MATHEMATICAL MODEL FOR THE TRANSMISSION DYNAMICS OF COVID-19 PANDEMIC WITH CONTACT TRACING AND FULL RECOVERY

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA IN PARTIAL FULFILLMENT OF THE REQIUREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF TECHNOLOGY (MTech) IN MATHEMATICS

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ABSTRACT

In this work, a Mathematical Model for the Transmission Dynamics of Covid-19 Pandemic with Contact Tracing and Full Recovery was formulated and carefully analyzed. The total population is divided into six compartments that reflects Covid-19 dynamics. The equilibrium points of the model were determined and analyzed for stability. The analysis of the disease-free equilibrium state shows that it is stable under certain conditions. The equilibrium states were obtained and analyzed for their stability relatively to the effective reproduction number. The result shows that, the disease-free equilibrium state was stable and the criteria for stability of the endemic equilibrium state are established. The study showed that the Covid-19 infectious free equilibrium is locally and globally asymptotically stable $R_0 < 1$. The analytical solution was obtained using Homotopy perturbation Method (HPM) and effective reproduction number was computed in order to measure the relative impact for individual or combined intervention for effective disease control. The result of the numerical simulation shows that at high vaccination rate of the Human the Covid-19 virus can be eradicated completely which will also eradicate the tracing of the disease from Human.

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ABBREVIATION, GLOSSARIES AND SYMBOLS

γ_1 — A proportion of Hospitalized individuals who leave the compartment to recovered class

δ_H - — Death due to COVID-19 of Hospitalized individuals

Λ — Recruitment Rate

β — Contact rate between susceptible Individuals and Infected Individuals

μ — Natural Death rate

θ — Proportion of Exposed Individuals who are successfully quarantined

ε_H — Modification factor representing the reduction of contacts between susceptible

individuals and hospitalized individuals

$(1-\theta_e)$ — Fraction of exposed individuals who show infections at the end of the incubationperiod.

$(1-\theta_q)$ — Fraction of quarantine individuals who show symptoms at the end of the incubationperiod.

$\theta_q \rho_q$ — Proportion of individuals Quarantined who leave the compartment to susceptible class

$\theta_e \rho_e$ Proportion of untraced individuals who leave the compartment to susceptible class

γ_H The rate at which infected individuals are hospitalized

δ_I Death due to COVID-19 of Infected but not hospitalized individuals

$E(t)$ Untraced individuals who are exposed to COVID-19 at time t

$Q(t)$ Traced Individuals who are exposed to COVID-19 at time t.

$R(t)$ Fully Recovered individuals at time t.

$S(t)$ Susceptible individuals at time t.

$I(t)$ Infected individuals at time t.

$H(t)$ Hospitalized individuals at time t.

CDC Centre for Disease Control

DSF Disease-Free Equilibrium

HPM Homotopy Perturbation Method

N Total Population

WHO World Health Organization

EE Endemic Equilibrium

CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

Coronavirus disease 2019 (COVID-19) is defined as illness caused by a novel coronavirus now called severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2; formerly called 2019-nCoV), which was first identified amid an outbreak of respiratory illness cases in Wuhan City, Hubei Province, China Center for Disease Control (CDC, 2019a). Since December 2019, many unexplained cases of pneumonia with cough, dyspnea, fatigue, and fever as the main symptoms have occurred in Wuhan, China in a short period of time, (Huang *et al*. 2020).

China's health authorities and CDC quickly identified the pathogen of such cases as a new type of coronavirus, which the World Health Organization (WHO) named COVID-19 on January 10, 2020, (WHO, 2020). On January 22, 2020, the Information Office of the State Council of the People's Republic of China held a press conference introduced the relevant situation of pneumonia prevention and control of new coronavirus infection. On the same day, the People's Republic of China's CDC released a plan for the prevention and control of pneumonitis of new coronavirus infection, including the COVID-19 epidemic Research, specimen collection and testing, tracking and management of close contacts, and propaganda, education and risk communication to the public National Health Commission of the People's Republic of China. 2020.

Wuhan, China is the origin of COVID-19 and one of the Cities most affected by it. The Mayor of Wuhan stated at a press conference on January 31, 2020 that Wuhan is urgently building Vulcan Mountain Hospital and Thunder Mountain Hospital patients will be officially admitted on February 3 and February 6, (Health Commission of Hubei Province. 2020).

February 6, 2020, a total of 31,161 confirmed cases, including 636 deaths, were reported in the Chinese mainland, 22,112 confirmed cases, including 618 deaths, were reported in Hubei province, and 11,618 confirmed cases, including 478 deaths, and were reported in Wuhan city. The spread of COVID-19 and various interventions have had an incalculable negative impact on People's daily lives and the normal functioning of society. Cities in China's Hubei Province have issued varying degrees of closures and traffic restrictions (Chan *et al.*, 2020).

The coronavirus disease 2019 (COVID-19) has led to high morbidity and mortality in China, Europe, America and the Africa, triggering unprecedented public health crises throughout the world. On March 11, 2020, the World Health Organization (WHO) declared COVID-19 as a global pandemic. COVID-19 is caused by a novel coronavirus which is now named severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2). SARS-CoV-2 is regarded as the third zoonotic human coronavirus emerging in the current century, after SARS-CoV in 2002 and the Middle East respiratory syndrome coronavirus (MERSCoV) in 2012.

