MATHEMATICAL MODELLING FOR THE TRANSMISSION DYNAMICS OF RIFT VALLEY FEVER VIRUS WITH HUMAN HOST

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ABSTRACT

In this research, transmission dynamics of Rift Valley Fever (RVF) with mosquito, livestock and human host using ordinary differential equation was studied and analyzed. RVF is a viral zoonosis fundamentally transmitted by mosquitoes and primarily affects livestock but has the ability to affect humans. It has become a public worry due to its potential to spread rapidly and become an epidemic. The Effective Reproduction Number R_c was computed using next generation matrix and used to investigate the local and global stability of the equilibrium, the disease-free equilibrium state was found to be locally asymptotically stable if $R_c < 1$. And by constructing a function using Castillo-Chavez's method, the disease-free state is found to be globally asymptotically stable if $R_c \leq 1$. This implies that rift valley fever could be put under control in a population where the Reproduction Number is less than 1. Numerical simulations using Adomian Decomposition Method (ADM) gives insightful analytical results to further explore the dynamics of the disease.

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

Rift Valley Fever (RVF) is a viral illness of people and livestock that can cause mild to serious side effects. RVF is also known as enzootic hepatitis of sheep and cattle (Adeyeye *et al*., 2011). It is an acute, infectious and zoonotic disease of predominantly cattle, sheep, goat, camels, African buffalo (*Syncerus caffer*) and humans. The disease is caused by an arbovirus and is associated with periodic outbreaks that mostly occur on the African continent. It is a febrile disease that is accompanied by abortion in livestock and a severe fatal haemorrhagic syndrome in humans has been observed (Evans *et al*., 2008). The disease was first reported among sheep in Kenya by Montgomery in 1912 and Stordy in 1913 (Anon, 2010), but the disease was not isolated until 1931 (Morril, 2001).

RVF is transmitted by mosquitoes and infects domestic livestock and humans in Africa and the Middle East (Abdo-Salam *et al*., 2006). The mild indications may include: fever, muscle pains, and migraines which frequently keep going for as long as seven days. The serious side effects may include: loss of sight starting three weeks after the contamination, diseases of the cerebrum creating extreme migraines and confusion, and bleeding along with liver issues which may happen within the first few days. The individuals who have bleeding have a 50% chance of death (WHO, 2010).

RVF is a viral zoonosis that essentially influences animals yet in addition has the ability to infect humans. The sickness additionally brings about huge monetary misfortunes because of death and early termination among RVF-infected animals. The infection was first recognized in 1931 during an examination concerning a scourge among sheep on a ranch in the Rift Valley of Kenya. From that point forward, episodes have been accounted for in sub-Saharan Africa. In 1977 a hazardous episode was accounted for in Egypt, the RVF infection was introduced to Egypt through infected animals exchange along the Nile irrigation system framework. In 1997–1998, a significant outbreak occurred in Kenya, Somalia and Tanzania following El Niño occasion and broad flooding. Following infected animals exchange from the horn of Africa, RVF spread in September 2000 to Saudi Arabia and Yemen, denoting the primary detailed event of the virus outside the African landmass and raising worries that it could stretch out to different parts of Asia and Europe (WHO, 2018).

The outbreak on the Arabian Peninsula represents the first cases of RVF outside Africa. In 2007, an outbreak occurred in Kenya and Somalia where over 404 human cases, including 118 deaths, were reported (Centers for Disease control and Prevention (CDC), 2007). In South Africa, the last outbreak occurred in May 2010; preliminary investigation revealed that 186 humans were confirmed RVF cases out of which 18 died (WHO, 2010). An overview of the disease is necessary given climate changes that favour possible outbreaks (Gould and Higgs, 2009) and the warning signals dispatched to countries in Africa by the Food and Agriculture Organization and World Health organization (Food and Agriculture Organization, 2008).

In Nigeria, Ferguson first isolated the virus from animals (Ferguson, 1959). Subsequent serological evidence suggests that the virus may be circulating at low levels in domestic livestock and in the human population, particularly among livestock workers and wildlife rangers (Olaleye *et al.*,1996). Cattle, sheep, goats and camels in the states of Kaduna and Sokoto have revealed significant antibody titres in their serum (Ezeifeka *et al.*, 1982).

Serological prevalence of the disease in these animal species in Ile-Ife and Ibadan was observed by Olaleye *et al*. (1996) who confirmed the existence of the disease in Nigeria. Apart from these observations, experimental infection with different strains of the disease in three indigenous breeds of sheep in Nigeria, namely: The West African dwarf, Yankasa and Ouda have resulted in fatal disease (Fagbami *et al.*, 1975). Further studies are therefore required to determine the present status of the disease in Nigeria.

1.2 Statement of the Research Problem

RVF has been a major course of concern, presently, virological and serological evidence suggests that the RVF virus exists throughout sub-Saharan Africa and Madagascar and, in the light of its recurrence in Egypt in 1993 and 2003 (Anon, 2010), it may be extending its range even further. In September 2000, cases of unexplained haemorrhagic fever in humans and associated animal deaths in south-western Saudi Arabia and Yemen were confirmed as RVF and, by mid-January 2001, the disease had claimed several human lives in these countries (Abdo-Salam *et al*., 2006).

Hence, we formulated a model to take a closer look at this silent but deadly disease, by incorporating control and trapping of these infectious vectors and livestock to curb the spread of the disease.

1.3 Aim and Objectives of the Study

The aim of this work is to develop and analyse a mathematical model for the transmission dynamics of Rift Valley Fever (RVF) virus with human host. The objectives of the study are to:

1. formulate a mathematical model for RVF virus.

2. determine the criteria for positivity of the model equations.
3. determine the Disease free and endemic equilibria of the model equations and conditions for their stability.
4. solve analytically using Adomian Decomposition Method.
5. carry out numerical simulation of the model using maple software.

1.4 Motivation for the Study

Nigeria is at high risk of RVF and it would be great to take precautionary measures to prevent any occurrence of this plague. The outbreak of diseases in the world recently has proven that no stone must be left unturned; every communicable disease must be treated as a matter of urgency. In order to save lives and prevent any other pandemic from occurring, the study of subject matters like this becomes a necessity for our general well being and for generations yet to come.

1.5 Justification for the Study

RVF has plagued a lot of nations of the world and it is still spreading its tentacles. The singular factor that RVF can be transmitted by mosquitoes raises a lot of concern, neglect of this disease can cause future outbreaks. RVF is a mosquito-borne disease (Abdulkadir, 1989). *Aedes* is the species of mosquito that is incriminated in biological transmission (Turell *et al*., 2008), although *Glossina*, *Culicoides*, *Culex* species and sand flies may play limited roles in biological and mechanical transmission (Hoch *et al*., 1985). Apart from these vectors, the disease has been reported to spread through needle inoculation, contact with infected animals or humans with high prevalence during periods of heavy rainfall (Zeller *et al*.,1997).

1.6 Scope and Limitation of the Study

The model is limited to the transmission dynamics of Rift Valley Fever with human host incorporating culling rate (control) and trapping of mosquitoes.

1.7 Definition of Terms

Effective reproduction number R_c: This is the average number of secondary cases per infection in a population made up of both susceptible and non-susceptible hosts.

Disease free equilibrium: This is a steady state solution of a system, when there is no disease present in a given population.

Trapping of Mosquitoes: This is the control of mosquito population to reduce damages caused to human health, economies and general well being.

Culling livestock: This is the elimination of undesired animals from the herd for reasons of uneconomic, poor production, sterility problems and incurable diseases.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Mathematical Models of Rift Valley Fever

Farida *et al.* (2016) developed vaccination models for live and killed vaccines. A ruminant population at time t ($N(t)$) was divided into classes of susceptible ($S(t)$), infectious ($I(t)$), recovered ($R(t)$) and vaccinated by live vaccines ($V_1(t)$) or vaccinated by killed vaccines ($V_2(t)$) ruminants. A population of adult female mosquitoes at time t ($M(t)$) was divided into susceptible ($U(t)$) and infectious ($W(t)$) classes.

Ruminants (livestock) have a very high immunity, hence they concluded that RVFV remains endemic at a very low level, when an outbreak occurs. The model equations were given as:

$$\left.\begin{aligned}
\dot{S}(t) &= \Lambda_r(S, I, R, V) + (1 - \rho_{12} - \rho_{13})\lambda V_1 - \beta WS - \rho_{11}\phi_1 S - \mu S \\
\dot{I}(t) &= \beta WS + \rho_{13}\lambda V_1 - (\mu + d + \gamma) I \\
\dot{R}(t) &= \gamma I + \rho_{12}\lambda V_1 - \mu R \\
\dot{V}_1(t) &= \rho_{11}\phi_1 S - (\mu + \lambda) V_1 \\
\dot{U}(t) &= \Lambda_m(M) - \alpha IU - (1 - \delta)\alpha V_1 U - \eta U \\
\dot{W}(t) &= \alpha IU + (1 - \delta)\alpha V_1 U - \eta W
\end{aligned}\right\} \qquad (2.1)$$

Although live vaccines induce early and long-term immunity, they may cause viraemia in ruminants and have a potential for virulence reversion. Hence, they were not recommended in non-endemic areas or during the breeding season of mosquitoes or during disease outbreaks (Ikegami and Makino, 2009; Kamal, 2011). Susceptible ruminants were vaccinated at a rate $\rho_{11}\varphi_1$, where $1/\varphi_1$ is the time period that ruminants remain susceptible before being vaccinated and only a fraction ρ_{11} of ruminants was actually vaccinated. Vaccinated ruminants leave the vaccination class at a rate λ with a probability of ρ_{12} to successfully acquire a life-long immunity, a probability of ρ_{13} that reversion to virulence occurred, and a probability of $1 - \rho_{12} - \rho_{13}$ for vaccine failure.

Although killed vaccines are safer than live vaccines, they may have poor immunogenicity by not inducing long-term immunity and often requiring multiple vaccination doses (Ikegami and Makino, 2009; Bird, 2012). It was assumed that susceptible ruminants are vaccinated at rate $\rho_{21}\varphi_2$, where $1/\varphi_2$ is the time period that ruminants remain susceptible before being vaccinated by killed vaccine and only a fraction ρ_{21} of ruminants was actually vaccinated. Vaccinated ruminants leave the vaccination class at rate ν with a probability of ρ_{22} to receive booster vaccines and successfully acquire long-term immunity, and a probability of $1-\rho_{22}$ for individuals to become susceptible again due to vaccine failure or not receiving booster vaccines.

The model described in (Tianchan *et al.*, 2012) was constructed to describe the transmission of RVFV between three prototypes: two mosquito populations and one livestock population. The model considered both individual-to-individual transmission of virus between species (called "horizontal transmission") and mother-to-offspring transmission of virus (vertical transmission) in one mosquito species. The mosquitoes that can transmit RVFV both horizontally to livestock and vertically to their progeny "floodwater *Aedes*" mosquitoes were labelled "species 1". Livestock were labelled "species 2", and mosquitoes that transmit RVFV only horizontally to livestock "*Culex*" were labelled "species 3". Considering populations of these species distributed throughout a large but finite two-dimensional region. The general model allowed for travel among any pair of patches in the simulated region.

Example of this movement is livestock that is transported from a farm to a different farm or auction house. Such travel need not be between adjacent patches; transportation may move individuals between one patch and a geographically disconnected patch. Species living on a given patch may have patch-specific epidemiologic and demographic characteristics.

Gaff *et al*. (2007) constructed a compartmental, ordinary differential equation (ODE) model of RVFV, it considers two populations of mosquitoes (one exhibiting vertical transmission and the other not) and a population of livestock animals with disease-dependent mortality. One population of vectors represented Aedes mosquitoes, which can be infected through either vertically or via a blood meal from an infectious host. The other vector population is able to transmit RVFV to hosts but not to their offspring; here they considered it to be a population of Culex mosquitoes. Once infectious, mosquito vectors remain infectious for the remainder of their lifespan. Infection is assumed not to affect mosquito behavior or longevity significantly. Hosts, which represent various livestock animals, can become infected when fed upon by infectious vectors. Hosts may then die from RVFV infection or recover, whereupon they have lifelong immunity to reinfection (Wilson, 1994). Neither age structure nor spatial effects were incorporated into this model. Populations contain a number of susceptible (S_i), incubating (infected, but not yet infectious) (E_i) and infectious (I_i) individuals, $i = 1, 2, 3$. Infected livestock will either die from RVFV or will recover with immunity (R_2). To reflect the vertical transmission in the Aedes species, compartments for uninfected (P_1) and infected(Q_1) eggs are included. As the Culex species cannot transmit RVF vertically, only uninfected eggs (P_3) are included. Adult vectors emerge from these compartments at the appropriate maturation rates. The size of each adult mosquito population is $N_i = S_i + E_i + I_i$, for $_i$ = 1 and 3. The livestock population was modeled using a logistic population model with a given carrying capacity, K_2. The total livestock population size is $N_2 = S_2 + E_2 + I_2 + R_2$.

The system of ODEs that represented the populations is given below:

Aedes vector

$$\left.\begin{aligned}
\frac{dP_1}{dt} &= b_1\left(N_1 - q_1 I_1\right) - \theta_1 P_1 \\
\frac{dQ_1}{dt} &= b_1 q_1 I_1 - \theta_1 Q_1 \\
\frac{dS_1}{dt} &= \theta_1 P_1 - d_1 S_1 - \frac{\beta_{21} S_1 I_2}{N_2} \\
\frac{dE_1}{dt} &= -d_1 E_1 + \frac{\beta_{21} S_1 I_2}{N_2} - \varepsilon_1 E_1 \\
\frac{dI_1}{dt} &= \theta_1 Q_1 - d_1 I_1 + \varepsilon_1 E_1 \\
\frac{dN_1}{dt} &= \left(b_1 - d_1\right) N_1
\end{aligned}\right\} \quad (2.2)$$

Livestock

$$\left.\begin{aligned}
\frac{dS_2}{dt} &= b_2 N_2 - \frac{d_2 S_2 N_2}{K_2} - \frac{\beta_{12} S_2 I_1}{N_1} - \frac{\beta_{32} S_2 I_3}{N_3} \\
\frac{dE_2}{dt} &= -\frac{d_2 E_2 N_2}{K_2} + \frac{\beta_{12} S_2 I_1}{N_1} + \frac{\beta_{32} S_2 I_3}{N_3} - \varepsilon_2 E_2 \\
\frac{dI_2}{dt} &= -\frac{d_2 I_2 N_2}{K_2} + \varepsilon_2 E_2 - \gamma_2 I_2 - \mu_2 I_2 \\
\frac{dR_2}{dt} &= -\frac{d_2 R_2 N_2}{K_2} + \gamma_2 I_2 \\
\frac{dN_2}{dt} &= -N_2\left(b_2 - \frac{d_2 N_2}{K_2}\right) - \mu_2 I_2
\end{aligned}\right\} \quad (2.3)$$

Culex mosquito Vector

$$\left.\begin{aligned}
\frac{dP_3}{dt} &= b_3N_3 - \theta_3P_3 - \frac{\beta_{23}S_3I_2}{N_2}\\
\frac{dS_3}{dt} &= \theta_3P_3 - d_3S_3 - \frac{\beta_{23}S_3I_2}{N_2}\\
\frac{dE_3}{dt} &= -d_3E_3 - \frac{\beta_{23}S_3I_2}{N_2} - \varepsilon_3E_3\\
\frac{dI_3}{dt} &= -d_3I_3 + \varepsilon_3E_3\\
\frac{dN_3}{dt} &= \left(b_3 - d_3\right)N_3
\end{aligned}\right\} \qquad (2.4)$$

The model presented was a simplified representation of the complex biology involved in the epidemiology of RVF.

Our model is based on the following assumptions;

i. That the population is heterogeneous.

ii. That people, animals and vectors (mosquitoes) have equal natural death rate in their respective compartments.

iii. The only way of entry into the population is through birth and the only way of exit is through death from natural causes or culling (elimination of infected animals) for livestock only.

iv. That trapping of mosquitoes controls the spread of the disease.

v. That RVF virus can be spread from humans to mosquitoes and vice versa; mosquitoes to livestock and vice versa.

2.2 Effective Reproduction Number

The effective reproduction number (R_c) is the average number of secondary cases per infectious case in a population made up of both susceptible and non-susceptible hosts. If $R_c > 1$, the number of cases will increase, such as at the start of an epidemic. Where $R_c=1$, the disease is endemic, and where $R_c < 1$ there will be a decline in the number of cases. The effective reproduction number can be estimated as the product of the basic reproduction number and the fraction of the host population that is susceptible (x). So:

$$R_c = R_0 x \tag{2.5}$$

The next generation approach described by Van Driessche and Watmough (2002) is an acceptable method to compute Basic reproduction number.

We used this approach to determine our Effective Reproduction Number of the next generation matrix FV^{-1}.

$V_i^+(x)$ is the rate of transfer of individuals into compartment i by every means except the epidemic.

$V_i^-(x)$ is the transfer of individuals out of compartment i.

$$V_i = V_i^-(x) - V_i^+(x) \tag{2.6}$$

Given the DFE, R_c is calculated thus:

$$F = \frac{\partial F_i}{\partial x_j}(E_0) \tag{2.7}$$

$$V = \frac{\partial V_i}{\partial x_j}(E_0) \tag{2.8}$$

$$R_c = \rho\left(FV^{-1}\right) \tag{2.9}$$

2.3 Global Stability of Disease Free Equilibrium

Global stability means that the system will come to the equilibrium point from any possible starting point (i.e., there is no "nearby" condition). Castillo Chavez stability theorem was used to determine the global stability of disease free equilibrium in this study.

We can write the model system as:

$$\frac{dX_s}{dt} = A\left(X_s - X_{D.F.E_S}\right) + A_1 X_i \tag{2.10}$$

$$\frac{dX_i}{dt} = A_2 X_i \tag{2.11}$$

2.4 Local Stability of Disease Free Equilibrium

Local stability of an equilibrium point means that if you put the system somewhere nearby the equilibrium point then it will move itself to the equilibrium point in some time.

2.5 Adomian Decomposition Method (ADM)

The Adomian decomposition method (ADM) is a semi-analytical method for solving ordinary and partial nonlinear differential equations. The method was first introduced by an American mathematician and aerospace engineer of Armenian descent George Adomian in 1981 and developed by him in 198l. The method employs the use of the "Adomian polynomials" to represent the nonlinear portion of the equation as a convergent series with respect to these polynomials, without actual linearization of the system. These polynomials mathematically generalize Maclaurin series about an arbitrary external parameter, which gives the solution method more flexibility than direct Taylor series expansion.

This method has been applied to solve differential and integral equations of linear and non-linear problems in mathematics, physics, biology and chemistry and now a large number of research papers have been published to show the feasibility of the decomposition method (Nhawu *et al.*, 2015).

We show how the method works by considering the derivative operator: $D = d/dx$

$$D^{-1}y(x) = y_0 + \int_0^x y(s)\,ds \tag{2.12}$$

Where *s=0* was chosen for simplicity

$$y(0) = y_0 \tag{2.13}$$

The derivative operator is defined on the space of smooth functions and its inverse acts in the space of integrable functions.

$$D^{-1}Dy = D^{-1}y'(x) = y'(0) + \int_0^x y'(s)\,ds = y'(0) + y(x) - y(0) \tag{2.14}$$

Assuming the solution is the infinite sum

$$y(x) = y_0 + y_1 + y_2 + ... + \sum_{k\geq 0} y_k \tag{2.15}$$

We substitute the series into either the differential equation or the formula;

$$\begin{gathered} D\sum_{k\geq 0} y_k + a\sum_{k\geq 0} y_k = \sum_{k\geq 1} Dy_k + a\sum_{k\geq 0} y_k = 0 \\ or \\ \sum_{k\geq 0} y_k = -aD^{-1}\sum_{k\geq 0} y_k = -a\sum_{k\geq 0} D^{-1}y_k \end{gathered} \tag{2.16}$$

Comparing like terms, we get the recurrence relation

$$Dy_{k+1} + ay_k = 0 \\ or \\ y_{k+1} = -aD^{-1}y_k, k = 0,1,2 \tag{2.17}$$

Each of these recurrences is a linear difference-differential equation;

$$y_1 = -aD^{-1}y_0 = -ay_0 - axy_0 \tag{2.18}$$

with $y_1(0) = -ay_0$,

$$y_2 = -aD^{-1}y_1 = -a^2D^{-1}x = -a^2y_0 + \mathrm{a}^2\frac{x^2}{2}\mathrm{y}_0 \tag{2.19}$$

with $y_2(0) = -a^2y_0$

$$y_3 = -aD^{-1}y_2 = -a^3y_0 - a^3\frac{x^3}{3!}y_0 \tag{2.20}$$

with $y_3(0) = -a^3y_0$

We get the solution;

$$y(x) = -y_0a(1 - a + a^2 - ...) + y_0\left(1 - ax + \frac{a^2x^2}{2!} - ...\right) = y_0 - \frac{a}{a+1} + y_0e^{-ax} \tag{2.21}$$

For the initial value problems for non-linear equations such as;

$$\frac{dy}{dt} + a(t)y(t) + b(t)N_y = f(t), y(0) = y_0 \tag{2.22}$$

The ADM method is to represent the non-linear term as the sum of Adomian polynomials

$$N_y = \sum_{n \geq 0} A_n \left(y_0, y_1, y_2, ..., y_n \right), \tag{2.23}$$

Where A_n are the Adomian polynomials specifically generated for each non-linear operator according to the formula

$$A_n \left(y_0, y_1, y_2, ..., y_n \right) = \frac{1}{n!} \frac{d^n}{d\lambda^n} N \left[\sum_{k=0}^{n} y_k \lambda^k \right] \Bigg|_{\lambda=0}, n = 0,1,2,... \tag{2.24}$$

$$\begin{aligned} A_0 &= N_{y0} \\ A_1 &= \frac{y_1 dN_{y0}}{dy_0}, ... \end{aligned} \tag{2.25}$$

$$N_y = A_0 + A_1 + A_2 + ... \tag{2.26}$$

correspond to ordinary generating functions.

2.6 Sensitivity Analysis

Sensitivity analysis confirms the effect each parameter has on the disease transmission. The objective of sensitivity analysis is to give rise to uncertainties of the model outputs (Leon *et al.*,2009).

To determine sensitivity index with respect to a parameter value q, we have;

$$\mathrm{I}_q^{R_c} = \frac{\partial R_c}{\partial q} \times \frac{q}{R_c} \qquad (2.27)$$

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Formulation of the Model

The model equations of Rift Valley Fever (RVF) are formulated using first order ordinary differential equation. Features such as vaccination to susceptible class, treatment of infected class and culling livestock, which is the elimination of infected livestock, control and spread of the disease among livestock and humans using vaccination and treatment respectively.

In formulating the model, we considered horizontal transmission in mosquitoes; control (culling rate) vector population was also considered. Humans were considered to be a source of infection to mosquitoes (contact rate from humans to vectors was assumed to be almost negligible). We also assumed that livestock and humans get infected when they come in contact with infectious vectors. Natural death rate occurs in all three groups.

The model was divided into three populations; the susceptible, S_i and infected, I_i classes, for $i = h, l, m$ for, human (h), livestock (l) and mosquitoes (m), respectively. The two susceptible populations (humans and livestock) become infected via an infectious mosquito bite at per capita rates β_i. The newborns in each category are recruited at the per capita birth rate of Λ_i and hosts die naturally at per capita rates μ_i. Recovery in livestock is introduced at a constant rate γ_l; recovery in humans at a constant rate γ_h. The rates for treatment are; livestock τ_l, treated humans τ_h and the vector is trapped at a constant rate δ_m. Since a population dynamics model is considered, all the state variables and parameters are assumed to be non-negative. The model assumes that individuals mix homogeneously in the human and livestock population where all individuals have equal chance of getting the infection if they come into contact with infectious mosquitoes, and that transmission of the infection occurs with a standard incidence. It is the assumption of the model that there is natural mortality, thus there is no disease induced death, but rather culling of infected livestock

The human population is sub-divided into the following subgroups; susceptible S_h, exposed E_h, infected I_h, and recovered R_h. The size of the human population is therefore given by;

$$N_h = S_h + E_h + \mathrm{I}_h + R_h \tag{3.1}$$

The livestock population is given by;

$$N_l = S_l + E_l + \mathrm{I}_l + R_l \tag{3.2}$$

And mosquito population is given by;

$$N_m = S_m + \mathrm{I}_m \tag{3.3}$$

The disease occurs with equal probability across all age groups, hence the natural death rate μ_h is the same across all stages of the disease in humans.

The schematic representation of the model is given in the Figure 3.1.

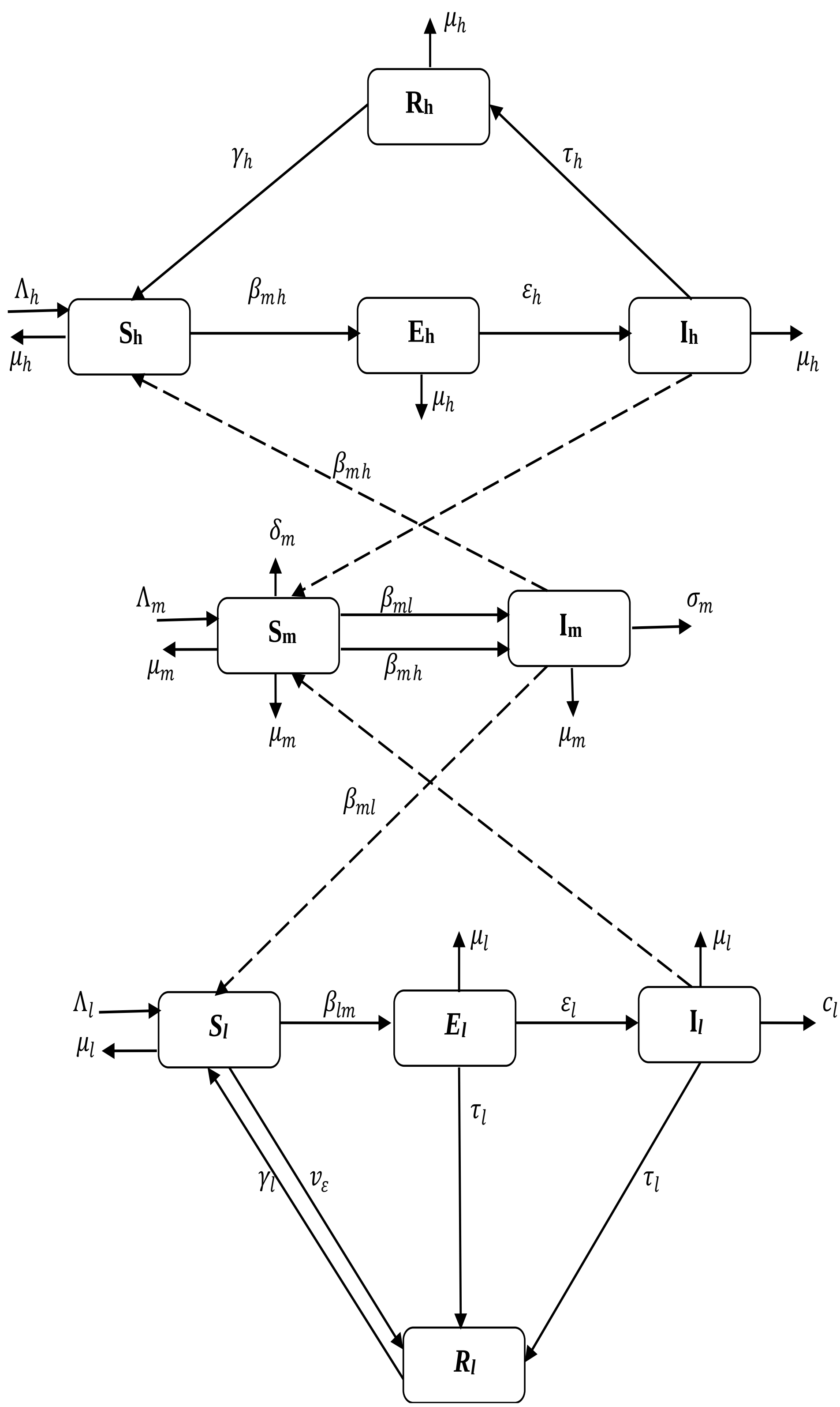

Fig. 3.1 Schematic Diagram of the Model

The model equations derived from schematic diagram in figure (3.1) are given thus:

Human Population

$$\frac{dS_h}{dt} = \Lambda_h - \frac{\beta_{hm} \mathrm{I}_m S_h}{N_h} - \mu_h S_h + \gamma_h R_h \qquad (3.4)$$

$$\frac{dE_h}{dt} = \frac{\beta_{hm} \mathrm{I}_m S_h}{N_h} - (\varepsilon_h + \mu_h) E_h \qquad (3.5)$$

$$\frac{d\mathrm{I}_h}{dt} = \varepsilon_h E_h - (\mu_h + \tau_h) \mathrm{I}_h \qquad (3.6)$$

$$\frac{dR_h}{dt} = \tau_h \mathrm{I}_h - (\mu_h + \gamma_h) R_h \qquad (3.7)$$

Livestock Population

$$\frac{dS_l}{dt} = \Lambda_l - \frac{\beta_{lm} \mathrm{I}_m S_l}{N_l} - (\mu_l + v_\varepsilon) S_l + \gamma_l R_l \qquad (3.8)$$

$$\frac{dE_l}{dt} = \frac{\beta_{lm} \mathrm{I}_m S_l}{N_l} - (\mu_l + \tau_l + \varepsilon_l) E_l \qquad (3.9)$$

$$\frac{d\mathrm{I}_l}{dt} = \varepsilon_l E_l - (\mu_l + c_l + \tau_l) \mathrm{I}_l \qquad (3.10)$$

$$\frac{dR_l}{dt} = v_\varepsilon S_l + \tau_l E_l + \tau_l \mathrm{I}_l - (\mu_l + \gamma_l) R_l \qquad (3.11)$$

