APPLICATION OF THE IMPLICIT FUNCTION THEOREM TO THE STUDY OF ZEROS OF CHARACTERISTIC EQUATIONS

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CERTIFICATION

This thesis titled "APPLICATION OF THE IMPLICIT FUNCTION THEOREM TO THE STUDY OF ZEROS OF CHARACTERISTIC EQUATIONS" by Mathew Oladele Alemoru, meets the regulations governing the award of the degree of Masters of Technology in Mathematics. Federal University of Minna and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

The project is dedicated to the Almighty God who made it possible for me to attend this course and finish it successfully. It is also dedicated to my late Dad Mr. Stephen Alemoru and my Mum Madam Alice Alemoru.

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ABSTRACT

The aim of this work is to consolidate the application of the Implicit function theorem to a specific epidemiological problem (Yellow Fever Epidemics) whose basic equations satisfies the axioms under which such theorem can be applied.

Also, to determine the amount of resources consumed by a certain individual, we do make a tacit appeal to the above –mentioned theorem, which is thus applied in the field of economics theory. We have restricted ourselves to these two applications for the sake of brevity.

A word about the structure of the work. The first chapter contains enough notions to comprehend the remaining work, while chapter two and three supplies the necessary foundation that motivate and anticipates formulation of specific problems in chapter four and relates them to the principal theorems in chapter two and three where it is stated and proved rigorously. Chapter Five, the closing one is basically the interpretation of the result followed by concluding remarks.

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CHAPTER 1

SOME BASIC CONCEPTS

1.1. Introduction

The world is full of systems such as physical, chemical, biological or economic to name a few, which depend on one or many factors [10]. This relationship could either be linear or non-linear when considered in mathematical context.

In economics for example, the price of a commodity depends strongly on demand as well as cost of production. These various variables can be mathematically related together by functions. The subject of this work is centered around this notion of functions with emphasis on the Implicit Function, and the application is based on a special theorem namely the Implicit Function Theorem, which is a very important theorem in Mathematical Analysis.

In order to state and prove this main theorem in the next chapter, some basic concepts such as relations and functions will be useful in the sequel and we shall define them in this chapter.

The primary objective of this chapter is to introduce and define some classical terms and give a brief review of the Implicit Function Theorem, which is the key tool we use in this work.

1.2. Definition [3]

Let X and Y be any two non-empty sets. The Cartesian product $X \times Y$ of X and Y is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$.

1.3. Definition (Relation) [14]

A relation usually denoted by R is any subset of the Cartesian product $X \times Y$ of any two non-empty sets X and Y.

Note that if R is a relation on $X \times Y$ i.e. $R \subset X \times Y$, we say that R is a relation from X to Y. If $X = Y$, we simply say that R is a relation in X.

1.4. Definition

The domain of a relation R is the set Dom(R)

$$\text{Dom}(R) = \{ x : (x, y) \in R, \text{ for some } y\}$$

The range of R is the set

$$\text{Rng}(R) = \{ y : (x, y) \in R, \text{ for some } x\}$$

1.5. Definition (Function)

Let X and Y be two non-empty sets. A function is a rule that associates to each element $x \in X$ one and only one element $y \in Y$. In the notational form, we write

$$f : X \rightarrow Y$$

$$f(x) = y$$

Also Dom(f) $\subset X$ and Rng(f) $\subset Y$ and the terms transformation, operation and mapping are all synonymouswith function. Without any ambiguity and without loss of generality, we shall adopt the convention of using small letters f, g for functions, but sometimes F, G as the case may be.

There are in general many kinds of functions such as polynomials, trigonometric, exponential, logarithmic functions to name a few.

Some concepts such as one-to-one, onto, and composition of functions are of importance in the study of functions but we shall not dwell on them here. Our focus in this work being the application of the Implicit Function Theorem as stated in the title.

Next we shall define the inverse of a function and thereby introduce the Inverse Function Theorem which will help us to have a good understanding of the subject of this work.

1.6. Definition (Inverse Function)

Given a function $f(x)$, the function $g(x)$ is called the inverse function of $f(x)$ if it has the following properties.

(i) $f(g(x)) = g(f(x)) = x$

(ii) the domain of $g(x)$ is the range of $f(x)$

(iii) the range of $g(x)$ is the domain of $f(x)$

1.7. Remark

The Inverse Function Theorem states roughly in a layman's language that a continuously differentiable mapping f is invertible in a neighbourhood of any point x at which the linear function $f'(x)$ is invertible (i.e. capable of having an inverse).

1.8. Inverse Function Theorem

Suppose that f is a continuously differentiable mapping of a subset G of $\Re^n$ i.e. $G \subset \Re^n$, $f : G \to \Re^n$ with non-zero Jacobian at all points $x \in G$ and $f'(x)$ is invertible for some $x_0 \in G$, $y_0 = f(x_0)$. Then there exists neighbourhood $U = U(x_0)$ and $V = V(y_0)$ i.e.

(a) **There exists open sets U and V in $\Re^n$ such that $x_0 \in U$ and $y_0 \in V$, f is one-to-one on U and $f(U) = V$;**

(b) **If g is the inverse of f (which exists by (a), defined in V by $g(f(x)) = x$, $x \in U$, then $g \in C'(U)$, i.e g is continuously differentiable on V)**

If we write the equation $y = f(x)$ in the component form, we arrive at the following interpretation of the conclusion of the theorem:

The system of n equations

$$y_i = f_i(x_1, \ldots, x_n); \; 1 \le i \le n$$

can be solved for $x_1,...,x_n$ in terms of $y_1,....y_n$ if we restrict X and Y to small enough neighbourhoods of x and y; the solution are unique and continuously differentiable.

Although it is the statement of the inverse function theorem that was only supplied above, proof shall however be given in the next chapter also. With these notions in mind, we are now well equipped to introduce the idea of Implicit Function Theorem.

The Implicit Function Theorem as part of the bedrock of Mathematical Analysis and Geometry and Geometrical Analysis [8]. Its history is complex and is intimately bound up with the development of fundamental ideas in Analysis and Geometry. There are many forms of the theorem and they are stated below:

(i) The classical formulation

(ii) Formulation in other function spaces (abstract spaces)

(iii) Formulation for non-smooth functions and

(iv) Formulation for functions with degenerate Jacobian

Particularly, powerful Implicit Function Theorems, such as the Nash Moser Theorem (often referred to as the hard Implicit Function Theorem) have been

developed for specific applications (e.g. the imbedding of Riemannian manifolds) [15].

Inverse Function Theorem is a special case of the Implicit Function Theorem where the dimension of each variable is the same. In the next chapter we discuss the Implicit Function Theorem. In chapter 3, we introduce and prove a modified version of the theorem and then applied it to the study of the nature of the roots of characteristics equation arising from a mathematical model of yellow fever disease epidemics.

CHAPTER 2

THE IMPLICIT FUNCTION THEOREM

2.1 Introduction

Functions can primarily be classified as either explicit or implicit. In what follows, attempts shall be made to explain the idea of explicit function briefly before we dwell on implicit functions.

We speak of explicit functions whenever we are certain about the direction of correspondence between differet variables. Given two variables x and y, say, if the value of y varies in some definite form with respect to the value which is arbitrarily assigned to x, then the variable y is said to be an explicit function of x. When this happens the variable x is said to determine the variable y. For example the functions,

$$f(x) = y = 64x^3$$

$$y = x^2$$

and

$$y = \sin x$$

are all explicit functions [4].

On the other hand, an implicit relationship is said to characterize two or more variables whenever the direction of correspondence between them is not obvious.

Thus an implicit function exists between the variables x and y, say if the values that the two variables can take are interdependent or linked in some definite form. For instance, the algebraic process indicated by the following:

$$x^2 - y + 6 = 0 \qquad (2.1)$$

represents an implicit function of two variables x and y.

2.2 Implicit Functions

The function

$$x^2 y^2 + \sin xy = 0 \qquad (2.2)$$

represents an implicit function between x and y, because it is not obvious which f the two variables x and y is the dependent or independent variable.But there is a mutual relationship between x and y. Thus as equation (2.2) suggests, implicit functions usually takes the form whereby the functions are usually written with both variables on the same side of the equality sign while zero or a constant term which is not dependent on both x and y is on the other side. An example is the equation of a circle center origin, with radius a, its equation will be given by:

$$x^2 + y^2 = a^2$$

From the foregoing, from an equation of the form

$$F(x, y) = 0 \qquad (2.3)$$

we can determine y as a function of x. If that happens, we have

$$y = g(x)$$

is non-zero. Note that the determinant of a square matrix $A = [a_{i,j}]$ is denoted by *det* A or *det* $[a_{i,j}]$. If *det* $[a_{i,j}] \neq 0$, the solution of (2.5) above can be obtained by the Cramer's rule which expresses each x_k as a quotient of two determinants, say $x_k = A_k / D$ where $D =$ *det* $[a_{i,j}]$ and A_k is the determinant of the matrix obtained by replacing the k^{th} column of $[a_{i,j}]$ by $t_1, t_2, \ldots t_n$ [7].