Nigeria, a country with approximately 207 000 000 populace located in West Africa recorded its first case of COVID-19 in February 27, 2020 (NCDC, 2020) and as at 30th of March, 2020 (22:00 WAT) 131 individuals have been infected with the virus. The total number of recoveries being 8 and deaths, 2. Lagos State (Nigeria) has a larger number of infected persons followed by the country's capital Abuja. Meanwhile lock down was declared in major cities of the country (Mbah, 2020), and entry flights from countries with over 1000 cases have been banned (Stephanie & Adebayo, 2020).

In fact, there are many imminent questions about the spread of COVID-19. How many people will be infected tomorrow? When will the inflection point of the infection rate appear? How many people will be infected during the peak period? Can existing interventions effectively control the COVID-19? What mathematical models are available to help us answer these questions? The COVID-19 is a novel coronavirus that was only discovered in December 2019, so data on the outbreak is still insufficient, and medical means such as clinical trials are still in a difficult exploratory stage (Wang *et al.*, 2020). So far, epidemic data have been difficult to apply directly to existing mathematical models, and questions need to be addressed as to how effective the existing emergency response has been and how to invest medical resources more scientifically in the future and so on.

1.2 Statement of the Research Problem

COVID-19 is one of the world infectious virus. According to World Health Organization WHO (2020), the morbidity is estimated to be above 26,000,000 cases and the mortality is above 1,000,000 globally, of which 30% are in Africa. There is a need to understand the transmission, prevention and prediction of the outbreak of the virus and the fact that the population growth is an important factor that contributes to the increase of spread of some vector-borne virus in developing countries. Most of the mathematical modelling of COVID-19 considered spreading with asymptomatic infected and interacting people, but in this research work, tracing and full Recovery will be consider.

1.3 Aim and Objectives of the Study

The aim of this research work is to develop and analyze a mathematical model for the transmission dynamics of COVID-19 pandemic with contact tracing and full recovery.

The objectives are to:

i. formulate a mathematical model for the transmission dynamics of the COVID-19 virus.
ii. examine the epidemiological well posesness of the model.
iii. obtain the Disease-Free Equilibrium (DFE) and the Endemic Equilibrium (EE).
iv. compute the basic reproduction number of the model
v. analyze the stability of DFE and EE by using basic reproduction number
vi. solve the model equations analytically use (HPM) to solve the system of six ordinary differential equations.
vii. obtain the numerical simulations of the model using computer software

1.4 Motivation for the Study

This research work is motivated owing to the unique spread dynamics of COVID-19 and seeks to look for the prevention of the virus through Mathematical Modeling. The researcher is also inspired to make vital information available for policy makers in the fight against COVID-19 virus in Nigeria.

1.5 Justification of the Study

The need for detailed and qualitative scholarly work on COVID-19 virus justifies the study. The thesis may also aid mathematicians and research scientists to further develop suitable models to help public health professionals to make better strategies for controlling the virus.

1.6 Scope and Limitations of the Study

The model subdivides the population into six mutually-exclusive classes namely; Susceptible (S), Traced individuals who are exposed to COVID-19 (Q), Untraced individuals who are exposed to COVID-19 (E), Infected (I), Hospitalized (H) and Full Recovered (R)

Limitations;

The study is limited to the mathematical modeling of the COVID-19 pandemic with contact tracing and full recovery individual class.

1.7 Definition of Terms

Equilibrium: means a state of rest or balancing due to equal action of opposing force of a body.

Stable Equilibrium: is the state of a system such that when slightly moved tends to come back to its original state of rest.

Susceptible: These are individuals who are not yet infected but can still be infected.

Infected: These are group of persons who have Covid-19 infection.

Recovered: These are group of persons who have been treated and recovered from the Covid-19 illness.

Epidemiology: is the study of Disease in population**.**

Mathematical Model: is the process of representing a phenomenon in mathematical term.

Differential Equation: is a mathematical equation that relates some function with its derivatives.

Disease Free Equilibrium (D.F.E): is globally asymptotically stable when the reproduction number is less than one.

Endemic Equilibrium (E.E): is globally asymptotically stable when the reproduction number is greater than one.

Endemic: is when an infection in a population is maintained in the population without the need for external input.

CHAPETR TWO

2.0 LITERATURE REVIEW

2.1 Overview of COVID-19 Virus

Coronavirus disease 2019 (COVID-19) is defined as illness caused by a novel coronavirus now called severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2; formerly called 2019-nCoV), which was first identified amid an outbreak of respiratory illness cases in Wuhan City, Hubei Province, China (CDC, 2019b).

It was initially reported to the WHO on December 31, 2019. On January 30, 2020, the WHO declared the COVID-19 outbreak a global health emergency (Ramzy, 2020). On March 11, 2020, the WHO declared COVID-19 a global pandemic, its first such designation since declaring H1N1 influenza a pandemic in 2009 (WHO, 2020). Illness caused by SARS-CoV-2 was termed COVID-19 by the WHO, the acronym derived from " coronavirus disease 2019. " The name was chosen to avoid stigmatizing the virus's origins in terms of populations, geography, or animal associations.

On February 11, 2020, the Coronavirus Study Group of the International Committee on Taxonomy of Viruses issued a statement announcing an official designation for the novel virus: severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2), (Gorbalenya *et al.*, 2020).