Mosquito Population

$$\frac{dS_m}{dt} = \Lambda_m - \frac{\beta_{ml} \mathrm{I}_l S_m}{N_m} - \frac{\beta_{mh} \mathrm{I}_h S_m}{N_m} - (\mu_m + \delta_m) S_m \qquad (3.12)$$

$$\frac{dI_m}{dt} = \frac{\beta_{ml} I_l S_m}{N_m} + \frac{\beta_{mh} I_h S_m}{N_m} - (\mu_m + \delta_m) I_m \quad (3.13)$$

And summing (3.4) - (3.7), (3.8)- (3.11) and (3.12) - (3.13) gives;

$$\frac{dN_h}{dt} = \Lambda_h - (S_h + E_h + I_h + R_h)\mu_h \quad (3.14)$$

$$\frac{dN_l}{dt} = \Lambda_l - (S_l + E_l + I_l + R_l)\mu_l - c_l I_l \quad (3.15)$$

$$\frac{dN_m}{dt} = \Lambda_m - (\mu_m + \delta_m)(S_m + I_m) \quad (3.16)$$

Table 3.1: Notation and definition of variables and parameters

Symbol	**Description**
$N_h(t)$	Total population of humans at time t
$N_l(t)$	Total population of livestock at time t
$N_m(t)$	Total population of mosquitoes at time t
$S_h(t)$	Susceptible humans at time t
$S_l(t)$	Susceptible livestock at time t
$S_m(t)$	Susceptible mosquitoes at time t
$E_h(t)$	Exposed humans at time t
$E_l(t)$	Exposed livestock at time t
$I_h(t)$	Infected humans at time t
$I_l(t)$	Infected livestock at time t

$I_m(t)$ Infected mosquitoes at time t

$R_l(t)$ Recovered livestock at time t

$R_h(t)$ Recovered humans at time t

Λ_h Recruitment rate of human

Λ_l Recruitment rate of livestock

Λ_m Recruitment rate of mosquitoes

v_ε Efficacy of vaccination

c_l Culling rate of livestock (control)

β_{lm} Adequate contact rate from livestock to mosquito

β_{ml} Adequate contact rate from mosquito to livestock

β_{mh} Adequate contact rate from mosquito to humans

β_{hm} Adequate contact rate from humans to mosquitoes

ε_l Disease incubation period in livestock

ε_h Disease incubation period in humans

γ_h Rate at which humans recover

γ_l Rate at which livestock recover

τ_h Treatment rate in humans

τ_l Treatment rate in livestock

δ_m Trapping rate of mosquitoes

μ_m Natural death rate of mosquitoes

μ_l Natural death rate of livestock

μ_h Natural death rate of humans

The following assumptions were considered in constructing the model:

1. The recruitment rate into the susceptible class is at constant rate.

For the model equations, let;

$$k_1 = \varepsilon_h + \mu_h \tag{3.17}$$

$$k_2 = \mu_h + \tau_h \tag{3.18}$$

$$k_3 = \mu_h + \gamma_h \tag{3.19}$$

$$k_4 = \mu_l + v_\varepsilon \tag{3.20}$$

$$k_5 = \mu_l + \tau_l + \varepsilon_l \tag{3.21}$$

$$k_6 = \mu_l + c_l + \tau_l \tag{3.22}$$

$$k_7 = \mu_l + \gamma_l \tag{3.23}$$

$$k_8 = \mu_m + \delta_m \tag{3.24}$$

Thus, equations (3.4) to (3.13) becomes

$$\frac{dS_h}{dt} = \Lambda_h - \frac{\beta_{hm} I_m S_h}{N_h} - \mu_h S_h + \gamma_h R_h \tag{3.25}$$

$$\frac{dE_h}{dt} = \frac{\beta_{hm} I_m S_h}{N_h} - k_1 E_h \tag{3.26}$$

$$\frac{dI_h}{dt} = \varepsilon_h E_h - k_2 I_h \tag{3.27}$$

$$\frac{dR_h}{dt} = \tau_h I_h - k_3 R_h \tag{3.28}$$

$$\frac{dS_l}{dt} = \Lambda_l - \frac{\beta_{lm} I_m S_l}{N_l} - k_4 S_l + \gamma_l R_l \tag{3.29}$$

$$\frac{dE_l}{dt} = \frac{\beta_{lm} I_m S_l}{N_l} - k_5 E_l \tag{3.30}$$

$$\frac{dI_l}{dt} = \varepsilon_l E_l - k_6 I_l \tag{3.31}$$

$$\frac{dR_l}{dt} = v_\varepsilon S_l + \tau_l E_l + \tau_l I_l - k_7 R_l \tag{3.32}$$

$$\frac{dS_m}{dt} = \Lambda_m - \frac{\beta_{ml} I_l S_m}{N_m} - \frac{\beta_{mh} I_h S_m}{N_m} - k_8 S_m \tag{3.33}$$

$$\frac{dI_m}{dt} = \frac{\beta_{ml} I_l S_m}{N_m} + \frac{\beta_{mh} I_h S_m}{N_m} - k_8 I_m \tag{3.34}$$

3.2 Basic Properties of the Model

3.2.1 Feasible region of the model

Theorem 3.1: The system (3.25) - (3.34) has solutions which are contained in the feasible region ξ.

Let $\phi = (S_h, E_h, I_h, R_h) \in \Re_+^4$ and $\varphi = (S_l, E_l, I_l, R_l) \in \Re_+^4$ and $V = (S_m, I_m) \in \Re_+^2$ be any solution of the system with non-negative initial conditions, then adding the equations together (3.25)-(3.28) and (3.29)-(3.32) and (3.33)-(3.34), we have

$$\frac{dS_h}{dt} + \frac{dE_h}{dt} + \frac{dI_h}{dt} + \frac{dR_h}{dt} = \Lambda_h - \mu_h (S_h + E_h + I_h + R_h)$$

$$\frac{dN_h}{dt} = \Lambda_h - \mu_h N_h \tag{3.35}$$

$$\frac{dN_h}{dt} + \mu_h N_h = \Lambda_h \tag{3.36}$$

$$IF = e^{\mu_h t}$$

$$N_h(t) e^{\mu_h t} = \int \Lambda_h e^{\mu_h (t) dt} + C$$

$$N_h(t) = \frac{\Lambda_h}{\mu_h} + Ce^{-\mu_h t}$$

Using the initial conditions

$$t = 0, \mathrm{N}_h(0) = N_{h0}$$

$$C = N_{h0} - \frac{\Lambda_h}{\mu_h}$$

$$N(t) = \frac{\Lambda_h}{\mu_h} + \left(N_{h0} - \frac{\Lambda_h}{\mu_h}\right)e^{-\mu_h t} \tag{3.37}$$

Applying Birkoff and Rota’s theorem on differential inequality (Birkoff & Rota, 1982), we obtain:

As $t \to \infty$. The total population approaches $\frac{\Lambda_h}{\mu_h}$

Thus, $0 \le N_h \le \frac{\Lambda_h}{\mu_h}$

Again,

$$\frac{dS_l}{dt} + \frac{dE_l}{dt} + \frac{d\mathrm{I}_l}{dt} + \frac{dR_l}{dt} = \Lambda_l - \mu_l\left(S_l + E_l + \mathrm{I}_l + R_l\right) - c_l\mathrm{I}_l$$

$$\frac{dN_l}{dt} = \Lambda_l - \mu_l N_l - c_l\mathrm{I}_l \tag{3.38}$$

$$\frac{dN_l}{dt} + \mu_l N_l = \Lambda_l - c_l\mathrm{I}_l \tag{3.39}$$

$$IF = e^{\int e^{\mu_l dt}} = e^{\mu_l t}$$

$$N_l\left(t\right)e^{\mu_l t} = \int \frac{\left(\Lambda_l - c_l \mathrm{I}_l\right)}{\mu_l} e^{\mu_l t} + C$$

$$N_l = \frac{\left(\Lambda_l - c_l \mathrm{I}_l\right)}{\mu_l} + Ce^{-\mu_l t}$$

Using the initial conditions, $t = 0, N_l\left(0\right) = N_{l0}$

$$N_{l0} = \frac{\left(\Lambda_l - c_l \mathrm{I}_l\right)}{\mu_l} + C$$

$$C = N_{l0} - \frac{\left(\Lambda_l - c_l \mathrm{I}_l\right)}{\mu_l}$$

$$N_l = \frac{\left(\Lambda_l - c_l \mathrm{I}_l\right)}{\mu_l} + \left(N_{l0} - \frac{\left(\Lambda_l - c_l \mathrm{I}_l\right)}{\mu_l}\right)e^{-\mu_l t} \qquad (3.40)$$

Also,

$$\frac{dN_m}{dt} = \Lambda_m - \left(\mu_m + \delta_m\right)\left(S_m + \mathrm{I}_m\right)$$

$$\frac{dN_m}{dt} = \Lambda_m - \left(\mu_m + \delta_m\right)N_m \qquad (3.41)$$

$$\frac{dN_m}{dt} + \left(\mu_m + \delta_m\right)N_m = \Lambda_m \qquad (3.42)$$

$$IF = e^{\int (\mu_m + \delta_m)dt} = e^{(\mu_m + \delta_m)t}$$

$$N_m e^{\mu_m t} = \int \left(\frac{\Lambda_m}{\mu_m + \delta_m} \right) e^{(\mu_m + \delta_m)t} + C$$

$$N_m = \frac{\Lambda_m}{\mu_m + \delta_m} + Ce^{-(\mu_m + \delta_m)t}$$

Applying initial conditions at $t = 0, N_m(0) = N_{m0}$

$$N_m = \frac{\Lambda_m}{\mu_m + \delta_m} + C$$

$$C = N_{m0} - \frac{\Lambda_m}{\mu_m + \delta_m}$$

$$N_m = \frac{\Lambda_m}{\mu_m + \delta_m} + \left(N_{m0} - \frac{\Lambda_m}{\mu_m + \delta_m} \right) e^{-(\mu_m + \delta_m)t} \qquad (3.43)$$

Equations (3.36)- (3.38) satisfy the conditions of the theorem above. Therefore, they exist in the feasible region.

3.2.2 Positivity of the solutions

The model monitors human, livestock and vector population. We show that all the variables are non-negative always.

Theorem 3.2: Let

$$\Psi = \left\{ \begin{array}{l} \left(S_h, E_h, \mathrm{I}_h, R_h\right) \in \Re^4 : S_h\left(0\right) > 0, \mathrm{E}_h\left(0\right) > 0, \mathrm{I}_h\left(0\right) > 0, R_h\left(0\right) > 0 \\ S_h + E_h + \mathrm{I}_h + R_h \leq \dfrac{N_h}{\mu_h} \\ \left(S_l, E_l, \mathrm{I}_l, R_l\right) \in \Re^4 : S_l\left(0\right) > 0, \mathrm{E}_l\left(0\right) > 0, \mathrm{I}_l\left(0\right) > 0, R_l\left(0\right) > 0 \\ S_l + E_l + \mathrm{I}_l + R_l \leq \dfrac{N_l}{\mu_l} \\ \left(S_m, \mathrm{I}_m\right) \in \Re^2 : S_m\left(0\right) > 0, \mathrm{I}_m\left(0\right) > 0 \\ S_m + \mathrm{I}_m \leq \dfrac{N_m}{\mu_m} \end{array} \right\} \tag{3.44}$$

Then the solutions of $\left\{S_h\left(t\right), \mathrm{E}_h\left(t\right), \mathrm{I}_h\left(t\right), R_h\left(t\right), S_l\left(t\right), \mathrm{E}_l\left(t\right), \mathrm{I}_l\left(t\right), R_l\left(t\right), S_m\left(t\right), \mathrm{I}_m\left(t\right)\right\}$ are positive for all $t \geq 0$.

Proof:

Applying the method used by Wiah (Wiah *et al*., 2014)

From (3.25), we have;

$$\frac{dS_h}{dt} = \Lambda_h - \frac{\beta_{hm}\mathrm{I}_m S_h}{N_h} - \mu_h S_h + \gamma_h R_h$$

$$\frac{dS_h}{dt} \geq -\mu_h S_h$$

$$\frac{dS_h}{S_h} \geq -\mu_h dt \tag{3.45}$$

$$\int \frac{dS_h}{S_h} \geq \int -\mu_h dt$$

$$S_h(t) \geq S_h(0)e^{-\mu_h t} \geq 0$$

From (3.26), we have;

$$\frac{dE_h}{dt} = \frac{\beta_{hm}\mathrm{I}_m S_h}{N_h} - k_1 E_h$$

$$\frac{dE_h}{dt} \geq -k_1 E_h$$

$$\frac{dE_h}{E_h} \geq -k_1 dt \tag{3.46}$$

$$\int \frac{dE_h}{E_h} \geq \int -k_1 dt$$

$$E_h(t) \geq E_h(0)e^{-k_1 t} \geq 0$$

From equation (3.27), we have;

$$\frac{d\mathrm{I}_h}{dt} = \varepsilon_h E_h - k_2 \mathrm{I}_h$$

$$\frac{d\mathrm{I}_h}{dt} \geq -k_2 \mathrm{I}_h$$

$$\frac{d\mathrm{I}_h}{\mathrm{I}_h} \geq -k_2 dt \tag{3.47}$$

$$\int \frac{dI_h}{I_h} \geq \int -k_2 dt$$

$$I_h(t) \geq I_h(0) e^{-k_2 t} \geq 0$$

From equation (3.28), we have;

$$\frac{dR_h}{dt} = \tau_h I_h - k_3 I_h$$

$$\frac{dR_h}{dt} \geq -k_3 R_h$$

$$\frac{dR_h}{R_h} \geq -k_3 dt \tag{3.48}$$

$$\int \frac{dR_h}{R_h} \geq \int -k_3 dt$$

$$R_h(t) \geq R_h(0) e^{-k_3 t} \geq 0$$

From equation (3.29), we have;

$$\frac{dS_l}{dt} = \Lambda_l - \frac{\beta_{lm} I_m S_l}{N_l} - k_4 S_l + \gamma_l R_l$$

$$\frac{dS_l}{dt} \geq -k_4 S_l$$

$$\frac{dS_l}{S_l} \geq -k_4 dt \tag{3.49}$$

$$\int \frac{dS_l}{S_l} \geq \int -k_4 dt$$

$$S_l(t) \geq S_l(0) e^{-k_4 t} \geq 0$$

From equation (3.30), we have;

$$\frac{dE_l}{dt} = \frac{\beta_{lm} \mathrm{I}_m S_l}{N_l} - k_5 E_l$$

$$\frac{dE_l}{dt} \geq -k_5 E_l$$

$$\frac{dE_l}{E_l} \geq -k_5 dt \tag{3.50}$$

$$\int \frac{dE_l}{E_l} \geq \int -k_5 dt$$

$$E_l(t) \geq E_l(0) e^{-k_5 t} \geq 0$$

From equation (3.31), we have;

$$\frac{d\mathrm{I}_l}{dt} = \varepsilon_l E_l - k_6 \mathrm{I}_l$$

$$\frac{d\mathrm{I}_l}{dt} \geq -k_6 \mathrm{I}_l$$

$$\frac{d\mathrm{I}_l}{\mathrm{I}_l} \geq -k_6 dt \tag{3.51}$$

$$\int \frac{d\mathrm{I}_l}{\mathrm{I}_l} \geq \int -k_6 dt$$

$$\mathrm{I}_l(t) \geq \mathrm{I}_l(0) e^{-k_6 t} \geq 0$$

From equation (3.32), we have;

$$\frac{dR_l}{dt} = v_\varepsilon S_l + \tau_l E_l + \tau_l \mathrm{I}_l - k_7 R_l$$

$$\frac{dR_l}{dt} \geq -k_7 R_l$$

$$\frac{dR_l}{R_l} \geq -k_7 dt \tag{3.52}$$

$$\int \frac{dR_l}{R_l} \geq \int -k_7 dt$$

$$R_l(t) \geq R_l(0) e^{-k_7 t} \geq 0$$

From equation (3.33), we have;

$$\frac{dS_m}{dt} = \Lambda_m - \frac{\beta_{ml} \mathrm{I}_l S_m}{N_m} - \frac{\beta_{mh} \mathrm{I}_h S_m}{N_m} - k_8 S_m$$

$$\frac{dS_m}{dt} \geq -k_8 S_m$$

$$\frac{dS_m}{S_m} \geq -k_8 dt \tag{3.53}$$

$$\int \frac{dS_m}{S_m} \geq \int -k_8 dt$$

$$S_m(t) \geq S_m(0) e^{-k_8 t} \geq 0$$

From equation (3.34), we have;

$$\frac{d\mathrm{I}_m}{dt} = \frac{\beta_{ml} \mathrm{I}_l S_m}{N_m} + \frac{\beta_{mh} \mathrm{I}_h S_m}{N_m} - k_8 \mathrm{I}_m$$

$$\frac{d\mathrm{I}_m}{dt} \geq -k_8 \mathrm{I}_m$$

$$\frac{d\mathrm{I}_m}{\mathrm{I}_m} \geq -k_8 dt \tag{3.54}$$

$$\int \frac{d\mathrm{I}_m}{\mathrm{I}_m} \geq \int -k_8 dt$$

$$\mathrm{I}_m(t) \geq \mathrm{I}_m(0)e^{-k_8 t} \geq 0$$

Therefore $S_h, E_h, \mathrm{I}_h, R_h, S_l, E_l, \mathrm{I}_l, R_l, S_m, \mathrm{I}_m$ remain positive at every given time. The solutions of the model equations (3.25) to (3.34) are all positive. Hence the model is valid.

3.3 Equilibrium States of the Model

At equilibrium, let

$$\frac{dS_h}{dt} = \frac{dS_l}{dt} = \frac{dS_m}{dt} = \frac{dE_h}{dt} = \frac{dE_l}{dt} = \frac{d\mathrm{I}_h}{dt} = \frac{d\mathrm{I}_l}{dt} = \frac{d\mathrm{I}_m}{dt} = \frac{dR_h}{dt} = \frac{dR_l}{dt} = 0 \qquad (3.55)$$

At any arbitrary equilibrium state, let

$$E^* = \begin{pmatrix} S_h \\ S_l \\ S_m \\ E_h \\ E_l \\ \mathrm{I}_h \\ \mathrm{I}_l \\ \mathrm{I}_m \\ R_h \\ R_l \end{pmatrix} = \begin{pmatrix} S_h^* \\ S_l^* \\ S_m^* \\ E_h^* \\ E_l^* \\ \mathrm{I}_h^* \\ \mathrm{I}_l^* \\ \mathrm{I}_m^* \\ R_h^* \\ R_l^* \end{pmatrix} \qquad (3.56)$$

Then, the steady states of (3.25) - (3.34) satisfy the following algebraic system:

$$\Lambda_h - \frac{\beta_{hm}\mathrm{I}_m^* S_h^*}{N_h} - \mu_h S_h^* + \gamma_h R_h^* = 0 \qquad (3.57)$$

$$\frac{\beta_{hm}\mathrm{I}_m^* S_h^*}{N_h} - k_1 E_h^* = 0 \tag{3.58}$$

$$\varepsilon_h E_h^* - k_2 \mathrm{I}_h^* = 0 \tag{3.59}$$

$$\tau_h \mathrm{I}_h^* - k_3 R_h^* = 0$$

(3.60)

$$\Lambda_l - \frac{\beta_{lm}\mathrm{I}_m^* S_l^*}{N_l} - k_4 S_l^* + \gamma_l R_l^* = 0 \tag{3.61}$$

$$\frac{\beta_{lm}\mathrm{I}_m^* S_l^*}{N_l} - k_5 E_l^* = 0 \tag{3.62}$$

$$\varepsilon_l E_l^* - k_6 \mathrm{I}_l^* = 0 \tag{3.63}$$

$$v_\varepsilon S_l^* + \tau_l E_l^* + \tau_l \mathrm{I}_l^* - k_7 R_l^* = 0 \tag{3.64}$$

$$\Lambda_m - \frac{\beta_{ml}\mathrm{I}_l^* S_m^*}{N_m} - \frac{\beta_{mh}\mathrm{I}_h^* S_h^*}{N_m} - k_8 S_m^* = 0 \tag{3.65}$$

$$\frac{\beta_{ml}\mathrm{I}_l^* S_m^*}{N_m} + \frac{\beta_{mh}\mathrm{I}_h^* S_m^*}{N_m} - k_8 \mathrm{I}_m^* = 0 \tag{3.66}$$

From equation (3.59)

$$\mathrm{I}_h^* = \frac{\varepsilon_h E_h^*}{k_2} \tag{3.67}$$

From equation (3.63)

$$I_l^* = \frac{\varepsilon_l E_l^*}{k_6} \tag{3.68}$$

From equation (3.58), we have;

$$E_h^* = \frac{\beta_{hm} I_m^* S_h^*}{k_1 N_h} \tag{3.69}$$

From equation (3.62), we have;

$$E_l^* = \frac{\beta_{lm} I_m^* S_l^*}{k_5 N_l} \tag{3.70}$$

Substituting (3.69) into (3.67) gives;

$$\mathrm{I}_h^* = \frac{\beta_{hm} \varepsilon_h S_h^* \mathrm{I}_m^*}{k_1 k_2 N_h} = \frac{k_9 S_h^* \mathrm{I}_m^*}{N_h} \tag{3.71}$$

Where,

$$k_9 = \frac{\beta_{hm} \varepsilon_h}{k_1 k_2} \tag{3.72}$$

Substituting (3.70) into (3.68) gives;

$$\mathrm{I}_l^* = \frac{\beta_{lm} \varepsilon_l S_l^* \mathrm{I}_m^*}{k_5 k_6 N_l} = \frac{k_{10} S_l^* \mathrm{I}_m^*}{N_l} \tag{3.73}$$

Where,

$$k_{10} = \frac{\beta_{lm} \varepsilon_l}{k_5 k_6} \tag{3.74}$$

Substituting (3.71) and (3.73) into (3.66) gives;

$$\left(\frac{\beta_{ml}S_m^*}{N_m}\right)\left(\frac{k_{10}S_l^*\mathrm{I}_m^*}{N_l}\right)+\left(\frac{\beta_{mh}S_m^*}{N_m}\right)\left(\frac{k_9S_h^*\mathrm{I}_m^*}{N_h}\right)-k_8\mathrm{I}_m^*=0$$

$$\frac{\beta_{ml}S_m^*k_{10}S_l^*\mathrm{I}_m^*N_h^*+\beta_{mh}S_m^*k_9S_h^*\mathrm{I}_m^*N_l^*-k_8N_h^*N_l^*N_m^*\mathrm{I}_m^*}{N_h^*N_l^*N_m^*}=0$$

$$\left(\beta_{ml}S_m^*k_{10}S_lN_h^*+\beta_{mh}S_m^*k_9S_h^*N_l^*-k_8N_h^*N_l^*N_m^*\right)\mathrm{I}_m^*=0$$

$$\left(\frac{\beta_{ml}\Lambda_h^2\beta_{lm}\varepsilon_l\Lambda_lk_7k_1k_2+\beta_{mh}\Lambda_h^2\beta_{hm}\varepsilon_h\Lambda_l\left(k_7+v_\varepsilon\right)k_5k_6-\Lambda_h^2\Lambda_l\left(k_7+v_\varepsilon\right)k_1k_2k_5k_6k_8}{k_1k_2k_5k_6k_8\mu_h\left(k_4k_7-\gamma_lv_\varepsilon\right)}\right)\mathrm{I}_m^*=0$$

(3.75)

From equation (3.75),

$$\mathrm{I}_m^*=0 \tag{3.76}$$

or,

$$\frac{\beta_{ml}\Lambda_h^2\beta_{lm}\varepsilon_l\Lambda_lk_7k_1k_2+\beta_{mh}\Lambda_h^2\beta_{hm}\varepsilon_h\Lambda_l\left(k_7+v_\varepsilon\right)k_5k_6-\Lambda_h^2\Lambda_l\left(k_7+v_\varepsilon\right)k_1k_2k_5k_6k_8}{k_1k_2k_5k_6k_8\mu_h\left(k_4k_7-\gamma_lv_\varepsilon\right)}=0$$

(3.77)

3.4 Disease Free Equilibrium State (DFE)

Disease free equilibrium states are steady when all the infectious classes in a population are zero, that is; the population comprises of susceptible humans and vectors only.

At Disease Free Equilibrium, let;

$$\begin{pmatrix} S_h \\ E_h \\ I_h \\ R_h \\ S_l \\ E_l \\ I_l \\ R_l \\ S_m \\ I_m \end{pmatrix} = \begin{pmatrix} S_h^0 \\ E_h^0 \\ I_h^0 \\ R_h^0 \\ S_l^0 \\ E_l^0 \\ I_l^0 \\ R_l^0 \\ S_m^0 \\ I_m^0 \end{pmatrix} \tag{3.78}$$

Substituting (3.76) into (3.69) and (3.73) gives;

$$I_h^* = I_l^* = 0 \tag{3.79}$$

Substituting (3.76) into (3.65) and (3.66) gives;

$$E_h^* = E_l^* = 0 \tag{3.80}$$

Substituting (3.76), (3.79) and (3.80) gives into equation (3.56) gives;

$$R_h^* = 0 \tag{3.81}$$

From (3.57);

$$S_l^* = \frac{\Lambda_l + \gamma_l R_l}{k_4} \tag{3.82}$$

From (3.60);

$$R_l^* = \frac{v_\varepsilon S_l}{k_7} \tag{3.83}$$

Substituting (3.83) into (3.82) gives,

$$S_l^* = \frac{\Lambda_l k_7 + v_\varepsilon \gamma_l S_l^*}{k_4 k_7}$$

$$S_l^* = \frac{\Lambda_l k_7}{k_4 k_7 - v_\varepsilon \gamma_l} \tag{3.84}$$

Substituting (3.84) into (3.83) gives,

$$R_l^* = \frac{v_\varepsilon \Lambda_l}{k_4 k_7 - v_\varepsilon \gamma_l} \tag{3.85}$$

From (3.61);

$$S_m^* = \frac{\Lambda_m}{k_8} \tag{3.86}$$

Thus for human population, the Disease Free Equilibrium state is given by;

$$\begin{pmatrix} S_h \\ E_h \\ I_h \\ R_h \end{pmatrix} = \begin{pmatrix} \frac{\Lambda_h}{\mu_h} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.87}$$

Thus for Livestock population, the Disease Free Equilibrium state is given by;

$$\begin{pmatrix} S_l \\ E_l \\ I_l \\ R_l \end{pmatrix} = \begin{pmatrix} \dfrac{k_7 \Lambda_l}{k_4 k_7 - v_\varepsilon \gamma_l} \\ 0 \\ 0 \\ \dfrac{v_\varepsilon \Lambda_l}{k_4 k_7 - v_\varepsilon \gamma_l} \end{pmatrix} \tag{3.88}$$

Thus for Mosquito population, the Disease Free Equilibrium state is given by;

$$\begin{pmatrix} S_m \\ I_m \end{pmatrix} = \begin{pmatrix} \dfrac{\Lambda_m}{k_8} \\ 0 \end{pmatrix} \tag{3.89}$$

Disease Free Equilibrium (DFE) Point

$$\begin{pmatrix} S_h \\ E_h \\ I_h \\ R_h \\ S_l \\ E_l \\ I_l \\ R_l \\ S_m \\ I_m \end{pmatrix} = \begin{pmatrix} \dfrac{\Lambda_h}{\mu_h} \\ 0 \\ 0 \\ 0 \\ \dfrac{\Lambda_l k_7}{k_4 k_7 - \nu_\varepsilon \gamma_l} \\ 0 \\ 0 \\ \dfrac{\nu_\varepsilon \Lambda_l}{k_4 k_7 - \nu_\varepsilon \gamma_l} \\ \dfrac{\Lambda_m}{k_8} \\ 0 \end{pmatrix} \tag{3.90}$$

Equation (3.90) shows the disease free equilibrium of the population.

3.5 Effective Reproduction Number, R_c

Effective reproduction number is the number of secondary infections caused by an infected individual during his entire time of infectiousness (Diekmann *et al.*,1990). If the reproduction ratio is greater than one, the disease will spread throughout the entire population and if it is less than one, the disease will die out with time. The basic reproduction number determines the direction of the disease (Oguntolu *et al*.,2019).