In particular, if each $t_i = 0$, $i = 1, 2, \ldots, n$, then each $x_k = 0$ which is the trivial solution. Now the system (2.5) can be written in the form of (2.3). So each equation in (2.5) then takes the form:

$$f_i(X,t) = 0 \text{ where } X = (x_1, x_2, \ldots, x_n)$$

and

$$f_i(X,t) = \sum_{j=1}^{n} a_{i,j} x_j - t_i \qquad \textbf{(2.6)}$$

Therefore the system in (2.5) can be expressed as one vector equation $f(X,t) = 0$ Where $f = (f_1, f_2, \ldots f_n)$. If $D_j f_i$ denote the partial derivative of f_i with respect to the j^{th} coordinate x_j then $D_j f_i(X,t) = a_{i,j}$ Thus the coefficient matrix $A = [a_{i,j}]$ in the system (2.5) is a Jacobian matrix. As noted earlier, linear algebra instructs us that the system (2.5) has a unique solution if the determinant of this Jacobian matrix

is non-zero.

In the general Implicit Function Theorem, the non-vanishing of the determinant of a Jacobian matrix also plays an important role. This comes about by approximating *f* by a linear function. The equation *f(X, t) = 0* gets replaced by a system of linear equations whose coefficient matrix is the Jacobian matrix of *f*.

2.3 Notation and Definition (Jacobian)

If $f = (f_1, f_2, ... f_n)$ **and** $X = (x_1, x_2 .... x_n)$**, the Jacobian matrix** $Df(X) = \left[D_j f_i(X)\right]$ **is an n x n matrix where as noted earlier** $D_j f_i$ **denotes the partial derivative of** f_i **with respect to the** j^{th} **coordinate** x_j **and that** $D_j f_i(X, t) = a_{ij}$**. The determinant of the Jacobian matrix is called a Jacobian determinant and is denoted by** $J_f(X)$**. Thus**

$$J_f(X) = \det Df(X) = \det\left[D_j f_i(X)\right] \tag{2.7}$$

The notation

$$\frac{\partial(f_1, ... f_n)}{\partial(x_1, ... x_n)} \tag{2.8}$$

Is also used to denote the Jacobian determinant $J_f(X)$ **i.e.**

$$J_f(X) = \det Df(X) = \det\left[D_j f_i(X)\right] = \frac{\partial(f_1, ... f_n)}{\partial(x_1, ... x_n)} \tag{2.9}$$

or

$$J_f(X) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & & . & . & \frac{\partial f_n}{\partial x_1} \\ . & \frac{\partial f_2}{\partial x_2} & . & . & . \\ . & . & . & . & . \\ . & . & . & . & \\ \frac{\partial f_1}{\partial x_n} & & . & . & \frac{\partial f_n}{\partial x_n} \end{vmatrix} \tag{2.10}$$

Our aim in this chapter is to study the Implicit Function Theorem and some examples where it applies. To do this, we first introduce the notion of implicit differentiation.

2.4 Implicit Differentiation

Assume $f : \Omega \subset \Re^n \to \Re^n$ is differentiable, and $y = f(x) \in C^r(\Omega)$ and consider

$$y + \ln y = x + \ln x \tag{2.11}$$

We say that the equation (2.11) implicitly defines y as a function of x. By differentiating each side, i.e. obtain after re-arranging

$$\frac{dy}{dx} = \frac{1 + \frac{1}{x}}{1 + \frac{1}{y}} \tag{2.12}$$

Now, suppose that we have an equation of the form

$$F(x, y) = 0 \tag{2.13}$$

Where x is an exogenous variable (independent) and y an endogenous (dependent) variable, An illustrative example is given by:

$$F(x, y) = y + \ln y - x - \ln x = 0 \tag{2.14}$$

Taking the total differential,

$$dF = F_x dx + F_y dy \tag{2.15}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} \tag{2.16}$$

provided that $F_y \neq 0$. **In the special case when *F(x, y)* is given by (2.14),**

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{1+\frac{1}{x}}{1+\frac{1}{y}} \tag{2.17}$$

Thus the result achieved by implicit differentiation could equally well have been achieved by total differential. At times rather than there being a single exogenous variable x, there may be many. Suppose

$$F(x_1, x_2, \ldots\ldots, x_n, y) = 0 \tag{2.18}$$

Where $x_1, x_2, \ldots\ldots, x_n$ are exogenous variables and y is once more an endogenous variable. Taking the total differential of equation (2.18) gives:

$$dF = \frac{\partial F}{\partial x_1} dx_1 + ... + \frac{\partial F}{\partial x_n} dx_n + \frac{\partial F}{\partial y} dy = 0 \qquad (2.19)$$

At times we may be interested in the comparative static effects of how an endogenous variable y changes as an exogenous variable x_k say, changes and all other x's are kept constant. In this case

$$\left. \begin{aligned} & dF = \frac{\partial F}{\partial x_k} dx_k + \frac{\partial F}{\partial y} dy = 0 \\ & \Rightarrow \frac{\partial y}{\partial x_k} = -\frac{\dfrac{\partial F}{\partial x_k}}{\dfrac{\partial F}{\partial y}} = -\frac{F_{x_k}}{F_y} \end{aligned} \right\} \qquad (2.20)$$

provided

$$F_y \neq 0 \qquad (2.21).$$

It is important to note the use of the partial derivative symbol on the left hand side of the equation (2.21). This is because we are finding the rate of change of y with respect to x_k when all other exogenous variables other than x_k are kept constant. By contrast, in (2.16), we wrote $\frac{dy}{dx}$ because y is a function of a single variable x in this case. Equation (2.6) in the case of single exogenous variable or equation (2.9) in the case of many exogenous variable are referred to as the "Implicit Function Rule".

The following examples will illustrate the above discussion very vividly. Let

$$F(x,y) = x^2 + y^2 - 9 = 0 \tag{2.22}$$

This is the circle with radius 3 center at the origin *(0, 0)*. This implicitly defines *y* as a function of *x* is obviously incorrect because the value of *y* that satisfies (2.22) is not a function of *x*. However, the relation defined by (2.22) can be split into two separate parts both of which involve *y* being defined as a function of *x*, that is

$$y = \pm\sqrt{9 - x^2}$$

The two separate parts are

$$y = \sqrt{9 - x^2} \tag{2.23}$$

and

$$y = -\sqrt{9 - x^2} \tag{2.24}$$

Equation (2.23) being the top half of the circle and (2.24) the lower half.

From (2.23)

$$\frac{dy}{dx} = \frac{-x}{\sqrt{9 - x^2}} \tag{2.25}$$

While (2.24) gives:

$$\frac{dy}{dx} = \frac{x}{\sqrt{9 - x^2}} \tag{2.26}$$

Using the Implicit Function rule, we take the total differential of (2.22)

to obtain

$$dF = 2xdx + 2ydy = 0 \tag{2.27}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}; \quad y \neq 0 \tag{2.28}$$

By considering equation (2.25) and (2.26) we can write:

when

$$y = \sqrt{9 - x^2}, \ \frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{9 - x^2}}$$

and when

$$y = -\sqrt{9 - x^2}, \ \frac{dy}{dx} = -\frac{x}{y} = \frac{x}{\sqrt{9 - x^2}}$$

Thus the equation (2.28) which is the Implicit Function rule captures both cases simultaneously.

2.5 Simultaneous Equation Rule of the Implicit Function Rule

So far we have only considered in the previous section one single equation for the implicit function and a single endogenous (independent) variable.

Suppose we have a system of well defined n equations in x endogenous variables and m exogenous (dependent) variables.

$$\left.\begin{aligned} F^1(x_1,\ldots,x_n,\alpha_1,\ldots,\alpha_m) &= 0 \\ &\;\;\vdots \\ F^n(x_1,\ldots,x_n,\alpha_1,\ldots,\alpha_m) &= 0 \end{aligned}\right\} \qquad (2.29)$$

Taking total differentials, we have

$$\left.\begin{aligned} &\frac{\partial F^1}{\partial x_1}dx_1+\ldots+\frac{\partial F^1}{\partial x_n}dx_n+\frac{\partial F^1}{\partial \alpha_1}d\alpha_1+\ldots+\frac{\partial F^1}{\partial \alpha_m}d\alpha_m \\ &\qquad\qquad\vdots \\ &\frac{\partial F^n}{\partial x_1}dx_1+\ldots+\frac{\partial F^n}{\partial x_n}dx_n+\frac{\partial F^n}{\partial \alpha_1}d\alpha_1+\ldots+\frac{\partial F^n}{\partial \alpha_m}d\alpha_m \end{aligned}\right\} \qquad (2.30)$$

or equivalently

$$\begin{pmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{pmatrix} \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} = \begin{pmatrix} \frac{\partial F^1}{\partial \alpha_1}d\alpha_1 & \cdots & \frac{\partial F^1}{\partial \alpha_m}d\alpha_m \\ \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial \alpha_1}d\alpha_1 & \cdots & \frac{\partial F^n}{\partial \alpha_m}d\alpha_m \end{pmatrix} \qquad (2.31)$$

Equation (2.31) is in the form $Ax = b$ and the Crmer's rule can be used to solve for provided $|A| \neq 0$, that is

$$\begin{vmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{vmatrix} \neq 0 \qquad (2.32)$$

This determinant is referred to as the Jacobian determinant. A unique solution exists for $dx_1,\dots,dx_n$ if $|A| \neq 0$ and the equations are said to be consistent; and if $|A| = 0$ they are inconsistent as the equations have infinitely many solutions.