The Centers for Disease Control and Prevention (CDC) has estimated that SARS-CoV-2 entered the United States in late January or early February, establishing low-level community spread before being noticed (CDC, 2020). Since that time, the United States has experienced widespread infections, with nearly 200,000 deaths reported as of September 20, 2020.

On April 3, 2020, the CDC issued a recommendation that the general public, even those without symptoms, should begin wearing face coverings in public settings where social-distancing measures are difficult to maintain in order to abate the spread of COVID-19.

The CDC had postulated that this situation could result in large numbers of patients requiring medical care concurrently, resulting in overloaded public health and healthcare systems and, potentially, elevated rates of hospitalizations and deaths. The CDC advised that non-pharmaceutical inter venations (NPIs) will serve as the most important response strategy in attempting to delay viral spread and to reduce disease impact (CDC, 2020).

2.1.1 Signs and symptoms of COVID-19

Presentations of COVID-19 have ranged from asymptomatic/mild symptoms to severe illness and mortality. Symptoms may develop 2 days to 2 weeks following exposure to the virus.

A pooled analysis of 181 confirmed cases of COVID-19 outside Wuhan, China, found the mean incubation period to be 5.1 days and that 97.5% of individuals who developed symptoms did so within 11.5 days of infection, (Lauer *et al.*, 2020)

Wu and McGoogan reported that, among 72,314 COVID-19 cases reported to the Chinese Center for disease Control and Prevention (CCDC), 81% were mild (absent or mild pneumonia), 14% were severe (hypoxia, dyspnea, >50% lung involvement within 24-48 hours), 5% were critical (shock, respiratory failure, multi organ dysfunction), and 2.3% were fatal, (Wu and McGoogan, 2020).

The following symptoms may indicate COVID-19

i. Fever or chills

ii. Cough

iii. Shortness of breath or difficulty breathing

iv. Fatigue

v. Muscle or body aches

vi. Headache

vii. New loss of taste or smell

viii. Sore throat

ix. Congestion or runny nose

x. Nausea or vomiting

xi. Diarrhea

Other reported symptoms have included the following:

a. Sputum production

b. Malaise

c. Respiratory distress

d. Neurologic (eg, headache, altered mentality)

The most common serious manifestation of COVID-19 appears to be pneumonia.

A complete or full loss of the sense of smell (anosmia) has been reported as a potential history finding in patients eventually diagnosed with COVID-19, (Rabin, 2020). A phone survey of outpatients with mildly symptomatic COVID-19 found that 64.4% (130 of 202) reported any altered sense of smell or taste (Spinato *et al.,* 2020)

2.1.2 Diagnosis of COVID-19

COVID-19 should be considered a possibility in (1) patients with respiratory tract symptoms and newly onset fever or (2) in patients with severe lower respiratory tract symptoms with no clear cause. Suspicion is increased if such patients have been in an area with community transmission of SARSCoV-2 or have been in close contact with an individual with confirmed or suspected COVID-19 in the preceding 14 days.

Microbiologic (PCR) testing is required for definitive diagnosis. At present, such testing is of limited availability.

Patients who do not require emergency care are encouraged to contact their healthcare provider over the phone. Patients with suspected COVID-19 who present to a healthcare facility should prompt infection-control measures. They should be evaluated in a private room with the door closed (an airborne infection isolation room is ideal) and asked to wear a surgical mask. All other standard contact and airborne precautions should be observed, and treating healthcare personnel should wear eye protection (CDC, 2020).

2.1.3 Prevention of infection of COVID-19

Amid human COVID-19 outbreaks, close contact with different patients is the most noteworthy hazard factor for COVID-19 infection disease. Without particular treatment or vaccine, the best way to diminish contamination in individuals is by bringing issues to light of the hazard factors and instructing individuals about the measures they can take to lessen presentation to the infection. Surveillance measures and rapid identification of new cases is critical for outbreak containment (WHO, 2020).

Public health educational messages should focus on the following risks:

i. Clean your hand often. Use soap and water, or an alcohol-based hand rub.

ii. Maintain a safe distance from anyone who is coughing or sneezing.

iii. Wear a mask when physical distancing is not possible

iv. Don’t touch your eyes, nose or mouth.

v. Cover your nose and mouth with your bent elbow or a tissue when you cough or sneeze.

vi. Stay home if you feel unwell

vii. If you have a fever, cough and difficulty breathing, seek medical attention.

2.1.4 Treatment of COVID-19 positive patients

There are now series of vaccines among these are AstraZeneca, Covaxin, Sinovac, Sinopharm, Johnson and Johnson (J&J), Janssen and Moderna, Pfizer which are found appropriate to control COVID-19 (WHO, 2021).