Using Next Generation Matrix (Diekmann and Heesterbeek, 2000);

$$R_c = \rho\left(FV^{-1}\right)$$

where ρ is the spectral radius of the Next Generation Matrix $\left(FV^{-1}\right)$, $F_i\left(x\right)$ is the rate of appearance of new infections in compartment i, $V_i^+\left(x\right)$ is the rate of transfer of individuals out of compartment i

$$V_i = V_i^-\left(x\right) - V_i^+\left(x\right) \tag{3.91}$$

Given the DFE, R_c is calculated thus:

$$F = \frac{\partial F_i}{\partial x_j}\left(E_0\right) \tag{3.92}$$

$$V = \frac{\partial V_i}{\partial x_j}\left(E_0\right) \tag{3.93}$$

$$F_i = \begin{pmatrix} \dfrac{\beta_{hm} \mathrm{I}_m^* S_h^o}{N_h^o} \\ 0 \\ \dfrac{\beta_{lm} \mathrm{I}_m^* S_l^o}{N_l^o} \\ 0 \\ \dfrac{\beta_{mh} \mathrm{I}_h^* S_m^o}{N_m^o} + \dfrac{\beta_{ml} \mathrm{I}_l^* S_m^o}{N_m^o} \end{pmatrix} \tag{3.94}$$

$$F=\begin{bmatrix} 0 & 0 & 0 & 0 & \beta_{hm} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{\beta_{lm}k_7}{k_7+v_\varepsilon} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_{mh} & 0 & \beta_{ml} & 0 \end{bmatrix} \tag{3.95}$$

$$V=V^- - V^+ = \begin{pmatrix} -k_1E_h^* \\ \varepsilon_h E_h^* - k_2 \mathrm{I}_h^* \\ -k_5 E_l^* \\ \varepsilon_l E_l^* - k_6 \mathrm{I}_l^* \\ -k_8 \mathrm{I}_m^* \end{pmatrix} \tag{3.96}$$

Where,

$$V^- = \begin{pmatrix} -k_1E_h^* \\ -k_2 \mathrm{I}_h^* \\ -k_5 E_l^* \\ -k_6 \mathrm{I}_l^* \\ -k_8 \mathrm{I}_m^* \end{pmatrix} \tag{3.97}$$

$$V^+ = \begin{pmatrix} 0 \\ \varepsilon_h E_h^* \\ 0 \\ \varepsilon_l E_l^* \\ 0 \end{pmatrix} \tag{3.98}$$

$$V = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 \\ -\varepsilon_h & k_2 & 0 & 0 & 0 \\ 0 & 0 & k_5 & 0 & 0 \\ 0 & 0 & -\varepsilon_l & k_6 & 0 \\ 0 & 0 & 0 & 0 & k_8 \end{bmatrix} \tag{3.99}$$

$$V^{-1} = \begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 & 0 \\ \frac{\varepsilon_h}{k_1 k_2} & \frac{1}{k_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{k_5} & 0 & 0 \\ 0 & 0 & \frac{\varepsilon_l}{k_5 k_6} & \frac{1}{k_6} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{k_8} \end{bmatrix} \tag{3.100}$$

$$FV^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\beta_{hm}}{k_8} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\beta_{lm} k_7}{(k_7 + v_\varepsilon) k_8} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_{mh}\varepsilon_h}{k_1 k_2} & \frac{\beta_{mh}}{k_2} & \frac{\beta_{ml}\varepsilon_l}{k_5 k_6} & \frac{\beta_{ml}}{k_6} & 0 \end{bmatrix} \tag{3.101}$$

From (3.101), we calculate the eigenvalues to determine the effective reproduction number R_c.

Taking the dominant eigenvalue of the matrix FV^{-1} and computing $|A - \lambda I| = 0$, gives;

$$\begin{vmatrix} -\lambda & 0 & 0 & 0 & \dfrac{\beta_{hm}}{k_8} \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & \dfrac{\beta_{lm}k_7}{(k_7+v_\varepsilon)k_8} \\ 0 & 0 & 0 & -\lambda & 0 \\ \dfrac{\beta_{mh}\varepsilon_h}{k_1k_2} & \dfrac{\beta_{mh}}{k_2} & \dfrac{\beta_{ml}\varepsilon_l}{k_5k_6} & \dfrac{\beta_{ml}}{k_6} & -\lambda \end{vmatrix} = 0 \qquad (3.102)$$

$$\lambda^5 - \left(\frac{\beta_{ml}\beta_{ml}k_1k_2k_7\varepsilon_l + \beta_{mh}\beta_{hm}k_5k_6k_7\varepsilon_h + \beta_{mh}\beta_{hm}k_5k_6v_\varepsilon\varepsilon_h}{k_1k_2k_5k_6k_8\left(k_7+v_\varepsilon\right)} \right)\lambda^3 = 0 \qquad (3.103)$$

$$\lambda^5 - \left(\frac{\beta_{ml}\beta_{ml}k_1k_2k_7\varepsilon_l + \beta_{mh}\beta_{hm}k_5k_6k_7\varepsilon_h + \beta_{mh}\beta_{hm}k_5k_6v_\varepsilon\varepsilon_h}{k_1k_2k_5k_6k_8\left(k_7+v_\varepsilon\right)} \right)\lambda^3 = 0 \qquad (3.104)$$

Implies;

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$$

$$\lambda_4 = \frac{\sqrt{k_1k_2k_5k_6k_8\left(k_7+v_\varepsilon\right)\left(\beta_{ml}\beta_{lm}k_1k_2k_7\varepsilon_l + \beta_{mh}\beta_{hm}k_5k_6k_7\varepsilon_h + \beta_{mh}\beta_{hm}k_5k_6v_\varepsilon\varepsilon_h\right)}}{k_1k_2k_5k_6k_8\left(k_7+v_\varepsilon\right)} \qquad (3.105)$$

$$\lambda_5 = -\frac{\sqrt{k_1k_2k_5k_6k_8\left(k_7+v_\varepsilon\right)\left(\beta_{ml}\beta_{lm}k_1k_2k_7\varepsilon_l + \beta_{mh}\beta_{hm}k_5k_6k_7\varepsilon_h + \beta_{mh}\beta_{hm}k_5k_6v_\varepsilon\varepsilon_h\right)}}{k_1k_2k_5k_6k_8\left(k_7+v_\varepsilon\right)} \qquad (3.106)$$

Clearly, λ_4 is the dominant eigenvalue

Therefore;

$$R_c = \frac{\sqrt{k_1k_2k_5k_6k_8\left(k_7+v_\varepsilon\right)\left(\beta_{lm}\beta_{ml}k_1k_2k_7\varepsilon_l+\beta_{mh}\beta_{hm}k_5k_6k_7\varepsilon_h+\beta_{mh}\beta_{hm}k_5k_6v_\varepsilon\varepsilon_h\right)}}{k_1k_2k_5k_6k_8\left(k_7+v_\varepsilon\right)} \quad (3.107)$$

Thus our effective Reproduction number is given by equation (3.107). This is the average number of secondary cases generated by an infected individual in this model.

3.6 Local Stability of Disease Free Equilibrium State

We investigate the local stability of the equilibrium points by the theorem below:

Theorem 3.3: The disease free equilibrium of the model equations (3.25)-(3.34) is locally asymptotically stable if $R_c < 1$.

Proof:

Linearizing the model equations (3.25)-(3.34) at any arbitrary equilibrium point (E^*) gives the jacobian

$$J\left(E^*\right)=\begin{bmatrix} -c_1 & 0 & 0 & \gamma_h & 0 & 0 & 0 & 0 & 0 & -c_2 \\ c_3 & -k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 \\ 0 & \varepsilon_l & -k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_h & -k_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_4 & 0 & 0 & \gamma_l & 0 & -c_5 \\ 0 & 0 & 0 & 0 & c_6 & -k_5 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_l & -k_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_\varepsilon & \tau_l & \tau_l & -k_7 & 0 & 0 \\ 0 & 0 & -c_7 & 0 & 0 & 0 & -c_8 & 0 & -c_9 & 0 \\ 0 & 0 & c_7 & 0 & 0 & 0 & c_8 & 0 & 0 & -k_8 \end{bmatrix} \quad (3.108)$$

where;

$$c_1 = \frac{\beta_{hm} I_m}{N_h} + \mu_h \ ,$$

$$c_2 = \frac{\beta_{hm} S_h}{N_h} \ ,$$

$$c_3 = \frac{\beta_{hm} I_m}{N_h} \ ,$$

$$c_4 = \frac{\beta_{lm} I_m}{N_l} + k_4 \ ,$$

$$c_5 = \frac{\beta_{lm} S_l}{N_l} \ ,$$

$$c_6 = \frac{\beta_{lm} I_m}{N_l} \ ,$$

$$c_7 = \frac{\beta_{mh} S_m}{N_m} \ ,$$

$$c_8 = \frac{\beta_{lm} S_m}{N_m} \ ,$$

$$c_9 = \frac{\beta_{ml} I_l}{N_m} + \frac{\beta_{mh} I_h}{N_m} + k_8$$

We evaluated the Jacobian at the disease free equilibrium to determine the local stability of the system and obtained;

$$J\left(E^{o}\right)=\begin{bmatrix} -\mu_h & 0 & 0 & \gamma_h & 0 & 0 & 0 & 0 & 0 & -\beta_{hm} \\ 0 & -k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_{hm} \\ 0 & \varepsilon_h & -k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_h & -k_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_4 & 0 & 0 & \gamma_l & 0 & \frac{-\beta_{lm}k_7}{k_7+v_\varepsilon} \\ 0 & 0 & 0 & 0 & 0 & -k_5 & 0 & 0 & 0 & \frac{\beta_{lm}k_7}{k_7+v_\varepsilon} \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_l & -k_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_\varepsilon & \tau_l & \tau_l & -k_7 & 0 & 0 \\ 0 & 0 & -\beta_{mh} & 0 & 0 & 0 & -\beta_{ml} & 0 & -k_8 & 0 \\ 0 & 0 & \beta_{mh} & 0 & 0 & 0 & \beta_{ml} & 0 & 0 & -k_8 \end{bmatrix} \tag{3.109}$$

Using elementary row transformation, the matrix above becomes

$$J\left(E^o\right)=\begin{bmatrix} -\mu_h & 0 & 0 & \gamma_h & 0 & 0 & 0 & 0 & 0 & -\beta_{hm} \\ 0 & -k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_{hm} \\ 0 & 0 & -k_2 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{\beta_{hm}\varepsilon_h}{k_1} \\ 0 & 0 & 0 & -k_3 & 0 & 0 & 0 & 0 & 0 & A_1 \\ 0 & 0 & 0 & 0 & -k_4 & 0 & 0 & \gamma_l & 0 & -A_2 \\ 0 & 0 & 0 & 0 & 0 & -k_5 & 0 & 0 & 0 & A_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_6 & 0 & 0 & A_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -A_4 & 0 & A_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_8 & -A_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_7 \end{bmatrix} \quad (3.110)$$

$$A_1=\frac{\beta_{hm}\tau_h\varepsilon_h}{k_1k_2} \ ,$$

$$A_2=\frac{\beta_{lm}k_7}{\Lambda_l\left(k_7+\nu_\varepsilon\right)} \ ,$$

$$A_3 = \frac{\beta_{lm} k_7 \varepsilon_l}{k_5 \Lambda_l \left(k_7 + v_\varepsilon\right)} \ ,$$

$$A_4 = \frac{k_4 k_7 - \gamma_l v_\varepsilon}{k_4} \ ,$$

$$A_5 = \frac{\beta_{lm} k_7 \left(k_4 k_6 \tau_l + k_5 \tau_l \varepsilon_l - k_5 k_6 v_\varepsilon\right)}{k_4 k_5 k_6 \left(k_7 + v_\varepsilon\right)} \ ,$$

$$A_6 = \frac{k_5 k_6 k_7 \beta_{hm} \beta_{mh} + k_1 k_7 \varepsilon_l \beta_{lm} \beta_{ml} + k_5 k_6 v_\varepsilon \beta_{hm} \beta_{mh}}{k_1 k_5 k_6 \left(k_7 + v_\varepsilon\right)}$$

$$A_7 = \frac{k_5 k_6 k_7 \beta_{hm} \beta_{mh} + k_1 k_7 \varepsilon_l \beta_{lm} \beta_{ml} + k_5 k_6 v_\varepsilon \beta_{hm} \beta_{mh} + k_1 k_5 k_6 k_7 k_8 + k_1 k_5 k_6 k_8 v_\varepsilon}{k_1 k_5 k_6 \left(k_7 + v_\varepsilon\right)}$$

Therefore, the Characteristics equation of the upper triangular Jacobian is

$$J\left(E^{o}\right)=\begin{bmatrix} -\left(\mu_h+\lambda\right) & 0 & 0 & \gamma_h & 0 & 0 & 0 & 0 & 0 & -\beta_{hm} \\ 0 & -\left(k_1+\lambda\right) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_{hm} \\ 0 & 0 & -\left(k_2+\lambda\right) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_{hm}\varepsilon_h}{k_1} \\ 0 & 0 & 0 & -\left(k_3+\lambda\right) & 0 & 0 & 0 & 0 & 0 & A_1 \\ 0 & 0 & 0 & 0 & -\left(k_4+\lambda\right) & 0 & 0 & \gamma_l & 0 & -A_2 \\ 0 & 0 & 0 & 0 & 0 & -\left(k_5+\lambda\right) & 0 & 0 & 0 & A_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\left(k_6+\lambda\right) & 0 & 0 & A_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\left(A_4+\lambda\right) & 0 & A_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\left(k_8+\lambda\right) & -A_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_7-\lambda \end{bmatrix}$$

(3.111)

Therefore, the eigenvalues are

$$\lambda_1=-\mu_h<0$$

(3.112)

$$\lambda_2=-k_1=-\left(\mu_h+\varepsilon_h\right)<0 \quad (3.113)$$

$$\lambda_3=-k_2=-\left(\mu_h+\tau_h\right)<0$$

(3.114)

$$\lambda_4=-k_3=-\left(\mu_h+\gamma_h\right)<0$$

(3.115)

$$\lambda_5 = -k_4 = -\left(\mu_l + v_\varepsilon\right) < 0 \tag{3.116}$$

$$\lambda_6 = -k_5 = -\left(\mu_l + \tau_l + \varepsilon_l\right) < 0 \tag{3.117}$$

$$\lambda_7 = -k_6 = -\left(\mu_l + c_l + \tau_l\right) < 0 \tag{3.118}$$

$$\lambda_8 = -A_4 = -\left(\frac{k_4 k_7 - \gamma_l v_\varepsilon}{k_4}\right) = \frac{-\mu_l\left(\mu_l + \gamma_l + v_\varepsilon\right)}{\mu_l + v_\varepsilon} < 0 \tag{3.119}$$

$$\lambda_9 = -k_8 = -\left(\mu_m + \delta_m\right) < 0 \tag{3.120}$$

$$\lambda_{10} = A_7 = \frac{k_5 k_6 k_7 \beta_{hm}\beta_{mh} + k_1 k_7 \varepsilon_l \beta_{lm}\beta_{ml} + k_5 k_6 v_\varepsilon \beta_{hm}\beta_{mh} + k_1 k_5 k_6 k_7 k_8 + k_1 k_5 k_6 k_8 v_\varepsilon}{k_1 k_5 k_6\left(k_7 + v_\varepsilon\right)} \tag{3.121}$$

For λ_{10} to be negative, then

$$\frac{k_5 k_6 k_7 \beta_{hm}\beta_{mh} + k_1 k_7 \varepsilon_l \beta_{lm}\beta_{ml} + k_5 k_6 v_\varepsilon \beta_{hm}\beta_{mh} + k_1 k_5 k_6 k_7 k_8 + k_1 k_5 k_6 k_8 v_\varepsilon}{k_1 k_5 k_6\left(k_7 + v_\varepsilon\right)} < 0 \tag{3.122}$$

$$\frac{k_2 k_5 k_6 k_7 \beta_{hm}\beta_{mh} + k_1 k_2 k_7 \varepsilon_l \beta_{lm}\beta_{ml} + k_2 k_5 k_6 v_\varepsilon \beta_{hm}\beta_{mh}}{k_1 k_2 k_5 k_6 k_8\left(k_7 + v_\varepsilon\right)} < 1$$

$$\Rightarrow R_c < 1$$

This implies that, $\lambda_{10} < 0$ if $R_c < 1$, Hence, the disease free equilibrium E^o of the equation (3.26) and (3.35) is locally asymptotically stable (LAS) if $R_c < 1$.

3.7 Global Stability of Disease-Free Equilibrium State $\left(E^0\right)$

Theorem 3.4: The D.F.E $\left(E^0\right)$ of the model system is globally asymptotically stable (GAS) in the feasible region ξ of $R_c < 1$ and unstable if $R_c > 1$.

Proof: To establish the global stability of the D.F.E, the two conditions as in (Castillo-chavez *et al*., 2002) for $R_c < 1$ were used for the model system. The conditions are

$(H1)$ For $\frac{dX}{dt} = F(X,0), X$ is globally asymptotically stable (g.a.s),

$(H2)$ $G(X,Z) = AZ \quad \hat{G}(X,Z)$, $\hat{G}(X,Z) \geq 0$ for $(X,Z) \in \xi$, where $A = D_z G(X^*,0)$ is an M-matrix (the off-diagonal elements of *A* are nonnegative) and ξ is the region where the model makes biological sense.

We can write the model system as:

$$\frac{dX_s}{dt} = A\left(X_s - X_{D.F.E_S}\right) + A_1 X_i \tag{3.123}$$

$$\frac{dX_s}{dt} = A_2 X_i \tag{3.124}$$

Where

$$X_s = \left(S_h^0, S_l^0, R_l^0, S_m^0\right)^T \tag{3.125}$$

are the non-infectious compartments,

$$X_i = \left(I_h^0, I_l^0, I_m^0, E_h^0, E_m^o, R_h^0\right)^T \tag{3.126}$$

denote the infectious compartments. The disease-free equilibrium is denoted as

$$E^0 = \left(X_s^*, 0\right) \tag{3.127}$$

Where,

$$X_s^* = \left(N_h^0, N_l^0, N_m^0\right)$$

(3.128)

$$\frac{dX_s}{dt} = F\left(X_s, 0\right) \tag{3.129}$$

$$\frac{dS_h}{dt} = \Lambda_h - \mu_h S_h \tag{3.130}$$

$$\frac{dS_l}{dt} = \Lambda_l - k_1 S_l + \gamma_l R_l \tag{3.131}$$

$$\frac{dR_l}{dt} = V_\varepsilon S_l - k_2 R_l \tag{3.132}$$

$$\frac{dS_m}{dt} = \Lambda_m - k_3 S_m \tag{3.133}$$

From (3.130)

$$S_h(t)=\frac{\Lambda_h}{\mu_h}\left(1-e^{-\mu_h t}\right)+S_h(0)e^{-\mu_h t} \quad (3.134)$$

From (3.131), we have;

$$\frac{dS_l}{dt}=\Lambda_l-k_1S_l+\gamma_l\frac{V_\varepsilon\Lambda_l}{k_4k_7} \quad (3.135)$$

$$\frac{dS_l}{dt}=\Lambda_l+\frac{\gamma_lV_\varepsilon\Lambda_l}{k_4k_7}-k_1S_l \quad (3.136)$$

$$S_l(t)=\frac{\Lambda_l\left(k_4k_7+\gamma_lV_\varepsilon\right)}{k_1k_4k_7}-\frac{\Lambda_l\left(k_4k_7+\gamma_lV_\varepsilon\right)}{k_1k_4k_7}e^{-k_1t}+S_l(0)e^{-k_1t} \quad (3.137)$$

From (3.132)

$$\frac{dR_l}{dt}=V_\varepsilon S_l-k_2R_l \quad (3.138)$$

$$\frac{dR_l}{dt}=\frac{V_\varepsilon\Lambda_l}{k_4}-k_2R_l \quad (3.139)$$

$$R_l(t)=\frac{V_\varepsilon\Lambda_l}{k_2k_4}-\frac{V_\varepsilon\Lambda_l}{k_2k_4}e^{-k_2t}+R_l(0)e^{-k_2t} \quad (3.140)$$

From (3.133)

$$\frac{dS_m}{dt}=\Lambda_m-k_3S_m$$

$$S_m(t) = \frac{\Lambda_m}{k_3} - \frac{\Lambda_m}{k_3} e^{-k_3 t} + S_m(0) e^{-k_3 t} \tag{3.141}$$

Hence,

$S_h^0(t) \rightarrow N_h^0(t)$ as $t \rightarrow 0$

$S_l^0(t) + R_l^0(t) \rightarrow N_l^0(t)$ as $t \rightarrow 0$

$S_m^0(t) \rightarrow N_m^0(t)$ as $t \rightarrow 0$

Irrespective of the value of

$S_h^0(0), S_l^0(0), R_l^0(0), S_m^0(0)$

Thus

$X_s^{**} = \left(N_h^0, N_l^0, N_m^0, 0\right)$ is globally asymptotically stable.

Given that;

$G(x, y) = C_y - \hat{G}(x, y)$

$\hat{G}(x, y) = C_y - G(x, y)$

Where $C = \dfrac{\partial G(x, 0)}{\partial t}$

$$y' = G(x,y) = \begin{pmatrix} \dfrac{\beta_{hm} I_m S_h}{N_h} - k_1 E_h \\ \varepsilon_h E_h - k_2 I_h \\ \dfrac{\beta_{lm} I_m S_l}{N_l} - k_5 E_l \\ \varepsilon_l E_l - k_6 I_l \\ \dfrac{\beta_{ml} I_l S_m}{N_m} + \dfrac{\beta_{mh} I_h S_m}{N_m} - k_8 I_m \end{pmatrix} \tag{3.142}$$

$$C = \begin{bmatrix} -k_1 & 0 & 0 & 0 & \dfrac{\beta_{hm} S_h}{N_h} \\ \varepsilon_h & -k_2 & 0 & 0 & 0 \\ 0 & 0 & -k_5 & 0 & \dfrac{\beta_{lm} S_l}{N_l} \\ 0 & 0 & \varepsilon_l & -k_6 & 0 \\ 0 & \dfrac{\beta_{mh} S_h}{N_m} & 0 & \dfrac{\beta_{ml} S_l}{N_m} & -k_8 \end{bmatrix} \tag{3.143}$$

$$C_y = \begin{bmatrix} -k_1 & 0 & 0 & 0 & \dfrac{\beta_{hm}S_h}{N_h} \\ \varepsilon_h & -k_2 & 0 & 0 & 0 \\ 0 & 0 & -k_5 & 0 & \dfrac{\beta_{lm}S_l}{N_l} \\ 0 & 0 & \varepsilon_l & -k_6 & 0 \\ 0 & \dfrac{\beta_{mh}S_h}{N_m} & 0 & \dfrac{\beta_{ml}S_l}{N_m} & -k_8 \end{bmatrix} \begin{bmatrix} E_h \\ I_h \\ E_l \\ I_l \\ I_m \end{bmatrix} \tag{3.144}$$

$$C_y = \begin{pmatrix} -k_1 E_h + \dfrac{\beta_{hm}S_h I_m}{N_h} \\ \varepsilon_h E_h - k_2 I_h \\ -k_5 E_l + \dfrac{\beta_{lm}S_l I_m}{N_l} \\ \varepsilon_l E_l - k_6 I_l \\ \dfrac{\beta_{mh}S_h I_h}{N_m} + \dfrac{\beta_{ml}S_l I_l}{N_m} - k_8 I_m \end{pmatrix} \tag{3.145}$$

$$\hat{G}(x,y) = C_y - G(x,y)$$

$$= \begin{pmatrix} -k_1E_h + \dfrac{\beta_{hm}S_hI_m}{N_h} \\ \varepsilon_hE_h - k_2I_h \\ -k_5E_l + \dfrac{\beta_{lm}S_lI_m}{N_l} \\ \varepsilon_lE_l - k_6I_l \\ \dfrac{\beta_{mh}S_hI_h}{N_m} + \dfrac{\beta_{ml}S_lI_l}{N_m} - k_8I_m \end{pmatrix} - \begin{pmatrix} \dfrac{\beta_{hm}I_mS_h}{N_h} - k_1E_h \\ \varepsilon_hE_h - k_2I_h \\ \dfrac{\beta_{lm}I_mS_l}{N_l} - k_5E_l \\ \varepsilon_lE_l - k_6I_l \\ \dfrac{\beta_{ml}I_lS_m}{N_m} + \dfrac{\beta_{mh}I_hS_m}{N_m} - k_8I_m \end{pmatrix} \tag{3.146}$$

$$\hat{G}(x,y) = \begin{pmatrix} \hat{G}_1(x,y) \\ \hat{G}_2(x,y) \\ \hat{G}_3(x,y) \\ \hat{G}_4(x,y) \\ \hat{G}_5(x,y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.147}$$

Then $\hat{G}(x,y) = 0$

This satisfies the conditions H1 and H2, therefore the disease free equilibrium is globally asymptotically stable when $R_0 < 1$.

3.8 Endemic Equilibrium Point (EEP) in terms of Force of Infection

The endemic Equilibrium Point (EEP) in terms of forces of infection are computed;

Let

$$E^1 = \left(S_h, E_h, \mathrm{I}_h, \mathrm{R}_h, \mathrm{S}_l, E_l, \mathrm{I}_l, \mathrm{R}_l, \mathrm{S}_m, \mathrm{I}_m\right) = \left(S_h^{**}, \mathrm{E}_h^{**}, \mathrm{I}_h^{**}, \mathrm{R}_h^{**}, S_l^{**}, \mathrm{E}_l^{**}, \mathrm{I}_l^{**}, \mathrm{R}_l^{**}, S_m^{**}, \mathrm{I}_m^{**}\right) \quad (3.148)$$

are the Endemic Equilibrium Points.

$$\Lambda_h - \lambda_{hm}^{**} S_h^{**} - \mu_h S_h^{**} + \gamma_h R_h^{**} = 0 \quad (3.149)$$

$$\lambda_{hm}^{**} S_h^{**} - k_1 E_h^{**} = 0 \quad (3.150)$$

$$\varepsilon_h E_h^{**} - k_2 \mathrm{I}_h^{**} = 0 \quad (3.151)$$

$$\tau_h \mathrm{I}_h^{**} + k_3 R_h^{**} = 0 \quad (3.152)$$

$$\Lambda_l - \lambda_{lm}^{**} S_l^{**} - k_4 S_l^{**} + \gamma_l R_l^{**} = 0 \quad (3.153)$$

$$\lambda_{lm}^{**} S_l^{**} - k_5 E_l^{**} = 0 \quad (3.154)$$

$$\varepsilon_l E_l^{**} - k_6 \mathrm{I}_l^{**} = 0 \quad (3.155)$$

$$v_\varepsilon S_l^{**} + \tau_l E_l^{**} + \tau_l \mathrm{I}_l^{**} - k_7 R_l^{**} = 0 \quad (3.156)$$

$$\Lambda_m - \lambda_{ml}^{**} S_m^{**} - \lambda_{mh}^{**} S_m^{**} - k_8 S_m^{**} = 0 \quad (3.157)$$

$$\lambda_{ml}^{**} S_m^{**} + \lambda_{mh}^{**} S_m^{**} - k_8 \mathrm{I}_m^{**} = 0 \quad (3.158)$$

Where,

$$\lambda_{hm}^{**} = \frac{\beta_{hm}\mathrm{I}_m}{N_h}, \lambda_{lm}^{**} = \frac{\beta_{lm}\mathrm{I}_m}{N_l}, \lambda_{mh}^{**} = \frac{\beta_{mh}\mathrm{I}_h}{N_m}, and\, \lambda_{ml}^{**} = \frac{\beta_{ml}\mathrm{I}_l}{N_m} \quad (3.159)$$

λ_{hm}^{**} : is the force of infection of humans to mosquitoes.

λ_{lm}^{**}: is the force of infection of livestock to mosquitoes.

λ_{mh}^{**}: is the force of infection of mosquitoes to humans.

λ_{ml}^{**}: is the force of infection of mosquitoes to livestock.