If only one of the α terms changes, say α_j we write

$$\begin{pmatrix} \frac{\partial F^1}{\partial x_1} & \cdots & \frac{\partial F^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x_1} & \cdots & \frac{\partial F^n}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial \alpha_j} \\ \vdots \\ \frac{\partial x_n}{\partial \alpha_j} \end{pmatrix} = \begin{pmatrix} \frac{\partial F^1}{\partial \alpha_j} \\ \vdots \\ \frac{\partial F^n}{\partial \alpha_j} \end{pmatrix} \qquad (2.33)$$

We can solve for $\frac{\partial xi}{\partial \alpha_j}$ terms using the Cramer's rule, provided the Jacobian determinant is non-zero. Equation (2.33) is sometimes referred to as the "simultaneous version of the Implicit Function Rule."

In general it is more convenient to work with equation (2.33) and for this reason,

it seems normal to work with the total differential variant.

In order to forge ahead, we need some definitions and basic theorems before stating our main theorem, which will aid in the understanding of the latter.

2.6 Definition

Assume that $f:\Omega\subset\Re^n\rightarrow\Re^n$ is differentiable at *c*. The Jacobian of *f* at *c* is

$$J_f(c)=\frac{\partial(f_1,\ldots.f_n)}{\partial(x_1,\ldots.x_n)}(c)= det\begin{bmatrix}\frac{\partial f_1}{\partial x_1}(c) & \cdots & \frac{\partial f_1}{\partial x_n}(c)\\ \vdots & \cdots & \vdots\\ \vdots & \cdots & \vdots\\ \frac{\partial f_n}{\partial x_1}(c) & \cdots & \frac{\partial f_n}{\partial x_n}(c)\end{bmatrix} \quad \textbf{(2.34)}$$

A function $f:\Omega\subset\Re^n\rightarrow\Re^n$ is locally 1 – 1 on $D\subset\Omega$ if for every $p\in D$, there exists $\varepsilon>0$ such that *f* is 1 – 1 on $B(p,\varepsilon)$ (the ball center *p* radius ε). The next theorem proves this. We first state without proof a result which is used thereafter.

2.7 Mean Value Theorem [11]

Let *f* be a function such that:

(i) it is continuous in the closed interval [*a*, *b*];

(ii) it is differentiable in the open interval (*a*, *b*);

Then there is a number *c* in the open interval (*a*, *b*) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{2.35}$$

2.8 Lemma

Assume that $f:\Omega \subset \Re^n \to \Re^n$ **is in** $C^n(\Omega)$ **and that** $J_f(p) \neq 0$. **Then there is an** $\varepsilon > 0$ **such that** f **is 1 - 1 on** $B(p, \varepsilon)$.

Proof

Since the partials are continuous at p **and** $J_f(p) \neq 0$ **we can find** $\varepsilon > 0$ **such that if** $p_1, \ldots, p_n \in B(p, \varepsilon)$, **then**

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1}(p_1) & \cdots & \frac{\partial f_1}{\partial x_n}(p_1) \\ \frac{\partial f_2}{\partial x_1}(p_2) & \cdots & \frac{\partial f_2}{\partial x_n}(p_2) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(p_n) & \cdots & \frac{\partial f_n}{\partial x_n}(p_n) \end{vmatrix} \neq 0 \tag{2.36}$$

Let $a \neq b \in B(p, \varepsilon)$ **applying the mean value theorem to each component function** f_i **to get a point** p_i **on the line segment joining** a **and** b **we have**

$$f_i(b) - f_i(a) = \left(\frac{\partial f_i}{\partial x_1}(p_i), \frac{\partial f_i}{\partial x_2}(p_i), \ldots, \frac{\partial f_i}{\partial x_n}(p_i) \right) . (b - a) \tag{2.37}$$

and assuming $f(b) = f(a)$ **we have**

$$
0 = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(p_1) & . & . & . & \frac{\partial f_1}{\partial x_n}(p_1) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{\partial f_n}{\partial x_1}(p_n) & . & . & . & \frac{\partial f_m}{\partial x_n}(p_m) \end{vmatrix} .(b-a) \qquad \textbf{(2.38}
$$

which is impossible.

2.9 Theorem (Injective Mapping Theorem)

Assume that $f:\Omega\subset\Re^n \to \Re^n$ is $C^n(\Omega)$ and that rank $\left(f'(p)\right) = n$ for every $p \in \Omega$.

Then is locally 1 – 1 on Ω.

Proof

Let $p_0 \in \Omega$. Since rank $\left(f'(p)\right) = n \le m$, we can assume by re-arranging the component functions if necessary that

$$
\frac{\partial(f_1,\ldots.f_n)}{\partial(x_1,\ldots.x_n)}(p_0) \neq 0 \qquad \textbf{(2.39)}
$$

Let $f:\Omega\subset\Re^n \to \Re^n$ be defined by $f = (f_1,\ldots.f_n)$ It follows from the previous lemma that there is $\varepsilon > 0$ such that f is 1 – 1 on $B(p_0,\varepsilon)$ Hence, it is clear that f is also 1 – 1 on $B(p_0,\varepsilon)$. Next, we consider an example on the above notions.

2.10 Example

Let $f(x,y)=(e^x\cos y, e^x\sin y)$. **Then**

$$J_f(x,y)=\begin{vmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{vmatrix}=e^{2x}\neq 0$$

It follows that ***f*** **is locally 1 – 1 on** $\Re^2$. **However** ***f*** **is not globally 1 – 1 on** $\Re^2$ **since** $f(x,y)=f(x,y-2\pi)$.

2.11 Lemma

Let $f:\Omega\subset\Re^n\rightarrow\Re^n$ **and assume that** $\Omega\subset\Re^n$ **is open,** $f\in C'(\Omega)$ **and** $J_f(p)\neq 0$ **for every** $p\in\Omega$. **Then** $f(\Omega)$ **is open.**

Proof

Let $q_0=f(p_0)\in f(\Omega)$ **Then,** $\exists \quad >0 \ni \overline{B}(p_0,\delta)\in\Omega$ **and** ***f*** **is 1 – 1 on** $\overline{B}(p_0,\delta)$.

Let $C=\{p\in\Omega\ni\|p-p_0\|=\delta\}$ be the boundary of $\overline{B}(p_0,\delta)$ **where** $\|.\|$ **denotes the usual** $\Re^n$ **norm. Then** ***f(c)*** **is compact and** $q_0=f(p_0)\notin f(c)$. **Let**

$r=\min\{\|q-q_0\|\ni q\in f(c)\}$, $q_1\in B(q_0,\frac{r}{3})$ **and define** $\varphi:\overline{B}(p_0,\delta)\rightarrow\Re$ **by**

$$\varphi(p)=\|f(p)-q_1\|^2 \qquad \textbf{(2.40)}$$

Since φ **is continuous on ,** $\overline{B}(p_0,\delta)$ **there exists a** p^* **such that** $\varphi(p^*)\leq\varphi(p)$. **Now if**

$p^* \in C$, then $\sqrt{\varphi(p^*)} \le \frac{2}{3}r \ge \frac{1}{3}r > \sqrt{\varphi(p_0)}$ **which is impossible. Thus,** $p^* \notin C$, so p^* **is a local minimum for** φ **and hence** $\nabla\varphi(p^*) = 0$. **If** $p = (x_1,....x_n)$, **and** $q = (y_1,....y_n)$ **then**

$$\varphi(p) = \sum_{i=1}^{n} \left(f_i(p) - y_i\right)^2 \qquad \textbf{(2.41)}$$

Hence

$$0 = 2\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(p^*)\left(f_i(p^*) - y_i\right);\ j = 1,2....n \qquad \textbf{(2.42)}$$

Since

$$\left|\frac{\partial f_i}{\partial x_j}(p^*)\right| \neq 0;\ f_i(p^*) = y_i \qquad \textbf{(2.43)}$$

and $f(p^*) = q_i$. **Thus** $\overline{B}(q_0, \frac{r}{3}) \subset f(\Omega)$.

2.12 Theorem (Open Mapping Theorem)

Let $f:\Re^n \to \Re^m$. **Assume that** $\Omega \subset \Re^n$ **is open, that** $f \in C'(\Omega)$ **and that rank** $\left(f'(p)\right) = m$ **for every** $p \in \Omega$ **Then** ***f(D)*** **is open.**

Proof

Let $c = (c_1,....c_n) \in \Omega$ **We may assume that**

$$\frac{\partial(f_1,....,f_m)}{\partial(x_1,....,x_m)}(c) \neq 0 \qquad \textbf{(2.44)}$$

and since $f \in C'(\Omega)$, we can find an open neighbourhood U of c such that

$$\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_m)}(p) \neq 0 \qquad (2.45)$$

for each $p \in U$. We let $g:\Re^n \to \Re^m$ be defined by

$$g(x_1,\ldots x_m) = f(x_1,\ldots x_m, c_{m+1},\ldots c_n) \qquad (2.46)$$

on the open set

$$V = \{(x_1,\ldots x_m) | (x_1,\ldots x_m, c_{m+1}\ldots c_n) \in U\} \subset \Re^m \qquad (2.47)$$

Now $g \in C(V)$ and $J_g(x_1,\ldots x_m) \neq 0$ for any $(x_1,\ldots x_m) \in V$. The previous lemma shows that $g(V)$ is open. Also $f(c) \in g(V) \subset f(U)$. This shows that $f(c) \in f(U)^0$, i.e. the interior of $f(U)$.

