A FIFTH ORDER SIX-STAGE EXPLICIT RUNGE-KUTTA METHOD FOR THE SOLUTION OF INITIAL VALUE PROBLEMS.

BY

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CERTIFICATION

This thesis titled "A FIFTH ORDER SIX-STAGE EXPLICIT RUNGE-KUTTA METHOD FOR SOLVING INITIAL VALUE PROBLEMS" by Abraham Ochoche meets the regulations governing the award of the degree of Masters of Technology in Mathematics, Federal University of Technology Minna and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This thesis is dedicated to my mother, Mrs. Dorcas A. Ocholi, for her love, support encouragement, and understanding.

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TABLE OF CONTENTS

ABSTRACT

The goal, the target, the objective, and indeed, the very essence of any Numerical method, is to replicate the Exact solution, or at the least produce solutions that are very close to the exact solution. Hence, the closer such a solution is to the exact solution, the better the method. In the light of this, we develop in this work, a new six-stage Runge-Kutta method, of order five, for the solution of Initial Value Problems. The strength of the new scheme is that it gives solutions that are very close to the exact solutions, even closer than some popular existing methods which are known to be highly efficient. Some Initial Value Problems were solved using the new scheme and the results help to establish its very high degree of accuracy.

CHAPTER ONE

GENERAL INTRODUCTION

1.1 INTRODUCTION

Historically, differential equations have originated in chemistry, physics and engineering. More recently, they have also arisen in medicine, biology, anthropology, and the like. However, we are going to restrict ourselves to Ordinary Differential Equations (ODE), with special emphasis on *Initial Value Problems* (IVP) ; so called because the condition on the solution of the differential equation, are all specified at the start of the trajectory i.e. they are initial conditions.

Numerical solution of ODEs is the most important technique in continues time dynamics. Since most ODEs are not soluble analytically, numerical integration is the only way to obtain information about the trajectory. Many different methods have been proposed and used in an attempt to solve accurately, various types of ODEs. However, there is a handful of methods known and used universally (i.e. Runge-Kutta, Adam-Bashforth-Moulton and Backward Difference Formulae). All these, discretise the differential system, to produce a difference equation or map.

The methods, obtain different maps from the same equation, but they have the same aim; that the dynamics of the maps, should correspond closely, to the dynamics of the differential equation. From the Runge-Kutta family of algorithms, come (arguably) the most well-known and used methods for numerical integration.

As stated earlier, mathematical modeling of physical everyday problems in different fields of human endeavours, often results in differential equations. With a differential equation, we can associate initial conditions, boundary, or auxiliary conditions on the unknown function and its derivatives. If these conditions are specified at a single value of the independent variable, they are referred to as initial conditions and the combination of the differential equation and an appropriate umber of interval conditions is called an Initial Value Problem, and these are the ones of particular interest to us in this work.

In elementary treatment of differential equations, it is assumed that the IVP has a unique solution that exist throughout the interval of interest and which can be obtained, by analytical techniques. However, many of the differential equations encountered in practice, cannot be solved explicitly, so we are led to methods for obtaining approximations to solutions. Such solutions are usually called numerical solutions. In finding numerical solutions to differential equations, the goal is to get a method, which will produce results that will (possibly) be the same as the exact solution. While this goal may not be easy to achieve, we aim for a numerical solution that is as close to the exact solution as possible.

With the advent of computers, numerical methods are now an increasingly attractive and efficient way to obtain approximate solutions to differential equations that had hitherto proved difficult, even impossible to solve analytically.

As was earlier noted, there exist a number of methods for solving differential equations this way. These methods can be broadly grouped as: one-step methods, and multi-step methods.

However, for this work, we are particularly interested in the class of methods first proposed by David Runge (1856-1927), a German mathematician and physicist, and further extended by another German mathematician called Wilhelm Kutta (1867-1944); a method commonly referred to as the Runge-Kutta methods.

1.2 LITERATURE REVIEW

The dynamics of the Runge-Kutta methods can be described as highly flexible. This is because the slightest change in any of the unknown parameters (b_r, c_r, a_{ij}), in course of formulating a Runge-Kutta scheme, would quite naturally result in a new scheme.

As a general example, if we consider the general S-stage Runge-Kutta method, a change in any of the free parameters (the free parameters results from the difference between the number of equations and the number of unknowns, during the Taylor series expansion), for a method of a particular stage number, would give rise to a different scheme of the same stage, and possibly the same order. As a specific example, let $S = 2$, we would arrive at a set of three equations in four unknowns, and thus, there would exist one (free) parameter family of solutions (i.e. one degree of freedom). Since there exists an infinite number of values that this free parameter can assume, it implies that there is an infinite number of two-stage Runge-tta methods of order two, that can be so derived by altering the free parameter. Lambert, '73)

The fundamental idea of the Runge-Kutta method is to avoid the computation of higher order derivatives that the Taylor method involves, when employed in obtaining solutions for Initial Value Problems (IVP).

DAVID RUNGE [1895], in his paper on the numerical solutions of differential equations, put forward a method for solving first order differential equations (specifically, IVP), that achieved a higher order than the Linear Multi-step Methods (LMM), by sacrificing the linearity of the algorithm while preserving its one-step nature. His method involves extending the approximations of the improved of the improved Euler method further, so as to obtain a one-step method having a higher order of accuracy. This is because one-step methods, have the advantage of permitting a change of mesh length at any step, since no starting process is required. Since the time of Runge, many researchers have taken advantage of the flexibility of the method to derive schemes either to improve accuracy or error control strategies.

HEUN [1900], put forward the following third-order formula for a three-stage method

$$y_{n+1} - y_n = \frac{h}{4}(k_1 + 3k_3)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{h}{3}, y_n + \frac{h}{3}k_1)$$

$$k_3 = f(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}k_2)$$

He reckoned that Runge's work could be further extended to include terms up to order h^3 previously ignored by Runge.

We however observe that the computational advantage in choosing $b_2 = 0$, in the above method, is somewhat illusory since, although k_2 does not appear in the first equation of the scheme, it must nevertheless be calculated at each step, because we need k_2 to obtain k_3.

WILHELM KUTTA [1901], extended the method of Runge further, to systems of equations. Thus, this method has come to be known as the Runge-Kutta method. Kutta's third order rule is given by

$$
\begin{aligned}
y_{n+1} - y_n &= \frac{h}{4}(k_1 + 4k_2 + k_3) \\
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \\
k_3 &= f(x_n + h, y_n - h\,k_1 + 2hk_2)
\end{aligned}
$$

According to Lambert [1973]; "it is the most popular third-order Runge-Kutta method, for desk computations (largely because the coefficient $\frac{1}{2}$ is preferable to $\frac{1}{3}$, which appears frequently in Heun's method)."

MERSON [1957], was the first to propose the idea of deriving a special R-K method, which would admit an easily calculated error estimate, which does not depend on quantities calculated at previous steps. Merson's method is:

$$y_{n+1} - y_n = \frac{h}{6}(k_1 + 4k_4 + k_5)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{h}{3}, y_n + \frac{h}{3}k_1)$$

$$k_3 = f[x_n + \frac{h}{3}, y_n + \frac{h}{6}(k_1 + k_2)]$$

$$k_4 = f[x_n + \frac{h}{2}, y_n + \frac{h}{8}k_1 + \frac{3}{8}hk_3]$$

and it is defined by the Butcher tableau below:

$$\begin{array}{c|ccccc}
0 & & & & & \\
\frac{1}{3} & \frac{1}{3} & & & & \\
\frac{1}{3} & \frac{1}{6} & \frac{1}{6} & & & \\
\frac{1}{2} & \frac{1}{8} & \frac{3}{8} & & & \\
1 & \frac{1}{2} & 0 & \frac{-3}{2} & 2 & \\
\hline
 & \frac{1}{6} & 0 & 0 & \frac{2}{3} & \frac{1}{6}
\end{array}$$

The above method, has order four and an estimate for the local truncation error given by:

$$30T_{n+1} = h(-2k_1 + 9k_3 + 8k_4 + k_5)$$

This method, has been widely used for non-linear problems, although the error estimate is valid only when the differential equation is linear in both x and y, that is of the form:

$$y' = ax + by + c$$

Merson's idea, is to derive R-K methods of order r and r+1, which share the same set of vectors $\{k_i\}$. This process is known as embedding.

With a slight modification to the Butcher tableau, embedded methods following Merson's idea can be represented in the following form:

$$
\begin{array}{c|c}
c & A \\
\hline
 & b^T \\
 & \\
 & \hat{b}^T \\
 & \\
\hline
 & E^T
\end{array}
$$

This notation is to be interpreted to mean that the method defined by c, A and b^T has order r and the method defined by c, A, and $\hat{b}^T$ has order r+1. the difference between the values for y_{n+1} generated by these two methods, is then taken as an estimate for the local truncation error.

The vector E^T is $\hat{b}^T$-b^T, so that the error estimate is given by $h\sum E_i k_i$, where $E^T = [E_1, E_2, ... E_r]$. The label (r, r+1), is usually attached to such an embedded method.

In the light of Butcher's theorem (that there is no five-stage method of order five), it becomes obvious that for a fourth order embedded method, a minimum of six stages will be needed. This explains why Merson's proposed error estimator could not be a valid one. Since this method, without the error estimator, is a five-stage method of order four and with the error estimator, it is a five-stage method of order five (which Butcher has since shown, to be impossible).

Nevertheless, nothing should be taken away from Merson's method, (represented by the modified Butcher tableau below), for it did play an important role, in pointing the way to future developments.

0					
1/3	1/3				
1/3	1/6	1/6			
1/2	1/8	3/8			
1	1/2	0	−3/2	2	
	1/6	0	0	2/3	1/6
	1/10	0	3/10	2/5	1/5
	−1/15	0	3/10	−4/15	1/30

HAMMING [1962], went a step further to derive and implement a fourth-order Runge-Kutta scheme in solving differential equations.

BUTCHER J.C. [1963, 1976], in a long series of papers starting in the mid-sixties, has developed various theories out of the Runge-Kutta method. Notable among his theories are;

i. An s-stage explicit R-K method, cannot have order greater than s,

ii. There exists no five-stage explicit R-K method of order five.

He also established the order condition for all class of Runge-Kutta method.

is the representation of a Runge-Kutta scheme, in matrix notation; a form known as the *Butcher Tableau.* Recall the general s-stage Runge-Kutta method

$$y_{n+1} - y_n = h\sum_{i=1}^{s} b_i k_i$$

$$k_i = f(x_n + c_i h, y_n + h\sum_{i=1}^{s} a_{ij} k_j), i = 1,2,3,...,s$$

Call the b_is the weights, the c_is the abscissae, and the k_is the slopes. Butcher defined the s – dimensional vectors c and b and the s x s matrix A, by $c = [c_1, c_2, ..., c_s]^T$, and $b = [b_1, b_2 ..., b_s]^T$ and $A = [a_{ij}]$. Then method expressed conveniently as Butcher tableau

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \quad = \quad \begin{array}{c|ccccc} c_1 & a_{11} & a_{12} & a_{13} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & a_{23} & \cdots & a_{2s} \\ c_3 & a_{31} & a_{32} & a_{33} & \cdots & a_{3s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_r & a_{s1} & a_{s2} & a_{s3} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & b_3 & \cdots & b_s \end{array}$$

will assume

$$c_i = \sum_{j=1}^{s-1} a_{ij}, i = 1,2,...,s$$

One important use, to which the Butcher tableau could be put, is in determining the type of the method (i.e. explicit, implicit, and semi-implicit).

- If 'A' is strictly lower triangular $\Rightarrow$ explicit method; calculate k_i explicitly, then k_2, etc, up to k_s;
- If $\exists\, a_{ij} \neq 0$, j>I $\Rightarrow$ implicit method;

  Requires a system of s x s (non-linear) equations be solved per step.
- If $a_{ij} = 0, j > i \text{ and } \exists\, a_{ii} \neq 0 \Rightarrow$ semi-implicit;

  Require s scalar (non-linear) equations be solved per step.

UTCHER J. C. [1964], derived an m-stage implicit Runge-Kutta method, making suitable ıoices of the m(m+1) free parameters which has the maximal attainable order 2m, for all m.

ne demonstrated further, that the implicit Runge-Kutta methods are not attractive for general usage; because each integration step requires the solution of a system of equations, that is in general non-linear for the m-unknowns.

SCRATON [1964], derived a fourth-order estimate which admits an error which is valid for a non-linear differential equation, unlike Merson's. the method is as below:

$$y_{n+1} - y_n = h[\frac{17}{162}k_1 + \frac{81}{170}k_3 + \frac{32}{135}k_4 + \frac{250}{1377}k_5]$$
$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \frac{2h}{9}, y_n + \frac{2h}{9}k_1)$$
$$k_3 = f(x_n + \frac{h}{3}, y_n + \frac{h}{12}k_1 + \frac{h}{4}k_2)$$
$$k_4 = f[x_n + \frac{3h}{4}, y_n + \frac{3h}{128}(23k_1 - 81k_2 + 90k_3)]$$
$$k_5 = f[x_n + \frac{9h}{10}, y_n + \frac{9h}{10000}(-345k_1 + 2025k_2 - 1220k_3 + 544k_4)]$$

He gave the estimate for the local truncation error as:

$$T_{n+1} = hqr/s$$

where

$$q = \frac{-1}{18}k_1 + \frac{27}{170}k_3 - \frac{4}{15}k_4 + \frac{25}{153}k_5$$

$$r = \frac{19}{24}k_1 - \frac{27}{8}k_2 + \frac{57}{20}k_3 - \frac{4}{15}k_4$$
$$s = k_4 - k_1$$

hough, Scraton's estimate was more realistic than Merson's when applied to a general n-linear differential equation, it has the disadvantage that it is not linear in the k_rs. As a result, it is applicable only to a single differential equation, and does not extend to a system of equations. As noted by Lambert (1973); "in order to find a method which admits an error estimate which is linear in the k_r, and thus holds for a general non-linear differential equation, or system of equations, it is necessary to make further sacrifices in the form of additional function evaluations."

ENGLAND [1969], made the necessary sacrifices in the form of additional function evaluations, and thus, came up with the following fourth-order six-stage method:

$$y_{n+1} - y_n = \frac{h}{6}[k_1 + 4k_3 + k_4]$$
$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$
$$k_3 = f[x_n + \frac{h}{2}, y_n + \frac{h}{4}(k_1 + k_2)]$$
$$k_4 = [x_n + h, y_n - hk_2 + 2hk_3)]$$
$$k_5 = f[x_n + \frac{2h}{3}, y_n + \frac{h}{27}(7k_1 + 10k_2 + k_4)]$$
$$k_6 = f[x_n + \frac{h}{5}, y_n + \frac{h}{625}(28k_1 - 125k_2 + 546k_3 + 54k_4 - 378k_5)]$$

He gave the associated estimate for the local truncation error as:

$$T_{n+1} = \frac{h}{336}(-42k_1 - 224k_3 - 21k_4 + 162k_5 + 125k_6)$$

must be noted that, if the method is used without the error estimate, it is essentially a four-ıge method. The modified Butcher tableau for the England's method is as below:

0					
1/2	1/2				
1/2	1/4	1/4			
1	0	−1	2		
1/3	7/27	10/27	0		
1/5	28/625	−1/5	546/625	−378/625	
	1/6	0	1/6	0	0
	1/24	0	5/48	27/56	125/336
	−1/8	0	−1/6	27/56	125/336

A feature of England's method, is that (unlike Merson's method), the last two elements of b^T are zero, implying that if the error estimate is not required, then only four stages (the minimum possible for fourth-order) need be computed. The method, is thus, economical if only occasional estimation of the error is intended.

SHAMPINE and ALLEN [1973], developed a subroutine for solving the fourth-order R-K method which was different from Ralston's fourth-order R-K method.

HAIRER and WANNER [1981], showed that R-K methods could be extended to orders five and six which have the properties of order, stability and efficiency of implementation to a high extent. These authors classified all algebraically stable methods of an arbitrary order and give various relationships between contractivity and order of implicit methods.

)NUMANYI, et al [1981], developed software for a method of finite approximations for the umerical solution of differential equation, which was based on the Tau method. According) them, problems with complex initial boundary conditions or mixed conditions involving ombinations of functions and derivatives values, can be dealt with by means of their program. Accordingly, encouraging results have been obtained in the solutions of problems with regions of rapid variation, oscillatory behaviour and in the presence of stiffness.

ONUMANYI and ORTIZ [1982], presented a method known as *Numerical Solutions of High Order Boundary Value Problems* for ordinary differential equations with an estimation of error. According to the authors, results of remarkably high accuracy and computational simplicity can be obtained by using Ortiz recursive formulation of Tau method. Besides, an error estimate of the number presented can be produced at a low computational extra cost.

ASCHER and BADER [1985], discussed the stability of collocation at Gaussian points. Symmetric R-K schemes according to them are particularly useful for solving stiff two-point boundary value problems. They observed that unlike initial value ODEs, the Jacobian of a well-conditioned problem may have both eigen values with a large negative real part and eigen values with a large positive real part. Hence, invariance with respect to the direction of integration is a very desirable property; which symmetric schemes possess.

GUPTA [1985], used the finite difference methods which combine features of both R-K process and Gap schemes to develop an adaptivity code for the solution of first order differential equations with two boundary conditions. He found an eighth-order, A-stable method that has second, fourth, and sixth order A-stable methods embedded in it. He then went on to describe a variable order, variable step difference solver using the embedded methods.

BURRAGE [1987], examined the stability properties of some special class of multi-valued methods known as multi-step R-K methods. He further constructed some families of algebraically stable methods of arbitrarily high order for the solution of the first order initial value problems. In particular, Burrage has studied the order conditions of these methods, and has shown that one can always construct methods of order 2s+r-1, where 2s denotes the highest order possible, and r-1, the number of free parameters existing in the methods.

SANNUGI and EVANS [1988], put forward a method, that surpassed that of England. They presented a modified version of the fourth-order Runge-Kutta formula, which required no extra function evaluation, yet provides estimation of the local truncation error. The basic idea of the modification, follows from the fact that numerical solutions of similar order can be obtained by using Arithmetic Mean (A. M) and the Geometric Mean (G. M) averaging of the functional values. The method is also suitable for the estimation of the local discretization error of one step methods known as embedding methods. Each step is integrated twice, using the p^{th}-order and the $(p+1)^{th}$-order methods, then the difference between the values obtained, gives the estimate of the error.

DORMAND, et al [1989], considered the applications of Runge-Kutta interpolation to global error estimation. They brought out some special formulae of orders two, four, and six and went on to show that a pseudo-problem, which is based on dense output values within any one step and reliable global error estimates could be mesh-points, by using the special R-K formulae.

HUNDORFER and SHNEID [1989], made a joint discovery of the fact that among the several stability and consistency concepts for R-K methods applied to stiff initial value problems (IVP), B-stability and B-consistency turned out to be equivalent for IVP with a one-sided Lipschitz constant $K \geq 0$. They guarantee stability with respect to perturbations of the IVP for $m \leq 0$.

JAIN, et al [1989], have shown that by using the well-known properties of the s-stage implicit R-K method for the first order differential equations, it is possible to obtain almost super stable methods of arbitrary order, for the direct integration of the general second order IVP by increasing the number of stages s. the method, when used successfully, can solve singular perturbation problems for which $\partial f / \partial y$ and/or $\partial f / \partial x$ are negative and large.

JAZCILEVICH and TEWARSON [1989], constructed functions characterizing the stability of explicit boundary value R-K methods. The method is based on the generalization of the algebraic stability criterion and can also be used to design methods with better stability and the selection of mesh-points. The criterion obtained, was found useful in the study study of explicit boundary value Runge-Kutta method.

KEELING [1989], constructed an implicit Runge-Kutta method with a stability function having distinct real poles. Such methods offer a computational speed-up when used on parallel machines (multiprocessor computers) with a modest number of processors. Sometimes, the method is called Multiple Implicit Runge-Kutta (MIRK) and hence due to the so-called order reduction phenomenon, the poles of the MIRK are required to be real.

He went further, to prove that the necessary condition for a q-stage real MIRK to be A-stable, with a maximal order q+1, is that q must be either 1, 2, 3 or 5. he showed that for every positive integer q, there exists a q-stage, real MIRK which is strongly A_0-stable with order still q+1 and for every even q, there is a q-stable real MIRK which is L-stable with order q.

MUIR and BEAME [1989], introduced a method called "AN Error Expression for Reflected and Averaged Implicit Runge-Kutta method." This method is useful in the numerical solution of initial value problems as well as the solutions of two-point boundary value problems. In fact, the main result of this method relate the error expression of an averaged method, to that of the method upon which it is bsed, since it is derived from another method by applying the results obtained, they showed that for each member of the class of averaged methods, there exists an embedded lower order method, which can be used for error estimations, in a formula-pair fashion.

BUTCHER and CASH [1990], derived a special class of implicit R-K methods for the numerical solution of stiff IVP. They derived the formulae from single implicit methods by adding one or more extra diagonally implicit stages. For the derivation, they considered singly implicit methods and in particular diagonally implicit methods.

They established that each class of methods offers some advantages over other methods as well as some disadvantages. For diagonally implicit methods, their limitation of the stage-order to 1, and the difficulty of finding high order for the methods as a whole, or of constructing realistic local error estimates, makes these methods unlikely candidates for incorporating into highly accurate and efficient software.

CALVO, et al [1990], developed a new pair of embedded Runge-Kutta formulae of orders five and six. This method is derived from a family of Runge-Kutta methods depending on the eight parameters by using certain measures of accuracy and stability.

When this method is compared with the other methods of the same order, greater accuracy is achieved, especially when used with an extra function evaluation per-step, a C^1-continuous interpolant of order five can be obtained.

SOMMEIJER [1990], considered a method based on the simplest well known classical Runge-Kutta method. The main characteristic of the resulting scheme of this integration rule, is that the computational complexity is hardly increased. This means that the first spatial operators are replaced by the finite difference or the finite element approximations that termed the semi-discretization. Then the time-continuous system of the ordinary differential equations, is integrated in time, by using the classical R-K method or by the forward Euler scheme. Following this technique, several choices have been made for the semi-discretization as well as for the time integration.

SOWA [1990], investigated the linear stability properties of a R-K method for solving the compressible Navier-Stokes equations and was able to produce another method. His method was based on the Fourier-transformation of the linearized spatial operation in which he fully considered unsplit spatial operator, resulting from a second order central difference approximation of the spatial derivatives. He also compared the theoretical stability limit with that encountered in numerical simulations of an IVP, as well as with the practical stability limit is slightly more restrictive than the one theoretically derived. He made further attempts to obtain an analytical expression of the stability limit, which was not possible, due to the complexity of the eigen-values and the difficulty of solving the high degree polynomial equation for the time step.

JULYAN and PIRO [1992], investigated the dynamics of a continuous time system, described by an ordinary differential equation. They attempted to elucidate the dynamics of the Runge-Kutta methods, by the application of the techniques of dynamical systems theory to the maps produced in the numerical analysis. Their aim, was to investigate what pitfalls there may be, in the integration of non-linear and chaotic systems.

HALL, G. [1992], was able to make a modification to the usual algorithm of codes for non-stiff problems, which overcomes the difficulties usually experienced in the use of such codes. Usually, codes for non-stiff problems can exhibit unnecessary roughness in the behavior of the step size, when stability, rather than accuracy, is the determining factor. This is inefficient, usually involving many rejected steps. Hall's modification however, caused the step size, to behave smoothly, and the new algorithm appears to be remarkably robust and provides the optimal use of a given R-K formula.

VAN DER HOUWEN and SOMMEIJER [1995], in their work, titled, "Iteration of Runge-Kutta Methods with Block Triangular Jacobians." They considered iteration processes for solving the implicit relations associated with implicit Runge-Kutta methods applied to stiff

IVPs. The conventional approach, for solving the R-K equations uses Newton iteration employing the full right-hand side Jacobian. They noted that for IVPs of large dimensions, this method is not attractive because of the high cost involved in the LU-decomposition of the Jacobian of the R-K equations. They outlined an alternative approach which directly replaces the R-K Jacobian by a block-diagonal or block-triangular matrix whose block themselves, are block triangular matrices. Such a grossly 'simplified' Newton iteration process, allows for a considerable amount of parallelism. They then aimed to investigate the effects on the convergence of block-triangular Jacobian approximations.

ADEWALE [1998], derived a new five-stage explicit one-step R-K methodof order four for the numerical solution of IVPs. The new method aid computation through the use of whole numbers instead of fractions as observed in existing methods of this form. This is helpful, when the computations are performed manually, as it reduces the number of operations involved in the evaluation of the k_rs. He also provided a computer program, that uses the new scheme, to solve IVPs. The new method with its corresponding Butcher tableau is as below:

$$y_{n+1} - y_n = \frac{h}{12}[2k_1 + 8k_3 + k_4 + k_5]$$
$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \frac{h}{3}, y_n + \frac{h}{3}k_1)$$
$$k_3 = f(x_n + \frac{h}{2}k_1, y_n + \frac{h}{2}k_2)$$
$$k_4 = f[x_n + h, y_n + h(-3k_1 + 5k_2 - k_3)]$$
$$k_5 = f[x_n + h, y_n + h(3k_1 - 3k_3 + k_4)]$$

0					
1/3	1/3				
1/2	0	1/2			
1	−3	5	−1		
1	3	0	−3	1	
	1/6	0	2/3	1/12	1/12

GARBA and YAKUBU [1999], derived a new R-K formula of order five, which does not require the use of error control strategy, but has better approximations than some existing R-K formulae.

Earlier on, we did mention that when a Runge-Kutta method of desired order is derived, there are in general, a number of free parameters which cannot be used to increase the order, any further. Lambert (1973), give a number of uses to which these free parameters could be put:

(i) These free parameters, could be chosen in such a way that the resulting method have simple coefficients, convenient for desktop computations,

(ii) Perhaps, the most important tasks to which free parameters can be applied, is the reduction of the local truncation error,

(iii) There are other ways in which we may attempt to use the free parameters in order to improve local accuracy,

(iv) Another area where we can look for some advantage from a judicious choice of the free parameters, concerns the weak stability characteristics of R-K methods, particularly for stages grater than four.

With regards to explicit Runge-Kutta methods of order greater than four, Julyan and Piro (1992), identifies some unresolved issues:

(a) What is the minimum number of stages necessary for an explicit method to attain order p? This is still an open problem.

(b) Exactly how many stages are required to obtain a ninth-order or tenth-order explicit method? We only know that somewhere between twelve and seventeen stages will give us ninth-order explicit method, and somewhere between that number and seventeen stages will give us a tenth-order explicit method.

(c) Nothing is known for explicit methods of order higher than ten.

We must note that for explicit Runge-Kutta methods of order five, it is quite obvious that the minimum number of stages necessary, is six. This will become clearer, when we consider the following general results, as put forward by Butcher (1963, 1976):

(i) An explicit q-stage method, cannot have order greater than q; for $q \leq 4$,

(ii) There is no five-stage explicit Runge-Kutta method of order five.

From the above, our assertion follows quite naturally.

A number of computer software have been developed for a system of differential equations using the Runge-Kutta method.

For example, the C-XSC program was developed for a system of differential equations to be solved by the Runge-Kutta method. The C-XSC program is very similar to the mathematical notation. Dynamic vectors are used to make the program independent of the size of the system of differential equations to be solved.

RKSUITE is an excellent collection of codes based on Runge-Kutta methods for the numerical solution of an IVP for the first order system of ordinary differential equations. It supersedes some very widely used codes, namely RKF45 code and its descendent DDERKF in the SLATEC library and DO2PAF and associated codes in the NAG Fortran library. RKSUITE is written in standard Fortran 77 and is distributed in source form. RKSUITE implements three Runge-Kutta pairs: (2,3), (4,5), and (7,8). The (4,50 pair, for example, uses both a 4^{th} and a 5^{th} order approximation to estimate the error in the 4^{th} formula; using extrapolation, it then produces a formula of order five. Similarly, the (2,3) pair produces a formula of order three, and the (7,8) pair, a formula of order eight.

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1.3 DEFINITIONS

Differential Equations:

A differential equation is an equation involving an unknown function and one or more of its derivatives. It is a relationship between an independent variable x, a dependent variable y, and one or more differential coefficients of y with respect to x.

E.g. $\frac{dy}{dx} = 2x + y.$

Order of a Differential Equation:

The order of a differential equation is given by the order of the highest derivative involved in the equation. For example $\frac{du}{dt} = F(t)G(t)$ is of the first order.

Ordinary Differential Equations (ODE):

An ODE, is an equation that contains an independent variable x, an unknown function y(x) and certain derivatives of y such as $y'(x), y''(x), \cdots, y^n(x)$. For example, $y' = x + 2y,$ is an ODE. In general, any equation of the form:

$$F(x, y', y'', \cdots, y^n) = 0$$

is an ODE of order n, n>0.

Linear Equations:

An equation of order n is said to be linear if it has the special form:

$$a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x) \qquad (\Delta)$$

where the $a_i(x)$ are arbitrary functions of x only. Also, we note that in this form, the unknown function y and all its derivatives appear linearly.

xplicit Runge-Kutta (R-K) Methods:

Given that in a R-K method of order s;

$$k_i = f(x_n + c_i h, y_n + h\sum a_{ij} k_i), i = 1(1)s$$

If we have that $a_{ij} = 0$, whenever $j \geq i$, I = 1(1)s, then each k_i is given explicitly in terms of previously computed k_js, j = 1(1)i-1, and the method is then an explicit or classical R-K method.

Semi-implicit R-K Methods:

If on the other hand (from above), we have that $a_{ij} = 0$ for j>i, then the method is a semi-implicit R-K method.

Implicit R-K Methods:

If we have a situation where $a_{ij} \neq 0 \ for \ j > i$, then the R-K method is an implicit method and each k_i is not given in terms of previously computed k_j, j=1(1)i-1. Rather a system of non-linear equations results.

Local Truncation Error (lte):

The local truncation error (lte) t_{n+1} of the one-step scheme is given by

$$t_{n+1} = y(x_{n+1}) - y(x_n) - h\phi(x_n, y(x_n); h)$$

where y(x) is the true solution to the IVP.