2.2 Mathematical Models of COVID-19

Mustapha and Hanane (2020) developed a Mathematical modeling of COVID-19 spreading with asymptomatic infected and interacting peoples. It takes account on the asymptomatic people and the strategies involving hospital isolation of the confirmed infected person, quarantine of people contacting them, and the home containment of all population to restrict mobility. They establish a relationship between the containment control coefficient c_0 and the basic reproduction number R_0. Different scenarios are tested with different values of c_0 , for which the stability of a Disease Free Equilibrium (DFE) point is correlated with the condition linking R_0 and c_0

The model Equations is given as Equations (2.1) to (2.6)

$$\frac{dS}{dt} = -\beta S\left(A + \alpha I\right) + \xi Q \tag{2.1}$$

$$\frac{dI}{dt} = \theta Q + \delta A - \left(\mu + d_I\right) I \tag{2.2}$$

$$\frac{dA}{dt} = \beta SA - (\delta + \lambda) A \quad (2.3)$$

$$\frac{dQ}{dt} = \alpha\beta SI - (\xi + \theta) Q \qquad \text{on } [0, t_f] \quad (2.4)$$

$$\frac{dR}{dt} = \mu I + \lambda A \quad (2.5)$$

$$\frac{dD}{dt} = d_I I \quad (2.6)$$

Yichi *et al.* (2020), developed a Mathematical Modeling and Epidemic Prediction of COVID-19 and Its Significance to Epidemic Prevention and Control Measures. They establish the dynamics model of infectious diseases and time series model to predict the trend and short-term prediction of the transmission of COVID-19, which will be conducive to the intervention and prevention of COVID-19 by departments at all levels in mainland China and buy more time for clinical trials.

Based on the transmission mechanism of COVID-19 in the population and the implemented prevention and control measures, they establish the dynamic models of the six chambers, and establish the time series models based on different mathematical formulas according to the variation law of the original data.

The model equations is given as equations (2.7) to (2.12)

$$\frac{dS}{dt} = d_{qs} Q - f - d_{sq} S \quad (2.7)$$

$$\frac{dE}{dt} = f - \varepsilon E - d_{eq} - E \quad (2.8)$$

$$\frac{dD}{dt} = d_{qd}Q + d_{id}I - (\gamma + \delta)D \tag{2.9}$$

$$\frac{dQ}{dt} = d_{eq}E + d_{sq}S - d_{qs}Q - d_{qd}Q \tag{2.10}$$

$$\frac{dI}{dt} = \varepsilon E - d_{id}I - \delta I \tag{2.11}$$

$$\frac{dR}{dt} = \gamma D \tag{2.12}$$

Ivorra *et al.* (2020) developed a mathematical model for the spread of the coronavirus disease 2019 (COVID-19). It is a new $\theta - SEIHRD$ model (not a SIR, SEIR or other general purpose model), which takes into account the known special characteristics of this disease, as the existence of infectious undetected cases and the different sanitary and infectiousness conditions of hospitalized people. In particular, it includes a novel approach that considers the fraction θ of detected cases over the real total infected cases, which allows to study the importance of this ratio on the impact of COVID-19. The model is also able to estimate the needs of beds in hospitals. It is complex enough to capture the most important effects, but also simple enough to allow an affordable identification of its parameters, using the data that authorities report on this pandemic.

They study the particular case of China (including Chinese Mainland, Macao, Hong-Kong and Taiwan, as done by the World Health Organization in its reports on COVID-19), the country spreading the disease, and use its reported data to identify the model parameters, which can be of interest for estimating the spread of COVID-19 in other countries. They show a good agreement between the reported data and the estimations given by our model. They also study the behavior of the outputs returned by our model when considering incomplete reported data (by truncating them at some dates before and after the peak of daily reported cases). By comparing those results, we can estimate the error produced by the model when identifying the parameters at early stages of the pandemic. Finally, taking into account the advantages of the novelties introduced by our model, we study different scenarios to show how different values of the percentage of detected cases would have changed the global magnitude of COVID-19 in China, which can be of interest for policy makers.

Berge *et al.* (2018) proposed a simple mathematical model that incorporates imperfect contact tracing, quarantine and hospitalization (or isolation). The control reproduction number R_c of each sub-model and for the full model are computed. Theoretically, they prove that when R_c is less than one, the corresponding model has a unique globally asymptotically stable disease-free equilibrium. Conversely, when R_c is greater than one, the disease-free equilibrium becomes unstable and a unique globally asymptotically stable endemic equilibrium arises. Furthermore, we numerically support the analytical results and assess the efficiency of different control strategies. Our main observation is that, to eradicate EVD, the combination of high contact tracing (up to 90%) and effective isolation is better than all other control measures, namely: (1) perfect contact tracing, (2) effective isolation or full hospitalization, (3) combination of medium contact tracing and medium isolation.

The model Equations is given as Equations (2.13) to (2.18)

$$\frac{dS}{dt} = \pi - \beta_0 S\left(I + \varepsilon H\right) + p_{qs}\lambda_q Q + p_{us}\lambda\mu U - \mu S \qquad (2.13)$$

$$\frac{dQ}{dt} = p\beta_0 S\left(I + \varepsilon H\right) - \lambda_q Q - \mu Q \qquad (2.14)$$

$$\frac{dU}{dt} = (1 - p)\beta_0 S(I + \varepsilon H) - \lambda_u U - \mu U \tag{2.15}$$

$$\frac{dI}{dt} = (1 - p_{us})\lambda_u U - (\eta_i + d_i + \mu) I \tag{2.16}$$

$$\frac{dH}{dt} = (1 - p_{qs})\lambda_q Q + \eta_i I - (d_h + \mu + \gamma) H \tag{2.17}$$

$$\frac{dR}{dt} = \gamma H - \mu R \tag{2.18}$$

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Development of the Model

In this chapter, we developed and analyzed a mathematical model of transmission dynamics of COVID-19 pandemic with contact tracing and full recovery. Hence, the mathematical model for human transmission dynamics of COVID-19 pandemic were formulated.