From (3.149)

$$S_h^{**} = \frac{\Lambda_h + \gamma_h R_h^{**}}{\lambda_{hm}^{**} + \mu_h} \tag{3.160}$$

From (3.150)

$$E_h^{**} = \frac{\lambda_{hm}^{**} S_h^{**}}{k_1} \tag{3.161}$$

From (3.151)

$$I_h^{**} = \frac{\varepsilon_h E_h^{**}}{k_2} \tag{3.162}$$

From (3.152)

$$R_h^{**} = \frac{\tau_h I_h^{**}}{k_3} \tag{3.163}$$

From (3.153)

$$S_l^{**} = \frac{\Lambda_l + \gamma_l R_l^{**}}{\lambda_{lm}^{**} + k_4} \tag{3.164}$$

From (3.154)

$$E_l^{**} = \frac{\lambda_{lm}^{**} S_l^{**}}{k_5} \tag{3.165}$$

From (3.155)

$$\mathrm{I}_l^{**} = \frac{\varepsilon_l E_l^{**}}{k_6} \tag{3.166}$$

From (3.156)

$$R_l^{**} = \frac{v_\varepsilon S_l^{**} + \tau_l E_l^{**} + \tau_l \mathrm{I}_l^{**}}{k_7} \tag{3.167}$$

From (3.157)

$$S_m^{**} = \frac{\Lambda_m}{\lambda_{mh}^{**} + \lambda_{ml}^{**} + k_8} \tag{3.168}$$

From (3.158)

$$\mathrm{I}_m^{**} = \frac{\left(\lambda_{mh}^{**} + \lambda_{ml}^{**}\right) S_m^{**}}{k_8} \tag{3.169}$$

Substituting (3.163) into (3.160) gives

$$S_h^{**} = \frac{\Lambda_h + \gamma_h \tau_h \mathrm{I}_h^{**}}{\left(\lambda_{hm}^{**} + \mu_h\right) k_3} \tag{3.170}$$

Substituting (3.170) into (3.161), gives

$$E_h^{**} = \frac{\lambda_{hm}^{**}\left(\Lambda_h + \gamma_h \tau_h \mathrm{I}_h^{**}\right)}{k_1 k_3 \left(\lambda_{hm}^{**} + \mu_h\right)} \tag{3.171}$$

Substituting (3.171) into (3.162), gives

$$I_h^{**} = \frac{\varepsilon_h \Lambda_h \lambda_{hm}^{**}}{k_1 k_2 k_3 \left(\lambda_{hm}^{**} + \mu_h\right) - \varepsilon_h \gamma_h \tau_h \lambda_{hm}^{**}} \tag{3.172}$$

Let,

$$I_h^{**} = \frac{\varepsilon_h \Lambda_h \lambda_{hm}^{**}}{k_1 k_2 k_3 \left(\lambda_{hm}^{**} + \mu_h\right) - \varepsilon_h \gamma_h \tau_h \lambda_{hm}^{**}} = A_1 \tag{3.173}$$

Substituting (3.173) into (3.170), gives

$$S_h^{**} = \frac{\Lambda_h + \gamma_h \tau_h A_1}{\left(\lambda_{hm}^{**} + \mu_h\right) k_3} \tag{3.174}$$

Substituting (3.173) into (3.171), gives

$$E_h^{**} = \frac{\lambda_{hm}^{**}\left(\Lambda_h + \gamma_h \tau_h A_1\right)}{k_1 k_3 \left(\lambda_{hm}^{**} + \mu_h\right)} \tag{3.175}$$

Substituting (3.173) into (3.163), gives

$$R_h^{**} = \frac{\tau_h A_1}{k_3} \tag{3.176}$$

From (3.166), we have;

$I_l^{**} = \frac{\varepsilon_l E_l^{**}}{k_6}$ making E_l^{**} the subject of formula,

$$E_l^{**} = \frac{k_6 I_l^{**}}{\varepsilon_l} \tag{3.177}$$

Substituting (3.164) and (3.177) into (3.167), gives

$$R_l^{**}=\frac{v_\varepsilon \Lambda_l+\left(\tau_l k_6+\tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**}+k_4\right)\mathrm{I}_l^{**}}{k_7\left(\lambda_{lm}^{**}+k_4\right)\left(\left(\lambda_{lm}^{**}+k_4\right)-v_\varepsilon \gamma_l\right)} \tag{3.178}$$

Substituting (3.178) into (3.164), gives

$$S_l^{**}=\frac{\Lambda_l+\gamma_l v_\varepsilon \Lambda_l+\left(\tau_l k_6+\tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**}+k_4\right)\gamma_l \mathrm{I}_l^{**}}{k_7\left(\lambda_{lm}^{**}+k_4\right)^2\left(\left(\lambda_{lm}^{**}+k_4\right)-v_\varepsilon \gamma_l\right)} \tag{3.179}$$

Substituting (3.179) into (3.165) gives,

$$E_l^{**}=\lambda_{lm}^{**}\left(\frac{\Lambda_l+\gamma_l v_\varepsilon \Lambda_l+\left(\tau_l k_6+\tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**}+k_4\right)\gamma_l \mathrm{I}_l^{**}}{k_5 k_7\left(\lambda_{lm}^{**}+k_4\right)^2\left(\left(\lambda_{lm}^{**}+k_4\right)-v_\varepsilon \gamma_l\right)}\right) \tag{3.180}$$

Substituting (3.180) into (3.166) gives,

$$\mathrm{I}_l^{**}=\frac{\varepsilon_l \Lambda_l \lambda_{lm}^{**}+\varepsilon_l v_\varepsilon \gamma_l \Lambda_l \lambda_{lm}^{**}}{\left(\left(k_5 k_6 k_7\left(\lambda_{lm}^{**}+k_4\right)^2\left(\left(\lambda_{lm}^{**}+k_4\right)-v_\varepsilon \gamma_l\right)\right)-\left(\left(\tau_l k_6+\tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**}+k_4\right)\varepsilon_l \gamma_l \lambda_{lm}^{**}\right)\right)\left(k_5 k_6 k_7\left(\lambda_{lm}^{**}+k_4\right)^2\left(\left(\lambda_{lm}^{**}+k_4\right)-v_\varepsilon \gamma_l\right)\right)} \tag{3.181}$$

Let,

$$\mathrm{I}_l^{**}=A_2 \tag{3.182}$$

Substituting (3.182) into (3.179), gives

$$S_l^{**}=\frac{\Lambda_l+\gamma_l v_\varepsilon \Lambda_l+\left(\tau_l k_6+\tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**}+k_4\right)\gamma_l A_2}{k_7\left(\lambda_{lm}^{**}+k_4\right)^2\left(\left(\lambda_{lm}^{**}+k_4\right)-v_\varepsilon \gamma_l\right)} \tag{3.183}$$

Substituting (3.182) into (3.180), gives

$$E_l^{**} = \lambda_{lm}^{**} \left(\frac{\Lambda_l + \gamma_l v_\varepsilon \Lambda_l + \left(\tau_l k_6 + \tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**} + k_4\right)\gamma_l A_2}{k_5 k_7 \left(\lambda_{lm}^{**} + k_4\right)^2 \left(\left(\lambda_{lm}^{**} + k_4\right) - v_\varepsilon \gamma_l\right)} \right) \tag{3.184}$$

Substituting (3.182) into (3.178), gives

$$R_l^{**} = \frac{v_\varepsilon \Lambda_l + \left(\tau_l k_6 + \tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**} + k_4\right) A_2}{k_7 \left(\lambda_{lm}^{**} + k_4\right)\left(\left(\lambda_{lm}^{**} + k_4\right) - v_\varepsilon \gamma_l\right)} \tag{3.185}$$

Substituting (3.169) into (3.168), we have;

$$\mathrm{I}_m^{**} = \frac{\Lambda_m \left(\lambda_{mh}^{**} + \lambda_{ml}^{**}\right)}{k_8 \left(\lambda_{mh}^{**} + \lambda_{ml}^{**} + k_8\right)} \tag{3.186}$$

$$\begin{pmatrix} S_h^{**} \\ E_h^{**} \\ \mathrm{I}_h^{**} \\ R_h^{**} \\ S_l^{**} \\ E_l^{**} \\ \mathrm{I}_l^{**} \\ R_l^{**} \\ S_m^{**} \\ \mathrm{I}_m^{**} \end{pmatrix} = \begin{pmatrix} \dfrac{\Lambda_h + \gamma_h \tau_h A_1}{\left(\lambda_{hm}^{**} + \mu_h\right) k_3} \\ \dfrac{\lambda_{hm}^{**}\left(\Lambda_h + \gamma_h \tau_h A_1\right)}{k_1 k_3 \left(\lambda_{hm}^{**} + \mu_h\right)} \\ \dfrac{\varepsilon_h \Lambda_h \lambda_{hm}^{**}}{k_1 k_2 k_3 \left(\lambda_{hm}^{**} + \mu_h\right) - \varepsilon_h \gamma_h \tau_h \lambda_{hm}^{**}} \\ \dfrac{\tau_h A_1}{k_3} \\ \dfrac{\Lambda_l + \gamma_l v_\varepsilon \Lambda_l + \left(\tau_l k_6 + \tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**} + k_4\right) \gamma_l A_2}{k_7 \left(\lambda_{lm}^{**} + k_4\right)^2 \left(\left(\lambda_{lm}^{**} + k_4\right) - v_\varepsilon \gamma_l\right)} \\ \lambda_{lm}^{**} \left(\dfrac{\Lambda_l + \gamma_l v_\varepsilon \Lambda_l + \left(\tau_l k_6 + \tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**} + k_4\right) \gamma_l A_2}{k_5 k_7 \left(\lambda_{lm}^{**} + k_4\right)^2 \left(\left(\lambda_{lm}^{**} + k_4\right) - v_\varepsilon \gamma_l\right)} \right) \\ A_2 \\ \dfrac{v_\varepsilon \Lambda_l + \left(\tau_l k_6 + \tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**} + k_4\right) A_2}{k_7 \left(\lambda_{lm}^{**} + k_4\right)\left(\left(\lambda_{lm}^{**} + k_4\right) - v_\varepsilon \gamma_l\right)} \\ \dfrac{\Lambda_m}{\lambda_{mh}^{**} + \lambda_{ml}^{**} + k_8} \\ \dfrac{\Lambda_m \left(\lambda_{mh}^{**} + \lambda_{ml}^{**}\right)}{k_8 \left(\lambda_{mh}^{**} + \lambda_{ml}^{**} + k_8\right)} \end{pmatrix} \tag{3.187}$$

The total population of humans at endemic equilibrium in terms of forces of infection is given as;

$$\left.\begin{aligned} N_h^{**} &= S_h^{**} + E_h^{**} + \mathrm{I}_h^{**} + R_h^{**} \\ N_h^{**} &= \frac{\Lambda_h + \gamma_h \tau_h A_1}{\left(\lambda_{hm}^{**} + \mu_h\right) k_3} + \frac{\lambda_{hm}^{**}\left(\Lambda_h + \gamma_h \tau_h A_1\right)}{k_1 k_3 \left(\lambda_{hm}^{**} + \mu_h\right)} + \frac{\varepsilon_h \Lambda_h \lambda_{hm}^{**}}{k_1 k_2 k_3 \left(\lambda_{hm}^{**} + \mu_h\right) - \varepsilon_h \gamma_h \tau_h \lambda_{hm}^{**}} + \frac{\tau_h A_1}{k_3} \end{aligned}\right\} \tag{3.188}$$

The total population of livestock at endemic equilibrium in terms of forces of infection is given as;

$$
\left.\begin{aligned}
&N_l^{**} = S_l^{**} + E_l^{**} + \mathrm{I}_l^{**} + R_l^{**} \\
&N_l = \frac{\Lambda_l + \gamma_l v_\varepsilon \Lambda_l + \left(\tau_l k_6 + \tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**} + k_4\right)\gamma_l A_2}{k_7\left(\lambda_{lm}^{**} + k_4\right)^2\left(\left(\lambda_{lm}^{**} + k_4\right) - v_\varepsilon \gamma_l\right)} + \lambda_{lm}^{**}\left(\frac{\Lambda_l + \gamma_l v_\varepsilon \Lambda_l + \left(\tau_l k_6 + \tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**} + k_4\right)\gamma_l A_2}{k_5 k_7\left(\lambda_{lm}^{**} + k_4\right)^2\left(\left(\lambda_{lm}^{**} + k_4\right) - v_\varepsilon \gamma_l\right)}\right) + \\
&\frac{\varepsilon_l \Lambda_l \lambda_{lm}^{**} + \varepsilon_l v_\varepsilon \gamma_l \Lambda_l \lambda_{lm}^{**}}{\left(\left(k_5 k_6 k_7\left(\lambda_{lm}^{**} + k_4\right)^2\left(\left(\lambda_{lm}^{**} + k_4\right) - v_\varepsilon \gamma_l\right)\right) - \left(\left(\tau_l k_6 + \tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**} + k_4\right)\varepsilon_l \gamma_l \lambda_{lm}^{**}\right)\right)\left(k_5 k_6 k_7\left(\lambda_{lm}^{**} + k_4\right)^2\left(\left(\lambda_{lm}^{**} + k_4\right) - v_\varepsilon \gamma_l\right)\right)} + \\
&\frac{v_\varepsilon \Lambda_l + \left(\tau_l k_6 + \tau_l \varepsilon_l\right)\left(\lambda_{lm}^{**} + k_4\right) A_2}{k_7\left(\lambda_{lm}^{**} + k_4\right)\left(\left(\lambda_{lm}^{**} + k_4\right) - v_\varepsilon \gamma_l\right)}
\end{aligned}\right\}
$$

(3.189)

The total population of mosquitoes at endemic equilibrium in terms of forces of infection is given as;

$$
\left.\begin{aligned}
&N_m^{**} = S_m^{**} + \mathrm{I}_m^{**} \\
&N_m^{**} = \frac{\Lambda_m}{\lambda_{mh}^{**} + \lambda_{ml}^{**} + k_8} + \frac{\Lambda_m\left(\lambda_{mh}^{**} + \lambda_{ml}^{**}\right)}{k_8\left(\lambda_{mh}^{**} + \lambda_{ml}^{**} + k_8\right)}
\end{aligned}\right\} \quad (3.190)
$$

Hence, from equation (3.187) the endemic equilibrium point in terms of force of infection is determined.

3.9 Numerical Solution of the Model Equations using Adomian Decomposition Method (ADM)

This method has been applied to solve differential and integral equations of linear and non-linear problems in mathematics, physics, biology and chemistry and up to now a large number of research papers have been published to show the feasibility of the decomposition method (Nhawu et al.,2015). It was first introduced by George Adomian in 1981 and developed by him in 1988 (Adomian, 1988).

The numerical solution of the model equations was solved and gave the following results.

3.9.1 Analytical solution of the model equation using adomian decomposition method

Considering equations (3.4) - (3.13) with the following initial conditions;

$$\left.\begin{array}{l} S_h(0)=S_{ho},\quad E_h(0)=E_{ho},\ I_h(0)=I_{ho},\quad R_h(0)=R_{ho},\quad S_l(0)=S_{lo}, \\ E_l(0)=E_{lo},\ I_l(0)=I_{lo},\quad R_l(0)=R_{lo},\quad S_m(0)=S_{mo},\quad I_m(0)=I_{mo} \end{array}\right\} \tag{3.191}$$

Integrating both sides of (3.4) through (3.13) with respect to $'t'$ and applying the initial conditions (3.191) gives;

$$S_h(t)=S_{h0}+\Lambda_h t-\int_0^t \frac{\beta_{hm}I_m S_h}{N_m}dt-\mu_h\int_0^t S_h dt-\gamma_h\int_0^t R_h dt \tag{3.192}$$

$$E_h(t)=E_{h0}+\int_0^t \frac{\beta_{hm}I_m S_h}{N_h}dt-\int_0^t(\varepsilon_h+\mu_h)E_h dt \tag{3.193}$$

$$\mathrm{I}_h(t)=\mathrm{I}_{h0}+\varepsilon_h\int_0^t E_h dt-\int_0^t(\mu_h+\tau_h)\mathrm{I}_h dt \tag{3.194}$$

$$R_h(t)=R_{h0}+\tau_h\int_0^t \mathrm{I}_h dt-\int_0^t(\mu_h+\gamma_h)R_h dt \tag{3.195}$$

$$S_l(t)=S_{l0}+\Lambda_l t-\int_0^t \frac{\beta_{lm}\mathrm{I}_m S_l}{N_l}dt-\int_0^t(\mu_l+v_\varepsilon)S_l dt+\gamma_l\int_0^t R_l dt \tag{3.196}$$

$$E_l(t)=\int_0^t \frac{\beta_{lm}I_m S_l}{N_l}dt-\int_0^t(\mu_l+\tau_l+\varepsilon_l)E_l dt \tag{3.197}$$

$$\mathrm{I}_l(t)=\varepsilon_h\int_0^t E_h dt-\int_0^t(\mu_l+c_l+\tau_l)\mathrm{I}_l dt \tag{3.198}$$

$$R_l(t) = \nu_\varepsilon \int_0^t S_l dt + \tau_l \int_0^t E_l dt + \tau_l \int_0^t \mathrm{I}_l dt - \int_0^t (\mu_l + \gamma_l) R_l dt \tag{3.199}$$

$$S_m(t) = S_{m0} + \Lambda_m t - \int_0^t \frac{\beta_{ml}\mathrm{I}_l S_m}{N_m} dt - \int_0^t \frac{\beta_{mh}\mathrm{I}_h S_m}{N_m} dt - \int_0^t (\mu_m + \delta_m) S_m dt \tag{3.200}$$

$$\mathrm{I}_m(t) = \mathrm{I}_{mo} + \int_0^t \frac{\beta_{ml}\mathrm{I}_l S_m}{N_m} dt - \int_0^t \frac{\beta_{mh}\mathrm{I}_h S_m}{N_m} dt - \int_0^t (\mu_m + \sigma_m) \mathrm{I}_m dt \tag{3.201}$$

Using Adomian Decomposition method, the solution of equation (3.192) through (3.201) are given in the series of the form;

$$\left.\begin{array}{l} S_h = \sum_{n=0}^{\infty} S_{hn}, \quad E_h = \sum_{n=0}^{\infty} E_{hn}, \quad \mathrm{I}_h = \sum_{n=0}^{\infty} \mathrm{I}_{hn}, \quad R_h = \sum_{n=0}^{\infty} R_{hn}, \; S_l = \sum_{n=0}^{\infty} S_{ln}, \\ E_l = \sum_{n=0}^{\infty} E_{ln}, \quad \mathrm{I}_l = \sum_{n=0}^{\infty} \mathrm{I}_{ln}, \quad R_l = \sum_{n=0}^{\infty} R_{ln}, \quad S_m = \sum_{n=0}^{\infty} S_{mn}, \quad \mathrm{I}_m = \sum_{n=0}^{\infty} \mathrm{I}_{mn} \end{array}\right\} \tag{3.202}$$

The integrands in equation (3.192) through (3.202) are expressed thus;

$$\left.\begin{array}{l} A = \mathrm{I}_m S_h, \quad B = S_h, \quad C = R_h, \quad D = E_h, \quad F = \mathrm{I}_h, \quad G = \mathrm{I}_m S_l, \quad H = S_l, \\ J = R_l, \; K = E_l, \quad L = \mathrm{I}_l, \quad M = R_l, \quad N = \mathrm{I}_l S_m, \; P = \mathrm{I}_h S_m, \; Q = S_m, \; T = \mathrm{I}_m \end{array}\right\} \tag{3.203}$$

The linear and non-linear operators in equation (3.203) are decomposed in series form as

$$\left.\begin{array}{l} A = \sum_{n=0}^{\infty} A_n, \quad B = \sum_{n=0}^{\infty} B_n, \quad C = \sum_{n=0}^{\infty} C_n, \quad D = \sum_{n=0}^{\infty} D_n, \; F = \sum_{n=0}^{\infty} F_n, \\ G = \sum_{n=0}^{\infty} G_n, \quad H = \sum_{n=0}^{\infty} H_n, \quad J = \sum_{n=0}^{\infty} J_n, \quad K = \sum_{n=0}^{\infty} K_n, \quad L = \sum_{n=0}^{\infty} L_n, \\ M = \sum_{n=0}^{\infty} M_n, \quad N = \sum_{n=0}^{\infty} N_n, \quad P = \sum_{n=0}^{\infty} P_n, \quad Q = \sum_{n=0}^{\infty} Q_n, \quad T = \sum_{n=0}^{\infty} T_n \end{array}\right\} \tag{3.204}$$

Substituting (3.202) into equation (3.192) to (3.201), gives;

$$\sum_{n=0}^{\infty} S_{hn} = S_{h0} + \Lambda_h t - \sum_{n=0}^{\infty}\int_0^t \frac{\beta_{hm} I_m S_h}{N_m} dt - \mu_h \sum_{n=0}^{\infty}\int_0^t S_h dt - \gamma_h \sum_{n=0}^{\infty}\int_0^t R_h dt \qquad (3.205)$$

$$\sum_{n=0}^{\infty} E_{hn} = E_{h0} + \sum_{n=0}^{\infty}\int_0^t \frac{\beta_{hm} I_m S_h}{N_h} dt - \sum_{n=0}^{\infty}\int_0^t (\varepsilon_h + \mu_h) E_h dt \qquad (3.206)$$

$$\sum_{n=0}^{\infty} \mathrm{I}_{hn} = \mathrm{I}_{h0} + \sum_{n=0}^{\infty} \varepsilon_h \int_0^t E_h dt - \sum_{n=0}^{\infty}\int_0^t (\mu_h + \tau_h) \mathrm{I}_h dt \qquad (3.207)$$

$$\sum_{n=0}^{\infty} R_{hn} = R_{h0} + \tau_h \sum_{n=0}^{\infty}\int_0^t \mathrm{I}_h dt - \sum_{n=0}^{\infty}\int_0^t (\mu_h + \gamma_h) R_h dt \qquad (3.208)$$

$$\sum_{n=0}^{\infty} S_{ln} = S_{l0} + \Lambda_l t - \sum_{n=0}^{\infty}\int_0^t \frac{\beta_{lm} \mathrm{I}_m S_l}{N_l} dt - \sum_{n=0}^{\infty}\int_0^t (\mu_l + v_\varepsilon) S_l dt + \gamma_l \sum_{n=0}^{\infty}\int_0^t R_l dt \qquad (3.209)$$

$$\sum_{n=0}^{\infty} E_{ln} = \sum_{n=0}^{\infty}\int_0^t \frac{\beta_{lm} I_m S_l}{N_l} dt - \sum_{n=0}^{\infty}\int_0^t (\mu_l + \tau_l + \varepsilon_l) E_l dt \qquad (3.210)$$

$$\sum_{n=0}^{\infty} I_{ln} = \varepsilon_h \sum_{n=0}^{\infty}\int_0^t E_h dt - \sum_{n=0}^{\infty}\int_0^t (\mu_l + c_l + \tau_l) \mathrm{I}_l dt \qquad (3.211)$$

$$\sum_{n=0}^{\infty} R_{ln} = v_\varepsilon \sum_{n=0}^{\infty}\int_0^t S_l dt + \tau_l \sum_{n=0}^{\infty}\int_0^t E_l dt + \tau_l \sum_{n=0}^{\infty}\int_0^t \mathrm{I}_l dt - \sum_{n=0}^{\infty}\int_0^t (\mu_l + \gamma_l) R_l dt \qquad (3.212)$$

$$\sum_{n=0}^{\infty} S_{mn} = S_{m0} + \Lambda_m t - \sum_{n=0}^{\infty}\int_0^t \frac{\beta_{ml} \mathrm{I}_l S_m}{N_m} dt - \sum_{n=0}^{\infty}\int_0^t \frac{\beta_{mh} \mathrm{I}_h S_m}{N_m} dt - \sum_{n=0}^{\infty}\int_0^t (\mu_m + \delta_m) S_m dt \qquad (3.213)$$

$$\sum_{n=0}^{\infty} \mathrm{I}_{mn} = \mathrm{I}_{mo} + \sum_{n=0}^{\infty}\int_0^t \frac{\beta_{ml} \mathrm{I}_l S_m}{N_m} dt - \sum_{n=0}^{\infty}\int_0^t \frac{\beta_{mh} \mathrm{I}_h S_m}{N_m} dt - \sum_{n=0}^{\infty}\int_0^t (\mu_m + \sigma_m) \mathrm{I}_m dt \qquad (3.214)$$

Equation (3.204) to (3.214) can be written as;

$$\sum_{n=0}^{\infty} S_{hn} = S_{h0} + \Lambda_h t - \frac{\beta_{hm}}{N_m}\int_0^t \sum_{n=0}^{\infty} A_n dt - \mu_h \int_0^t \sum_{n=0}^{\infty} B_n dt + \gamma_h \int_0^t \sum_{n=0}^{\infty} C_n dt \tag{3.215}$$

$$\sum_{n=0}^{\infty} E_{hn} = E_{h0} + \frac{\beta_{hm}}{N_m}\int_0^t \sum_{n=0}^{\infty} A_n dt - (\varepsilon_h + \mu_h)\int_0^t \sum_{n=0}^{\infty} D_n dt \tag{3.216}$$

$$\sum_{n=0}^{\infty} I_{hn} = I_{ho} + \varepsilon_h \int_0^t \sum_{n=0}^{\infty} D_n dt - (\mu_h + \tau_h)\int_0^t \sum_{n=0}^{\infty} F_n dt \tag{3.217}$$

$$\sum_{n=0}^{\infty} R_{hn} = R_{h0} + \tau_h \int_0^t \sum_{n=0}^{\infty} F_n dt - (\mu_h + \tau_h)\int_0^t \sum_{n=0}^{\infty} C_n dt \tag{3.218}$$

$$\sum_{n=0}^{\infty} S_{ln} = S_{l0} + \Lambda_l t - \frac{\beta_{lm}}{N_l}\int_0^t \sum_{n=0}^{\infty} G_n dt - (\mu_l + v_\varepsilon)\int_0^t \sum_{n=0}^{\infty} H_n dt + \gamma_l \int_0^t \sum_{n=0}^{\infty} J_n dt \tag{3.219}$$

$$\sum_{n=0}^{\infty} E_{ln} = E_{l0} + \frac{\beta_{lm}}{N_l}\int_0^t \sum_{n=0}^{\infty} G_n dt - (\mu_l + \tau_l + \varepsilon_l)\int_0^t \sum_{n=0}^{\infty} K_n dt \tag{3.220}$$

$$\sum_{n=0}^{\infty} \mathrm{I}_{ln} = \mathrm{I}_{l0} + \varepsilon_h \int_0^t \sum_{n=0}^{\infty} K_n dt - (\mu_l + c_l + \tau_l)\int_0^t \sum_{n=0}^{\infty} L_n dt \tag{3.221}$$

$$\sum_{n=0}^{\infty} R_{ln} = R_{l0} + v_\varepsilon \int_0^t \sum_{n=0}^{\infty} H_n dt + \tau_l \int_0^t \sum_{n=0}^{\infty} K_n dt + \tau_l \int_0^t \sum_{n=0}^{\infty} L_n dt - (\mu_l + \gamma_l)\int_0^t \sum_{n=0}^{\infty} M_n dt \tag{3.222}$$

$$\sum_{n=0}^{\infty} S_{mn} = S_{m0} + \Lambda_m t - \frac{\beta_{ml}}{N_m}\int_0^t \sum_{n=0}^{\infty} N_n dt - \frac{\beta_{mh}}{N_m}\int_0^t \sum_{n=0}^{\infty} P_n dt - (\mu_m + \delta_m)\int_0^t \sum_{n=0}^{\infty} Q_n dt \tag{3.223}$$

$$\sum_{n=0}^{\infty} \mathrm{I}_{mn} = \mathrm{I}_{m0} + \frac{\beta_{ml}}{N_m}\int_0^t \sum_{n=0}^{\infty} N_n dt + \frac{\beta_{mh}}{N_m}\int_0^t \sum_{n=0}^{\infty} P_n dt - (\mu_m + \sigma_m)\int_0^t \sum_{n=0}^{\infty} T_n dt \tag{3.224}$$

From equation (3.215) through (3.224), we define the following scheme;

$$\begin{aligned} &S_{H0} = S_{h0} + \Lambda_h t && E_{L0} = E_{l0} \\ &E_{H0} = E_{h0} && \mathrm{I}_{L0} = \mathrm{I}_{l0} \\ &\mathrm{I}_{H0} = \mathrm{I}_{h0} && R_{L0} = R_{l0} \\ &R_{H0} = R_{h0} && S_{M0} = S_{m0} + \Lambda_m t \\ &S_{H0} = S_{h0} + \Lambda_h t && \mathrm{I}_{M0} = \mathrm{I}_{m0} \end{aligned} \tag{3.225}$$

Hence;

$$S_{Hn+1} = -\frac{\beta_{hm}}{N_m}\int_0^t A_n dt - \mu_h \int_0^t B_n dt - \gamma_h \int_0^t C_n dt \tag{3.226}$$

$$E_{Hn+1} = \frac{\beta_{hm}}{N_m}\int_0^t A_n dt - (\varepsilon_h + \mu_h)\int_0^t D_n dt \tag{3.227}$$

$$\mathrm{I}_{Hn+1} = \varepsilon_h \int_0^t D_n dt - (\mu_h + \tau_h)\int_0^t F_n dt \tag{3.228}$$

$$R_{Hn+1} = \tau_h \int_0^t F_n dt - (\mu_h + \tau_h)\int_0^t C_n dt \tag{3.229}$$

$$S_{Ln+1} = -\frac{\beta_{lm}}{N_l}\int_0^t G_n dt - (\mu_l + v_\varepsilon)\int_0^t H_n dt + \gamma_l \int_0^t J_n dt \tag{3.230}$$

$$E_{Ln+1} = \frac{\beta_{lm}}{N_l}\int_0^t G_n dt - (\mu_l + \tau_l + \varepsilon_l)\int_0^t K_n dt \tag{3.231}$$

$$\mathrm{I}_{Ln+1} = \varepsilon_l \int_0^t K_n dt - (\mu_l + c_l + \tau_l)\int_0^t L_n dt \tag{3.232}$$

$$R_{Ln+1} = v_\varepsilon \int_0^t H_n dt + \tau_l \int_0^t K_n dt + \tau_l \int_0^t L_n dt - (\mu_l + \gamma_l)\int_0^t M_n dt \tag{3.233}$$

$$S_{Mn+1} = -\frac{\beta_{ml}}{N_m}\int_0^t N_n dt - \frac{\beta_{mh}}{N_m}\int_0^t P_n dt - \left(\mu_m + \delta_m\right)\int_0^t Q_n dt \tag{3.234}$$

$$\mathrm{I}_{Mn+1} = \frac{\beta_{ml}}{N_m}\int_0^t N_n dt + \frac{\beta_{mh}}{N_m}\int_0^t P_n dt - \left(\mu_m + \sigma_m\right)\int_0^t T_n dt \tag{3.235}$$

$$\left.\begin{aligned} A_0 &= I_{M0}S_{H0} \\ A_1 &= I_{M1}S_{H0} + I_{M0}S_{H1} \\ A_2 &= I_{M2}S_{H0} + I_{M1}S_{H1} + I_{M0}S_{H2} \end{aligned}\right\} \tag{3.236}$$