The local truncation error simply put, is the amount by which the true solution of the IVP fails to satisfy the first order difference equation, under the simplying assumption that the previous solutions are exact (i.e. $y_n = y(x_n)$).

Initial Value Problems (IVPs):

If with a difference equation, we specify conditions at a single value of the independent variable, these conditions are referred to as initial conditions. The combination of the differential equation and an appropriate number of initial conditions is called an Initial Value Problem (IVP). E.g. $y' = 2x + y; y(0) = 1$.

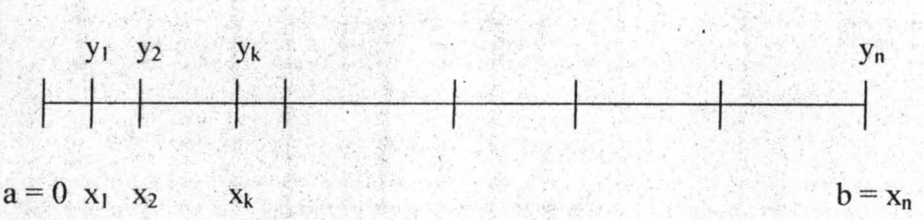
CHAPTER TWO

NUMERICAL SOLUTION METHODS

We recall the first order differential equation

$$y' = f(x,y); y(a) = y_0, f:\Re\times\Re^m \to \Re^m \qquad (i)$$

over some interval [a,b], where $a < \infty, b < \infty$



The usual numerical method for solving (i) are referred to as discrete variable methods, because they discretise the interval [a,b] into subintervals and then generate a sequence of approximate solutions for y(x) i.e. $y_1, y_2, y_3, \ldots$ at points $x_1, x_2, x_3, \ldots$. No attempt is made to approximate the exact solution, y(x), over a continuous range of the independent variable x.

Apparently, only a small class of differential equations possess analytical solutions y(x), expressible in terms of known tabulated transcendental functions that satisfy the differential equation, as well as the initial conditions. Kamke, (1943). As an illustration, consider the well-known Van der Pol oscillator

$$y'' + \mu(1-y^2)y' + \lambda y = 0; y(a), y'(a)\ given \qquad (ii)$$

for some real positive numbers $\mu\ and\ \lambda$. This problem was first formulated by B. Van der Pol in 1926. The differential equation (ii), has attracted a lot of research attention both in nonlinear mechanics and in control theory. To date, this problem has no solution in terms of known tabulated transcendental function. Even when the analytical solutions to certain differential equations are available, their numerical evaluation may be quite intractable.

So, for such differential equations that are not soluble analytically, numerical integration is the only way to obtain information about the trajectory. As stated in section 1, there are many different methods that have been proposed and used in an attempt to solve accurately, various types of ODEs. Such methods, are known as numerical methods and they can be broadly grouped into two, viz:

(a) One-step Methods, and

(b) Multi-step Methods.

2.1 **One-step Methods**

A differential equation has no "memory". That is the values of y(x) for x before x_n, do not directly affect the values of y(x)for x after x_n. Some numerical methods have memory, and some do not. The class of methods known as one-step methods, have no memory; given y_n, there is a recipe for y_{n+1} that depends only on information at x_n, n = 1,2,...,k.

So for one-step methods, (or single-step methods) only the information from one previous point (mesh point), is used to compute the successive point. For example, only the initial point (x_0, y_0) is used to compute (x_1, y_1), while (x_1, y_1) is used to compute (x_2, y_2), and so on. One-step methods are self-starting, and permit a change of step-length, in the course of computation. A general one-step method can then be written in the form

$$y_{n+1} - y_n = h\phi(x_n, y_n; h); y_0 = y(x_0) \qquad (iii)$$

where ϕ is the increment function that characterizes the one-step method, h is the steplenght.

The goal would be to obtain algorithms for which the true solution, y(x)

almost satisfies (iii) i.e.

$$y(x_{n+1}) = y(x_n) + h\phi(x_n, y(x_n)) + h\tau_n \qquad (iv)$$

with τ_n "small". The quantity $h\tau$, is called the local (truncation) error.

2.1.1 **Taylor series Method**

Taylor series method is a straight forward adaptation of classic calculus to develop the solution as an infinite series. The catch is that a computer usually cannot be programmed to construct the terms and one does not know how many terms should be used.

Perhaps the simplest one-step methods of order p are based on Taylor series expansion (e.g. Euler, Runge-Kutta) of the solution y(x). If $y^{(p+1)}(x)$ is continuous on [a,b], then Taylor's formula gives

$$y(x_{n+1}) = y(x_n) + h[y'(x_n) + \cdots y^{(p)}(x_n)\frac{h^{p-1}}{p!}] + y^{p+1()}(\varphi_n)\frac{h^{p+1}}{(p+1)!} \qquad (v)$$

where $x_n \leq \varphi_n \leq x_{n+1}$

The continuity of $y^{(p+1)}(x)$ implies that it is bounded on [a,b]

and so ,

$$y^{(p+1)}(\varphi_n)\frac{h^{(p+1)}}{(p+1)!} = o(h^{p+1}) = ho(h^p)$$

Using the fact that $y' = f(x, y)$, (v) can be written in the form

$$y(x_{n+1}) = y(x_n) + h[f(x_n) + \cdots f^{(p-1)}(x_n, y(x_n))\frac{h^{p-1}}{p!}] + ho(h^p) \qquad (vi)$$

where the total derivatives of f are defined recursively by

$$f'(x, y) = f_x(x, y) + f_y(x, y),$$
$$f^k(x, y) = f_x^{(k-1)}(x, y) + f_y^{(k-1)}(x, y)f(x, y); k = 2,3,\cdots$$

Comparison of (iv) with (vi), shows that to obtain a method of order p, we can let

$$\phi(x_n, y(x_n)) = f(x_n, y(x_n)) + \cdots + f^{(p-1)}(x_n, y(x_n))\frac{h^{p-1}}{p!} \qquad (vii)$$

This choice leads to a family of methods known as the Taylor series methods, given in the following algorithm.

Taylor-series Algorithm

To obtain an approximate solution of order p to the IVP (i) on [a,b], we will need to let $h = (b-a)/n$ and generate the sequence

$$y_{n+1} = y_n + h[f(x_n, y_n) + \cdots + f^{(p-1)}(x_n, y_n)\frac{h^{p-1}}{p!}] \qquad (viii)$$

$$x_{n+1} = x_n + h, n = 0,1,2,\cdots,k-1$$

where $x_0 = a, and\ y_0 = A$

We can easily observe from (viii) that the Taylor series method of order p = 1, is in fact the Euler's method:

$$\left.\begin{aligned} y_{n+1} &= y_n + hf(x_n, y_n) \\ x_{n+1} &= x_n + h \end{aligned}\right\} \qquad (ix)$$

Taylor series can be quite effective if the total derivatives of f are not too difficult to evaluate.

Software packages, are available that perform exact differentiation, facilitating their use (e.g. ADIFOR, MAPLE, MATHEMATICA, etc). However, most of today's software packages for solving IVPs, such as (i), do not employ Taylor series methods.

As stated earlier in this section, Taylor series method is the foundation for some of the simplest and appealingly effective one-step methods, notably of these is the Runge-Kutta methods.

2.1.2 **Runge- Kutta Methods**

The Runge-Kutta or R-k methods, are extensions of the basic idea of Euler's method using approximations which agree with more terms of the Taylor series. The Basic steplenght is h as with Euler's method, but some intermediate points are also computed and the slopes at these points, are used to improve the overall change between x_n *and* $x_n + h \approx x_{n+1}$. Start from (x_n,y_n), take one step of Euler's Rule of length c_2h and evaluate the derivative vector at the point so reached; the result is k_2. We now have two samples for the derivative, k_1 and k_2, a weighted mean of k_1 and k_2 is used as the initial slope in another Euler step (from (x_n,y_n)) of length c_3h, the derivative at the point so reached is then evaluated; the result is k_3. Continuing in this manner, we obtain a set k_i, I = 1, 2, ..., s of samples of the derivatives. The final step ($y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i$) is yet another Euler step from (x_n,y_n) to (x_{n+1},y_{n+1}), using as initial slope a weighted mean of the samples $k_1, k_2, \ldots, k_s$. Thus an explicit Runge-Kutta method sends out feelers into the solution space, to gather samples of the derivative, before deciding in which direction to take an Euler step.

Runge-Kutta methods are designed to approximate Taylor series methods, but have the advantage of not requiring explicit evaluations of the derivatives of f(x,y). The basic idea, is to use a linear combination of values of f(x,y) to approximate y(x). This linear combination is matched up as closely as possible with a Taylor series for y(x) to obtain methods of the highest possible order p.

So an S-stage Runge-Kutta process can thus be viewed as an extension of the Taylor expansion scheme whereby the evaluation of the first and higher order derivatives, of f(x,y) is replaced by S function evaluations within every interval of integration $[x_n, x_{n+1}]$. The R-K scheme is basically a substitution method of the form

$$y_{n+1} = y_n + \phi_{RK}(x_n, y_n; h) \qquad (x)$$

with the increment function ϕ_{RK} given as a weighted mean of the slopes at specific points.

The number of coefficients for each class of R-K method can be ascertained, as shown below:

TYPE	NUMBER OF COEFFICIENTS
Explicit	s(s+1)/2
Semi-implicit	s(s+3)/2
Implicit	s(s+1)

As discussed in section 1.2, various R-k schemes have been proposed. However, according to Lambert (1991) the four-stage classical R-K scheme of order four, has proven to be the most popular of them all. Therefore, it is only fitting that we illustrate the use of R-K methods, by using the classical scheme, to solve the differential equation

$$y' = x + y; y(0) = 1$$

vith steplenght h = 0.1 and $x_{n+1} = x_n + h$

The classical four-stage scheme is given as

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

here

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$

$$k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

or n = 0, x = 0.1

$$= y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

th

$$k_1 = f(x_0, y_0) = 0 + 1 = 1$$

$$k_2 = f(x_0 + 1/2h, y_0 + 1/2h\,k_1)$$
$$= f(0.05, 1.05)$$
$$= 0.05 + 1.05$$
$$\therefore k_2 = 1.1$$

$$k_3 = f(x_0 + 1/2h, y_0 + 1/2h\,k_2)$$
$$= f(0.05, 1.055)$$
$$= 0.05 + 1.055$$
$$\therefore k_3 = 1.105$$

$$k_4 = f(x_0 + h, y_0 + hk_3)$$
$$= f(0.1, 1.11055)$$
$$= 0.1 + 1.1105$$
$$\therefore k_4 = 1.2105$$

$$y_1 = 1 + \frac{0.1}{6}(1 + 2.2 + 2.21 + 1.2105)$$
$$= 1.110341667$$

n = 1, x = 0.2

$$y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

with

$$k_1 = f(x_1, y_1)$$
$$= f(0.1, 1.110341667)$$
$$= 1.210341667$$

$$k_2 = f(x_1 + \frac{1}{2}h, y_1 + 0.05k_1)$$
$$= f[0.1 + 0.05, 1.110341667 + 0.05(1.210341667)]$$
$$= 1.32085875$$

$$k_3 = f(x_1 + \frac{1}{2}h, y_1 0.05k_2)$$
$$= f[0.1 + 0.05, 1.11034166 + 0.05(1.32085875)]$$
$$= 1.326384605$$

$$k_4 = f(x_1 + h, y_1 + hk_3)$$
$$\therefore k_4 = 1.442980128$$
$$\Rightarrow y_2 = 1.242805142$$

$$y_3 = y_2 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_2, y_2)$$
$$= f(0.2, 1.242805142)$$
$$= 0.2 + 1.242805142$$
$$= 1.442805142$$

$$k_2 = f(x_1 + \frac{1}{2}h, y_1 + 0.05k_1)$$
$$= 1.564945399$$

$$k_3 = f(x_1 + \frac{1}{2}h, y_1 + 0.05k_2)$$
$$= 1.571052412$$

$$k_4 = f(x_2 + 0.1, y_1 + 0.1k_3)$$
$$\therefore k_4 = 1.699910383$$
$$\Rightarrow y_3 = 1.399716995$$

n = 3, x = 0.3

$$y_4 = y_3 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_3, y_3)$$
$$= 1.699716995$$

$$k_2 = f(x_3 + 0.1h, y_3 + 0.1k_1)$$
$$= 1.784702844$$

$$k_3 = f(x_3 + 0.1h, y_3 + 0.1k_2)$$
$$= 1.838952137$$

$$k_4 = (x_3 + h, y_3 + hk_3)$$
$$= 1.983612208$$

$$\therefore y_4 = 1.581894314$$

Solving the differential equation analytically we obtain

$$y_E(x) = 2e^x - x - 1$$

$$\therefore y_E(0.1) = 1.1103418$$
$$y_E(0.2) = 1.2428055$$
$$y_E(0.3) = 1.3997176$$
$$y_E(0.4) = 1.5836494$$

2.2 Multi-step Methods

The numerical methods for the solution of the differential equation

$$y' = f(x, y), y(x_0) = y_0; f : \Re \times \Re^m \to \Re^m \qquad (xi)$$

are called multi-step methods, if the value of y(x) at $x = x_{n+1}$ uses the values of the dependent variable and its derivatives at more than one grid or mesh point. Suppose the approximate values of y and $y' = f(x, y)$ at the points $x_m = x_0 + mh$, m = 1, 2, ...,n. We denote the approximate values of these points by

$$y(x_m) = y_m, f(x_m, y(x_m)) = f_m; m = 0,1,\cdots,n$$

Thus the general multi-step or k-step method for the solution of the IVP may be written as

$$y_{n+1} = a_1 y_n + a_2 y_{n-1} + \cdots + a_k y_{n-k+1} + h\phi(x_{n+1}, x_n, \cdots, x_{n-k+1}, y'_{n+1}, y'_n, \cdots, y_{n-k+1}; h) \qquad (xii)$$

where h is the constant step size and $a_1, a_2, \cdots, a_k$ are real given constants. If ϕ is independent of y_{n+1} then the general multi-step method, is called an explicit, open, or predictor method; otherwise an implicit, closed or corrector method. The k-1 values $y_1, y_2, \cdots, y_{k-1}$ required to start the computation are obtained, using the single-step methods. The special cases of the linear multi-step method are used for solving the IVP.

Explicit Multi-step Methods

Explicit multi-step methods, are obtained by integrating the differential equation

$$y' = f(x, y)$$

between the limits x_{n-j} *and* x_{n+1}, to get

$$y(x_{n+1}) = y(x_{n-j}) + \int_{x_{n-j}}^{x_{n+1}} f(x, y)dx \qquad (xiii)$$

This is then integrated by approximating f(x, y) by a polynomial which interpolates f(x, y) at k points $x_n, x_{n-1}, \cdots, x_{n-k+1}$. The Newton backward formula of degree (k-1) could be used for this purpose. This will give us

$$y(x_{n+1}) = y(x_{n-j}) + h\sum_{m=0}^{k-1} \gamma_m^{(j)} \nabla^m f_n + T_k^{(j)} \qquad (xiv)$$

where:

$$\left.\begin{aligned} T_k^{(j)} &= h^{k+1} \int (-1)^k \binom{-u}{k} f^{(k)}(\varphi)du \\ \gamma_m^{(j)} &= \int_{-j}^{1} (-1)^m \binom{-u}{m} du \end{aligned}\right\} \qquad (xv)$$

If we ignore the remainder term $T_k^{(j)}$ in (xiv) we get

$$y_{n+1} = y_{n-j} + h\sum_{m=0}^{k-1} \gamma_m^{(j)} \nabla^m f_n \qquad (xvi)$$

If the difference $\nabla^m f_n$ re expressed in terms of the function values f_m, from the definition of the backwards difference operator ∇, we find predictor method; otherwise an implicit, closed or corrector method. The k-1 values $y_1, y_2, \cdots, y_{k-1}$ required to start the computation are obtained, using the single-step methods. The special cases of the linear multi-step method are used for solving the IVP.

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$$y(x_{n+1}) = y(x_{n-j}) + h\sum_{m=0}^{k-1} \gamma_m^{(j)} \nabla^m f_n + T_k^{(j)} \qquad (xiv)$$

where:

$$\left.\begin{aligned} T_k^{(j)} &= h^{k+1} \int (-1)^k \binom{-u}{k} f^{(k)}(\varphi) du \\ \gamma_m^{(j)} &= \int_{-j}^{1} (-1)^m \binom{-u}{m} du \end{aligned}\right\} \qquad (xv)$$

If we ignore the remainder term $T_k^{(j)}$ in (xiv) we get

$$y_{n+1} = y_{n-j} + h\sum_{m=0}^{k-1} \gamma_m^{(j)} \nabla^m f_n \qquad (xvi)$$

If the difference $\nabla^m f_n$ re expressed in terms of the function values f_m, from the definition of the backwards difference operator ∇, we find

$$\nabla^m f_n = \sum (-1)^p \binom{m}{p} f_{n-p} \qquad (xvii)$$

By substituting (xvii) into (xvi) and regrouping, we obtain

$$y_{n+1} = y_{n-j} + h\sum_{m=0}^{k-1} \gamma_m^{*(j)} f_{n-m} \qquad (xviii)$$

A number of interesting formulae can be obtained for various integer values of k in (xvi), which is the general explicit multi-step method.

Implicit Multi-step Methods

As we pointed out previously, explicit methods involve expressing y'_{n+1} in terms of previously calculated ordinates and slopes. Implicit multi-step methods on the other hand, involves the unknown slope y'_{n+1} on the right hand side, and are obtained by replacing $f(x,y)$ in (xiii) by a polynomial which interpolates $f(x,y)$ at $x_n, x_{n-1}, \cdots, x_{n-k+1}$ for an integer k > 0. The Newton backward difference formula which interpolates at these k + 1 points in terms of $u = (x - x_n)/h$, when substituted into (xiii) yields

$$y(x_{n+1}) = y(x_{n-1}) + h\sum_{m=0}^{k} \sigma_m^{(j)} \nabla^m f_{n+1} + T_{k+1}^{*(j)} \qquad (xix)$$

where:

$$\left.\begin{aligned} T_{k+1}^{*(j)} &= h^{k+2} \int_{-j}^{1} (-1)^{k+1} \binom{1-u}{k+1} f^{(k+1)}(\varphi)du \\ \sigma_m^{(j)} &= \int (-1)^m \binom{1-u}{m} du \end{aligned}\right\} \qquad (xx)$$

If we ignore $T_{k+1}^{*(j)}$ in (xix), we get

$$y_{n+1} = y_{n-j} + h\sum_{m=0}^{k} \sigma_m^{(j)} \nabla^m f_{n+1} \qquad (xxi)$$

where

$$\sigma_0^{(j)} = 1 + j$$

$$\sigma_1^{(j)} = -\frac{1}{2}(1+j)^2$$

$$\sigma_2^{(j)} = -\frac{1}{12}(1+j)^2(1-2j)$$

$$\sigma_3^{(j)} = -\frac{1}{24}(1+j)^2(1-j)^2$$

$$\sigma_4^{(j)} = -\frac{1}{720}(1+j)^2(19-38j+27j^2-6j^3)$$

$$\sigma_5^{(j)} = -\frac{1}{1440}(1+j)^2(27-54j+45j^2-16j^3+2j^4)$$

If we replace the difference operator $\nabla^m f_{n+1}$ in terms of the function values, we obtain

$$y_{n+1} = y_{n-j} + h\sum_{m=0}^{k}\sigma_m^{*(j)} f_{n-m+1} \qquad (xxii)$$

From (xxi) or (xxii), it is possible to obtain a number of multi-step formulae for various integer values of j. It is obvious from (xix) that the implicit multi-step methods are of one order higher than the corresponding explicit multi-step methods with the same number of ordinates and slopes.

2.2.1 Adam-Bashforth Formulae (j = 0)

As observed in section 2.2, a number of interesting explicit formulae can be obtained for various integer values of k. One of such formula is the Adam-Bashforth formula, which results from equation (xvi) for j = 0;

$$y_{n+1} = y_{n-j} + h\sum_{m=0}^{k-1}\gamma_m^{(0)}\nabla^m f_n \qquad (xxiii)$$

Calculating a few of $\gamma_m^{(j)}$ from (xv), we obtain

$$\gamma_0^{(j)} = \int_{-j}^{1} du = 1 + j \; ; \gamma_0^{(0)} = 1$$

$$\gamma_1^{(j)} = \int_{-j}^{1} u du = 1/2(i - j)(1 + j) \; ; \gamma_1^{(0)} = 1/2$$

$$\gamma_2^{(j)} = \int_{-j}^{1} 1/2 u(u+1) du = 1/12(5 - 3j^2 + 2j^3) \; ; \gamma_2^{(0)} = 5/12$$

$$\gamma_3^{(j)} = \int_{-j}^{1} 1/6 u(u+1)(u+2) du = 1/24(3 - j)(3 + j - j^2 + j^3) \; ; \gamma_3^{(0)} = 3/8$$

$$\gamma_4^{(j)} = \int_{-j}^{1} \frac{1}{24} u(u+1)(u+2)(u+3) du = \frac{1}{720}(251 - 90j^2 + 110j^3 - 45j^4 + 6j^5) \; ; \gamma_4^{(0)} = \frac{251}{720}$$

$$\gamma_5^{(j)} = \int_{-j}^{1} \frac{1}{120} u(u+1)(u+2)(u+3)(u+4) du = \frac{1}{1440}(5 - j)(95 + 19j - 25j^2 + 35j^3 - 14j^4 + 2j^5);$$

$$\gamma_5^{(0)} = \frac{475}{1440}$$

Replacing the coefficients $\gamma_m^{(0)}$ by their values in (xxiii), we get

$$y_{n+1} = y_n + h[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n \frac{3}{8}\nabla^3 f_n + \frac{251}{720}\nabla^4 f_n + \frac{475}{1440}\nabla^5 f_n + \cdots]$$

The coefficients $\gamma_m^{*(0)}$ from (xxii) are given below:

k	$\gamma_0^{*(0)}$	$\gamma_1^{*(0)}$	$\gamma_2^{*(0)}$	$\gamma_3^{*(0)}$	$\gamma_4^{*(0)}$	$\gamma_5^{*(0)}$
1	1					
2	$\frac{3}{2}$	$-\frac{1}{2}$				
3	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$			
4	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{9}{24}$		
5	$\frac{1901}{720}$	$-\frac{2774}{720}$	$\frac{2616}{720}$	$-\frac{1274}{720}$	$\frac{251}{720}$	
6	$\frac{4277}{1440}$	$-\frac{7923}{1440}$	$\frac{9982}{1440}$	$-\frac{7298}{1440}$	$\frac{2877}{1440}$	$-\frac{475}{1440}$

It is obvious from that with k computed values, we obtain Adam-Bashforth formulae of order k, since the truncation error is of the form ch^{k+1}, where c is independent of h.

To illustrate how the Adam-Bashforth formulae are used, we shall solve the IVP below:

$$y' = x + y, y(0) = 1, \text{ with } h = 0.1$$

using the fifth order Adam-Bashforth method. The fifth order Adam-Bashforth method is given by:

$$y_{n+1} = y_n + \frac{h}{720}[1901 f_n - 2774 f_{n-1} + 2616 f_{n-2} - 1274 f_{n-3} + 251 f_{n-4}], n \geq 4$$

The values for y_1, y_2, y_3 *and* y_4 are obtained using the Taylor series method of order five

$$y_{n+1} = y_n + hy' + \frac{h^2}{2}{}_n\, y_n'' + \frac{h^3}{6} y_n''' + \frac{h^4}{24} y^{iv}{}_n + \frac{h^5}{120} y^{v}{}_n$$

where

$$y_n' = x_n + y_n$$
$$y_n'' = 1 + y_n' = 1 + x_n + y_n$$
$$y_n''' = y_n'' = 1 + x_n + y_n$$
$$y^{iv}{}_n = y_n''' = 1 + x_n + y_n$$
$$y^{v}{}_n = y^{iv}{}_n = 1 + x_n + y_n$$

hence, we have

$$y_{n+1} = y_n + h(x_n + y_n) + (1 + x_n + y_n)[\frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{120}], n = 0,1,2,3$$

n = 0

$$y_1 = y_0 + h(x_0 + y_0) + (1 + x_0 + y_0)[\frac{0.01}{2} + \frac{0.001}{6} + \frac{0.0001}{24} + \frac{0.00001}{120}]$$
$$= 1 + 0.1 + 2(\frac{0.01}{2} + \frac{0.001}{6} + \frac{0.0001}{24} + \frac{0.00001}{120})$$
$$\therefore y_1 = 1.1103418 \Rightarrow y_1' = x_1 + y_1 = 0.1 + 1.1103418 = 1.2103418 \equiv f_1$$

n = 1

$$y_2 = y_1 + h(x_1 + y_1) + (1 + x_1 + y_1)[\frac{0.01}{2} + \frac{0.001}{6} + \frac{0.0001}{24} + \frac{0.00001}{120}]$$
$$= 1.1103418 + 0.1(1.2103418) + 2.2103418(0.0051709167)$$
$$\therefore y_2 = 1.2428055 \Rightarrow y_2' = x_2 + y_2 = 0.2 + 1.2428055 = 1.4428055 \equiv f_2$$

$n = 2$

$$y_3 = y_2 + h(x_2 + y_2) + (1 + x_2 + y_2)[\frac{0.01}{2} + \frac{0.001}{6} + \frac{0.0001}{24} + \frac{0.00001}{120}]$$
$$= 1.2428055 + 0.1(1.4428055) + 1.2428055(0.0051709167)$$
$$\therefore y_3 = 1.3997176 \Rightarrow y_3' = x_3 + y_3 = 0.3 + 1.3997176 = 1.6997176 \equiv f_3$$

n = 3

$$y_4 = y_3 + h(x_3 + y_3) + (1 + x_3 + y_3)[\frac{0.01}{2} + \frac{0.001}{6} + \frac{0.0001}{24} + \frac{0.00001}{120}]$$
$$= 1.3997176 + 0.16997176 + 0.03196$$
$$\therefore y_4 = 1.5836494 \Rightarrow y_4' = x_4 + y_4 = 0.4 + 1.5836494 = 1.9836494 \equiv f_4$$

Thus the starting values are:

$y_1 = 1.1103418$; $f_1 = 1.2103418$
$y_2 = 1.2428055$; $f_2 = 1.4428055$
$y_3 = 1.3997176$; $f_3 = 1.6997176$
$y_4 = 1.5836494$; $f_4 = 1.9836494$

Now, we will use these starting values in the Adam-Bashforth formula above;

For n = 4

$$y_5 = y_4 + \frac{h}{720}[1901f_4 - 2774f_3 + 2616f_2 - 1274f_1 + 251f_0]$$

$$= 1.5836494 + \frac{0.1}{720}[1901(1.9836494) - 2774(1.6997176) + 2616(1.4428055)$$

$$-1274(1.2103418) + 251(1)]$$

$$= 1.5836494 + 0.2137923$$

$$\therefore y_5 = 1.7974417 \Rightarrow y_5' = x_5 + y_5 = 0.5 + 1.7974417 = 2.2974417 \equiv f_5$$

For n = 5

$$y_6 = y_5 + \frac{h}{720}[1901f_5 - 2774f_4 + 2616f_3 - 1274f_2 + 251f_1]$$

$$= 1.7974417 + \frac{0.1}{720}[1901(2.297447) - 2774(1.9836494) + 2616(1.6997176)$$

$$-1274(1.4428055) + 251(1.2103418)]$$

$$\therefore y_6 = 2.0442356 \Rightarrow y_5' = x_6 + y_6 = 0.6 + 2.0442356 = 2.6442356 \equiv f_6$$

$n = 6$

$$y_7 = y_6 + \frac{h}{720}[1901f_6 - 2774f_5 + 2616f_4 - 1274f_3 + 251f_2]$$

$$= 2.0442356 + \frac{0.1}{720}[1901(2.6442356) - 2774(2.2974412) + 2616(1.9836494)$$

$$-1274(1.6997176) + 251(1.4428055)]$$

$$\therefore y_7 = 2.3275055 \Rightarrow y_7' = x_7 + y_7 = 0.7 + 2.3275055 = 3.0275022 \equiv f_7$$

$n = 7$

$$y_8 = y_7 + \frac{h}{720}[1901f_7 - 2774f_6 + 2616f_5 - 1274f_4 + 251f_3]$$

$$= 2.3275055 + \frac{0.1}{720}[1901(3.0275022) - 2774(2.6442356) + 2616(2.2974417)$$

$$-1274(1.9836494) + 251(1.4428055)]$$

$$\therefore y_8 = 2.6421209 \Rightarrow y_8' = x_8 + y_8 = 0.8 + 2.6421209 = 3.4421209 \equiv f_8$$

As pointed out in section 2, there are two types of multi-step methods; Explicit multi-step methods, and Implicit multi-step methods. Adam-Bashforth formula is an example of an explicit multi-step method, with j = 0. We will now consider the Adam-Moulton formula, which is an example of implicit multi-step methods, with j = 0.

2.2.2 **Adam-Moulton Formula (j = 0)**

If we substitute j = 0 into Equation (xxi) we obtain

$$y_{n+1} = y_n + h[f_{n+1} - \frac{1}{2}\nabla f_{n+1} - \frac{1}{12}\nabla^2 f_{n+1} - \frac{1}{24}\nabla^3 f_{n+1} - \frac{19}{720}\nabla^4 f_{n+1} - \frac{27}{1440}\nabla^5 f_{n+1}]$$

The coefficients $\sigma_m^{*(0)}$ in Equation (xxii) are given below:

k	$\sigma_0^{(0)}$	$\sigma_1^{(0)}$	$\sigma_2^{(0)}$	$\sigma_3^{(0)}$	$\sigma_4^{(0)}$	$\sigma_5^{(0)}$
0	1					
1	$\frac{1}{2}$	$\frac{1}{2}$				
2	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$			
3	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$		
4	$\frac{251}{720}$	$\frac{646}{720}$	$-\frac{264}{720}$	$\frac{106}{720}$	$-\frac{19}{720}$	
5	$\frac{475}{1440}$	$\frac{1427}{1440}$	$-\frac{798}{1440}$	$\frac{482}{1440}$	$-\frac{173}{1440}$	$\frac{27}{1440}$

We will illustrate the use of this formula, by solving the IVP below:

$$y' = x + y, y(0) = 1, \text{ with } h = 0.1$$

The formula is as below

$$y_{n+1} = y_n + \frac{h}{24}[9f_{n+1} + 19f_{n+1} - 5f_{n-1} + f_{n-2}]; \quad Note: f_{n+1} = y'_{n+1}$$

We note that y_{n+1} is contained in both sides of the equation above. In other words the unknown y_{n+1}, cannot be calculated directly, since it is contained within f_{n+1} (i.e. ϕ is dependent on y_{n+1}). Help is required and to this, we engage the services of the *predictor-corrector* method. The Adam-Bashforth method

$$y_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$$

is used as a predictor, while the Adam-Moulton Method given above is used as the corrector. Both methods will now be used to solve the above IVP.

