THE DYNAMIC RESPONSE OF A FLAT PLATE UNDER A MOVING LOAD BY FINITE ELEMENT METHOD

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CERTIFFICATION

This thesis titled: *THE DYNAMIC RESPONSE OF A FLATPLATE UNDER A MOVING LOAD BY FINITE ELEMENT METHOD* by Baba Galadima Agaie (M – Tech/SSSE/2003/2004/973) meets the regulations governing the award of the degree of (M – Tech.) of the Federal University of technology, Minna and is approved for its contribution to scientific knowledge and literary presentation.

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DEDICATION

This research work is dedicated to my parents and teachers present and past.

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ALHAMDULLAHI RABIL ALAMIN

(PRASIE BEEN TO ALLAH LORD OF THE WORLD)

Abstract

It has been observed that collapses of structures are based on the way they are constructed and the impacted force on them. It is therefore necessary to study the design of their structural materials and the maximum external force the structure can withstand on it which forms the basis of this research work. For more approximate results, finite element method is used as a piecewise approximating which reduces error in the numerical results to the governing equation. The dynamic response of a flat plate subjected to moving load which impact some forces on it at any instance of time is considered.

Literature review

If a load is acting on solid and structure, then is a function of both time and space are involved which is called moving loads.

Inglis, Fryba steele and Timosheko Young (1994), have solved by analytic method the dynamic responses of a uniform bean subjected to moving load. The dynamic Analysis of Elastic Bean on visco elastic foundation subject to oscillation constant and variable magnitude load was investigated by Aiyesimi Y. M (1989) The bean under the action of a variable travelling transverse load by Oni S.T. (1997). Theory and problems of finite element analysis by Buchana G.R. (1995). Finite element method for engineers by Huebner K. Hill (1975). A simplified green element analysis of static and dynamic foundation by Oniyekwe O.O (2002). In this reach the Dynamic response of a flat plate under a moving load is investigated at instant time. In which we assume that the plate is thin and the deformation small, also the load is always keeping contact with the plate and impact force at any instant time.

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CHAPTER ONE: INTRODUCTION

1.0 INTRODUCTION

By the experience of the collapsed of Tacoma Narrow bridge in USA in 1940, a bridge in Noma in Nigeria in 2005,throbbing of the deck of a passenger ship, damage of constructed roads in Nigeria with high traffic, the frequently earth quake occurrences in populate countries of the world it was observed that these problems as different courses.

In over coming all these type of problems, it is necessary to study the dynamic components of the structures involve in the construction of any above. Before a bridge is designs the result of this research will be useful, in the analysis of loads expected to pass through it to avoid its collapse or crack as what happens in Nigeria bridges and roads.

In this research work we shall study the response of a flat plate under a moving load. The load shall be assumed to be impacting some forces on the plate at any instant of time.

1.1 Differential Equation

1.1.1 Introduction: Differential equations have their origin in mechanics. The aim of mechanics is to explain and predict the motion of bodies. Newton's laws of motion led to the study of differential equation, whose solution can be used to predict the position of a body at some later time.

Differential equation has been closely associated with the rise of physical science in previous centuries and they are now being used as models for real world problem in variety of other disciplines. Scientists, engineers, and economists working on a wide variety of problems, find it useful to set up mathematical models of the systems, which they are investigating. These models often yield an equation that contains some derivatives of unknown functions. Such equation is referred to as a **differential equation**.

Example of such models includes the free fall of a body, the absorption of drugs into the body tissues, the decay of radioactive substances etc.

These differential equations basically fall into two classes' **ordinary differential equation** and **partial differential equation** depending on the number of independent variable present in the differential equation.

1.1.2 Linear and Non-Linear: If it does not occur any where the product of the dependent variable $y(t)$ with itself or any of its derivatives in the equation then the equation, is said to be linear otherwise it is non-linear.

$$L(y) = \sum_{P=0}^{m} f_P(t) y^{(P)}(t) = r(t), \; f_P(t) \ldots\ldots\ldots (1.1.2.0)$$

is a know function and its of order p.

For non-linear d.e the general equation is written as $F\left(t, y, y^{1} \ldots, y^{(m-1)}, y^{(m)}\right) = 0$

or

$$y^{(m)}(t) = F\left(t, y, y^{1} \ldots, y^{(m-1)}\right) \ldots\ldots.(1.1.2.1) \text{ of order m}$$

While solving, m arbitrary constant are involved which are to be determined.

If m conditions are prescribed at one point then the differential equation together these conditions is called Initial Value Problem (IVP) and the condition is initial condition, expressed as

$$y^{(m)}(t) = F\left(t, y, y^{1} ..., y^{(m-1)}\right), \; y^{(P)}(t_0) = y_p \quad P = 0,1,2...,m-1 ......(1.1.2.2)$$

If prescribed at more than one point these are called boundary condition. Then differential equation of the system together with its Boundary condition is called Boundary Value Problem (BVP).

1.1.3 Classification of Differential Equation: Differential equations are classified into two classes ordinary and partial differential equations based on the number of independent variables involve in the differential equation. It is the ordinary differential equation if one independent is involved and partial differential equation if more than one independent variable is involved.

1.2 Ordinary Differential Equation: The general form of writing O.D.E is $L(y) = r$ .... (1.2.0) were L is the differential operator and r is the function of independent variable. It is used to describe any physical substances that involve one independent variable such as spring mass system, resistor capacity inductance circuits, bending of beams, chemical reactions and so on.

The order of any differential equation is the order of its highest derivative and like wise the degree is the degree of its highest order.

1.3 Partial Differential Equation (PDE): Quantities that may be evaluated at a specific location in space at a given time require more that one independent variable, for their specification. The Differential Equation to be solved to determine these quantities among the partial derivative with respect to these independent variables are referred to as Partial Differential Equation

Mathematically physical system can be modeled e.g.

$$L[u] = A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \delta y} + C\frac{\partial^2 u}{\partial y^2} = D\frac{\partial u}{\partial y} + E\frac{\partial u}{\partial x} + u(x,y) \ldots\ldots. \ (1.3.0)$$

is a second order PDE where x, y are independent variables and its solution is of the form $u = u(x,y) \ldots\ldots\ldots \ (1.3.1)$

If $D\frac{\partial u}{\partial y} + E\frac{\partial u}{\partial x} + u(x,y) = 0$

Then it is said to be homogeneous and if

$$D\frac{\partial u}{\partial y} + E\frac{\partial u}{\partial x} + u(x,y) \neq 0$$

It is non-homogeneous.

1.3.1 Methods of Solving Partial Differential Equation: There are two methods of finding solution to a PDE, Analytical and Numerical methods.

1.3.2 Analytical Methods: These are methods that give the exact solution to the Partial Differential Equation.

1.3.3 Numerical Methods: These are methods that give the approximate solution numerically to the partial differential equation.

1.4 Analytical Methods

1.4.0 Method of General Solution: In this method we first find the general solution and then the particular solution which satisfies the boundary and the initial condition.

1.4.1 Separation of Variables: In this method it is assumed that a solution can be expressed as a product of unknown function each of which depends only on one of the independent variables. The success of the method hinges on being able to write the resulting equation so that one side depends on the remaining variables, each side must be a constant. By repetition of this unknown functions can then be determined. Super position of these solutions can then be used to find the actual solution. The method often makes used of Fourier series, Fourier integral, Bessel series and Legendre series.

As an illustration of the use of separation of variable, consider heat in a slab of thickness L. the governing equation for temperature, denoted by u is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad t > 0,\ 0 < x < L \qquad \ldots (1.4.1.0)$$

If we let the boundaries be kept at the same temperature, say zero, then the boundary conditions are homogeneous

$$U(0,t) = U(L,t) = 0, \quad t > 0 \text{ and } U(x,0) = f(x) \qquad \ldots (1.4.1.1)$$

Using separation of variables

U=X(x)T(t) $\frac{X''}{x} = \frac{T'}{KT} = -\lambda^2$ where λ is real. This leads to two ordinary

Differential equation $\frac{X''}{x} = -\lambda^2$ and $\frac{T'}{kT} = -\lambda$

The general solutions is

$T = e^{-k+\lambda^2 t}$ $X = A\sin\lambda x = 0 + B\cos\lambda x$

The boundary conditions for x are x(0)=x(L)=0

Consequently B=0 and the Eigen value condition is $\sin\lambda L = 0$

Which gives $\lambda_n = \frac{n\pi}{L}, n = 1,2,3,...$

The eigen functions are

$X_n = \sin\frac{n\pi}{L}x, n = 1,2,3,...$

and the corresponding time factor is

$T_n = \exp\left[-\left(\frac{n\pi}{L}\right)^2 kt\right], n = 1,2,3,...$

Hence the final solution is of the form

$$U = \sum_{n=0}^{\infty} a_n \exp\left[\left(-\frac{n\lambda}{L}\right)^2 kt\right]\sin\frac{n\pi}{L} \qquad \text{... (1.4.1.2)}$$

Now the initial condition implies that

$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$

For a pair of sine, the following identities hold

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0, m \neq n \\ L/2, m = n \end{cases}$$

Multiplying both sides by $\sin\frac{n\pi x}{L}$ and integrating from 0 to L we get

$$a_0 = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, n = 1,2,3...$$

Thus the solution is completely determined as

$$U(x,t) = \sum_{n=1}^{\infty}[\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx]\sin\frac{n\pi x}{L}\exp\left[-\left(\frac{n\lambda}{L}\right)^2 kt\right]\sin\frac{n\pi}{L}kt \quad ... (1.4.1.3)$$

1.4.2 Laplace Transformation method: The PDE is written in its Laplace transform and associated with the boundary condition which is first obtained with respect to one of the independent variable. We then solve the resulting equation for the Laplace transform of the required solution which is then found by taking the inverse Laplace transform. In cases where Laplace inversion is difficult the complex inversion formula can be used.

We now as an example by using Laplace transform in solving PDE

consider the concentration c(x,t) of a difference surface of a given substance governed by $\frac{\partial c}{\partial t} = k\frac{\partial^2 c}{\partial x^2}$ 0<x<L, t>0 ....(1.4.2.0)

subject to the boundary conditions $\frac{\partial c}{\partial t}(0,t) = 0, c(L,t) = C_0$ and the initial condition c(x, 0) =0 0<x<L

Taking the Laplace transform with respect to t and using the initial condition, we get from the LHS $\frac{\overline{\partial c}}{\partial x} = s\overline{c}$

From the RHS we get $\frac{\partial^2 c}{\partial x^2} = \frac{\partial^2 \overline{c}}{\partial x^2}$

therefore, the PDE becomes

$$\frac{\partial^2 c}{\partial x^2} = \frac{s^2 \overline{c}}{k} = 0 \quad 0 < x < L$$

the boundary conditions are

$$\frac{\partial^2 \overline{c}}{\partial x}(0, s) = 0$$

at the left end and

$$\overline{c}(L, s) = \int_0^\infty e^{-st} c(L, t) dt = \frac{c_0}{s}$$

at the right end. Thus the PDE has been reduced to ODE the solution is

$$\frac{c}{c_0} = \frac{\cosh \sqrt{s/k} * ds}{s \cosh \sqrt{s/kL}}$$

Inverse transform is

$$\frac{c}{c_0} = \frac{1}{2\pi . i} \int \frac{e^{st} \cosh \sqrt{s/k} * ds}{s \cosh \sqrt{s/kL}}$$

The final result is

$$\frac{c}{c_0} = 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos\left[\left(\frac{n+1}{2}\right)\pi x\right]^2 \exp\left\{kt\left[\left(n + \frac{1}{2}\right)\frac{\pi}{L}\right]^2\right\} \quad \text{... (1.4.2.1)}$$

1.5 Numerical method: The numerical methods of solving differential equation are algorithms which produce a series of approximate solutions to the differential equation at certain spaced points called grid, nodal, met, or mesh points, along the co-ordinate of integral interval.

1.5.0 Single Step Methods: A single step method for solving the DE is one in which the solution is approximated by calculating the solution of a related first order difference equation. Thus a general single step method can be written in the form $y_{n+1} = y_n + h\Phi(t_m, y_n, h), n = 0,1,2,...., N-1$ .... (1.5.0.1)

where $\Phi(t, y, h)$ is a function of the argument t, y, h called increment function which depend on $\frac{dy}{dt} = f(t, y), y(t_0) = y_0, t \in [t_0, b]$

If y_{n+1} can be obtained simply by evaluating the right hand side of (1.5.0.1) it is called explicit otherwise it is implicit

1.5.1 Taylor Series Method: Let us assume that the Differential Equation has a unique solution $u(t)$ on $[t_0, b]$ and that $y(t) \in C^{(P+1)}[t_0, b]$ for $P \geq 1$ (1.5.0.1)

The Solution $y(t)$ can be expanded in a Taylor series about any point t_n.

$$y(t) = y(t_0) + (t - t_0)y'(t_0) + \frac{1}{2}(t - t_0)^2 y''(t_0) + \frac{1}{3}(t - t_0)^3 y'''(t_0)$$

$$+ ... + \frac{1}{p!}(t - t_0)^P y^{(P)}(t_0) + \frac{(t - t_0)^{P+1}}{(p+1)!} y^{(P+1)}(\xi)$$

$$y(t_{n+1}) = y(t_0) + hy'(t_0) + \frac{h^2}{2!} y''(t_0) + ... + \frac{h^P}{P!} y^P(t_0) + \frac{h^{(P+1)}}{(P+1)!} y^{(P+1)}(\xi)$$

if $t = t_{n+1}$ $h = t_{n+1} - t_n$ ... (1.5.1.1)

1.5.2 Runge-Kutta Methods: This is the method that used the Mean Value Theorem that is $y' = f(t, y), y(t_0) = y_0 \ t \in (t_0, b)$ satisfies $y(t_{n+1}) = y(t_n) + hy'(\xi_n)$ where $\xi_n = t_n + Q_n h, \ O < Q_n < I,$ for 1st order methods. For 2nd order method,

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf(t_n + c_2 h, y_n + a_{21} k_1)$$

$$y_{n+1} = y_n + w_1 k_1 + w_2 k_2 \qquad \text{.......... (1.5.2.0)}$$

3rd order method

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf(t_n + c_3 h_1, y_n + a_{21} k_1)$$

$$k_2 = h(t_n + c_3 h, y_n + a_{31} k_1 + a_{32} k_2)$$

$$y_{n+1} = y_n + w_1 k_1 + w_2 k_2 + w_3 k_3$$

Therefore,

$$k_1 = hf\left(t_n + c_i h, y_n + \sum_{j=1} a_{ij} k_j \right) \ c_1 = 0, \ i = 1,2,...v$$

$$k_1 = hf(t_n, y_n) \ i = 1$$

$$k_2 = hf(t_n + c_2 h, y_n + a_{21} h) k_3 = hf(t_n + c_2 h, y_n + a_{31} k_1 + a_3 k_2)$$

$$y_{n+1} = y_n + \sum_{i=1}^{m} w_i k_i \qquad \text{................. (1.5.2.1)}$$

1.5.3 Implicit Runge-Kutta: Is defined with v slope by the following equations $k_i = hf(t_n + c_i h, y + \sum_{j=1}^{v} a_{ij} k_j)$ .... (1.5.3.0)

$$y_{n+1} = y_n + \sum_{i=1} w_i k_i \qquad \text{.......... (1.5.3.1)}$$

Where $c_i = \sum_{j=1}^{v} a_{ij}, i = 1,2,\ldots,v,$

and, $a_{ij}, 1 \le i, j \le v \quad w_1, w_2, \ldots, w_v$ are arbitrary constants

1.5.4 Multi-step Methods: The method is called multi-step method if the value of $y(t)$ at $t = t_{n+1}$ uses the values of the dependent variables and its derivative at more than one mesh points. This is written generally as

$$y_{n+1} = a_1 y_n + a_2 y_{n+1} + \ldots + a_k y_{n-k+1} + h\phi(t_{n+1}, t_n, \ldots, t_{n-1} t_{n-k+1} + y'_{n+1} + y'_n + \ldots + y'_{n-k+1}) \ldots (1.5.4.0)$$

If ϕ is independent of y'_{n+1} it is explicit (Predictor) otherwise an implicit (corrector).