Following Berge *et al.* (2018), we denote Λ as the number of Susceptible recruited per unit time (day) as result of birth or immigration and μ the common natural death rate of all individuals. The disease is transmitted through direct contact between Susceptible Individuals (S), Infected Individuals (I), Hospitalized Individuals (H), Untraced Individuals who are exposed to COVID-19 (E), Traced Individuals who are exposed to COVID-19 (Q) and Fully Recovered Individuals (R). Let β be the contact rate between Susceptible Individuals and Infected Individuals (I) and ε_H modification factor represent the reduction of contact between Susceptible Individuals and hospitalized Individuals (H). Since Hospitalized Individuals are isolated, it is reasonable to assume that $0 \le \varepsilon_H \le 1$. After contact between Susceptible and Infective, $\theta_q \rho_q$ is proportion of Individuals Quarantined who leave the compartment to Susceptible class. Based on the medical tests, these quarantined individuals are either hospitalized (positive test) or considered Susceptible after 14 days (negative test). Let $\theta_q \rho_q$ be the exit rate from the quarantine class Q to both classes S and H. A proportion $\theta_q \rho_q$ of the quarantined goes back to the S - Compartment, whereas the remaining proportion $(1-\theta_q)\rho_q$ enters class H. Similarly $(1-\theta)$ is the proportion of untraced individuals who enter the E-Compartment individuals exits the $E-$compartment so that $\theta_e\rho_e E$ and $(1-\theta_e)\rho_e E$ are the number of untraced individuals who enter classes S and I, respectively. Individuals in class I move to class H at rate γ_H and die due to COVID-19 at rate δ_I. Note that γ_H can increase as the number of traced people increases. Since the infected individuals who die out of the hospital can still transmit the disease, we assume that among the $(\delta_I+\mu)I$ dead individuals, a proportion θ is safely buried, while the remaining proportion $(1-\theta)$ can transmit the infection during funerals. Let β be the effective contact rate between Susceptible and COVID-19 virus. We assume $\beta^{'}=\beta\theta$ where $\theta(0\le\theta\le1)$ is modification parameter which accounts for the fact that the number of contacts with a dead individuals is less than that with an active person. Individuals of class H die to COVID-19 virus at rate δ_H and recover at rate γ_I per unit time. We assume that the model parameters are non-negative.

Basic Assumptions

a. We assumed that a susceptible individual can only contact COVID-19 through human to human transmission.

b. We assumed that the population of the infected can be hospitalized at the rate γ_H

c. The natural death rate is time constant, that is at any point in time any individual can die naturally.

d. The recovered individuals after being vaccinated cannot have COVID-19 again

e. There is permanent immunity on recovery

f. Treatment is introduced to the infected population

g. Controls are implemented continuously.

3.2 Model Formulation

We combine the Basic assumptions, model parameters, state the variables and the COVID-19 infection processes to formulate a schematic diagram for COVID-19 infection as show in Figure 3.1.