Using the classical four-stage Runge-Kutta method, we get the starting values as:

$y_1 = 1.110342$; $f_1 = y_1' = 1.210342$

$y_2 = 1.242806$; $f_2 = y_2' = 1.442806$
$y_3 = 1.399718$; $f_3 = y_3' = 1.699718$

To determine y_4 we will use the predictor.

At n = 3

$$y_4 = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$\therefore \hat{y}_4 = 1.583641292$; $\hat{f}_4 = y_4' = 1.983641292$

We will now make use of the corrector

$$y_4 = y_3 + \frac{h}{24}[9f_4 + 19f_3 - 5f_2 + f_1];$$
$$\therefore y_4 = 1.58365019; \ f_4 = 1.98365019$$

At n = 4

$$y_4 = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$\hat{y}_5 = 1.797443843$; $\hat{f}_5 = 2.297434117$

$\Rightarrow y_5 = 1.797443843\,; \hat{f}_5 = 2.297434117$
$\therefore y_5 = 1.797443843\,;$

By similar computations, we get

$y_6 = 2.0442397$
$y_7 = 2.3275082$
$y_8 = 2.6510854$

As we stated earlier, Linear Multi-step Methods (LMM) sacrifice the one-step nature of the algorithm, but retain linearity with the advantage that it is easy to estimate errors, but difficult to change steplenght. On the other hand, R-K methods appears to have gone in the opposite direction; sacrificing linearity while retaining the one-step nature of the algorithm; with the advantage of easy change of steplenght, but difficulty in error estimation.

So, we are left with an ironical situation: with LMMs it is easy to ascertain when a change in steplenght ir required, but difficult to change steplenght. While with Runge-Kutta methods, it is hard to determine when a change in steplenght is required, but easy to change the steplenght. Another disadvantage of LMMs is that they are not self starting. They rely on one-step methods to obtain initial values to begin the computation.

A major advantage of multi-step methods over R-K methods, is that for the R-K methods many function evaluations are required in taking one step (six in the case of a six-stage method). On the other hand using the Adam-Moulton method as an example, the predictor requires only the evaluations of f_n and the use of the corrector requires the additional evaluation of f_{n+1} for each iteration performed. There is obviously, a reduction of computational time.

CHAPTER THREE

DERIVATION OF A NEW SIX-STAGE RUNGE-KUTTA SCHEME

An explicit s-stage Runge-Kutta (R-K) method for the numerical integration of a dynamical system

$$\frac{dy}{dx} = f(x, y) \qquad (i)$$

with step size h, is a map (where f and y are vectors)

$$(x, y) \to (x + h, y + h * b[1] * k[1] + ... + h * b[s] * k[s]) \qquad (ii)$$

with "intermediate stages" k[1],...k[s], given by

$$\left.\begin{aligned} &k[1] = f(x, y), \\ &k[2] = f(x + c[2] * h, y + h * a[2,1] * k[1]), \\ &\ldots \\ &k[s] = f(x + c[s*] * h, y + h * a[s,1] * k[1] + \cdots + h * a[s, s - 1 * k[s - 1]) \end{aligned}\right\} \qquad \text{(iii)}$$

Various numerical schemes arises from different choices of the Butcher parameters: the (sxs)-matrix a[i,j], the weights b=[b[1]...b[s]], and the abscissae c=[0, c[2],...,c[s]].

3.1 The Philosophy Behind R-K Methods

Recall the IVP

$$y' = f(x, y), \qquad y(a) = \alpha \qquad (iv)$$

of all computational methods for the numerical solution of this problem, the easiest to implement is Euler's rule

$$\left.\begin{aligned} y_{n+1} &= y_n + hf(x_n, y_n) \\ &\equiv y_n + hf_n \end{aligned}\right\} \qquad \text{(v)}$$

It is explicit and being a one-step method, it requires no additional starting values (i.e. it is self-starting) and readily permits a change of step length during the computation. Its low order of accuracy of course makes it of limited practical value. Linear Multi-step Methods (LMM) achieve higher orders, by sacrificing the one-step nature of the algorithm, while retaining linearity with respect to $j_{n+j}, f_{n+j}, j = 0,1,\cdots,k$. However, it is possible to achieve an even higher order, by sacrificing linearity but preserving the one-step nature of the algorithm. This in essence, is the philosophy behind the methods first proposed by David Runge and subsequently expanded by Wilhelm Kutta, and Heun.

Runge-Kutta methods thus, retain the advantages of one-step methods and may be regarded as a particular case of the general explicit one-step method

$$y_{n+1} = y_n + h\phi(x_n, y_n; h) \qquad (vi)$$

Simply put, R-K methods are designed to approximate Taylor's series methods, but have the advantage of not requiring explicit evaluations of the derivatives of f(x,y), where x often represents time (t). the basic idea is to use a linear combination of values of f(x,y) to approximate y(x). this linear combination is matched up as closely as possible with a Taylor series for y(x) to obtain methods of the highest possible order q.

We note that an s-stage R-K method involves s function evaluations per step. Each of the functions $k_r(x, y; h)$, r= 1, 2 ,... , s, may be interpreted as an approximation to the derivative $y'(x)$, and the function $\phi(x, y; h)$ as a weighted mean of these approximations. Consistency demands that $\sum_{r=1}^{s} b_r = 1$.

If we can choose values for the constants b_r, c_r, a_{rs}, such that the expansion of the function $\phi(x, y; h)$ *as*

$$
\left.
\begin{aligned}
\phi(x, y; h) &= \sum_{r=1}^{s} b_r k_r, \\
k_1 &= f(x, y), \\
&\vdots \\
k_r &= f(x + c_r h, y + h\sum_{s=1}^{r-1} a_{rs} k_s), r = 2,3,\cdots, R \\
c_r &= \sum_{s=1}^{r-1} a_{rs}, r = 2,3,\cdots, s
\end{aligned}
\right\} \quad \text{(vii)}
$$

in powers of h differs from the expansion of the function $\phi_T(x, y; h)$ given by

$$
\begin{aligned}
\phi_T(x, y; h) &\equiv f(x, y) + \frac{h}{2!} f'(x, y) + \cdots + \frac{h^{p-1}}{p!} f^{(p-1)}(x, y) \\
&= \sum_{r=0}^{p-1} \frac{h^r}{(r+1)!} f^{(r)}(x_n, y_n) \qquad (viii)
\end{aligned}
$$

where

$$
f^{(q)}(x, y) = \frac{d^q}{dx^q} f(x, y), q = 1,2,\cdots,(p-1)
$$

only in the p^{th} and higher powers of h, then the method clearly has order p. In (viii), we are assuming that $y(x) \in C^p[a, b]$; that is y(x) possesses p continuous derivatives for $x \in [a, b]$.

There is a good deal of tedious manipulations involved in deriving Runge-Kutta methods of higher orders. The process for deriving a given R-K scheme, can be summarised into the following three steps:

Step1:

Obtain the Taylor series expansion of k_r (the slopes) defined by

$$k_r = f(z_r, y_n + h\sum_{j=1}^{s} a_{rj} k_j), \qquad (ix)$$

where

$$z_r = x_n + c_r h, r = 1(1)s$$

about the point (x_n,y_n) in the solution space.

Step 2:

Insert these expansions and $c_r (c_r = \sum_{j=1}^{s} a_{rj}, r = 1(1)s)$ into the expression for the general s-stage R-K method, given as

$$\phi_{RK} = \sum_{j=1}^{s} b_j k_j, s \geq 1 \qquad (x)$$

Step 3:

Compare the coefficients in powers of h for both the increment function ϕ_{RK} of the Runge-Kutta method given by (x) above and the increment function ϕ_T for the Taylor expansion method specified by (viii).

The totality of the unknown coefficients $\{b_j, c_r, a_{rj}, j = 1(1s)\}$ normally exceeds the number of equations, so some can be chosen so as to attain some desired goals. Some of these goals are:

(i) to minimize a bound of the local truncation error (lte) (Raltson 1962),

(ii) to maximize the attainable order of the scheme (King, 1966, achieved this for the differential systems $y' = f(x)$),

(iii) to optimize the interval of absolute stability (Lawson 1966, 1967b),

(iv) to reduce storage requirements (Gill 1951, Conte and Reeves, 1956, Blum 1962, and Fyfe 1966,) and

(v) to achieve methods that uses whole numbers for computation instead of fractions as with other methods (Adewale, 1998).

3.2 TAYLOR SERIES EXPANSION

The general 6-stage explicit Runge-Kutta method for the solution of the Initial Value Problem (IVP)

$$y' = f(x, y), \quad y(a_0) = \omega_0; f : \Re \times \Re^m \to \Re^m \qquad (1)$$

is defined by

$$y_{n+1} - y_n = h\Phi(x, y; h)$$

where

$$\Phi(x, y; h) = \sum_{i=1}^{6} b_r k_r$$

$$\Rightarrow \quad y_{n+1} = y_n + h[b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6] \qquad (2)$$

where

$$\left.\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + c_2 h, y_n + ha_{21} k_1), \\ k_3 &= f(x_n + c_3 h + y_n + h(a_{31} k_1 + a_{32} k_2)), \\ k_4 &= f(x_n + c_4 h, y_n + h(a_{41} k_1 + a_{42} k_2 + a_{43} k_3)), \\ k_5 &= f(x_n + c_5 h, y_n + h(a_{51} k_1 + a_{52} k_2 + a_{53} k_3 + a_{54} k_4)), \\ k_6 &= f\{x_n + c_5 h, y_n + h(a_{61} k_1 + a_{62} k_2 + a_{63} k_3 + a_{64} k_4 + a_{65} k_5)). \end{aligned}\right\} \qquad (3)$$

and

$$\left.\begin{aligned} c_2 &= a_{21} \\ c_3 &= a_{31} + a_{32} \\ c_4 &= a_{41} + a_{42} + a_{43} \\ c_5 &= a_{51} + a_{52} + a_{53} + a_{54} \\ c_6 &= a_{61} + a_{62} + a_{63} + a_{64} + a_{65} \end{aligned}\right\} \qquad (4)$$

Equation (4) can be re-written as:

$$\left.\begin{aligned} a_{21} &= c_2 \\ a_{31} &= c_3 - a_{32} \\ a_{41} &= c_4 - (a_{42} + a_{43}) \\ a_{51} &= c_5 - (a_{52} + a_{53} + a_{54}) \\ a_{61} &= c_6 - (a_{62} + a_{63} + a_{64} + a_{65)} \end{aligned}\right\} \quad (5)$$

By substituting Equation (5) into Equation (3), we

$$\left.\begin{aligned} k_1 &= f(x_n, y_n) = f \\ k_2 &= f(x_n + c_2 h, y_n + a_{21} h k_1) \\ k_3 &= f(x_n + c_3 h, y_n + h[(c_3 - a_{32})k_1 + a_{32} k_2] \\ k_4 &= f(x_n + c_4 h, y_n + h[(c_4 - (a_{42} + a_{43}))k_1 + a_{42} k_2 + a_{43} k_3)] \\ k_5 &= f(x_n + c_5 h, y_n + h[(c_5 - (a_{52} + a_{53} + a_{54}))k_1 + a_{52} k_2 + a_{53} k_3 + a_{54} k_4)] \\ k_6 &= f(x_n + c_6 h, y_n + h[(c_5 - (a_{62} + a_{63} + a_{64} + a_{65}))k_1 + a_{62} k_2 + a_{63} k_3 + a_{64} k_4 + a_{65} k_5)] \end{aligned}\right\} \quad (6)$$

Call the k_r's the slopes, b_r's the weights, and c_r's the abscissae, r=1(1)6

We proceed, to expand each k_r in Equation (6) in turn, by Taylor's theorem:

$$f(x+m, y+n) = f(x,y) + Df(x,y) + \frac{1}{2}D^2 f(x,y) + \frac{1}{3!}D^3 f(x,y) + \ldots + \frac{1}{n!}D^n f(x,y)$$

where D is the differential operator, defined as:

$$\left.\begin{aligned} &D = \frac{m\partial}{\partial x} + \frac{n\partial}{\partial y} \text{ and} \\ &D^n f(x,y) = \left(\frac{m\partial}{\partial x} + \frac{n\partial}{\partial y}\right)^n f(x,y) \end{aligned}\right\} \quad (7)$$

$$\Rightarrow Df = mf_x + nf_y$$

$$D^2 f = \left(\frac{m\partial}{\partial x} + \frac{n\partial}{\partial y}\right)^2 f(x,y)$$

$$= \left(\frac{m\partial}{\partial x} + \frac{n\partial}{\partial y}\right)\left[\left(\frac{m\partial}{\partial x} + \frac{n\partial}{\partial y}\right) f(x,y)\right]$$

$$\therefore D^2 f = \left(\frac{m\partial}{\partial x} + \frac{n\partial}{\partial y}\right)(mf_x + nf_y) = m^2 f_{xx} + 2mnf_{xy} + n^2 f_{yy}$$

$$D^3 f = \left(\frac{m\partial}{\partial x} + \frac{n\partial}{\partial y}\right)(D^2 f)$$

$$= m^3 f_{xxx} + 3m^2 n f_{xxy} + 3mn^2 f_{xyy} + n^3 f_{yyy}$$

$$D^4 f = \left(\frac{m\partial}{\partial x} + \frac{n\partial}{\partial y}\right)(D^3 f)$$

$$= m^4 f_{xxxx} + 4m^3 n f_{xxxy} + 4m^2 n^2 f_{xxyy} + 4mn^3 f_{xyyy} + n^4 f_{yyyy}$$

Therefore,

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + c_2 h + y_n + hc_2 k_1)$$

The Taylor series expansion of k_2 about the point (x_n, y_n) in the solution space yields:

$$k_2 = f + c_2 h f_x + c_2 h k_1 f_y + \frac{1}{2}(c_2{}^2 h^2 f_{xx} + 2c_2{}^2 h^2 k_1 f_{xy} + (c_2 h k_1)^2 f_{yy}) + \frac{1}{6}((c_2 h)^3 f_{xxx}$$

$$+ 3((c_2 h)^2 (c_2 h) k_1) f_{xxy} + 3c_2 h (c_2 h k_1)^2 f_{xyy} + (c_2 h k_1)^3 f_{yyy}) + \frac{1}{24}((c_2 h)^4 f_{xxxx}$$

$$+ 4(c_2 h)^4 k_1 f_{xxxy} + 4(c_2 h)^2 (c_2 h k_1)^2 f_{xxyy} + 4c_2 h (c_2 h k_1)^3 f_{xyyy} + (c_2 h k_1)^4 f_{yyyy}) + o(h^5)$$

with all the terms evaluated at (x_n, y_n)

Replacing k_1 with f, we now have:

$$k_2 = f + c_2 h f_x + c_2 h f f_y + \frac{1}{2}(c_2{}^2 h^2 f_{xx} + 2c_2{}^2 h^2 f f_{xy} + (c_2 h)^2 f^2 f_{yy}) + \frac{1}{6}((c_2 h)^3 f_{xxx}$$

$$+ 3((c_2 h)^3) f f_{xxy} + 3(c_2 h)^3 f^2 f_{xyy} + (c_2 h_1)^3 f^3 f_{yyy}) + \frac{1}{24}((c_2 h)^4 f_{xxxx}$$

$$+ 4(c_2 h)^4 f f_{xxxy} + 4(c_2 h)^4 f^2 f_{xxyy} + 4(c_2 h)^4 f^3 f_{xyyy} + (c_2 h)^4 f^4 f_{yyyy}) + o(h^5)$$

Collecting like terms together, we obtain

$$\therefore k_2 = f + c_2 h (f_x + f f_y) + \frac{1}{2}(c_2 h)^2 (f_{xx} + 2 f f_{xy} + f^2 f_{yy}) + \frac{1}{6}(c_2 h)^3 (f_{xxx} + 3 f f_{xxy} + 3 f^2 f_{xyy} + f^3 f_{yyy})$$

$$+ \frac{1}{24}(c_2 h)^4 (f_{xxxx} + 4 f f_{xxxy} + 4 f^2 f_{xxyy} + 4 f^3 f_{xyyy} + f^4 f_{yyyy}) + o(h^5)$$

Setting

$$F = f f_x + f f_y \quad \Bigg\}$$

$$G = f_{xx} + 2ff_{xy} + f^2 f_{yy}$$

$$H = f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy} \tag{8}$$

$$I = f_{xxxx} + 4ff_{xxxy} + 4f^2 f_{xxyy} + 4f^3 f_{xyyy} + f^4 f_{yyyy}$$

From Equation (8)

$$\Rightarrow k_2 = f + c_2hF + \frac{1}{2}(c_2h)^2 G + \frac{1}{6}(c_2h)^3 H + \frac{1}{24}(c_2h)^4 I$$

Now from Equation (3),

$$k_3 = f(x_n + c_3h, y_n + h((c_3 - a_{32})k_1 + a_{32}k_2)$$

By expanding k_3 in Taylor series about the point (x_n, y_n) in the solution space, and substituting f for k_1, yields

$$\begin{aligned}
k_3 = f &+ c_3hf_x + h((c_5 - a_{32})f + a_{32}k_2)f_y + \frac{1}{2}((c_3h)^2 f_{xx} + 2(c_3h)(h(c_3 - a_{32})f + a_{32}k_2)f_{xy} \\
&+ (h(c_3 - a_{32})f + a_{32}k_2)^2 f_{yy}) + \frac{1}{6}((c_3h)^3 f_{xxx} + 3(c_3h)^2 (h((c_3 - a_{32})f + a_{32}k_2))f_{xxy} \\
&+ 3(c_3h)(h((c_3 - a_{32})f + a_{32}k_2))^2 f_{xyy} + (h((c_3 - a_{32})f + a_{32}k_2))^3 f_{yyy}) + \frac{1}{24}((c_3h)^4 f_{xxxx}) \\
&+ 4(c_3h)^3 (h((c_3 - a_{32})f + a_{32}k_2))f_{xxxy} + 4(c_3h)^2 (h((c_3 - a_{32})f + a_{32}k_2))^2 f_{xxyy} \\
&+ 4(c_3h)(h((c_3 - a_{32})f + a_{32}k_2))^3 f_{xyyy} + (h((c_3 - a_{32})f + a_{32}k_2))^4 f_{yyyy}) + o(h^5)
\end{aligned}$$

with all the terms evaluated at (x_n, y_n).

By collecting like powers of h together, we obtain

$$k_3 = f + h(c_3 f_x + ((c_3 - a_{32})f + a_{32}k_2)f_y) + \frac{1}{2}h^2(c_3^2 f_{xx} 2c_3((c_3 - a_{32})f + a_{32}k_2)f_{xy}$$

$$+((c_3 - a_{32})f + a_{32}k_2)^2 f_{yy}) + \frac{1}{6}h^3(c_3^3 f_{xxx} + 3c_3^2((c_3 - a_{32})f + a_{32}k_2)f_{xxy}$$

$$+3c_3((c_3 - a_{32})f + a_{32}k_2)^2 f_{xyy} + ((c_3 - a_{32})f + a_{32}k_2)^3 f_{yyy})$$

$$+\frac{1}{24}h^4(c_3^4 f_{xxxx} + 4c_3^3((c_3 - a_{32})f + a_{32}k_2)f_{xyyy} + 4c_3^2((c_3 - a_{32})f + a_{32}k_2)^2 f_{xxyy}$$

$$+4c_3((c_3 - a_{32})f + a_{32}k_2)^3 f_{xyyy} + ((c_3 - a_{32})f + a_{32}k_2)^4 f_{yyyy}) + o(h^5)$$

Substituting for k_2 in k_3 we now have:

$$k_3 = f + h(c_3 f_x + ((c_3 - a_{32})f + a_{32}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G + \frac{1}{6}(c_2 h)^3 H))f_y)$$

$$+\frac{1}{2}h^2(c_3^2 f_{xx} 2c_3((c_3 - a_{32})f + a_{32}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G))f_{xy} + ((c_3 - a_{32})f$$

$$+a_{32}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G))^2 f_{yy}) + \frac{1}{6}h^3(c_3^3 f_{xxx} + 3c_3^2((c_3 - a_{32})f + a_{32}(f + c_2 hF))f_{xxy}$$

$$+3c_3((c_3 - a_{32})f + a_{32}(f + c_2 hF))^2 f_{xyy} + ((c_3 - a_{32})f + a_{32}(f + c_2 hF))^3 f_{yyy})$$

$$+\frac{1}{24}h^4(c_3^4 f_{xxxx} + 4c_3^3((c_3 - a_{32})f + a_{32}f)f_{xyyy} + 4c_3^2((c_3 - a_{32})f + a_{32}f)^2 f_{xxyy}$$

$$+4c_3((c_3 - a_{32})f + a_{32}f)^3 f_{xyyy} + ((c_3 - a_{32})f + a_{32}f)^4 f_{yyyy}) + o(h^5)$$

On expanding, we get

$$k_3 = f + hc_3 f_x + hc_3 ff_y - ha_{32} ff_y + ha_{32} ff_y + h^2 c_2 a_{32} Ff_y + \frac{1}{2} h^3 c_2^2 a_{32} Gf_y + \frac{1}{6} h^4 c_2^3 a_{32} Hf_y + \frac{1}{2}(hc_3)^2 f_{xx}$$

$$+ (hc_3)^2 ff_{xy} - h^2 c_3 a_{32} ff_{xy} + h^2 c_3 a_{32} ff_{xy} + h^3 c_2 c_3 a_{32} Ff_{xy} + \frac{1}{2} h^4 c_2^2 c_3 a_{32} Gf_{xy} + \frac{1}{2} h^2 (c_3 f - a_{32} f$$

$$+ a_{32} f + hc_2 a_{32} F + \frac{1}{2} h^2 c_2^2 a_{32} G)^2 f_{yy} + \frac{1}{6}(hc_3)^3 f_{xxx} + \frac{1}{2} h^3 c_3^2 (c_3 f - a_{32} f + a_{32} f + hc_2 a_{32} F) f_{xxy}$$

$$+ \frac{1}{2} h^3 c_3 (c_3 f - a_{32} f + a_{32} f + hc_2 a_{32} F)^2 f_{xyy} + \frac{1}{6} h^3 (c_3 f - a_{32} f + a_{32} f + hc_2 a_{32} F)^3 f_{yyy}$$

$$+ \frac{1}{24}(hc_3)^4 f_{xxxx} + \frac{1}{6} h^4 c_3^3 (c_3 f - a_{32} f + a_{32} f) f_{xxxy} + \frac{1}{6} h^4 c_3^2 (c_3 f - a_{32} f + a_{32} f)^2 f_{xxyy}$$

$$+ \frac{1}{6} h^4 c_3 (c_3 f - a_{32} f + a_{32} f)^3 f_{xyyy} + \frac{1}{24} h^4 (c_3 f - a_{32} f + a_{32} f) f_{yyyy} + o(h^5)$$

$$k_3 = f + hc_3 (f_x + ff_y) + h^2 c_2 a_{32} Ff_y + \frac{1}{2} h^3 c_2^2 a_{32} Gf_y + \frac{1}{6} h^4 c_2^3 a_{32} Hf_y + \frac{1}{2}(hc_3)^2 f_{xx}$$

$$+ (hc_3)^2 ff_{xy} + h^3 c_2 c_3 a_{32} Ff_{xy} + \frac{1}{2} h^4 c_2^2 c_3 a_{32} Gf_{xy} + \frac{1}{2} h^2 (c_3^2 f^2 + 2hc_2 c_3 a_{32} Ff + h^2 c_2^2 c_3 a_{32} Gf) f_{yy}$$

$$+ \frac{1}{6}(hc_3)^3 f_{xxx} + \frac{1}{2} h^3 c_3^3 ff_{xxy} + \frac{1}{2} h^4 c_2 c_3^2 a_{32} Ff_{xxy} + \frac{1}{2} h^3 c_3 (c_3^2 f^2 + 2hc_2 c_3 a_{32} Ff) f_{xyy}$$

$$+ \frac{1}{6} h^3 (c_3^3 f^3 + 3hc_2 c_3^2 a_{32} Ff^2) f_{yyy} + \frac{1}{24}(hc_3)^4 f_{xxxx} + \frac{1}{6} h^4 c_3^4 ff_{xxxy} + \frac{1}{6} h^4 c_3^4 f^2 f_{xxyy}$$

$$+ \frac{1}{6} h^4 c_3^4 f^3 f_{xyyy} + \frac{1}{24} h^4 c_3^4 f^4 f_{yyyy} + o(h^5)$$

$$k_3 = f + hc_3 F + h^2 c_2 a_{32} Ff_y + \frac{1}{2} h^3 c_2^2 a_{32} Gf_y + \frac{1}{6} h^4 c_2^3 a_{32} Hf_y + \frac{1}{2}(hc_3)^2 (f_{xx} + 2ff_{xy} + f^2 f_{yy})$$

$$+ h^3 c_2 c_3 a_{32} Ff_{xy} + \frac{1}{2} h^4 c_2^2 c_3 a_{32} Gf_{xy} + \frac{1}{2} h^2 (2hc_2 c_3 a_{32} Ff + h^2 c_2^2 c_3 a_{32} Gf) f_{yy}$$

$$+ \frac{1}{6}(hc_3)^3 (f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy}) + \frac{1}{2} h^4 c_2 c_3^2 a_{32} Ff_{xxy} + h^4 c_2 c_3^2 a_{32} Fff_{xyy}$$

$$+ \frac{1}{2} h^4 c_2 c_3^2 a_{32} Ff^2 f_{yyy} + \frac{1}{24}(hc_3)^4 (f_{xxxx} + 4ff_{xxxy} + 4f^2 f_{xxyy} + 4f^3 f_{xyyy} + f^4 f_{yyyy}) + o(h^5)$$

From Equation (8), we now have

$$k_3 = f + hc_3 F + h^2 c_2 a_{32} Ff_y + \frac{1}{2} h^3 c_2^2 a_{32} Gf_y + \frac{1}{6} h^4 c_2^3 a_{32} Hf_y + \frac{1}{2}(hc_3)^2 G$$

$$+ h^3 c_2 c_3 a_{32} Ff_{xy} + \frac{1}{2} h^4 c_2^2 c_3 a_{32} Gf_{xy} + \frac{1}{2} h^2 (2hc_2 c_3 a_{32} Ff + h^2 c_2^2 c_3 a_{32} Gf) f_{yy}$$

$$+ \frac{1}{6}(hc_3)^3 H + \frac{1}{2} h^4 c_2 c_3^2 a_{32} Ff_{xxy} + h^4 c_2 c_3^2 a_{32} Fff_{xyy} + \frac{1}{2} h^4 c_2 c_3^2 a_{32} Ff^2 f_{yyy}$$

$$+ \frac{1}{24}(hc_3)^4 I + o(h^5)$$

Collecting like powers of h together, we arrive at

$$k_3 = f + hc_3F + h^2[c_2a_{32}Ff_y + \frac{1}{2}c_3^2G] + h^3[\frac{1}{2}c_2^2a_{32}Gf_y + c_2c_3a_{32}Ff_{xy} + +c_2c_3a_{32}Fff_{yy} + \frac{1}{6}c_3^3H]$$
$$+ h^4[\frac{1}{6}c_2^3a_{32}Hf_y + \frac{1}{2}c_2^2c_3a_{32}Gf_{xy} + \frac{1}{2}c_2^2c_3a_{32}Gff_{yy} + \frac{1}{2}c_2c_3^2a_{32}Ff_{xxy} + c_2c_3^2a_{32}Fff_{xyy}$$
$$+ \frac{1}{2}c_2c_3^2a_{32}Ff^2f_{yyy} + \frac{1}{24}(hc_3)^4I] + o(h^5) \tag{10}$$

Next, we have;

$$k_4 = f[x_n + hc_4, y_n + h((c_4 - (a_{42} + a_{43}))k_1 + a_{42}k_2 + a_{43}k_3)]$$

When we expand K_4 in Taylor series about the point (x_n, y_n) in the solution space, and replace k_1 with f, we obtain:

$$k_4 = f + hc_4f_x + h((c_4 - (a_{42} + a_{43}))f + a_{42}k_2 + a_{43}k_3)f_y + \frac{1}{2}[(hc_4)^2f_{xx} + 2(hc_4)h((c_4 - (a_{42} + a_{43}))f$$
$$+ a_{42}k_2 + a_{43}k_3)f_{xy} + h^2((c_4 - (a_{42} + a_{43}))f + a_{42}k_2 + a_{43}k_3)^2 fyy] + \frac{1}{6}[(hc_4)^3f_{xxx}$$
$$+ 3(hc_4)^2h((c_4 - (a_{42} + a_{43}))f + a_{42}k_2 + a_{43}k_3)f_{xxy} + 3(hc_4)h^2((c_4 - (a_{42} + a_{43}))f + a_{42}k_2$$
$$+ a_{43}k_3)^2f_{xyy} + h^3((c_4 - (a_{42} + a_{43}))f + a_{42}k_2 + a_{43}k_3)^3f_{yyy}] + \frac{1}{24}[(hc_4)^4f_{xxxx}$$
$$+ 4(hc_4)^3h((c_4 - (a_{42} + a_{43}))f + a_{42}k_2 + a_{43}k_3)f_{xxxy} + 4(hc_4)^2h^2((c_4 - (a_{42} + a_{43}))f$$
$$+ a_{42}k_2 + a_{43}k_3)^2f_{xxyy} + 4(hc_4)h^3((c_4 - (a_{42} + a_{43}))f + a_{42}k_2 + a_{43}k_3)^3f_{xyyy}$$
$$+ h^4((c_4 - (a_{42} + a_{43}))f + a_{42}k_2 + a_{43}k_3)^4f_{yyyy}] + o(h^5)$$

all the terms being evaluated at (x_n, y_n).