1.5.5 Explicit Multi Step: By integrating the differential equation $y' = f(t, y)$ between the limit t_{n-1} and t_{n+1} we get

$$y(t_{n+1}) = y(t_{n-1}) + \int_{t_{n-1}}^{t_{n+1}} f(t, y)dt \qquad \text{........ (1.5.4.0)}$$

Integrating (1.5.4.0) by approximating $f(t, y)$ by polynomial which interpolate $f(t, y)$ at k points $t_n, t_{n-1}, \ldots, t_{n-k+1}$. We shall use the Newton backward difference formula of degree (k-1) for this purpose

If $f(t, y)$ has k continuous derivatives then we have

$$p_{k-1}(t) = f_n + (t - t_n)\frac{\quad}{h} + (t - t_n)(t - t_{n-1})\frac{\quad}{2!h^2} + ... + (t - t_n)(t - t_{n-1})...(t - t_{n-k+2})\frac{\quad}{(k-1)!h^{k-1}} +$$

$$(t - t_n)(t - t_{n-1})...(t - t_{n-k+1})\frac{f^{(k)}(\xi)}{k!} ....(1.5.4.1)$$

where $f^{(k)}(\xi)$ is the k^{th} derivatives of f evaluated at some ξ in an interval

containing t, t_{n-k+1}, and, t_n substituting $u = \frac{(t - t_n)}{h}$ in (1.5.4.1)

$$p_{k-1}(t) = f_n + u\nabla f_n + u(u+1)\frac{\nabla^2 f_n}{2!} + ... + u(u+1)..(u+k-2)\frac{\nabla^{k-1} f_n}{(k-1)!} +$$

$$u(u+1)...(t - t_{n-k+1})\frac{h^k f^{(k)}(\xi)}{k!} = \sum_{m=0}^{k-1}(-1)^m \binom{-u}{m}\nabla^m f_n + (-1)\binom{-u}{k} h^k f^{(k)}(\xi) ....(1.5.4.2)$$

Substituting this in (1.5.4.0) and putting $dt = hdu$ we obtain

$$y(t_{n+1}) = y(t_{n-1}) + h\int_{-1}^{1}\left[\sum_{m=0}^{k-1}(-1)^m \binom{-u}{m}\nabla^m f_n + (-1)^k \binom{-u}{k} h^k f^{(k)}(\xi)\right]du$$

$$= y(t_{n-1}) + h\sum_{n=0}^{k-1}\tau_m^{(j)}\nabla^m f_n + T_k^{(j)} \qquad .... (1.5.4.3)$$

$$\tau_m^{(j)} = \int_{-1}^{1}(-1)^m \binom{-u}{m} du, T_k^{(j)} = h^{k+1}\int_{-1}^{1}(-1)^k \binom{-u}{k} f^{(k)}(\xi)du \qquad ....(1.5.4.4)$$

If remainder term $T_m^{(j)}$ is ignore we get

$$y_{n+1} = y_{n-1} + h\sum_{m=0}^{k-1}\tau_m^{(j)}\nabla^m f_n \qquad .... (1.5.4.5)$$

Calculating few terms of $\tau_m^{(j)}$ we obtain

$$\tau_0^{(j)} = 1 + j, \tau_1^{(j)} = \tfrac{1}{2}(1 - j)(1 + j), \tau_2^{(j)} = \tfrac{1}{12}(5 - 3j^2 + 2j^3), \tau_0^{(j)} = \tfrac{1}{24}(3 - j)(3 + j - j^2 + j^3)...(1.5.4.6)$$

1.5.5 Adams-Bashforth formulas (j=0) If $\tau_m^{(j)}$ is evaluate then

equation (1.5.4.5) becomes

$$y_{n+1} = y_n + h[2f_n + \tfrac{1}{2}\nabla f_n + \tfrac{5}{12}\nabla^2 f_n + \tfrac{3}{8}\nabla^3 f_n + \tfrac{251}{720}\nabla^4 f_n + \tfrac{475}{1440}\nabla^5 f_n ...] ....(1.5.5.0)$$

1.5.6 Nystrom formulas (j=1) If $\tau_m^{(j)}$ is evaluate then equation (1.5.4.5) becomes

$$y_{n+1} = y_{n-1} + h[2f_n + \tfrac{1}{3}\nabla^2 f_n + \tfrac{1}{3}\nabla^3 f_n + \tfrac{29}{90}\nabla^4 f_n + \tfrac{11}{15}\nabla^5 f_n + \ldots] \ldots\ldots\ldots(1.5.6.0)$$

1.5.7 Implicit Multi Step: As in explicit method we expressed y_{n+1} in term of previously

Implicit multi Step method:

$$y(t_{n+1}) = y(t_{n-1}) + h\int_{-1}^{1}\left[\sum_{m=0}^{k}(-1)^m\binom{1-u}{m}\nabla^m f_{n+1} + (-1)^{k+1}\binom{1-u}{k+1}h^{k+1}f^{(K+1)}(\xi)\right] \ldots (1.5.7.0)$$

$$y(t_{n+1}) = y(t_{n-1}) + h\sum_{m=0}^{k}\delta_m(j)\nabla^m f_{n+1} + T^{\bullet}_{k+1}(j) \qquad \ldots\ldots\ldots (1.5.7.1)$$

were $T^{\bullet}_{k+1}(j) = h^{k+2}\int_{-j}^{1}(-1)\binom{1-u}{k+1}f^{(k+1)}(\xi)du \quad \delta_m(j) = \int_{-1}^{1}(-1)\binom{1-u}{m}du$ ....(1.5.7.2)

replacing the difference operator $\nabla^m f_{n+1}$ in terms of the function values

we obtain $y_{n+1} = y_{n-j} + h\sum_{m=0}^{k}\delta^{\bullet}_m(j)f_{n-m+1}$ ....(1.5.7.3)

1.5.8 Adams-Moulton formulas (j=0) then equation (1.5.7.3) becomes

$$y_{n+1} = y_n + h[f_{n+1} - \tfrac{1}{2}\nabla f_{n+1} - \tfrac{1}{12}\nabla^2 f_{n+1} - \tfrac{1}{24}\nabla^3 f_{n+1} - \tfrac{19}{720}\nabla^4 f_{n+1} - \tfrac{72}{1440}\nabla^5 f_{n+1}\ldots]\ldots(1.5.7.5)$$

1.5.9 Milne-Simpson formulas (j=1)

$$y_{n+1} = y_{n-1} + h[2f_{n+1} - 2\nabla f_{n+1} + \tfrac{1}{3}\nabla^2 f_{n+1} + \nabla^3 f_{n+1} - \tfrac{1}{90}\nabla^4 f_{n+1} - \tfrac{1}{90}\nabla^5 f_{n+1}\ldots]\ldots(1.5.7.6)$$

1.6 Finite Difference Method (FDM): This is an approximated numerical method that gives a point wise solution to the PDE. This method improved as more points are used.

For various values of γ_1, γ_2 we obtain

(a) $(\nabla_t + \frac{1}{2}\nabla_t^2)u_m^{n+1} = \tau_1 \partial_x^2 u_m^{n+1}$ which is called the Richtmyer formula. For

$\gamma_1 = 0, \gamma_2 = 0$

(b) $(\nabla_t + \frac{1}{2}\nabla_t^2)u_m^{n+1} = \tau_1 \partial_x^2 (1 - \frac{1}{4}\nabla_t^2)u_m^{n+1}$ for $\gamma_1 = 0, \gamma_2 = -\frac{1}{4}$

1.7 Finite Element Method (FEM): This is approximated numerical method that gives a piece wise solution to the PDE, which is an improved of FDM (see fig 1.7.1). The methods used in finding the parameters involve in FDM are as follows.

1.7.1 Weighted Residual Methods: The approximating methods which provide analytical approximation. These are methods which gives analytical procedure of obtaining solutions in form of functions which are close some sense to the exact solution of BVP or IVP.

Some few well known methods are least square method, Partition methods, Galerkin solution. Moment method and collocation

1.7.1.1 Least square: Chose weighting function to be $w_i = \frac{\partial \varepsilon[x.a]}{\partial}$ to provide N simultaneous equation for the determination of parameters $a_1, a_2 ... a_n$

1.7.1.2 Partition method: When the domain R is divided into N non-overlapping sub domain $R_j = j = 1,2...N$ and weighted Residual are chosen as

$w_j = \begin{cases} 1, x \in R_j \\ 0, x \notin R_j \end{cases}$ then the DE is satisfied on the average in each of the N sub

1.6.3 Multilevel explicit difference scheme: The general three level scheme in which seven points are involve and can be written as

$(1-\gamma_t V_t)^{-1}(\nabla_t + \tau_t'' V_t^2)u_m^n = \tau\partial_x^2 u_m^n$ or

$$(1+\tau_t^{\bullet})u_m^{n+1} = \left[1+2\tau_t^{\bullet}+\tau(1-\gamma_t^{\bullet})\partial x^2\right]u_m^n - (\tau_t^{\bullet} - \tau\gamma_t^{\bullet}\partial_x^2)u_m^{n+1} \quad ......(1.6.3.1)$$

where $\tau_t^{\bullet}.and.\gamma_t^{\bullet}$ are arbitrary parameters. The truncation error is given by

$T^{\bullet}{}_m^n = (\nabla_t + \tau_t^{\bullet}\nabla_t^2)u(x_m, t_n) - \tau\partial_x^2(1 - \gamma_t^{\bullet}\nabla_t)u(x_m, t_n)$ where $u(x_m, t_n)$ satisfies (1.6.1.1)

1.6.4 Two level implicit differences: The general two level implicit difference scheme involving six points is obtained as

$(1-\lambda_t\nabla_t)^{-1}\nabla_t u_m^{n+1} = \tau\left[(1+\sigma\partial_x^2)^{-1}\partial_x^2\right]u_m^{n+1}$ which on

$$\left[1+(\sigma-\tau(1-\gamma_t))\partial_x^2\right]u_m^{n+1} = \left[1+(\sigma+\tau\gamma_t)\partial_x^2\right]u_m^n$$

For various values of σ, γ_1 we get the following unconditionally stable methods

(a) $\nabla_t u_m^{n+1} = \tau\partial_x^2 u_m^{n+1}$ for σ=0, γ_1=0 which is Laesonen formula

(b) $\nabla_t u_m^{n+1} = \frac{\tau}{2}\partial_x^2(u_m^{n+1} + v_m^n)$ for σ=0, γ_1=½ called Crank-Neoson formula.

1.6.5 Multilevel implicit difference scheme: The three level difference scheme base upon nine points

$(1-\gamma_1\nabla_t + \gamma_2\nabla_t^2)^{-1}(\nabla_t + \tau_t\nabla_t^2)u_m^{n+1} = \tau(1+\sigma\partial_x^2)^{-1}\partial x u_m^{n+1}$ where $\tau_1, \gamma_1, \gamma_2, and \sigma$ are arbitrary which can be simplified and written as

$$\begin{gathered}\left[(1+\tau_1)+[\sigma(1+2\tau_1)-\tau(1-\gamma_1+\gamma_2)]\partial_x^2\right]u_m^{n+1} = \\ \left[(1+2\tau_1)+[\sigma(1+2\tau_1)+\tau(\gamma_1-2\gamma_2)]\partial_x^2\right]u_m^n - \left[\tau_1+(\sigma\tau_1-\tau_1\gamma_2)\partial_x^2\right]u_m^{n+1}\end{gathered} \quad ...(1.6.5.1)$$

The basic idea of FDM is to replace the derivatives by finite difference operator. An important feature of FDM is it approximates PDE by set of algebraic equations. This makes it very suitable for application in computer. With FDM we can treat some fairly difficult problem in practical problem. See fig 1.6.1.

1.6.1 Difference methods: These are difference equation obtained from a given DE which are them solve directly indirectly. The scheme can be formed depending on the level which is either explicit or implicit.

Example of difference schemes are these:

Consider the heat flow equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ldots\ldots\ldots(1.6.1.1)$$

Were t and x are time and space co-ordinate respectively, in the region $\Re = [a \le x \le b].[t \ge 0]$ For $x = a + mh, t = nk.$

(the quantities h and k are mesh size in space and time respectively) the difference approximation at nodal point (m, n) can be written as

$$G(\nabla_t)u_m^{n+1}. = \tau\partial^2 u_m^n \text{ were } \tau = k/h^2, \left(\frac{\partial^2 u}{\partial x^2}\right)_m^n = h^{-1}\partial_x^2 u(x_m, t_n) + o(h^2)\ldots\ldots(1,6.1.2)$$

1.6.2 Two level explicit difference scheme: Choosing $G(\nabla_t) = \nabla_t$ then

(1.6.1.1) becomes $\nabla_t u_m^{n+1} = \tau\partial x^2 u_m^n$ which is known as the Schmidt formula.

domain R_j. The required equation for Partition method becomes

$$\int_{R_j} \varepsilon[x,a]dx = 0, j = 1,2,\ldots N$$

1.7.1.3 Galerkin Method: If the weighting function is chosen to be

$w_j = \dfrac{\partial w(x,a)}{\partial a_j}, j = 1,2,..N$ where $w(x,a)$ is the approximated solution of the

problem. $\int_{R_j} \psi_j(x)\varepsilon[x,a]dx = 0$

1.7.1.4 Moment Method: The weighting function is chosen to be $w_j = p_j(x)$

Where $p_j(x)$ are polynomials equal $\int_{R_j} p_j(x)\varepsilon[x,a]dx = 0$

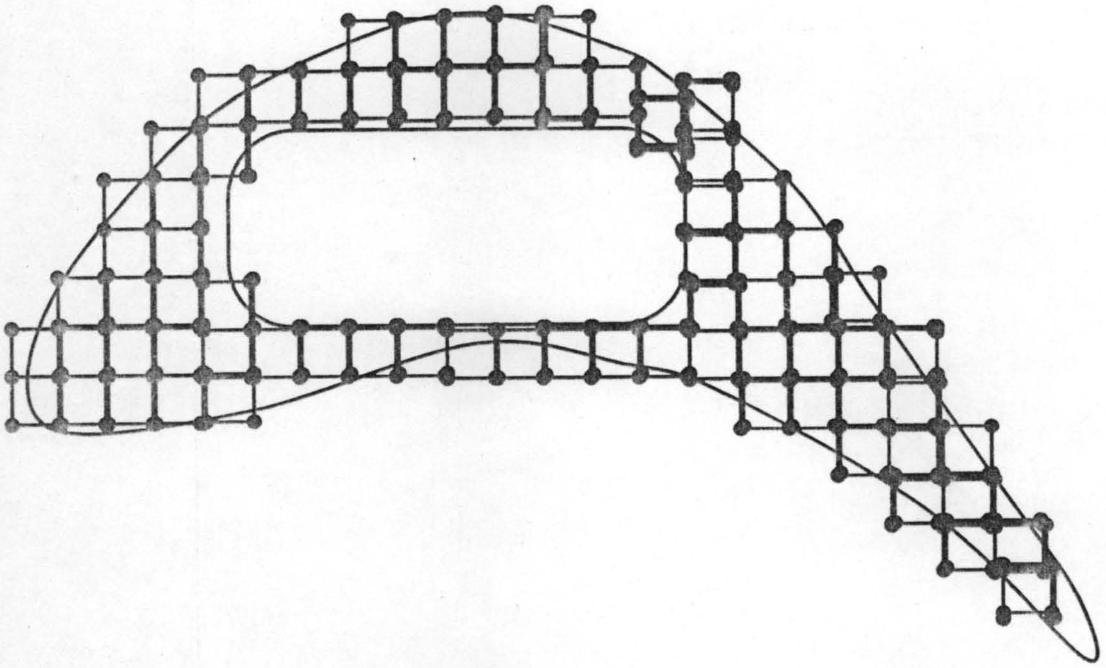


Fig. 1.6.1 Finite Difference Method

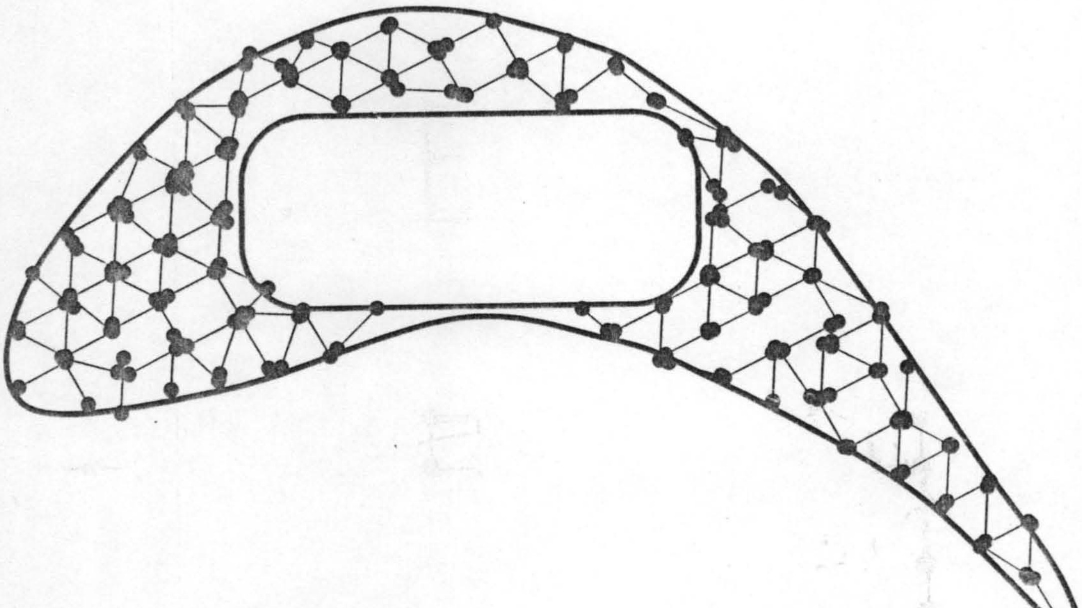


Fig. 1.7.1 Finite Element Method

CHAPTER TWO: MATHEMATICAL FORMULATION

2.1 Deformation of plate theory (Vibration of plates). From fig 2.1 let the plate be of uniform thickness r, which is small compare to the dimensions of the plate, on $x-y$ plane at middle plane of the plate. Assume that the deflection in the z-direction is small compare with the thickness and middle plane is normal and remain normal even after deflection on middle surface during vibration.

If the strains in thin layer of a typical element are considered at the distance z from the middle surface the following are the strains representation:

$$\ell_x = -z\frac{\partial^2 w}{\partial x^2},\ \ell_y = -z\frac{\partial^2 w}{\partial y^2}, r_{x,y} = -2z\frac{\partial^2 w}{\partial x \partial y} \qquad \text{....... (2.1.0)}$$

Where w denotes the deflection of the plate in z direction $e_x, e_y, r_{x,y}$ are normal strains and shear strain in the thin layer. The corresponding stresses by the relationship are known as:

$$\sigma_x = \frac{\varepsilon}{1-v^2}(\ell_x + v\ell_y),\ \sigma_y = \frac{\varepsilon}{1-v^2}(\ell_y + v\ell_x),\ \tau_{xy} = \frac{\varepsilon z}{(1-v)}\frac{\partial^2 w}{\partial x \partial y} \qquad \text{....... (2.1.1)}$$

The potential energy accumulated in the shaded layer of the element during deformation will be:

$$dU = \frac{1}{2}(\ell_x\sigma_x + \ell_y\sigma_y + r_{xy}\tau_{xy})dxdydz \qquad \text{............ (2.1.2)}$$

$$U = \frac{1}{2}\iiint(\ell_x\sigma_x + \ell_y\sigma_y + r_{xy}\tau_{xy})dxdydz \qquad \text{............. (2.1.3)}$$

$$\frac{\ell_x 6_x}{2} = \frac{EZ^2}{2(1-V^2)}\left[\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right]\cdot\frac{\partial^2 w}{\partial x^2}$$

$$\frac{\ell_y 6_y}{2} = \frac{EZ^2}{2(1-V^2)}\left[\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right]\cdot\frac{\partial^2 w}{\partial y^2}$$

$$\frac{r_{xy}\tau_{xy}}{2} = \frac{2E(1-V)}{1-V^2}Z\left[2\frac{\partial^2 w}{\partial x\partial y}\right]^2$$

$$dU = \frac{EZ^2}{2(1-V^2)}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2v\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} + 2(1-V)\frac{\partial^2 w}{\partial x\partial y}\right]$$

$$= \frac{EZ^2}{2(1-V^2)}\left[\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2}\right]$$

$$U = \frac{EZ^3}{\sigma(1-V^2)}\iint\left[\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2}\right]dxdy \qquad \text{... (2.1.4)}$$

Since $Z = t$ (thickness)

$$U = D\iint\Delta^4 dxdy,\ D = \frac{Et^2}{12(1-v^2)} \qquad \text{...(2.1.5)}$$

is the flexural rigidity of the plate. According to Smith[7]

2.2 Boundary Condition: The boundary condition for rectangular or square plate is of three cases simply supported edges, clamped edges and the free edges. See fig. 2.2.1

2.2.1 Simply Supported Edge: Two conditions must be satisfied that is the displacement $w = 0$ and the movement with the consigning direction of the edge is zero.

$$\therefore D\left(\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2}\right)=0, \quad \ldots\ldots\ldots\ldots (2.2.1.0)$$

and since $D \neq 0$ for edge x=constant

$$\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2}=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.2.1.1)$$

Similarly for edge y = constant

$$\frac{\partial^2 w}{\partial y^2}+v\frac{\partial^2 w}{\partial x^2}=0, \quad \ldots\ldots\ldots\ldots (2.2.1.2)$$

Since w=0 and along the edge x=constant it follows that $\frac{\partial^2 w}{\partial x^2}=0$. It is possible to replace the boundary condition above with $\left[w=0, \frac{\partial^2 w}{\partial x^2}=0\right]_B$

.....(2.2.1.3)

Similarly for edge y=constant the boundary condition can be replaced by

$\left[w=0, \frac{\partial^2 w}{\partial y^2}=0\right]_B$ .... (2.2.1.4)

2.2.2 Clamped Edge (fixed edge): The two conditions that must be satisfied are that the displacement w must be zero and the slope of the line perpendicular to the edge must be zero, i.e. for edge y,

$w=0, \frac{\partial w}{\partial y}=0$ For edge y=constant

$w=0, \frac{\partial w}{\partial x}=0$ For edge x=constant ...... (2.2.2.0)

2.2.3 Free edge: For $x = constant$ the two conditions to be satisfied are that the moment M_x must be zero and that the quantity $V_x = Q_x - \frac{\partial M_{xy}}{\delta y}$ on that edge must be zero.