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$$\left.\begin{aligned} B_0 &= S_{H0} \\ B_1 &= S_{H1} \\ B_2 &= S_{H2} \end{aligned}\right\} \tag{3.237}$$

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$$\left.\begin{aligned} C_0 &= R_{H0} \\ C_1 &= R_{H1} \\ C_2 &= R_{H2} \end{aligned}\right\} \tag{3.238}$$

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$$\left.\begin{aligned} D_0 &= E_{H0} \\ D_1 &= E_{H1} \\ D_2 &= E_{H2} \end{aligned}\right\} \tag{3.239}$$

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$$\left.\begin{aligned} F_0 &= \mathrm{I}_{H0} \\ F_1 &= \mathrm{I}_{H1} \\ F_2 &= \mathrm{I}_{H2} \end{aligned}\right\} \qquad (3.240)$$

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$$\left.\begin{aligned} G_0 &= I_{M0}S_{L0} \\ G_1 &= I_{M1}S_{L0} + I_{M0}S_{L1} \\ G_2 &= I_{M2}S_{L0} + I_{M1}S_{L1} + I_{M0}S_{L2} \end{aligned}\right\} \qquad (3.241)$$

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$$\left.\begin{aligned} H_0 &= S_{L0} \\ H_1 &= S_{L1} \\ H_2 &= S_{L2} \end{aligned}\right\} \qquad (3.242)$$

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$$\left.\begin{aligned} J_0 &= R_{L0} \\ J_1 &= R_{L1} \\ J_2 &= R_{L2} \end{aligned}\right\} \qquad (3.243)$$

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$$\left.\begin{aligned} K_0 &= E_{L0} \\ K_1 &= E_{L1} \\ K_2 &= E_{L2} \end{aligned}\right\} \tag{3.244}$$

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$$\left.\begin{aligned} L_0 &= \mathrm{I}_{L0} \\ L_1 &= \mathrm{I}_{L1} \\ L_2 &= \mathrm{I}_{L2} \end{aligned}\right\} \tag{3.245}$$

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$$\left.\begin{aligned} M_0 &= R_{L0} \\ M_1 &= R_{L1} \\ M_2 &= R_{L2} \end{aligned}\right\} \tag{3.246}$$

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$$\left.\begin{aligned} N_0 &= I_{L0}S_{M0} \\ N_1 &= I_{L1}S_{M0} + I_{L0}S_{M1} \\ N_2 &= I_{L2}S_{M0} + I_{L1}S_{M1} + I_{L0}S_{M2} \end{aligned}\right\} \tag{3.247}$$

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$$\left.\begin{aligned} P_0 &= \mathrm{I}_{H0} S_{M0} \\ P_1 &= \mathrm{I}_{H1} S_{M0} + \mathrm{I}_{H0} S_{M1} \\ P_2 &= \mathrm{I}_{H2} S_{M0} + \mathrm{I}_{H1} S_{M1} + \mathrm{I}_{H0} S_{M2} \end{aligned}\right\} \tag{3.248}$$

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$$\left.\begin{aligned} Q_0 &= S_{M0} \\ Q_1 &= S_{M1} \\ Q_2 &= S_{M2} \end{aligned}\right\} \tag{3.249}$$

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$$\left.\begin{aligned} T_0 &= \mathrm{I}_{M0} \\ T_1 &= \mathrm{I}_{M1} \\ T_2 &= \mathrm{I}_{M2} \end{aligned}\right\} \tag{3.250}$$

For $n=0$, equation (3.226) gives;

$$S_{H1} = -\frac{\beta_{hm}}{N_m}\int_0^t A_0 dt - \mu_h \int_0^t B_0 dt + \gamma_n \int_0^t C_0 dt \tag{3.251}$$

Substituting (3.236) through (3.238) into (3.251) gives

$$S_{H1} = -\frac{\beta_{hm}}{N_m}\int_0^t \mathrm{I}_{M0} S_{H0} dt - \mu_h \int_0^t S_{H0} dt + \gamma_n \int_0^t R_{H0} dt \tag{3.252}$$

Substituting equation (3.225) into (3.252) gives

$$S_{H1}=-\frac{\beta_{hm}}{N_m}\int_0^t I_{M0}\left[S_{h0}+\Lambda_h t\right]dt-\mu_h\int_0^t\left[S_{h0}+\Lambda_h t\right]dt+\gamma_n\int_0^t R_{h0}dt \qquad (3.253)$$

$$S_{H1}=-\frac{\beta_{hm}}{N_m}\int_0^t\left[\mathrm{I}_{m0}S_{h0}+\mathrm{I}_{m0}\Lambda_h t\right]dt-\mu_h\int_0^t\left[S_{h0}+\Lambda_h t\right]dt+\gamma_n\int_0^t R_{h0}dt \qquad (3.254)$$

Integrating and collecting like terms, we obtained;

$$S_{H1}=-\frac{\beta_{hm}}{N_m}\left[I_{m0}S_{h0}t+\frac{I_{mo}\Lambda_h t^2}{2}\right]-\mu_h\left[S_{ho}t+\frac{\Lambda_h t^2}{2}\right]+\gamma_h\left[R_{ho}t\right]$$

$$S_{H1}=-\frac{\beta_{hm}I_{m0}S_{h0}t}{N_m}-\frac{\beta_{hm}I_{m0}\Lambda_h t^2}{2N_m}-\mu_h S_{h0}t-\frac{\mu_h\Lambda_h t^2}{2}+\gamma_n R_{ho}t$$

$$S_{H1}=-\frac{\beta_{hm}\mathrm{I}_{m0}S_{h0}t}{N_m}-\mu_h S_{h0}t+\gamma_h R_{ho}t-\frac{\beta_{hm}\mathrm{I}_{m0}\Lambda_h t^2}{2N_m}-\frac{\mu_h\Lambda_h t^2}{2}$$

$$S_{H1}=\left[-\frac{\beta_{hm}I_{m0}S_{h0}}{N_m}-\mu_h S_{h0}+\gamma_h R_{ho}\right]t-\left[\frac{\beta_{hm}I_{m0}\Lambda_h}{N_m}-\mu_h\Lambda_h\right]\frac{t^2}{2} \qquad (3.255)$$

For $n=0$, equation (3.227) gives;

$$E_{H1}=\frac{\beta_{hm}}{N_m}\int_0^t A_0 dt-\left(\varepsilon_h+\mu_h\right)\int_0^t D_0 dt \qquad (3.256)$$

Substituting (3.236) through (3.239) into (3.256) gives

$$E_{H1}=\frac{\beta_{hm}}{N_m}\int_0^t \mathrm{I}_{M0}S_{H0}dt-\left(\varepsilon_h+\mu_h\right)\int_0^t E_{H0}dt \qquad (3.257)$$

Substituting equation (3.225) into (3.257) gives;

$$E_{H1} = \frac{\beta_{hm}}{N_m}\int_0^t \mathrm{I}_{m0}\left[S_{ho} + \Lambda_h t\right]dt - \left(\varepsilon_h + \mu_h\right)\int_0^t E_{h0}dt \tag{3.258}$$

Integrating and collecting like terms, we obtained;

$$E_{H1} = \frac{\beta_{hm}}{N_m}\left[\mathrm{I}_{m0}S_{h0}t + \frac{I_{m0}\Lambda_h t^2}{2}\right] - \left(\varepsilon_h + \mu_h\right)\left[E_{h0}t\right]$$

$$E_{H1} = \left[\frac{\beta_{hm}\mathrm{I}_{m0}S_{h0}}{N_m} - \left(\varepsilon_h + \mu_h\right)E_{h0}\right]t + \left[\frac{\beta_{hm}I_{m0}\Lambda_h}{N_m}\right]\left[\frac{t^2}{2}\right] \tag{3.259}$$

For $n = 0$, equation (3.228) gives;

$$\mathrm{I}_{H1} = \varepsilon_h\int_0^t D_0 dt - \left(\mu_h + \tau_h\right)\int_0^t F_0 dt \tag{3.260}$$

Substituting (3.239) through (3.240) into (3.260) gives

$$I_{H1} = \varepsilon_h\int_0^t E_{H0}dt - \left(\mu_h + \tau_h\right)\int_0^t \mathrm{I}_{H0}dt \tag{3.261}$$

Substituting equation (3.225) into (3.261) gives;

$$\mathrm{I}_{H1} = \varepsilon_h\int_0^t E_{h0}dt - \left(\mu_h + \tau_h\right)\int_0^t \mathrm{I}_{H0}dt \tag{3.262}$$

Integrating and collecting like terms, we obtained;

$$\mathrm{I}_{H1} = \varepsilon_h\left[E_{h0}t\right] - \left(\mu_h + \tau_h\right)\left[\mathrm{I}_{h0}t\right]$$

$$I_{H1} = \left[\varepsilon_h E_{h0} - \left(\mu_h + \tau_h\right)\left[\mathrm{I}_{h0}\right]\right]t \tag{3.263}$$

For $n=0,$ equation (3.229) gives;

$$R_{H1}=\tau_h\int_0^t F_0dt-\left(\mu_h+\tau_h\right)\int_0^t C_0dt \tag{3.264}$$

Substituting (3.238) through (3.240) into (3.264) gives;

$$R_{H1}=\tau_h\int_0^t \mathrm{I}_{H0}dt-\left(\mu_h+\tau_h\right)\int_0^t R_{H0}dt \tag{3.265}$$

Substituting equation (3.225) into (3.265) gives;

$$R_{H1}=\tau_h\int_0^t \mathrm{I}_{h0}dt-\left(\mu_h+\tau_h\right)\int_0^t R_{h0}dt \tag{3.266}$$

Integrating and collecting like terms, we obtained;

$$R_{H1}=\tau_h\mathrm{I}_{h0}t-\left(\mu_h+\tau_h\right)R_{h0}t$$

$$R_{H1}=\left[\tau_h\mathrm{I}_{h0}-\left(\mu_h+\tau_h\right)R_{h0}\right]t \tag{3.267}$$

For $n=0,$ equation (3.230) gives;

$$S_{L1}=-\frac{\beta_{lm}}{N_l}\int_0^t G_0dt-\left(\mu_l+v_\varepsilon\right)\int_0^t H_0dt+\gamma_l\int_0^t J_0dt \tag{3.268}$$

Substituting (3.241) through (3.243) into (3.268) gives;

$$S_{L1}=-\frac{\beta_{lm}}{N_l}\int_0^t I_{M0}S_{L0}dt-\left(\mu_l+v_\varepsilon\right)\int_0^t S_{L0}dt+\gamma_L\int_0^t R_{L0} \tag{3.269}$$

Substituting equation (3.225) into (3.265) gives;

$$S_{L1} = -\frac{\beta_{lm}}{N_l}\int_0^t \mathrm{I}_{m0}\left(S_{l0} + \Lambda_l t\right)dt - \left(\mu_l + v_\varepsilon\right)\int_0^t \left(S_{l0} + \Lambda_l t\right)dt + \gamma_l \int_0^t R_{lo}dt \qquad (3.270)$$

Integrating and collecting like terms, we obtained;

$$S_{L1} = -\frac{\beta_{lm}}{N_l}\left[I_{m0}S_{l0}t + \frac{I_{m0}\Lambda_l t^2}{2}\right] - \mu_L\left[S_{l0}t + \frac{\Lambda_l t^2}{2}\right] - v_\varepsilon\left[S_{l0}t + \frac{\Lambda_l t^2}{2}\right] + \gamma_l\left[R_{l0}t\right]$$

$$S_{L1} = \left[-\frac{\beta_{lm}\mathrm{I}_{m0}\mathrm{S}_{l0}}{N_l} - \left(\mu_L + v_\varepsilon\right)S_{l0} + \gamma_l R_{l0}\right]t - \left[\frac{\beta_{lm}\mathrm{I}_{m0}\Lambda_l}{N_l} + \left(\mu_L + v_\varepsilon\right)\Lambda_L\right]\frac{t^2}{2} \qquad (3.271)$$

For $n = 0$, equation (3.230) becomes;

$$E_{L1} = \frac{\beta_{lm}}{N_l}\int_0^t G_0 dt - \left(\mu_l + \tau_l + \varepsilon_l\right)\int_0^t K_0 dt \qquad (3.272)$$

Substituting (3.241) through (3.244) into (3.272) gives;

$$E_{L1} = \frac{\beta_{lm}}{N_l}\int_0^t I_{M0}S_{L0}dt - \left(\mu_l + \tau_l + \varepsilon_l\right)\int_0^t E_{L0}dt \qquad (3.273)$$

Substituting equation (3.225) into (3.273) gives;

$$E_{L1} = \frac{\beta_{lm}}{N_l}\int_0^t \mathrm{I}_{m0}\left(S_{l0} + \Lambda_l t\right)dt - \left(\mu_l + \tau_l + \varepsilon_l\right)\int_0^t E_{l0}dt \qquad (3.274)$$

Integrating and collecting like terms, we obtained;

$$E_{L1} = \frac{\beta_{lm}}{N_l}\left[\mathrm{I}_{m0}S_{l0}t + \frac{\mathrm{I}_{m0}\Lambda_l t^2}{2}\right] - \left(\mu_l + \tau_l + \varepsilon_l\right)E_{l0}t$$

$$E_{L1} = \left[\frac{\beta_{lm} \mathrm{I}_{m0} S_{l0}}{N_l} - \left(\mu_l + \tau_l + \varepsilon_l \right) E_{l0} \right] t + \left[\frac{\beta_{lm} \mathrm{I}_{m0} \Lambda_l}{N_l} \right] \frac{t^2}{2} \qquad (3.275)$$

For $n = 0$, equation (3.232) becomes;

$$\mathrm{I}_{L1} = \varepsilon_l \int_0^t K_0 dt - \left(\mu_l + c_l + \tau_l \right) \int_0^t L_0 dt \qquad (3.276)$$

Substituting (3.244) through (3.245) into (3.276) gives;

$$\mathrm{I}_{L1} = \varepsilon_L \int_0^t E_{L0} dt - \left(\mu_l + c_l + \tau_l \right) \int_0^t \mathrm{I}_{L0} dt \qquad (3.277)$$

Substituting equation (3.225) into (3.277) gives;

$$\mathrm{I}_{L1} = \varepsilon_l \int_0^t E_{L0} dt - \left(\mu_l + c_l + \tau_l \right) \int_0^t \mathrm{I}_{L0} dt \qquad (3.278)$$

Integrating and collecting like terms, we obtained;

$$I_{L1} = \varepsilon_l \left(E_{L0} t \right) - \left(\mu_l + c_l + \tau_l \right) \mathrm{I}_{L0} t$$

$$\mathrm{I}_{L1} = \left[\varepsilon_l E_{L0} - \left(\mu_l + c_l + \tau_l \right) \mathrm{I}_{L0} \right] t \qquad (3.279)$$

For $n = 0$, equation (3.234) becomes;

$$R_{L1} = v_\varepsilon \int_0^t H_0 dt + \tau_l \int_0^t K_0 dt + \tau_l \int_0^t L_0 dt - \left(\mu_l + \gamma_l \right) \int_0^t M_0 dt$$

(3.280)

Substituting (3.242) through (3.244) to (3.246) into (3.280) gives;

$$R_{L1} = v_\varepsilon \int_0^t S_{L0} dt + \tau_l \int_0^t E_{L0} dt + \tau_l \int_0^t \mathrm{I}_{L0} dt - (\mu_l + \gamma_l) \int_0^t R_{L0} dt \tag{3.281}$$

Substituting equation (3.225) into (3.281) gives;

$$R_{L1} = v_\varepsilon \int_0^t (S_{l0} + \Lambda_l t) dt + \tau_l \int_0^t E_{l0} dt + \tau_l \int_0^t \mathrm{I}_{l0} dt - (\mu_l + \gamma_l) \int_0^t R_{l0} dt \tag{3.282}$$

Integrating and collecting like terms, we obtained;

$$R_{L1} = v_\varepsilon \left[S_{l0} t + \frac{\Lambda_l t^2}{2} \right] + \tau_l [E_{l0} t] + \tau_l [\mathrm{I}_{l0} t] - (\mu_l + \gamma_l)[R_{l0} t]$$

$$R_{L1} = \left[v_\varepsilon S_{l0} + \tau_l E_{l0} + \tau_l \mathrm{I}_{l0} - (\mu_l + \gamma_l) R_{l0} \right] + (v_\varepsilon \Lambda_l) \frac{t^2}{2} \tag{3.283}$$

For $n = 0$, equation (3.235) becomes;

$$S_{M1} = -\frac{\beta_{ml}}{N_m} \int_0^t N_0 dt - \frac{\beta_{mh}}{N_m} \int_0^t P_0 dt - (\mu_m + \delta_m) \int_0^t Q_0 dt \tag{3.284}$$

Substituting (3.247) through (3.249) into (3.284) gives;

$$S_{M1} = -\frac{\beta_{ml}}{N_m} \int_0^t \mathrm{I}_{L0} S_{M0} dt - \frac{\beta_{mh}}{N_m} \int_0^t \mathrm{I}_{Ho} S_{Mo} dt - (\mu_m + \delta_m) \int_0^t S_{M0} dt \tag{3.285}$$

Substituting equation (3.225) into (3.285) gives;

$$S_{M1} = -\frac{\beta_{ml}}{N_m} \int_0^t \mathrm{I}_{l0} (S_{m0} + \Lambda_m t) dt - \frac{\beta_{mh}}{N_m} \int_0^t \mathrm{I}_{h0} (S_{m0} + \Lambda_m t) dt - (\mu_m + \delta_m) \int_0^t (S_{m0} + \Lambda_m t) dt \tag{3.286}$$

Integrating and collecting like terms, we obtained;

$$S_{M1}=-\frac{\beta_{ml}}{N_m}\left(I_{l0}S_{m0}t+\frac{I_{l0}\Lambda_m t^2}{2}\right)-\frac{\beta_{mh}}{N_m}\left(I_{h0}S_{m0}t+\frac{I_{h0}\Lambda_m t^2}{2}\right)-(\mu_m+\delta_m)\left(S_{m0}t+\frac{\Lambda_m t^2}{2}\right)$$

(3.287)

$$S_{M1}=\left(-\frac{\beta_{ml}I_{l0}S_{m0}}{N_m}-\frac{\beta_{mh}I_{h0}S_{m0}}{N_m}-(\mu_m+\delta_m)S_{m0}\right)t+\left(-\frac{\beta_{ml}I_{l0}\Lambda_m}{N_m}-\frac{\beta_{mh}I_{h0}\Lambda_m}{N_m}-(\mu_m+\delta_m)\Lambda_m\right)\frac{t^2}{2} \quad (3.288)$$

For $n=0,$ equation (3.235) becomes;

$$I_{M1}=\frac{\beta_{ml}}{N_m}\int_0^t N_0 dt+\frac{\beta_{mh}}{N_m}\int_0^t P_0 dt-(\mu_m+\delta_m)\int_0^t T_0 dt \quad (3.289)$$

Substituting (3.247), (3.248) and (3.250) into (3.289) gives;

$$I_{M1}=\frac{\beta_{ml}}{N_m}\int_0^t I_{L0}S_{M0}dt+\frac{\beta_{mh}}{N_m}\int_0^t I_{H0}S_{M0}dt-(\mu_m+\sigma_m)\int_0^t I_{M0}dt \quad (3.290)$$

Substituting equation (3.228) into (3.292) gives;

$$I_{M1}=\frac{\beta_{ml}}{N_m}\int_0^t I_{l0}(S_{m0}+\Lambda_m t)dt+\frac{\beta_{mh}}{N_m}\int_0^t I_{h0}(S_{m0}+\Lambda_m t)dt-(\mu_m+\sigma_m)\int_0^t I_{M0}dt \quad (3.291)$$

Integrating and collecting like terms, we obtained;

$$I_{M1}=\frac{\beta_{ml}}{N_m}\left(I_{l0}S_{m0}t+\frac{I_{l0}\Lambda_m t^2}{2}\right)+\frac{\beta_{mh}}{N_m}\left(I_{ho}S_{mo}t+\frac{I_{h0}\Lambda_m t^2}{2}\right)-(\mu_m+\sigma_m)(I_{m0}t)$$

$$I_{M1}=\left(\frac{\beta_{ml}I_{l0}S_{m0}}{N_m}+\frac{\beta_{mh}I_{h0}S_{m0}}{N_m}-(\mu_m+\sigma_m)I_{m0}\right)t+\left(\frac{\beta_{ml}I_{l0}\Lambda_m}{N_m}+\frac{\beta_{mh}I_{h0}\Lambda_m}{N_m}\right)\frac{t^2}{2} \quad (3.292)$$

For $n=1,$ equation (3.226) gives;

$$S_{H2} = -\frac{\beta_{mh}}{N_h}\int_0^t A_1 dt - \mu_h \int_0^t B_1 dt + \gamma_h \int_0^t C_1 dt \tag{3.293}$$

Substituting (3.236) to (3.237) into (3.293) gives;

$$S_{H2} = -\frac{\beta_{mh}}{N_m}\int_0^t \left(I_{M1}S_{Ho} + I_{M0}S_{H1}\right) dt - \mu_h \int_0^t S_{H1} dt + \gamma_h \int_0^t R_{H1} dt \tag{3.294}$$

Integrating and collecting like terms, S_{H2} gives;

$$-\frac{1}{N_h}\left(\beta_{hm}\left(\frac{1}{4}\left(\frac{1}{2}\frac{\beta_{ml} I_{l0}\Lambda_m}{N_m} + \frac{1}{2}\frac{\beta_{mh} I_{h0}\Lambda_m}{N_m}\right)\Lambda_h t^4 + \frac{1}{3}\left(I_{m0}\left(-\frac{1}{2}\frac{\beta_{hm} I_{m0}\Lambda_h}{N_h} - \frac{1}{2}\mu_h \Lambda_h\right) + \left(\frac{\beta_{ml} I_{l0} S_{m0}}{N_m} + \frac{\beta_{mh} I_{h0} S_{m0}}{N_m} - \left(\mu_m + \sigma_m\right) I_{m0}\right)\Lambda_h + \left(\frac{1}{2}\frac{\beta_{ml} I_{l0}\Lambda_m}{N_m} + \frac{1}{2}\frac{\beta_{mh} I_{h0}\Lambda_m}{N_m}\right) S_{h0}\right) t^3 + \frac{1}{2}\left(I_{m0}\left(-\frac{\beta_{hm} I_{m0} S_{h0}}{N_h} - \mu_h S_{h0} + \gamma_h R_{h0}\right) + \left(\frac{\beta_{ml} I_{l0} S_{m0}}{N_m} + \frac{\beta_{mh} I_{h0} S_{m0}}{N_m} - \left(\mu_m + \sigma_m\right) I_{m0}\right) S_{h0}\right) t^2\right)\right) - \mu_h\left(\frac{1}{3}\left(-\frac{1}{2}\frac{\beta_{hm} I_{m0}\Lambda_h}{N_h} - \frac{1}{2}\mu_h \Lambda_h\right) t^3 + \frac{1}{2}\left(-\frac{\beta_{hm} I_{m0} S_{h0}}{N_h} - \mu_h S_{h0} + \gamma_h R_{h0}\right) t^2\right) + \frac{1}{2}\gamma_h\left(\tau_h I_{h0} - \left(\mu_h + \gamma_h\right) R_{h0}\right) t^2 \tag{3.295}$$

For $n = 1$, equation (3.227) becomes;

$$E_{H2} = \frac{\beta_{hm}}{N_h}\int_0^t A_1 dt - \left(\varepsilon_h + \mu_h\right)\int_0^t D_1 dt \tag{3.296}$$

Substituting (3.239) and (3.242) into (3.298) gives;

$$E_{H2} = \frac{\beta_{hm}}{N_m}\int_0^t \left(\mathrm{I}_{M1}S_{H0} + \mathrm{I}_{M0}S_{H1}\right)dt - \left(\varepsilon_h + \mu_h\right)\int_0^t E_{H1}dt \tag{3.297}$$

Integrating and collecting like terms, E_{H2} gives;

$$\frac{1}{N_h}\left(\beta_{hm}\left(\frac{1}{4}\left(\frac{1}{2}\,\frac{\beta_{ml}\,\mathrm{I}_{l0}\,\Lambda_m}{N_m} + \frac{1}{2}\,\frac{\beta_{mh}\,\mathrm{I}_{h0}\,\Lambda_m}{N_m}\right)\Lambda_h\,t^4 + \frac{1}{3}\left(\mathrm{I}_{m0}\left(-\frac{1}{2}\,\frac{\beta_{hm}\,\mathrm{I}_{m0}\,\Lambda_h}{N_h} - \frac{1}{2}\,\mu_h\,\Lambda_h\right) + \left(\frac{\beta_{ml}\,\mathrm{I}_{l0}\,S_{m0}}{N_m} + \frac{\beta_{mh}\,\mathrm{I}_{h0}\,S_{m0}}{N_m} - \left(\mu_m + \sigma_m\right)\mathrm{I}_{m0}\right)\Lambda_h + \left(\frac{1}{2}\,\frac{\beta_{ml}\,\mathrm{I}_{l0}\,\Lambda_m}{N_m} + \frac{1}{2}\,\frac{\beta_{mh}\,\mathrm{I}_{h0}\,\Lambda_m}{N_m}\right)S_{h0}\right)t^3 + \frac{1}{2}\left(\mathrm{I}_{m0}\left(-\frac{\beta_{hm}\,\mathrm{I}_{m0}\,S_{h0}}{N_h} - \mu_h\,S_{h0} + \gamma_h\,R_{h0}\right) + \left(\frac{\beta_{ml}\,\mathrm{I}_{l0}\,S_{m0}}{N_m} + \frac{\beta_{mh}\,\mathrm{I}_{h0}\,S_{m0}}{N_m} - \left(\mu_m + \sigma_m\right)\mathrm{I}_{m0}\right)S_{h0}\right)t^2\right)\right) - \left(\varepsilon_h + \mu_h\right)\left(\frac{1}{6}\,\frac{\beta_{hm}\,\mathrm{I}_{m0}\,\Lambda_h\,t^3}{N_h} + \frac{1}{2}\left(\frac{\beta_{hm}\,\mathrm{I}_{m0}\,S_{h0}}{N_h} - \left(\varepsilon_h + \mu_h\right)E_{h0}\right)t^2\right) \tag{3.298}$$

For $n = 1$, equation (3.228) becomes;

$$\mathrm{I}_{H2} = \varepsilon_h\int_0^t D_1 dt - \left(\mu_h + \tau_h\right)\int_0^t F_1 dt \tag{3.399}$$

Substituting (3.239) and (3.240) into (3.299) gives;

$$\mathrm{I}_{H2} = \varepsilon_h\int_0^t E_{H1} dt - \left(\mu_h + \tau_h\right)\int_0^t \mathrm{I}_{H1} dt \tag{3.300}$$

Integrating and collecting like terms, I_{H2} gives;

$$\varepsilon_h\left(\frac{1}{6}\frac{\beta_{hm}\mathrm{I}_{m0}\Lambda_h t^3}{N_h}+\frac{1}{2}\left(\frac{\beta_{hm}\mathrm{I}_{m0}S_{h0}}{N_h}-\left(\varepsilon_h+\mu_h\right)E_{h0}\right)t^2\right)$$
$$-\frac{1}{2}\left(\tau_h+\mu_h\right)\left(\varepsilon_h E_{h0}-\left(\tau_h+\mu_h\right)\mathrm{I}_{h0}\right)t^2 \quad (3.301)$$

When $n=1$, equation (3.229) becomes;

$$R_{H2}=\tau_h\int_0^t F_1 dt-\left(\mu_h+\tau_h\right)\int_0^t C_1 dt \quad (3.302)$$

Substituting (3.238) and (3.240) into (3.302) gives;

$$R_{H2}=\tau_h\int_0^t \mathrm{I}_{H1} dt-\left(\mu_h+\tau_h\right)\int_0^t R_{H1} dt \quad (3.303)$$

Integrating and collecting like terms, R_{H2} gives;

$$\frac{1}{2}\left(\varepsilon_h E_{h0}-\left(\tau_h+\mu_h\right)\mathrm{I}_{h0}\right)t^2-\frac{1}{2}\left(\gamma_h+\mu_h\right)\left(\tau_h\mathrm{I}_{h0}-\left(\gamma_h+\mu_h\right)R_{h0}\right)t^2 \quad (3.304)$$

For $n=1$, equation (3.230) becomes;

$$S_{L2}=-\frac{\beta_{lm}}{N_l}\int_0^t G_1 dt-\left(\mu_l+v_\varepsilon\right)\int_0^t H_1 dt+\gamma_l\int_0^t J_1 dt \quad (3.305)$$

Substituting (3.241) and (3.242) into (3.305) gives;

$$S_{L2}=-\frac{\beta_{lm}}{N_l}\int_0^t\left(\mathrm{I}_{M1}S_{L0}+\mathrm{I}_{M0}S_{L1}\right)dt-\left(\mu_l+v_\varepsilon\right)\int_0^t S_{L1}dt+\gamma_l\int_0^t R_{L1}dt \quad (3.306)$$