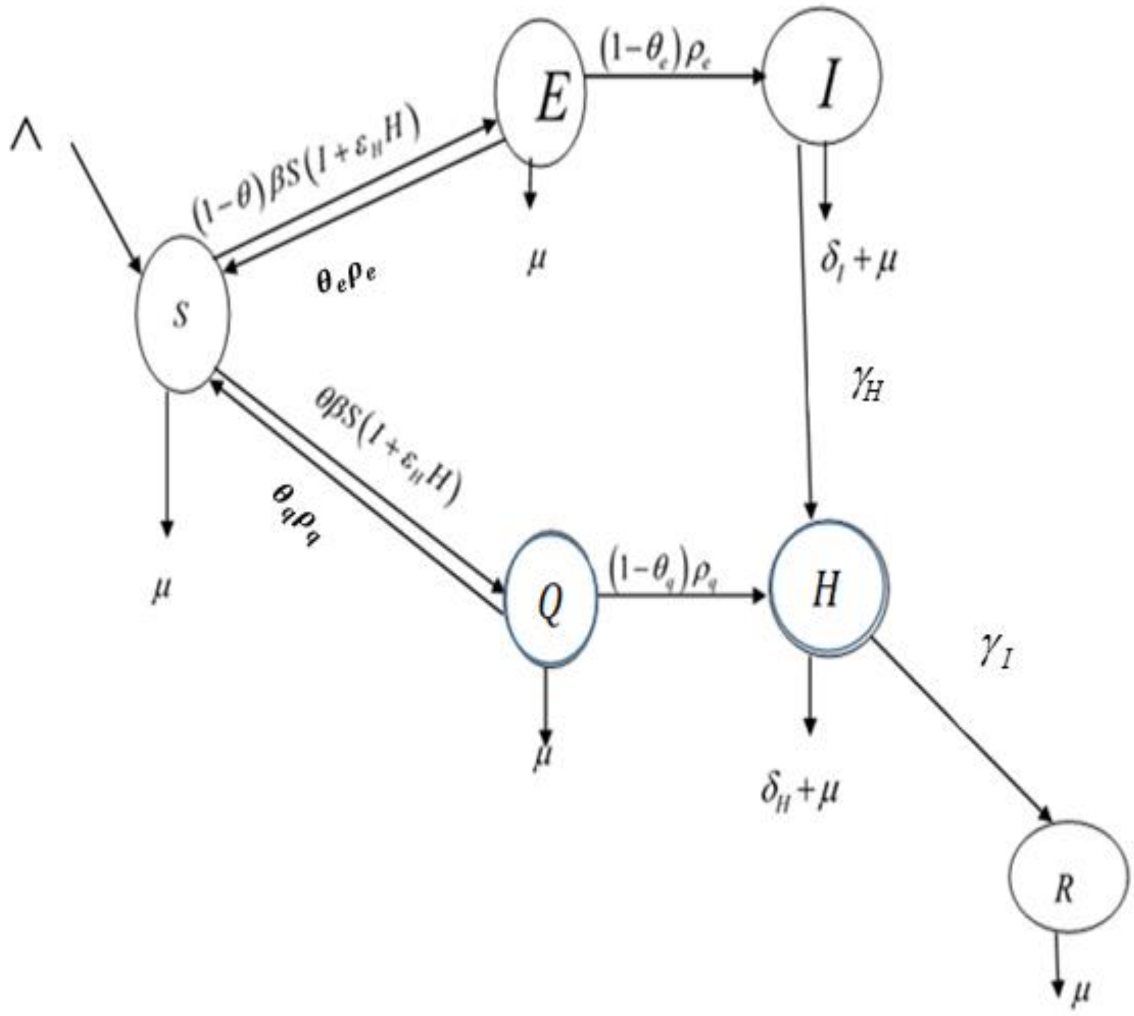


Figure 3.1: Schematic Diagram of the Model

From Figure 3.1 above, we represent this mathematically; we have the following system of differential Equations is given as Equations (3.1) to (3.6).

$$\frac{dS}{dt} = \Lambda - \beta\theta\left(I + \varepsilon_H H\right)S + \theta_e\rho_e E + \theta_q\rho_q Q - \mu S \tag{3.1}$$

$$\frac{dQ}{dt} = \theta\beta S(I + \varepsilon_H H) - \left(1 - \theta_q\right)\rho_q Q - \left(\theta_q\rho_q + \mu\right)Q \tag{3.2}$$

$$\frac{dE}{dt} = (1-\theta)\beta S(I + \varepsilon_H H) - \left(1 - \theta_e\right)\rho_e E - \left(\theta_e\rho_e + \mu\right)E \tag{3.3}$$

$$\frac{dI}{dt} = (1-\theta_e)\rho_e E - (\gamma_H + \delta_I + \mu)I \tag{3.4}$$

$$\frac{dH}{dt} = (1-\theta_q)\rho_q Q - (\gamma_1 + \delta_H + \mu)H + \gamma_H I \tag{3.5}$$

$$\frac{dR}{dt} = \gamma_1 H - \mu R \tag{3.6}$$

3.3 The Positive Invariant Region

The entire population size N can be determined from Equations (3.1) to (3.6) which yield (3.7)

The total population size is $N = S + Q + E + I + H + R$ (3.7)

Adding equation (3.1) into equation (3.6), yield (3.8) and (3.9)

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dQ}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dH}{dt} + \frac{dR}{dt} \tag{3.8}$$

$$\frac{dN}{dt} = \Lambda - \mu S - \mu Q - \mu I - \mu H - \mu R - \delta_I I - \delta_H H \tag{3.9}$$

In the absence of the disease $\left(\delta_I I = \delta_H H = 0\right)$, then equations (3.8) gives the positive invariant region can be obtained by using the following theorem.

Theorem 3.1

The solutions of the system of Equations (3.1) to (3.6) are feasible for $t > 0$ if they are within the invariant region D as in (3.10).

Proof

$$\text{Let } D = (S, Q, E, I, H, R) \varepsilon R_{+}^{6} \tag{3.10}$$

be any solution of the system of Equations (3.1) to (3.6) with non- zero initial conditions.

Assuming there is no disease – induced deaths, Equation (3.9) yield (3.11)

$$\left.\begin{aligned} \frac{dN}{dt} &\leq \Lambda - \mu N \\ \frac{dN}{dt} &+ \mu N \leq \Lambda \end{aligned}\right\} \tag{3.11}$$

The integrating factor for Equation (3.11) gives (3.12) and (3.13)

$$\frac{dN}{dt} + \mu N e^{\mu t} \leq \Lambda e^{\mu t} \tag{3.12}$$

$$\frac{dN}{dt} \leq \mu \tag{3.13}$$

Integrating both sides (3.13), yielded (3.14) and (3.15)

$$N(t) \tag{3.14}$$

$$N(t) = \frac{\Lambda}{\mu} + c \tag{3.15}$$

Applying the initial condition $t=0, N(0)=N_0$ gives (3.16) and (3.17)

$$N_0 \le \frac{\Lambda}{\mu}+c \Rightarrow N_0-\frac{\Lambda}{\mu}\le c \tag{3.16}$$

$$\Rightarrow N \le \frac{\Lambda}{\mu}+\left(N_0-\frac{\Lambda}{\mu}\right)e^{-\mu t} \tag{3.17}$$