Where $Q_x = \frac{\partial M_{xy}}{\partial y} - \frac{\partial M_x}{\partial x}$ ...... (2.2.3.0)

Since $M_{xy} = -M_{yx}, V_x = \frac{\partial M_x}{\partial x} - 2\frac{\partial M_{xy}}{\partial y}$

$$\Rightarrow V_x = -D\left(\frac{\partial^3 w}{\partial x^3} + (2-v)\frac{\partial^3 w}{\partial x \partial y^2}\right) \text{ for } M_x = D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right)$$

By the condition of free edge $M_x = 0, V_x = 0$

For free edge y

$$V_y = -D\left(\frac{\partial^3 w}{\partial y^3} + (2-v)\frac{\partial^3 w}{\partial y \partial x^2}\right) = 0 \text{ for } M_y = D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right) = 0$$

In order to analyze the displacements and the stresses in a rectangular plate the differential equation

$D\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) = U$ is solved........(2.2.3.1)

The above deformation is the so-called Kirchhoff-Love plate theory if the load moves and keeps contact with the plate impacting force on it, at any instant of times according to Smith[7] and William[8].

2.3 Finite Element Method (FEM): The finite element method is a numerical technique for obtaining approximate solutions to differential equation used in wide variety of engineering problems. It is difficult to obtain exact solution to some differential equations so it becomes necessary to use approximate solution rather than the exact which is a close solution. FEM is a piece wise approximation to the equation.

The basic premise of FEM is that a solution region can be replaced by assemblage of all the discretized element solutions. These elements can be put in variety of ways to represent complex shapes which are later put together. Figure (1.6.1) and (1.7.1) shows the FDM and FEM.

2.3.2 History of the Method: The method first appeared in 1960 when Clough used it in the paper presented on plane elasticity paper, but it idea date back on two you asked a applied mathematician, physicist, or an engineer about the origination of the method. Each has some justification for the claims.

The applied mathematician are interested on boundary value problem, physicist on solving continuum problems and engineer searching for way in which to find the stiffness influence coefficients of shall type of Structure reinforced by ribs and spars.

2.3.3 How the Method Works: The problem is discretized into finite number of elements by dividing the solution region into elements. Then express the unknown field variable in terms of assumed approximating function within each element called interpolation function, which are defined in term of the values of the field variables at specified points called nodal or nodes points. With finite element representation of problem, nodal values of the field variables becomes the new unknown. Once these unknown are found, the interpolation function define the field variable throughout the assemblage of elements. Finite element method has the ability to formulate solutions for individual elements. Before adding them together to represents the entire problem.

The process is in six steps

(a) Discretize the continuum

(b) Select interpolation

(c) Find the elements properties

(d) Assemble the elements properties to obtain the system equation

(e) Solve the equation

(f) Make additional computation if desired see Kenneth.

2.3.4 Applications: The application of Fem can be divided in to three categories depending on the problem, to solve. They are

(a) Equilibrium problem

(b) Eigen value problem of solid and fluid mechanics

(c) Multitude of time dependent or propagation.

(a) Equilibrium problem (Time independent problem) in solid mechanics problem to find the displacement distribution or stress or temperature distribution for a given mechanical or thermal loading. This could sometime be finding pressure velocity, temperature and concentration.

(b) Eigen Value Problem: These is problems whose solution often require the determination of natural frequencies e.g. problem involving both solid and fluid.

Multitude of time-dependent or propagation problem of continuum mechanics. In this category time dimension is added to the problem of the first two categories.

But in the mere fact this method (FEM) can be used to solve particular problem does not mean it is the most practical solution techniques.

The range of FEM application extends to all engineering disciplines.

2.4 Finite Element methods: we divide the domain R into M finite elements with piecewise approximate solutions as $w = u^{(1)} + u^{(2)} + ... + u^{(e)} + ... + u^{(m)}$ were $u^{(1)}, u^{(2)}, ... u^{(m)}$ are approximate solutions of each element

The function $u^{(e)}$ is express as $u^{(e)} = N^{(e)}\phi^{(e)}$ where $N^{(e)}$ is the shape function and $\phi^{(e)}$ is the column vector which depends on the nodal values of the function u or its derivatives.

The nodal parameters ϕ may be determined by using any one of the method used to determine the nodal parameters such as least square finite element, Galerkin finite element e. t. c

2.5 Choice of Method: The numerical method gives the approximate solution to PDE. Becomes of its piecewise approximation to the PDE, FEM is used to find the solution to the governing equation.

The cubic Hermite Polynomial will be used as the interpolation function, since the plate is assumes to be of rectangular shape see Kenneth.

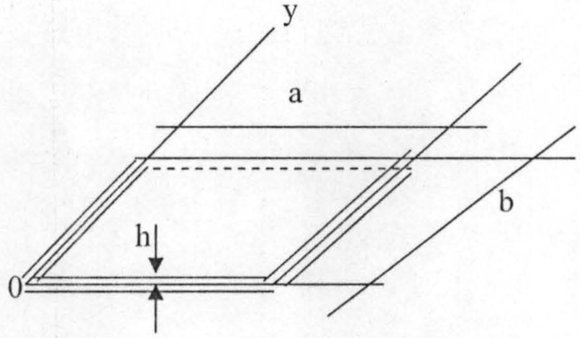


Fig. 2.1.1

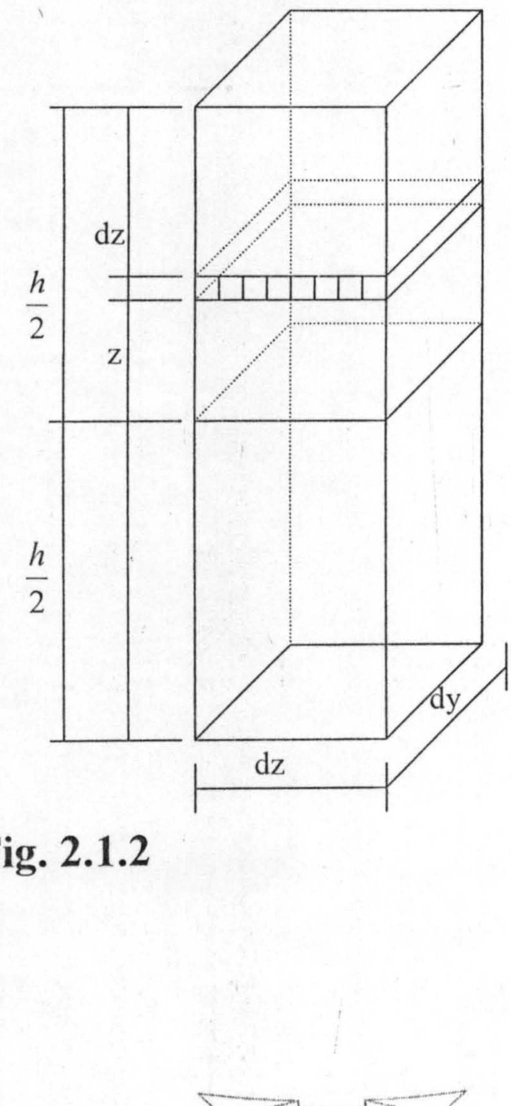


Fig. 2.1.2

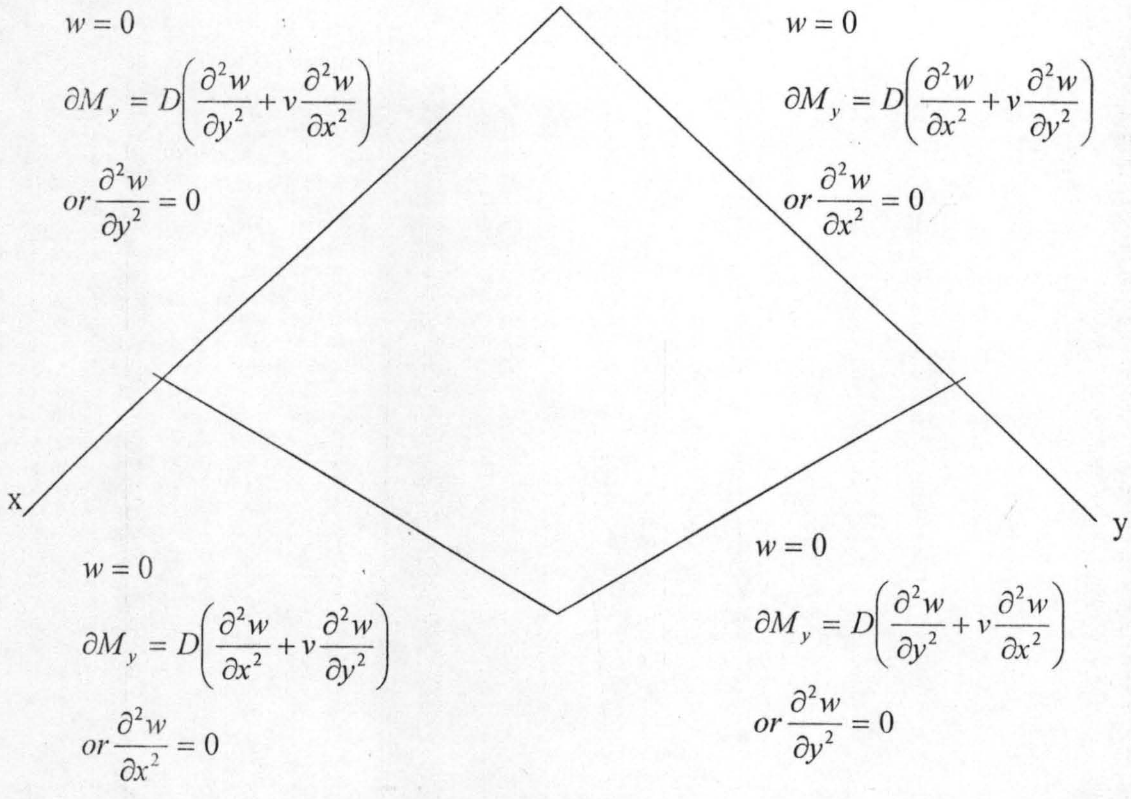


Fig. 2.2.1 Free Edge Boundary Condition

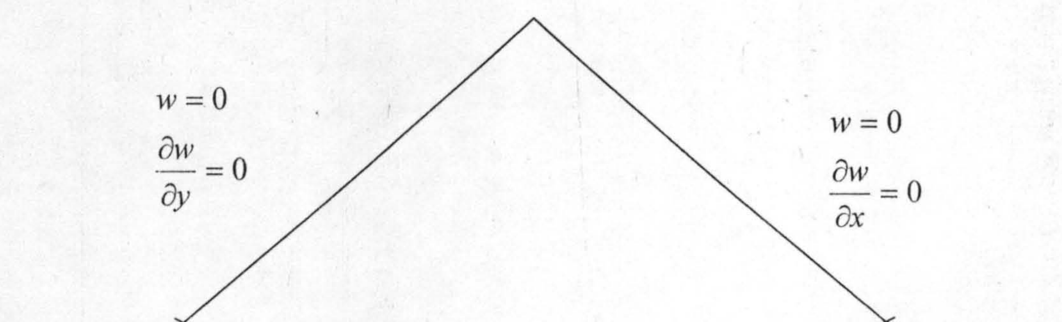


Fig. 2.2.2 Fixed Edge Boundary Condition

Since $\nabla^4 w(x,y,r) = \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2}$ .... (3.1.4)

Then (3.1.3a) becomes:

$$\left[\frac{\partial^4[N]}{\partial x^4} + \frac{\partial^4[N]}{\partial y^4} + 2\frac{\partial^4[N]}{\partial x^2 \partial y^2}\right]\{U_i\} = \frac{P}{D} \quad .... \ (3.1.5)$$

Multiple by the weight function $[N]^T$ and integrate both x, y axes

$$\iint\left[[N]^T \frac{\partial^4[N]}{\partial x^4} + [N]^T \frac{\partial^4[N]}{\partial y^4} + 2[N]^T \frac{\partial^4[N]}{\partial x^2 \partial y^2}\right]\{U_i\}dxdy = \iint \frac{P}{D}[N]^T dxdy \ .. \quad (3.1.6)$$

$$\iint [N]^T \frac{\partial^4[N]}{\partial x^4} dxdy, \quad .... \ (3.1.7)$$

using integration by part we have (twice)

$$= \int\left[[N]^T \frac{\partial^3[N]}{\partial x^3} - \int \frac{\partial[N]^T}{\partial x} \cdot \frac{\partial^3[N]}{\partial x^3} dx\right]dy$$

$$= \int [N]^T \frac{\partial^3[N]}{\partial x^3} dy - \iint \frac{\partial[N]^T}{\partial x} \cdot \frac{\partial^3[N]}{\partial x^3} dxdy$$

$$= \int [N]^T \frac{\partial^3[N]}{\partial x^3} dy - \int\left[\frac{\partial[N]^T}{\partial x} \cdot \frac{\partial^3[N]}{\partial x^3} - \int \frac{\partial^2[N]^T}{\partial x^2} \cdot \frac{\partial^2[N]}{\partial x^2} dx\right]dy$$

$$= \int [N]^T \frac{\partial^3[N]}{\partial x^3} dy - \int \frac{\partial[N]^T}{\partial x} \cdot \frac{\partial^3[N]}{\partial x^2} dy + \iint \frac{\partial^2[N]^T}{\partial x^2} \cdot \frac{\partial^2[N]}{\partial x^2} dxdy \ .... \ (3.1.8)$$

Similarly

$$\iint [N]^T \frac{\partial^4[N]}{\partial y^4} dxdy = \int [N]^T \frac{\partial^3[N]}{\partial y^3} dx - \int \frac{\partial[N]^T}{\partial y} \cdot \frac{\partial^3[N]}{\partial y^3} dy + \iint \frac{\partial^2[N]^T}{\partial y^2} \cdot \frac{\partial^2[N]}{\partial y^2} dxdy ...(3.1.9)$$

Also

$$\iint [N]^T \frac{\partial^4 [N]}{\partial x^2 \partial y^2} dxdy \quad \text{........... (3.1.10)}$$

$$= \int \left[[N]^T \frac{\partial^3 [N]}{\partial x \partial y^2} dy - \int \frac{\partial [N]^T}{\partial x} \cdot \frac{\partial^3 [N]}{\partial x \partial y^2} - \int \frac{\partial^2 [N]^T}{\partial x^2} \cdot \frac{\partial^2 [N]}{\partial y^2} dx \right] dy$$

$$= \int [N]^T \frac{\partial^3 [N]}{\partial x \partial y^2} dy - \int \frac{\partial [N]^T}{\partial x} \cdot \frac{\partial^3 [N]}{\partial x \partial y^2} dy + \iint \frac{\partial^2 [N]^T}{\partial x^2} \cdot \frac{\partial^2 [N]}{\partial y^2} dxdy$$

By applying boundary condition of deformation which are

$$\left[w = 0, \frac{\partial w}{\partial x} = 0 \right]_B, \left[w = 0, \frac{\partial w}{\partial y} = 0 \right]_B$$

Equation (3.1.6) becomes

$$\left[\iint \frac{\partial^2 [N]^T}{\partial x^2} \cdot \frac{\partial^2 [N]}{\partial x^2} dxdy + \iint \frac{\partial^2 [N]^T}{\partial y^2} \cdot \frac{\partial^2 [N]}{\partial y^2} dxdy + 2 \iint \frac{\partial^2 [N]^T}{\partial x^2} \cdot \frac{\partial^2 [N]}{\partial y^2} dxdy \right] *$$

$$\{U_i\} = \frac{\rho}{D} \iint [N]^T dxdy \text{...(3.1.11)}$$

Let

$$[M] = \iint \frac{\partial^2 [N]^T}{\partial x^2} \cdot \frac{\partial^2 [N]}{\partial x^2} dxdy + \iint \frac{\partial^2 [N]^T}{\partial y^2} \cdot \frac{\partial^2 [N]}{\partial y^2} dxdy + 2 \iint \frac{\partial^2 [N]^T}{\partial x^2} \cdot \frac{\partial^2 [N]}{\partial y^2} dxdy \text{...(3.1.12)}$$

(3.1.6) becomes:

$$[M]\{U_i\} = \frac{\rho}{D} \iint [N]^T dxdy \quad \text{........... (3.1.13)}$$

Using Cubic Hermitian polynomial as the interpolation function since the plate is assume to be a rectangular plate then