Integrating and collecting like terms, S_{L2} gives;

$$
\begin{aligned}
&-\frac{1}{N_l}\Bigg(\beta_{lm}\Bigg(-\frac{1}{20}\,\frac{I_{m0}\,\beta_{lm}\left(\frac{1}{2}\,\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}+\frac{1}{2}\,\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\right)\Lambda_l\,t^5}{N_l}+\frac{1}{4}\Bigg(I_{m0}\Bigg(\\
&-\frac{1}{N_l}\Bigg(\beta_{lm}\Bigg(\frac{1}{3}\,I_{m0}\Bigg(-\frac{1}{2}\,\frac{\beta_{lm}\,I_{m0}\,\Lambda_l}{N_l}-\frac{1}{2}\,(\mu_l+v_\varepsilon)\,\Lambda_l\Bigg)+\frac{1}{3}\Bigg(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}\\
&+\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-(\mu_m+\sigma_m)\,I_{m0}\Bigg)\Lambda_l+\frac{1}{3}\Bigg(\frac{1}{2}\,\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}+\frac{1}{2}\,\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\Bigg)S_{l0}\Bigg)\Bigg)\\
&-(\mu_l+v_\varepsilon)\Bigg(-\frac{1}{6}\,\frac{\beta_{lm}\,I_{m0}\,\Lambda_l}{N_l}-\frac{1}{6}\,(\mu_l+v_\varepsilon)\,\Lambda_l\Bigg)+\frac{1}{6}\,\gamma_l\,v_\varepsilon\,\Lambda_l\Bigg)+\Bigg(\frac{1}{2}\,\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}\\
&+\frac{1}{2}\,\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\Bigg)\Lambda_l\Bigg)t^4+\frac{1}{3}\Bigg(I_{m0}\Bigg(-\frac{1}{N_l}\Bigg(\beta_{lm}\Bigg(\frac{1}{2}\,I_{m0}\Bigg(-\frac{\beta_{lm}\,I_{m0}\,S_{l0}}{N_l}-(\mu_l\\
&+v_\varepsilon)\,S_{l0}+\gamma_l\,R_{l0}\Bigg)+\frac{1}{2}\Bigg(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}+\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-(\mu_m+\sigma_m)\,I_{m0}\Bigg)S_{l0}\Bigg)\Bigg)-(\mu_l\\
&+v_\varepsilon)\Bigg(-\frac{1}{2}\,\frac{\beta_{lm}\,I_{m0}\,S_{l0}}{N_l}-\frac{1}{2}\,(\mu_l+v_\varepsilon)\,S_{l0}+\frac{1}{2}\,\gamma_l\,R_{l0}\Bigg)+\gamma_l\Bigg(\frac{1}{2}\,v_\varepsilon\,S_{l0}+\frac{1}{2}\,\tau_l\,E_{l0}\\
&+\frac{1}{2}\,\tau_l\,I_{l0}-\frac{1}{2}\,(\mu_l+\gamma_l)\,R_{l0}\Bigg)\Bigg)+\Bigg(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}+\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-(\mu_m+\sigma_m)\,I_{m0}\Bigg)\Lambda_l\\
&+\Bigg(\frac{1}{2}\,\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}+\frac{1}{2}\,\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\Bigg)S_{l0}\Bigg)t^3+\frac{1}{2}\Bigg(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}+\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-(\mu_m\\
&+\sigma_m)\,I_{m0}\Bigg)S_{l0}\,t^2\Bigg)\Bigg)-(\mu_l+v_\varepsilon)\Bigg(-\frac{1}{20}\,\frac{\beta_{lm}\left(\frac{1}{2}\,\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}+\frac{1}{2}\,\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\right)\Lambda_l\,t^5}{N_l}\\
&+\frac{1}{4}\Bigg(-\frac{1}{N_l}\Bigg(\beta_{lm}\Bigg(\frac{1}{3}\,I_{m0}\Bigg(-\frac{1}{2}\,\frac{\beta_{lm}\,I_{m0}\,\Lambda_l}{N_l}-\frac{1}{2}\,(\mu_l+v_\varepsilon)\,\Lambda_l\Bigg)+\frac{1}{3}\Bigg(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}\\
&+\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-(\mu_m+\sigma_m)\,I_{m0}\Bigg)\Lambda_l+\frac{1}{3}\Bigg(\frac{1}{2}\,\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}+\frac{1}{2}\,\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\Bigg)S_{l0}\Bigg)\Bigg)\\
&-(\mu_l+v_\varepsilon)\Bigg(-\frac{1}{6}\,\frac{\beta_{lm}\,I_{m0}\,\Lambda_l}{N_l}-\frac{1}{6}\,(\mu_l+v_\varepsilon)\,\Lambda_l\Bigg)+\frac{1}{6}\,\gamma_l\,v_\varepsilon\,\Lambda_l\Bigg)t^4+\frac{1}{3}\Bigg(\\
&-\frac{1}{N_l}\Bigg(\beta_{lm}\Bigg(\frac{1}{2}\,I_{m0}\Bigg(-\frac{\beta_{lm}\,I_{m0}\,S_{l0}}{N_l}-(\mu_l+v_\varepsilon)\,S_{l0}+\gamma_l\,R_{l0}\Bigg)+\frac{1}{2}\Bigg(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}\\
&+\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-(\mu_m+\sigma_m)\,I_{m0}\Bigg)S_{l0}\Bigg)\Bigg)-(\mu_l+v_\varepsilon)\Bigg(-\frac{1}{2}\,\frac{\beta_{lm}\,I_{m0}\,S_{l0}}{N_l}-\frac{1}{2}\,(\mu_l\\
&+v_\varepsilon)\,S_{l0}+\frac{1}{2}\,\gamma_l\,R_{l0}\Bigg)+\gamma_l\Bigg(\frac{1}{2}\,v_\varepsilon\,S_{l0}+\frac{1}{2}\,\tau_l\,E_{l0}+\frac{1}{2}\,\tau_l\,I_{l0}-\frac{1}{2}\,(\mu_l+\gamma_l)\,R_{l0}\Bigg)\Bigg)t^3\Bigg)\\
&+\gamma_l\Bigg(\frac{1}{6}\,v_\varepsilon\,\Lambda_l\,t^3+\frac{1}{2}\,\big(v_\varepsilon\,S_{l0}+\tau_l\,E_{l0}+\tau_l\,I_{l0}-(\mu_l+\gamma_l)\,R_{l0}\big)\,t^2\Bigg)
\end{aligned}
$$

(3.307)

For $n=1$, equation (3.231) becomes;

$$E_{L2}=\frac{\beta_{lm}}{N_l}\int_0^t G_1 dt-\left(\mu_l+\tau_l+\varepsilon_l\right)\int_0^t K_1 dt \qquad (3.308)$$

Substituting (3.241) and (3.244) into (3.308) gives;

$$E_{L2}=\frac{\beta_{lm}}{N_l}\int_0^t \left(\mathrm{I}_{M1}S_{L0}+\mathrm{I}_{M0}S_{L1}\right)dt-\left(\mu_l+\tau_l+\varepsilon_l\right)\int_0^t E_{L1}dt$$

(3.309)

Integrating and collecting like terms, E_{L2} gives;

$$
\begin{aligned}
&\frac{1}{N_l}\Bigg(\beta_{lm}\Bigg(-\frac{1}{20}\frac{I_{m0}\beta_{lm}\Bigg(\frac{1}{2}\frac{\beta_{ml}I_{l0}\Lambda_m}{N_m}+\frac{1}{2}\frac{\beta_{mh}I_{h0}\Lambda_m}{N_m}\Bigg)\Lambda_l t^5}{N_l}+\frac{1}{4}\Bigg(I_{m0}\Bigg(\\
&\quad-\frac{1}{N_l}\Bigg(\beta_{lm}\Bigg(\frac{1}{3}I_{m0}\Bigg(-\frac{1}{2}\frac{\beta_{lm}I_{m0}\Lambda_l}{N_l}-\frac{1}{2}(\mu_l+v_\varepsilon)\Lambda_l\Bigg)+\frac{1}{3}\Bigg(\frac{\beta_{ml}I_{l0}S_{m0}}{N_m}\\
&\quad+\frac{\beta_{mh}I_{h0}S_{m0}}{N_m}-(\mu_m+\sigma_m)I_{m0}\Bigg)\Lambda_l+\frac{1}{3}\Bigg(\frac{1}{2}\frac{\beta_{ml}I_{l0}\Lambda_m}{N_m}+\frac{1}{2}\frac{\beta_{mh}I_{h0}\Lambda_m}{N_m}\Bigg)S_{l0}\Bigg)\Bigg)\\
&\quad-(\mu_l+v_\varepsilon)\Bigg(-\frac{1}{6}\frac{\beta_{lm}I_{m0}\Lambda_l}{N_l}-\frac{1}{6}(\mu_l+v_\varepsilon)\Lambda_l\Bigg)+\frac{1}{6}\gamma_l v_\varepsilon\Lambda_l\Bigg)+\Bigg(\frac{1}{2}\frac{\beta_{ml}I_{l0}\Lambda_m}{N_m}\\
&\quad+\frac{1}{2}\frac{\beta_{mh}I_{h0}\Lambda_m}{N_m}\Bigg)\Lambda_l\Bigg)t^4+\frac{1}{3}\Bigg(I_{m0}\Bigg(-\frac{1}{N_l}\Bigg(\beta_{lm}\Bigg(\frac{1}{2}I_{m0}\Bigg(-\frac{\beta_{lm}I_{m0}S_{l0}}{N_l}-(\mu_l\\
&\quad+v_\varepsilon)S_{l0}+\gamma_l R_{l0}\Bigg)+\frac{1}{2}\Bigg(\frac{\beta_{ml}I_{l0}S_{m0}}{N_m}+\frac{\beta_{mh}I_{h0}S_{m0}}{N_m}-(\mu_m+\sigma_m)I_{m0}\Bigg)S_{l0}\Bigg)\Bigg)-(\mu_l\\
&\quad+v_\varepsilon)\Bigg(-\frac{1}{2}\frac{\beta_{lm}I_{m0}S_{l0}}{N_l}-\frac{1}{2}(\mu_l+v_\varepsilon)S_{l0}+\frac{1}{2}\gamma_l R_{l0}\Bigg)+\gamma_l\Bigg(\frac{1}{2}v_\varepsilon S_{l0}+\frac{1}{2}\tau_l E_{l0}\\
&\quad+\frac{1}{2}\tau_l I_{l0}-\frac{1}{2}(\mu_l+\gamma_l)R_{l0}\Bigg)\Bigg)+\Bigg(\frac{\beta_{ml}I_{l0}S_{m0}}{N_m}+\frac{\beta_{mh}I_{h0}S_{m0}}{N_m}-(\mu_m+\sigma_m)I_{m0}\Bigg)\Lambda_l\\
&\quad+\Bigg(\frac{1}{2}\frac{\beta_{ml}I_{l0}\Lambda_m}{N_m}+\frac{1}{2}\frac{\beta_{mh}I_{h0}\Lambda_m}{N_m}\Bigg)S_{l0}\Bigg)t^3+\frac{1}{2}\Bigg(\frac{\beta_{ml}I_{l0}S_{m0}}{N_m}+\frac{\beta_{mh}I_{h0}S_{m0}}{N_m}-(\mu_m\\
&\quad+\sigma_m)I_{m0}\Bigg)S_{l0}t^2\Bigg)\Bigg)-(\mu_l+\tau_l+\varepsilon_l)\Bigg(\frac{1}{6}\frac{\beta_{lm}I_{m0}\Lambda_l t^3}{N_l}+\frac{1}{2}\Bigg(\frac{\beta_{lm}I_{m0}S_{l0}}{N_l}-(\mu_l+\tau_l\\
&\quad+\varepsilon_l)E_{l0}\Bigg)t^2\Bigg)
\end{aligned}
\tag{3.310}
$$

For $n=1$, equation (3.232) becomes;

$$
I_{L2}=\varepsilon_l\int_0^t K_1\,dt-(\mu_l+c_l+\tau_l)\int_0^t L_1\,dt \tag{3.311}
$$

Substituting (3.244) and (3.245) into (3.311) gives;

$$\mathrm{I}_{L2} = \varepsilon_l \int_0^t E_{L1} dt - \left(\mu_l + c_l + \tau_l\right) \int_0^t \mathrm{I}_{L1} dt \tag{3.312}$$

Integrating and collecting like terms, I_{L2} gives;

$$\begin{aligned} &\varepsilon_l \left(\frac{1}{6} \frac{\beta_{lm} \mathrm{I}_{m0} \Lambda_l t^3}{N_l} + \frac{1}{2} \left(\frac{\beta_{lm} \mathrm{I}_{m0} S_{l0}}{N_l} - \left(\varepsilon_l + \mu_l + \tau_l\right) E_{l0} \right) t^2 \right) \\ &- \frac{1}{2} \left(\tau_l + \mu_l + c_l\right) \left(\varepsilon_l E_{l0} - \left(\tau_l + \mu_l + c_l\right) \mathrm{I}_{l0}\right) t^2 \end{aligned} \tag{3.313}$$

For $n = 1$, equation (3.233) becomes;

$$R_{L2} = v_\varepsilon \int_0^t H_1 dt + \tau_l \int_0^t K_1 dt + \tau_1 \int_0^t L_1 dt - \left(\mu_l + \gamma_l\right) \int_0^t M_1 dt \tag{3.314}$$

Substituting (3.242) through (3.246) into (3.314) gives;

$$R_{L2} = v_\varepsilon \int_0^t S_{L1} dt + \tau_l \int_0^t E_{L1} dt + \tau_1 \int_0^t \mathrm{I}_{L1} dt - \left(\mu_l + \gamma_l\right) \int_0^t R_{L1} dt \tag{3.315}$$

Integrating and collecting like terms, R_{L2} gives;

$$v_\varepsilon\Bigg(-\frac{1}{20}\frac{\beta_{lm}\left(\frac{1}{2}\frac{\beta_{ml}I_{l0}\Lambda_m}{N_m}+\frac{1}{2}\frac{\beta_{mh}I_{h0}\Lambda_m}{N_m}\right)\Lambda_l t^5}{N_l}+\frac{1}{4}\Bigg(-\frac{1}{N_l}\Bigg(\beta_{lm}\Bigg(\frac{1}{3}I_{m0}\Bigg($$

$$-\frac{1}{2}\frac{\beta_{lm}I_{m0}\Lambda_l}{N_l}-\frac{1}{2}\left(\mu_l+v_\varepsilon\right)\Lambda_l\Bigg)+\frac{1}{3}\Bigg(\frac{\beta_{ml}I_{l0}S_{m0}}{N_m}+\frac{\beta_{mh}I_{h0}S_{m0}}{N_m}-\Big(\mu_m$$

$$+\sigma_m\Big)I_{m0}\Bigg)\Lambda_l+\frac{1}{3}\left(\frac{1}{2}\frac{\beta_{ml}I_{l0}\Lambda_m}{N_m}+\frac{1}{2}\frac{\beta_{mh}I_{h0}\Lambda_m}{N_m}\right)S_{l0}\Bigg)\Bigg)-\left(\mu_l+v_\varepsilon\right)\Bigg($$

$$-\frac{1}{6}\frac{\beta_{lm}I_{m0}\Lambda_l}{N_l}-\frac{1}{6}\left(\mu_l+v_\varepsilon\right)\Lambda_l\Bigg)+\frac{1}{6}\gamma_l v_\varepsilon\Lambda_l\Bigg)t^4+\frac{1}{3}\Bigg(-\frac{1}{N_l}\Bigg(\beta_{lm}\Bigg(\frac{1}{2}I_{m0}\Bigg($$

$$-\frac{\beta_{lm}I_{m0}S_{l0}}{N_l}-\left(\mu_l+v_\varepsilon\right)S_{l0}+\gamma_l R_{l0}\Bigg)+\frac{1}{2}\Bigg(\frac{\beta_{ml}I_{l0}S_{m0}}{N_m}+\frac{\beta_{mh}I_{h0}S_{m0}}{N_m}-\Big(\mu_m$$

$$+\sigma_m\Big)I_{m0}\Bigg)S_{l0}\Bigg)\Bigg)-\left(\mu_l+v_\varepsilon\right)\left(-\frac{1}{2}\frac{\beta_{lm}I_{m0}S_{l0}}{N_l}-\frac{1}{2}\left(\mu_l+v_\varepsilon\right)S_{l0}+\frac{1}{2}\gamma_l R_{l0}\right)$$

$$+\gamma_l\left(\frac{1}{2}v_\varepsilon S_{l0}+\frac{1}{2}\tau_l E_{l0}+\frac{1}{2}\tau_l I_{l0}-\frac{1}{2}\left(\mu_l+\gamma_l\right)R_{l0}\right)\Bigg)t^3\Bigg)+\tau_l\Bigg(\frac{1}{6}\frac{\beta_{lm}I_{m0}\Lambda_l t^3}{N_l}$$

$$+\frac{1}{2}\left(\frac{\beta_{lm}I_{m0}S_{l0}}{N_l}-\left(\mu_l+\tau_l+\varepsilon_l\right)E_{l0}\right)t^2\Bigg)+\frac{1}{2}\tau_l\left(\varepsilon_l E_{l0}-\left(\mu_l+c_l+\tau_l\right)I_{l0}\right)t^2-\Big(\mu_{}$$

$$+\gamma_l\Big)\left(\frac{1}{6}v_\varepsilon\Lambda_l t^3+\frac{1}{2}\left(v_\varepsilon S_{l0}+\tau_l E_{l0}+\tau_l I_{l0}-\left(\mu_l+\gamma_l\right)R_{l0}\right)t^2\right)$$

(3.316)

For $n=1$, equation (3.234) becomes;

$$S_{M2}=-\frac{\beta_{ml}}{N_m}\int_0^t N_1 dt-\frac{\beta_{ml}}{N_m}\int_0^t P_1 dt-\left(\mu_m+\delta_m\right)\int_0^t Q_1 dt \qquad (3.317)$$

Substituting (3.247) through (3.249) into (3.317) gives;

$$S_{M2}=-\frac{\beta_{ml}}{N_m}\int_0^t\left(I_{L1}S_{M0}+I_{L0}S_{M1}\right)dt-\frac{\beta_{mh}}{N_m}\int_0^t\left(I_{H1}S_{M0}+I_{H0}S_{M1}\right)dt-\left(\mu_m+\delta_m\right)\int_0^t S_{M1}dt \qquad (3.318)$$

$$S_{M2}=-\frac{\beta_{ml}}{N_m}\int_0^t\left(I_{L1}S_{M0}+I_{L0}S_{M1}\right)dt-\frac{\beta_{mh}}{N_m}\int_0^t\left(I_{H1}S_{M0}+I_{H0}S_{M1}\right)dt-\left(\mu_m+\delta_m\right)\int_0^t S_{M1}dt \qquad (3.319)$$

Integrating and collecting like terms, S_{M2} gives;

$$-\frac{1}{N_m}\left(\beta_{ml}\left(\frac{1}{3}\left(\mathrm{I}_{l0}\left(-\frac{1}{2}\frac{\beta_{ml}\mathrm{I}_{l0}\Lambda_m}{N_m}-\frac{1}{2}\frac{\beta_{mh}\mathrm{I}_{h0}\Lambda_m}{N_m}-\frac{1}{2}\left(\mu_m+\delta_m\right)\Lambda_m\right)+\left(\varepsilon_l E_{l0}-\left(\mu_l+c_l+\tau_l\right)\mathrm{I}_{l0}\right)\Lambda_m\right)t^3+\frac{1}{2}\left(\mathrm{I}_{l0}\left(-\frac{\beta_{ml}\mathrm{I}_{l0}S_{m0}}{N_m}-\frac{\beta_{mh}\mathrm{I}_{h0}S_{m0}}{N_m}-\left(\mu_m+\delta_m\right)S_{m0}\right)+\left(\varepsilon_l E_{l0}-\left(\mu_l+c_l+\tau_l\right)\mathrm{I}_{l0}\right)S_{m0}\right)t^2\right)\right)-\frac{1}{N_m}\left(\beta_{mh}\left(\frac{1}{3}\left(\mathrm{I}_{h0}\left(-\frac{1}{2}\frac{\beta_{ml}\mathrm{I}_{l0}\Lambda_m}{N_m}-\frac{1}{2}\frac{\beta_{mh}\mathrm{I}_{h0}\Lambda_m}{N_m}-\frac{1}{2}\left(\mu_m+\delta_m\right)\Lambda_m\right)+\left(\varepsilon_h E_{h0}-\left(\mu_h+\tau_h\right)\mathrm{I}_{h0}\right)\Lambda_m\right)t^3+\frac{1}{2}\left(\mathrm{I}_{h0}\left(-\frac{\beta_{ml}\mathrm{I}_{l0}S_{m0}}{N_m}-\frac{\beta_{mh}\mathrm{I}_{h0}S_{m0}}{N_m}-\left(\mu_m+\delta_m\right)S_{m0}\right)+\left(\varepsilon_h E_{h0}-\left(\mu_h+\tau_h\right)\mathrm{I}_{h0}\right)S_{m0}\right)t^2\right)\right)-\left(\mu_m+\delta_m\right)\left(\frac{1}{3}\left(-\frac{1}{2}\frac{\beta_{ml}\mathrm{I}_{l0}\Lambda_m}{N_m}-\frac{1}{2}\frac{\beta_{mh}\mathrm{I}_{h0}\Lambda_m}{N_m}-\frac{1}{2}\left(\mu_m+\delta_m\right)\Lambda_m\right)t^3+\frac{1}{2}\left(-\frac{\beta_{ml}\mathrm{I}_{l0}S_{m0}}{N_m}-\frac{\beta_{mh}\mathrm{I}_{h0}S_{m0}}{N_m}-\left(\mu_m+\delta_m\right)S_{m0}\right)t^2\right)$$

(3.320)

For $n=1$, equation (3.235) becomes;

$$\mathrm{I}_{M2}=\frac{\beta_{ml}}{N_m}\int_0^t N_1dt+\frac{\beta_{mh}}{N_m}\int_0^t P_1dt-\left(\mu_m+\sigma_m\right)\int_0^t T_1dt \qquad (3.321)$$

Substituting (3.247), (3.248) and (3.250) into (3.321) gives;

$$\mathrm{I}_{M2}=\frac{\beta_{ml}}{N_m}\int_0^t\left(\mathrm{I}_{L1}S_{M0}+\mathrm{I}_{L0}S_{M1}\right)dt-\frac{\beta_{mh}}{N_m}\int_0^t\left(\mathrm{I}_{H1}S_{M0}+\mathrm{I}_{H0}S_{M1}\right)dt-\left(\mu_m+\sigma_m\right)\int_0^t\mathrm{I}_{M1}dt \qquad (3.322)$$

Integrating and collecting like terms, I_{M2} gives;

$$
\frac{1}{N_m}\left(\beta_{ml}\left(\frac{1}{3}\left(I_{l0}\left(-\frac{1}{2}\,\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}-\frac{1}{2}\,\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}-\frac{1}{2}\left(\mu_m+\delta_m\right)\Lambda_m\right)+\left(\varepsilon_l E_{l0}-\left(\mu_l\right.\right.\right.\right.\right.
$$

$$
\left.\left.+c_l+\tau_l\right)I_{l0}\right)\Lambda_m\Bigg)t^3+\frac{1}{2}\left(I_{l0}\left(-\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}-\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-\left(\mu_m+\delta_m\right)S_{m0}\right)\right.
$$

$$
\left.\left.\left.+\left(\varepsilon_l E_{l0}-\left(\mu_l+c_l+\tau_l\right)I_{l0}\right)S_{m0}\right)t^2\right)\right)+\frac{1}{N_m}\left(\beta_{mh}\left(\frac{1}{3}\left(I_{h0}\left(-\frac{1}{2}\,\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}\right.\right.\right.\right.
$$

$$
\left.-\frac{1}{2}\,\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}-\frac{1}{2}\left(\mu_m+\delta_m\right)\Lambda_m\right)+\left(\varepsilon_h E_{h0}-\left(\mu_h+\tau_h\right)I_{h0}\right)\Lambda_m\Bigg)t^3+\frac{1}{2}\left(I_{h0}\left(\right.\right.
$$

$$
\left.\left.\left.\left.-\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}-\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-\left(\mu_m+\delta_m\right)S_{m0}\right)+\left(\varepsilon_h E_{h0}-\left(\mu_h+\tau_h\right)I_{h0}\right)S_{m0}\right)t^2\right)\right)
$$

$$
-\left(\mu_m+\sigma_m\right)\left(\frac{1}{3}\left(\frac{1}{2}\,\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}+\frac{1}{2}\,\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\right)t^3+\frac{1}{2}\left(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}\right.\right.
$$

$$
\left.\left.+\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-\left(\mu_m+\sigma_m\right)I_{m0}\right)t^2\right)
$$

(3.323)

For the final results of $S_H, E_H, I_H, R_H, S_L, E_L, I_L, R_L, S_M, I_M$, check equations (3.324)-(3.333) in appendix A.

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Variables and Parameter Values Estimation

Table 4.1: Values for variables/parameters used for analytical solutions of the Model

Symbol	Value	Sources
S_h	500	Assumed
E_h	220	Assumed
	100	Assumed
I_h		Assumed
R_h	200	Assumed
S_l		Assumed
	890	Assumed
E_l	450	Assumed
I_l	200	Assumed
R_l	120	Assumed
S_m	700	
I_m	500	Estimated
N_h	1020	
		Estimated
N_l	1660	Estimated
N_m	1200	
μ_l	0.5	(Majok *et al*.,1991)
Λ_l	0.9	Assumed
γ_l	0.25	Assumed
ε_l	0.25	Assumed
v_ε	0.25	Assumed
β_{ml}	0.25	Assumed
Λ_h	0.8	Assumed
Λ_m	0.25	Assumed
c_l	0.25	Assumed
ε_h	0.25	Assumed
β_{lm}	0.39	Assumed
β_{mh}	0.25	Assumed

β_{hm}	0.001	Assumed
τ_h	0.25	Assumed
τ_l	0.25	Assumed
δ_m	0.25	Luguoye *et al.*, 2016
μ_h	0.01	Assumed
μ_m	0.67	Assumed

4.2 Sensitivity Analysis of the Model

Sensitivity analysis confirms the effect each parameter has on the disease transmission. The objective of sensitivity analysis is to give rise to uncertainties of the model outputs (Leon *et al.*,2009).

Table 4.2 Sensitivity indices of the model parameters on R_c

Parameters	Value	Sensitivity Index
β_{lm}	0.61	0.4843392736
ε_l	0.25	0.2421696368
β_{ml}	0.25	0.4843392736
γ_l	0.25	0.04036160614
ε_h	0.25	0.0006023356211
μ_l	0.5	-0.4036160614
v_ε	0.25	-0.1210848184
β_{mh}	0.25	0.01566072602
β_{hm}	0.001	0.01566072602

Table 4.2 shows the sensitivity indices of the effective Reproduction number, R_c for the model equations (3.4)-(3.13). The parameters have both positive and negative effects on R_c. The most sensitive parameters are contact rates from livestock to mosquito, β_{lm} mosquito to livestock, β_{ml}. The next important parameter is incubation period in livestock (ε_l). The parameters with the least effect on R_c are; incubation period o the disease in humans, ε_h and natural death rate of livestock μ_l. The values shown in table 4.2 were computed with maple and codes were shown in appendix B.

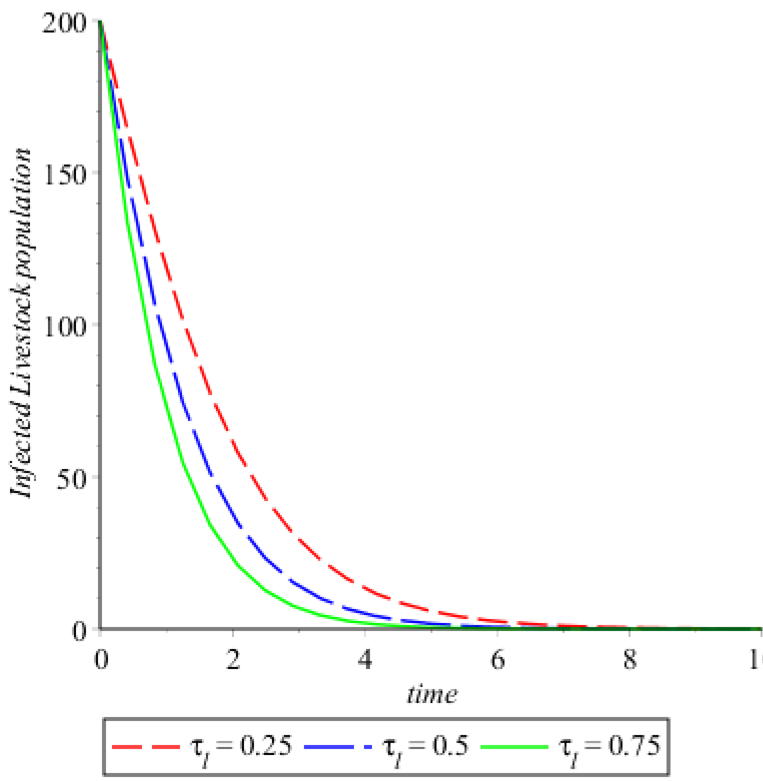

Figure 4.1: Effect of Treatment on Infected Livestock Population

Figure 4.1 shows the effect of treating livestock, this causes a decrease in the infected class of livestock. The more livestock are treated and precautionary measures are put in place, the less the spread of RVF virus among the livestock population.

