if player C plays a fixed choice while R uses a random choice, the expected pay-off for various probability mixes of R_1 and R_2 is as shown in figure 2.5.4 (a)

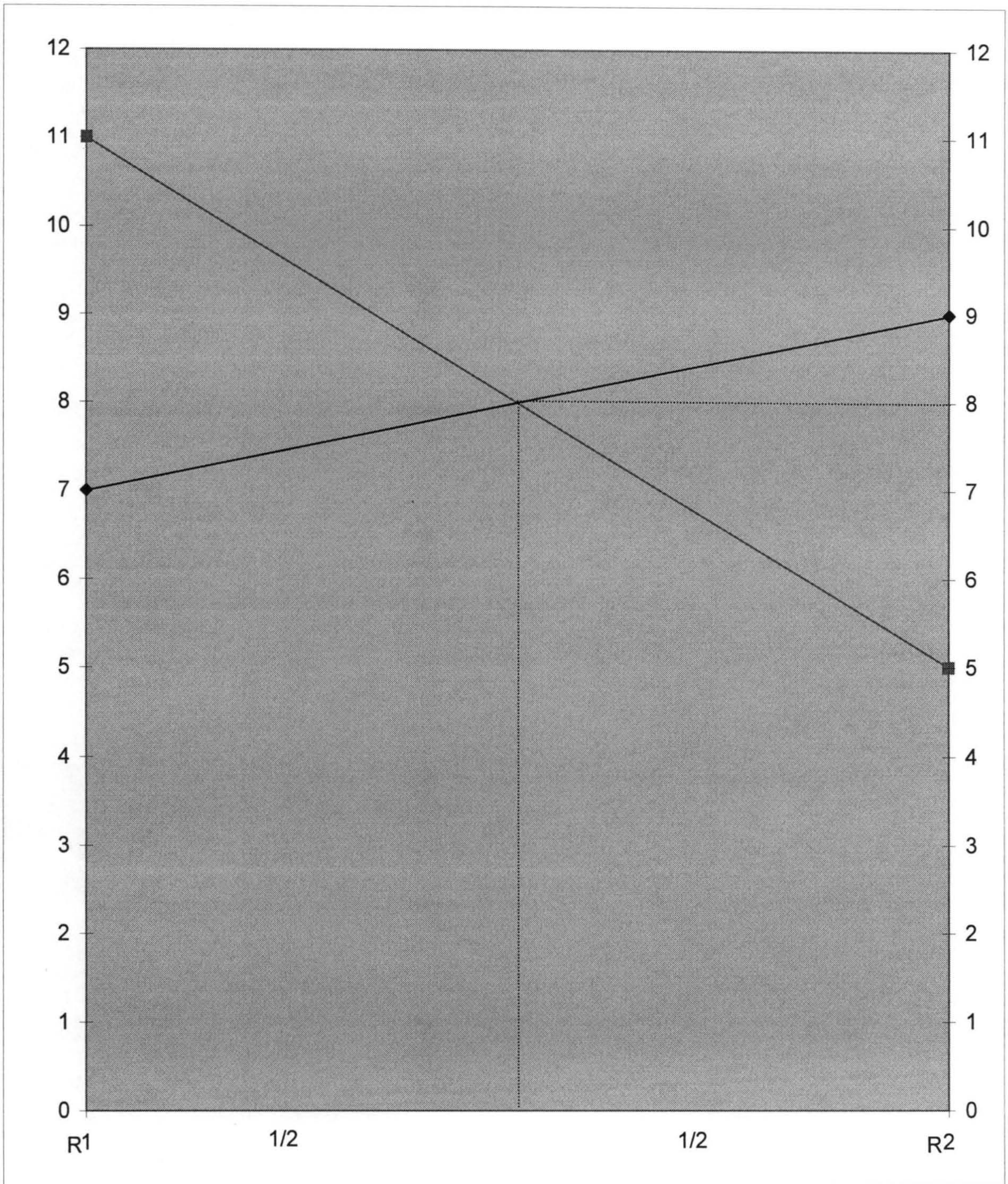


Figure 2.5.4(a) Illustration of the minimax solution of figure 2.5.4

In figure 2.5.4(a) for player R to realize maximal expected pay-off R must play the probability mix half of the time R_1 and of the time R_2.