$[N] = [N_1(x,y), N_2(x,y), \ldots, N_{12}(x,y)]$ And

$$\{U_i\} = \left\{U_1, \frac{\partial U_1}{\partial x}, \frac{\partial U_1}{\partial y}, U_2, \frac{\partial U_2}{\partial x}, \frac{\partial U_2}{\partial y}, \ldots, U_4, \frac{\partial U_4}{\partial x}, \frac{\partial U_4}{\partial y}\right\} \ldots\ldots(3.1.14)$$

Such that

$$N_1(x,y) = AQ(x)Q(y),\ N_2(x,y) = AQ(x)R(y)b$$

$$N_3(x,y) = AR(x)Q(y)a,\ N_4(x,y) = AQ(x)S(y)$$

$$N_5(x,y) = AQ(x)T(y)b,\ N_6(x,y) = AR(x)S(y)a$$

$$N_7(x,y) = AS(x)S(y),\ N_8(x,y) = AS(x)T(y)b$$

$$N_8(x,y) = AT(x)S(y)a,\ N_{10}(x,y) = AS(x)Q(y)$$

$$N_{11}(x,y) = AS(x)R(y)b,\ N_{12}(x,y) = AT(x)Q(y)a \ldots(3.1.15)$$

Where $Q(x) = a^3 - 3ax^2 + 2x^3,\ Q(y) = b^3 - 3by^2 + 2y^3$

$$R(x) = a^2x - 2ax^2 + 3x^3\quad R(y) = b^2y - 2by^2 + y^3$$

$$T(x) = x^3 - ax^2\quad T(y) = y^3 - by^2,\ S(x) = 3ax^2 - 2x^3$$

$$A = \frac{1}{(ab)^3} \qquad \ldots\ldots\ldots\ldots (3.1.17)$$

$$\frac{\partial^2 N_1}{\partial x^2} = 6AQ(y)n(x),\ \frac{\partial^2 N_2}{\partial x^2} = 6AR(y)n(x)b,\ \frac{\partial^2 N_3}{\partial x^2} = 2AR(y)m(x)a,$$

$$\frac{\partial^2 N_4}{\partial x^2} = 6AS(y)n(x), \frac{\partial^2 N_5}{\partial x^2} = 6AbT(y)n(x), \frac{\partial^2 N_6}{\partial x^2} = -2AaR(y)m(x)$$

$$\frac{\partial^2 N_7}{\partial x^2} = 6AS(y)n(x), \frac{\partial^2 N_8}{\partial x^2} = -6AbT(y)n(x), \frac{\partial^2 N_9}{\partial x^2} = -2AaS(y)o(x)$$

$$\frac{\partial^2 N_{10}}{\partial x^2} = -6AQ(y)n(x), \frac{\partial^2 N_{11}}{\partial x^2} = -6AbR(y)n(x), \frac{\partial^2 N_{12}}{\partial x^2} = 2AaQ(y)o(x) \ldots(3.1.18)$$

$$\frac{\partial^2 N_1}{\partial y^2} = 6AQ(x)n(y), \frac{\partial^2 N_2}{\partial y^2} = -2AbM(y)Q(x), \frac{\partial^2 N_3}{\partial y^2} = -6AaR(x)n(y)$$

$$\frac{\partial^2 N_4}{\partial y^2} = -6AQ(x)n(y), \frac{\partial^2 N_5}{\partial y^2} = -2AbQ(x)n(y), \frac{\partial^2 N_6}{\partial y^2} = 6AaR(x)n(y)$$

$$\frac{\partial^2 N_7}{\partial y^2} = -6AS(x)n(y), \frac{\partial^2 N_8}{\partial y^2} = -2AbS(x)o(y), \frac{\partial^2 N_9}{\partial y^2} = 6AaT(x)n(y)$$

$$\frac{\partial^2 N_{10}}{\partial y^2} = 6AS(x)n(y), \frac{\partial^2 N_{11}}{\partial y^2} = -2AbS(x)n(y), \frac{\partial^2 N_{12}}{\partial y^2} = -6AaT(x)o(y) \ldots (3.1.19)$$

Where

$m(x) = 2x - a \quad n(y) = 2y - b \quad m(x) = 2a - 3x \quad m(y) = 2b - 3y \quad o(x) = a - 3x$

$o(y) = b - 3y$ .... (3.1.20)

$$[H]^T = \begin{bmatrix} Q(y)n(x) \\ R(y)n(x)b \\ \frac{1}{3}Q(y)m(x)a \\ S(y)n(x) \\ T(y)n(x)b \\ -\frac{1}{3}R(y)m(x)a \\ -S(y)n(x) \\ -T(y)n(x)b \\ -\frac{1}{3}S(y)o(x)a \\ -Q(y)n(x) \\ -R(y)n(x)b \\ \frac{1}{3}Q(y)o(x)a \end{bmatrix} \times 6A \qquad [H^1]^T = \begin{bmatrix} Q(x)n(y) \\ -\frac{1}{3}Q(x)m(y)b \\ -R(x)n(y)a \\ -Q(x)n(y) \\ -\frac{1}{3}Q(x)o(y)b \\ R(x)n(y)a \\ -S(x)n(y) \\ -\frac{1}{3}S(x)o(y)b \\ T(x)n(y)a \\ S(x)n(y) \\ -\frac{1}{3}S(x)m(y)b \\ -T(x)n(y)a \end{bmatrix} \times 6A$$

Therefore equation (3.1.12) becomes:

$$[M^1] = 36A^2 \iint [H]^T.[H]dxdy + 36A^2 \iint [H^1]^T.[H^1]dxdy + 72A^2 \iint [H]^T.[H^1]dxdy \text{..(3.1.21)}$$

Where $[H]^T.[H].[H^1]^T.[H^1]^T$ are $[H]^T[H^1]$ are first evaluated before taking the integral values of each element in the matrix.

Therefore

$$\{U_i\} = \frac{\rho}{D}[M^1]^{-1}[N^1] \qquad \text{.............. (3.1.22)}$$

Where

$$[N^1] = \iint [N]^T dxdy \qquad \text{.................. (3.1.23)}$$

The effect of moving point load can be felt on $x-axis$, $y-axis$ or both

For the load moving along $x-axis$ parallel to $y-axis$ equation 3.1.2 becomes

$$D\nabla^4 w(w,y) = \rho\delta(x-(x_0+\chi Cos\beta t)) \qquad \text{........ (3.1.24)}$$

For the load moving along $y-axis$ parallel to $x-axis$ equation 3.1.2 becomes

$$D\nabla^4 w(x,y) = \rho\delta(y-(y_0+\chi Sin\beta t)) \qquad \text{....... (3.1.25)}$$

CHAPTER THREE: PROBLEM SOLUTION

We assume that the flat plate is thin and deformation is small, also the load is always keeping contact with the plate and impact force at any instant of time. Based on the above the plate is of the type so-called Kirchhoff-Love plate theory. Therefore the governing equation subjected to the concentrated moving load with magnitude P, mass m and μ is the mass of the plate per unit area and is given by:

$$D\nabla^4 w(x,y,t) - \mu\frac{\partial^2 w(x,y,t)}{\partial t^2} = \delta(x-(x_0+\chi Cos\beta t))\delta(y-(y_0+\chi Sin\beta t))\left[\left(\rho - m\frac{\partial^2 w}{\partial t^2}(x,y,t)\right)\right] \text{...(3.1.1)}$$

Taking thickness t to be constant, (3.1.1) becomes according Oni[4]

$$D\nabla^4 w(x,y) = \rho\delta(x-(x_0+\chi\cos\beta t))\delta(y-(y_0+\chi\sin\beta t)) \text{ .... (3.1.2)}$$

The effect of the moving load is felt at the points $[(x_0+\chi\cos\beta t),(y_0+\chi\sin\beta t)]$ Aiyesimi[1]

At the points $x = x_0 + \chi\cos\beta t$ and $y = y_0 + \chi\sin\beta t$ equation (3.1.2) becomes

$$D\nabla^4 w(x,y) = \rho \qquad \text{............. (3.1.3)}$$

Let $w = [N]\{U_i\}$ where $[N]$ is a shape function and $\{U_i\}$ is the nodal displacement vector of the plate element.

$$\therefore \nabla^4 w(x,y) = \frac{\rho}{D} \qquad \text{...... (3.1.3a)}$$

Where $D = \frac{\varepsilon r^2}{12(1-\nu^2)}$ with ε = young's Modulus, and ν = Poisson ratio of the plate material respectively and r is the thickness of the plate.

For a variable moving load along y-axis keeping x-axis constant

$$D\nabla^4 w(x,y) = \rho Cos\omega t\delta(y-(y_0+\chi Sin\beta t)) \qquad \text{......... (3.1.26)}$$

For a circular moving load equation 3.1.2 becomes

$$D\nabla^4 w(x,y) = \rho\delta(x-(x_0-(x+\chi\cos\beta t))\delta(y-(y_0+\chi Sin\beta t)) \quad \text{..... (3.1.27)}$$

For a variable circular moving load equation 3.1.2 becomes

$$D\nabla^4 w(x,y) = \rho Cos\omega t\delta(x-(x_0+\chi\cos\beta t))\delta(y-(y_0+\chi Sin\beta t)) \quad \text{.... (3.1.28)}$$

See Aiyesimi[1]

Solving (3.1.24), (3.1.25), (3.1.26),(3.1.27), (3.1.28) we have

$$[M']\{U_i\} = \frac{\rho}{D}[N(x_0+\chi\cos\beta t)]^T \int[N(y)]^T dy \qquad \text{.......... (3.1.27)}$$

$$[M']\{U_i\} = \frac{\rho}{D}[N(y_0+\chi\sin\beta t)]^T \int[N(x)]^T dx \qquad \text{......... (3.1.28)}$$

$$[M']\{U_i\} = \frac{\rho}{D}\cos\omega t[N(y_0+\chi\sin\beta t)]^T \int[N(x)]^T dx \qquad \text{.......... (3.1.29)}$$

$$[M']\{U_i\} = \frac{\rho}{D}[N(x_0+\chi\cos\beta t)]^T[N(y_0+\chi\sin\beta t)]^T \qquad \text{...... (3.1.30)}$$

$$[M']\{U_i\} = \frac{\rho}{D}\cos\omega t[N(y_0+\chi\sin\beta t)]^T[N(x_0+\chi\cos\beta t]^T \qquad \text{...... (3.1.31)}$$

Since $[N] = [Q(x)Q(y), Q(x)Q(y)b, -R(x)Q(y)a, Q(x)S(y), -Q(x)T(y)b,$

$$-R(x)S(y)a, S(x)S(y), -S(x)T(y)b, T(x)S(y)a, S(x)Q(y), S(x)R(y)b, T(x)Q(y)a]$$

$$\text{.............. (3.1.32)}$$

Then equations (3.1.27), (3.1.28), (3.1.29), (3.1.30), (3.1.31) becomes

$$[M']\{U_i\} = \frac{\rho}{D}[N_1'] \qquad \cdots (3.1.32)$$

$$[M']\{U_i\} = \frac{\rho}{D}[N_2'] \qquad \cdots (3.1.33)$$

$$[M']\{U_i\} = \frac{\rho}{D}[N_3'] \qquad \cdots (3.1.33)$$

$$[M']\{U_i\} = \frac{\rho}{D}[N_4'] \qquad \cdots (3.1.34)$$

$$[M']\{U_i\} = \frac{\rho}{D}[N_5'] \qquad \cdots (3.1.35)$$

vhere $[N_1'],[N_2'],[N_3'],\ [N_4']$and $[N_5']$ are:

$$']=\begin{bmatrix} Q(x_0+\chi Cos\beta t)\int Q(y)dy \\ Q(x_0+\chi Cos\beta t)b\int R(y)dy \\ R(x_0+\chi Cos\beta t)a\int Q(y)dy \\ Q(x_0+\chi Cos\beta t)\int S(y)dy \\ -Q(x_0+\chi Cos\beta t)b\int T(y)dy \\ -R(x_0+\chi Cos\beta t)a\int T(y)dy \\ S(x_0+\chi Cos\beta t)\int S(y)dy \\ -S(x_0+\chi Cos\beta t)b\int T(y)dy \\ T(x_0+\chi Cos\beta t)a\int S(y)dy \\ S(x_0+\chi Cos\beta t)\int Q(y)dy \\ S(x_0+\chi Cos\beta t)b\int R(y)dy \\ T(x_0+\chi Cos\beta t)a\int Q(y)dy \end{bmatrix} \qquad [N_2']=\begin{bmatrix} Q(y_0+\chi Sin\beta t)\int Q(x)dx \\ R(y_0+\chi Sin\beta t)b\int Q(x)dx \\ Q(y_0+\chi Sin\beta t)a\int Q(x)dx \\ S(y_0+\chi Sin\beta t)\int Q(x)dx \\ -T(y_0+\chi Sin\beta t)b\int Q(x)dx \\ -S(y_0+\chi Sin\beta t)a\int R(x)dx \\ S(y_0+\chi Sin\beta t)\int S(x)dx \\ -T(y_0+\chi Sin\beta t)b\int S(x)dx \\ S(y_0+\chi Sin\beta t)a\int T(x)dx \\ Q(y_0+\chi Sin\beta t)\int S(x)dx \\ R(y_0+\chi Sin\beta t)b\int S(x)dx \\ Q(y_0+\chi Sin\beta t)a\int T(x)dx \end{bmatrix}$$

$$[N_3'] = \begin{bmatrix} Q(y_0 + \chi Sin\beta t)\int Q(x)dx \\ R(y_0 + \chi Sin\beta t)b\int Q(x)dx \\ Q(y_0 + \chi Sin\beta t)a\int Q(x)dx \\ S(y_0 + \chi Sin\beta t)\int Q(x)dx \\ -T(y_0 + \chi Sin\beta t)b\int Q(x)dx \\ -S(y_0 + \chi Sin\beta t)a\int R(x)dx \\ S(y_0 + \chi Sin\beta t)\int S(x)dx \\ -T(y_0 + \chi Sin\beta t)b\int S(x)dx \\ S(y_0 + \chi Sin\beta t)a\int T(x)dx \\ Q(y_0 + \chi Sin\beta t)\int S(x)dx \\ R(y_0 + \chi Sin\beta t)b\int S(x)dx \\ Q(y_0 + \chi Sin\beta t)a\int T(x)dx \end{bmatrix} Cos\omega t. \quad [N_4'] = \begin{bmatrix} Q(y_0 + \chi Sin\beta t)Q(x_0 + \chi\cos\beta t) \\ R(y_0 + \chi Sin\beta t)bQ(x_0 + \chi\cos\beta t) \\ Q(y_0 + \chi Sin\beta t)aQ(x_0 + \chi\cos\beta t) \\ S(y_0 + \chi Sin\beta t)Q(x_0 + \chi\cos\beta t) \\ -T(y_0 + \chi Sin\beta t)bQ(x_0 + \chi\cos\beta t) \\ -S(y_0 + \chi Sin\beta t)aR(x_0 + \chi\cos\beta t) \\ S(y_0 + \chi Sin\beta t)S(x_0 + \chi\cos\beta t) \\ -T(y_0 + \chi Sin\beta t)bS(x_0 + \chi\cos\beta t) \\ S(y_0 + \chi Sin\beta t)aT(x_0 + \chi\cos\beta t) \\ Q(y_0 + \chi Sin\beta t)S(x_0 + \chi\cos\beta t) \\ R(y_0 + \chi Sin\beta t)bS(x_0 + \chi\cos\beta t) \\ Q(y_0 + \chi Sin\beta t)aT(x_0 + \chi\cos\beta t) \end{bmatrix}$$

$$[N_5'] = \begin{bmatrix} Q(y_0 + \chi Sin\beta t)Q(x_0 + \chi\cos\beta t) \\ R(y_0 + \chi Sin\beta t)bQ(x_0 + \chi\cos\beta t) \\ Q(y_0 + \chi Sin\beta t)aQ(x_0 + \chi\cos\beta t) \\ S(y_0 + \chi Sin\beta t)Q(x_0 + \chi\cos\beta t) \\ -T(y_0 + \chi Sin\beta t)bQ(x_0 + \chi\cos\beta t) \\ -S(y_0 + \chi Sin\beta t)aR(x_0 + \chi\cos\beta t) \\ S(y_0 + \chi Sin\beta t)S(x_0 + \chi\cos\beta t) \\ -T(y_0 + \chi Sin\beta t)bS(x_0 + \chi\cos\beta t) \\ S(y_0 + \chi Sin\beta t)aT(x_0 + \chi\cos\beta t) \\ Q(y_0 + \chi Sin\beta t)S(x_0 + \chi\cos\beta t) \\ R(y_0 + \chi Sin\beta t)bS(x_0 + \chi\cos\beta t) \\ Q(y_0 + \chi Sin\beta t)aT(x_0 + \chi\cos\beta t) \end{bmatrix} \cos\omega t$$

The evaluation of $[H]^T \otimes [H]$, $[H^1]^T \otimes [H^1]$and $[H]^T \otimes [H^1]$are as in appendix A, B, C before the integration is performed and the boundary of the plate used.

We then obtained $[T]$ and substituted in terms of $[M']$

Such that (3.1.22), (3.1.33), (3.1.34), (3.1.35), (3.1.36), and (3.1.37) becomes:

$$\{U_i\} = \frac{\rho}{D}[T]^{-1}[N'] \qquad \ldots (3.1.38)$$

$$\{U_i\} = \frac{\rho}{D}[T]^{-1}[N_1'] \qquad \ldots (3.1.39)$$

$$\{U_i\} = \frac{\rho}{D}[T]^{-1}[N_2'] \qquad \ldots (3.1.40)$$

$$\{U_i\} = \frac{\rho}{D}[T]^{-1}[N_3'] \qquad \ldots (3.1.41)$$

$$\{U_i\} = \frac{\rho}{D}[T]^{-1}[N_4'] \qquad \ldots (3.1.42)$$

$$\{U_i\} = \frac{\rho}{D}[T]^{-1}[N_5'] \qquad \ldots (3.1.43)$$

Respectively

CHAPTER FOUR NUMERICAL SIMUTIONS

One element is used to cover the entire domain of the rectangular plate this is assumed to be unit plate.

With these the limit of our integral values will be from 0 to 1 on both axes.

Each element in appendix A,B,C where evaluated by a software math Computer Aided Design, also to other matrices involves. Attached are printed out results for each matrices.