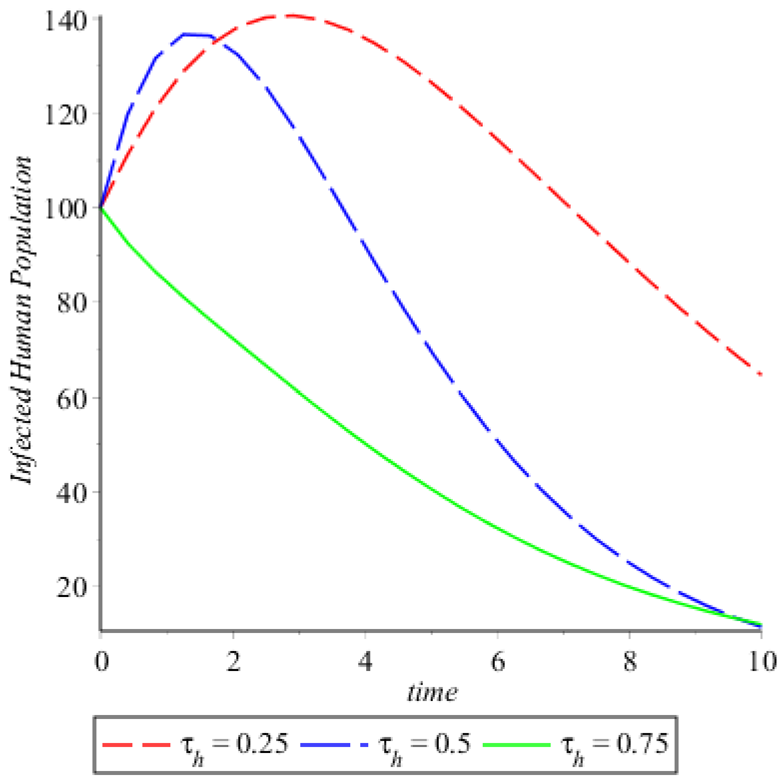

Figure 4.2: Effect of Treatment of Humans on Infected Human Population

Figure 4.2 shows the relationship between treatment of humans and infected human population. The higher the rate of treatment, the lower the infected human population. This means the more humans are treated, the less the spread of this disease in the population.

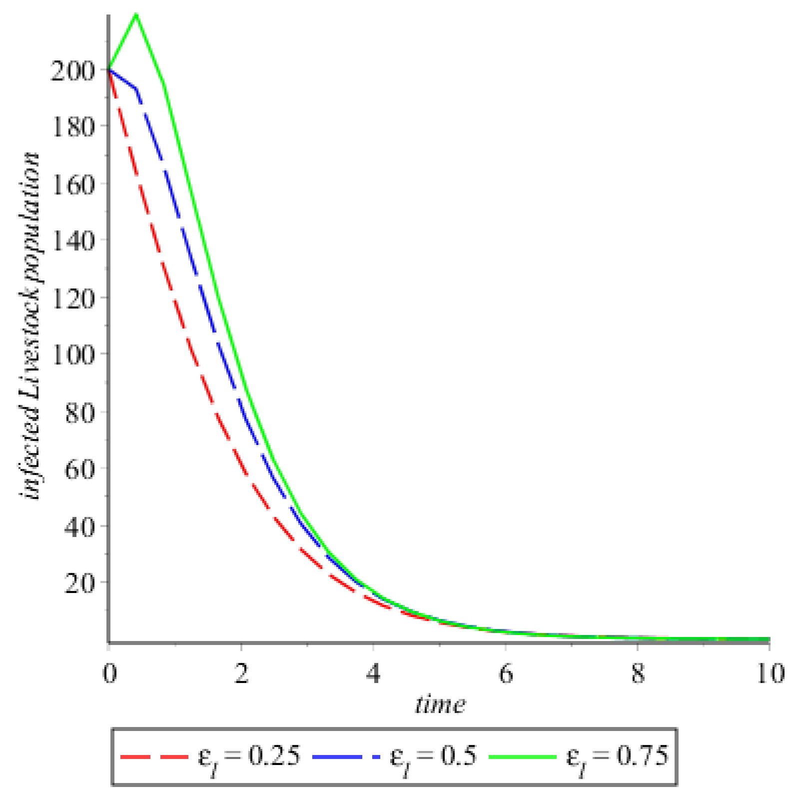

Figure 4.3: Effect of Incubation Period on Infected Livestock Population

Figure 4.3 shows the relationship between incubation period and infected livestock. The longer the incubation period, the higher the spread of RVF virus among the vectors and eventually increase in infected livestock population.

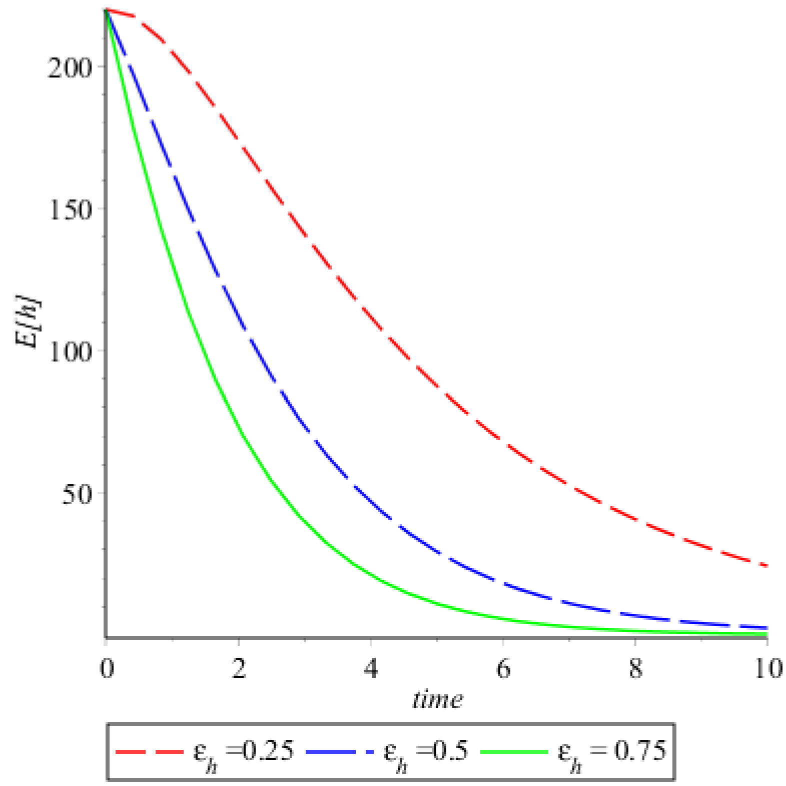

Figure 4.4: Effect of Incubation Period of Humans on Exposed Human Population

Figure 4.4 shows the relationship between exposed humans and incubation period. The longer the incubation period of the disease, the lower the exposed human population. Extended incubation period, decreases the exposed human class because RVF virus cannot be transmitted from one human to another. Infected humans when treated recover and as such the exposed class reduces.

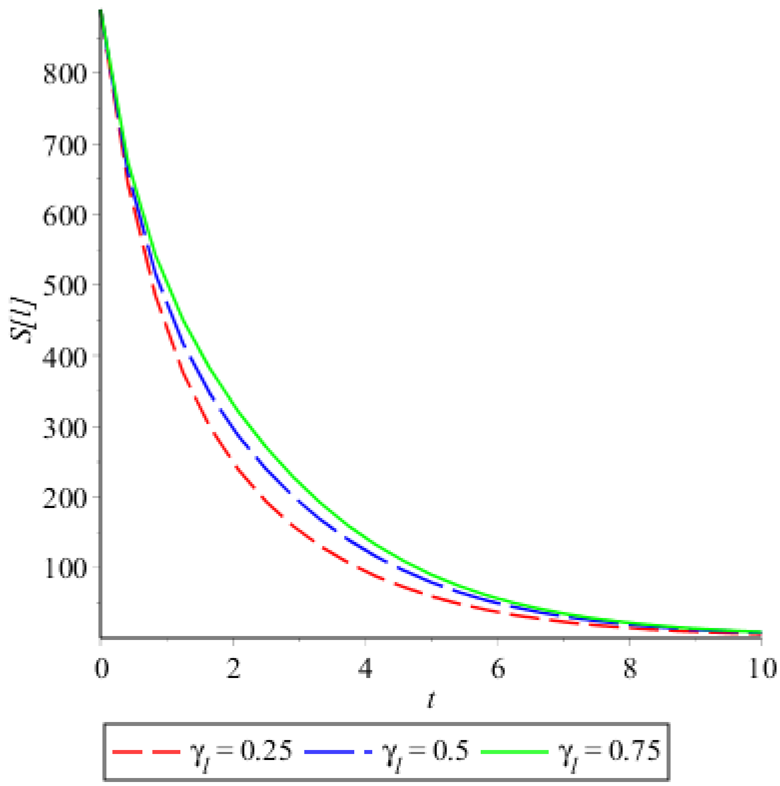

Figure 4.5: Effect of Recovery of Livestock on Susceptible Livestock Population

This relationship shows that an increase in recovery of livestock, increases the susceptible livestock population. The more livestock are recovered, the higher the susceptible livestock class.

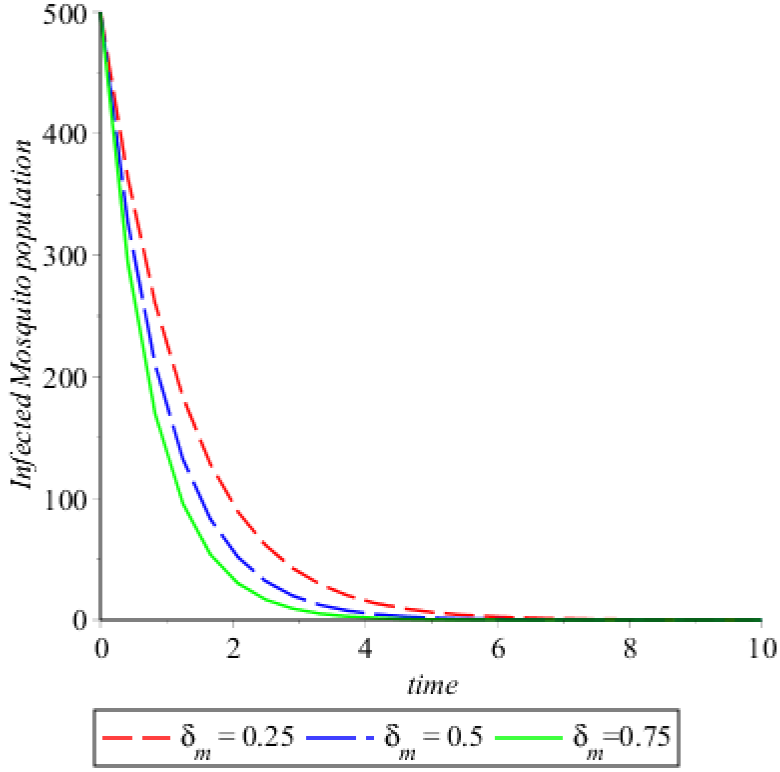

Figure 4.6 Trapping Rate of Mosquitoes on Infected Mosquito Population

Figure 4.6 shows the relationship between trapping rate of mosqitoes on infected mosquito Population. An increase in the trapping rate, results in decrease of infected mosqitoes; this is because they have little or no interaction with either infected humans or livestock.

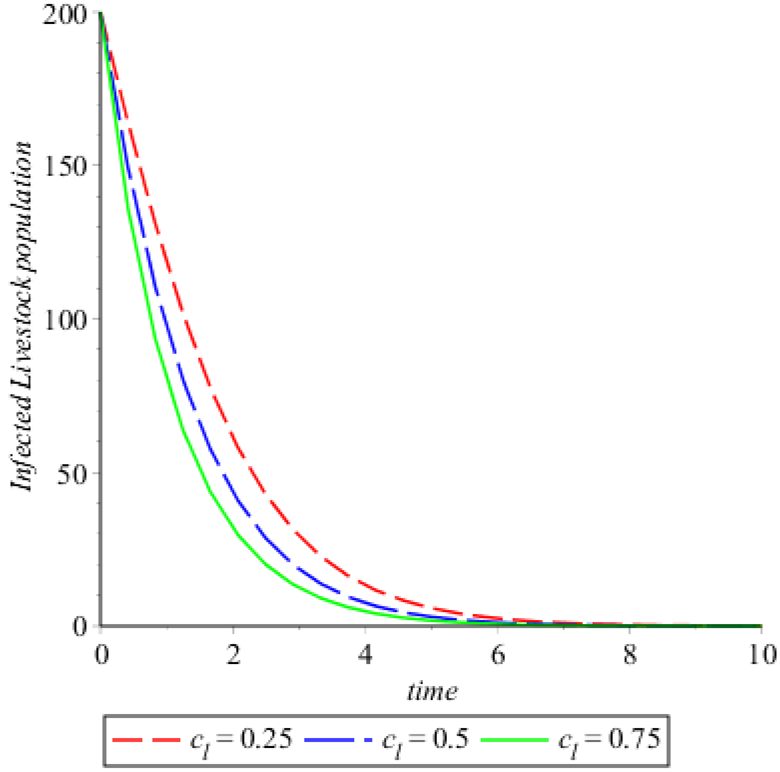

Figure 4.7 Culling Rate of Livestock on Infected Livestock Population

This relationship shows that the increase in the elimination of infected livestock (culling rate), decreases the infected livestock population. The more infected livestock are eliminated, the less the transmission of RVF virus.

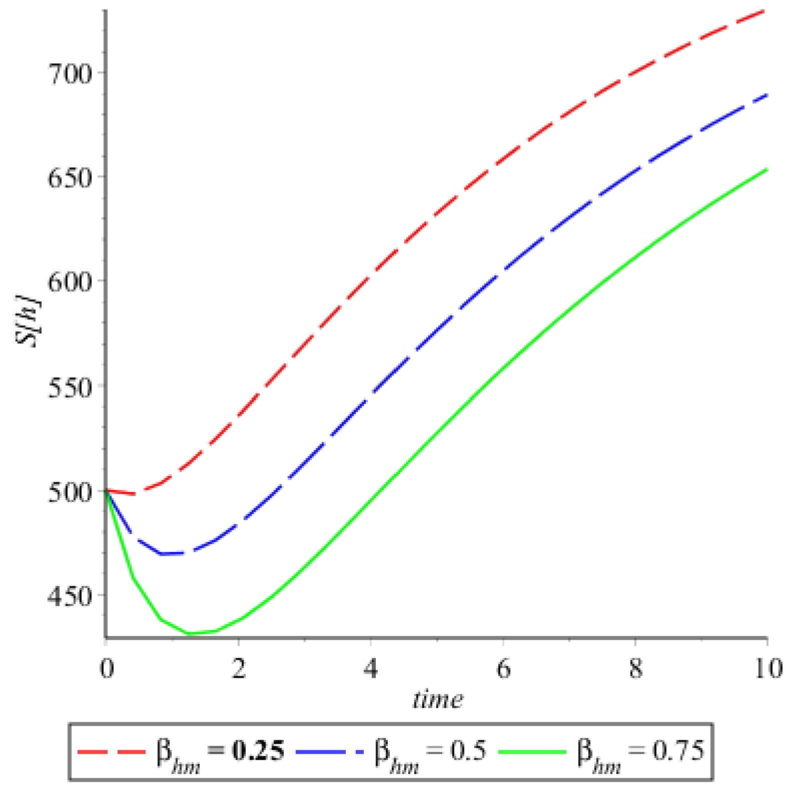

Figure 4.8 Effect of Contact Rate from Humans to Mosquitoes on Susceptible Population

This relationship shows that an increase in contact rate from humans to mosquitoes, yields a decrease in susceptible human population. The less humans come in contact with vectors, the less the spread of the disease and the higher the susceptible population.

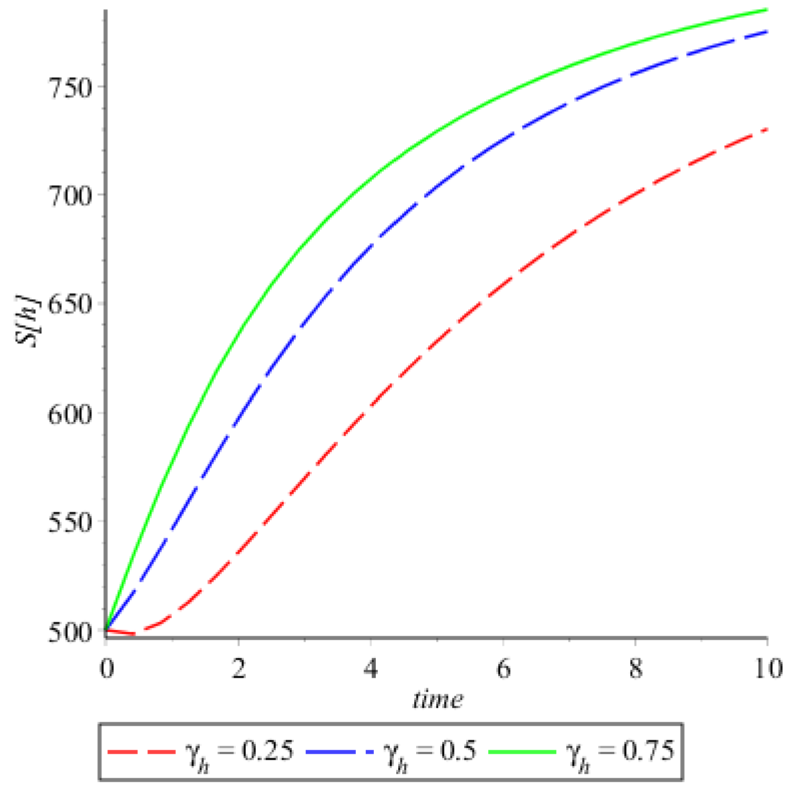

Figure 4.9 Effect of Recovery of Humans on Susceptible Human Population

This relationship shows that an increase in recovery rate, yields an increase in susceptible human population. The more humans recover, the higher the susceptible population of humans.

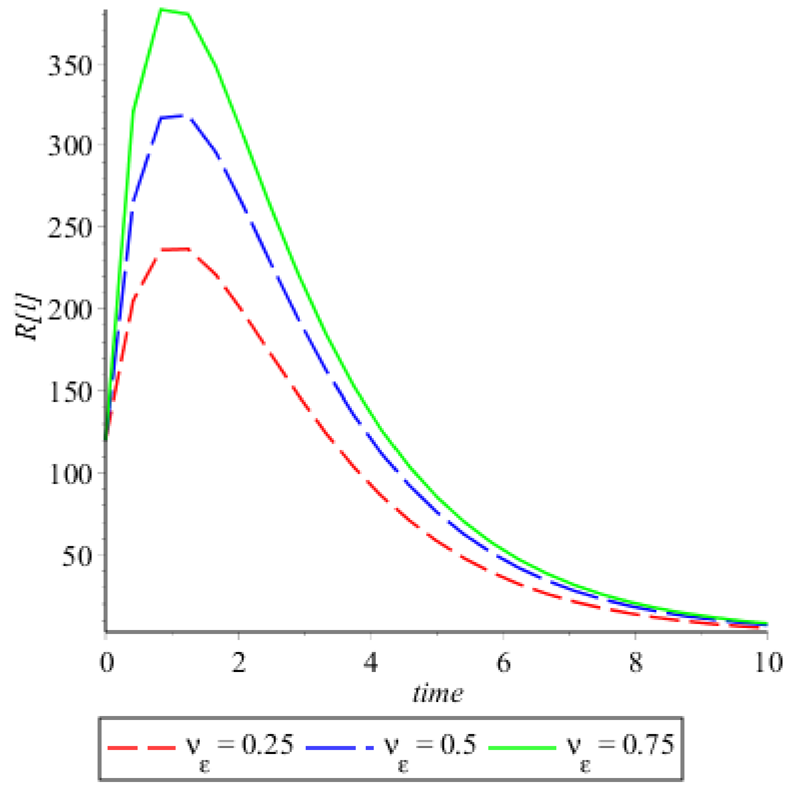

Figure 4.10 Effect of Vaccination of Livestock on Recovered Livestock Population

This figure shows the efficacy of vaccination on livestock; the more livestock are vaccinated, the more the population of recovered livestock. In other words, an increase in the rate of vaccination yields an increase in the recovered livestock class and eventually the spread of the disease reduces overtime.

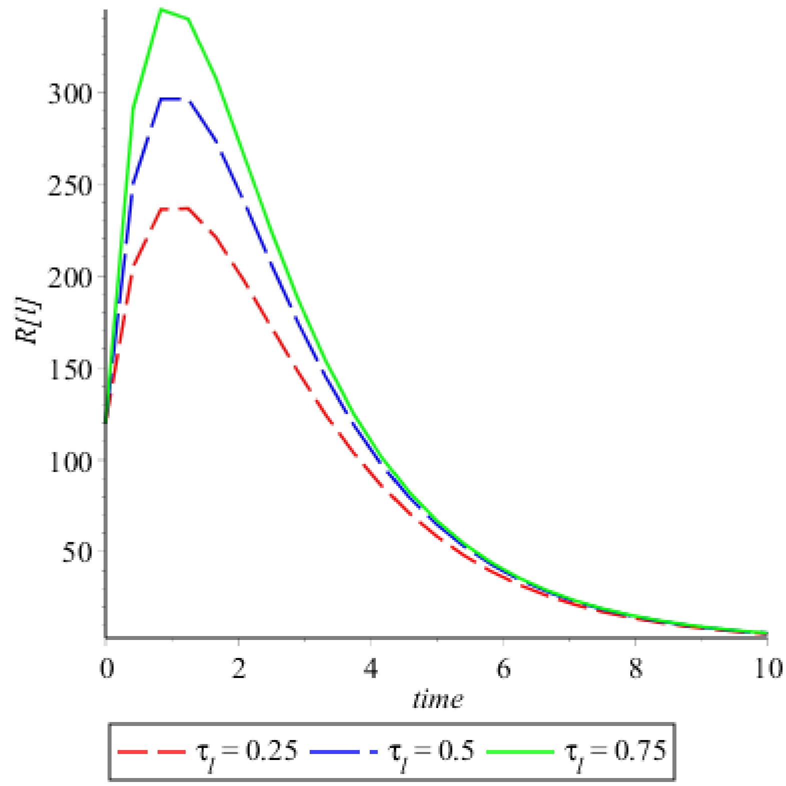

Figure 4.11 Effect of Treatment of Livestock on Recovered Livestock Population

This relationship shows that an increase in treatment of livestock, yields an increase in the recovered class of livestock. The more livestock are treated, the less the spread of RVF virus in the population.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

The Rift Valley Fever Model formulated in this work exists in a feasible region where disease free and endemic equilibrium points are obtained and the local and global stability of disease-free equilibrium was investigated. The positivity of solutions using Wiah's method was also determined. The model has three interventions; efficacy of vaccination, culling of livestock and trapping of mosquitoes. The model analysis showed that disease free equilibrium exists and is locally asymptotically stable whenever its effective reproduction number is less than 1, and it has a unique endemic equilibrium when $R_c>1$. These results have important public health implications, since they determine the severity and outcome of the epidemic that is, clearance or persistence of infection) and provide a framework for the design of control strategies. Further analysis showed that the disease-free point is locally stable implying that small perturbations and fluctuations on the disease state will always result in the eradication of the disease if $R_c<1$. In the final analysis efficacy of vaccination, culling of livestock and mosquito trapping intervention program will effectively control the spread of rift valley fever.

Adomian Decomposition method was used to solve the model equations. The sensitivity analysis of the parameters was also investigated and sensitivity indices were obtained.

5.2 Contribution to Knowledge

In this work a model was developed for the transmission dynamics of Rift Valley Fever virus, we incorporated effective vaccination, culling rate (control), trapping of mosquitoes, contact rates; humans to mosquitoes and vice versa, livestock to mosquitoes and vice versa, and incubation period of humans and livestock.

5.3 Recommendations

The system of equations can further be looked into by incorporating isolation of infected livestock. Limit and/or prohibit movements of animals from affected areas to disease-free areas to reduce the spread of the disease.

Use of personal protective equipment (PPE): particularly important for veterinarians (care, autopsies) and PPE in slaughter houses or during slaughtering animals should be used to prevent livestock to human transmission.

More vector control programmes using insecticide and mosquito treated nets, can be adopted in addition to trapping vectors.

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APPENDICES

APPENDIX A: Maple Code for Adomian Decomposition Method (ADM)

> $S_H := collect(S_{H0} + S_{H1} + S_{H2}, t)$

$$
\begin{aligned}
S_H := & -\frac{1}{4}\,\frac{\beta_{hm}\left(\frac{1}{2}\,\frac{\beta_{ml}\,\mathrm{I}_{l0}\,\Lambda_m}{N_m} + \frac{1}{2}\,\frac{\beta_{mh}\,\mathrm{I}_{h0}\,\Lambda_m}{N_m}\right)\Lambda_h\,t^4}{N_h} + \Bigg(-\frac{1}{N_h}\Bigg(\beta_{hm}\Bigg(\frac{1}{3}\,\mathrm{I}_{m0}\Bigg(\\
& -\frac{1}{2}\,\frac{\beta_{hm}\,\mathrm{I}_{m0}\,\Lambda_h}{N_h} - \frac{1}{2}\,\mu_h\,\Lambda_h\Bigg) + \frac{1}{3}\left(\frac{\beta_{ml}\,\mathrm{I}_{l0}\,S_{m0}}{N_m} + \frac{\beta_{mh}\,\mathrm{I}_{h0}\,S_{m0}}{N_m} - \left(\mu_m + \sigma_m\right)\mathrm{I}_{m0}\right)\Lambda_h \\
& + \frac{1}{3}\left(\frac{1}{2}\,\frac{\beta_{ml}\,\mathrm{I}_{l0}\,\Lambda_m}{N_m} + \frac{1}{2}\,\frac{\beta_{mh}\,\mathrm{I}_{h0}\,\Lambda_m}{N_m}\right)S_{h0}\Bigg)\Bigg) - \mu_h\left(-\frac{1}{6}\,\frac{\beta_{hm}\,\mathrm{I}_{m0}\,\Lambda_h}{N_h} - \frac{1}{6}\,\mu_h\,\Lambda_h\right)\Bigg) \\
& t^3 + \Bigg(-\frac{1}{2}\,\frac{\beta_{hm}\,\mathrm{I}_{m0}\,\Lambda_h}{N_h} - \frac{1}{2}\,\mu_h\,\Lambda_h - \frac{1}{N_h}\Bigg(\beta_{hm}\Bigg(\frac{1}{2}\,\mathrm{I}_{m0}\Bigg(-\frac{\beta_{hm}\,\mathrm{I}_{m0}\,S_{h0}}{N_h} - \mu_h\,S_{h0} \\
& + R_{h0}\,\gamma_h\Bigg) + \frac{1}{2}\left(\frac{\beta_{ml}\,\mathrm{I}_{l0}\,S_{m0}}{N_m} + \frac{\beta_{mh}\,\mathrm{I}_{h0}\,S_{m0}}{N_m} - \left(\mu_m + \sigma_m\right)\mathrm{I}_{m0}\right)S_{h0}\Bigg)\Bigg) - \mu_h\Bigg(\\
& -\frac{1}{2}\,\frac{\beta_{hm}\,\mathrm{I}_{m0}\,S_{h0}}{N_h} - \frac{1}{2}\,\mu_h\,S_{h0} + \frac{1}{2}\,R_{h0}\,\gamma_h\Bigg) + \frac{1}{2}\,\gamma_h\left(\tau_h\,\mathrm{I}_{h0} - \left(\mu_h + \gamma_h\right)R_{h0}\right)\Bigg)t^2 + \Bigg(\Lambda_h \\
& - \frac{\beta_{hm}\,\mathrm{I}_{m0}\,S_{h0}}{N_h} - \mu_h\,S_{h0} + R_{h0}\,\gamma_h\Bigg)t + S_{h0}
\end{aligned}
\tag{3.324}
$$

> $E_H := collect(E_{H0} + E_{H1} + E_{H2}, t)$

$$E_H := \frac{1}{4}\,\frac{\beta_{hm}\left(\frac{1}{2}\,\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m} + \frac{1}{2}\,\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\right)\Lambda_h\,t^4}{N_h} + \left(\frac{1}{N_h}\left(\beta_{hm}\left(\frac{1}{3}\,I_{m0}\left(-\frac{1}{2}\,\frac{\beta_{hm}\,I_{m0}\,\Lambda_h}{N_h} - \frac{1}{2}\,\mu_h\,\Lambda_h\right) + \frac{1}{3}\left(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m} + \frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m} - \left(\mu_m + \sigma_m\right) I_{m0}\right)\Lambda_h + \frac{1}{3}\left(\frac{1}{2}\,\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m} + \frac{1}{2}\,\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\right)S_{h0}\right)\right) - \frac{1}{6}\,\frac{\left(\varepsilon_h + \mu_h\right)\beta_{hm}\,I_{m0}\,\Lambda_h}{N_h}\right)t^3 + \left(\frac{1}{2}\,\frac{\beta_{hm}\,I_{m0}\,\Lambda_h}{N_h} + \frac{1}{N_h}\left(\beta_{hm}\left(\frac{1}{2}\,I_{m0}\left(-\frac{\beta_{hm}\,I_{m0}\,S_{h0}}{N_h} - \mu_h\,S_{h0} + R_{h0}\,\gamma_h\right) + \frac{1}{2}\left(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m} + \frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m} - \left(\mu_m + \sigma_m\right) I_{m0}\right)S_{h0}\right)\right) - \left(\varepsilon_h + \mu_h\right)\left(\frac{1}{2}\,\frac{\beta_{hm}\,I_{m0}\,S_{h0}}{N_h} - \frac{1}{2}\left(\varepsilon_h + \mu_h\right)E_{h0}\right)\right)t^2 + \left(\frac{\beta_{hm}\,I_{m0}\,S_{h0}}{N_h} - \left(\varepsilon_h + \mu_h\right)E_{h0}\right)t + E_{h0} \tag{3.325}$$