Similarly if player R plays a fixed choice while C uses a random choice, the expected pay-off for
various probability mixes of c_1 and c_2 is as shown in figure 2.5.4(b)

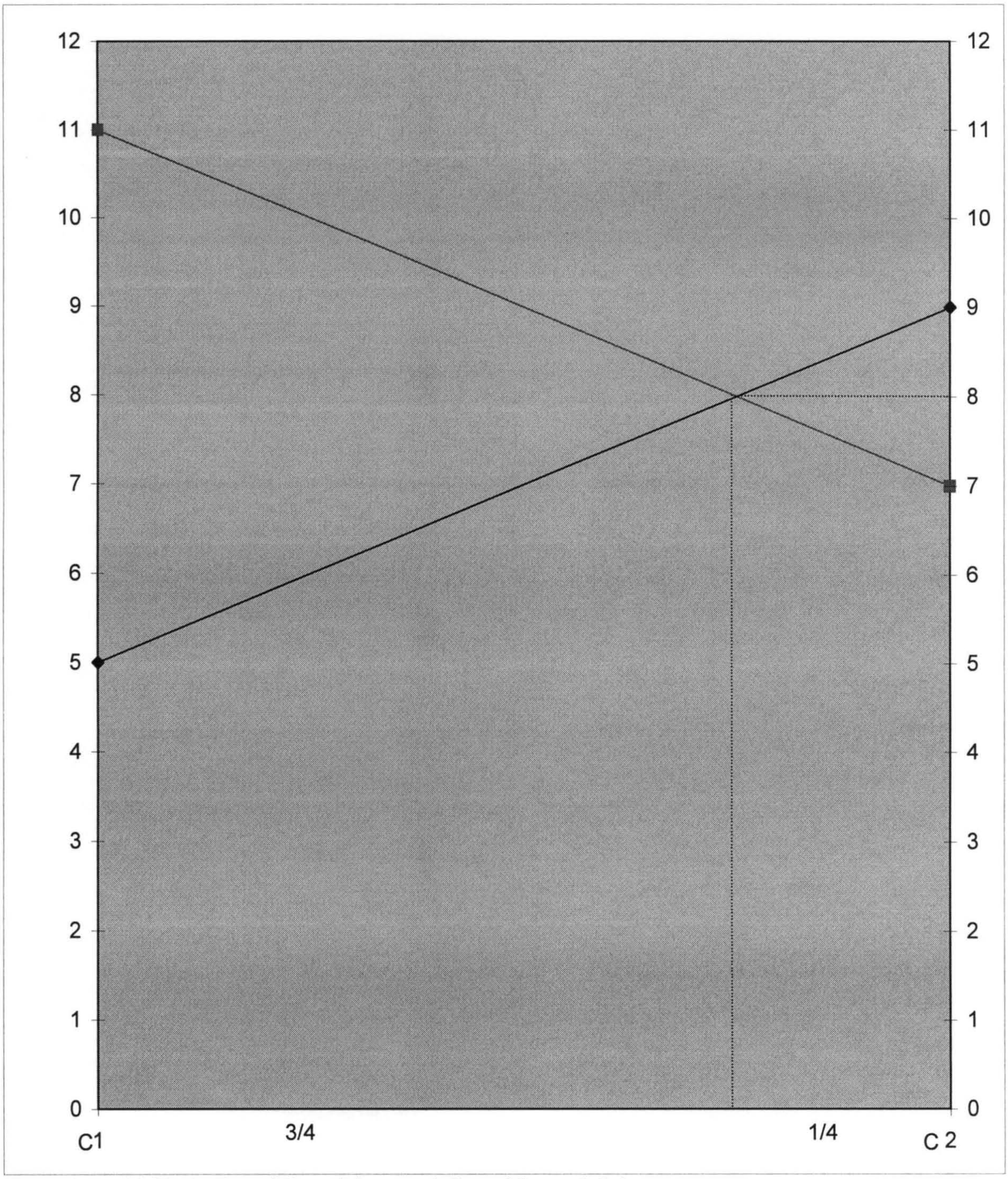


Figure 2.5.4(b) Illustration of the minimax solution of figure 2.5.4

Similarly in figure 2.5.4(b) player C must play the probability mix 1/4 of the time C_1 and 3/4 of the time C_2 to realize maximal expected pay-off.

We then have that;

$$E[\underset{p}{\text{minimax}}\ \underset{q}{}\ G_{ij}] = 8 = E[\underset{p}{\text{maxmin}}\ \underset{q}{}\ G_{ij}]$$

where E is the expected pay-off, p, q the probability mixes.

Example 2.5.2
Let us consider a matrix game with a pay-off matrix as shown in figure 2.5.5

Player C

		C_1	C_2
Player R	R1	$G_{11}=12$	$G_{12}=3$
	R2	$G_{21}=4$	$G_{22}=7$

Figure 2.5.5 A (2x2 matrix) discrete game with mixed strategies.

From the above matrix player C is maximizing, while player R is minimizing.