2.13 Lemma

Assume that $L:\Re^n \to \Re^m$ is linear and that *ker* $L = \{0\}$ Then there exists an $M > 0$ such that

$$\|L(u)\| \geq M\|u\| \qquad (2.48)$$

for all $u \in \Re^n$.

Proof

Let $S = \{u \in \Re^n : \|u\| = 1\}$ Then *L(S)* is compact and does not contain *0*. Let

$M = \min\{\|L(u)\| : u \in S\}$.

2.14 Lemma

Let $f:\Re^n \to \Re^m$ If f is C^1 at p_0 and if $J_f(p_0) \neq 0$ then there exists an open neighbourhood U of p_0 and an $m > 0$ such that if $p \in U$

$$\|f(p) - f(p_0)\| \geq m\|p - p_0\| \qquad \textbf{(2.49)}$$

Proof

Since $J_f(p_0) \neq 0$, $Df(p_0):\Re^n \to \Re^n$ is 1 – 1 By the previous lemma we can find an $M > 0$ such that $\|Df(p_0)\| \geq M\|u\|$. Let $m = \dfrac{M}{2}$.

By the definition of differentiability there exists an open set U containing p_0 such that if $p \in U$

$$\|f(p) - f(p_0) - Df(p_0)(p - p_0)\| \geq m\|p - p_0\| \qquad \textbf{(2.50)}$$

But

$$\|f(p) - f(p_0) - Df(p_0)(p - p_0)\| \geq Df(p_0)\|p - p_0\| - \|f(p) - f(p_0)\|$$

$$\geq 2m\|p - p_0\| - \|f(p) - f(p_0)\| \qquad \textbf{(2.51)}$$

This can only happen if

$$\|f(p) - f(p_0)\| \geq m\|p - p_0\| \qquad \textbf{(2.52)}$$

Next we state and prove a special case of the Inverse Function Theorem, where the dimensions of the variables are the same.

2.15 Theorem (Inverse Mapping Theorem)

Let $f:\Omega\subset \Re^n \to \Re^n$. Suppose that $f \in C^1(\Omega)$ and that $J_f(p_0) \neq 0$ Then there exists an open neighbourhood *U* of p_0 on which *f* is invertible. Moreover f^{-1} is differentiable at $f(p_0)$ with

$$Df^{-1}(p_0) = Df(p_0)^{-1} \tag{2.53}$$

Proof

It follows from the Injective Mapping Theorem, the Open Mapping Theorem and lemma (2.13) that there exists an open neighbourhood *U* of p_0 such that *f* is 1 – 1 on *U*, *W = f(U)* is open and

$$\|f(p) - f(p_0)\| \geq m\|p - p_0\| \text{ for all } p \in U \tag{2.54}$$

Let $g:\Re^n \to \Re^n$ be defined by

$$g(p) = \begin{cases} \dfrac{1}{\|p - p_0\|}\|f(p) - f(p_0) - Df(p_0)(p - p_0)\|; & p \neq p_0, p \in U \\ 0 & p = p_0 \end{cases} \tag{2.55}$$

Since $J_f(p_0) \neq 0$, $(Df(p_0))^{-1}$ exists and

$$\|p-p_0\|(Df(p_0))^{-1}(g(p)) = (Df(p_0))^{-1}(f(p)-f(p_0)-Df(p_0)(p-p_0))$$

$$= (Df(p_0))^{-1}(f(p)-f(p_0)-(p-p_0) \qquad \textbf{(2.56)}$$

Let $q_0 = f(p_0)$ and $q = f(p)$.**Then**

$$\frac{1}{m}\|q-q_0\|\|(Df(p_0))^{-1}(g(p))\| \geq \|f^{-1}(q)-f^{-1}(q_0)-(Df(p_0))^{-1}(q-q_0)\| \quad \textbf{(2.57)}$$

Now, we can find an $\varepsilon > 0$ **such that** $B(p_0,\varepsilon) \subset U$. **As** ***f*** **is continuous on** $B(p_0,\varepsilon)$,

f^{-1} **is continuous at** q_0.

It follows that

$$\lim_{q \to q_0} \frac{\|f^{-1}(q)-f^{-1}(q_0)-Df(p_0)^{-1}(q-q_0)\|}{\|q-q_0\|} = 0 \qquad \textbf{(2.58)}$$

Since as $q \to q_0$, $p \to p_0$ **and** ***g*** **is continuous at** p_0.

2.16 Example

Let $f(x,y) = (e^x \cos y,\ e^x \sin y)$. **Then as above**

$$J_f(x,y) = \det\begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} = e^{2x} \neq 0$$

It follows that ***f*** **is locally invertible on** $\Re^2$ **and**

$$|Df(x,y)|^{-1} = e^{-2x}\begin{bmatrix} e^x \cos y & e^x \sin y \\ -e^x \sin y & e^x \cos y \end{bmatrix}$$

Observe that ***F(x, y)*** **is 1 – 1 on the strip** $0 \leq y < 2\pi$. **If we let**

$(u,v) = (e^x \cos y, e^x \sin y)$ **on this strip, then we can explicitly solve for** u **and** v**. In particular**

$$u^2 + v^2 = e^{2x}$$

$$\tan y = \frac{v}{u}$$

Consequently,

$$x = \frac{1}{2}\ln(u^2 + v^2)$$

$$y = \tan^{-1}\left(\frac{u}{v}\right)$$

We now differentiate to get that

$$Df^{-1}(u,v) = \begin{bmatrix} \frac{u}{u^2+v^2} & \frac{v}{u^2+v^2} \\ -\frac{v}{u^2+v^2} & \frac{u}{u^2+v^2} \end{bmatrix}$$

$$= \frac{1}{u^2+v^2}\begin{bmatrix} u & v \\ -v & u \end{bmatrix}$$

$$= e^{2x}\begin{bmatrix} e^x \cos y & e^x \sin y \\ -e^x \sin y & e^x \cos y \end{bmatrix}$$

as expected.

2.17 Remarks (Implicit Function Theorem)

Assume that we are given an equation in two variables of the form

$$f(x, y) = 0$$

And a point (x_0, y_0) such that $f(x_0, y_0) = 0$. We say that we can solve for y in terms of x in a neighbourhood U of x_0 if there exists a function $\varphi : U \rightarrow \Re$ such that $\varphi(x_0) = y_0$ and $f(x, \varphi(x)) = 0$. The function φ is said to be implicitly defined by f.

We can use the chain rule to obtain

$$\varphi'(x) = \frac{dy}{dx} = \frac{-\dfrac{\partial f}{\partial x}(x,y)}{\dfrac{\partial f}{\partial y}(x,y)} \qquad \textbf{(2.60)}$$

provided that $\frac{\partial f}{\partial y}(x,y) \neq 0$ The previous question : Given f and (x_0, y_0) when does such a function exist? The answer to this will arise from the Implicit Function Theorem. It is a remarkable consequence of the theorem that under the mild assumption that $f(x, y)$ is continuously differentiable on a neighbourhood of (x_0, y_0) the fact that $\frac{\partial f}{\partial y}(x,y) \neq 0$ is sufficient to guarantee the existence of such a φ

Moreover φ is actually uniquely defined on a neighbourhood of x_0.

2.18 Theorem (Implicit Function Theorem)

Let $f = (f_1,\ldots\ldots,f_n)$, $p = (x_1,\ldots\ldots,x_m)$, $q = (y_1,\ldots\ldots,y_n)$. **Suppose that** $f:\Omega \subseteq \Re^{n+m} \to \Re^n$ **is such that** $f \in C^1(\Omega)$ **on the open set** Ω, **that** $(p_0, q_0) \in \Omega$ **is such that** $f(p_0, q_0)$ **and**

$$\frac{\partial(f_1,\ldots\ldots,f_n)}{\partial(y_1,\ldots\ldots,y_n)}(p_0,q_0) \neq 0 \qquad \textbf{(2.61)}$$

Then there exists a unique function $\varphi:\Re^m \to \Re^n$ **defined in a neighbourhood** ***U*** **of** p_0 **with** $\varphi \in C^1(U)$ **and**

(i) $\varphi(p_0) = q_0$

(ii) $f(p,\varphi(p)) = 0$ **for each** $p \in U$

Moreover, if $y_i = \varphi_i(p)$ **then**

$$\left[\frac{\partial f_i}{\partial x_j}\right] = -\left[\frac{\partial f_i}{\partial y_j}\right]\left[\frac{\partial \varphi_i}{\partial x_j}\right] \qquad \textbf{(2.62)}$$

In particular

$$\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial y_i}{\partial x_j} = \frac{\dfrac{\partial(f_1,\ldots\ldots,f_n)}{\partial(x_1,\ldots\ldots,x_m,y_1,\ldots\ldots,y_n)}}{\dfrac{\partial(f_1,\ldots\ldots,f_n)}{\partial(y_1,\ldots\ldots,y_n)}} \qquad \textbf{(2.63)}$$