Substituting k_2 and k_3 into k_4 we have

$$k_4 = f + hc_4 f_x + h[c_4 f - a_{42} f - a_{43} f + a_{42}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G + \frac{1}{6}(c_2 h)^3 H$$

$$+ a_{43}(f + hc_3 F + h^2 c_2 a_{32} Ff_y + \frac{1}{2} h^2 c_3^2 G + \frac{1}{2} h^3 c_2^2 a_{32} Gf_y + h^3 c_2 c_3 a_{32} Ff_{xy} + h^3 c_2 c_3 a_{32} Fff_{yy}$$

$$+ \frac{1}{6} h^3 c_3^3 H)] f_y + \frac{1}{2}(hc_4)^2 f_{xx} + h^2 c_4 [c_4 f - a_{42} f - a_{43} f + a_{42}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G)$$

$$+ a_{43}(f + hc_3 F + h^2 c_2 a_{32} Ff_y + \frac{1}{2} h^2 c_3^2 G)] f_{xy} + \frac{1}{2} h^2 [c_4 f - a_{42} f - a_{43} f$$

$$+ a_{42}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G) + a_{43}(f + hc_3 F + h^2 c_2 a_{32} Ff_y + \frac{1}{2} h^2 c_3^2 G)]^2 f_{yy} + \frac{1}{6}(hc_4)^3 f_{xxx}$$

$$+ \frac{1}{2}(hc_4)^2 h[c_4 f - a_{42} f - a_{43} f + a_{42}(f + c_2 hF) + a_{43}(f + hc_3 F)] f_{xxy}$$

$$+ \frac{1}{2}(hc_4)^2 h^2 [c_4 f - a_{42} f - a_{43} f + a_{42}(f + c_2 hF) + a_{43}(f + hc_3 F)]^2 f_{xyy}$$

$$+ \frac{1}{6} h^3 [c_4 f - a_{42} f - a_{43} f + a_{42}(f + c_2 hF) + a_{43}(f + hc_3 F)]^3 f_{yyy}$$

$$+ \frac{1}{24}(hc_4)^4 f_{xxxx} + \frac{1}{6}(hc_4)^3 h[c_4 f - a_{42} f - a_{43} f + a_{42} f + a_{43} f] f_{xxxy}$$

$$+ \frac{1}{6}(hc_4)^2 h^2 [c_4 f - a_{42} f - a_{43} f + a_{42} f + a_{43} f]^2 f_{xxyy} + \frac{1}{6}(hc_4) h^3 [c_4 f - a_{42} f - a_{43} f$$

$$+ a_{42} f + a_{43} f]^3 f_{xyyy} + \frac{1}{24} h^4 [c_4 f - a_{42} f - a_{43} f + a_{42} f + a_{43} f]^4 f_{yyyy} + o(h^5)$$

$$k_4 = f + hc_4 f_x + hc_4 ff_y + h^2 c_2 a_{42} Ff_y + \frac{1}{2} h^3 c_2^2 a_{42} Gf_y + \frac{1}{6} h^4 c_3^3 a_{42} Hf_y + h^2 c_3 a_{43} Ff_y + h^3 c_2 a_{32} a_{43} Ff^2{}_y$$

$$+ \frac{1}{2} h^3 c_3^2 a_{43} Gf_y + \frac{1}{2} h^4 c_2^2 a_{32} a_{43} Gf_y^2 + h^4 c_2 c_3 a_{32} a_{43} Ff_{xy} f_y + h^4 c_2 c_3 a_{32} a_{42} Fff_y f_{yy}$$

$$+ \frac{1}{6} h^4 c_3^3 a_{43} Hf_y + \frac{1}{2}(hc_4)^2 f_{xx} + h^2 c_4^2 ff_{xy} + h^3 c_2 c_4 a_{42} Ff_{xy} + \frac{1}{2} h^4 c_2^2 c_4 a_{42} Gf_{xy} + h^3 c_3 c_4 a_{43} Ff_{xy}$$

$$+ h^4 c_2 c_4 a_{32} a_{43} Ff_{xy} f_y + \frac{1}{2} h^4 c_3^2 c_4 a_{43} Gf_{xy} + \frac{1}{2} h^2 c_4^2 f^2 f_{yy} + h^3 c_2 c_4 a_{42} Fff_{yy} + \frac{1}{2} h^4 c_2^2 c_4 a_{42} Gff_{yy}$$

$$+ h^3 c_3 c_4 a_{43} Fff_{yy} + h^4 c_2 c_4 a_{32} a_{43} Fff_y f_{yy} + \frac{1}{2} h^4 c_3^2 c_4 a_{43} Gff_{yy} + \frac{1}{6}(hc_4)^3 f_{xxx} + \frac{1}{2}(hc_4)^3 ff_{xxy}$$

$$+ \frac{1}{2} h^4 c_2 c_4^2 a_{42} Ff_{xxy} + \frac{1}{2} h^4 c_3 c_4^2 a_{43} Ff_{xxy} + \frac{1}{2}(hc_4)^3 f^2 f_{xyy} + h^4 c_2 c_4^2 a_{42} Fff_{xyy} + h^4 c_3 c_4^2 a_{43} Fff_{xyy}$$

$$k_4 = f + hc_4(f_x + ff_y) + h^2(c_2 a_{42} Ff_y + c_3 a_{43} Ff_y) + h^3(\frac{1}{2}c_2^2 a_{42} Gf_y + h^3 c_2 a_{32} a_{43} Ff^2{}_y$$
$$+\frac{1}{2}h^3 c_3^2 a_{43} Gf_y + h^3 c_2 c_4 a_{42} Ff_{xy} + h^3 c_3 c_4 a_{43} Ff_{xy} + h^3 c_2 c_4 a_{42} Fff_{yy} + h^3 c_3 c_4 a_{43} Fff_{yy})$$
$$+h^4(\frac{1}{6}c_3^3 a_{42} Hf_y + \frac{1}{6}c_3^3 a_{43} Hf_y + \frac{1}{2}c_2^2 c_4 a_{42} Gf_{xy} + \frac{1}{2}c_2^2 a_{32} a_{43} Gf_y^2 + c_2 c_3 a_{32} a_{43} Ff_{xy} f_y$$
$$+c_2 c_4 a_{32} a_{43} Ff_{xy} f_y + \frac{1}{2}c_3^2 c_4 a_{43} Gf_{xy} + \frac{1}{2}c_2^2 c_4 a_{42} Gff_{yy} + \frac{1}{2}c_3^2 c_4 a_{43} Gff_{yy} + \frac{1}{2}c_2 c_4^2 a_{42} Ff_{xxy}$$
$$+h^4 c_2 c_3 a_{32} a_{42} Fff_y f_{yy} + c_2 c_4 a_{32} a_{43} Fff_y f_{yy} + \frac{1}{2}c_3 c_4^2 a_{43} Ff_{xxy} + c_2 c_4^2 a_{42} Fff_{xyy} + c_3 c_4^2 a_{43} Fff_{xyy}$$
$$+\frac{1}{2}c_2 c_4^2 a_{42} Ff^2 f_{yyy} + \frac{1}{2}c_3 c_4^2 a_{43} Ff^2 f_{yyy}) + \frac{1}{2}(hc_4)^2(f_{xx} + 2ff_{xy} + f^2 f_{yy})$$
$$+\frac{1}{6}(hc_4)^3(f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy}) + \frac{1}{24}(hc_4)^4(f_{xxxx} + 4ff_{xxxy} + 4f^2 f_{xxyy}$$
$$+4f^3 f_{xyyy} + f^4 f_{yyyy}) + o(h^5)$$

From equation (8) and gathering all like powers of h together:

$$k_4 = f + hc_4 F + h^2(\frac{1}{2}c_4^2 G + c_2 a_{42} Ff_y + c_3 a_{43} Ff_y) + h^3(\frac{1}{6}c_4^3 H + \frac{1}{2}c_2^2 a_{42} Gf_y$$
$$+h^3 c_2 a_{32} a_{43} Ff^2{}_y + \frac{1}{2}h^3 c_3^2 a_{43} Gf_y + h^3 c_2 c_4 a_{42} Ff_{xy} + h^3 c_3 c_4 a_{43} Ff_{xy}$$
$$+h^3 c_2 c_4 a_{42} Fff_{yy} + h^3 c_3 c_4 a_{43} Fff_{yy}) + h^4(\frac{1}{24}c_4^4 I + \frac{1}{6}c_3^3 a_{42} Hf_y$$
$$+\frac{1}{6}c_3^3 a_{43} Hf_y + \frac{1}{2}c_2^2 c_4 a_{42} Gf_{xy} + \frac{1}{2}c_2^2 a_{32} a_{43} Gf_y^2 + c_2 c_3 a_{32} a_{43} Ff_{xy} f_y$$
$$+c_2 c_4 a_{32} a_{43} Ff_{xy} f_y + \frac{1}{2}c_3^2 c_4 a_{43} Gf_{xy} + \frac{1}{2}c_2^2 c_4 a_{42} Gff_{yy} + \frac{1}{2}c_3^2 c_4 a_{43} Gff_{yy}$$
$$+\frac{1}{2}c_2 c_4^2 a_{42} Ff_{xxy} + h^4 c_2 c_3 a_{32} a_{42} Fff_y f_{yy} + c_2 c_4 a_{32} a_{43} Fff_y f_{yy}$$
$$+\frac{1}{2}c_3 c_4^2 a_{43} Ff_{xxy} + c_2 c_4^2 a_{42} Fff_{xyy} + c_3 c_4^2 a_{43} Fff_{xyy} + \frac{1}{2}c_2 c_4^2 a_{42} Ff^2 f_{yyy}$$
$$+\frac{1}{2}c_3 c_4^2 a_{43} Ff^2 f_{yyy}) + o(h^5) \quad (11)$$

$$k_5 = f[x_n + hc_5, y_n + h((c_5 - (a_{52} + a_{53} + a_{54}))k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)]$$

Expanding k_5 in Taylor series as before, and putting f for k_1 yields;

$$\begin{aligned}
k_5 = {} & f + hc_5 f_x + h[(c_5 - (a_{52} + a_{53} + a_{54}))f + a_{52}k_2 + a_{53}k_3 + a_{54}k_4]f_y \\
& + \frac{1}{2}[(hc_5)^2 f_{xx} + 2(hc_5)h((c_5 - (a_{52} + a_{53} + a_{54}))f + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)f_{xy} \\
& + h^2((c_5 - (a_{52} + a_{53} + a_{54}))f + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)^2 f_{yy}] + \frac{1}{6}[(hc_5)^3 \\
& + 3(hc_5)^2 h((c_5 - (a_{52} + a_{53} + a_{54}))f + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)f_{xxy} \\
& + 3(hc_5)h^2((c_5 - (a_{52} + a_{53} + a_{54}))f + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)^2 f_{xyy} \\
& + h^3((c_5 - (a_{52} + a_{53} + a_{54}))f + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)^3 f_{yyy}] \\
& + \frac{1}{24}[(hc_5)^4 f_{xxxx} + 4(hc_5)^3 h((c_5 - (a_{52} + a_{53} + a_{54}))f + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)f_{xxxy} \\
& + 4(hc_5)^2 h^2((c_5 - (a_{52} + a_{53} + a_{54}))f + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)^2 f_{xxyy} \\
& + 4(hc_5)h^3((c_5 - (a_{52} + a_{53} + a_{54}))f + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)^3 f_{xyyy} \\
& + h^4((c_5 - (a_{52} + a_{53} + a_{54}))f + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)f_{yyyy}] + o(h^5)
\end{aligned}$$

when we substitute for k_2, k_3, and k_4 into k_5 we would have

$$k_5 = f + hc_5 f_x + h[(c_5 f - a_{52} f - a_{53} f - a_{54} f + a_{52}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G + \frac{1}{6}(c_2 h)^3 H)$$

$$+ a_{53}(f + hc_3 F + \frac{1}{2}h^2 c_3^2 G + h^2 c_2 a_{32} Ff_y + \frac{1}{2}h^3 c_2^2 a_{32} Gf_y + h^3 c_2 c_3 a_{32} Ff_{xy} + + h^3 c_2 c_3 a_{32} Fff_{yy}$$

$$+ \frac{1}{6}h^3 c_3^3 H + a_{54}(f + hc_4 F + \frac{1}{2}h^2 c_4^2 G + h^2 c_2 a_{42} Ff_y + h^2 c_3 a_{43} Ff_y + \frac{1}{6}h^3 c_4^3 H + \frac{1}{2}h^3 c_2^2 a_{42} Gf_y$$

$$+ h^3 c_2 a_{32} a_{43} Ff_y^2 + \frac{1}{2}h^3 c_3^2 a_{43} Gf_y + h^3 c_2 c_4 a_{42} Ff_{xy} + h^3 c_3 c_4 a_{43} Ff_{xy} + h^3 c_2 c_4 a_{42} Fff_{yy}$$

$$+ h^3 c_3 c_4 a_{43} Fff_{yy}))]f_y + \frac{1}{2}[(hc_5)^2 f_{xx} + 2(hc_5)h((c_5 f - a_{52} f - a_{53} f - a_{54} f$$

$$+ a_{52}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G) + a_{53}(f + hc_3 F + \frac{1}{2}h^2 c_3^2 G + h^2 c_2 a_{32} Ff_y)$$

$$+ a_{54}(f + hc_4 F + \frac{1}{2}h^2 c_4^2 G + h^2 c_2 a_{42} Ff_y + h^2 c_3 a_{43} Ff_y))f_{xy}.$$

$$+ h^2(c_5 - a_{52} f - a_{53} f - a_{54} f + a_{52}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G) + a_{53}(f + hc_3 F + \frac{1}{2}h^2 c_3^2 G$$

$$+ h^2 c_2 a_{32} Ff_y) + a_{54}(f + hc_4 F + \frac{1}{2}h^2 c_4^2 G + h^2 c_2 a_{42} Ff_y + h^2 c_3 a_{43} Ff_y))^2 f_{yy}]$$

$$+ \frac{1}{6}[(hc_5)^3 f_{xxx} + 3(hc_5)^2 h(c_5 - a_{52} f - a_{53} f - a_{54} f + a_{52}(f + c_2 hF) + a_{53}(f + hc_3 F)$$

$$+ a_{54}(f + hc_4 F))f_{xxy} + 3(hc_5)h^2(c_5 - a_{52} f - a_{53} f - a_{54} f + a_{52}(f + c_2 hF)$$

$$+ a_{53}(f + hc_3 F) + a_{54}(f + hc_4 F))^2 f_{xyy} + h^3(c_5 - a_{52f} - a_{53} f - a_{54} f + a_{52}(f + c_2 hF)$$

$$+ a_{53}(f + hc_3 F) + a_{54}(f + hc_4 F))^3 f_{yyy}] + \frac{1}{24}[(hc_5)^4 f_{xxxx} + 4(hc_5)^3 h((c_5 - a_{52} f - a_{53} f$$

$$- a_{54} f + a_{52} f + a_{53} f + a_{54} f)f_{xxxy} + 4(hc_5)^2 h^2((c_5 - a_{52} f - a_{53} f - a_{54} f + a_{52} f + a_{53} f + a_{54} f)^2 f_{xxyy}$$

$$+ 4(hc_5)h^3((c_5 - a_{52} f - a_{53} f - a_{54} f + a_{52} f + a_{53} f + a_{54} f)^3 f_{xyyy}$$

$$+ h^4((c_5 - a_{52} f - a_{53} f - a_{54} f + a_{52} f + a_{53} f + a_{54} f)f_{yyyy}] + o(h^5)$$

On evaluating k_5 further, we obtain

$$k_5 = f + hc_5 f_x + h[(c_5 f + hc_2 a_{52} F + \frac{1}{2}h^2 c_2^2 a_{52} G + \frac{1}{6}h^3 c_2^3 a_{52} H + hc_3 a_{53} F + \frac{1}{2}h^2 c_3^2 a_{53} G$$

$$+ h^2 c_2 a_{32} a_{53} Ff_y + \frac{1}{2}h^3 c_2^2 a_{32} a_{53} Gf_y + h^3 c_2 c_3 a_{32} a_{53} Ff_{xy} + + h^3 c_2 c_3 a_{32} a_{53} Fff_{yy}$$

$$+ \frac{1}{6}h^3 c_3^3 a_{53} H + hc_4 a_{54} F + \frac{1}{2}h^2 c_4^2 a_{54} G + h^2 c_2 a_{42} a_{54} Ff_y + h^2 c_3 a_{43} a_{54} Ff_y + \frac{1}{6}h^3 c_4^3 a_{54} H$$

$$+ \frac{1}{2}h^3 c_2^2 a_{42} a_{54} Gf_y + h^3 c_2 a_{32} a_{43} a_{54} Ff_y^2 + \frac{1}{2}h^3 c_3^2 a_{43} a_{54} Gf_y + h^3 c_2 c_4 a_{42} a_{54} Ff_{xy}$$

$$+ h^3 c_3 c_4 a_{43} a_{54} Ff_{xy} + h^3 c_2 c_4 a_{42} a_{54} Fff_{yy} + h^3 c_3 c_4 a_{43} a_{54} Fff_{yy})]f_y + \frac{1}{2}[(hc_5)^2 f_{xx} + 2(hc_5)h(c_5 f$$

$$+c_2ha_{52}F+\frac{1}{2}c_2^2h^2a_{52}G+hc_3a_{53}F+\frac{1}{2}h^2c_3^2a_{53}G+h^2c_2a_{32}a_{53}Ff_y+hc_4a_{54}F+\frac{1}{2}h^2c_4^2a_{54}G$$

$$+h^2c_2a_{42}a_{54}Ff_y+h^2c_3a_{43}a_{54}Ff_y)f_{xy}+h^2(c_5+c_2a_{52}hF+\frac{1}{2}(c_2h)^2a_{52}G+hc_3a_{53}F$$

$$+\frac{1}{2}h^2c_3^2Ga_{53}+h^2c_2a_{32}a_{53}Ff_y+hc_4a_{54}F+\frac{1}{2}h^2c_4^2a_{54}G+h^2c_2a_{42}a_{54}Ff_y+h^2c_3a_{43}a_{54}Ff_y)^2f_{yy}]$$

$$+\frac{1}{6}[(hc_5)^3f_{xxx}+3(hc_5)^2h(c_5+c_2a_{52}hF+hc_3a_{53}F+hc_4a_{54}F)f_{xxy}+3(hc_5)h^2(c_5+c_2a_{52}hF$$

$$+hc_3a_{53}F+hc_4a_{54}F)^2f_{xyy}+h^3(c_5+c_2a_{52}hF+hc_3a_{53}F+hc_4a_{54}F)^3f_{yyy}]$$

$$+\frac{1}{24}[(hc_5)^4f_{xxxx}+4h^4c_5^4f_{xxxy}+4h^4c_5^4f_{xxyy}+4h^4c_5^4f_{xyyy}+h^4c_5^4f_{yyyy}]+o(h^5)$$

$$k_5=f+hc_5f_x+h[(c_5f+hc_2a_{52}F+\frac{1}{2}h^2c_2^2a_{52}G+\frac{1}{6}h^3c_2^3a_{52}H+hc_3a_{53}F+\frac{1}{2}h^2c_3^2a_{53}G$$

$$+h^2c_2a_{32}a_{53}Ff_y+\frac{1}{2}h^3c_2^2a_{32}a_{53}Gf_y+h^3c_2c_3a_{32}a_{53}Ff_{xy}++h^3c_2c_3a_{32}a_{53}Fff_{yy}$$

$$+\frac{1}{6}h^3c_3^3a_{53}H+hc_4a_{54}F+\frac{1}{2}h^2c_4^2a_{54}G+h^2c_2a_{42}a_{54}Ff_y+h^2c_3a_{43}a_{54}Ff_y+\frac{1}{6}h^3c_4^3a_{54}H$$

$$+\frac{1}{2}h^3c_2^2a_{42}a_{54}Gf_y+h^3c_2a_{32}a_{43}a_{54}Ff_y^2+\frac{1}{2}h^3c_3^2a_{43}a_{54}Gf_y+h^3c_2c_4a_{42}a_{54}Ff_{xy}$$

$$+h^3c_3c_4a_{43}a_{54}Ff_{xy}+h^3c_2c_4a_{42}a_{54}Fff_{yy}+h^3c_3c_4a_{43}a_{54}Fff_{yy})]f_y+\frac{1}{2}[(hc_5)^2f_{xx}+2(hc_5)h(c_5f$$

$$+c_2ha_{52}F+\frac{1}{2}c_2^2h^2a_{52}G+hc_3a_{53}F+\frac{1}{2}h^2c_3^2a_{53}G+h^2c_2a_{32}a_{53}Ff_y+hc_4a_{54}F+\frac{1}{2}h^2c_4^2a_{54}G$$

$$+h^2c_2a_{42}a_{54}Ff_y+h^2c_3a_{43}a_{54}Ff_y)f_{xy}+h^2(c_5+c_2a_{52}hF+\frac{1}{2}(c_2h)^2a_{52}G+hc_3a_{53}F$$

$$+\frac{1}{2}h^2c_3^2Ga_{53}+h^2c_2a_{32}a_{53}Ff_y+hc_4a_{54}F+\frac{1}{2}h^2c_4^2a_{54}G+h^2c_2a_{42}a_{54}Ff_y+h^2c_3a_{43}a_{54}Ff_y)^2f_{yy}]$$

$$+\frac{1}{6}[(hc_5)^3f_{xxx}+3(hc_5)^2h(c_5+c_2a_{52}hF+hc_3a_{53}F+hc_4a_{54}F)f_{xxy}+3(hc_5)h^2(c_5+c_2a_{52}hF$$

$$+hc_3a_{53}F+hc_4a_{54}F)^2f_{xyy}+h^3(c_5+c_2a_{52}hF+hc_3a_{53}F+hc_4a_{54}F)^3f_{yyy}]$$

$$+\frac{1}{24}(hc_5)^4[f_{xxxx}+4f_{xxxy}+4f_{xxyy}+4f_{xyyy}+f_{yyyy}]+o(h^5)$$

$$k_5 = f + hc_5(f_x + ff_y) + h^2c_2a_{52}Ff_y + \frac{1}{2}h^3c_2^2a_{52}Gf_y + \frac{1}{6}h^4c_2^3a_{52}Hf_y + h^2c_3a_{53}Ff_y$$

$$+ h^3c_2a_{32}a_{53}Ff_y^2 + \frac{1}{2}h^3c_3^2a_{53}Gf_y + \frac{1}{2}h^4c_2^2a_{32}a_{53}Gf_y^2 + h^4c_2c_3a_{32}a_{53}Ff_{xy}f_y$$

$$+ h^4c_2c_3a_{32}a_{53}Fff_yf_{yy} + \frac{1}{6}h^4c_3^3a_{53}Hf_y + h^2c_4a_{54}Ff_y + \frac{1}{2}h^3c_4^2a_{54}Gf_y$$

$$+ + h^3c_2a_{42}a_{54}Ff_y^2 + + h^3c_3a_{43}a_{54}Ff_y^2 + \frac{1}{6}h^4c_4^3a_{54}Hf_y + \frac{1}{2}h^4c_2^2a_{42}a_{54}Gf_y^2$$

$$+ \frac{1}{2}h^4c_3^2a_{43}a_{54}Gf_y^2 + h^4c_2a_{32}a_{43}a_{54}Ff_y^3 + h^4c_2c_4a_{42}a_{54}Ff_{xy}f_y + h^4c_3c_4a_{43}a_{54}Ff_{xy}f_y$$

$$+ h^4c_2c_4a_{42}a_{54}Fff_yf_{yy} + h^4c_3c_4a_{43}a_{54}Fff_yf_{yy} + \frac{1}{2}(hc_5)^2f_{xx} + (hc_5)^2ff_{xy} + h^3c_2c_5a_{52}Ff_{xy}$$

$$+ \frac{1}{2}h^4c_2^2c_5a_{52}Gf_{xy} + h^3c_3c_5a_{53}Ff_{xy} + h^4c_2c_5a_{32}a_{53}Ff_{xy}f_y + \frac{1}{2}h^4c_3^2c_5a_{53}Gf_{xy}$$

$$+ h^3c_4c_5a_{54}Ff_{xy} + \frac{1}{2}h^4c_4^2c_5a_{54}Gf_{xy} + h^4c_2c_5a_{42}a_{54}Ff_{xy}f_y + h^4c_3c_5a_{43}a_{54}Ff_{xy}f_y$$

$$+ \frac{1}{2}(hc_5)^2f_{yy}^2 + h^3c_2c_5a_{52}Fff_{yy} + \frac{1}{2}h^4c_2^2c_5a_{52}Gff_{yy} + h^3c_3c_5a_{53}Fff_{yy} + h^4c_2c_5a_{32}a_{53}Fff_yf_{yy}$$

$$+ h^3c_4c_5a_{54}Fff_{yy} + h^4c_2c_5a_{42}a_{54}Fff_yf_{yy} + h^4c_3c_5a_{43}a_{54}Fff_yf_{yy} + \frac{1}{2}h^4c_3^2c_5a_{53}Gff_{yy}$$

$$+ \frac{1}{2}h^4c_4^2c_5a_{54}Gff_{yy} + \frac{1}{6}(hc_5)^3f_{xxx} + \frac{1}{2}(hc_5)^3ff_{xxy} + \frac{1}{2}h^4c_2c_5^2a_{52}Ff_{xxy} + \frac{1}{2}h^4c_3c_5^2a_{53}Ff_{xxy}$$

$$+ \frac{1}{2}h^4c_4c_5^2a_{54}Ff_{xxy} + \frac{1}{2}(hc_5)^3f^2f_{xyy} + h^4c_2c_5^2a_{52}Fff_{xyy} + h^4c_3c_5^2a_{53}Fff_{xyy} + h^4c_4c_5^2a_{54}Fff_{xyy}$$

$$+ \frac{1}{6}(hc_5)^3f^3f_{yyy} + \frac{1}{2}h^4c_2c_5^2a_{52}Ff^2f_{yyy} + \frac{1}{2}h^4c_3c_5^2a_{53}Ff^2f_{yyy} + \frac{1}{2}h^4c_4c_5^2a_{54}Ff^2f_{yyy}$$

$$+ \frac{1}{24}(hc_5)^4[f_{xxxx} + 4f_{xxxy} + 4f^2f_{xxyy} + 4f^3f_{xyyy} + f^4f_{yyyy}] + o(h^5)$$

$$k_5 = f + hc_5F + h^2(c_2a_{52}Ff_y + c_3a_{53}Ff_y + c_4a_{54}Ff_y) + h^3(\frac{1}{2}c_2^2a_{52}Gf_y + \frac{1}{2}c_3^2a_{53}Gf_y + \frac{1}{2}c_4^2a_{54}Gf_y$$
$$+ c_2a_{32}a_{53}Ff_y^2 + c_2a_{42}a_{54}Ff_y^2 + c_3a_{43}a_{54}Ff_y^2 + c_2c_5a_{52}Ff_{xy} + c_3c_5a_{53}Ff_{xy} + h^3c_4c_5a_{54}Ff_{xy}$$
$$+ c_2c_5a_{52}Fff_{yy} + c_3c_5a_{53}Fff_{yy} + c_4c_5a_{54}Fff_{yy}) + h^4(\frac{1}{6}c_2^3a_{52}Hf_y + \frac{1}{6}c_3^3a_{53}Hf_y$$
$$+ \frac{1}{6}c_4^3a_{54}Hf_y + \frac{1}{2}c_2^2a_{32}a_{53}Gf_y^2 + \frac{1}{2}c_2^2a_{42}a_{54}Gf_y^2 + \frac{1}{2}c_3^2a_{43}a_{54}Gf_y^2 + c_2c_3a_{32}a_{53}Ff_{xy}f_y$$
$$+ c_2c_4a_{42}a_{54}Ff_{xy}f_y + c_2c_5a_{32}a_{53}Ff_{xy}f_y + c_3c_4a_{43}a_{54}Ff_{xy}f_y + c_2c_5a_{42}a_{54}Ff_{xy}f_y$$
$$+ c_3c_5a_{43}a_{54}Ff_{xy}f_y + c_2c_3a_{32}a_{53}Fff_yf_{yy} + c_2c_4a_{42}a_{54}Fff_yf_{yy} + c_3c_4a_{43}a_{54}Fff_yf_{yy}$$
$$+ c_2c_5a_{32}a_{53}Fff_yf_{yy} + c_2a_{32}a_{43}a_{54}Ff_y^3 + c_2c_5a_{42}a_{54}Fff_yf_{yy} + c_3c_5a_{43}a_{54}Fff_yf_{yy}$$
$$+ \frac{1}{2}c_2^2c_5a_{52}Gf_{xy} + \frac{1}{2}c_3^2c_5a_{53}Gf_{xy} + \frac{1}{2}c_4^2c_5a_{54}Gf_{xy} + \frac{1}{2}c_2^2c_5a_{52}Gff_{yy} + \frac{1}{2}c_3^2c_5a_{53}Gff_{yy}$$
$$+ \frac{1}{2}c_4^2c_5a_{54}Gff_{yy} + \frac{1}{2}c_2c_5^2a_{52}Ff_{xxy} + \frac{1}{2}c_3c_5^2a_{53}Ff_{xxy} + \frac{1}{2}c_4c_5^2a_{54}Ff_{xxy} + c_2c_5^2a_{52}Fff_{xyy}$$
$$+ c_3c_5^2a_{53}Fff_{xyy} + c_4c_5^2a_{54}Fff_{xyy} + \frac{1}{2}c_2c_5^2a_{52}Ff^2f_{yyy} + \frac{1}{2}c_3c_5^2a_{53}Ff^2f_{yyy} + \frac{1}{2}c_4c_5^2a_{54}Ff^2f_{yyy}$$
$$c_2a_{32}a_{43}a_{54}Ff_y^3) + \frac{1}{2}(hc_5)^2(f_{xx} + 2ff_{xy} + f_{yy}^2) + \frac{1}{6}(hc_5)^3(f_{xxx} + 3ff_{xxy} + 3f_{xyy} + f^3f_{yyy})$$
$$+ \frac{1}{24}(hc_5)^4[f_{xxxx} + 4f_{xxxy} + 4f^2f_{xxyy} + 4f^3f_{xyyy} + f^4f_{yyyy}] + o(h^5)$$