If C(the maximizer) plays first, he should pick C1 since this is the column with the larger minimum, namely 4. If R (the minimizer) plays first, he should pick R2 since this is the row with the smaller maximum namely 7. Thus it makes a difference who plays first.

i.e $\underset{C\ R}{\text{maxmin}}\ G_{ij} = 4 \leq \underset{R\ C}{\text{minmaz}}\ G_{ij} = 7$
(C plays first) (R plays first)

The above dilemma may be resolved by having each player make a random selection of strategies on each play according to some fixed probability.

Thus if player C plays a fixed choice while R uses a random choice, the expected pay-off for various probability mixes of R_1 and R_2 is as shown in figure 2.5.6 (a)

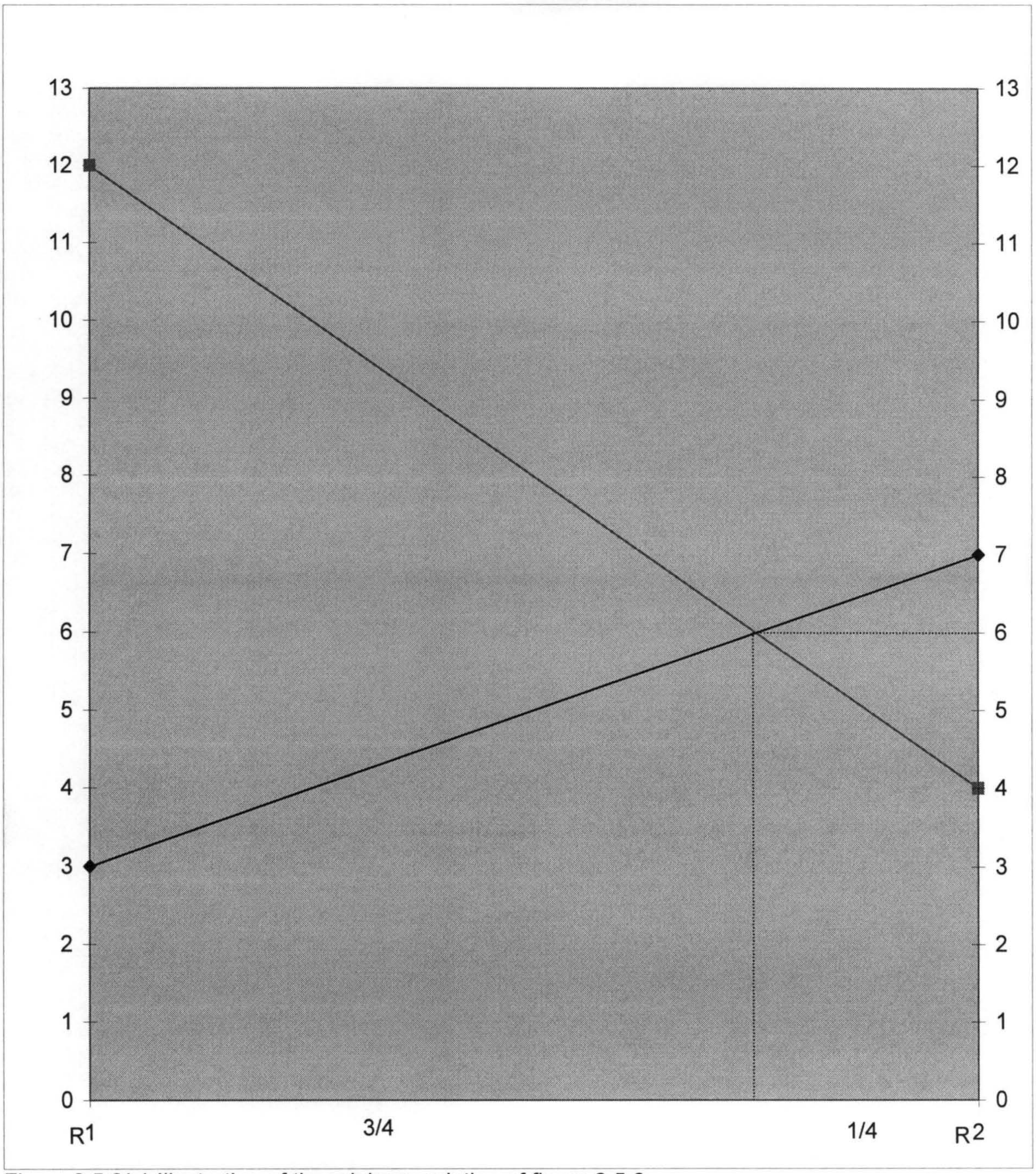


Figure 2.5.6(a) Illustration of the minimax solution of figure 2.5.6

In figure 2.5.6(a) for player R to realize maximal expected pay-off R must play the probability mix 1/4 of the time R_1 and 3/4 the time R_2.

Similarly if player R plays a fixed choice while C uses a random choice, the expected pay-off for
various probability mixes of C_1 and C_2 is as shown in figure 2.5.6(b)

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