Proof

Let $F:\Re^{n+m} \to \Re^{n+m}$ **be defined by**

$$F(p,q) = (p, f(p,q))$$

$$= \left(x_1,\ldots,x_m, f_1(x_1,\ldots,x_m,y_1,\ldots,y_n),\ldots\ldots, f_m(x_1,\ldots\ldots,x_m,y_1,\ldots,y_n)\right)$$

$$= \left(x_1,\ldots,x_m, f_1(x_1,\ldots,x_m,y_1,\ldots,y_n),\ldots\ldots, f_n(x_1,\ldots\ldots,x_m,y_1,\ldots,y_n)\right) \qquad \textbf{(2.64)}$$

Then $F \in C^1(\Omega)$ and

$$J_F(p_0,q_0) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \dfrac{\partial f_1}{\partial x_1} & \cdot & \cdot & \cdot & \dfrac{\partial f_1}{\partial x_m} & \dfrac{\partial f_1}{\partial y_1} & \cdot & \cdot & \cdot & \dfrac{\partial f_1}{\partial y_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial f_n}{\partial x_1} & \cdot & \cdot & \cdot & \dfrac{\partial f_n}{\partial x_m} & \dfrac{\partial f_n}{\partial y_1} & \cdot & \cdot & \cdot & \dfrac{\partial f_n}{\partial y_n} \end{bmatrix}_{(p_0,q_0)}$$

$$= \frac{\partial(f_1,\ldots,f_n)}{\partial(y_1,\ldots,y_n)}(p_0,q_0) \neq 0 \qquad \textbf{(2.65)}$$

It follows from the Inverse Mapping Theorem that *F* has an inverse in an open neighbourhood *W* of (p_0,q_0) with *F(W)* an open neighbourhood of $F(p_0,q_0)$ and $F^{-1} \in C^{-1}(F(W))$ where $F^{-1}(.)$ is a hypersurface in a neighbourhood of(p_0,q_0) [7].

Now

$$F(p,q) = (p, f(p,q)) = (r.s) \qquad \textbf{(2.66)}$$

implies that

$$(p,q) = F^{-1}(r,s) = (r,\theta(r,s)) \qquad (2.67)$$

Hence

$$(r,s) = F(F^{-1}(r,s)) = F(r,\theta(r,s)) = (r,f(r,\theta(r,s))) \qquad (2.68)$$

for all *(r, s)* in *F(W)*. This shows that

$$s = f(r,\theta(r,s)). \qquad (2.69)$$

and thus

$$0 = f(p,\theta(p,0)) \qquad (2.70)$$

for all p in a neighbourhood U of p_0 We now let $\varphi(p) = \theta(p,0)$ for $p \in U$

Note that

$$\varphi(p_0) = \theta(p_0,0) = q_0 \qquad (2.71)$$

Since $F^{-1} \in C^{-1}(F(W))$ it is clear that $\varphi \in C^{-1}(U)$.

If we apply the chain rule to

$$0 = f(p,\varphi(p)) \qquad (2.72)$$

we get

$$0 = \frac{\partial f_i}{\partial x_i} + \sum_{k=1}^{n} \frac{\partial f_i}{\partial y_k} \cdot \frac{\partial \varphi_k}{\partial x_i} \qquad (2.73)$$

To solve the system we note that

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_i} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f_n}{\partial x_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial \varphi_i}{\partial x_i} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial \varphi_k}{\partial x_i} \end{bmatrix} \tag{2.74}$$

By the Cramer's rule, we obtain

$$\frac{\partial \varphi_k}{\partial x_i} = \frac{\det \begin{bmatrix} \frac{\partial f_i}{\partial y_1} & \cdots & \frac{\partial f_i}{\partial y_{i-1}} & -\frac{\partial f_i}{\partial y_k} & \frac{\partial f_i}{\partial y_{i+1}} & \cdots & \frac{\partial f_i}{\partial y_n} \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{\partial f_i}{\partial y_k} & \cdots & \frac{\partial f_i}{\partial y_k} & -\frac{\partial f_i}{\partial y_k} & \frac{\partial f_i}{\partial y_k} & \cdots & \frac{\partial f_i}{\partial y_k} \end{bmatrix}}{\frac{\partial (f_1, \ldots, f_n)}{\partial (y_1, \ldots y_n)}}$$

$$= \frac{\dfrac{\partial (f_1, \ldots, f_n)}{\partial (y_1, \ldots y_{i-1}, y_i, y_{i+1}, \ldots y_n)}}{\dfrac{\partial (f_1, \ldots, f_n)}{\partial (y_1, \ldots y_n)}} \tag{2.75}$$

Finally, φ is unique in the sense that if

$$f(p, \varphi_1(p)) = f(p, \varphi_2(p)) = 0 \tag{2.76}$$

for some function φ_2 that is continuously differentiable in a neighbourhood of p_0 with $\varphi_2(p_0) = q_0$ for p near sufficiently to p_0 so

$$F^{-1}(p, \varphi_1(p)) = F^{-1}(p, \varphi_2(p)) = (p, 0)$$

$F^{-1}(p, \varphi_1(p)) = F^{-1}(p, \varphi_2(p)) = (p, 0)$ **(2.77)**

However, since F^{-1} is 1 – 1, $\varphi_1(p) = \varphi_2(p)$.

2.19 Remarks

The fundamental question that the Implicit Function Theorem attempts to supply answer to is : when is the relation defined by $f(x, y) = 0$ also a function? In other words, when can the equation $f(x, y) = 0$ be solved explicitly for y in terms of x, yielding a unique solution? The Implicit Function Theorem deals with this question locally. It tells us that, given a point (x_0, y_0) such that $f(x_0, y_0) = 0$ under certain conditions there will be a neighbourhood of (x_0, y_0) such that in this neighbourhood the relation defined by $f(x, y) = 0$ is also a function.

2.20 Corllary

If $f : \Re^2 \to \Re$ is C^1 on a neighbourhood of (x_0, y_0) and $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ then the equation $f(x, y) = 0$ has a unique solution $y = \varphi(x_0) = y_0$ which exists and is continuously differentiable in a neighbourhood U of x_0 with

$$\varphi'(x) = \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}(x,y)}{\frac{\partial f}{\partial y}(x,y)} \qquad (2.78)$$

2.21 One – Dimensional Case

When $n = m = 1$ the theorem reduces to: Let F be a continuously differentiable, real valued function defined on an open set $E \subset \Re^2$ and let (x_0, y_0) be a point on E for which $F(x_0, y_0) = 0$ and such that

$$\left.\frac{\partial F}{\partial y}\right|_{(x_0, y_0)} \neq 0 \qquad (2.79)$$

Then there exists an open interval I containing y_0 and a unique function $g: I \to \Re$ which is continuously differentiable and such that $g(y_0) = x_0$ and $F(g(y), y) = 0$ for all $y \in I$.

2.22 Example

Show that in a neighbourhood of the point *(1, -1)* the equation

$$x^3 + y^3 - 3x + 3y = -6$$

defines y uniquely as a function $\varphi(x)$ of x and find $\varphi'(x)$ and $\varphi'(1)$.

Solution

We let $f(x,y) = x^3 + y^3 - 3x + 3y + 6$. Since f is clearly C^1 and since *f(1, -1) = 0*, the

Implicit Function Theorem guarantees the existence of φ provided that

$$\frac{\partial f}{\partial y}(1,-1) \neq 0$$

But $\frac{\partial f}{\partial y}(1,-1) = 3y^2 + 3 = 3(-1)^2 + 3 = 6 \neq 0$, **we have that**

$$\varphi'(x) = \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}(x,y)}{\frac{\partial f}{\partial y}(x,y)}$$

$$= \frac{-(3x^2 - 3)}{3y^2 + 3}$$

and

$$\varphi'(1) = -\frac{\frac{\partial f}{\partial x}(1,-1)}{\frac{\partial f}{\partial y}(1,-1)} = \frac{-(3(1)^2 - 3)}{3(-1)^2 + 3} = \frac{0}{6} = 0$$

2.23 Corollary

If $f:\Re^3 \to \Re$ is C^1 in a neighbourhood of (x_0, y_0, z_0), $f(x_0, y_0, z_0) = 0$ and $\frac{\partial f}{\partial z}(x_0, y_0, z_0) \neq 0$ then the equation *f(x, y, z) = 0* has a unique solution $\varphi(x, y) = z$, $\varphi(x_0, y_0) = z_0$ which exists and is continuously differentiable in a neighbourhood *U* of (x_0, y_0) with

$$\frac{\partial \varphi}{\partial x} = \frac{\partial z}{\partial x} = \frac{-\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial z}}$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial z}{\partial y} = \frac{-\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial z}}$$

2.24 Theorem

The Implicit Function Theorem can be stated as: Let $f = (f_1, \ldots. f_n)$ be a vector valued function defined on an open set S in $\Re^{n+k}$ with values in $\Re^n$. Suppose $f \in C^1$ Let (X_0, t_0) be a point in S for which $f(X_0, t_0) = 0$ and for which the n x n determinant $\left[D_j f_i(X_0, t_0)\right] \neq 0$.

Then there exists a k – dimensional open set T_0 containing t_0 and one and only one vector valued function g defined on T_0 and having values on $\Re^n$ such that

(a) $g \in C^1$ **on** T_0

(b) $g(T_0) = t_0$

(c) $f(g(t), t) = 0$ **for every t on** T_0.