From Equation (8) and further grouping like powers of h together we get

$$k_5 = f + hc_5F + h^2[\frac{1}{2}c_5^2G + (c_2a_{52} + c_3a_{53} + c_4a_{54})Ff_y] + h^3[\frac{1}{6}c_5^3H + (\frac{1}{2}c_2^2a_{52}$$
$$+ \frac{1}{2}c_3^2a_{53} + \frac{1}{2}c_4^2a_{54})Gf_y + (c_2a_{32}a_{53} + c_2a_{42}a_{54} + c_3a_{43}a_{54})Ff_y^2$$
$$+ (c_2c_5a_{52} + c_3c_5a_{53} + c_4c_5a_{54})Ff_{xy} + (c_2c_5a_{52} + c_3c_5a_{53} + c_4c_5a_{54})Fff_{yy}]$$
$$+ h^4[\frac{1}{24}c_5^4I + \frac{1}{6}Hf_y(c_2^3a_{52} + c_3^3a_{53} + c_4^3a_{54}) + \frac{1}{2}Gf_y^2(c_2^2a_{32}a_{53} + c_2^2a_{42}a_{54}$$
$$+ c_3^2a_{43}a_{54}) + (c_2c_3a_{32}a_{53} + c_2c_4a_{42}a_{54} + c_2c_5a_{32}a_{53} + c_3c_4a_{43}a_{54}$$
$$+ c_2c_5a_{42}a_{54} + c_3c_5a_{43}a_{54})Ff_{xy}f_y + (c_2c_3a_{32}a_{53} + c_2c_4a_{42}a_{54} + c_3c_4a_{43}a_{54}$$

$$+c_2c_5a_{32}a_{53}+c_2c_5a_{42}a_{54}+c_3c_5a_{43}a_{54})Fff_yf_{yy}+\frac{1}{2}Gf_{xy}(c_2^2c_5a_{52}+c_3^2c_5a_{53}$$

$$+c_4^2c_5a_{54})+\frac{1}{2}Gff_{yy}(c_2^2c_5a_{52}+c_3^2c_5a_{53}+c_4^2c_5a_{54})$$

$$+\frac{1}{2}Ff_{xxy}(c_2c_5^2a_{52}+c_3c_5^2a_{53}+c_4c_5^2a_{54})+(c_2c_5^2a_{52}+c_3c_5^2a_{53}+c_4c_5^2a_{54})Fff_{xyy}$$

$$+\frac{1}{2}Ff^2f_{yyy}(c_2c_5^2a_{52}+c_3c_5^2a_{53}+c_4c_5^2a_{54})+c_2a_{32}a_{43}a_{54}Ff_y^3]+o(h^5) \quad (12)$$

$$k_6=f(x_n+hc_6f_x,y_n+h((c_6-(a_{62}+a_{63}+a_{64}+a_{65}))k_1+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5))$$

By expanding k_6 as before and substituting f for k_1 we have

$$k_6=f+hc_6f_x+h((c_6-(a_{62}+a_{63}+a_{64}+a_{65}))f+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)f_y$$

$$+\frac{1}{2}(hc_6)^2f_{xx}+h^2c_6((c_6-(a_{62}+a_{63}+a_{64}+a_{65}))f+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)f_{xy}$$

$$+\frac{1}{2}h^2((c_6-(a_{62}+a_{63}+a_{64}+a_{65}))f+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)^2f_{yy}+\frac{1}{6}(hc_6)^3f_{xxx}$$

$$+\frac{1}{2}h^3c_6^2((c_6-(a_{62}+a_{63}+a_{64}+a_{65}))f+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)f_{xxy}$$

$$+\frac{1}{2}h^3c_6((c_6-(a_{62}+a_{63}+a_{64}+a_{65}))f+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)^2f_{xyy}$$

$$+\frac{1}{6}h^3((c_6-(a_{62}+a_{63}+a_{64}+a_{65}))f+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)^3f_{yyy}+\frac{1}{24}(hc_6)^4f_{xxxx}$$

$$+\frac{1}{6}h^4c_6^3((c_6-(a_{62}+a_{63}+a_{64}+a_{65}))f+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)f_{xxxy}$$

$$+\frac{1}{6}h^4c_6^2((c_6-(a_{62}+a_{63}+a_{64}+a_{65}))f+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)^2f_{xxyy}$$

$$+\frac{1}{6}h^4c_6((c_6-(a_{62}+a_{63}+a_{64}+a_{65}))f+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)^3f_{xyyy}$$

$$+\frac{1}{24}h^4((c_6-(a_{62}+a_{63}+a_{64}+a_{65}))f+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)^4f_{yyyy}+o(h^5)$$

with all the terms evaluated at (x_n, y_n)

Substituting for k_1, k_2, k_3, k_4, and k_5 into k_6:

$$k_6 = f + hc_6 f_x + h[(c_6 - (a_{62} + a_{63} + a_{64} + a_{65}))f + a_{62}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G + \frac{1}{6}(c_2 h)^3 H)$$

$$+ a_{63}(f + hc_3 F + \frac{1}{2}h^2 c_3^2 G + h^2 c_2 a_{32} Ff_y + \frac{1}{2}h^3 c_2^2 a_{32} Gf_y + h^3 c_2 c_3 a_{32} Ff_{xy} + + h^3 c_2 c_3 a_{32} Fff_{yy}$$

$$+ \frac{1}{6}h^3 c_3^3 H) + a_{64}(f + hc_4 F + \frac{1}{2}h^2 c_4^2 G + h^2 c_2 a_{42} Ff_y + h^2 c_3 a_{43} Ff_y + \frac{1}{6}h^3 c_4^3 H + \frac{1}{2}h^3 c_2^2 a_{42} Gf_y$$

$$+ h^3 c_2 a_{32} a_{43} Ff_y^2 + \frac{1}{2}h^3 c_3^2 a_{43} Gf_y + h^3 c_2 c_4 a_{42} Ff_{xy} + h^3 c_3 c_4 a_{43} Ff_{xy} + h^3 c_2 c_4 a_{42} Fff_{yy}$$

$$+ h^3 c_3 c_4 a_{43} Fff_{yy}) + a_{65}(f + hc_5 F + h^2 \frac{1}{2}h^2 c_5^2 G + h^2 c_2 a_{52} Ff_y + h^2 c_3 a_{53} Ff_y + h^2 c_4 a_{54} Ff_y$$

$$+ \frac{1}{6}h^3 c_5^3 H + \frac{1}{2}h^3 c_2^2 a_{52} Gf_y + \frac{1}{2}h^3 c_3^2 a_{53} Gf_y + \frac{1}{2}h^3 c_4^2 a_{54} Gf_y + h^3 c_2 a_{32} a_{53} Ff_y^2 + h^3 c_2 a_{42} a_{54} Ff_y^2$$

$$+ h^3 c_3 a_{43} a_{54} Ff_y^2 + h^3 c_2 c_5 a_{52} Ff_{xy} + h^3 c_3 c_5 a_{53} Ff_{xy} + h^3 c_4 c_5 a_{54} Ff_{xy} + h^3 c_2 c_5 a_{52} Fff_{yy}$$

$$+ h^3 c_3 c_5 a_{53} Fff_{yy} + h^3 c_4 c_5 a_{54} Fff_{yy})]f_y + \frac{1}{2}(hc_6)^2 f_{xx} + h^2 c_6 [(c_6 - (a_{62} + a_{63} + a_{64} + a_{65}))f$$

$$+ a_{62}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G) + a_{63}(f + hc_3 F + \frac{1}{2}h^2 c_3^2 G + h^2 c_2 a_{32} Ff_y)$$

$$+ a_{64}(f + hc_4 F + \frac{1}{2}h^2 c_4^2 G + h^2 c_2 a_{42} Ff_y + h^2 c_3 a_{43} Ff_y) + a_{65}(f + hc_5 F + h^2 \frac{1}{2}h^2 c_5^2 G + h^2 c_2 a_{52} Ff_y$$

$$+ h^2 c_3 a_{53} Ff_y + h^2 c_4 a_{54} Ff_y)]f_{xy} + \frac{1}{2}h^2 [(c_6 - (a_{62} + a_{63} + a_{64} + a_{65}))f$$

$$+ a_{62}(f + c_2 hF + \frac{1}{2}(c_2 h)^2 G) + a_{63}(f + hc_3 F + \frac{1}{2}h^2 c_3^2 G + h^2 c_2 a_{32} Ff_y) + a_{64}(f + hc_4 F +$$

$$\frac{1}{2}h^2 c_4^2 G + h^2 c_2 a_{42} Ff_y + h^2 c_3 a_{43} Ff_y) + a_{65}(f + hc_5 F + h^2 \frac{1}{2}h^2 c_5^2 G + h^2 c_2 a_{52} Ff_y + h^2 c_3 a_{53} Ff_y$$

$$+ h^2 c_4 a_{54} Ff_y)]^2 f_{yy} + \frac{1}{6}(hc_6)^3 f_{xxx} + \frac{1}{2}h^3 c_6^2 [(c_6 - (a_{62} + a_{63} + a_{64} + a_{65}))f$$

$$+ a_{62}(f + c_2 hF) + a_{63}(f + hc_3 F) + a_{64}(f + hc_4 F) + a_{65}(f + hc_5 F)]f_{xxy}$$

$$+ \frac{1}{2}h^3 c_6 [(c_6 - (a_{62} + a_{63} + a_{64} + a_{65}))f + a_{62}(f + c_2 hF) + a_{63}(f + hc_3 F) + a_{64}(f + hc_4 F)$$

$$+ a_{65}(f + hc_5 F)]^2 f_{xyy} + \frac{1}{6}h^3 [(c_6 - (a_{62} + a_{63} + a_{64} + a_{65}))f + a_{62}(f + c_2 hF) + a_{63}(f + hc_3 F)$$

$$+ a_{64}(f + hc_4 F) + a_{65}(f + hc_5 F)]^3 f_{yyy} + \frac{1}{24}(hc_6)^4 f_{xxxx} + \frac{1}{6}h^4 c_6^3 [(c_6 - (a_{62} + a_{63} + a_{64} + a_{65}))f + a_{62}$$

$$+ a_{63} f + a_{64} f + a_{65} f]f_{xxxy} + \frac{1}{6}h^4 c_6^2 ((c_6 - (a_{62} + a_{63} + a_{64} + a_{65}))f + a_{62} f + a_{63} f + a_{64} f + a_{65} f)^2 f_{xxy}$$

$$+ \frac{1}{6}h^4 c_6 ((c_6 - (a_{62} + a_{63} + a_{64} + a_{65}))f + a_{62} f + a_{63} f + a_{64} f + a_{65} f)^3 f_{xyyy}$$

$$+ \frac{1}{24}h^4 ((c_6 - (a_{62} + a_{63} + a_{64} + a_{65}))f + a_{62} f + a_{63} f + a_{64} f + a_{65} f)^4 f_{yyyy} + o(h^5)$$

$$k_6 = f + hc_6 f_x + h[c_6 - a_{62} f - a_{63} f - a_{64} f - a_{65} f + a_{62} f + c_2 a_{62} hF + \frac{1}{2} c_2^2 h^2 a_{62} G + \frac{1}{6} c_2^3 h^3 a_{62} H$$

$$+ a_{63} f + hc_3 a_{63} F + \frac{1}{2} h^2 c_3^2 a_{63} G + h^2 c_2 a_{32} a_{63} Ff_y + \frac{1}{2} h^3 c_2^2 a_{63} a_{32} Gf_y + h^3 c_2 c_3 a_{63} a_{32} Ff_{xy}$$

$$+ h^3 c_2 c_3 a_{32} a_{63} Fff_{yy} + \frac{1}{6} h^3 c_3^3 a_{63} H + a_{64} f + hc_4 a_{64} F + \frac{1}{2} h^2 c_4^2 a_{64} G + h^2 c_2 a_{42} a_{64} Ff_y + h^2 c_3 a_{43} a_{64} Ff_y$$

$$+ \frac{1}{6} h^3 c_4^3 a_{64} H + \frac{1}{2} h^3 c_2^2 a_{42} a_{64} Gf_y + h^3 c_2 a_{32} a_{43} a_{64} Ff_y^2 + \frac{1}{2} h^3 c_3^2 a_{43} a_{64} Gf_y + h^3 c_2 c_4 a_{42} a_{64} Ff_{xy}$$

$$+ h^3 c_3 c_4 a_{43} a_{64} Ff_{xy} + h^3 c_2 c_4 a_{42} a_{64} Fff_{yy} + h^3 c_3 c_4 a_{43} a_{64} Fff_{yy} + a_{65} f + hc_5 a_{65} F + \frac{1}{2} h^2 c_5^2 a_{65} G$$

$$+ h^2 c_2 a_{52} a_{65} Ff_y + h^2 c_3 a_{53} a_{65} Ff_y + h^2 c_4 a_{54} a_{65} Ff_y + \frac{1}{6} h^3 c_5^3 a_{65} H + \frac{1}{2} h^3 c_2^2 a_{52} a_{65} Gf_y$$

$$+ \frac{1}{2} h^3 c_3^2 a_{53} a_{65} Gf_y + \frac{1}{2} h^3 c_4^2 a_{54} a_{65} Gf_y + h^3 c_2 a_{32} a_{53} a_{65} Ff_y^2 + h^3 c_2 a_{42} a_{54} a_{65} Ff_y^2$$

$$+ h^3 c_3 a_{43} a_{54} a_{65} Ff_y^2 + h^3 c_2 c_5 a_{52} a_{65} Ff_{xy} + h^3 c_3 c_5 a_{53} a_{65} Ff_{xy} + h^3 c_4 c_5 a_{54} a_{65} Ff_{xy}$$

$$+ h^3 c_2 c_5 a_{52} a_{65} Fff_{yy} + h^3 c_3 c_5 a_{53} a_{65} Fff_{yy} + h^3 c_4 c_5 a_{54} a_{65} Fff_{yy}] f_y + \frac{1}{2} (hc_6)^2 f_{xx} + h^2 c_6 [c_6 - a_{62} f$$

$$- a_{63} f - a_{64} f - a_{65} f + a_{62} f + hc_2 a_{62} F + \frac{1}{2} c_2^2 h^2 a_{62} G + a_{63} f + hc_3 a_{63} F + \frac{1}{2} h^2 c_3^2 a_{63} G$$

$$+ h^2 c_2 a_{32} a_{63} Ff_y + a_{64} f + hc_4 a_{64} F + \frac{1}{2} h^2 c_4^2 a_{64} G + h^2 c_2 a_{42} a_{64} Ff_y + h^2 c_3 a_{43} a_{64} Ff_y + a_{65} f + hc_5 a_{65} F$$

$$+ h^2 \frac{1}{2} h^2 c_5^2 a_{65} G + h^2 c_2 a_{52} a_{65} Ff_y + h^2 c_3 a_{53} a_{65} Ff_y + h^2 c_4 a_{54} a_{65} Ff_y] f_{xy} + \frac{1}{2} h^2 [c_6 - a_{62} f - a_{63} f$$

$$- a_{64} f - a_{65} f + a_{62} f + hc_2 a_{62} F + \frac{1}{2} c_2^2 h^2 a_{62} G + a_{63} f + hc_3 a_{63} F + \frac{1}{2} h^2 c_3^2 a_{63} G + h^2 c_2 a_{32} a_{63} Ff_y$$

$$+ a_{64} f + hc_4 a_{64} F + \frac{1}{2} h^2 c_4^2 a_{64} G + h^2 c_2 a_{42} a_{64} Ff_y + h^2 c_3 a_{43} a_{64} Ff_y + a_{65} f + hc_5 a_{65} F$$

$$+ h^2 \frac{1}{2} h^2 c_5^2 a_{65} G + h^2 c_2 a_{52} a_{65} Ff_y + h^2 c_3 a_{53} a_{65} Ff_y + h^2 c_4 a_{54} a_{65} Ff_y]^2 f_{yy} + \frac{1}{6} (hc_6)^3 f_{xxx}$$

$$+ \frac{1}{2} h^3 c_6^2 [c_6 - a_{62} f - a_{63} f - a_{64} f - a_{65} f + a_{62} f + c_2 a_{62} hF + a_{63} f + hc_3 a_{63} F + a_{64} f$$

$$+ hc_4 a_{64} F + a_{65} f + hc_5 a_{65} F] f_{xxy} + \frac{1}{2} h^3 c_6 [c_6 - a_{62} f - a_{63} f - a_{64} f - a_{65} f + a_{62} f + hc_2 a_{62} F$$

$$+ a_{63} f + hc_3 a_{63} F + a_{64} f + hc_4 a_{64} F + a_{65} f + hc_5 a_{65} F]^2 f_{xyy} + \frac{1}{6} h^3 [(c_6 - a_{62} f - a_{63} f$$

$$- a_{64} f - a_{65} f + a_{62} f + c_2 a_{62} hF + a_{63} f + hc_3 a_{63} F + a_{64} f + hc_4 a_{64} F + a_{65} f + hc_5 a_{65} F]^3 f_{yyy}$$

$$+ \frac{1}{24} (hc_6)^4 f_{xxxx} + \frac{1}{6} h^4 c_6^3 [c_6 - a_{62} f - a_{63} f - a_{64} f - a_{65} f + a_{62} f + a_{63} f + a_{64} f + a_{65} f] f_{xxxy}$$

$$+ \frac{1}{6} h^4 c_6^2 [c_6 - a_{62} f - a_{63} f - a_{64} f - a_{65} f + a_{62} f + a_{63} f + a_{64} f + a_{65} f]^2 f_{xxyy}$$

$$+ \frac{1}{6} h^4 c_6 [c_6 - a_{62} f - a_{63} f - a_{64} f - a_{65} f + a_{62} f + a_{63} f + a_{64} f + a_{65} f]^3 f_{xyyy}$$

$$
\begin{aligned}
k_6 = {} & f + hc_6 f_x + h[c_6 f + c_2 a_{62} hF + \frac{1}{2} c_2^2 h^2 a_{62} G + \frac{1}{6} c_2^3 h^3 a_{62} H + hc_3 a_{63} F + \frac{1}{2} h^2 c_3^2 a_{63} G \\
& + h^2 c_2 a_{32} a_{63} F f_y + \frac{1}{2} h^3 c_2^2 a_{63} a_{32} G f_y + h^3 c_2 c_3 a_{63} a_{32} F f_{xy} + h^3 c_2 c_3 a_{32} a_{63} F f f_{yy} + \frac{1}{6} h^3 c_3^3 a_{63} H \\
& + hc_4 a_{64} F + \frac{1}{2} h^2 c_4^2 a_{64} G + h^2 c_2 a_{42} a_{64} F f_y + h^2 c_3 a_{43} a_{64} F f_y + \frac{1}{6} h^3 c_4^3 a_{64} H + \frac{1}{2} h^3 c_2^2 a_{42} a_{64} G f_y \\
& + h^3 c_2 a_{32} a_{43} a_{64} F f_y^2 + \frac{1}{2} h^3 c_3^2 a_{43} a_{64} G f_y + h^3 c_2 c_4 a_{42} a_{64} F f_{xy} + h^3 c_3 c_4 a_{43} a_{64} F f_{xy} + h^3 c_2 c_4 a_{42} a_{64} F f f_{yy} \\
& + h^3 c_3 c_4 a_{43} a_{64} F f f_{yy} + hc_5 a_{65} F + \frac{1}{2} h^2 c_5^2 a_{65} G + h^2 c_2 a_{52} a_{65} F f_y + h^2 c_3 a_{53} a_{65} F f_y + h^2 c_4 a_{54} a_{65} F f_y \\
& + \frac{1}{6} h^3 c_5^3 a_{65} H + \frac{1}{2} h^3 c_2^2 a_{52} a_{65} G f_y + \frac{1}{2} h^3 c_3^2 a_{53} a_{65} G f_y + \frac{1}{2} h^3 c_4^2 a_{54} a_{65} G f_y + h^3 c_2 a_{32} a_{53} a_{65} F f_y^2 \\
& + h^3 c_2 a_{42} a_{54} a_{65} F f_y^2 + h^3 c_3 a_{43} a_{54} a_{65} F f_y^2 + h^3 c_2 c_5 a_{52} a_{65} F f_{xy} + h^3 c_3 c_5 a_{53} a_{65} F f_{xy} \\
& + h^3 c_4 c_5 a_{54} a_{65} F f_{xy} + h^3 c_2 c_5 a_{52} a_{65} F f f_{yy} + h^3 c_3 c_5 a_{53} a_{65} F f f_{yy} + h^3 c_4 c_5 a_{54} a_{65} F f f_{yy}] f_y \\
& + \frac{1}{2} (hc_6)^2 f_{xx} + h^2 c_6 [c_6 f + hc_2 a_{62} F + \frac{1}{2} c_2^2 h^2 a_{62} G + hc_3 a_{63} F + \frac{1}{2} h^2 c_3^2 a_{63} G + h^2 c_2 a_{32} a_{63} F f_y \\
& + hc_4 a_{64} F + \frac{1}{2} h^2 c_4^2 a_{64} G + h^2 c_2 a_{42} a_{64} F f_y + h^2 c_3 a_{43} a_{64} F f_y + hc_5 a_{65} F + h^2 \frac{1}{2} h^2 c_5^2 a_{65} G \\
& + h^2 c_2 a_{52} a_{65} F f_y + h^2 c_3 a_{53} a_{65} F f_y + h^2 c_4 a_{54} a_{65} F f_y] f_{xy} + \frac{1}{2} h^2 [c_6 f + hc_2 a_{62} F + \frac{1}{2} c_2^2 h^2 a_{62} G + hc_3 a_{63} F \\
& + \frac{1}{2} h^2 c_3^2 a_{63} G + h^2 c_2 a_{32} a_{63} F f_y + hc_4 a_{64} F + \frac{1}{2} h^2 c_4^2 a_{64} G + h^2 c_2 a_{42} a_{64} F f_y + h^2 c_3 a_{43} a_{64} F f_y + hc_5 a_{65} F \\
& + \frac{1}{2} h^2 c_5^2 a_{65} G + h^2 c_2 a_{52} a_{65} F f_Y + h^2 c_3 a_{53} a_{65} F f_y + h^2 c_4 a_{54} a_{65} F f_y]^2 f_{yy} + \frac{1}{6} (hc_6)^3 f_{xxx} \\
& + \frac{1}{2} h^3 c_6^2 [c_6 f + c_2 a_{62} hF + hc_3 a_{63} F + hc_4 a_{64} F + hc_5 a_{65} F] f_{xxy} + \frac{1}{2} h^3 c_6 [c_6 f + hc_2 a_{62} F + hc_3 a_{63} F \\
& + hc_4 a_{64} F + hc_5 a_{65} F]^2 f_{xyy} + \frac{1}{6} h^3 [(c_6 f + c_2 a_{62} hF + hc_3 a_{63} F + hc_4 a_{64} F + hc_5 a_{65} F]^3 f_{yyy} \\
& + \frac{1}{24} (hc_6)^4 f_{xxxx} + \frac{1}{6} h^4 c_6^4 f f_{xxxy} + \frac{1}{6} h^4 c_6^4 f^2 f_{xxyy} + \frac{1}{6} h^4 c_6^4 f^3 f_{xyyy} + \frac{1}{24} h^4 c_6^4 f^4 f_{yyyy} + o(h^5)
\end{aligned}
$$

$$k_6 = f + hc_6(f_x + ff_y) + h^2[(c_2a_{62} + c_3a_{63} + c_4a_{64} + c_5a_{65})Ff_y] + h^3[\frac{1}{2}(c_2^2a_{62} + c_3^2a_{63} + c_4^2a_{64} + c_5^2a_{65})Gf_y$$

$$+ (c_2a_{32}a_{63} + c_2a_{42}a_{64} + c_3a_{43}a_{64} + c_2a_{52}a_{65} + c_3a_{53}a_{65} + c_4a_{54}a_{65})Ff_Y^2$$

$$+ (c_2c_6a_{62} + c_3c_6a_{63} + c_4c_6a_{64} + c_5c_6a_{65})Ff_{xy} + (c_2c_6a_{62} + c_3c_6a_{63} + c_4c_6a_{64} + c_5c_6a_{65})Fff_{yy}]$$

$$+ h^4[\frac{1}{2}(c_2^2a_{63}a_{32} + c_2^2a_{42}a_{64} + c_3^2a_{43}a_{64} + c_2^2a_{52}a_{65} + c_3^2a_{53}a_{65} + c_4^2a_{54}a_{65})Gf_Y^2 + \frac{1}{6}(c_2^3a_{62} + c_3^3a_{63}$$

$$+ c_4^3a_{64} + c_5^3a_{65})Hf_y + (c_2c_3a_{32}a_{63} + c_2c_4a_{42}a_{64} + c_3c_4a_{43}a_{64} + c_2c_5a_{52}a_{65} + c_3c_5a_{53}a_{65} + c_4c_5a_{54}a_{65}$$

$$+ c_2c_6a_{32}a_{63} + c_2c_6a_{42}a_{64} + c_3c_6a_{43}a_{64} + c_2c_6a_{52}a_{65} + c_3c_6a_{53}a_{65} + c_4c_6a_{54}a_{65})Ff_{xy}f_y$$

$$+ (c_2c_3a_{32}a_{63} + c_2c_4a_{42}a_{64} + c_3c_4a_{43}a_{64} + c_2c_5a_{52}a_{65} + c_3c_5a_{53}a_{65} + c_4c_5a_{54}a_{65}$$

$$+ c_2c_6a_{32}a_{63} + c_2c_6a_{42}a_{64} + c_3c_6a_{43}a_{64} + c_2c_6a_{52}a_{65} + c_3c_6a_{53}a_{65} + c_4c_6a_{54}a_{65})Fff_yf_{yy}$$

$$+ (c_2a_{32}a_{43}a_{64} + c_2a_{32}a_{53}a_{65} + c_2a_{42}a_{54}a_{65} + c_3a_{43}a_{54}a_{65})Ff_y^3 + \frac{1}{2}(c_2^2c_6a_{62} + c_3^2c_6a_{63} + c_4^2c_6a_{64} + c_5^2c_6a_{65})Gf_{xy}$$

$$+ \frac{1}{2}(c_2^2c_6a_{62} + c_3^2c_6a_{63} + c_4^2c_6a_{64} + c_5^2c_6a_{65})Gff_{yy} + \frac{1}{2}(c_2c_6^2a_{62} + c_3c_6^2a_{63} + c_4c_6^2a_{64} + c_5c_6^2a_{65})Ff_{xxy}$$

$$+ (c_2c_6^2a_{62} + c_3c_6^2a_{63} + c_4c_6^2a_{64} \quad h^4c_5c_6^2a_{65})Fff_{xyy} + \frac{1}{2}(c_2c_6^2a_{62} + c_3c_6^2a_{63} + c_4c_6^2a_{64} + c_5c_6^2a_{65})Ff^2f_{yyy}]$$

$$+ \frac{1}{2}(hc_6)^2(f_{xx} + 2ff_{xy} + f^2f_{yy}) + \frac{1}{6}(hc_6)^3(f_{xxx} + 3ff_{xxy} + 3f^2f_{xyy} + f^3f_{yyy})$$

$$+ \frac{1}{24}(hc_6)^4(f_{xxxx} + 4c_6^4ff_{xxxy} + 4c_6^4f^2f_{xxyy} + 4f^3f_{xyyy} + f^4f_{yyyy}) + o(h^5)$$

$$k_6 = f + hc_6(f_x + ff_y) + h^2[(c_2a_{62} + c_3a_{63} + c_4a_{64} + c_5a_{65})Ff_y] + h^3[\frac{1}{2}(c_2^2a_{62} + c_3^2a_{63} + c_4^2a_{64} + c_5^2a_{65})Gf_y$$

$$+ (c_2a_{32}a_{63} + c_2a_{42}a_{64} + c_3a_{43}a_{64} + c_2a_{52}a_{65} + c_3a_{53}a_{65} + c_4a_{54}a_{65})Ff_Y^2$$

$$+ (c_2c_6a_{62} + c_3c_6a_{63} + c_4c_6a_{64} + c_5c_6a_{65})Ff_{xy} + (c_2c_6a_{62} + c_3c_6a_{63} + c_4c_6a_{64} + c_5c_6a_{65})Fff_{yy}]$$

$$+ h^4[\frac{1}{2}(c_2^2a_{63}a_{32} + c_2^2a_{42}a_{64} + c_3^2a_{43}a_{64} + c_2^2a_{52}a_{65} + c_3^2a_{53}a_{65} + c_4^2a_{54}a_{65})Gf_Y^2 + \frac{1}{6}(c_2^3a_{62} + c_3^3a_{63}$$

$$+ c_4^3a_{64} + c_5^3a_{65})Hf_y + (c_2c_3a_{32}a_{63} + c_2c_4a_{42}a_{64} + c_3c_4a_{43}a_{64} + c_2c_5a_{52}a_{65} + c_3c_5a_{53}a_{65} + c_4c_5a_{54}a_{65}$$

$$+ c_2c_6a_{32}a_{63} + c_2c_6a_{42}a_{64} + c_3c_6a_{43}a_{64} + c_2c_6a_{52}a_{65} + c_3c_6a_{53}a_{65} + c_4c_6a_{54}a_{65})Ff_{xy}f_y$$

$$+ (c_2c_3a_{32}a_{63} + c_2c_4a_{42}a_{64} + c_3c_4a_{43}a_{64} + c_2c_5a_{52}a_{65} + c_3c_5a_{53}a_{65} + c_4c_5a_{54}a_{65}$$

$$+ c_2c_6a_{32}a_{63} + c_2c_6a_{42}a_{64} + c_3c_6a_{43}a_{64} + c_2c_6a_{52}a_{65} + c_3c_6a_{53}a_{65} + c_4c_6a_{54}a_{65})Fff_yf_{yy} + (c_2a_{32}a_{43}a_{64}$$

$$+ c_2a_{32}a_{53}a_{65} + c_2a_{42}a_{54}a_{65} + c_3a_{43}a_{54}a_{65})Ff_y^3 + \frac{1}{2}(c_2^2c_6a_{62} + c_3^2c_6a_{63} + c_4^2c_6a_{64} + c_5^2c_6a_{65})Gf_{xy}$$