Therefore, as t→∞ in (3.17) the humans N approaches $K=\frac{\Lambda}{\mu}$ (That is, $N \to K=\frac{\Lambda}{\mu}$)

the parameter $K=\frac{\Lambda}{\mu}$ is called the carrying capacity. Hence all feasible solution set of the humans of the model Equation (3.1) to Equation (3.6) enter the region, which yield for (3.18)

$$D=\left\{(S,\mathrm{Q},\mathrm{E}\,I,\mathrm{H},R,)\in R^6 : S>0,\mathrm{Q}>0,\mathrm{E}\ge 0,\ge 0,\mathrm{H}>0,R\ge 0,N\le\frac{\Lambda}{\mu}\right\} \tag{3.18}$$

Therefore, where $N(0)$ are initial population of non- human primates and humans respectively. Therefore $0\le N\le\frac{\Lambda}{\mu}$ as t→∞ .This implies that, $\frac{\Lambda}{\mu}$ are upper bounds for $N(t)$ respectively, as long $N(0)\le\frac{\Lambda}{\mu}$. Hence, the feasible solution of the model Equations in (3.1)-(3.6) enters the region D which is a positively invariant set. Thus, the system is mathematically and epidemiologically well-posed. Therefore, for an initial starting point $x\in D$, the trajectory lies in D, and so it is sufficient to restrict our analysis on D. Clearly, under the dynamics described by the model equations, the closed set D is hence a positively invariant set.

3.4 Positivity of Solutions

Since Equations (3.5) - (3.6) represent the population in each compartment and all model parameters are all positive, then if lies in a region D defined by

Theorem 3.2 Let the initial data for the model Equation be given as (3.19)

$$S(0) \geq 0, \mathrm{Q}(0) \geq 0, E(0) \geq 0, I(0) \geq 0, \mathrm{H}(0) \geq 0, R(0) \geq 0 \tag{3.19}$$

Then the solutions $\left(S(t), \mathrm{Q}(t), E(t), I(t), \mathrm{H}(t), R(t)\right)$ of the model equation with non-negative initial data will remain non-negative for all time $t > 0$.

Proof

From the first Equation (3.1)

$$\frac{dS}{dt} = \Lambda - \beta SI - \beta S\varepsilon_H H + \theta_e \rho_e E + \theta_q \rho_q Q - \mu S \geq \mu S(t) \tag{3.20}$$

$$\frac{dS}{dt} \geq -\mu S \mu S(t) \tag{3.21}$$

Separating the variables and integrating both side we have

$$\frac{dS}{S} \geq -\mu dt \tag{3.22}$$

$$InS(t) \geq -\mu t + c \tag{3.23}$$

$$S(t) = e^{-\mu t + c} \tag{3.24}$$

$$S(t) = e^{-\mu t} \tag{3.25}$$

Where $K = e^{c}$

Using the initial condition $t = 0 \Rightarrow S(0) \geq K$

Therefore, $S(t) \geq S(0)e^{-\mu(t)} \geq 0$ (3.26)

From equation (3.2)

$$\frac{dQ}{dt} = \theta\beta S(I + \varepsilon_H H) - (1-\theta_q)\rho_q Q(t) - (\rho_q + \mu)Q(t) \geq -(\rho_q + \mu)Q(t) \tag{3.27}$$

$$\frac{dQ}{dt} \geq -(\rho_q + \mu)Q(t)(\rho_q + \mu)Q(t) \tag{3.28}$$

Separating the variable and integrating both side

$$\frac{dQ}{Q(t)} \geq -(\rho_q + \mu)dt \tag{3.29}$$

$$InQ(t) \geq -(\rho_q + \mu)t + c \tag{3.30}$$

$$Q(t) = e^{-(\rho_q + \mu)t + c} \tag{3.31}$$

$$Q(t) = e^{-(\rho_q + \mu)t} \tag{3.32}$$

Where $k = e^{c}$ (3.33)

Using the initial condition $t = 0 \Rightarrow Q(0) \geq k$

Therefore, $Q(t) \geq Q(0)e^{-(\rho_q + \mu)t} \geq 0$ (3.34)

From equation (3.3)

$$\frac{dE}{dt} = (1-\theta)\beta S(I+\varepsilon_H H) - (1-\theta_e)\rho_e E - (\rho_e+\mu)E \geq -(\rho_e+\mu)E \tag{3.35}$$

$$\frac{dE}{dt} \geq -(\rho_e+\mu)E(\rho_e+\mu)E(t) \tag{3.36}$$

Separating the variable and integrating both side

$$\frac{dE}{E} \geq -(\rho_e+\mu)dt \tag{3.37}$$

$$InE(t) \geq -(\rho_e+\mu)t+c \tag{3.38}$$

$$E(t) = e^{-(\rho_e+\mu)t+c} \tag{3.39}$$

$$E(t) = e^{-(\rho_e+\mu)t} \tag{3.40}$$

Where $k = e^c$ (3.41)

Using the initial condition $t = 0 \Rightarrow E(t) \geq k$

Therefore, $E(t) \geq E(0)e^{-(\rho_e+\mu)t} \geq 0$ (3.42)