Proof

The Inverse Function Theorem shall be applied to a certain vector valued function

$$F = (F_1, \ldots F_n, F_{n+1}, \ldots F_{n+k}) \qquad \textbf{(2.80)}$$

defined on *S* and having values in $\Re^{n+k}$. The function *F* is defined as follows: For $\leq m \leq n$, let $F_m(X,t) = f_m(X,t)$ and for $\leq m \leq k$, $F_{n+m}(X,t) = t_m$ We can write

$$F = (f, I) \qquad \textbf{(2.81)}$$

where $f = (f_1, \ldots f_n)$ and when *I* is the identity function defined by *I(t) = t* for each *t* in $\Re^k$. The Jacobian $J_F(X,t)$ then has the same value as the n x n determinant $\det\left[D_j f_i(X,t)\right]$ because the terms which appear in the last *k* rows and also in the last *k* columns of $J_F(X,t)$ form a *k* x *k* determinant with ones along the main diagonal and zeros elsewhere; the intersection of the first n rows and n columns consists of the determinant $\det\left[D_j f_i(X,t)\right]$ and

$$\left[D_i f_{n+j}(X,t)\right] = 0 \text{ for } \leq i \leq n, \ 1 \leq j \leq k$$

Hence the Jacobian $J_F(X_0, t_0) \neq 0$ and also $F(X_0, t_0) = (0, t_0)$.

Therefore by the Inverse Function theorem, there exists open sets *X* and *Y* containing (X_0, t_0) and $(0, t_0)$ respectively, such that *F* is 1 – 1 on *X*; and $X = F^{-1}(Y)$. Also there exists a local inverse function *G*, defined on *Y* and having values on *X*, such that

$$G(F(X, t)) = (X, t) \qquad \textbf{(2.82)}$$

and such that $G \in C^1$ on *Y*. Now *G* can be reduced to components as follows:

***G = (V, W)* where $= (V_1,\ldots,V_n)$ is a vector valued function defined on *Y* with values in $\Re^n$ and $= (w_1,\ldots,w_k)$ is also defined on *Y* but has values in $\Re^k$. We can now determine *V* and *W* explicitly.**

The equation *G(F(X, t)) = (X, t)*, when written in terms of the components *V* and *W* gives the two equations:

$$V(F(X,t)) = X \quad \text{and} \quad W(F(X,t)) = t \qquad \textbf{(2.83)}$$

But now, every point *(X, t)* in *Y* can be written uniquely in the form $(X,t) = F(X',t')$ for some (X',t') in *X* and the inverse image $F^{-1}(Y)$ contains *X*. Furthermore, by the manner in which *F* was defined, when we write $(X,t) = F(X',t')$ we must have $t'=t$. Therefore,

$$(X,t) = V(F(X,t)) = X' \qquad \textbf{(2.84)}$$

and

$$(X,t) = W(F(X',t')) = t' \qquad \textbf{(2.85)}$$

Hence the function *G* can be described as follows: Given a point *(X, t)* in *Y*, we have

$$G(X,t) = (X',t) \qquad \textbf{(2.86)}$$

where X' is a point in $\Re^n$ such that

$$(X,t) = F(X',t) \qquad \textbf{(2.87)}$$

This statement implies that:

$$F(V(X, t), t) = (X, t) \text{ in } Y \quad (2.88)$$

Now, if we define the set T_0 and the function g in the theorem as follows: Let

$$T_0 = \{t \in \Re^k : (0, t) \in Y\} \quad (2.89)$$

and for each t in T_0 define $g(t) = V(0, t)$. The set T_0 is open in $\Re^k$ Moreover, $g \in C^1$ on T_0 because $G \in C^1$ on Y and the components of g are taken from the components of G. Also

$$g(t_0) = V(0, t_0) = X_0 \quad (2.90)$$

because $(0, t_0) = F(X_0, t_0)$. Finally, the equation $F(V(X, t), t) = (X, t)$, which holds for every (X, t) in Y yields (by considering the components in $\Re^n$) the equation

$$f(V(X, t), t) = X \quad (2.91)$$

Taking $X = 0$, we see that for every t in T_0 we have $f(g(t), t) = 0$, and this completes the proof of statements (a), (b) and (c). It remains to prove the uniqueness of the function g. This follows from the 1 – 1 property of f. If there were another function, say h, which satisfies (c), then we have

$$(g(t), t) = (h(t), t), \text{ or } g(t) = h(t) \quad (2.92)$$

for every t in T_0.

2.25 Remarks

From the above theorem, the Implicit Function Theorem takes the form: Suppose that X and Y are subsets of the real line $\Re$ let $x_0 \in X$, $y_0 \in Y$ and let (x_0, y_0) be an interior point of the plane set X x Y; if F is continuous in some neighbourhood of (x_0, y_0) if $F(x_0, y_0) = 0$ and if there exist $\delta > 0$ and $\varepsilon > 0$ such that $F(x, y)$ for any fixed $x \in (x_0 - \delta, x_0 + \delta)$ is strictly monotone on $(y_0 - \varepsilon, y_0 + \varepsilon)$ as a function of y, then there is a $\delta_0 > 0$ such that there is a unique function

$$f:(x_0 - \delta_0, x_0 + \delta_0) \to (y_0 - \varepsilon, y_0 + \varepsilon) \qquad \textbf{(2.93)}$$

for which

$$F(x, f(x)) = 0 \qquad \textbf{(2.94)}$$

For all $x \in (x_0 - \delta_0, x_0 + \delta_0)$. Moreover, f is continuous and $f(x_0) = y_0$.

The hypothesis of this theorem are satisfied if F is continuous in a neighbourhood of (x_0, y_0) the partial derivative F_y exists and is continuous at (x_0, y_0); $F(x_0, y_0) = 0$ and $F_y(x_0, y_0) \neq 0$. If in addition the partial derivative F_x exists and is continuous at (x_0, y_0) then the implicit function f is differentiable at x_0 and

$$\frac{df(x_0)}{dx} = -\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)} \qquad \textbf{(2.95)}$$

This theorem has been generalized to the case of a system of equations, when F is a

vector – valued function. Let $\Re^n$ and $\Re^m$ be n- and m- dimensional Euclidean spaces with fixed coordinate systems and points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ respectively. Suppose that F maps a certain neighbourhood W of $(x_0, y_0) \in \Re^n \times \Re^m$ into $\Re^m$ and that $F_i, i = 1, \ldots, m$ are the coordinate functions (of the $n + m$ variables $x_1, \ldots, x_n, y_1, \ldots, y_m$) of F, that is $F = (F_1, \ldots, F_m)$. If F is differentiable on W, if $F(x_0, y_0) = 0$ and if the Jacobian:

$$\left.\frac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)}\right|_{(x_0, y_0)} \neq 0 \qquad \textbf{(2.96)}$$

then there are neighbourhoods U and V of $x_0 \in \Re^n$ and $y_0 \in \Re^m$ respectively, $U \times V \subset W$ and a unique mapping $f : U \to V$ such that $F(x, f(x)) = 0 \in \Re^m$ for all $x \in U$. Here $f(x_0) = y_0$, f is differentiable on U and if $f = (f_1, \ldots, f_m)$ then the explicit expression for the partialderivatives $\frac{\partial f_j}{\partial x_i}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ can be found from the system of m linear equations in these derivatives:

$$\frac{\partial F_k}{\partial x_i} + \sum_{j=1}^{m} \frac{\partial F_k}{\partial y_j} \frac{\partial f_j}{\partial x_i} = 0 \qquad \textbf{(2.97)}$$

$k = 1, \ldots, m$, $i =$ is fixed ($i = 1, \ldots, n$).

CHAPTER 3

MODIFIED VERSION OF IMPLICIT FUNCTION THEOREM

In this chapter we introduce the modified version of the implicit function theorem

which is used in chapter four to discuss the roots of the characteristics equation of a population model.

3.1 Theorem

Let $F(y,x)=0$ **be an implicit function such that**

(i) $x \geq 0$

(ii) $F(0,0)=0$

(iii) $F_y(0,0)=0$ **and** $F_x(0,0)=0$ **have the same sign with the former strictly non-vanishing.**

(iv) $y_x(0) = -\dfrac{F_x(0,0)}{F_y(0,0)} < 0$

Then $y=\phi(x)$ **exists in the neighbourhood of** $(x,y)=(0,0)$ **such that** $y<0$.

Proof

With assumptions (i), (ii) and (iii) the Implicit Function Theorem guarantees the existence of a unique solution $y=\phi(x)$ **in the neighbourhood of the origin.**

Let us assume that (iv) exists, i.e.

$$\frac{dy}{dx} = -\frac{F_x(0,0)}{F_y(0,0)} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}}$$

where $\dfrac{\partial F}{\partial y} \neq 0$

Let $y = f(x)$ **be a continuous explicit solution of the implicit function** $F(x,y) = 0$ **in**

the

neighbourhood of the origin. Then for any $x \in (-\delta, \delta)$, **where** $\delta > 0$, **there holds the**

identity $F(x, f(x)) \equiv 0$. **Hence**

$$\frac{\partial F(x, f(x))}{\partial x} = 0$$

for any point in this neighbourhood. By chain rule of differentiating a composite

function, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

hence

$$\frac{\partial F}{\partial x} = -\frac{\partial F}{\partial y}\frac{dy}{dx}$$

and

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}\frac{dy}{dx}} \neq 0$$

3.2 Remark

In the neighbourhood of the origin, if then the graph of the function lies below the x-axis that is and the gradient of the function is negative, because the function is a decreasing function, that is slanting from top left to right as shown below in figure 3.1.