$$+ \frac{1}{2}(c_2^2c_6a_{62} + c_3^2c_6a_{63} + c_4^2c_6a_{64} + c_5^2c_6a_{65})Gff_{yy} + \frac{1}{2}(c_2c_6^2a_{62} + c_3c_6^2a_{63} + c_4c_6^2a_{64} + c_5c_6^2a_{65})Ff_{xxy}$$

$$+(c_2c_6^2a_{62}+c_3c_6^2a_{63}+c_4c_6^2a_{64}\ +h^4c_5c_6^2a_{65})Fff_{xyy}+\frac{1}{2}(c_2c_6^2a_{62}+c_3c_6^2a_{63}+c_4c_6^2a_{64}+c_5c_6^2a_{65})Ff^2f_{yyy}]$$

$$+\frac{1}{2}(hc_6)^2(f_{xx}+2ff_{xy}+f^2f_{yy})+\frac{1}{6}(hc_6)^3(f_{xxx}+3ff_{xxy}+3f^2f_{xyy}+f^3f_{yyy})$$

$$+\frac{1}{24}(hc_6)^4(f_{xxxx}+4c_6^4ff_{xxxy}+4c_6^4f^2f_{xxyy}+4f^3f_{xyyy}+f^4f_{yyyy})+o(h^5)$$

$$k_6=f+hc_6(f_x+ff_y)+h^2[\frac{1}{2}(hc_6)^2G+(c_2a_{62}+c_3a_{63}+c_4a_{64}+c_5a_{65})Ff_y]+h^3[\frac{1}{6}(hc_6)^3H$$

$$+\frac{1}{2}(c_2^2a_{62}+c_3^2a_{63}+c_4^2a_{64}+c_5^2a_{65})Gf_y+(c_2a_{32}a_{63}+c_2a_{42}a_{64}+c_3a_{43}a_{64}+c_2a_{52}a_{65}$$

$$+c_3a_{53}a_{65}+c_4a_{54}a_{65})Ff_Y^2+(c_2c_6a_{62}+c_3c_6a_{63}+c_4c_6a_{64}+c_5c_6a_{65})Ff_{xy}\ +(c_2c_6a_{62}$$

$$+c_3c_6a_{63}+c_4c_6a_{64}+c_5c_6a_{65})Fff_{yy}]+h^4[\frac{1}{24}(hc_6)^4I+\frac{1}{2}(c_2^2a_{63}a_{32}+c_2^2a_{42}a_{64}$$

$$+c_3^2a_{43}a_{64}+c_2^2a_{52}a_{65}+c_3^2a_{53}a_{65}+c_4^2a_{54}a_{65})Gf_Y^2+\frac{1}{6}(c_2^3a_{62}+c_3^3a_{63}+c_4^3a_{64}$$

$$+c_5^3a_{65})Hf_y+(c_2c_3a_{32}a_{63}+c_2c_4a_{42}a_{64}+c_3c_4a_{43}a_{64}+c_2c_5a_{52}a_{65}\ +c_3c_5a_{53}a_{65}$$

$$+c_4c_5a_{54}a_{65}+c_2c_6a_{32}a_{63}+c_2c_6a_{42}a_{64}+c_3c_6a_{43}a_{64}+c_2c_6a_{52}a_{65}+c_3c_6a_{53}a_{65}$$

$$+c_4c_6a_{54}a_{65})Ff_{xy}f_y+(c_2c_3a_{32}a_{63}+c_2c_4a_{42}a_{64}+c_3c_4a_{43}a_{64}+c_2c_5a_{52}a_{65}$$

$$+c_3c_5a_{53}a_{65}+c_4c_5a_{54}a_{65}+c_2c_6a_{32}a_{63}+c_2c_6a_{42}a_{64}+c_3c_6a_{43}a_{64}+c_2c_6a_{52}a_{65}$$

$$+c_3c_6a_{53}a_{65}+c_4c_6a_{54}a_{65})Fff_yf_{yy}+(c_2a_{32}a_{43}a_{64}+c_2a_{32}a_{53}a_{65}+c_2a_{42}a_{54}a_{65}$$

$$+c_3a_{43}a_{54}a_{65})Ff_y^3+\frac{1}{2}(c_2^2c_6a_{62}+c_3^2c_6a_{63}+c_4^2c_6a_{64}+c_5^2c_6a_{65})Gf_{xy}+\frac{1}{2}(c_2^2c_6a_{62}$$

$$+c_3^2c_6a_{63}+c_4^2c_6a_{64}+c_5^2c_6a_{65})Gff_{yy}+\frac{1}{2}(c_2c_6^2a_{62}+c_3c_6^2a_{63}+c_4c_6^2a_{64}+c_5c_6^2a_{65})Ff_{xxy}$$

$$+(c_2c_6^2a_{62}+c_3c_6^2a_{63}+c_4c_6^2a_{64}\ +h^4c_5c_6^2a_{65})Fff_{xyy}+\frac{1}{2}(c_2c_6^2a_{62}+c_3c_6^2a_{63}+c_4c_6^2a_{64}$$

$$+c_5c_6^2a_{65})Ff^2f_{yyy}]+o(h^5) \qquad (13)$$

3.3 Generation of Systems of Equations and their Solutions

We will now slot in the expressions for k_1, k_2, k_3, k_4, k_5 *and* k_6 into (2), to obtain an expression for y_{n+1}

$$\therefore y_{n+1} = y_n + h[b_1 f + b_2(f + c_2 hF + \frac{1}{2}c_2^2 h^2 G + \frac{1}{6}c_2^3 h^3 H + \frac{1}{24}c_2^4 h^4 I)$$

$$+ b_3(f + hc_3 F + h^2 c_2 a_{32} Ff_y + \frac{1}{2}h^3 c_2^2 a_{32} Gf_y + \frac{1}{6}h^4 c_2^3 a_{32} Hf_y + \frac{1}{2}h^2 c_3^2 G$$

$$+ h^3 c_2 c_3 a_{32} Ff_{xy} + \frac{1}{2}h^4 c_2^2 c_3 a_{32} Gf_{xy} + h^3 c_2 c_3 a_{32} Fff_{yy} + \frac{1}{2}h^4 c_2^2 c_3 a_{32} Gff_{yy}$$

$$+ \frac{1}{6}h^3 c_3^3 H + \frac{1}{2}h^4 c_2 c_3^2 a_{32} Ff_{xxy} + h^4 c_2 c_3^2 a_{32} Fff_{xyy} + \frac{1}{2}h^4 c_2 c_3^2 a_{32} Ff^2 f_{yyy}$$

$$+ \frac{1}{24}h^4 c_3^4 I) + b_4(f + hc_4 F + h^2(\frac{1}{2}c_4^2 G + c_2 a_{42} Ff_y + c_3 a_{43} Ff_y) + h^3(\frac{1}{6}c_4^3 H$$

$$+ \frac{1}{2}c_2^2 a_{42} Gf_y + h^3 c_2 a_{32} a_{43} Ff^2{}_y + \frac{1}{2}h^3 c_3^2 a_{43} Gf_y + h^3 c_2 c_4 a_{42} Ff_{xy} + h^3 c_3 c_4 a_{43} Ff_{xy}$$

$$+ h^3 c_2 c_4 a_{42} Fff_{yy} + h^3 c_3 c_4 a_{43} Fff_{yy}) + h^4(\frac{1}{24}c_4^4 I + \frac{1}{6}c_3^3 a_{42} Hf_y$$

$$+ \frac{1}{6}c_3^3 a_{43} Hf_y + \frac{1}{2}c_2^2 c_4 a_{42} Gf_{xy} + \frac{1}{2}c_2^2 a_{32} a_{43} Gf_y^2 + c_2 c_3 a_{32} a_{43} Ff_{xy} f_y$$

$$+ c_2 c_4 a_{32} a_{43} Ff_{xy} f_y + \frac{1}{2}c_3^2 c_4 a_{43} Gf_{xy} + \frac{1}{2}c_2^2 c_4 a_{42} Gff_{yy} + \frac{1}{2}c_3^2 c_4 a_{43} Gff_{yy}$$

$$+ \frac{1}{2}c_2 c_4^2 a_{42} Ff_{xxy} + h^4 c_2 c_3 a_{32} a_{42} Fff_y f_{yy} + c_2 c_4 a_{32} a_{43} Fff_y f_{yy}$$

$$+ \frac{1}{2}c_3 c_4^2 a_{43} Ff_{xxy} + c_2 c_4^2 a_{42} Fff_{xyy} + c_3 c_4^2 a_{43} Fff_{xyy} + \frac{1}{2}c_2 c_4^2 a_{42} Ff^2 f_{yyy}$$

$$+ \frac{1}{2}c_3 c_4^2 a_{43} Ff^2 f_{yyy})) + b_5(f + hc_5 F + h^2[\frac{1}{2}c_5^2 G + (c_2 a_{52} + c_3 a_{53} + c_4 a_{54}) Ff_y]$$

$$+ h^3[\frac{1}{6}c_5^3 H + (\frac{1}{2}c_2^2 a_{52} + \frac{1}{2}c_3^2 a_{53} + \frac{1}{2}c_4^2 a_{54}) Gf_y$$

$$+ (c_2 a_{32} a_{53} + c_2 a_{42} a_{54} + c_3 a_{43} a_{54}) Ff_y^2 + (c_2 c_5 a_{52} + c_3 c_5 a_{53} + c_4 c_5 a_{54}) Ff_{xy}$$

$$+ (c_2 c_5 a_{52} + c_3 c_5 a_{53} + c_4 c_5 a_{54}) Fff_{yy}] + h^4[\frac{1}{24}c_5^4 I + \frac{1}{6}Hf_y(c_2^3 a_{52} + c_3^3 a_{53}$$

$$+ c_4^3 a_{54}) + \frac{1}{2}Gf_y^2(c_2^2 a_{32} a_{53} + c_2^2 a_{42} a_{54} + c_3^2 a_{43} a_{54}) + (c_2 c_3 a_{32} a_{53} + c_2 c_4 a_{42} a_{54}$$

$$+ c_2 c_5 a_{32} a_{53} + c_3 c_4 a_{43} a_{54} + c_2 c_5 a_{42} a_{54} + c_3 c_5 a_{43} a_{54}) Ff_{xy} f_y$$

$$+ (c_2 c_3 a_{32} a_{53} + c_2 c_4 a_{42} a_{54} + c_3 c_4 a_{43} a_{54} + c_2 c_5 a_{32} a_{53} + c_3 c_4 a_{43} a_{54} + c_2 c_5 a_{42} a_{54}$$

$$+c_3c_5a_{43}a_{54})Ff_{xy}f_y+(c_2c_3a_{32}a_{53}+c_2c_4a_{42}a_{54}+c_3c_4a_{43}a_{54}$$

$$+c_2c_5a_{32}a_{53}+c_2c_5a_{42}a_{54}+c_3c_5a_{43}a_{54})Fff_yf_{yy}+\frac{1}{2}Gf_{xy}(c_2^2c_5a_{52}$$

$$+c_3^2c_5a_{53}+c_4^2c_5a_{54})+\frac{1}{2}Gff_{yy}(c_2^2c_5a_{52}+c_3^2c_5a_{53}+c_4^2c_5a_{54})$$

$$+\frac{1}{2}Ff_{xxy}(c_2c_5^2a_{52}+c_3c_5^2a_{53}+c_4c_5^2a_{54})+(c_2c_5^2a_{52}+c_3c_5^2a_{53}+c_4c_5^2a_{54})Fff_{xyy}$$

$$+\frac{1}{2}Ff^2f_{yyy}(c_2c_5^2a_{52}+c_3c_5^2a_{53}+c_4c_5^2a_{54})+c_2a_{32}a_{43}a_{54}Ff_y^3])$$

$$+b_6(f+hc_6(f_x+ff_y)+h^2[\frac{1}{2}(hc_6)^2G+(c_2a_{62}+c_3a_{63}+c_4a_{64}+c_5a_{65})Ff_y]$$

$$+h^3[\frac{1}{6}(hc_6)^3H+\frac{1}{2}(c_2^2a_{62}+c_3^2a_{63}+c_4^2a_{64}+c_5^2a_{65})Gf_y+(c_2a_{32}a_{63}+c_2a_{42}a_{64}$$

$$+c_3a_{43}a_{64}+c_2a_{52}a_{65}+c_3a_{53}a_{65}+c_4a_{54}a_{65})Ff_Y^2+(c_2c_6a_{62}+c_3c_6a_{63}+c_4c_6a_{64}$$

$$+c_5c_6a_{65})Ff_{xy}+(c_2c_6a_{62}+c_3c_6a_{63}+c_4c_6a_{64}+c_5c_6a_{65})Fff_{yy}]$$

$$+h^4[\frac{1}{24}(hc_6)^4I+\frac{1}{2}(c_2^2a_{63}a_{32}+c_2^2a_{42}a_{64}+c_3^2a_{43}a_{64}+c_2^2a_{52}a_{65}+c_3^2a_{53}a_{65}$$

$$+c_4^2a_{54}a_{65})Gf_Y^2+\frac{1}{6}(c_2^3a_{62}+c_3^3a_{63}+c_4^3a_{64}+c_5^3a_{65})Hf_y$$

$$+(c_2c_3a_{32}a_{63}+c_2c_4a_{42}a_{64}+c_3c_4a_{43}a_{64}+c_2c_5a_{52}a_{65}+c_3c_5a_{53}a_{65}$$

$$+c_4c_5a_{54}a_{65}+c_2c_6a_{32}a_{63}+c_2c_6a_{42}a_{64}+c_3c_6a_{43}a_{64}+c_2c_6a_{52}a_{65}+c_3c_6a_{53}a_{65}$$

$$+c_4c_6a_{54}a_{65})Ff_{xy}f_y+(c_2c_3a_{32}a_{63}+c_2c_4a_{42}a_{64}+c_3c_4a_{43}a_{64}+c_2c_5a_{52}a_{65}$$

$$+c_3c_5a_{53}a_{65}+c_4c_5a_{54}a_{65}+c_2c_6a_{32}a_{63}+c_2c_6a_{42}a_{64}+c_3c_6a_{43}a_{64}+c_2c_6a_{52}a_{65}$$

$$+c_3c_6a_{53}a_{65}+c_4c_6a_{54}a_{65})Fff_yf_{yy}+(c_2a_{32}a_{43}a_{64}+c_2a_{32}a_{53}a_{65}+c_2a_{42}a_{54}a_{65}$$

$$+c_3a_{43}a_{54}a_{65})Ff_y^3+\frac{1}{2}(c_2^2c_6a_{62}+c_3^2c_6a_{63}+c_4^2c_6a_{64}+c_5^2c_6a_{65})Gf_{xy}+\frac{1}{2}(c_2^2c_6a_{62}$$

$$+c_3^2c_6a_{63}+c_4^2c_6a_{64}+c_5^2c_6a_{65})Gff_{yy}+\frac{1}{2}(c_2c_6^2a_{62}+c_3c_6^2a_{63}+c_4c_6^2a_{64}+c_5c_6^2a_{65})Ff_{xxy}$$

$$+(c_2c_6^2a_{62}+c_3c_6^2a_{63}+c_4c_6^2a_{64}\quad+h^4c_5c_6^2a_{65})Fff_{xyy}+\frac{1}{2}(c_2c_6^2a_{62}+c_3c_6^2a_{63}+c_4c_6^2a_{64}$$

$$+c_5c_6^2a_{65})Ff^2f_{yyy}])]$$

By opening brackets and collecting like powers of h together, we obtain

$$
\begin{aligned}
y_{n+1} = {} & y_n + hf(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) + h^2 F(c_2 b_2 + c_3 b_3 + c_4 b_4 + c_5 b_5 + c_6 b_6) \\
& + \frac{1}{2} h^3 G(c_2^2 b_2 + c_3^2 b_3 + c_4^2 b_4 + c_5^2 b_5 + c_6^2 b_6) + h^3 F f_y (c_2 b_3 a_{32} + c_2 b_4 a_{42} \\
& + c_3 b_4 a_{43} + c_2 b_5 a_{52} + c_3 b_5 a_{53} + c_4 b_5 a_{54} + c_2 b_6 a_{62} + c_3 b_6 a_{63} + c_4 b_6 a_{64} + c_5 b_6 a_{65}) \\
& + \frac{1}{6} h^4 H(c_2^3 b_2 + c_3^3 b_3 + c_4^3 b_4 + c_5^3 b_5 + c_6^3 b_6) + \frac{1}{2} h^4 G f_y (c_2^2 b_3 a_{32} + c_2^2 b_4 a_{42} \\
& + c_3^2 b_4 a_{43} + c_2^2 b_5 a_{52} + c_3^2 b_5 a_{53} + c_4^2 b_5 a_{54} + c_2^2 b_6 a_{62} + c_3^2 b_6 a_{63} + c_4^2 b_6 a_{64} + c_5^2 b_6 a_{65}) \\
& + h^4 F f_{xy} (c_2 c_3 b_3 a_{32} + c_2 c_4 b_4 a_{42} + c_3 c_4 b_4 a_{43} + c_2 c_5 b_5 a_{52} + c_3 c_5 b_5 a_{53} + c_4 c_5 b_5 a_{54} \\
& + c_2 c_6 b_6 a_{62} + c_3 c_6 b_6 a_{63} + c_4 c_6 b_6 a_{64} + c_5 c_6 b_6 a_{65}) + h^4 F f f_{yy} (c_2 c_3 b_3 a_{32} + c_2 c_4 b_4 a_{42} \\
& + c_3 c_4 b_4 a_{43} + c_2 c_5 b_5 a_{52} + c_3 c_5 b_5 a_{53} + c_4 c_5 b_5 a_{54} + c_2 c_6 b_6 a_{62} + c_3 c_6 b_6 a_{63} \\
& + c_4 c_6 b_6 a_{64} + c_5 c_6 b_6 a_{65}) + h^4 F f_y^2 (c_2 b_4 a_{32} a_{43} + c_2 b_5 a_{32} a_{53} + c_2 b_5 a_{42} a_{54} + c_3 b_5 a_{43} a_{54} \\
& + c_2 b_6 a_{32} a_{63} + c_2 b_6 a_{42} a_{64} + c_3 b_6 a_{43} a_{64} + c_2 b_6 a_{52} a_{65} + c_3 b_6 a_{53} a_{65} + c_4 b_6 a_{54} a_{65}) \\
& + \frac{1}{2} h^5 I(c_2^4 b_2 + c_3^4 b_3 + c_4^4 b_4 + c_5^4 b_5 + c_6^4 b_6) + \frac{1}{6} h^5 H f_y (c_2^3 b_3 a_{32} + c_2^3 b_4 a_{42} \\
& + c_3^3 b_4 a_{43} + c_2^3 b_5 a_{52} + c_3^3 b_5 a_{53} + c_4^3 b_5 a_{54} + c_2^3 b_6 a_{62} + c_3^3 b_6 a_{63} + c_4^3 b_6 a_{64} + c_5^3 b_6 a_{65}) \\
& + \frac{1}{2} h^5 G f_{xy} (c_2^2 c_3 b_3 a_{32} + c_2^2 c_4 b_4 a_{42} + c_3^2 c_4 b_4 a_{43} + c_2^2 c_5 b_5 a_{52} + c_3^2 c_5 b_5 a_{53} + c_4^2 c_5 b_5 a_{54} \\
& + c_2^2 c_6 b_6 a_{62} + c_3^2 c_6 b_6 a_{63} + c_4^2 c_6 b_6 a_{64} + c_5^2 c_6 b_6 a_{65}) + \frac{1}{2} h^5 G f f_{yy} (c_2^2 c_3 b_3 a_{32} + c_2^2 c_4 b_4 a_{42} \\
& + c_3^2 c_4 b_4 a_{43} + c_2^2 c_5 b_5 a_{52} + c_3^2 c_5 b_5 a_{53} + c_4^2 c_5 b_5 a_{54} + c_2^2 c_6 b_6 a_{62} + c_3^2 c_6 b_6 a_{63} + c_4^2 c_6 b_6 a_{64} \\
& + c_5^2 c_6 b_6 a_{65}) + \frac{1}{2} h^5 F f_{xxy} (c_2 c_3^2 b_3 a_{32} + c_2 c_4^2 b_4 a_{42} + c_3 c_4^2 b_4 a_{43} + c_2 c_5^2 b_5 a_{52} + c_3 c_5^2 b_5 a_{53} \\
& + c_4 c_5^2 b_5 a_{54} + c_2 c_6^2 b_6 a_{62} + c_3 c_6^2 b_6 a_{63} + c_4 c_6^2 b_6 a_{64} + c_5 c_6^2 b_6 a_{65}) \\
& + h^5 F f f_{xyy} (c_2 c_3^2 b_3 a_{32} + c_2 c_4^2 b_4 a_{42} + c_3 c_4^2 b_4 a_{43} + c_2 c_5^2 b_5 a_{52} + c_3 c_5^2 b_5 a_{53} + c_4 c_5^2 b_5 a_{54} \\
& + c_2 c_6^2 b_6 a_{62} + c_3 c_6^2 b_6 a_{63} + c_4 c_6^2 b_6 a_{64} + c_5 c_6^2 b_6 a_{65}) + \frac{1}{2} h^5 F f^2 f_{yyy} (c_2 c_3^2 b_3 a_{32} + c_2 c_4^2 b_4 a_{42} \\
& + c_3 c_4^2 b_4 a_{43} + c_2 c_5^2 b_5 a_{52} + c_3 c_5^2 b_5 a_{53} + c_4 c_5^2 b_5 a_{54} + c_2 c_6^2 b_6 a_{62} + c_3 c_6^2 b_6 a_{63} + c_4 c_6^2 b_6 a_{64} \\
& + c_5 c_6^2 b_6 a_{65}) + \frac{1}{2} h^5 G f_y^2 (c_2^2 b_4 a_{32} a_{43} + c_2^2 b_5 a_{32} a_{53} + c_2^2 b_5 a_{42} a_{54} + c_3^2 b_5 a_{43} a_{54} \\
& + c_2^2 b_6 a_{32} a_{63} + c_2^2 b_6 a_{42} a_{64} + c_3^2 b_6 a_{43} a_{64} + c_2^2 b_6 a_{52} a_{65} + c_3^2 b_6 a_{53} a_{65} + c_4^2 b_6 a_{54} a_{65}) \\
& + h^5 F f_{xy} f_y (c_2 c_3 b_4 a_{32} a_{43} + c_2 c_4 b_4 a_{32} a_{43} + c_2 c_3 b_5 a_{32} a_{53} + c_2 c_4 b_5 a_{42} a_{54} \\
& + c_3 c_4 b_5 a_{43} a_{54} + c_2 c_5 b_5 a_{32} a_{53} + c_2 c_5 b_5 a_{42} a_{54} + c_3 c_5 b_5 a_{43} a_{54} + c_2 c_3 b_6 a_{32} a_{63} \\
& + c_2 c_4 b_6 a_{42} a_{64} + c_3 c_4 b_6 a_{43} a_{64} + c_2 c_5 b_6 a_{52} a_{65} + c_3 c_5 b_6 a_{53} a_{65} + c_4 c_5 b_6 a_{54} a_{65} + c_2 c_6 b_6 a_{32} a_{63} \\
& + c_2 c_6 b_6 a_{42} a_{64} + c_3 c_6 b_6 a_{43} a_{64} + c_2 c_6 b_6 a_{52} a_{65} + c_3 c_6 b_6 a_{53} a_{65} + c_4 c_6 b_6 a_{54} a_{65})
\end{aligned}
$$

$$+h^5 Fff_y f_{yy}(c_2c_3b_4a_{32}a_{43}+c_2c_4b_4a_{32}a_{43}+c_2c_3b_5a_{32}a_{53}+c_2c_4b_5a_{42}a_{54}$$
$$+c_3c_4b_5a_{43}a_{54}+c_2c_5b_5a_{32}a_{53}+c_2c_5b_5a_{42}a_{54}+c_3c_5b_5a_{43}a_{54}+c_2c_3b_6a_{32}a_{63}$$
$$+c_2c_4b_6a_{42}a_{64}+c_3c_4b_6a_{43}a_{64}+c_2c_5b_6a_{52}a_{65}+c_3c_5b_6a_{53}a_{65}+c_4c_5b_6a_{54}a_{65}$$
$$+c_2c_6b_6a_{32}a_{63}+c_2c_6b_6a_{42}a_{64}+c_3c_6b_6a_{43}a_{64}+c_2c_6b_6a_{52}a_{65}+c_3c_6b_6a_{53}a_{65}$$
$$+c_4c_6b_6a_{54}a_{65})+h^5F_y^3(c_2b_5a_{32}a_{43}a_{54}+c_2b_6a_{32}a_{43}a_{64}+c_2b_6a_{32}a_{53}a_{65}$$
$$+c_2b_6a_{42}a_{54}a_{65}+c_3b_6a_{43}a_{54}a_{65})+o(h^5) \quad (14)$$

We now express the derivatives y', y'', y''', y^{iv} *and* y^v in the Taylor expansion:

$$y_{n+1}=y_n+hy'_n+\frac{1}{2}h^2y''_n+\frac{1}{3!}h^3y'''_n+\frac{1}{4!}h^4y_n^{iv}\frac{1}{5!}h^5y_n^{v}+o(h^6) \quad (15)$$

in terms of $f(x_n, y_n)$

From (i), we have that

$$y'=f$$

$$\therefore y''=f'=(\frac{\partial}{\partial x}+\frac{\partial y}{\partial x}.\frac{\partial}{\partial y})f=\frac{\partial f}{\partial x}+\frac{\partial y}{\partial x}.\frac{\partial f}{\partial y}=f_x+ff_y=F \quad (from\ Eq.(8))$$

$$y'''=F'=(\frac{\partial}{\partial x}+\frac{\partial y}{\partial x}.\frac{\partial}{\partial y})(f_x+ff_y)$$
$$=f_{xx}+f_xf_y+ff_{xy}+ff_{xy}+ff_y^2+f^2f_{yy}$$
$$=f_{xx}+2ff_{xy}+f^2f_{yy}+f_y(f_x+ff_y)$$
$$\Rightarrow y'''=G+Ff_y$$

$$y^{iv}=(G+Ff_y)'=(\frac{\partial}{\partial x}+\frac{\partial y}{\partial x}.\frac{\partial}{\partial y})(f_{xx}+f_xf_y+ff_{xy}+ff_{xy}+ff_yf_y+fff_{yy})$$
$$=f_{xxx}+2f_xf_{xy}+2ff_{xxy}+ff_xf_{yy}+ff_xf_{yy}+f^2f_{xyy}+f_{xx}f_y+f_xf_{xy}+f_xf_y^2+ff_{xy}f_y$$
$$+ff_{xy}f_y+ff_{xxy}+2ff_{xy}f_y+2f^2f_{xyy}+f^2f_yf_{yy}+f^2f_yf_{yy}+f^3f_{yyy}+ff_{xy}f_y$$
$$+ff_xf_{yy}+ff_y^3+f^2f_yf_{yy}+f^2f_yf_{yy}$$
$$\therefore y^{iv}=f_{xxx}+3f_xf_{xy}+3ff_{xxy}+3ff_xf_{yy}+3f^2f_{xyy}+f_{xx}f_y+f_xf_y^2+5ff_{xy}f_y+4f^2f_yf_{yy}$$
$$+f^3f_{yyy}+ff_y^3$$
$$=(f_{xxx}+3ff_{xxy}+3f^2f_{xyy}+f^3f_{yyy})+f_y(f_{xx}+2ff_{xy}+f^2f_{yy})+f_y^2(f_x+ff_y)$$
$$3ff_{yy}(f_x+ff_y)+3f_{xy}(f_x+ff_y)$$
$$\Rightarrow y^{iv}=H+Gf_y+Ff_y^2+3Fff_{yy}+3Ff_{xy}$$
$$\therefore y^{iv}=H+3F(ff_{yy}+f_{xy})+f_y(G+Ff_y)$$

$$y^{v} = [H + 3F(ff_{yy} + f_{xy}) + f_y(G + Ff_y)]'$$

$$= (\frac{\partial}{\partial x} + \frac{\partial y}{\partial x}.\frac{\partial}{\partial y})(f_{xxx} + 3f_x f_{xy} + 3ff_{xxy} + 3ff_x f_{yy} + 3fff_{xyy} + f_{xx}f_y + f_x f_y f_y + 5ff_{xy}f_y$$
$$+ 4fff_y f_{yy} + ffff_{yyy} + ff_y f_y f_y)$$

by adding same terms together, and re-arranging, we get

$$y^{v} = (f_{xxxx} + 4ff_{xxxy} + 4f^2 f_{xxyy} + 4f^3 f_{xyyy} + f^4 f_{yyyy}) + 2f^2 f_{xxyy} + 4f_{xx}f_{xy} + 6f_x f_{xxy}$$
$$+ 8ff_{xy}^2 + 12ff_x f_{xyy} + 9ff_y f_{xxy} 3f_x^2 f_{yy} + 4ff_{xx}f_{yy} + 13ff_x f_y f_{yy} + 12f_{xy}f^2 f_{yy}$$
$$+ 6f_x f^2 f_{yyy} + 15f_y f^2 f_{xyy} + f_{xxx}f_y + f_{xx}f_y^2 + 7f_x f_y f_{xy} + 9ff_{xy}f_y^2 + 11f^2 f_y^2 f_{yy}$$
$$+ 4f^3 f_{yy}^2 + 7f_y f^3 f_{yyy} + f_x f_y^3 + ff_y^4$$

$$\therefore y^{v} = I + f_y(f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy}) + 4f_{xy}(f_{xx} + 2ff_{xy} + f^2 f_{yy})$$
$$+ 4ff_{yy}(f_{xx} + 2ff_{xy} + f^2 f_{yy}) + f_y^2(f_{xx} + 2ff_{xy} + f^2 f_{yy}) + 2f^2 f_{xxyy} + Ff_y^3$$
$$+ 7f_{xy}f_y(f_x + ff_y) + 6f_{xxy}(f_x + ff_y) + 12ff_{xyy}(f_x + ff_y) + 6f^2 f_{yyy}(f_x + ff_y)$$
$$+ 10ff_y f_{yy}(f_x + ff_y) + 3f_x f_{yy}(f_x + ff_y) + 2f^2 f_{xxyy}$$

$$\Rightarrow y^{v} = I + Hf_y 4Gf_{xy} + 4Gff_{yy} + Gf_y^2 + Ff_y^3 + 7Ff_{xy}f_y + 6Ff_{xxy} + 12Fff_{xyy} + 6Ff^2 f_{yyy}$$
$$+ 10Fff_y f_{yy} + 3Ff_x f_{yy} + 2f^2 f_{xxyy}$$

by collecting like terms together, and factorizing we now have

$$y^{v} = I + Hf_y 4G(f_{xy} + ff_{yy}) + f_y^2(G + Ff_y^2) + 6F(f_{xxy} + 2ff_{xyy} + f^2 f_{yyy})$$
$$+ F(10ff_y f_{yy} + 3Ff_x f_{yy} + 7Ff_{xy}f_y) + 2f^2 f_{xxyy}$$

we now slot in the expressions for y', y'', y''', y^{iv} *and* y^{v} into Equation (15), to give

$$y_{n+1} = y_n + hy'f_n + \frac{1}{2}h^2 F + \frac{1}{6}h^3 G + \frac{1}{6}h^3 Ff_{y_n} + \frac{1}{24}h^4 H + \frac{1}{24}h^4 Gf_y + \frac{1}{24}h^4 Ff_y^2$$
$$+ \frac{1}{8}h^4 Fff_{yy} + \frac{1}{8}h^4 Ff_{xy} + \frac{1}{120}h^5 I + \frac{1}{120}h^5 Hf_y + \frac{1}{30}h^5 Gf_{xy} + \frac{1}{30}h^5 Gff_{yy} + \frac{1}{120}h^5 Gf_y^2$$
$$+ \frac{1}{120}h^5 Ff_y^3 + \frac{7}{120}h^5 Ff_{xy}f_y + \frac{1}{20}h^5 Ff_{xxy} + \frac{1}{10}h^5 Fff_{xyy} + \frac{1}{20}h^5 Ff^2 f_{yyy} + \frac{1}{12}h^5 Fff_y f_{yy}$$
$$+ \frac{1}{40}h^5 Ff_x f_{yy} + \frac{1}{60}h^5 f^2 f_{xxyy} + o(h^6) \qquad (16)$$