The constants involve are then varied to investigate their effect to the system. The boundary condition of the edge plate also is used before variation takes place.

Zero to eleven are numbers of columns and rows of matrices A, B and C

Were matrices $z, x^T, and\ w$ are numerical results for A,B, and C

$z =$

	0	1	2	3	4	5	6	7	8	9	10	11
0	4.457	0.629	-0.314	1.543	-1.371	0.314	-1.543	1.371	-0.771	-4.457	-0.629	-2.229
1	0.629	0.114	0.314	0.371	-0.286	0.057	-0.371	0.286	-0.186	-0.629	0.171	-0.314
2	-0.314	-0.314	1.486	-0.771	0.686	-0.21	0.771	-0.686	-0.257	2.229	0.314	0.743
3	1.543	0.371	-0.771	4.457	-1.629	0.186	-4.457	1.629	-2.229	-1.543	-0.371	-0.771
4	-1.371	-0.286	0.686	-1.629	0.914	-0.143	1.629	0.914	0.814	1.371	0.286	0.686
5	0.314	0.057	-0.21	0.186	-0.143	0.038	-0.186	0.143	-0.062	-0.057	-0.057	-0.105
6	-1.543	-0.371	0.771	-4.457	1.629	-0.186	4.457	-1.629	2.229	1.543	0.371	0.771
7	1.371	0.286	-0.686	1.629	-0.914	0.143	0.371	0.914	-0.814	-1.371	-0.286	-0.686
8	-0.771	-2.229	0.257	2.229	-0.814	-0.062	-2.229	-0.814	1.486	0.771	0.186	0.514
9	-4.457	-0.629	2.229	-1.543	1.371	-0.314	1.543	-1.371	0.771	4.457	0.629	2.229
10	-0.629	-0.114	0.314	-0.371	0.286	-0.057	0.371	-0.286	0.186	0.629	0.114	0.314
11	-2.229	-0.314	0.743	-0.771	0.686	-0.105	0.771	-0.686	0.514	2.229	0.314	1.486

$x^T =$

	0	1	2	3	4	5	6	7	8	9	10	11
0	2.88	0.24	-2.64	-2.88	0.24	0.22	2.88	-0.24	0.24	-2.88	-0.24	-0.24
1	2.64	0.32	-0.293	-0.24	-0.48	-0.027	0.24	0.48	0.02	-2.64	-0.32	-0.22
2	-0.24	-0.02	0.32	0.24	-0.24	-0.027	-0.24	0.02	0.08	0.24	0.02	-0.08
3	-2.88	-0.24	-2.64	2.88	-0.24	-0.22	-2.88	0.24	-0.24	2.88	0.24	0.24
4	-0.24	-0.08	-0.22	-2.64	0.72	-0.073	2.64	-0.72	0.22	-0.24	0.08	-0.02
5	0.24	0.02	-0.32	-0.24	0.02	0.027	0.24	-0.02	-0.08	0.24	-0.02	0.08
6	2.88	0.24	-0.24	-2.88	0.24	0.02	2.88	-0.24	0.08	-2.88	-0.24	-2.64
7	-0.24	0.08	0.02	2.64	-0.72	$6.667\cdot10^{-3}$	-2.64	0.72	-2.42	0.24	-0.08	0.22
8	0.24	0.02	0.48	-0.24	0.02	-0.04	0.24	-0.02	-0.72	-0.24	-0.02	-0.72
9	-2.88	0	0.48	0	-0.24	0	-2.88	0	0	2.88	0	2.64
10	-2.64	-0.32	0.22	0.24	0.48	-0.027	-0.24	0.48	-0.22	2.64	0.32	2.42
11	-0.24	0.02	-0.48	0.24	-0.02	0.04	0.02	-0.02	-0.72	-0.24	0.02	0.72

$w =$

	0	1	2	3	4	5	6	7	8	9	10	11
0	4.457	2.229	-0.629	-4.457	2.229	0.629	-1.543	0.771	1.371	1.543	0.771	1.371
1	2.229	1.486	-0.629	-2.229	0.743	0.314	-0.771	0.257	0.686	0.771	0.514	0.686
2	-0.629	-0.314	0.114	0.629	-0.314	-0.114	0.371	-0.186	0.286	-0.371	-0.186	-0.286
3	4.457	0.743	-0.943	4.457	-2.229	-0.629	1.543	-0.771	1.371	-1.543	0.771	-1.371
4	2.229	1.486	-0.314	-2.229	1.486	0.314	-0.771	0.514	-0.686	0.771	0.257	0.686
5	0.629	0.314	-0.114	-0.629	0.314	0.114	-0.371	0.186	-0.286	0.371	0.186	0.286
6	-1.543	-0.771	0.371	1.543	-0.771	-0.371	4.457	-2.229	1.629	-4.457	-2.229	-0.543
7	0.771	0.514	-0.186	-0.771	4.629	0.186	-2.229	1.486	-0.814	2.229	0.743	0.814
8	1.371	0.686	0.286	-1.371	-0.686	-0.286	1.629	-0.814	0.914	-1.629	-0.814	-0.914
9	1.543	-0.771	-0.371	-1.543	0.771	-0.286	-4.457	2.229	-1.629	4.457	2.229	1.629
10	0.771	0.514	-0.186	0.771	0.257	0.186	-2.229	0.743	-0.814	2.229	1.486	0.814
11	1.371	0.686	-0.286	-1.371	0.686	-0.286	-1.629	0.814	-0.914	1.629	0.814	0.914

$x^T + z + w =$

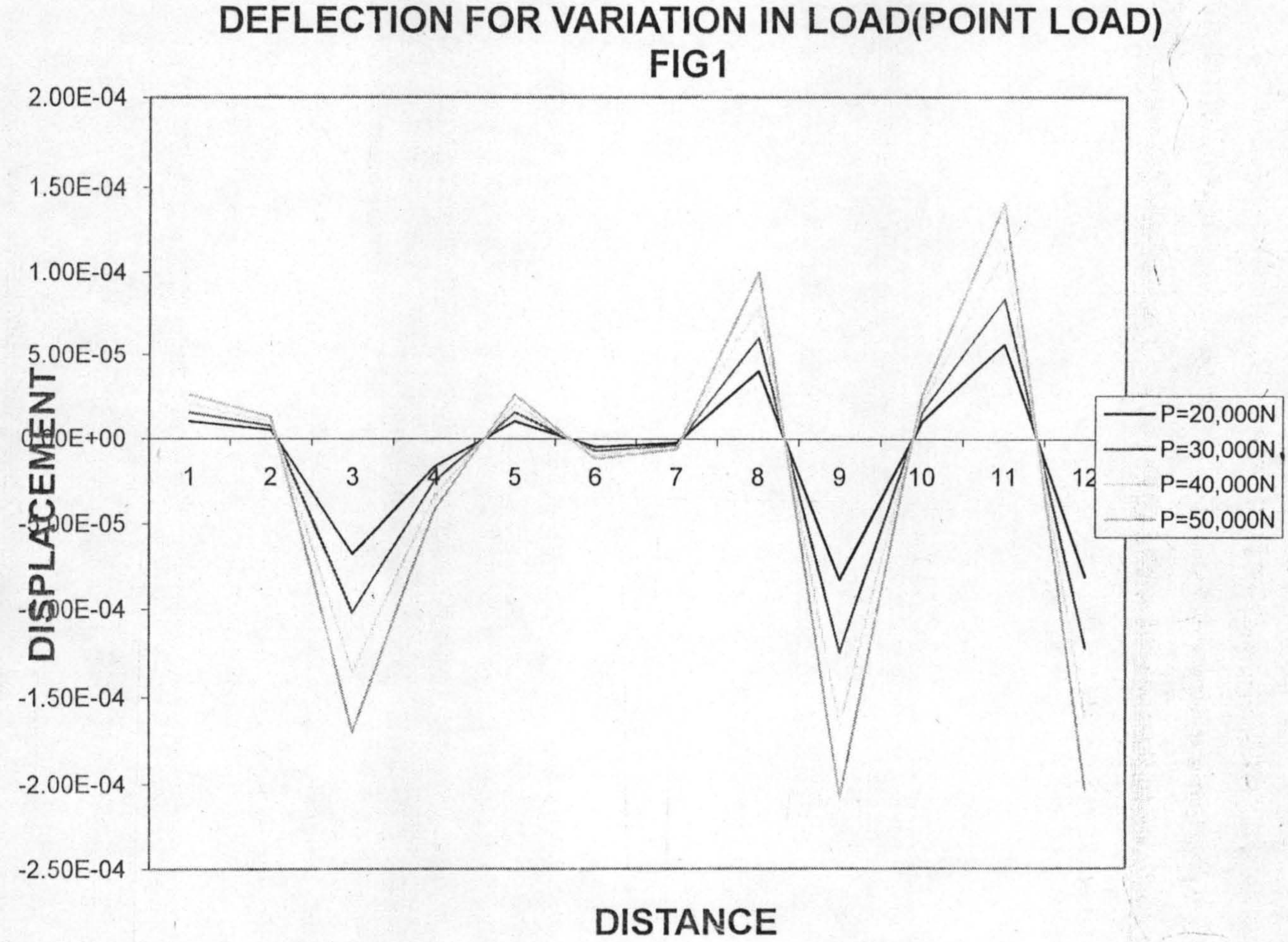
	0	1	2	3	4	5	6	7	8	9	10	11
0	11.794	3.097	-3.583	-5.794	1.097	1.163	-0.206	1.903	0.84	-5.794	-0.097	-1.097
1	5.497	1.92	-0.608	-2.097	-0.023	0.345	-0.903	1.023	0.52	-2.497	0.366	0.151
2	-1.183	-0.649	1.92	0.097	0.131	-0.35	0.903	-0.851	0.109	2.097	0.149	0.377
3	3.12	0.874	-4.354	11.794	-4.097	-0.663	-5.794	1.097	-1.097	-0.206	0.64	-1.903
4	0.617	1.12	0.151	-6.497	3.12	0.098	3.497	0.709	0.349	1.903	0.623	1.351
5	1.183	0.391	-0.644	-0.683	0.191	0.179	-0.317	0.309	-0.428	0.554	0.109	0.261
6	-0.206	-0.903	0.903	-5.794	1.097	-0.537	11.794	-4.097	3.937	-5.794	-2.097	-2.411
7	1.903	0.88	-0.851	3.497	2.994	0.335	-4.497	3.12	-4.049	1.097	0.377	0.349
8	0.84	-1.523	1.023	0.617	-1.48	-0.388	-0.36	-1.649	1.68	-1.097	-0.649	-1.12
9	-5.794	-1.4	2.337	-3.086	1.903	-0.6	-5.794	0.857	-0.857	11.794	2.857	6.497
10	-2.497	0.08	0.349	0.64	1.023	0.102	-2.097	0.937	-0.849	5.497	1.92	3.549
11	-1.097	0.391	-0.023	-1.903	1.351	-0.35	-0.837	0.109	-1.12	3.617	1.149	3.12

Sum of matrices A B and C

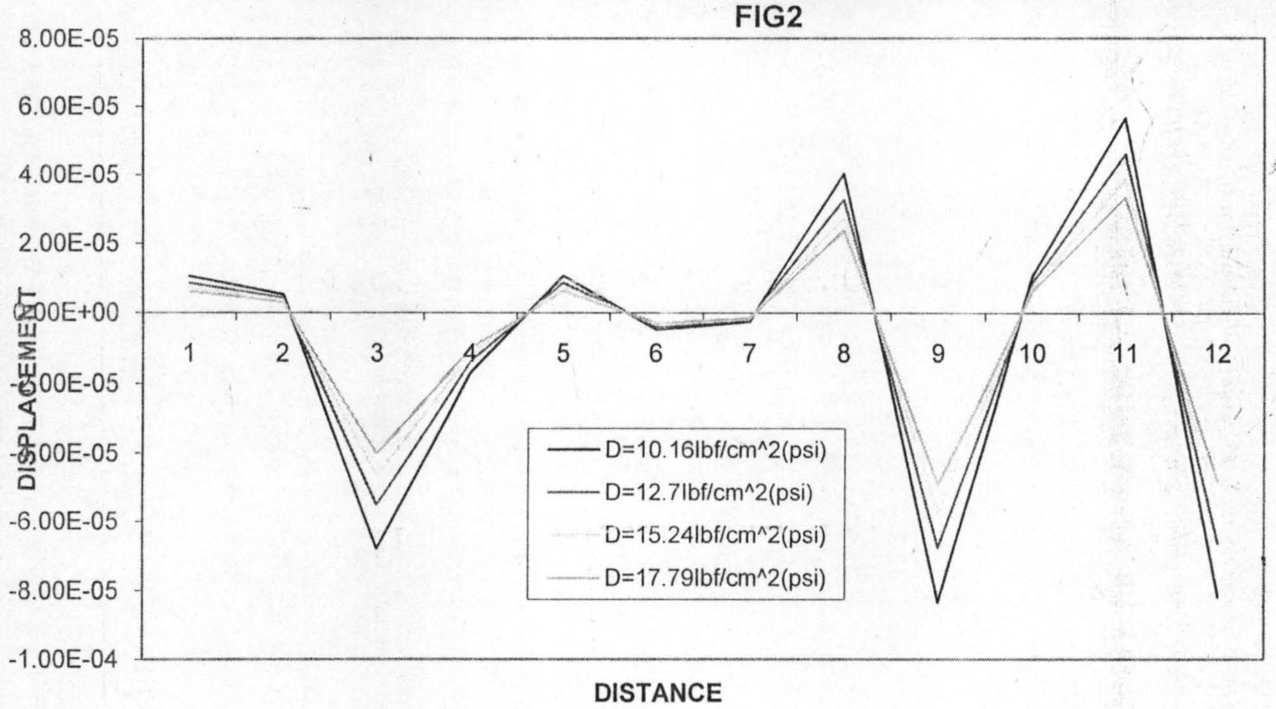
The matrix after the boundary condition of the plate

T =

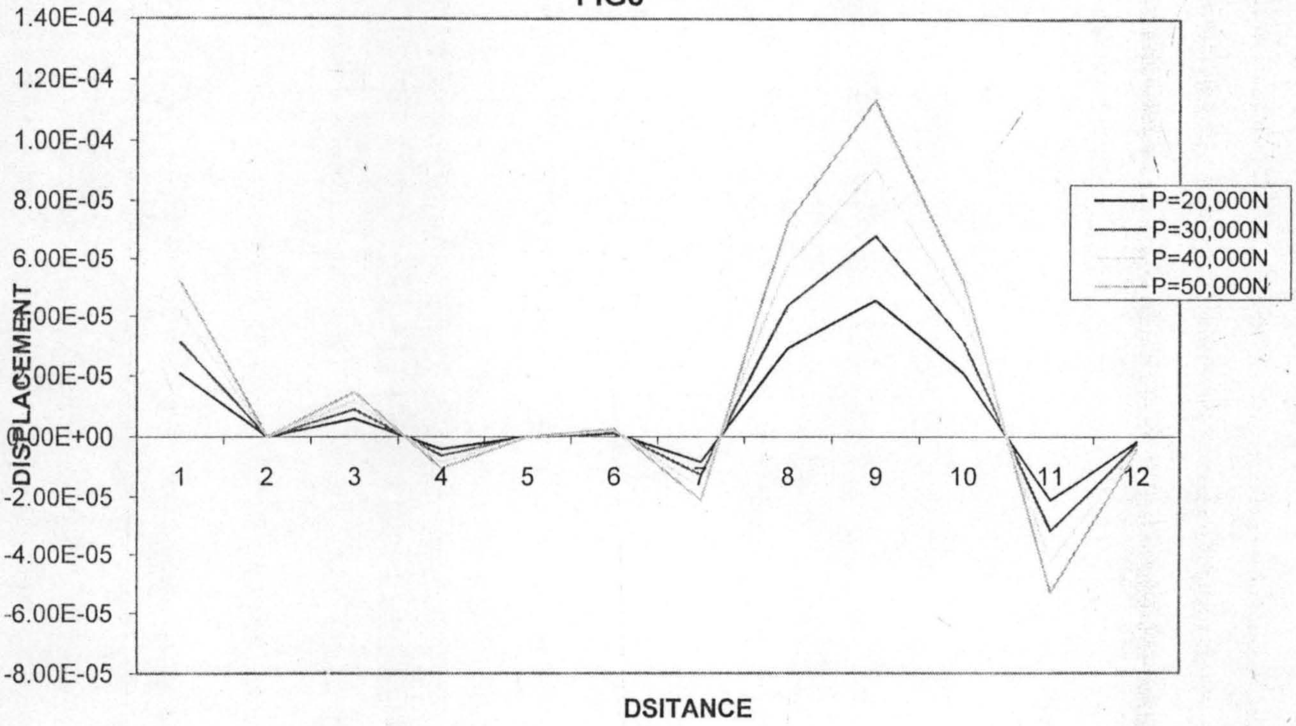
	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0
2	0	0	0.253	1.075	0	-2.209	-1.635	0.313	-0.183	0	0.248	0.277
3	0	0	0.187	7.83	0	-1.326	-10.345	1.983	-2.089	0	0.631	2.12
4	0	0	0	0	1	0	0	0	0	0	0	0
5	0	0	-0.144	-0.998	0	0.126	0.941	-0.264	0.194	0	-0.142	-0.197
6	0	0	-1.568	-10.776	0	1.297	10.169	-1.951	2.06	0	-0.542	-0.242
7	0	0	0.317	2.18	0	-0.258	-1.852	0.399	-0.415	0	0.037	0.061
8	0	0	-0.299	0.539	0	0.25	-0.436	-0.374	0.087	0	-0.296	-0.405
9	0	0	0	0	0	0	0	0	0	1	0	0
10	0	0	0.253	0.698	0	-0.402	-1.542	0.295	-0.313	0	0.234	0.318
11	0	0	0.317	2.183	0	-0.075	-2.061	0.394	-0.42	0	0.571	0.427