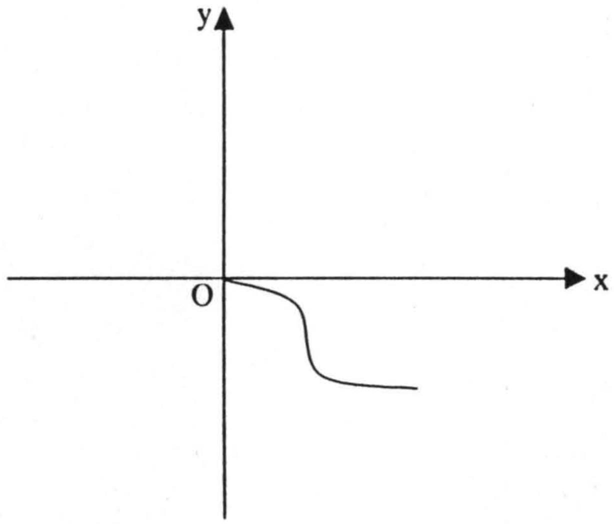

Fig. (3.1)

CHAPTER 4

APPLICATIONS OF THE IMPLICIT FUNCTION THEOREM

4.1 Yellow Fever Disease Dynamics Model

4.1.1 Introduction

Two of the most important problems faced by people studying mathematical models are numerical approximation of steady states and parameter identification. Both of these problems lead to large non-linear systems of equations that can sometimes be difficult to solve analytically. While there is an abundance of software available to tackle these types of problems, there are often multiple solutions, and many of these can be difficult to find without the aid of a careful strategy. One helpful strategy is to identify a salient parameter in the model and use the Implicit Function Theorem [12], in a modified version.

In the last chapter, we discussed the Inverse and Implicit Function Theorems as a motivation towards the applications presented in this chapter. In this section, we introduce a model of Yellow Fever Dynamics as proposed by Akinwande [1], and applied the modified version of the Implicit Function Theorem in chapter three to study the characteristics equation arising from the model with a view to analyse the stability or otherwise of the steady state of the model.

This model involves the interaction of two principal communities of hosts (human) and vectors (aedes egypti mosquitoes). The host community is partitioned

into three compartments of susceptibles class *S(t)*, Infected class *I(t)* and the Recovered or Immune *R(t)*. While the vector community is partitioned into two compartments of Non-Virus Carrying class *N(t)* and Virus Carrying class *M(t)* where $t \geq 0$ is the time.

The dynamics involve biting interactions between S and M, if effectivewill cause members of the class *S(t)* to move or flow into *I(t)*. Similarly, an effective biting interactions between *I(t)* and *N(t)* will cause the members of *N(t)* to flow into *M(t)* Here, effectiveness in the biting interactions is when there is a transfer of virus from the host to the vector or from the vector to the host.

It is assumed that the offsprings of the hosts' population are born immune, as they enjoy natural immunity for a period of one year. The transmission of virus among the vectors can however be vertical, so it is assumed that a proportion of the offsprings of the vectors are carriers from egg stage.

The model equations are therefore given by equations (4.1) to (4.5) below:

$$\frac{dS}{dt} = \beta_1 (S + I + R) - (\mu_1 + \gamma) S - \alpha_1 MS \qquad (4.1)$$

$$\frac{dR}{dt} = -\mu_1 R + \gamma S + \alpha I \qquad (4.2)$$

$$\frac{dI}{dt} = -(\mu_1 + \alpha + \delta) I + \alpha_1 MS \qquad (4.3)$$

$$\frac{dN}{dt} = \beta_2(N + (1-\theta)M) - \mu_2 N - \alpha_2 NI \qquad (4.4)$$

$$\frac{dM}{dt} = \theta\beta_2 M - \mu_2 M + \alpha_2 NI \qquad (4.5)$$

with the initial conditions

$$S(0) = S_0;\ R(0) = R_0;\ I(0) = I_0;\ N(0) = N_0;\ M(0) = M_0 \qquad (4.6)$$

The parameters are defined as follow:

β_1 = **natural birthrate for hosts**

β_2 = **natural birthrate for the vector**

μ_1 = **natural mortality rate for host**

μ_2 = **natural mortality rate for vector**

α = **recovery rate for infected host**

α_1 = **effective biting interaction rate between S and M compartments**

α_2 = **effective biting interaction rate between N and I compartments**

δ = **death rate among the infected host arising from infection**

θ = **proportion of the offsprings of M that get infected vertically**

γ = **immunization rate among host community**

4.1.2 Normalisation

Let

$$P = S + I + R;\; Q = N + M;\; P(0) = P_0;\; Q(0) = Q_0 \qquad (4.7)$$

To normalize the variables, we let

$$X = \frac{S}{P};\; Y = \frac{I}{P};\; Z = \frac{M}{Q} \qquad (4.8)$$

These transform equations (4.1) – (4.5) to of the system of equations (4.9) – (4.11) below:

$$\frac{dX}{dt} = \beta_1 - (\mu_1 + \gamma)X + \delta XY - \alpha_1 XM \qquad (4.9)$$

$$\frac{dY}{dt} = -(\alpha + \delta + \beta_1)Y + \alpha_1 XM \qquad (4.10)$$

$$\frac{dZ}{dt} = -\beta_2(1-\theta)Z + \alpha_2(1-Z)Y \qquad (4.11)$$

4.1.3 Steady States of the Model Equation

At steady state, let

$$(X(t), Y(t), Z(t)) = (x, y, z) \qquad (4.12)$$

with P and Q constants, then we obtain the following from (4.9) – (4.11):

$$\beta_1 - (\mu_1 + \gamma)x + \delta xy - \alpha_1 xzQ_0 = 0 \qquad \textbf{(4.13a)}$$

$$-(\alpha + \delta + \beta_1)y + \alpha_1 xzQ_0 = 0 \qquad \textbf{(4.13b)}$$

$$-\beta_2(1-\theta)z + \alpha_2(1-z)yP_0 = 0 \qquad \textbf{(4.13c)}$$

4.1.4 Definition (Steady or Equilibrium State)

The steady or equilibrium state of a mathematical model is the observable state as the system evolves. At steady state, the rate of change of the state with respect to time is set to zero in the continuous system as we are discussing in this section. In the discrete model, the steady state occurs when the variables are set to constants.

4.1.5 Definition (Characteristics Equation)

The characteristics equation of a mathematical model is the equation arising from the determinant Jacobian equation

$$|A - \lambda I| = 0 \qquad \textbf{(4.14)}$$

where A is the n-square matrix of the coefficients, λ is the eigenvalue and I is the n x n identity matrix.

4.1.6 Application of the Modified Implicit Function Theorem

In the above equations, x, y, z are the values of the variables at equilibrium state, $\beta_1, \beta_2, \mu_1, \mu_2, \alpha_1, \alpha_2, \theta, \delta, \alpha$ are non-negative parameters defined earlier in the

section.

A solutions of the simultaneous equations (4.13) is given by

$$(x,y,z)=\left(\frac{\beta_1}{\alpha_1+\gamma},0,0\right) \quad \textbf{(4.15)}$$

and the associated characteristics equation is given by

$$\lambda^2+\left(\alpha+\delta+\beta_1+\beta_2(1-\theta)\right)\lambda+\beta_2(1-\theta)\left\{\alpha+\delta-\frac{\alpha_1\alpha_2P_0Q_0\beta_1}{\alpha_1+\gamma}\right\} \quad \textbf{(4.16)}$$

For the purpose of our application, we shall use the variables λ, the eigenvalue and set $\omega=\beta_2(1-\theta)$ and then consider the Implicit Function

$$F(\lambda,\omega)=\lambda^2+\left(\alpha+\delta+\beta_1+\omega\right)\lambda+\omega\left\{\alpha+\delta-\frac{\alpha_1\alpha_2P_0Q_0\beta_1}{\alpha_1+\gamma}\right\}=0 \quad \textbf{(4.17)}$$

For $(\lambda,\omega)=(0,0)$;

$$F(0,0)=0 \quad \textbf{(4.18a)}$$

$$F_\lambda(\lambda,\omega)=\frac{\partial F}{\partial\lambda}(\lambda,\omega)\ ;\quad F_\omega(\lambda,\omega)=\frac{\partial F}{\partial\omega}(\lambda,\omega) \quad \textbf{(4.18b)}$$

and so

$$F_\lambda(\lambda,\omega)=2\lambda+\left(\alpha+\delta+\beta_1+\omega\right) \quad \textbf{(4.19a)}$$

$$F_\lambda(0,0)=\alpha+\delta+\beta_1>0 \quad \textbf{(4.19b)}$$

as α, δ, β_1 are non-negative parameters.

Also

$$F_{\omega}(\lambda,\omega) = \lambda + \alpha + \delta - \frac{\alpha_1 \alpha_2 P_0 Q_0 \beta_1}{\alpha_1 + \gamma} \qquad \textbf{(4.20)}$$

$$F_{\omega}(0,0) = \alpha + \delta - \frac{\alpha_1 \alpha_2 P_0 Q_0 \beta_1}{\alpha_1 + \gamma} \qquad \textbf{(4.21)}$$

So using the midified Implicit Function, we have that : $\lambda = \phi(\omega)$ exists in the neighbourhood of the origin $(\lambda,\omega) = (0,0)$, with $\lambda < 0$ i.e. the steady state given by equation (4.15) is locally stable.