> $I_H := collect\left(I_{H0} + I_{H1} + I_{H2}, t\right)$

$$I_H := \frac{1}{6}\,\frac{\varepsilon_h\,\beta_{hm}\,I_{m0}\,\Lambda_h\,t^3}{N_h} + \left(\varepsilon_h\left(\frac{1}{2}\,\frac{\beta_{hm}\,I_{m0}\,S_{h0}}{N_h} - \frac{1}{2}\left(\varepsilon_h + \mu_h\right)E_{h0}\right) - \frac{1}{2}\left(\mu_h + \tau_h\right)\left(\varepsilon_h\,E_{h0} - \left(\mu_h + \tau_h\right)I_{h0}\right)\right)t^2 + \left(\varepsilon_h\,E_{h0} - \left(\mu_h + \tau_h\right)I_{h0}\right)t + I_{h0} \tag{3.326}$$

> $R_H := collect\left(R_{H0} + R_{H1} + R_{H2}, t\right)$

$$R_H := \left(\frac{1}{2}\,\tau_h\left(\varepsilon_h\,E_{h0} - \left(\mu_h + \tau_h\right)I_{h0}\right) - \frac{1}{2}\left(\mu_h + \gamma_h\right)\left(\tau_h\,I_{h0} - \left(\mu_h + \gamma_h\right)R_{h0}\right)\right)t^2 + \left(\tau_h\,I_{h0} - \left(\mu_h + \gamma_h\right)R_{h0}\right)t + R_{h0} \tag{3.327}$$

> $S_L := collect\left(S_{L0} + S_{L1} + S_{L2}, t\right)$

$$S_L := -\frac{1}{4}\frac{\beta_{lm}\left(\frac{1}{2}\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}+\frac{1}{2}\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\right)\Lambda_l\,t^4}{N_l}+\left(-\frac{1}{N_l}\left(\beta_{lm}\left(\frac{1}{3}\,I_{m0}\left(-\frac{1}{2}\frac{\beta_{lm}\,I_{m0}\,\Lambda_l}{N_l}-\frac{1}{2}\left(\mu_l+\nu_\varepsilon\right)\Lambda_l\right)+\frac{1}{3}\left(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}+\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-\left(\mu_m+\sigma_m\right)I_{m0}\right)\Lambda_l+\frac{1}{3}\left(\frac{1}{2}\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}+\frac{1}{2}\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\right)S_{l0}\right)\right)-\left(\mu_l+\nu_\varepsilon\right)\left(-\frac{1}{6}\frac{\beta_{lm}\,I_{m0}\,\Lambda_l}{N_l}-\frac{1}{6}\left(\mu_l+\nu_\varepsilon\right)\Lambda_l\right)+\frac{1}{6}\gamma_l\,\nu_\varepsilon\,\Lambda_l\right)t^3+\left(-\frac{1}{2}\frac{\beta_{lm}\,I_{m0}\,\Lambda_l}{N_l}-\frac{1}{2}\left(\mu_l+\nu_\varepsilon\right)\Lambda_l-\frac{1}{N_l}\left(\beta_{lm}\left(\frac{1}{2}\,I_{m0}\left(-\frac{\beta_{lm}\,I_{m0}\,S_{l0}}{N_l}-\left(\mu_l+\nu_\varepsilon\right)S_{l0}+R_{l0}\,\gamma_l\right)+\frac{1}{2}\left(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}+\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-\left(\mu_m+\sigma_m\right)I_{m0}\right)S_{l0}\right)\right)-\left(\mu_l+\nu_\varepsilon\right)\left(-\frac{1}{2}\frac{\beta_{lm}\,I_{m0}\,S_{l0}}{N_l}-\frac{1}{2}\left(\mu_l+\nu_\varepsilon\right)S_{l0}+\frac{1}{2}R_{l0}\,\gamma_l\right)+\gamma_l\left(\frac{1}{2}S_{l0}\,\nu_\varepsilon+\frac{1}{2}E_{l0}\,\tau_l+\frac{1}{2}I_{l0}\,\tau_l-\frac{1}{2}\left(\mu_l+\gamma_l\right)R_{l0}\right)\right)t^2+\left(\Lambda_l-\frac{\beta_{lm}\,I_{m0}\,S_{l0}}{N_l}-\left(\mu_l+\nu_\varepsilon\right)S_{l0}+R_{l0}\,\gamma_l\right)t+S_{l0}$$

(3.328)

> $E_L := collect\left(E_{L0}+E_{L1}+E_{L2}, t\right)$

$$E_L := \frac{1}{4}\frac{\beta_{lm}\left(\frac{1}{2}\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}+\frac{1}{2}\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\right)\Lambda_l\,t^4}{N_l}+\left(\frac{1}{N_l}\left(\beta_{lm}\left(\frac{1}{3}\,I_{m0}\left(-\frac{1}{2}\frac{\beta_{lm}\,I_{m0}\,\Lambda_l}{N_l}-\frac{1}{2}\left(\mu_l+\nu_\varepsilon\right)\Lambda_l\right)+\frac{1}{3}\left(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}+\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-\left(\mu_m+\sigma_m\right)I_{m0}\right)\Lambda_l+\frac{1}{3}\left(\frac{1}{2}\frac{\beta_{ml}\,I_{l0}\,\Lambda_m}{N_m}+\frac{1}{2}\frac{\beta_{mh}\,I_{h0}\,\Lambda_m}{N_m}\right)S_{l0}\right)\right)-\frac{1}{6}\frac{\left(\mu_l+\tau_l+\varepsilon_l\right)\beta_{lm}\,I_{m0}\,\Lambda_l}{N_l}\right)t^3+\left(\frac{1}{2}\frac{\beta_{lm}\,I_{m0}\,\Lambda_l}{N_l}+\frac{1}{N_l}\left(\beta_{lm}\left(\frac{1}{2}\,I_{m0}\left(-\frac{\beta_{lm}\,I_{m0}\,S_{l0}}{N_l}-\left(\mu_l+\nu_\varepsilon\right)S_{l0}+R_{l0}\,\gamma_l\right)+\frac{1}{2}\left(\frac{\beta_{ml}\,I_{l0}\,S_{m0}}{N_m}+\frac{\beta_{mh}\,I_{h0}\,S_{m0}}{N_m}-\left(\mu_m+\sigma_m\right)I_{m0}\right)S_{l0}\right)\right)-\left(\mu_l+\tau_l+\varepsilon_l\right)\left(\frac{1}{2}\frac{\beta_{lm}\,I_{m0}\,S_{l0}}{N_l}-\frac{1}{2}\left(\mu_l+\tau_l+\varepsilon_l\right)E_{l0}\right)\right)t^2+\left(\frac{\beta_{lm}\,I_{m0}\,S_{l0}}{N_l}-\left(\mu_l+\tau_l+\varepsilon_l\right)E_{l0}\right)t+E_{l0}$$

(3.329)

> $I_L := collect(I_{L0} + I_{L1} + I_{L2}, t)N$

$$I_L := \frac{1}{6}\frac{\varepsilon_l \beta_{lm} I_{m0} \Lambda_l t^3}{N_l} + \left(\varepsilon_l\left(\frac{1}{2}\frac{\beta_{lm} I_{m0} S_{l0}}{N_l} - \frac{1}{2}(\mu_l + \tau_l + \varepsilon_l) E_{l0}\right) - \frac{1}{2}(\mu_l + c_l + \tau_l)(\varepsilon_l E_{l0} - (\mu_l + c_l + \tau_l) I_{l0})\right) t^2 + (\varepsilon_l E_{l0} - (\mu_l + c_l + \tau_l) I_{l0}) t + I_{l0}$$

(3.330)

> $R_L := collect(R_{L0} + R_{L1} + R_{L2}, t)$

$$R_L := \left(v_\varepsilon\left(-\frac{1}{6}\frac{\beta_{lm} I_{m0} \Lambda_l}{N_l} - \frac{1}{6}(\mu_l + v_\varepsilon)\Lambda_l\right) + \frac{1}{6}\frac{\tau_l \beta_{lm} I_{m0} \Lambda_l}{N_l} - \frac{1}{6}(\mu_l + \gamma_l) v_\varepsilon \Lambda_l\right) t^3 + \left(\frac{1}{2} v_\varepsilon \Lambda_l + v_\varepsilon\left(-\frac{1}{2}\frac{\beta_{lm} I_{m0} S_{l0}}{N_l} - \frac{1}{2}(\mu_l + v_\varepsilon) S_{l0} + \frac{1}{2} R_{l0}\gamma_l\right) + \tau_l\left(\frac{1}{2}\frac{\beta_{lm} I_{m0} S_{l0}}{N_l} - \frac{1}{2}(\mu_l + \tau_l + \varepsilon_l) E_{l0}\right) + \frac{1}{2}\tau_l(\varepsilon_l E_{l0} - (\mu_l + c_l + \tau_l) I_{l0}) - (\mu_l + \gamma_l)\left(\frac{1}{2} S_{l0} v_\varepsilon + \frac{1}{2} E_{l0}\tau_l + \frac{1}{2} I_{l0}\tau_l - \frac{1}{2}(\mu_l + \gamma_l) R_{l0}\right)\right) t^2 + (S_{l0} v_\varepsilon + E_{l0}\tau_l + I_{l0}\tau_l - (\mu_l + \gamma_l) R_{l0}) t + R_{l0}$$

(3.331)

> $S_M := collect(S_{M0} + S_{M1} + S_{M2}, t)$

$$
\begin{aligned}
S_M := \Bigg(& -\frac{1}{N_m}\Bigg(\beta_{ml}\bigg(\frac{1}{3}\,\mathrm{I}_{l0}\bigg(-\frac{1}{2}\,\frac{\beta_{ml}\,\mathrm{I}_{l0}\,\Lambda_m}{N_m}-\frac{1}{2}\,\frac{\beta_{mh}\,\mathrm{I}_{h0}\,\Lambda_m}{N_m}-\frac{1}{2}\,(\mu_m+\delta_m)\,\Lambda_m\bigg) \\
& +\frac{1}{3}\,\big(\varepsilon_l\,E_{l0}-(\mu_l+c_l+\tau_l)\,\mathrm{I}_{l0}\big)\,\Lambda_m\bigg)\Bigg)-\frac{1}{N_m}\Bigg(\beta_{mh}\bigg(\frac{1}{3}\,\mathrm{I}_{h0}\bigg(-\frac{1}{2}\,\frac{\beta_{ml}\,\mathrm{I}_{l0}\,\Lambda_m}{N_m} \\
& -\frac{1}{2}\,\frac{\beta_{mh}\,\mathrm{I}_{h0}\,\Lambda_m}{N_m}-\frac{1}{2}\,(\mu_m+\delta_m)\,\Lambda_m\bigg)+\frac{1}{3}\,\big(\varepsilon_h\,E_{h0}-(\mu_h+\tau_h)\,\mathrm{I}_{h0}\big)\,\Lambda_m\bigg)\Bigg)-(\mu_m \\
& +\delta_m)\bigg(-\frac{1}{6}\,\frac{\beta_{ml}\,\mathrm{I}_{l0}\,\Lambda_m}{N_m}-\frac{1}{6}\,\frac{\beta_{mh}\,\mathrm{I}_{h0}\,\Lambda_m}{N_m}-\frac{1}{6}\,(\mu_m+\delta_m)\,\Lambda_m\bigg)\Bigg)\,t^3+\Bigg(\\
& -\frac{1}{2}\,\frac{\beta_{ml}\,\mathrm{I}_{l0}\,\Lambda_m}{N_m}-\frac{1}{2}\,\frac{\beta_{mh}\,\mathrm{I}_{h0}\,\Lambda_m}{N_m}-\frac{1}{2}\,(\mu_m+\delta_m)\,\Lambda_m-\frac{1}{N_m}\Bigg(\beta_{ml}\bigg(\frac{1}{2}\,\mathrm{I}_{l0}\bigg(\\
& -\frac{\beta_{ml}\,\mathrm{I}_{l0}\,S_{m0}}{N_m}-\frac{\beta_{mh}\,\mathrm{I}_{h0}\,S_{m0}}{N_m}-(\mu_m+\delta_m)\,S_{m0}\bigg)+\frac{1}{2}\,\big(\varepsilon_l\,E_{l0}-(\mu_l+c_l+\tau_l)\,\mathrm{I}_{l0}\big)\,S_{m0}\bigg)\Bigg) \\
& -\frac{1}{N_m}\Bigg(\beta_{mh}\bigg(\frac{1}{2}\,\mathrm{I}_{h0}\bigg(-\frac{\beta_{ml}\,\mathrm{I}_{l0}\,S_{m0}}{N_m}-\frac{\beta_{mh}\,\mathrm{I}_{h0}\,S_{m0}}{N_m}-(\mu_m+\delta_m)\,S_{m0}\bigg)+\frac{1}{2}\,\big(\varepsilon_h\,E_{h0} \\
& -(\mu_h+\tau_h)\,\mathrm{I}_{h0}\big)\,S_{m0}\bigg)\Bigg)-(\mu_m+\delta_m)\bigg(-\frac{1}{2}\,\frac{\beta_{ml}\,\mathrm{I}_{l0}\,S_{m0}}{N_m}-\frac{1}{2}\,\frac{\beta_{mh}\,\mathrm{I}_{h0}\,S_{m0}}{N_m}-\frac{1}{2}\,(\mu_m \\
& +\delta_m)\,S_{m0}\bigg)\Bigg)\,t^2+\bigg(\Lambda_m-\frac{\beta_{ml}\,\mathrm{I}_{l0}\,S_{m0}}{N_m}-\frac{\beta_{mh}\,\mathrm{I}_{h0}\,S_{m0}}{N_m}-(\mu_m+\delta_m)\,S_{m0}\bigg)\,t+S_{m0}
\end{aligned}
$$

(3.332)

> $\mathrm{I}_M := collect(\mathrm{I}_{M0}+\mathrm{I}_{M1}+\mathrm{I}_{M2}, t)$

$$
\begin{aligned}
I_M := \Bigg(& \frac{1}{N_m} \bigg(\beta_{ml} \bigg(\frac{1}{3} I_{l0} \bigg(-\frac{1}{2} \frac{\beta_{ml} I_{l0} \Lambda_m}{N_m} - \frac{1}{2} \frac{\beta_{mh} I_{h0} \Lambda_m}{N_m} - \frac{1}{2} (\mu_m + \delta_m) \Lambda_m \bigg) \\
& + \frac{1}{3} \big(\varepsilon_l E_{l0} - (\mu_l + c_l + \tau_l) I_{l0} \big) \Lambda_m \bigg) \bigg) + \frac{1}{N_m} \bigg(\beta_{mh} \bigg(\frac{1}{3} I_{h0} \bigg(-\frac{1}{2} \frac{\beta_{ml} I_{l0} \Lambda_m}{N_m} \\
& - \frac{1}{2} \frac{\beta_{mh} I_{h0} \Lambda_m}{N_m} - \frac{1}{2} (\mu_m + \delta_m) \Lambda_m \bigg) + \frac{1}{3} \big(\varepsilon_h E_{h0} - (\mu_h + \tau_h) I_{h0} \big) \Lambda_m \bigg) \bigg) - (\mu_m \\
& + \sigma_m) \bigg(\frac{1}{6} \frac{\beta_{ml} I_{l0} \Lambda_m}{N_m} + \frac{1}{6} \frac{\beta_{mh} I_{h0} \Lambda_m}{N_m} \bigg) \Bigg) t^3 + \Bigg(\frac{1}{2} \frac{\beta_{ml} I_{l0} \Lambda_m}{N_m} + \frac{1}{2} \frac{\beta_{mh} I_{h0} \Lambda_m}{N_m} \\
& + \frac{1}{N_m} \bigg(\beta_{ml} \bigg(\frac{1}{2} I_{l0} \bigg(-\frac{\beta_{ml} I_{l0} S_{m0}}{N_m} - \frac{\beta_{mh} I_{h0} S_{m0}}{N_m} - (\mu_m + \delta_m) S_{m0} \bigg) + \frac{1}{2} \big(\varepsilon_l E_{l0} - (\mu_l \\
& + c_l + \tau_l) I_{l0} \big) S_{m0} \bigg) \bigg) + \frac{1}{N_m} \bigg(\beta_{mh} \bigg(\frac{1}{2} I_{h0} \bigg(-\frac{\beta_{ml} I_{l0} S_{m0}}{N_m} - \frac{\beta_{mh} I_{h0} S_{m0}}{N_m} - (\mu_m \\
& + \delta_m) S_{m0} \bigg) + \frac{1}{2} \big(\varepsilon_h E_{h0} - (\mu_h + \tau_h) I_{h0} \big) S_{m0} \bigg) \bigg) - (\mu_m + \sigma_m) \bigg(\frac{1}{2} \frac{\beta_{ml} I_{l0} S_{m0}}{N_m} \\
& + \frac{1}{2} \frac{\beta_{mh} I_{h0} S_{m0}}{N_m} - \frac{1}{2} (\mu_m + \sigma_m) I_{m0} \bigg) \Bigg) t^2 + \bigg(\frac{\beta_{ml} I_{l0} S_{m0}}{N_m} + \frac{\beta_{mh} I_{h0} S_{m0}}{N_m} - (\mu_m \\
& + \sigma_m) I_{m0} \bigg) t + I_{m0}
\end{aligned}
\tag{3.333}
$$

APPENDIX B: Maple Code for the Sensitivity Analysis of Effective Reproduction number.

> *restart*;

> $k_1 := \varepsilon_h + \mu_h;\ k_2 := \tau_h + \mu_h;\ k_3 := \gamma_h + \mu_h;\ k_4 := v_\varepsilon + \mu_l;\ k_5 := \varepsilon_l + \mu_l + \tau_l;\ k_6 := \varepsilon_l + \mu_l + c_l;\ k_7 := \gamma_l + \mu_l;\ k_8 := \delta_m + \mu_m;$

$$k_1 := \varepsilon_h + \mu_h$$
$$k_2 := \tau_h + \mu_h$$
$$k_3 := \gamma_h + \mu_h$$
$$k_4 := v_\varepsilon + \mu_l$$
$$k_5 := \varepsilon_l + \mu_l + \tau_l$$
$$k_6 := \varepsilon_l + \mu_l + c_l$$
$$k_7 := \gamma_l + \mu_l$$
$$k_8 := \delta_m + \mu_m$$

> $R_c := \frac{\sqrt{k_1 k_2 k_8 k_5 k_6 (k_7 + v_\varepsilon)(\beta_{hm}\beta_{mh} k_5 k_6 k_7 \varepsilon_h + \beta_{hm}\beta_{mh} k_5 k_6 v_\varepsilon \varepsilon_h + \beta_{lm}\beta_{ml} k_1 k_2 k_7 \varepsilon_l)}}{k_1 k_2 k_8 k_5 k_6 (k_7 + v_\varepsilon)}$

$$R_c := \left((\varepsilon_h + \mu_h)(\tau_h + \mu_h)(\delta_m + \mu_m)(\varepsilon_l + \mu_l + \tau_l)(\varepsilon_l + \mu_l + c_l)(\gamma_l + \mu_l + v_\varepsilon)\left(\beta_{hm}\beta_{mh}(\varepsilon_l + \mu_l + \tau_l)(\varepsilon_l + \mu_l + c_l)(\gamma_l + \mu_l)\varepsilon_h + \beta_{hm}\beta_{mh}(\varepsilon_l + \mu_l + \tau_l)(\varepsilon_l + \mu_l + c_l) v_\varepsilon \varepsilon_h + \beta_{lm}\beta_{ml}(\varepsilon_h + \mu_h)(\tau_h + \mu_h)(\gamma_l + \mu_l)\varepsilon_l\right)\right)^{1/2} \Big/ \left((\varepsilon_h + \mu_h)(\tau_h + \mu_h)(\delta_m + \mu_m)(\varepsilon_l + \mu_l + \tau_l)(\varepsilon_l + \mu_l + c_l)(\gamma_l + \mu_l + v_\varepsilon)\right)$$

> $A := \frac{\beta_{hm}}{R_c} \cdot \frac{d}{d\beta_{hm}} R_c;$

$$A := \frac{1}{2}\left(\left(\beta_{hm}\left(\varepsilon_h+\mu_h\right)\left(\tau_h+\mu_h\right)\left(\delta_m+\mu_m\right)\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)\left(\gamma_l+\mu_l+v_\varepsilon\right)\right)\Big/\left(\left(\varepsilon_h+\mu_h\right)\left(\tau_h+\mu_h\right)\left(\delta_m+\mu_m\right)\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)\left(\gamma_l+\mu_l+v_\varepsilon\right)\left(\beta_{hm}\beta_{mh}\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)\left(\gamma_l+\mu_l\right)\varepsilon_h+\beta_{hm}\beta_{mh}\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)v_\varepsilon\varepsilon_h+\beta_{lm}\beta_{ml}\left(\varepsilon_h+\mu_h\right)\left(\tau_h+\mu_h\right)\left(\gamma_l+\mu_l\right)\varepsilon_l\right)\right)^{1/2}\right)\cdot\left(\left(\beta_{mh}\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)\left(\gamma_l+\mu_l\right)\varepsilon_h+\beta_{mh}\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)v_\varepsilon\varepsilon_h\right)\Big/\left(\left(\varepsilon_h+\mu_h\right)\left(\tau_h+\mu_h\right)\left(\delta_m+\mu_m\right)\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)\left(\gamma_l+\mu_l+v_\varepsilon\right)\left(\beta_{hm}\beta_{mh}\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)\left(\gamma_l+\mu_l\right)\varepsilon_h+\beta_{hm}\beta_{mh}\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)v_\varepsilon\varepsilon_h+\beta_{lm}\beta_{ml}\left(\varepsilon_h+\mu_h\right)\left(\tau_h+\mu_h\right)\left(\gamma_l+\mu_l\right)\varepsilon_l\right)\right)^{1/2}\right)$$

> $eval\left(\boldsymbol{A}, \left[\beta_{hm}=0.001, \beta_{lm}=0.61, \beta_{ml}=0.25, \beta_{mh}=0.25, \mu_l=0.5, \varepsilon_h=0.25, \varepsilon_l=0.25, \gamma_l=0.25, k_1=0.26, k_2=0.26, k_3=0.26, k_4=0.75, k_5=1, k_6=1, k_7=0.75, k_8=0.92, v_\varepsilon=0.25\right]\right)$

$$0.01566072602$$

> $A1 := \frac{\beta_{lm}}{R_c} \cdot \frac{d}{d\beta_{lm}} R_c;$

$$A1 := \frac{1}{2}\left(\left(\beta_{lm}\left(\varepsilon_h+\mu_h\right)\left(\tau_h+\mu_h\right)\left(\delta_m+\mu_m\right)\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)\left(\gamma_l+\mu_l+v_\varepsilon\right)\right)\Big/\left(\left(\varepsilon_h+\mu_h\right)\left(\tau_h+\mu_h\right)\left(\delta_m+\mu_m\right)\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)\left(\gamma_l+\mu_l+v_\varepsilon\right)\left(\beta_{hm}\beta_{mh}\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)\left(\gamma_l+\mu_l\right)\varepsilon_h+\beta_{hm}\beta_{mh}\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)v_\varepsilon\varepsilon_h+\beta_{lm}\beta_{ml}\left(\varepsilon_h+\mu_h\right)\left(\tau_h+\mu_h\right)\left(\gamma_l+\mu_l\right)\varepsilon_l\right)\right)^{1/2}\right)\cdot\left(\left(\left(\varepsilon_h+\mu_h\right)\left(\tau_h+\mu_h\right)\beta_{ml}\left(\gamma_l+\mu_l\right)\varepsilon_l\right)\Big/\left(\left(\varepsilon_h+\mu_h\right)\left(\tau_h+\mu_h\right)\left(\delta_m+\mu_m\right)\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)\left(\gamma_l+\mu_l+v_\varepsilon\right)\left(\beta_{hm}\beta_{mh}\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)\left(\gamma_l+\mu_l\right)\varepsilon_h+\beta_{hm}\beta_{mh}\left(\varepsilon_l+\mu_l+\tau_l\right)\left(\varepsilon_l+\mu_l+c_l\right)v_\varepsilon\varepsilon_h+\beta_{lm}\beta_{ml}\left(\varepsilon_h+\mu_h\right)\left(\tau_h+\mu_h\right)\left(\gamma_l+\mu_l\right)\varepsilon_l\right)\right)^{1/2}\right)$$

> $eval\left(\boldsymbol{A1}, \left[\beta_{hm} = 0.001, \beta_{lm} = 0.61, \beta_{ml} = 0.25, \beta_{mh} = 0.25, \mu_l = 0.5, \varepsilon_h = 0.25, \varepsilon_l = 0.25, \gamma_l = 0.25, k_1 = 0.26, k_2 = 0.26, k_3 = 0.26, k_4 = 0.75, k_5 = 1, k_6 = 1, k_7 = 0.75, k_8 = 0.92, v_\varepsilon = 0.25\right]\right)$

$$0.4843392736$$

> $eval\left(\boldsymbol{A2}, \left[\beta_{hm} = 0.001, \beta_{lm} = 0.61, \beta_{ml} = 0.25, \beta_{mh} = 0.25, \mu_l = 0.5, \varepsilon_h = 0.25, \varepsilon_l = 0.25, \gamma_l = 0.25, k_1 = 0.26, k_2 = 0.26, k_3 = 0.26, k_4 = 0.75, k_5 = 1, k_6 = 1, k_7 = 0.75, k_8 = 0.92, v_\varepsilon = 0.25\right]\right)$

$$0.01566072602$$

> $eval\left(\boldsymbol{A3}, \left[\beta_{hm} = 0.001, \beta_{lm} = 0.61, \beta_{ml} = 0.25, \beta_{mh} = 0.25, \mu_l = 0.5, \varepsilon_h = 0.25, \varepsilon_l = 0.25, \gamma_l = 0.25, k_1 = 0.26, k_2 = 0.26, k_3 = 0.26, k_4 = 0.75, k_5 = 1, k_6 = 1, k_7 = 0.75, k_8 = 0.92, v_\varepsilon = 0.25\right]\right)$

$$0.4843392736$$

> $eval\left(\boldsymbol{A4}, \left[\beta_{hm} = 0.001, \beta_{lm} = 0.61, \beta_{ml} = 0.25, \beta_{mh} = 0.25, \mu_l = 0.5, \varepsilon_h = 0.25, \varepsilon_l = 0.25, \gamma_l = 0.25, k_1 = 0.26, k_2 = 0.26, k_3 = 0.26, k_4 = 0.75, k_5 = 1, k_6 = 1, k_7 = 0.75, k_8 = 0.92, v_\varepsilon = 0.25\right]\right)$

$$0.2421696368$$

> $eval\left(\boldsymbol{A5}, \left[\beta_{hm} = 0.001, \beta_{lm} = 0.61, \beta_{ml} = 0.25, \beta_{mh} = 0.25, \mu_l = 0.5, \varepsilon_h = 0.25, \varepsilon_l = 0.25, \gamma_l = 0.25, k_1 = 0.26, k_2 = 0.26, k_3 = 0.26, k_4 = 0.75, k_5 = 1, k_6 = 1, k_7 = 0.75, k_8 = 0.92, v_\varepsilon = 0.25\right]\right)$

$$0.04036160614$$

> $eval\left(\boldsymbol{A6}, \left[\beta_{hm} = 0.001, \beta_{lm} = 0.61, \beta_{ml} = 0.25, \beta_{mh} = 0.25, \mu_l = 0.5, \varepsilon_h = 0.25, \varepsilon_l = 0.25, \gamma_l = 0.25, k_1 = 0.26, k_2 = 0.26, k_3 = 0.26, k_4 = 0.75, k_5 = 1, k_6 = 1, k_7 = 0.75, k_8 = 0.92, v_\varepsilon = 0.25\right]\right)$

$$0.0006023356211$$

> $eval\left(\boldsymbol{A7}, \left[\beta_{hm} = 0.001, \beta_{lm} = 0.61, \beta_{ml} = 0.25, \beta_{mh} = 0.25, \mu_l = 0.5, \varepsilon_h = 0.25, \varepsilon_l = 0.25, \gamma_l = 0.25, k_1 = 0.26, k_2 = 0.26, k_3 = 0.26, k_4 = 0.75, k_5 = 1, k_6 = 1, k_7 = 0.75, k_8 = 0.92, v_\varepsilon = 0.25\right]\right)$

$$-0.4036160614$$

> $eval\left(\boldsymbol{A8}, \left[\beta_{hm} = 0.001, \beta_{lm} = 0.61, \beta_{ml} = 0.25, \beta_{mh} = 0.25, \mu_{l} = 0.5, \varepsilon_{h} = 0.25, \varepsilon_{l} = 0.25, \gamma_{l} = 0.25, k_{1} = 0.26, k_{2} = 0.26, k_{3} = 0.26, k_{4} = 0.75, k_{5} = 1, k_{6} = 1, k_{7} = 0.75, k_{8} = 0.92, v_{\varepsilon} = 0.25\right]\right)$

$$-0.1210848184$$