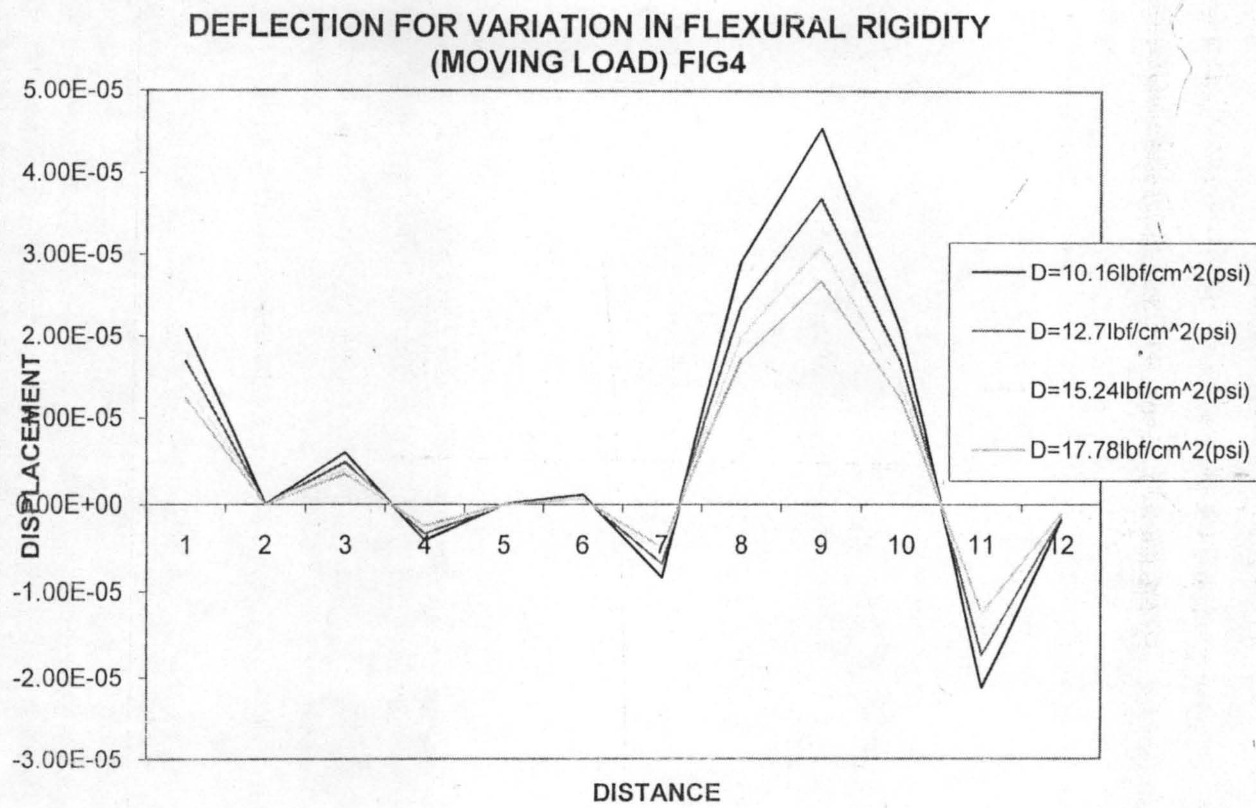

DEFLECTION FOR VARIATION IN LOAD(POINT LOAD)
FIG1
2.00E-04
1.50E-04
1.00E-04
5.00E-05
0.00E+00
-5.00E-05
-1.00E-04
-1.50E-04
-2.00E-04
-2.50E-04
DISPLACEMENT
1
2
3
4
5
6
7
8
9
10
11
12
DISTANCE
P=20,000N
P=30,000N
P=40,000N
P=50,000N

DEFLECTION FOR VARIATION IN FLEXURAL RIGIDITY(POINT LOAD)
FIG2

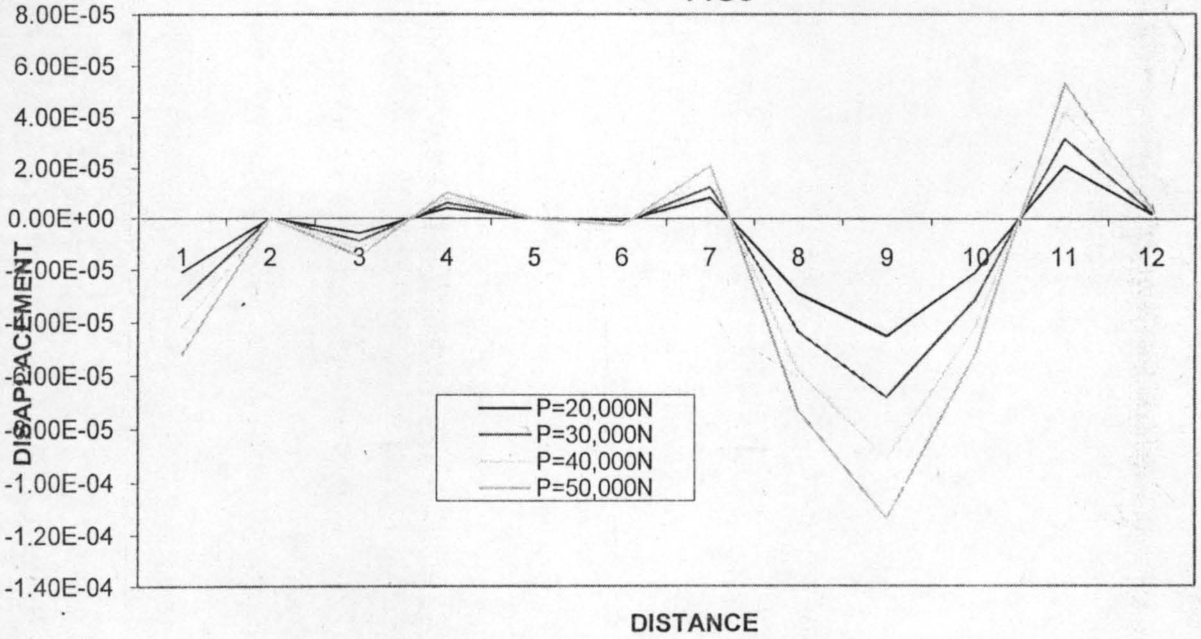


DEFLECTION FOR VARIATION IN LOAD(MOVING LOAD)
FIG3

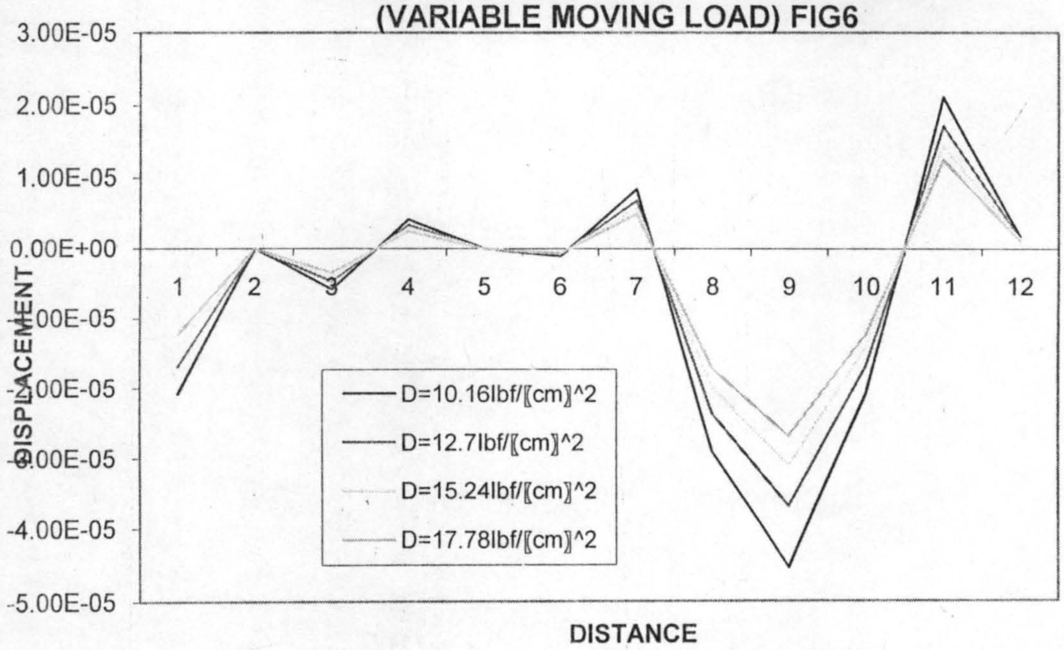



DEFLECTION FOR VARIATION IN FLEXURAL RIGIDITY
(MOVING LOAD) FIG4
5.00E-05
4.00E-05
3.00E-05
2.00E-05
1.00E-05
0.00E+00
-1.00E-05
-2.00E-05
-3.00E-05
DISPLACEMENT
1
2
3
4
5
6
7
8
9
10
11
12
DISTANCE
D=10.16lbf/cm^2(psi)
D=12.7lbf/cm^2(psi)
D=15.24lbf/cm^2(psi)
D=17.78lbf/cm^2(psi)