Next, we proceed to equate as many terms as possible in Equations (14) and (16), to obtain the coupled system below:

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = 1 \qquad (i)$$

$$b_2c_2 + b_3c_3 + b_4c_4 + b_5c_5 + b_6c_6 = \frac{1}{2} \qquad (ii)$$

$$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 + b_5c_5^2 + b_6c_6^2 = \frac{1}{3} \qquad (iii)$$

$$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 + b_5c_5^3 + b_6c_6^3 = \frac{1}{4} \qquad (iv)$$

$$b_2c_2^4 + b_3c_3^4 + b_4c_4^4 + b_5c_5^4 + b_6c_6^4 = \frac{1}{5} \qquad (v)$$

$${}_2b_3a_{32} + c_2b_4a_{42} + c_3b_4a_{43} + c_2b_5a_{52} + c_3b_5a_{53} + c_4b_5a_{54} + c_2b_6a_{62} + c_3b_6a_{63} + c_4b_6a_{64}$$
$$+ c_5b_6a_{65} = \frac{1}{6} \qquad (vi)$$

$$c_2^2b_3a_{32} + c_2^2b_4a_{42} + c_3^2b_4a_{43} + c_2^2b_5a_{52} + c_3^2b_5a_{53} + c_4^2b_5a_{54} + c_2^2b_6a_{62} + c_3^2b_6a_{63} + c_4^2b_6a_{64}$$
$$+ c_5^2b_6a_{65} = \frac{1}{12} \qquad (vii)$$

$$c_2^3b_3a_{32} + c_2^3b_4a_{42} + c_3^3b_4a_{43} + c_2^3b_5a_{52} + c_3^3b_5a_{53} + c_4^3b_5a_{54} + c_2^3b_6a_{62} + c_3^3b_6a_{63} + c_4^3b_6a_{64}$$
$$+ c_5^3b_6a_{65} = \frac{1}{20} \qquad (viii)$$

$$c_2c_3b_3a_{32} + c_2c_4b_4a_{42} + c_3c_4b_4a_{43} + c_2c_5b_5a_{52} + c_3c_5b_5a_{53} + c_4c_5b_5a_{54} + c_2c_6b_6a_{62}$$
$$+ c_3c_6b_6a_{63} + c_4c_6b_6a_{64} + c_5c_6b_6a_{65} = \frac{1}{8} \qquad (ix)$$

$$c_2^2c_3b_3a_{32} + c_2^2c_4b_4a_{42} + c_3^2c_4b_4a_{43} + c_2^2c_5b_5a_{52} + c_3^2c_5b_5a_{53} + c_4^2c_5b_5a_{54} + c_2^2c_6b_6a_{62}$$
$$+ c_3^2c_6b_6a_{63} + c_4^2c_6b_6a_{64} + c_5^2c_6b_6a_{65} = \frac{1}{15} \qquad (x)$$

$$c_2c_3^2b_3a_{32} + c_2c_4^2b_4a_{42} + c_3c_4^2b_4a_{43} + c_2c_5^2b_5a_{52} + c_3c_5^2b_5a_{53} + c_4c_5^2b_5a_{54} + c_2c_6^2b_6a_{62}$$
$$+ c_3c_6^2b_6a_{63} + c_4c_6^2b_6a_{64} + c_5c_6^2b_6a_{65} = \frac{1}{10} \qquad (xi)$$

$$c_2b_4a_{32}a_{43} + c_2b_5a_{32}a_{53} + c_2b_5a_{42}a_{54} + c_3b_5a_{43}a_{54} + c_2b_6a_{32}a_{63} + c_2b_6a_{42}a_{64} + c_3b_6a_{43}a_{64}$$
$$+ c_2b_6a_{52}a_{65} + c_3b_6a_{53}a_{65} + c_4b_6a_{54}a_{65} = \frac{1}{24} \qquad (xii)$$

$$c_2^2b_4a_{32}a_{43} + c_2^2b_5a_{32}a_{53} + c_2^2b_5a_{42}a_{54} + c_3^2b_5a_{43}a_{54} + c_2^2b_6a_{32}a_{63} + c_2^2b_6a_{42}a_{64} + c_3^2b_6a_{43}a_{64}$$
$$+ c_2^2b_6a_{52}a_{65} + c_3^2b_6a_{53}a_{65} + c_4^2b_6a_{54}a_{65} = \frac{1}{60}$$

$$c_2^2b_4a_{32}a_{43}+c_2^2b_5a_{32}a_{53}+c_2^2b_5a_{42}a_{54}+c_3^2b_5a_{43}a_{54}+c_2^2b_6a_{32}a_{63}+c_2^2b_6a_{42}a_{64}+c_3^2b_6a_{43}a_{64}$$
$$+c_2^2b_6a_{52}a_{65}+c_3^2b_6a_{53}a_{65}+c_4^2b_6a_{54}a_{65} \quad = \quad \frac{1}{60} \qquad (xiii)$$

$$c_2c_3b_4a_{32}a_{43}+c_2c_4b_4a_{32}a_{43}+c_2c_3b_5a_{32}a_{53}+c_2c_4b_5a_{42}a_{54}+c_3c_4b_5a_{43}a_{54}+c_2c_5b_5a_{32}a_{53}$$
$$+c_2c_5b_5a_{42}a_{54}+c_3c_5b_5a_{43}a_{54}+c_2c_3b_6a_{32}a_{63}+c_2c_4b_6a_{42}a_{64}+c_3c_4b_6a_{43}a_{64}+c_2c_5b_6a_{52}a_{65}$$
$$+c_3c_5b_6a_{53}a_{65}+c_4c_5b_6a_{54}a_{65}+c_2c_6b_6a_{32}a_{63}+c_2c_6b_6a_{42}a_{64}+c_3c_6b_6a_{43}a_{64}+c_2c_6b_6a_{52}a_{65}$$
$$+c_3c_6b_6a_{53}a_{65}+c_4c_6b_6a_{54}a_{65} \quad = \quad \frac{1}{12} \qquad (xiv)$$

$$c_2b_5a_{32}a_{43}a_{54}+c_2b_6a_{32}a_{43}a_{64}+c_2b_6a_{32}a_{53}a_{65}+c_2b_6a_{42}a_{54}a_{65}+c_3b_6a_{43}a_{54}a_{65}=\frac{1}{120} \quad (xv)$$

It is worth noting here that Equations (i)-(xv), are the necessary conditions for a Runge- Kutta method to have order five. We must also state here, that there are actually twenty equations, but as can easily be observed from Eq. (14), some of the equations have duplicates. So, to avoid solving the same equation twice or even thrice in some cases, we considered only one of such equations, in each case. Specifically, Eq. (ix) occurs twice, Eq. (x) occurs twice also, Eq. (xi) occurs thrice, and Eq. (xiv) occurs twice. These amounts to five equations. Hence, we are left with fifteen equations.

Now, to compute our list of equations, we recall Eq. (5):

$$a_{21}=c_2 \qquad (xvi)$$
$$a_{31}=c_3-a_{32} \qquad (xvii)$$
$$a_{41}=c_4-(a_{42}+a_{43}) \qquad (xviii)$$
$$a_{51}=c_5-(a_{52}+a_{53}+a_{54}) \qquad (xix)$$
$$a_{61}=c_6-(a_{62}+a_{63}+a_{64}+a_{65}) \qquad (xx)$$

So, altogether, we have twenty equations with twenty-six unknowns:

b_1	b_2	b_3	b_4	b_5	b_6
c_2	c_3	c_4	c_5	c_6	
a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	
a_{32}	a_{42}	a_{52}	a_{52}		
a_{43}	a_{53}	a_{63}			
a_{54}	a_{64}				
a_{65}					

The number of unknown coefficients can be determined from the simple formula $\frac{s(s+1)}{2}$ where s is the stage number of the process.

Thus we have six parameters family of solutions for a six stage method of order five; that is six degrees of freedom in assigning values to some of these variables. The twenty equations can be divided into three separate groups:

Group One

$$\left.\begin{aligned}
b_1+b_2+b_3+b_4+b_5+b_6 &= 1\\
b_2c_2+b_3c_3+b_4c_4+b_5c_5+b_6c_6 &= \frac{1}{2}\\
b_2c_2^2+b_3c_3^2+b_4c_4^2+b_5c_5^2+b_6c_6^2 &= \frac{1}{3}\\
b_2c_2^3+b_3c_3^3+b_4c_4^3+b_5c_5^3+b_6c_6^3 &= \frac{1}{4}\\
b_2c_2^4+b_3c_3^4+b_4c_4^4+b_5c_5^4+b_6c_6^4 &= \frac{1}{5}
\end{aligned}\right\} \qquad (17)$$

In the first group, we have five equations with eleven unknowns. Values will be assigned to b_1, c_2, c_3, c_4, c_5 *and* c_6. b_1 is chosen, because it occurs only in the first equation. c_2, c_3, c_4, c_5 *and* c_6, will be assigned values, so as to get a linear equation. When Eq. (17) is solved, we will have values for b_2, b_3, b_4, b_5 *and* b_6, in addition to b_1, c_2, c_3, c_4, c_5 *and* c_6.

Group Two

$$c_2b_3a_{32} + c_2b_4a_{42} + c_3b_4a_{43} + c_2b_5a_{52} + c_3b_5a_{53} + c_4b_5a_{54} + c_2b_6a_{62} + c_3b_6a_{63} + c_4b_6a_{64} + c_5b_6a_{65} = \frac{1}{6}$$

$$c_2^2b_3a_{32} + c_2^2b_4a_{42} + c_3^2b_4a_{43} + c_2^2b_5a_{52} + c_3^2b_5a_{53} + c_4^2b_5a_{54} + c_2^2b_6a_{62} + c_3^2b_6a_{63} + c_4^2b_6a_{64} + c_5^2b_6a_{65} = \frac{1}{12}$$

$$c_2^3b_3a_{32} + c_2^3b_4a_{42} + c_3^3b_4a_{43} + c_2^3b_5a_{52} + c_3^3b_5a_{53} + c_4^3b_5a_{54} + c_2^3b_6a_{62} + c_3^3b_6a_{63} + c_4^3b_6a_{64} + c_5^3b_6a_{65} = \frac{1}{20}$$

$$c_2c_3b_3a_{32} + c_2c_4b_4a_{42} + c_3c_4b_4a_{43} + c_2c_5b_5a_{52} + c_3c_5b_5a_{53} + c_4c_5b_5a_{54} + c_2c_6b_6a_{62} + c_3c_6b_6a_{63} + c_4c_6b_6a_{64} + c_5c_6b_6a_{65} = \frac{1}{8}$$

$$c_2^2c_3b_3a_{32} + c_2^2c_4b_4a_{42} + c_3^2c_4b_4a_{43} + c_2^2c_5b_5a_{52} + c_3^2c_5b_5a_{53} + c_4^2c_5b_5a_{54} + c_2^2c_6b_6a_{62} + c_3^2c_6b_6a_{63} + c_4^2c_6b_6a_{64} + c_5^2c_6b_6a_{65} = \frac{1}{15}$$

$$c_2c_3^2b_3a_{32} + c_2c_4^2b_4a_{42} + c_3c_4^2b_4a_{43} + c_2c_5^2b_5a_{52} + c_3c_5^2b_5a_{53} + c_4c_5^2b_5a_{54} + c_2c_6^2b_6a_{62} + c_3c_6^2b_6a_{63} + c_4c_6^2b_6a_{64} + c_5c_6^2b_6a_{65} = \frac{1}{10}$$

$$c_2b_4a_{32}a_{43} + c_2b_5a_{32}a_{53} + c_2b_5a_{42}a_{54} + c_3b_5a_{43}a_{54} + c_2b_6a_{32}a_{63} + c_2b_6a_{42}a_{64} + c_3b_6a_{43}a_{64} + c_2b_6a_{52}a_{65} + c_3b_6a_{53}a_{65} + c_4b_6a_{54}a_{65} = \frac{1}{24}$$

$$c_2^2b_4a_{32}a_{43} + c_2^2b_5a_{32}a_{53} + c_2^2b_5a_{42}a_{54} + c_3^2b_5a_{43}a_{54} + c_2^2b_6a_{32}a_{63} + c_2^2b_6a_{42}a_{64} + c_3^2b_6a_{43}a_{64} + c_2^2b_6a_{52}a_{65} + c_3^2b_6a_{53}a_{65} + c_4^2b_6a_{54}a_{65} = \frac{1}{60}$$

$$c_2c_3b_4a_{32}a_{43} + c_2c_4b_4a_{32}a_{43} + c_2c_3b_5a_{32}a_{53} + c_2c_4b_5a_{42}a_{54} + c_3c_4b_5a_{43}a_{54} + c_2c_5b_5a_{32}a_{53} + c_2c_5b_5a_{42}a_{54} + c_3c_5b_5a_{43}a_{54} + c_2c_3b_6a_{32}a_{63} + c_2c_4b_6a_{42}a_{64} + c_3c_4b_6a_{43}a_{64} + c_2c_5b_6a_{52}a_{65} + c_3c_5b_6a_{53}a_{65} + c_4c_5b_6a_{54}a_{65} + c_2c_6b_6a_{32}a_{63} + c_2c_6b_6a_{42}a_{64} + c_3c_6b_6a_{43}a_{64} + c_2c_6b_6a_{52}a_{65} + c_3c_6b_6a_{53}a_{65} + c_4c_6b_6a_{54}a_{65} = \frac{1}{12}$$

$$c_2b_5a_{32}a_{43}a_{54} + c_2b_6a_{32}a_{43}a_{64} + c_2b_6a_{32}a_{53}a_{65} + c_2b_6a_{42}a_{54}a_{65} + c_3b_6a_{43}a_{54}a_{65} = \frac{1}{120} \qquad (18)$$

For this second group of equations, we shall make use of values obtained from the first group, to solve for $a_{32}, a_{42}, a_{43}, a_{52}, a_{53}, a_{54}, a_{62}, a_{63}, a_{64}$ *and* a_{65}.

Group Three

$$\left.\begin{aligned} a_{21} &= c_2 \\ a_{31} &= c_3 - a_{32} \\ a_{41} &= c_4 - (a_{42} + a_{43}) \\ a_{51} &= c_5 - (a_{52} + a_{53} + a_{54}) \\ a_{61} &= c_6 - (a_{62} + a_{63} + a_{64} + a_{65}) \end{aligned}\right\} \quad (19)$$

In summary, values will be assigned to b_1, c_2, c_3, c_4, c_5 *and* c_6 to get $b_2, b_3, b_4, b_5\ b_6$, $a_{21}, a_{31}, a_{41}, a_{51}, a_{61} a_{32}, a_{42}, a_{43}, a_{52}, a_{53}, a_{54}, a_{62}, a_{63}, a_{64}$ *and* a_{65}. The values of all the unknowns, will then be substituted into Eq. (2) and (3) to get the desired scheme. As a reminder, Equations (2) and (3) are:

$$y_{n+1} = y_n + h[b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6]$$

and

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + c_2 h, y_n + h a_{21} k_1), \\ k_3 &= f(x_n + c_3 h + y_n + h(a_{31} k_1 + a_{32} k_2)), \\ k_4 &= f(x_n + c_4 h, y_n + h(a_{41} k_1 + a_{42} k_2 + a_{43} k_3)), \\ k_5 &= f(x_n + c_5 h, y_n + h(a_{51} k_1 + a_{52} k_2 + a_{53} k_3 + a_{54} k_4)), \\ k_6 &= f\{x_n + c_5 h, y_n + h(a_{61} k_1 + a_{62} k_2 + a_{63} k_3 + a_{64} k_4 + a_{65} k_5)). \end{aligned}$$

3.4 The New Six-Stage Runge-Kutta Method of Order Five

We will now proceed to assign the following values to some of the free parameters

$$\left.\begin{aligned}
b_1 &= \frac{7}{90} \\
c_2 &= 1 \\
c_3 &= \frac{1}{2} \\
c_4 &= \frac{1}{5} \\
c_5 &= \frac{1}{4} \\
c_6 &= \frac{3}{4}
\end{aligned}\right\} \qquad (20)$$

In choosing the above values, the goal was to get numbers which when substituted into Eq. (17), would produce a matrix that has a solution, and that would also combine well together to produce a scheme, that is of high accuracy, comparable to that produced by other schemes of the same order. Unlike the old days, when schemes were developed for easy desk top use, these days computers are at our disposal, to solve these schemes, so too much emphasis, was not placed on ease of desktop use.

Equation (20) would now be substituted into Equations (17), and the resulting systems of equations would be solved using MS-Excel *Paste Function* and *Numerical Solver* respectively. For the first group of equations we have the following augmented matrix:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & \frac{83}{90} \\
1 & \frac{1}{2} & \frac{1}{5} & \frac{1}{4} & \frac{3}{4} & \frac{1}{2} \\
1 & \frac{1}{4} & \frac{1}{25} & \frac{1}{16} & \frac{9}{16} & \frac{1}{3} \\
1 & \frac{1}{8} & \frac{1}{125} & \frac{1}{64} & \frac{27}{64} & \frac{1}{4} \\
1 & \frac{1}{16} & \frac{1}{625} & \frac{1}{256} & \frac{81}{256} & \frac{1}{5}
\end{bmatrix}$$

On solving the matrix above, the following results were arrived at

$$
\left.\begin{aligned}
b_2 &= 0.077777777778 = \frac{7}{90} \\
b_3 &= 0.133333333333 = \frac{2}{15} \\
b_4 &= -1.0516x10^{-12} \approx 0 \\
b_5 &= 0.355555555556 = \frac{16}{45} \\
b_6 &= 0.355555555556 = \frac{16}{45}
\end{aligned}\right\} \qquad (21)
$$

These values along with those assigned to b_1, c_2, c_3, c_4, c_5 *and* c_6, would now be substituted into Eq. (18) to solve the second group of equations. On substituting, we get the coupled system of equations below.

$$
\begin{aligned}
&0.13333333333a_{32} + 0.35555555556a_{52} + 0.17777777778a_{53} + 0.07111111112a_{54} \\
&+ 0.35555555556a_{62} + 0.17777777778a_{63} + 0.07111111112a_{64} + 0.08888888889a_{65} \\
&= 0.16666666667 \\
&0.13333333333a_{32} + 0.35555555556a_{52} + 0.08888888889a_{53} + 0.14222222222a_{54} \\
&+ 0.35555555556a_{62} + 0.08888888889a_{63} + 0.14222222222a_{64} + 0.02222222222a_{65} \\
&= 0.08333333333 \\
&0.13333333333a_{32} + 0.35555555556a_{52} + 0.04444444444a_{53} + 0.00284444448a_{54} \\
&+ 0.35555555556a_{62} + 0.04444444444a_{63} + 0.00284444448a_{64} + 0.00555555625a_{65} \\
&= 0.05 \\
&0.06666666666a_{32} + 0.08888888889a_{52} + 0.04444444444a_{53} + 0.01777777778a_{54} \\
&+ 0.26666666667a_{62} + 0.13333333333a_{63} + 0.05333333333a_{64} + 0.06666666667a_{65} \\
&= 0.125
\end{aligned}
$$

$$
\begin{aligned}
&0.06666666665a_{32} + 0.08888888889a_{52} + 0.02222222225a_{53} + 0.00355555556a_{54} \\
&+ 0.2666666667a_{62} + 0.06666666675a_{63} + 0.01066666668a_{64} + 0.01666666875a_{65} \\
&= 0.06666666667
\end{aligned}
$$

$$0.03333333325a_{32} + 0.22222222225a_{52} + 0.01111111125a_{53} + 0.04444444445a_{54}$$
$$+ 0.20000000025a_{62} + 0.10000000125a_{63} + 0.0400000005a_{64} + 0.050000000625a_{65}$$
$$= 0.1$$
$$0.35555555556a_{32}a_{53} + 0.35555555556a_{42}a_{54} + 0.17777777778a_{43}a_{54} + 0.35555555556a_{32}a_{63}$$
$$+ 0.35555555556a_{42}a_{64} + 0.17777777778a_{43}a_{64} + 0.35555555556a_{52}a_{65} + 0.17777777778a_{53}a_{65}$$
$$+ 0.07111111112a_{54}a_{65} = 0.041666666667$$
$$0.35555555556a_{32}a_{53} + 0.35555555556a_{42}a_{54} + 0.08888888889a_{43}a_{54} + 0.35555555556a_{32}a_{63}$$
$$+ 0.35555555556a_{42}a_{64} + 0.08888888889a_{43}a_{64} + 0.35555555556a_{52}a_{65} + 0.08888888889a_{53}a_{65}$$
$$+ 0.014222222224a_{54}a_{65} = 0.016666666667$$
$$0.26666666667a_{32}a_{53} + 0.16000000002a_{42}a_{54} + 0.080000000001a_{43}a_{54} + 0.4444444445a_{32}a_{63}$$
$$+ 0.33777777782a_{42}a_{64} + 0.168888888891a_{43}a_{64} + 0.35555555556a_{52}a_{65} + 0.17777777778a_{53}a_{65}$$
$$+ 0.07111111112a_{54}a_{65} = 0.08333333333$$
$$0.35555555556a_{32}a_{43}a_{54} + 0.35555555556a_{32}a_{43}a_{64} + 0.35555555556a_{32}a_{53}a_{65}$$
$$+ 0.35555555556a_{42}a_{54}a_{65} + 0.17777777778a_{43}a_{54}a_{65} = 0.008333333333 \quad (22)$$

On solving the coupled (non-linear)system of equations above, the following results were obtained

$$\left.\begin{aligned}
a_{32} &= -0.749655737 \\
a_{42} &= 0.560058106 \\
a_{43} &= 0.341486157 \\
a_{52} &= 0.045073568 \\
a_{53} &= 0.353037791 \\
a_{54} &= -0.405218409 \\
a_{62} &= 0.290909052 \\
a_{63} &= 0.331676697 \\
a_{64} &= 1.359792241 \\
a_{65} &= -0.477547722
\end{aligned}\right\} \quad (23)$$

Substituting Equations (20), (21), and (23) into Equations (2) and (3), we would have our new six-stage Runge-Kutta scheme to be:

$$y_{n+1} = y_n + h[\frac{7}{90}k_1 + \frac{7}{90}k_2 + \frac{2}{15}k_3 \frac{16}{45}k_5 + \frac{16}{45}k_6]$$
$$\therefore y_{n+1} = y_n + \frac{h}{90}[7k_1 + 7k_2 + 12k_3 + 32k_5 + 32k_6]$$

where

$$
\begin{aligned}
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + h, y_n + hk_1) \\
k_3 &= f[x_n + \frac{h}{2}, y_n + h(1.249655737k_1 - 0.749655737k_2)] \\
k_4 &= f[x_n + \frac{h}{5}, y_n - h(0.7015442631k_1 - 0.5600588106k_2 - 0.341486157k_3)] \\
k_5 &= f[x_n + \frac{h}{4}, y_n + h(0.25710705k_1 + 0.045073568k_2 + 0.353037791k_3 - 0.405218409k_4)] \\
k_6 &= f[x_n + \frac{3h}{4}, y_n - h(0.754830268k_1 - 0.290909052k_2 - 0.331676697k_3 - 1.359792241k_4 \\
&\quad + 0.477547722k_5)] \qquad (24)
\end{aligned}
$$

CHAPTER FOUR

APPLICATION AND COMPARISON OF RESULTS.

In this chapter, we use the new six-stage Runge-Kutta method to solve various differential equations, and also compare the solutions with those obtained using the Adam-Moulton method, Adam-Bashforth method, the classical four-stage Runge-Kutta method, and Lawson's six-stage method, of order five.

In instances where the exact solution exists, we would also compare the results obtained from the new scheme with that of the exact solution.