Another non-trivial solution (steady state) for the normalized system is given by

$$(x, y, z) = (f(y), y, g(y)) \qquad \textbf{(4.22)}$$

and the corresponding characteristics equation is thus given by

$$\left(-\beta_1 - \gamma + \delta y - \alpha_1 Q_0 g(y) - \lambda\right)\left[\left(-\alpha - \delta - \beta_1 + 2\delta y - \lambda\right)\right.$$

$$\left.\left(-\beta_2(1-\theta) - \alpha_2 y P_0 - \lambda\right) - \alpha_1 \alpha_2 P_0 Q_0 f(y)(1 - g(y))\right]$$

$$-\alpha_1 Q_0 g(y)\left[\delta f(y)\left(-\beta_2(1-\theta) - \alpha_2 y P_0 - \lambda\right) + \alpha_1 \alpha_2 P_0 Q_0 f(y)(1 - g(y))\right] = 0 \qquad \textbf{(4.23)}$$

Using λ and α_1 as salient parameters and the modified Implicit Function Theorem, we have that

$$F(\lambda,\alpha_1) = \left(-\beta_1 - \gamma + \delta y - \alpha_1 Q_0 g(y) - \lambda\right)\left[\left(-\alpha - \delta - \beta_1 + 2\delta y - \lambda\right)\right.$$

$$\left.\left(-\beta_2(1-\theta) - \alpha_2 y P_0 - \lambda\right) - \alpha_1\alpha_2 P_0 Q_0 f(y)(1-g(y))\right]$$

$$-\alpha_1 Q_0 g(y)\left[\delta f(y)\left(-\beta_2(1-\theta) - \alpha_2 y P_0 - \lambda\right) + \alpha_1\alpha_2 P_0 Q_0 f(y)(1-g(y))\right] = 0 \quad \textbf{(4.24)}$$

noting that for $-\alpha - \delta - \beta_1 + 2\delta y = 0$;

$$y = \frac{\alpha + \delta + \beta_1}{2\delta} \qquad \textbf{(4.25)}$$

We note that $F(0,0) = 0$.

Differentiating partially with respect to λ **and we obtain**

$$F_\lambda(\lambda,\omega) = -\left[-\lambda(\beta_2(1-\theta) - \alpha_2 y P_0 - \lambda) - \alpha_1\alpha_2 P_0 Q_0 f(y)(1-g(y))\right]$$

$$+\left(-\beta_1 - \gamma + \delta y - \alpha_1 Q_0 g(y) - \lambda\right)\left(-\lambda(\beta_2(1-\theta) - \alpha_2 y P_0 - \lambda + 1)\right)$$

$$= \lambda(\beta_2(1-\theta) - \alpha_2 y P_0 - \lambda) + \alpha_1\alpha_2 P_0 Q_0 f(y)(1-g(y)$$

$$+\left(-\beta_1 - \gamma + \delta y - \alpha_1 Q_0 g(y) - \lambda\right)\left(-\lambda(\beta_2(1-\theta) - \alpha_2 y P_0 - \lambda + 1)\right) \quad \textbf{(4.26)}$$

and

$$F_{\alpha_1}(\lambda,\omega) = \left(-\beta_1 - \gamma + \delta y - \alpha_1 Q_0 g(y) - \lambda\right).\left(\alpha_2 P_0 Q_0 f(y)(1-g(y))\right)$$

$$-Q_0 g(y)\left[-\lambda(\beta_2(1-\theta) - \alpha_2 y P_0 - \lambda) - \alpha_1\alpha_2 P_0 Q_0 f(y)(1-g(y))\right] \quad \textbf{(4.27)}$$

Hence

$$F_\lambda(0,0) = \left(-\beta_1 - \gamma + \delta y\right).\left(\beta_2(1-\theta) - \alpha_2 y P_0\right) \qquad \textbf{(4.28)}$$

and

$$F_{\alpha_1}(0,0) = \left(-\beta_1 - \gamma + \delta y - \alpha_1 Q_0 g(y) - \lambda\right).\left(\alpha_2 P_0 Q_0 f(y)(1 - g(y))\right)$$

$$+\delta Q_0 f(y) g(y)\left(\beta_2(1-\theta) - \alpha_2 y P_0\right) \qquad \textbf{(4.29)}$$

thus if

$$-\beta_1 - \gamma + \delta y > 0 \qquad \textbf{(4.30)}$$

then $F_\lambda > 0$ **and** $F_{\alpha_1} < 0$ **so the non-trivial equilibrium state is unstable.**

If $-\beta_1 - \gamma + \delta y < 0$ **then** $F_\lambda < 0$ **and** $F_{\alpha_1} > 0$ **so the non-trivial equilibrium state is yet unstable.**

4.2 Economics Model

4.2.1 The Model

Assume a consumer consumes varying amounts of two products ***X*** **and** ***Y*** **[6]**

Let X **= units of** ***A*** **consumed**

Y **= units of** ***B*** **consumed**

The consumer operates his utility by consuming these goods in accordance with the following utility function:

$$U = f(X, Y) \qquad \textbf{(4.32)}$$

And his purchasing decision is guided and limited by a budget constraint given by

$$P_B B + P_R R = I \qquad \textbf{(4.33)}$$

where P_i is the price of the product i and I is his total income. From (4.33) we obtain

$$R = \frac{-P_B B}{P_R} + \frac{I}{P_R} \tag{4.34}$$

The theory of utility maximization which states that the slope of the indifference curve equals the slope of the budget constaint, gives:

$$\frac{MU_B}{MU_R} = \frac{P_B}{P_R} \tag{4.35}$$

where MU_i = marginal utility of goods i which is the derivative of the utility function with respect to goods i. So we obtain

$$\frac{\partial U(B,R)}{\partial B} dB + \frac{\partial U(B,R)}{\partial R} dR = dK \tag{4.36}$$

where K is a non-negative constant and equals to some overall level of utility and.

$$\frac{\partial U(B,R)}{\partial B} + \frac{\partial U(B,R)}{\partial R} \cdot \frac{dR}{dB} = \frac{dK}{dB} \tag{4.37}$$

and since $\frac{dK}{dB} = 0$ we simplify to get

$$\frac{\partial U}{\partial B} + \frac{\partial U}{\partial R} \cdot \frac{dR}{dB} = 0 \tag{4.38}$$

and solving for $\frac{dR}{dB}$ gives

$$\frac{dR}{dB} = -\frac{\frac{\partial U}{\partial B}}{\frac{\partial U}{\partial R}} \quad ; \quad \frac{\partial U}{\partial R} \neq 0 \qquad \textbf{(4.39)}$$

4.2.2 Example

Let

$$U = \theta + \alpha \ln B + (1-\alpha)\ln R \qquad \textbf{(4.40)}$$

Differentiating gives

$$-\frac{\frac{\alpha}{B}}{\left(\frac{1-\alpha}{R}\right)} = \frac{P_B}{P_R} \qquad \textbf{(4.41)}$$

At equilibrium, B and R are given by

$$B^{*} = \frac{\alpha I}{P_B}, \quad R^{*} = \frac{(1-\alpha)I}{P_R} \qquad \textbf{(4.42)}$$

If for example $\alpha = 0.5$, $P_B = \$4$, $\theta = 100$, $P_R = \$2$, $I = \$200$, **give the values:**

$B^{*} = 25$ **and** $R^{*} = 50$.

4.3 Remarks

1. **The Implicit Function theorem as modified in this work can be utilized to determine the nature of the roots of a characteristics equation in a specified**

neighbourhood.

2. **The limitation of the theorem in the present state is that only one parameter can be considered at a time, also any parameter used will not feature in any constaint obtained in the analysis.**
3. **The theorem however helps in the analysis of the stability or otherwise of the steady state.**
4. **It will be of interest to develop the theorem to accommodate the use of more parameters simultaneously.**

CHAPTER 5

RESULTS AND CONCLUSION

5.1 Interpretation of the Result

The application of the Modified Implicit Function Theorem to the characteristics equations of the Mathematical Model of Yellow Fever Dynamics leads to the inequality

$$\frac{1}{2}(\alpha + \delta - \beta_1) < \gamma \qquad (5.1)$$

From this inequality, we observe that keeping the immunization rate slightly above the value $\frac{1}{2}(\alpha + \delta - \beta_1)$ could prevent the outbreak of epidemics.

We have seen therefore that this theorem is a powerful tool in the study of characteristics equations and should be explored further.

In the Economics model, the marginal utility of goods i can be expressed as the first derivative of the utility function taken with respect to the goods. The indifference curves slope is equal to the ratio of the marginal utilities. The equilibrium occurs when the slope of this curve is equal to the slope of the budget constraint.

5.2 Concluding Remarks

In this work, we have found that the Implicit Function Theorem can be a very useful tool in the analysis of certain complicated functions like the characteristics equation and therefore recommend further study of this theorem, especially the modified version used in this work [1].

It will be of great interest if the further study of the Implicit Function Theorem can take care of more than one parameter than it is in this work.

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