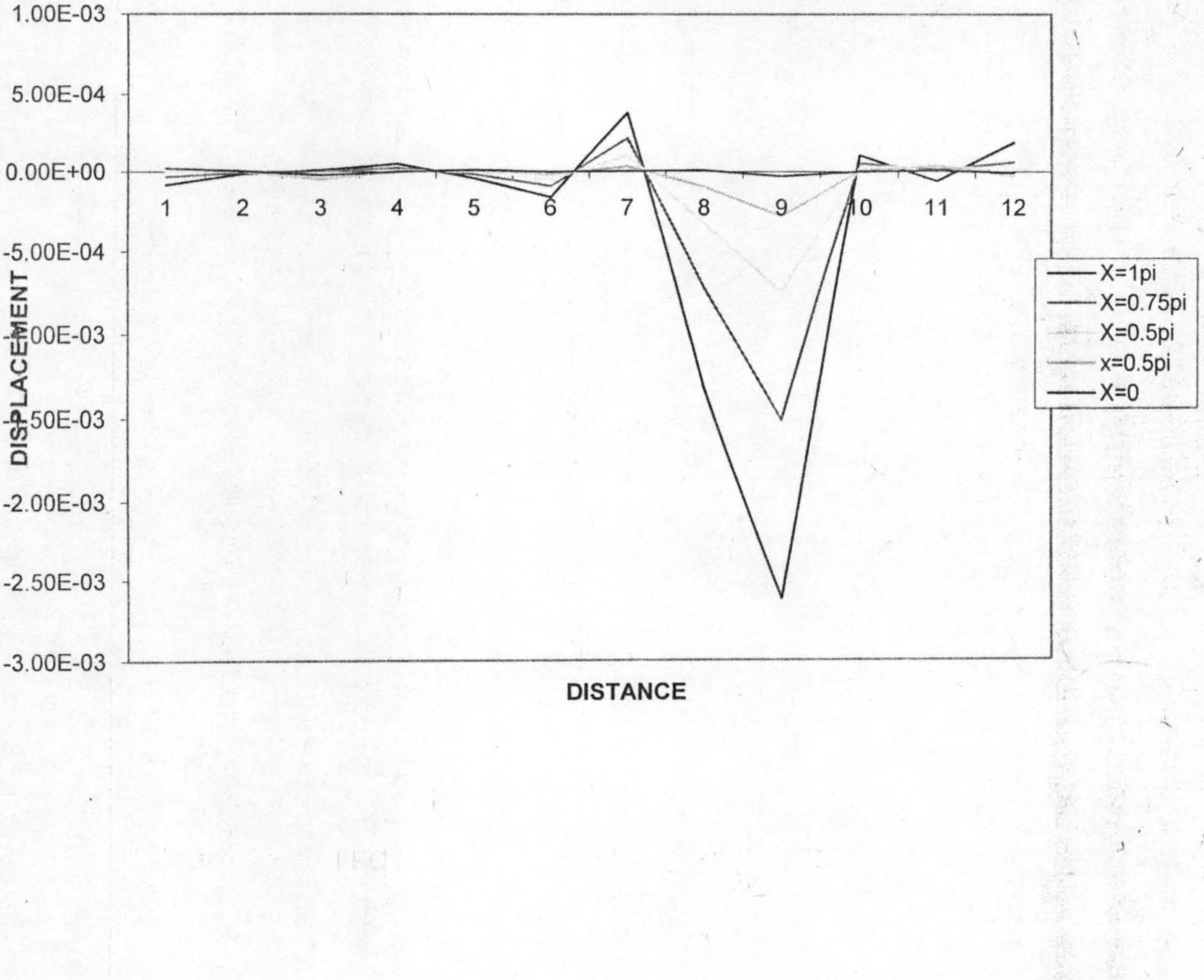
DEFLECTION FOR VARIATION IN LOAD (VARIABLE MOVING LOAD)
FIG5



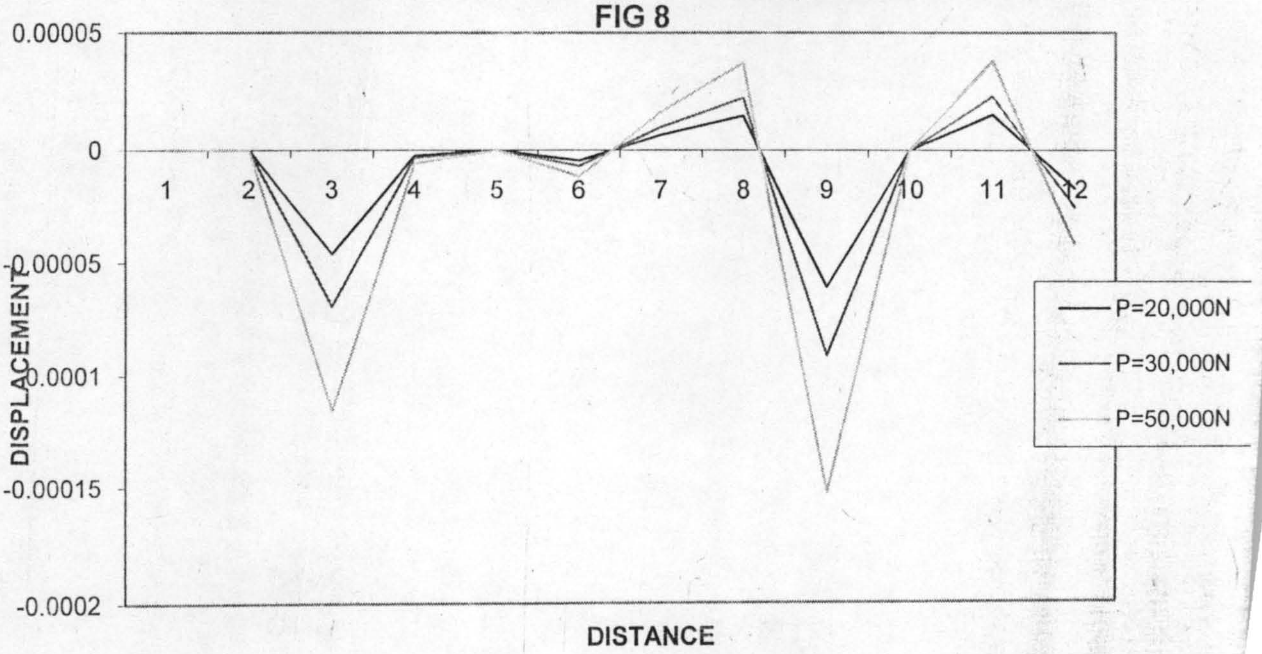
DEFLECTION FOR VARIATION IN FLEXURAL RIGIDITY (VARIABLE MOVING LOAD) FIG6



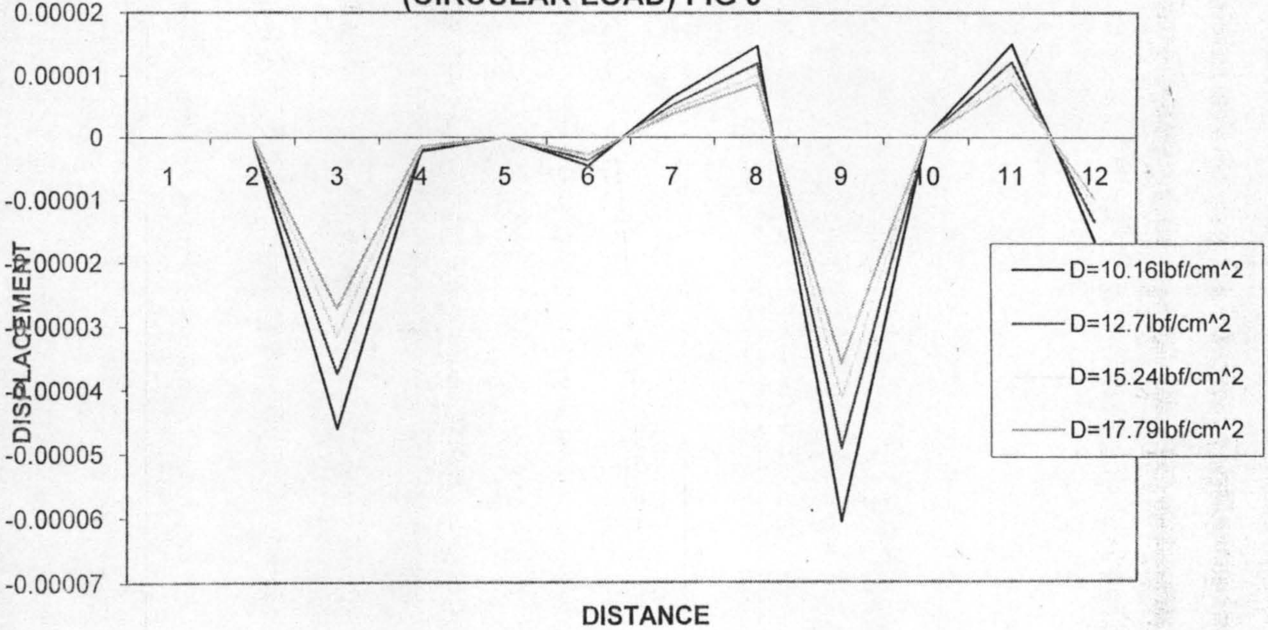
DEFLECTION FOR VARIATION IN RADIUS(X)(MOVING LOAD)
FIG 7

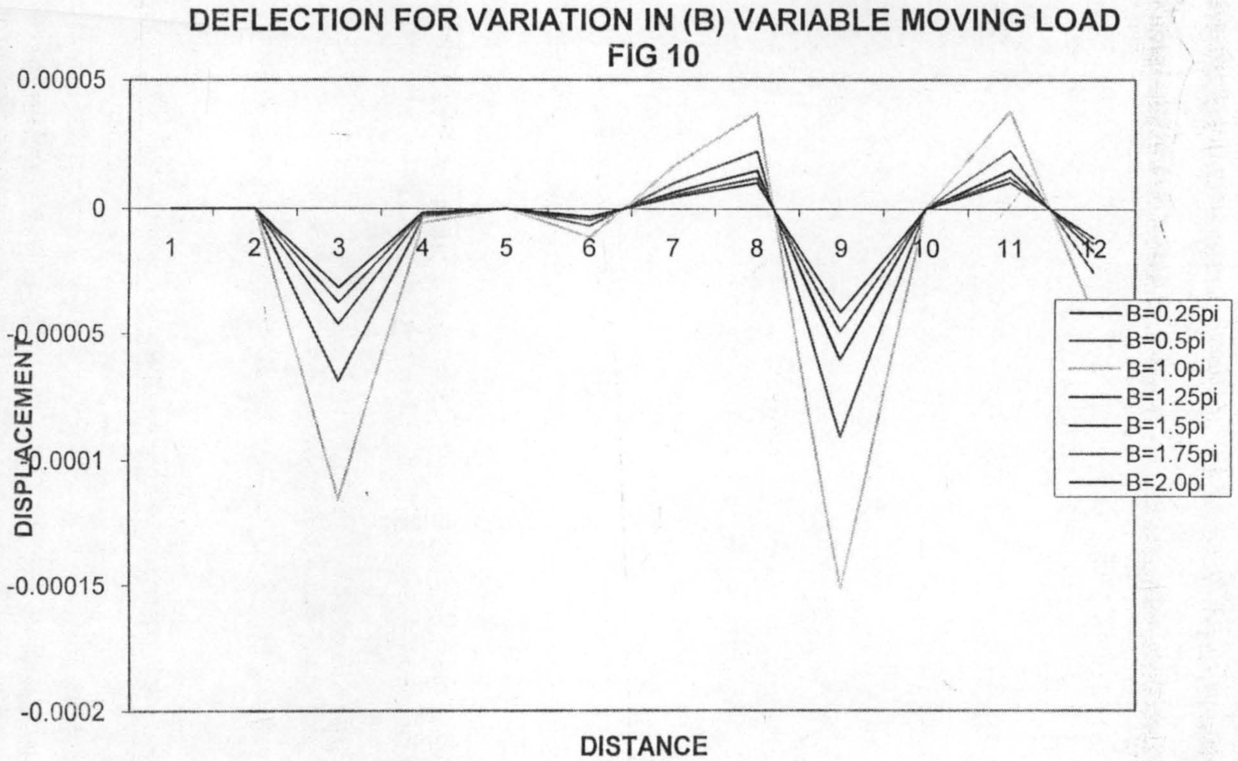


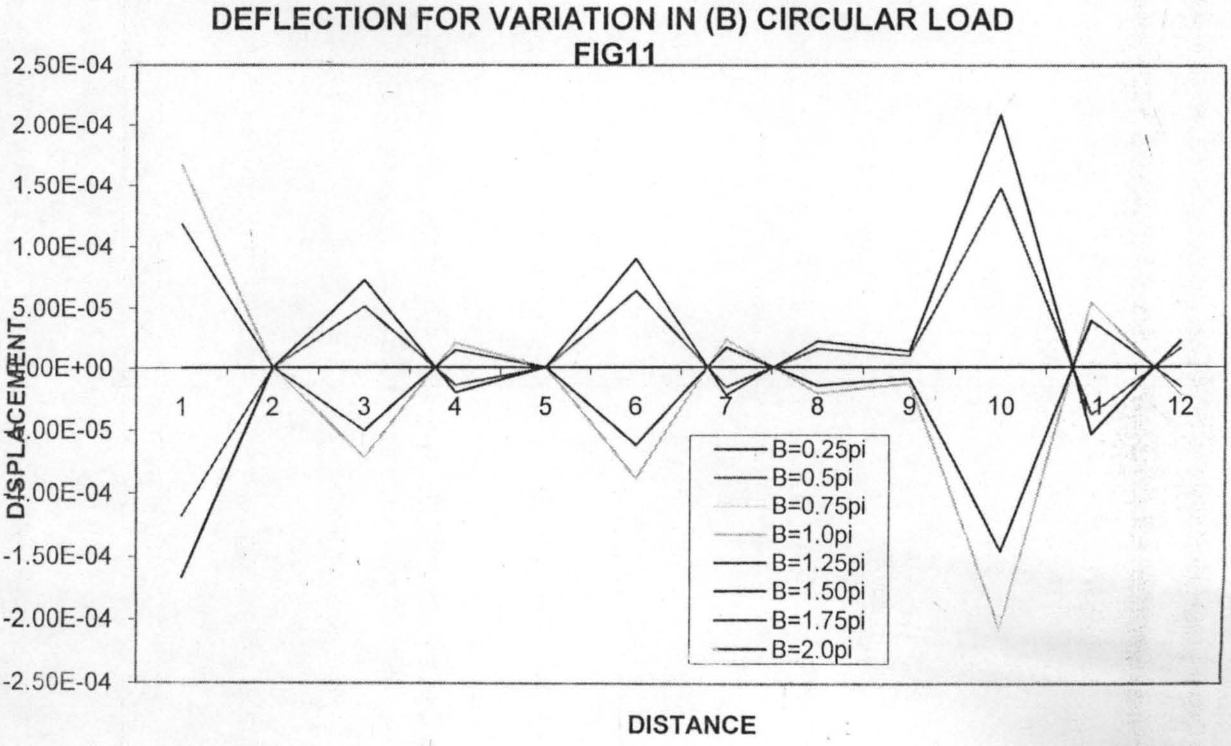
DEFLECTION FOR VARIATION IN LOAD (CIRCURAL LOAD)
FIG 8

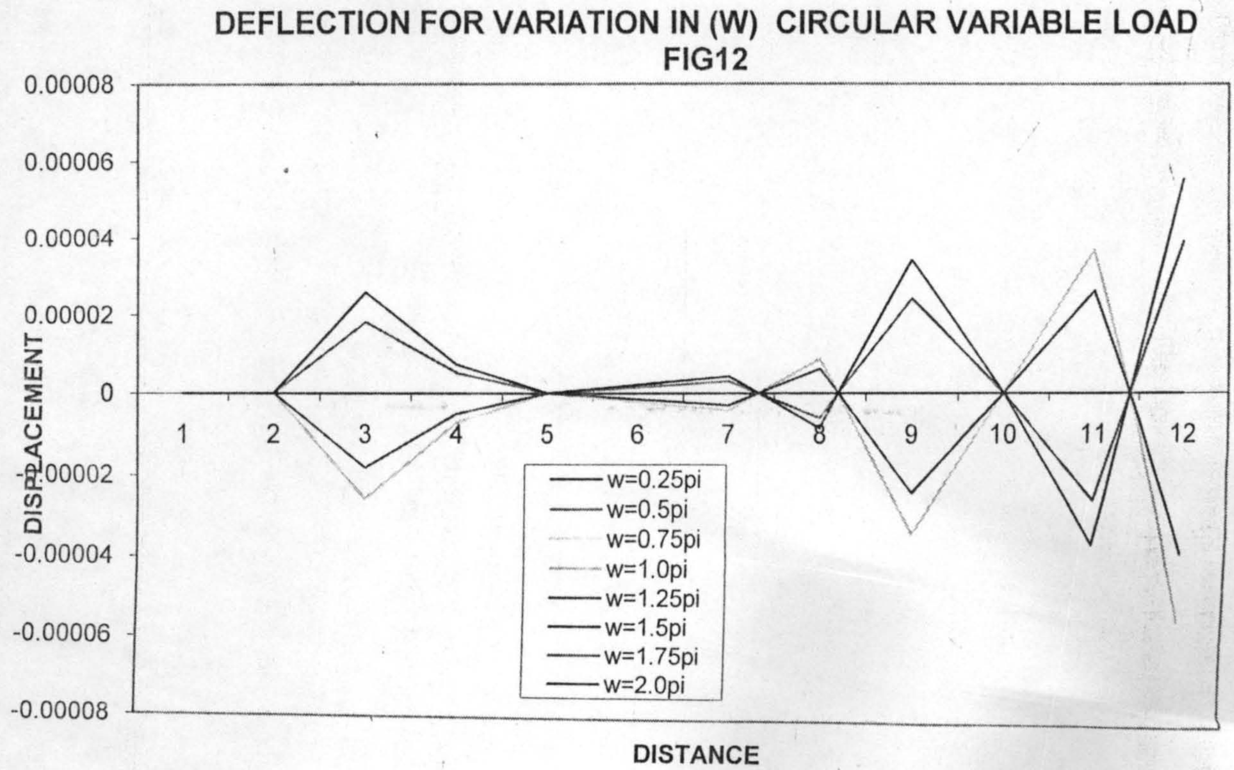


DEFLECTION FOR VARIATION IN FLEXURAL RIGIDITY (CIRCULAR LOAD) FIG 9




DEFLECTION FOR VARIATION IN (B) VARIABLE MOVING LOAD
FIG 10
0.00005
0
-0.00005
-0.0001
-0.00015
-0.0002
DISPLACEMENT
1
2
3
4
5
6
7
8
9
10
11
12
DISTANCE
B=0.25pi
B=0.5pi
B=1.0pi
B=1.25pi
B=1.5pi
B=1.75pi
B=2.0pi


DEFLECTION FOR VARIATION IN (B) CIRCULAR LOAD
FIG11
2.50E-04
2.00E-04
1.50E-04
1.00E-04
5.00E-05
0.00E+00
-5.00E-05
-1.00E-04
-1.50E-04
-2.00E-04
-2.50E-04
DISPLACEMENT
1
2
3
4
5
6
7
8
9
10
11
12
B=0.25pi
B=0.5pi
B=0.75pi
B=1.0pi
B=1.25pi
B=1.50pi
B=1.75pi
B=2.0pi
DISTANCE



DEFLECTION FOR VARIATION IN (W) CIRCULAR VARIABLE LOAD
FIG12
0.00008
0.00006
0.00004
0.00002
0
-0.00002
-0.00004
-0.00006
-0.00008
DISPLACEMENT
1
2
3
4
5
6
7
8
9
10
11
12
w=0.25pi
w=0.5pi
w=0.75pi
w=1.0pi
w=1.25pi
w=1.5pi
w=1.75pi
w=2.0pi
DISTANCE

CHAPTER FIVE

5.1 Discussion of Results

From Figures 1, 3, 5, and 8 we have shown the displacement against distance for various values of ρ (load) with $4.8 \times 10^8 N$. It can be observed that an increase in ρ leads to increase in the deflection. Also the displacement is sinusoidal with a maximum value at x = 10.8, 8.5, 10.5, and 10.5. The peak is at x = 8.5, 10.5, 8.5, and 8.5

In figures 2, 4, 6, and 9 we have shown the displacement against distance for various values of D flexural rigidity keeping ρ load constant. It can be observed that increase in D leads to decrease in deflection, which is also of sinusoidal with a maximum value at x = 10.8, 8.5, 6.5, and 10.5, the peak is at x = 8.510.5, 8.5, and 8.5 were ρ = 20000.

In fig 7, we have showed the displacement against distance for various variational radii χ it can be observe that the graph is almost linear at the point $\chi = 0$ deflection increases as radii increases with maximum point at 6.8 peak at 8.5 also sinusoidal.

In fig 10 we have shown the effect of variational angle on moving loads, the graph is sinusoidal. The deflection is higher at β=π and lower at β=π/4 with the same maximum and peak points at 8.5 and 10.5 respectively.

In fig 11 and 12 we look at the effect of variational angle of variable load, it was observed that the graph are anti symmetric in shape for some angle e.g $\pi/4$ and π it is linear at $\beta=3\pi/2$

5.2 Conlusion

In this research work we have studied the deflection of a plate subjected to the moving load using finite element method.

The system was interpolated in to a shape function by the used of cubic hermirtian interpolating polynomial. This is then solved by Galerkin`s procedure, multiplying through with weight residual function. The integral is then taken within the interval of plate dimension. This gives us the solution to the mass matrix of the problem. The nodal displacement is then determined after considering the natural boundary condition of the plate, by multiplying through with its inverse.

The magnitude of moving load, flexural rigidity of plate material, variational is then analyzed by plotting the graph of nodal displacement against the distance of deflection. It was also observed that the displacement increases for increase in load and decreases for increases flexural rigidity for any type of load. The variational angle also effect the displacement, well effect of variational radii is only felt when the load is moving along y direction for χ between zero and one. For load moving along x direction and circular load it is felt for χ from one upward. It is also observed that variational angle of variable loads affect the deflection and higher at π lower at 2π on both moving and circular variable loads.

5.3 Recommendation

With the above observation it is therefore recommended that the flexural rigidity of plate shall be proportional to the highest expected load passing through the plate. The radii should be proportional also, variational angle at $\pi/2$ is more sweet able for all problems. This reduces the problem of the plate collapse or crack. These ideas can be used in the construction of bridges and roads.

For further research the effect of eccencity, natural frequency variation of height of the plate i.e. on uniform can be considered. This will improve the result of this research work and structural reliability of the plate if applied in the construction.

REFERENCE:

Aiyesimi Y.M (1989) "On the Dynamics Analysis of elastic Beam on visco elastic foundation subjected to oscillating constant/variable magnitude load" M.Sc. Thesis University of Ilorin.

Buchana G.R. (1995) "Theory and problems of finite element analysis" Mc Graw Hill

Huebner K. H(1975) "Finite element method for engineers" John Wiley and sons

Oni S. T (1997) "On the thick Beams under the action of a variable transverses load" ABACU, The journal of Mathematics Association of Nigeria Vol. 25

Oniyekwe O. O. (2002) "A simplified green element analysis of static and dynamic foundation" New England College Hanniker, USA.

Timosheko S., Young D.H and Wesves W. (1994) "Vibration problems in engineering" Willey Inc.

Smith J. W (1996) "Vibration of structure application in Civil engineering design"

Williams B. B (1998) "Advanced mechanics of materials"

APPENDIX B

$36m(y)^2 Q(x)^2$	$-12bm(y)n(y) Q(x)^2$	$36am(y)^2 Q(x)R(x)$	$-36m(y)^2 Q(x)^2$	$-12bm(y)o(y) Q(x)^2$	$36am(y)^2 Q(x)R(x)$	$-36a^2m(y)^2 R(x)S(x)$	-12am(y)o(y) S(y) Q(x)	$36am(y)^2 Q(x)T(x)$	$-36m(y)^2 Q(x)S(x)$	- 12bn(y)m(y)Q (x)S(x)	$36am(y)^2 Q(x)T(x)$
- 12bm(y)n(y) $Q(x)^2$	$4 b^2 n(y)^2 Q(x)^2$	12abn(y)m(y) Q(x)R(x)	$12bm(y) n(y)Q(x)^2$	$4b^2n(y) o(y)Q(x)^2$	-12am(y)o(y) Q(x)R(x)	12bm(y) n(y)Q(x) S(x)	$4 b^2 n(y)^2 Q(x)S(x)$	- 12abn(y)m(y)Q(x)T(x)	12bm(y) n(y)Q(x) S(x)	$4 b^2 n(y)^2 Q(x)S(x)$	12abn(y)m(y)Q(x)T(x)
$-36am(y)^2 Q(x)R(x)$	12abn(y)m(y) Q(x)R(x)	$36a^2m(y)^2 R(x)^2$	$36am(y)^2 Q(x)R(x)$	12abm(y) o(y) Q(x)R(x)	$-36 a^2m(y)^2 R(x)^2$	$36am(y)^2 R(x)S(x)$	-12abn(y)o(y) S(y) R(x)	$-36a^2m(y)^2 R(x)T(x)$	$36am(y)^2 R(x)S(x)$	12abm(y) n(y)R(x) S(x)	$36a^2m(y)^2 R(x)T(x)$
$-36m(y)^2 Q(x)^2$	$12bn(y)m(y) Q(x)^2$	$36a^2m(y)^2 R(x)Q(x)$	$36m(y)^2 Q(x)^2$	$12bm(y) o(y) Q(x)^2$	$-36 am(y)^2 Q(x)R(x)$	$36m(y)^2 R(x)S(x)$	12bm(y)o(y) S(x) Q(x)	$-36a^2m(y)^2 Q(x)T(x)$	$36m(y)^2 Q(x)S(x)$	12bm(y) n(y)Q(x) S(x)	$36am(y)^2$ Q(x)T(x)
- 12bm(y)o(y) $Q(x)^2$	$4b^2n(y)o(y) Q(x)^2$	12abm(y)o(y) R(x)Q(x)	$12bm(y) o(y)Q(x)^2$	$4b^2o(y)^2 Q(x)^2$	-12abm(y) o(y) Q(x) R(x)	12bm(y) o(y)Q(x)S(x	$4 b^2 o(y)^2 Q(x)S(x)$	- 12abm(y)o(y) T(x) Q(x)	12bm(y)o(y) S(x) Q(x	$4 b^2 n(y)o(y) Q(x)S(x)$	12abo(y)m(y)Q(x)T(x)
$-36am(y)^2 Q(x)R(x)$	-12abn(y)m(y) Q(x)R(x)	$-36a^2m(y)^2 R(x)^2$	$-36m(y)^2 Q(x)R(x)$	-12abm(y) o(y) Q(x)R(x)	$36a^2m(y)^2 R(x)^2$	$-36m(y)^2 R(x)S(x)$	-12abm(y)o(y) R(x) S(x)	$36a^2m(y)^2 R(x)T(x)$	$-36am(y)^2 R(x)S(x)$	-12abm(y) n(y)R(x) S(x)	$-36a^2m(y)^2 R(x)T(x)$
$-36m(y)^2 Q(x)S(x)$	12bn(y)m(y) Q(x)S(x)	$36am(y)^2 R(x)S(x)$	$36m(y)^2 Q(x)S(x)$	12bo(y) m(y) Q(x)S(x)	$-36am(y)^2 S(x)R(x)$	$36m(y)^2 S(x)^2$	$12bm(y)o(y) S(x)^2$	$-36am(y)^2 S(x)T(x)$	$36m(y)^2 S(x)S(x)$	12bm(y) n(y)S(x) S(x)	$36am(y)^2 S(x)T(x)$
- 12bm(y)o(y) S(y) Q(x)	$4b^2n(y)o(y) Q(x)S(x)$	12abm(y)o(y) R(x)S(x)	12bm(y)o(y) Q(x)S(x)	$4b^2o(y)^2 Q(x)S(x)$	-12abm(y)o(y) R(x)S(x)	$12bm(y)o(y) S(x)^2$	$4 b^2 o(y)^2 S(x)^2$	-12abm(y) o(y) S(x)T(x)	$12bm(y)o(y) S(x)^2$	$4 b^2 n(y)o(y) S(x)S(x)$	12abo(y)m(y)S(x) T(x)
$36am(y)^2 Q(x)T(x)$	-12abn(y)m(y) Q(x)T(x)	$-36a^2m(y)^2 R(x)T(x)$	$-36am(y)^2 Q(x)T(x)$	-12abm(y) o(y) Q(x)T(x)	$36a^2m(y)^2 R(x)T(x)$	$-36 am(y)^2 S(x)T(x)$	-12abm(y)o(y) T(x) S(x)	$36a^2m(y)^2 T(x)^2$	$-36am(y)^2 T(x)S(x)$	-12abm(y) n(y)T(x) S(x)	$-36 a^2 m(y)^2 T(x)^2$
$-36m(y)^2 Q(x)S(x)$	12bn(y)m(y) Q(x)S(x)	$36am(y)^2 R(x)S(x)$	$36m(y)^2 Q(x)S(x)$	12bm(y) o(y) Q(x)S(x)	$-36 am(y)^2 R(x)S(x)$	$36m(y)^2 S(x)^2$	12b m(y)o(y) S(x) S(x)	$-36am(y)^2 S(x)T(x)$	$36m(y)^2 S(x)^2$	12bm(y) n(y)S(x) S(x)	$36am(y)^2 S(x)T(x)$
- 12bm(y)n(y) Q(x)S(x)	$4b^2n(y)^2 S(x)Q(x)$	12abm(y)n(y) R(x)S(x)	12bm(y)n(y) Q(x)S(x)	$4b^2o(y)n(y) Q(x)S(x)$	-12abm(y)n(y) R(x)S(x)	$12bm(y) S(x)^2$	$4b^2n(y)o(y) S(x)^2$	-12abm(y) n(y) S(x)T(x)	$12bm(y)n(y) S(x)^2$	$4b^2n(y)^2 S(x)^2$	12abn(y)m(y)S(x) T(x)
$-36m(y)^2 Q(x)T(x)a$	12abn(y)m(y) Q(x)T(y)	$36a^2m(y)^2 R(x)T(x)$	$36am(y)^2 Q(x)S(x)$	12abm(y) o(y) Q(x)T(x)	$-36a^2m(y)^2 R(x)T(x)$	$36am(y)^2 S(x)T(x)$	12abm(y)o(y) T(x) S(x)	$-36a^2m(y)^2 T(x)^2$	$12am(y)^2 S(x)T(x)$	12abm(y) n(y)T(x) S(x)	$36 a^2 m(y)^2 T(x)^2$