4.1 Comparison with Adam-Moulton and Adam-Bashforth Methods

We will now proceed to use the new six-stage Runge-Kutta method of order five to solve the differential equation:

$$y' = x + y; y(0) = 1, h = 0.1$$

$$y_{n+1} = y_n + \frac{h}{90}[7k_1 + 7k_2 + 12k_3 + 32k_5 + 32k_6]$$

$For\ n = 0$

$$k_1 = f(x_0, y_0)$$
$$= f(0,1)$$
$$k_1 = 0 + 1$$
$$\therefore k_1 = 1$$

$$k_2 = f(x_0 + h, y_0 + hk_1)$$
$$= f(0.1,1.1)$$
$$k_2 = 0.1 + 1.1$$
$$\therefore k_2 = 1.2$$

$$k_3 = f[x_0 + \frac{h}{2}, y_0 + h((0.5 + 0.749655737)k_1 - 0.749655737k_2)]$$

$$= f[0 + \frac{0.1}{2}, 1 + 0.1((0.5 + 0.749655737) - 0.749655737(1.2))]$$

$$= f[\frac{0.1}{2}, 1 + 0.03500688526]$$

$$= 0.05 + 1.03500688526$$

$$\therefore k_2 = 1.08500688526$$

$$k_3 = f[x_0 + \frac{h}{2}, y_0 + h((0.5 + 0.749655737)k_1 - 0.749655737(1.2))]$$

$$= f(\frac{0.1}{2}, 1 + 0.03500688526)$$

$$= 0.05 + 1.03500688526$$

$$\therefore k_3 = 1.08500688526$$

$$k_4 = f[x_0 + \frac{h}{5}, y_0 + h((0.2 - (0.560058106 + 0.341486157)(1)) + 0.560058106(1.2)$$
$$+ 0.341486157(1.085006885))]$$

$$= f(0.02, 1.03410402885)$$

$$= 0.02 + 1.03410402885$$

$$\therefore k_4 = 1.054104028$$

$$k_5 = f[x_0 + \frac{h}{4}, y_0 + h((0.25 - (0.045073568 + 0.353037791 - 0.405218409)(1))$$
$$+ 0.045073568(1.2) + 0.353037791(1.085006885)$$
$$- 0.405218409(1.054104028)]$$

$$\Rightarrow k_5 = f(\frac{0.1}{4}, 1 + 0.02671014084)$$

$$= 0.025 + 1.02671014084$$

$$k_5 = 1.0517104$$

$$k_6 = f[0 + \frac{0.3}{4}, 1 - 0.1(0.754830268(1) - 0.290909052(1.2) - 0.331676697(1.085006885)$$
$$- 1.3597922441(1.054104028) + 0.477547722(1.0517104))]$$

$$= f(0.075, 1.08852527906)$$

$$= 0.075 + 1.08852527906$$

$$\therefore k_6 = 1.163525279$$

$$\Rightarrow y_1 = y_0 + \frac{h}{90}[7(1) + 7(1.2) + 12(1.08500688526) + 32(1.05417104) + 32(1.163525279)]$$

$$= 1 + \frac{0.1}{90}[7 + 8.4 + 13.02008262 + 33.65472448 + 32.23280893]$$

$$\therefore y_1 = 1.1103417956$$

For n = 1

$$y_2 = y_1 + \frac{h}{90}[7k_1 + 7k_2 + 12k_3 + 32k_5 + 32k_6]$$

$$k_1 = f(x_1, y_1)$$

$$= f(0.1, 1.1103417956)$$

$$\therefore k_1 = 1.2103417956$$

$$k_2 = f(x_1 + h, y_1 + hk_1)$$

$$= f(0.2, 1.23137597516)$$

$$k_2 = 0.2 + 1.23137597516$$

$$\therefore k_2 = 1.43137597516$$

$$k_3 = f[x_1 + \frac{h}{2}, y_1 + h(1.249655737(1.2103417956) - 0.749655737(1.43137597516)]$$

$$= f(0.15, 1.1542889313)$$

$$= 0.15 + 1.1542889313$$

$$\therefore k_3 = 1.3042889313$$

$$k_4 = f[0.1 + \frac{h}{5}, y_1 - h(0.07015442631(1.2103417456) - 0.560058106(1.43137597516)$$

$$+ 0.341486157(1.3042889313))]$$

$$= f(0.1 + 0.02, 1.1103417956 + 0.0397942)$$

$$= 0.12 + 1.150136002272$$

$$\therefore k_4 = 1.270136022718$$

$$k_5 = f[0.1 + \frac{h}{4}, y_1 + 0.1(0.25710705(1.2103417956) + 0.045073568(1.43137597516)$$

$$+ 0.353037791(1.3042889313) - 0.405218409(1.270136022718))]$$

$$= f(0.125, 1.142490337999)$$

$$= 0.125 + 1.142490337999$$

$$\therefore k_5 = 1.267490337999$$

$$k_6 = f[0.1 + \frac{0.3}{4}, y_1 - 0.1(0.754830268(1.2103417956) - 0.290909052(1.43137597516)$$
$$- 0.331676697(1.3042889313) - 1.359792241(1.270136022718)$$
$$+ 0.477547722(1.267490337999))]$$
$$= f(0.175, 1.2160651764049)$$
$$\therefore k_6 = 1.3910651764049$$

$$\Rightarrow y_2 = y_1 + \frac{0.1}{90}[7(1.2103417596) + 7(1.4313759516) + 12(1.3042889313)$$
$$+ 32(1.267490338) + 32(1.3910651764)]$$
$$\therefore y_2 = 1.2428054267465$$

For n = 2

$$y_3 = y_2 + \frac{h}{90}[7k_1 + 7k_2 + 12k_3 + 32k_5 + 32k_6]$$

$$k_1 = f(x_2, y_2)$$
$$= f(0.2, 1.2428054267)$$
$$= 0.2 + 1.2428054267$$
$$\therefore k_1 = 1.4428054267$$
$$k_2 = f(x_2 + h, y_2 + hk_1)$$
$$= f(0.2 + 0.1, 1.2428054267 + 0.1442805427)$$
$$= 0.3 + 1.3870859694$$
$$\therefore k_2 = 1.6870859694$$
$$k_3 = f[x_2 + \frac{h}{2}, y_2 + 0.1(1.249655737(1.4428054267) - 0.749655737(1.6870859694))]$$
$$= f[0.2 + 0.05, 1.2428054267 + 0.1(1.8030100789 - 1.2647336758)]$$
$$= 0.25 + 1.2966330671$$
$$\therefore k_3 = 1.5466330671$$

$$k_4 = f[x_2 + \frac{1}{5}h, y_2 - (0.7015442631(1.4428054268) + 0.560058106(1.6870859694)$$
$$+ 0.341486157(1.5466330671))]$$
$$= f(0.22, 1.2888882353)$$
$$= 0.22 + 1.2888882353$$
$$\therefore k_4 = 1.5088882353$$

$$k_5 = f[x_2 + \frac{1}{4}h, y_2 + h(0.25710705(1.4428054268) + 0.045073568(1.6870859694)$$
$$+ 0.353037791(1.5466330671) - 0.405218409(1.5088882353))]$$
$$= f(0.225, 1.280964333)$$
$$= 0.225 + 1.280964333$$
$$\therefore k_5 = 1.505964333$$

$$k_6 = f[x_2 + \frac{3h}{4}, y_2 - 0.1(0.754830268(1.4428054267) - 0.290909052(1.6870859694)$$
$$- 0.331676697(1.5466330671) - 1.359792241(1.5088882353)$$
$$+ 0.477547722(1.505964333))]$$
$$= f(0.275, 1.3675356466)$$
$$= 0.275 + 1.3675356466$$
$$\therefore k_6 = 1.6425356466$$
$$y_3 = 1.24218054267 + \frac{0.1}{90}[7(1.4428054267) + 7(1.6870859694) + 12(1.5466330671)$$
$$+ 32(1.505964333) + 32(1.6425356466)]$$
$$\Rightarrow y_3 = 1.3997174667$$

For n = 3

$$y_4 = y_3 + \frac{h}{90}[7k_1 + 7k_2 + 12k_3 + 32k_5 + 32k_6]$$
$$k_1 = f(x_3, y_3)$$
$$= f(0.3, 1.3997174667)$$
$$= 0.3 + 1.3997174667$$
$$\therefore k_1 = 1.6997174667$$
$$k_2 = f(x_3 + h, y_3 + hk_1)$$
$$= f(0.3 + 0.1, 1.3997174667 + 0.1699717467)$$
$$= 0.4 + 1.5696892133$$

$k_2 = 1.9696892133$

$$k_3 = f[x_3 + \frac{h}{2}, y_3 + 0.1(1.249655737(1.6997174667) - 0.749655737(1.9696892133))]$$
$$= f(0.35, 1.4644647531)$$
$$= 0.35 + 1.4644647531$$
$$\therefore k_3 = 1.814464753$$

$$k_4 = f[x_3 + \frac{h}{5}, y_3 - 0.1(0.7015442631(1.6997174667) - 0.560058106(1.969689213)$$
$$- 0.341486157(1.814464753))]$$
$$= f(0.32, 1.4527502635)$$
$$= 0.32 + 1.4527502635$$

$$\therefore k_4 = 1.7727502635$$

$$k_5 = f[x_3 + \frac{h}{4}, y_3 + 0.1(0.25710705(1.6997174667) + 0.045073568(1.9696892133)$$
$$+ 0.353037791(1.8144464753) - 0.405218409(1.7727502635))]$$

$$= f(0.325, 1.4445188518)$$
$$= 0.325 + 1.4445188518$$
$$\therefore k_5 = 1.7695188518$$

$$k_6 = f[x_3 + \frac{3h}{4}, y_3 - 0.1(0.754830268(1.6997174667) - 0.290909052(1.969689213)$$
$$- 0.331676697(1.8144647531) - 1.35979224(1.7727502635)$$
$$+ 0.477547722(1.7695188518))]$$
$$= f(0.375, 1.5454534931)$$
$$= 0.375 + 1.5454534931$$
$$\therefore k_6 = 1.9204534931$$

$$\Rightarrow y_4 = 1.3997174666631 + \frac{0.1}{90}[7(1.699717466631) + 7(1.9696892133294)$$
$$+ 12(1.8144647531246) + 32(1.7695188517977) + 32(1.9204534930489)]$$
$$\therefore y_4 = 1.5836491764449$$

For n = 4

$$y_5 = y_4 + \frac{h}{90}[7k_1 + 7k_2 + 12k_3 + 32k_5 + 32k_6]$$

$$k_1 = f(x_4, y_4)$$
$$= f(0.4, 1.5836491764449)$$
$$= 0.4 + 1.5836491764449$$

$$\therefore k_1 = 1.9836491765$$

$$k_2 = f(x_4 + h, y_4 + hk_1)$$
$$= f(0.4 + 0.1, 1.5836491764449 + 0.19836491764449)$$
$$= f(0.5, 1.7820140940935)$$
$$= 0.5 + 1.7820140940935$$
$$\therefore k_2 = 2.2820140941$$

$$k_3 = f[x_4 + \frac{h}{2}, y_4 + 0.1(1.249655737(1.9836491765) - 0.749655737(2.2820140941))]$$
$$= f(0.45, 1.660464538)$$
$$= 0.45 + 1.660464538$$
$$\therefore k_3 = 2.110464538$$

$$k_4 = f[x_4 + \frac{h}{5}, y_4 - 0.1(0.7015442631(1.9836491765) - 0.560058106(2.2820140941) - 0.341486157(2.110464538))]$$
$$= f(0.42, 1.6443628981)$$
$$= 0.42 + 1.6443628981$$
$$\therefore k_4 = 2.0643628981$$

$$k_5 = f[x_4 + \frac{h}{4}, y_4 + 0.1(0.25710705(1.9836491765) + 0.045073568(2.2820140941) + 0.353037791(2.110464538) - 0.405218409(2.0643628981))]$$
$$= f(0.425, 1.6357916359)$$
$$= 0.425 + 1.6357916359$$
$$\therefore k_5 = 2.0607916359$$

$$k_6 = f[x_4 + \frac{3h}{4}, y_4 - 0.1(0.754830268(1.9836491765) - 0.290909052(2.2820140941) - 0.331676697(2.110464538) - 1.35979224(2.0643628981) + 0.477547722(2.0607916359))$$

$$= f(0.475, 1.752600209)$$
$$= 0.475 + 1.752600209$$
$$\therefore k_6 = 2.2276002089$$

$$\because y_5 = y_4 + \frac{h}{90}[7k_1 + 7k_2 + 12k_3 + 32k_5 + 32k_6]$$

$$\Rightarrow y_5 = 1.5836491765 + \frac{0.1}{90}[7(1.9836491765) + 7(2.2820140941) + 12(2.110464538)$$
$$+ 32(2.0607916359) + 32(2.2276002089)]$$
$$\therefore y_5 = 1.79744224$$

By similar computations we obtain

$$y_6 = 2.0442372$$
$$y_7 = 2.3275049$$
$$y_8 = 2.6510812$$

We now compute the result obtained for $y_1, y_2, y_3, y_4, y_5, y_6, y_7$ *and* y_8 by the new scheme, with those obtained for the Adam-Moulton and Adam-Bashforth methods, (see sections 2.2.1 and 2.2.2).

Ve will now compare the various results below

	Adam-Moulton	**Adam-Bashforth**	**New Scheme**	**Exact**
0.0	1.0	1.0	1.0	1.0
0.4	1.7974438	1.7974422	1.7974422	1.7974425
0.5	2.0442397	2.0442356	2.0442372	2.0442376
6	2.3275082	2.3275022	2.3275049	2.3275054
	510854	2.6510804	2.6510812	2.6510819

Absolute Errors

x	Adam-Moulton	Adam-Bashforth	New-Scheme
0.0	0.0	0.0	0.0
0.4	1.30E-06	8.00E-07	3.00E-07
0.5	2.10E-06	2.00E-06	4.00E-07
0.6	2.80E-06	3.20E-06	5.00E-07
0.7	3.50E-06	1.50E-06	7.00E-07

It is quite evident from above, that the new scheme is by far more accurate than the Adam-Moulton method, and even the Adam-Bashforth method of same order. This is not surprising, for accuracy is the intended target of the new scheme.

So, whenever there is a requirement for high degree of accuracy, the new scheme would be better, both in terms of accuracy, and ease of use.

2 Comparison with the Classical Four-Stage Runge-Kutta Method.

A comparison will now be made between the result obtained by the new six-stage Runge-Kutta method for y_1, y_2, y_3, y_4 *and* y_5, and those obtained using the classical four-stage Runge-Kutta method of order four (see sec. 2.1.2).

The classical Runge-Kutta method was used to solve the IVP

$$y' = x + y, y(0) = 1, h = 0.1$$

A comparison of both methods is shown below

x	Classical R-K Method	New Scheme	Exact
0.0	1.0	1.0	1.0
0.1	1.1103417	1.1103418	1.1103418
0.2	1.2428051	1.242805437	1.2428055
0.3	1.3997169	1.3997175	1.3997176
0.4	1.5818943	1.5836492	1.5836494

Absolute Errors

x	Classical R-K Method	New-Scheme
0.0	0.0	0.0
0.1	1.00E-07	0.00E+00
0.2	4.00E-07	1.00E-07
0.3	7.00E-07	1.00E-07
).4	1.76E-03	2.00E-07

It is again quite glaring, that the new six-stage R-K scheme performs better than the classical Runge-Kutta method. However, this is also not surprising, since the classical Runge-Kutta scheme is of order four, while the new six-stage scheme, is of order five.

4.3 Comparison with Lawson's Six-Stage Method of order five

Lawson's six-stage Runge-Kutta method of order five is given as:

$$y_{n+1} = y_n + \frac{h}{90}[7k_1 + 32k_2 + 12k_3 + 32k_5 + 7k_6]$$

where

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$
$$k_3 = f[x_n + \frac{3}{4}h, y_n + \frac{1}{8}h(k_1 + k_2)]$$
$$k_4 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_3)$$
$$k_5 = f[x_n + \frac{1}{4}h, y_n + \frac{3}{16}h(-k_2 + 2k_3 + 3k_4)]$$
$$k_6 = f[x_n + h, y_n + \frac{1}{7}h(k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5)]$$

The new six-stage R-K scheme, also of order five, is given by:

$$y_{n+1} = y_n + \frac{h}{90}[7k_1 + 7k_2 + 12k_3 + 32k_5 + 32k_6]$$

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + h, y_n + hk_1)$$
$$k_3 = f[x_n + \frac{h}{2}, y_n + h(1.249655737k_1 - 0.749655737k_2)]$$
$$= f[x_n + \frac{h}{5}, y_n - h(0.7015442631k_1 - 0.5600588106k_2 - 0.341486157k_3)]$$
$$k_5 = f[x_n + \frac{h}{4}, y_n + h(0.25710705k_1 + 0.045073568k_2 + 0.353037791k_3 - 0.405218409k_4)]$$
$$k_6 = f[x_n + \frac{3h}{4}, y_n - h(0.754830268k_1 - 0.290909052k_2 - 0.331676697k_3 - 1.359792241k_4 + 0.477547722k_5)]$$

To compare both methods given above, we shall apply each, to solve our following problems, i.e.

(i) $y' = x + y, y(0) = 1, h = 0.1$

(ii) $y' = 2y + x^2, y(0) = 1, h = 0.1$

MS-Excel was used to solve the above problems, employing both schemes and the results can be seen below.

For the first problem, both schemes had errors but as can be seen, th new scheme was by far more accurate than Lawson's method in solving this problem. For the second problem, both schemes recorded a much higher degree of errors, but once again, the new scheme proved to be by far more accurate than Lawson's scheme. It must be noted that Lawson's scheme performed quite poorly in handling this problem.

Also, it is observed that the error appears to grow with each step, for both schemes. This propagation of error, is one of the disadvantages of the Runge-Kutta process; errors are not so easy to watch. So, this behavior of the errors is to be expected. It is also observed that the error propagation for the new scheme is far lower than that of Lawson's scheme.

PROBLEM : $y' = x + y$; $y(0) = 1$; $h = 0.1$

EXACT : $Y_E(x) = 2e^x - x - 1$

X	NEW SCHEME	LAWSON'S SCHEME	EXACT	NEW SCHEME ERROR	LAWSON'S SCHEME ERROR
0.0	1.000000000	1.000000000	1.000000000	0.000000000	0.000000000
0.1	1.110341796	1.110365767	1.110341836	4.0E-08	2.3931E-05
0.2	1.242805427	1.242858412	1.242805516	8.9E-08	5.2896E-05
0.3	1.399717467	1.399805304	1.399717615	1.48E-07	8.7689E-05
0.4	1.583649177	1.583778611	1.583649395	0.000000218	0.000129216
0.5	1.79744224	1.797621049	1.797442541	3.01E-07	0.000178508
0.6	2.044237201	2.04447434	2.044237601	4E-07	0.000236739
0.7	2.327504899	2.32781066	2.327505415	5.16E-07	0.000305245
0.8	2.651081205	2.651467399	2.651081857	6.52E-07	0.000385542
0.9	3.019205412	3.019685576	3.019206222	8.1E-07	0.000479354
1.0	3.436562662	3.437152291	3.436563657	9.95E-07	0.000588634
1.1	3.908330838	3.909047647	3.908332048	1.21E-06	0.000715599
1.2	4.440232387	4.441096606	4.440233845	1.458E-06	0.000862761
1.3	5.038591589	5.039626297	5.038593335	1.746E-06	0.001032962
1.4	5.710397855	5.711629355	5.710399934	2.079E-06	0.001229421
1.5	6.463375679	6.46483392	6.463378141	2.462E-06	0.001455779
1.6	7.306061947	7.307781002	7.306064849	2.902E-06	0.001716153
1.7	8.247891376	8.249909977	8.247894784	3.408E-06	0.002015193
1.8	9.299290941	9.301653083	9.299294929	3.988E-06	0.002358154
1.9	10.47178423	10.47453985	10.471788885	4.655E-06	0.002750964
2.0	11.77810678	11.78131252	11.7781122	5.418E-06	0.003200318
2.1	13.23233354	13.23605359	13.232339825	6.285E-06	0.003713763
2.2	14.85001972	14.85432681	14.850027	7.279E-06	0.004299811
	16.6483565	16.65333296	16.64836491	8.41E-06	0.004968054
	18.64634306	18.65208206	18.646352761	9.701E-06	0.0057293
	20.86497675	20.87158364	20.864987921	1.1171E-05	0.006595719
	23.32746323	23.33505708	23.32747607	1.284E-05	0.007581013
	26.05944871	26.06816405	26.05946345	1.474E-05	0.008700604
	29.08927665	29.09926539	29.089293542	1.6892E-05	0.009971844
	32.4482714	32.45970499	32.448290739	1.9339E-05	0.011414246
	36.17105174	36.1841236	36.171073846	2.2106E-05	0.013049753
	40.29587732	40.31080559	40.295902562	2.5242E-05	0.014903028
	44.8650316	44.88206218	44.865060394	2.8794E-05	0.017001787
	49.92524502	49.94465501	49.925277841	3.2821E-05	0.019377169
	55.52816272	55.55026424	55.528200094	3.7374E-05	0.022064145
	61.7308614	61.7560059	61.730903916	4.2516E-05	0.025101983
	68.59642056	68.62500365	68.596468886	4.8326E-05	0.028534763
3.7	76.19455383	76.22702068	76.194608719	5.4889E-05	0.032411959
3.8	84.60230668	84.63915807	84.602368985	6.2305E-05	0.036789082
3.9	93.90482754	93.94662661	93.904898209	7.0669E-05	0.041728402
4.0	104.19622	104.2435998	104.196300060	8.006E-05	0.047299763
4.1	115.5804845	115.6446749	115.580575190	9.069E-05	0.064099755
4.2	128.1725594	128.2554663	128.172662080	0.00010268	0.082804232
4.3	142.0994712	142.1926971	142.099587390	0.00011619	0.093109688
4.4	157.5016059	157.6064043	157.501737330	0.00013143	0.104666923
4.5	174.534114	174.6518879	174.534262600	0.0001486	0.117625282

PROBLEM : $y' = 2y + x^2$; $y(0) = 1$; $h = 0.1$

EXACT : $Y_E(x) = 5/4e^{2x} - (x^2/2 + x/2 + 1/4)$

X	NEW SCHEME	LAWSON'S SCHEME	EXACT	NEW SCHEME ERROR	LAWSON'S SCHEME ERROR
0.0	1.000000000	1.000000000	1.000000000	0.000000000	0.000000000
0.1	1.221752187	1.221857630	1.221753448	1.261E-06	0.000104182
0.2	1.494777731	1.495040791	1.494780872	3.141E-06	0.000259919
0.3	1.832642647	1.833133435	1.83264855	5.903E-06	0.000484885
0.4	2.251916487	2.252728436	2.251926161	9.674E-06	0.000802275
0.5	2.772837326	2.774094083	2.772852286	1.496E-05	0.001241797
0.6	3.42012398	3.421988207	3.420146153	0.00002217	0.001842054
0.7	4.223968049	4.226652638	4.223999999	3.195E-05	0.002652639
0.8	5.221245600	5.225027941	5.22129053	0.000044930	0.003737411
0.9	6.456997116	6.462237227	6.457059331	6.2215E-05	0.005177896
1.0	7.986235113	7.993398631	7.986320124	8.5011E-05	0.007078507
1.1	9.876151962	9.885839265	9.87626687	0.000114908	0.009572395
1.2	12.208816527	12.22179958	12.20897048	0.000153953	0.012829099
1.3	15.08445785	15.101736747	15.08467254	0.000214695	0.017064207
1.4	18.62553803	18.648359775	18.62580846	0.000270432	0.022551315
1.5	22.98156585	23.011558408	22.98192115	0.000355305	0.029637258
1.6	28.335198178	28.374423818	28.33566275	0.000464572	0.038761068
1.7	34.909520206	34.960602905	34.91012506	0.000604854	0.050477845
1.8	42.977008509	43.043281624	42.97779305	0.000784541	0.065488574
1.9	52.870466399	52.956158164	52.87148062	0.001014221	0.084677544
2.0	64.996380348	65.106846750	64.99768754	0.001307192	0.10915921
2.1	79.851233556	79.993250462	79.85299138	0.001757824	0.140259082
2.2	98.041431361	98.223560729	98.04358583	0.002154469	0.179974899
	120.307638171	120.540686822	120.3103946	0.002756429	0.230292222
	147.554502626	147.852096615	147.5580219	0.003519274	0.294074715
	180.886963957	181.266267250	180.8914489	0.004484943	0.37481835
	221.654596630	222.137209825	221.6603023	0.005705670	0.476907525
	271.505772961	272.118856553	271.5130203	0.007247339	0.605836253
	2.453817416	333.231494987	332.4630093	0.009191884	0.768485687
	6.957807576	407.942917798	406.9694499	0.011642324	0.973467898
	98.021264558	499.267547691	498.0359919	0.014727342	1.231555791
	609.312693644	610.887519048	609.3313014	0.018607756	1.556217648
	745.312812794	747.300579841	745.3362973	0.023484506	1.964282541
	911.494377774	914.000754656	911.5239866	0.029608826	2.476768056
	114.541820841	1117.699025595	1114.579115	0.037294159	3.119910595
	62.619517777	1366.592895252	1362.666448	0.046930223	3.926447252
	5.699449630	1670.695659405	1665.758455	0.059005370	4.937204405
	.961409307	2042.238615478	2036.035537	0.074127693	6.203078478
	.281814479	2496.162362450	2488.374869	0.093054521	7.78749345
	330744429	3050.716926471	3040.947472	0.116727571	9.769454471
	1161770	3728.194817697	3715.947484	0.146322230	12.2473337
	9584960	4555.826463360	4540.482884	0.183299040	15.34357936
	3957081	5566.873984328	5547.663435	0.229477919	19.21054933
	2370421	6801.967224936	6777.929489	0.287118579	24.03773594
	972716	8310.735872911	8280.675008	0.359035284	30.06086491
	187310	10153.802709752	10116.22991	0.448722690	37.57279975

CHAPTER FIVE

ERROR ESTIMATION

5.1 Error Estimation

One major flaw in the Runge-Kutta methods is that it is quite difficult and complicated to watch errors. According to Lambert [1973], "bounds for the local truncation error, do not form a suitable basis for monitoring the local truncation error, with a view to constructing a step-control policy similar to that developed for Predictor-Corrector methods. What is needed, in place of a bound, is a readily computable estimate of the local truncation error, similar to that obtained by Milne's device for predictor-corrector pairs."

The estimate used for the new scheme, arises from an application of the process of *deferred approach to the limit,* i.e. Richardson extrapolation. This involves solving the problem twice using step sizes h and 2h.

der the localizing assumption that no previous errors have been made, we may write:

$$x_{n+1}) - y_{n+1} = T_{n+1} = \varphi(x_n, y(x_n))h^{p+1} + o(h^{p+2}) \qquad (i)$$

where p is the order of the Runge-Kutta method (i.e. p=5 in this case), $\varphi(x_n, y(x_n))h^{p+1}$ is principal local truncation error. Next, we will compute y^*_{n+1}, a second approximation $_{+1})$, obtained by applying the same method at x_{n-1} with steplenght 2h. Under the alizing assumption, it follows that:

$$- y^*_{n+1} = \varphi(x_{n-1}, y(x_{n-1}))(2h)^{p+1} + o(h^{p+2})$$

and on expanding $\varphi(x_{n-1}, y(x_{n-1}))$ about $(x_n, y(x_n))$,

$$y(x_{n+1}) - y^*_{n+1} = \varphi(x_n, y(x_n))(2h)^{p+1} + o(h^{p+2}) \qquad (ii)$$

On subtracting (i) from (ii), we obtain

$$y(x_{n+1}) - y^*_{n+1} = (2^{p+1} - 1)\varphi(x_n, y(x_n))h^{p+1} + o(h^{p+2})$$

Therefore, the principal local truncation error which is taken as an estimate for the local truncation error may be written as:

$$\varphi(x_n, y(x_n))h^{p+1} = T_{n+1} = (y(x_{n+1}) - y^*_{n+1})/(2^{p+1} - 1) \qquad (iii)$$

$$\Rightarrow T_{n+1} = (y(x_{n+1}) - y^*_{n+1})/(2^{p+1} - 1) \qquad (iv)$$

Equation (iv), is a means of obtaining quick estimates of the error involved in computations using the new scheme, without having to obtain the exact solution first.

Thus, to obtain an error estimate we will compute over two successive steps using steplenght h (at x_n i.e. y_{n+1}) and then recomputed over the double step using steplenght 2h (at x_{n-1} i.e. y^*_{n+1}).

The difference between the values for y so obtained, divided by 63 (obtained by substituting p = 5 in Eq. (iv)), is then an estimate of the local truncation error.

We will illustrate this, by solving the differential equations:

(i) $y' = x + y; y(0) = 1$

(ii) $y' = -y; y(0) = 1$

at steplenghts h = 0.1, and h = 0.2

(i) $y' = x + y; y(0) = 1$

at h = 0.1, and h = 0.2

The approximate solutions are as shown below.

PROBLEM 1 : y'=x + y ; y(0)=1;h=0.1

EXACT : $Y_E = 2e^X - x - 1$

h	X	NEW SCEME	EXACT	ACTUAL ERROR
0.1	0.1	1.110341796	1.110341836	4.00E-08
	0.2	1.242805427	1.242805516	8.90E-08
	0.3	1.399717467	1.399717615	1.48E-07
	0.4	1.583649177	1.583649395	2.18E-07
	0.5	1.79744224	1.797442541	3.01E-07
	0.6	2.044237201	2.044237601	4.00E-07
	0.7	2.327504899	2.327505415	5.16E-07
	0.8	2.651081205	2.651081186	6.52E-07
	0.9	3.019205412	3.019206222	8.10E-07
	1.0	3.436562662	3.436563657	9.95E-07
0.2	0.2	1.242803057	1.242805516	2.46E-06
	0.4	1.583643388	1.583649395	6.01E-06
	0.6	2.044226595	2.044237601	1.10E-05
	0.8	2.651063934	2.651081857	1.79E-05
	1.0	3.436536293	3.436563657	2.74E-05

Once again, we cannot over emphasize the accuracy of the new scheme, as can be observed from the table above, with none of the errors being greater than 10^{-7}. This shows that the new scheme gives approximations that nearly exact.

From problem (i), as can be seen from above, we compute an approximate solution to the problem, using the new scheme, we also computed the exact solution, and hence, the actual error. This is rather tasking. So, as we have repeatedly stated, the purpose for this section, is to obtain a means by which an *estimate* for the error, can be conveniently computed, without having to go through the rigors of computing the exact solution.

Next, we will make use of equation (iv) to obtain error estimates that do not depend on the exact solutions.

Recall Equation (iv)

$$T_{n+1} = (y_{n+1} - y^*_{n+1})/(2^{p+1} - 1)$$

where: y_{n+1} is the approximate solutions at h = 0.1

y^*_{n+1} is the approximate solutions at h = 0.2

p is the order of our method i.e. p = 5

hence, equation (iv) becomes

$$T_{n+1} = (y_{n+1} - y^*_{n+1})/63 \qquad (v)$$

At x = 0.2 :

$$T_{n+1} = (1.242805427 - 1.242803057)/63 = 3.7619E-08$$

At x = 0.4 :

$$T_{n+1} = (1.583649177 - 1.583643388)/63 = 9.189E-08$$

At x = 0.6 : $T_{n+1} = (2.044237201 - 2.044226595)/63$
$= 1.684E - 07$

At x = 0.8 : $T_{n+1} = (2.651081205 - 2.651063934)/63$
$= 2.74E - 07$

At x = 1.0 : $T_{n+1} = (2.651081205 - 2.651063934)/63$
$= 4.186E - 07$

We will now compare our error estimates, with the actual errors previously computed above, to see if our estimates are viable or not.

Problem 1 : y' = x + y; y(0) = 1, h = 0.1, h = 0.2

x	Actual Error	Error Estimate
0.2	8.90E-08	3.76E-08
0.4	2.18E-07	9.19E-08
0.6	4.00E-07	1.68E-07
0.8	6.52E-07	2.74E-07
1.0	9.95E-07	4.19E-07

From the above, we can see that the order of our estimates compare favourably with that of the actual errors, being of the same orders of between 10^{-7} *and* 10^{-8}. Therefore, we can conclude that we do not ned to compute the exact solution before we can compute errors, our error estimator (Eq. (v)) is capable of giving us a workable idea of the nature and order of the errors.

One important observation from the results above, is that the actual errors appear to increase with an increase in steplenght. So, we can safely say that by reducing the steplenght h, accuracy can be increased.

5.3 SUMMARY AND CONCLUSION

In this work, we have been able to develop a new six-stage classical (explicit) Runge-Kutta classical Runge-Kutta method of maximum order i.e. of order five.

We have also demonstrated the efficacy of the new scheme, by engaging it in solving a number of differential equations (i.e. IVPs), and the new scheme has been seen to be quite efficient and highly accurate.

An error estimate was also derived for the new scheme using Richardson extrapolation with which one can obtain an error estimate for the scheme, without having to obtain the exact solution first.

As we have observed previously, having used the new scheme to solve various IVPs, and comparing its results with other methods (namely Adam-Bashforth, Adam-Moulton, Classical four-stage R-K method, and Lawson six-stage R-K method). Our goal of an exceptionally accurate scheme has been achieved. Also we can therefore conclude that the new scheme satisfies our goal of deriving a scheme that attains the maximum possible order for a six-stage R-K method.

5.4 RECOMMENDATIONS

Although the new scheme was successfully derived and tested, it is by no means perfect. For one, it can still be improved upon so as to give even better estimates to numerical solutions of IVPs. Also, the new scheme as it is now, may not be easy to manipulate manually because of its decimal coefficients in the k_s.

Though, an error estimate has been derived for the new scheme, it is not built into the scheme. So, the new scheme could be improved upon so as to have a better error handling capability.

It must have been observed, that the derivation of higher-order R-K methods (i.e. orders greater than four) using the technique employed in this work, is a process involving a large amount of tedious algebraic manipulation which is both time consuming and error prone. It is recommended that future research in this area should make use of *Computer Algebra*, this would solve the latter problem, but not the former, as finding higher-order methods involves solving larger and larger coupled systems of polynomial equations.

The best way, to avoid this problem of tedious algebraic manipulations, is to make use of a very elegant theory developed by Butcher (1987, 2003, also Lambert (1991)), which enables one to easily establish the conditions for a R-K method, either explicit or implicit, to have a given order. This theory is based on the algebraic concept of rooted trees.

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