APPENDIX C

36m(x)m(y) Q(x)Q(y)	36bm(y)m(x) R(y)Q(x)	12am(x)n(y) Q(x)R(y)	36m(y)m(x) Q(x)S(y)	36bm(y)m(y) Q(x)T(y)	- 12am(y)n(x) Q(x)R(y)	- 36m(y)m(x) Q(x)S(y)	- 36bm(Y)m(x) Q(x)T(y)	- 12am(y)o(x) Q(x)S(y)	-36m(y)m(x) Q(x)Q(y)b	-36bm(y)m(x) Q(x)R(y)	12am(y)o(x) Q(x)Q(y)
- 12bn(y)m(x) Q(x)Q(y)	- $12b^2$m(y)m(x) Q(x)R(y)	-4abn(x)n(y) Q(y)Q(x)	- 12bn(y)m(x) Q(x)S(y)	- $12b^2$m(x)n(y) Q(x)T(y)	4abn(y)n(x) R(y)Q(x)	12bm(x)n(y) Q(x)S(y)	$12b^2$m(x)n(y) Q(x)T(Y)	4abn(y)o(x) S(y)Q(x)	12bm(x)n(y) Q(x)Q(y)	$12b^2$m(x)n(y) Q(x)R(y)	-4abo(x)n(y) Q(x)Q(y)
- 36am(x)m(y) Q(x)R(x)	-36am(x)R(y) m(y)R(x)b	- $12a^2$n(x)Q(y) m(y)R(x)	-36m(x)S(y) m(y)Q(x)	- 36abm(x)T(y) m(y)R(x)	$12a^2$n(x)R(y) m(y)R(x)	36am(x)S(y) m(y)R(x)	36abm(x)T(y) m(y)R(x)	- $12a^2$o(x)S(y) m(y)R(x)	36am(x)Q(y) m(y)R(x)	36abm(x)R(y) m(y)R(x)	- $12a^2$o(x)Q(y) m(y)R(x)
-36m(x)m(y) Q(x)Q(y)	-36bm(x)R(x) m(y)Q(x)	- 12an(x)Q(y) m(y)Q(x)	-36n(x)S(y) m(y)R(x)	-36bm(x)T(y) m(y)Q(x)	12an(x)R(y) m(y)Q(x)	36m(x)S(y) m(y)Q(x)	36bm(x)T(y) m(y)Q(x)	12ao(x)S(y) m(y)Q(x)	36m(x)Q(y) m(y)Q(x)	36bm(x)R(y) m(y)Q(x)	- 12ao(x)Q(y) m(y)Q(x)
-12m(x)o(y) Q(y)Q(x)	- $12b^2$m(x)R(y) o(y)Q(x)	-4abn(x)R(y) o(y)Q(x)	- 12bm(x)S(y) o(x)Q(y)	- $12b^2$m(x)T(y) o(y)Q(x)	4abn(x)R(y) o(y)Q(x)	12bm(x)S(y) o(y)Q(x)	$12b^2$m(x)T(y) o(y)Q(x)	4abo(x)S(y) o(y)Q(x)	12bm(x)Q(y) o(y)Q(x)	$12b^2$m(x)R(y) o(y)Q(x)	- 4abo(x)Q(y) o(y)Q(x)
36am(y)Q(y) m(y)S(x)	36abm(x)R(y) m(y)R(x)	$12a^2$n(x)R(y) m(y)R(x)	36am(x)S(y) m(y)R(x)	36abm(x)T(y) m(y)R(x)	- $12a^2$n(x)R(y) m(y)R(x)	- 36am(x)S(y) m(y)R(x)	- 36abm(x)T(y) m(y)R(x)	- $12a^2$o(x)S(y) m(y)R(x)	- 36abm(x)Q(y) m(y)R(x)	- 36abm(x)R(y) m(y)R(x)	- $12a^2$o(x)Q(y) m(y)R(x)
-36m(x)Q(y) m(y)S(x)	-36bm(x)R(y) m(y)S(x)	-12an(x)R(y) m(y)S(x)	-36m(x)S(y) m(y)S(x)	-36bm(x)T(y) m(y)S(x)	12an(x)R(y) m(y)S(x)	36m(x)S(y) m(y)S(x)	36bm(x)T(y) m(y)S(x)	12ao(x)S(y) m(y)S(x)	36m(x)Q(y) m(y)S(x)	36bm(x)R(y) m(y)S(x)	- 12ao(x)Q(y) m(y)S(x)
-12m(x)o(y) Q(y)S(x)b	- $12b^2$m(x)R(y) o(y)S(x)	- 4abn(x)Q(y) o(y)S(x)	- 12bm(x)S(y) o(y)S(x)	- $12b^2$m(x)T(y) o(y)S(x)	4abn(x)R(y) o(y)S(x)	12bm(x)S(y) o(y)S(x)	$12b^2$m(x)T(y) o(y)S(x)	4abo(x)S(y) o(y)S(x)	12bm(x)Q(y) o(y)S(x)	$12b^2$m(x)R(y) o(x)S(y)	- 4abo(x)Q(y) o(y)S(x)
36am(x)Q(y) m(y)T(x)	36abm(x)R(y) m(y)T(x)	$12a^2$n(x)Q(y) m(y)T(x)	36am(x)S(y) m(y)T(x)	36abm(x)T(y) m(y)T(x)	- $12a^2$n(x)R(y) m(y)T(x)	- 36am(x)S(y) m(y)T(x)	- 36abm(x)T(y) m(y)T(x)	- $12a^2$o(x)S(y) m(y)T(x)	-36am(x)Q(y) m(y)T(x)	- 36abm(x)R(y) m(y)T(x)	$12a^2$o(x)Q(y) m(y)T(x)
-36m(x)Q(y) m(y)S(x)	-36bm(x)R(y) m(y)S(x)	12an(x)Q(y) m(y)S(x)	36m(x)S(y) m(y)S(x)	36bm(x)T(y) m(y)S(x)	12an(x)R(y) m(y)S(x)	36m(x)S(y) m(y)S(x)	-36bm(x)T(y) m(y)S(x)	- 12ao(x)S(y) m(y)S(x)	-36m(x)Q(y) m(y)S(x)	-36bm(x)R(y) m(y)S(x)	12ao(x)Q(y) m(y)S(x)
- 12bm(x)Q(y) n(y)S(x)	- $12b^2$m(x)R(y) n(y)S(x)	-4n(y)n(x) R(y0S(x)b	- 12bm(x)S(y) n(y)S(x)	- $12b^2$m(x)T(y) n (x)S(y)	4abn(x)R(y) n(y)S(x)	12bm(x)S(y) n(y)S(x)	$12b^2$m(x)T(y) n(y)S(x)	4abo(x)S(y) n(y)S(x)	12bm(x)Q(y) n(x)S(x)	$12b^2$m(x)R(y) n(y)S(x)	- 4abo(x)Q(y) n(y)S(x)
- 36am(x)Q(y) m(y)T(x)	- 36abm(x)R(y) m(y)T(x)	- $12a^2$n(x)R(y) m(y)T(x)	- 36am(x)S(y) m(y)T(x)	- 36abm(x)T(y) m(y)R(x)	$12a^2$n(x)R(y) m(y)T(x)	36am(x)S(y) m(y)T(x)	36abm(x)T(y) m(y)T(x)	- $12a^2$o(x)S(y) m(y)T(x)	36am(x)Q(y) m(y)T(x)	36abm(x)R(y) m(y)T(x)	- $12a^2$o(x)Q(y) m(y)T(x)

APPENDIX A

$36m(x)^2Q(y)^2$	$36bm(x)^2Q(y)^2$ $R(y)$	$12am(x)n(x)$ $Q(y)R(y)$	$36m(x)^2Q(y)$ $S(y)$	$36bm(x)^2Q(y)$ $T(y)$	$-12am(x)n(x)$ $Q(y)R(y)$	$-36m(x)^2Q(y)$ $S(y)$	$36bm(x)^2Q(y)$ $T(y)$	$12am(x)o(x)$ $Q(y)S(y)$	$-36m(x)^2Q(y)^2$	$-36bm(x)^2Q(y)^2$ $R(y)$	$12am(x)o(x)$ $Q(y)^2$
$36bm(x)^2Q(y)^2$ $R(y)$	$36b^2m(x)^2$ $R(y)^2$	$12abm(x)n(x)$ $Q(y)R(y)$	$36bm(x)^2R(y)$ $S(y)$	$36b^2m(x)^2$ $R(y)T(Y)$	$-12abm(x)n(x)$ $R(y)^2$	$-36bm(x)R(y)$ $S(y)$	$-36b^2m(x)^2$ $R(y)T(Y)$	$12abm(x)o(x)$ $R(y) S(y)$	$-36bm(x)^2Q(y)$ $R(y)$	$-36b^2m(x)^2$ $R(y)^2$	$12abm(x)o(x)$ $R(y)Q(y)$
$12am(x)n(x)$ $Q(y)R(y)$	$12abm(x)n(x)$ $Q(y)R(y)$	$4a^2n(x)^2Q(y)^2$	$12am(x)n(x)$ $Q(y)S(y)$	$12abm(x)n(x)$ $Q(y)T(y)$	$-4a^2n(x)^2$ $Q(y)R(y)$	$-12am(x)n(y)$ $Q(y)S(y)$	$-12abm(x)n(x)$ $Q(y)T(y)$	$4a^2n(x)o(x)$ $Q(y)S(y)$	$-12am(x)n(x)$ $Q(y)^2$	$-12abm(x)n(x)$ $Q(y)R(y)$	$4a^2n(x)o(x)$ $Q(y)^2$
$36m(x)^2Q(y)$ $S(y)$	$36bm(x)^2R(y)^2$ $S(y)$	$12an(x)m(x)$ $Q(y)S(y)$	$36m(x)^2S(y)^2$	$36bm(x)^2S(y)$ $T(y)$	$-12am(x)n(x)$ $S(y)R(y)$	$-36m(x)^2S(y)^2$	$-36bm(x)^2$ $T(y)S(y)$	$12am(x)o(x)$ $S(y)^2$	$-36m(x)^2S(y)$ $Q(y)$	$-36bm(x)^2$ $S(y) R(y)$	$12am(x)o(x)$ $S(y)Q(y)$
$36bm(x)^2Q(y)$ $T(y)$	$36b^2m(x)^2$ $R(y)T(Y)$	$12abn(x)n(x)$ $Q(y)T(y)$	$36bm(x)^2S(y)$ $T(y)$	$36b^2m(x)^2$ $T(y)^2$	$-12abm(x)n(x)$	$-36bm(x)^2T(y)$ $S(y)$	$-36bm(x)^2$ $T(y)^2b^2$	$12abm(x)o(x)$ $S(y) T(y)$	$-36bm(x)^2$ $T(y) Q(y)$	$-36b^2m(x)^2$ $T(y)R(y)$	$12abm(x)o(x)$ $Q(y)T(y)$
$-12am(x)n(x)$ $Q(y)R(y)$	$-12abm(x)n(y)$ $R(y)^2$	$4a^2m(x)^2Q(y)$ $R(y)$	$-12am(x)n(x)$ $S(y)R(y)$	$-12abm(x)n(x)$ $T(y)R(y)$	$4a^2n(x)^2R(y)^2$	$12m(x)n(x)$ $R(y)S(y)$	$12abm(x)n(x)$ $R(y)T(y)$	$-4a^2n(x)o(x)$ $R(y)S(y)$	$12m(x)n(x)$ $R(y)Q(y)a$	$12abm(x)n(x)$ $R(y)^2$	$-4a^2n(x)o(x)$ $R(y)Q(y)$
$-36bm(x)^2S(y)^2$ $Q(y)$	$-36bm(x)^2R(y)$ $S(y)$	$-12am(x)n(x)$ $Q(y)S(y)$	$-36m(x)^2S(y)^2$	$-36bm(x)^2T(y)$ $S(y)$	$12am(x)n(x)$ $R(y)S(y)$	$36m(x)^2S(y)^2$	$36bm(x)^2S(y)$ $T(y)$	$-12am(x)o(x)$ $S(y)^2$	$36m(x)^2$ $S(y) Q(y)$	$36bm(x)^2$ $S(y)R(y)$	$-12am(x)o(x)$ $S(y)Q(y)$
$-36bm(x)^2Q(y)$ $T(y)$	$-36m(x)^2T(y)$ $R(y)b^2$	$-12a^2n(x)m(x)$ $T(y)Q(y)$	$-36bm(x)^2S(y)$ $T(y)$	$-36b^2m(x)^2$ $T(y)^2$	$12abm(x)n(x)$ $R(y)T(y)$	$36bm(x)^2S(y)$ $T(y)$	$36b^2m(x)^2$ $T(y)^2$	$-12abm(x)o(x)$ $T(y) S(y)$	$36bm(x)^2T(y)$ $Q(y)$	$36b^2m(x)^2$ $R(y)T(Y)$	$-12abm(x)o(x)$ $T(y)Q(y)$
$12am(x)o(x)$ $Q(y)S(y)$	$12abm(x)o(x)$ $R(y)S(y)$	$4a^2n(x)^2o(x)$ $Q(y)S(y)$	$12am(x)o(x)$ $S(y)^2$	$12abm(x)o(x)$ $S(y)T(y)$	$-4a^2n(x)^2o(x)$ $S(y)R(y)$	$-12am(x)o(x)$ $S(y)^2$	$-12abm(x)o(x)$ $T(y) S(y)$	$4a^2o(x)^2S(y)^2$	$-12am(x)o(x)$ $S(y)Q(y)$	$12abm(x)o(x)$ $S(y)R(y)$	$4a^2o(x)^2S(y)$ $Q(y)$
$-36m(x)^2Q(y)^2$	$-36bm(x)^2Q(y)$ $R(y)$	$-12am(x)n(x)$ $Q(y)^2$	$-36m(x)^2Q(y)$ $S(y)$	$-36bm(x)^2Q(y)$ $T(y)$	$12am(x)n(x)$ $R(y)Q(y)$	$36m(x)^2S(y)$ $Q(y)$	$36bm(x)^2$ $T(y) Q(y)$	$-12am(x)o(x)$ $S(y) Q(y)$	$36m(x)^2Q(y)^2$	$36b^2m(x)^2$ $R(y)Q(y)$	$-12am(x)o(x)$ $Q(y)^2$
$-36m(x)^2Q(y)$ $R(y)b$	$-36b^2m(x)^2$ $R(y)^2$	$-12abm(x)n(x)$ $Q(y)R(y)$	$-36m(x)^2R(y)$ $S(y)b$	$-36b^2m(x)^2$ $R(y)T(Y)$	$12abm(x)n(x)$ $S(y)R(y)$	$36m(x)^2S(y)$ $bR(y)$	$36b^2m(x)^2$ $T(y)R(y)$	$-12abm(x)o(x)$ $S(y) R(y)$	$36m(x)^2Q(y)$ $R(y)b$	$36b^2m(x)^2$ $R(y)^2$	$-12abm(x)o(x)$ $Q(y)R(y)$
$12m(x)o(x)$ $Q(y)^2a$	$12abm(x)o(x)$ $R(y)Q(y)$	$4a^2n(x)o(x)$ $Q(y)^2$	$12am(x)o(x)$ $S(y)Q(y)$	$12m(x)o(x)$ $Q(y)T(y)ab$	$-4a^2m(x)^2o(x)$ $R(y)Q(y)$	$-12am(x)o(x)$ $S(y)Q(y)$	$-12abm(x)o(x)$ $T(y) Q(y)$	$4a^2o(x)^2S(y)$ $Q(y)$	$-12am(x)o(x)$ $Q(y)^2$	$-12abm(x)o(x)$ $Q(y)R(y)$	$4a^2o(x)^2$ $Q(y)^2$