From equation (3.4)

$$\frac{dI}{dt} = (1-\theta_e)\rho_e E - (\gamma_H+\delta_I+\mu)I \geq -(\gamma_H+\delta_I+\mu)I \tag{3.43}$$

$$\frac{dI}{dt} \geq -(\gamma_H+\delta_I+\mu)I(\gamma_H+\delta_I+\mu)I(t) \tag{3.44}$$

$$\frac{dI}{I(t)} \geq -(\gamma_H + \delta_I + \mu)dt \tag{3.45}$$

Integrating

$$\ln I(t) \geq -(\gamma_H + \delta_I + \mu)t + c \tag{3.46}$$

$$\ln I(t) \geq ke^{-(\gamma_H + \delta_I + \mu)t + c} \tag{3.47}$$

Applying the initial condition $I(t) = 0 \Rightarrow I(0) \geq k$

Therefore, $I(t) \geq I(0)e^{-(\gamma_H + \delta_I + \mu)t} \geq 0$ (3.48)

From equation (3.5)

$$\frac{dH}{dt} = (1-\theta_q)\rho_q Q + \gamma_H I - (\gamma_I + \delta_H + \mu)H \geq -(\gamma_I + \delta_H + \mu)H \tag{3.49}$$

$$\frac{dH}{dt} \geq -(\gamma_I + \delta_H + \mu)H(\gamma_I + \delta_H + \mu)H(t) \tag{3.50}$$

$$\frac{dH}{H} \geq -(\gamma_I + \delta_H + \mu)dt \tag{3.51}$$

integrating

$$\ln H(t) \geq -(\gamma_H + \delta_I + \mu)t + c \tag{3.52}$$

$$H(t) \geq ke^{-(\gamma_H + \delta_I + \mu)t + c} \tag{3.53}$$

Applying the initial condition $H(t) = 0 \Rightarrow H(0) \geq k$ (3.54)

Therefore, $H(t) \geq H(0)e^{-(\gamma_H+\delta_I+\mu)t} \geq 0$ (3.55)

From equation (3.6)

$$\frac{dR}{dt} = \gamma_I H - \mu R \geq -\mu R \quad (3.56)$$

$$\frac{dR}{dt} \geq -\mu R \quad (3.57)$$

$$\frac{dR}{R(t)} \geq -\mu dt \quad (3.58)$$

integrating

$$\ln R(t) \geq -\mu t + c \quad (3.59)$$

$$R(t) \geq ke^{-\mu t + c} \quad (3.60)$$

Appling the initial condition $R(t) = 0 \Rightarrow R(0) \geq k$

Therefore, $R(t) \geq R(0)e^{-\mu t} \geq 0$ (3.61)

3.5 Equilibrium Point

An equilibrium point (fixed point) is a steady state, that is, a rest state, of a system. Thus, at any given equilibrium point, the rate of change of the model variables are equal.

That is,

$$\frac{dS}{dt} = \frac{dQ}{dt} = \frac{dE}{dt} = \frac{dI}{dt} = \frac{dH}{dt} = \frac{dR}{dt} = 0 \quad (3.62)$$

Let

$$(S, \mathrm{Q}, E, I, \mathrm{H}, R) = (S^*, \mathrm{Q}^*, E^*, I^*, \mathrm{H}^*, R^*) \quad (3.63)$$

Therefore, the system Equations (3.1) to (3.6) become

$$\Lambda-\beta S^{*}I^{*}-\beta S^{*}\varepsilon_{H}H^{*}+\theta_{e}\rho_{e}E^{*}+\theta_{q}\rho_{q}Q^{*}-\mu S^{*}=0 \tag{3.64}$$

$$\theta\beta S^{*}(I^{*}+\varepsilon_{H}H^{*})-\left(1-\theta_{q}\right)\rho_{q}Q^{*}-\left(\theta_{q}\rho_{q}+\mu\right)Q^{*}=0 \tag{3.65}$$

$$(1-\theta)\beta S^{*}(I^{*}+\varepsilon_{H}H^{*})-\left(1-\theta_{e}\right)\rho_{e}E^{*}-\left(\theta_{e}\rho_{e}+\mu\right)E^{*}=0 \tag{3.66}$$

$$(1-\theta_{e})\rho_{e}E^{*}-(\gamma_{H}+\delta_{I}+\mu)I^{*}=0 \tag{3.67}$$

$$(1-\theta_{q})\rho_{q}Q^{*}-(\gamma_{1}+\delta_{H}+\mu)H^{*}+\gamma_{H}I^{*}=0 \tag{3.68}$$

$$\gamma_{1}H^{*}-\mu R^{*}=0 \tag{3.69}$$

From equation (3.69)

$$R^{*}=\frac{\gamma_{I}H^{*}}{\mu} \tag{3.70}$$

From equation (3.68)

$$Q^{*}=\frac{\left(\gamma_{1}+\delta_{H}+\mu\right)H^{*}+\gamma_{H}I^{*}}{\rho_{q}\left(1-\theta_{q}\right)} \tag{3.71}$$

From equation (3.67)

$$E^{*}=\frac{\left(\gamma_{H}+\delta_{I}+\mu\right)I^{*}}{\left(1-\theta_{e}\right)\rho_{e}}$$

(3.72)

From equation (3.66)

$$I^{*}+\varepsilon_{H}H^{*}=\frac{\left(\left(1-\theta_{e}\right)\rho_{e}-\left(\theta_{e}\rho_{e}+\mu\right)\right)E^{*}}{\left(1-\theta\right)\beta S^{*}} \tag{3.73}$$

$\Rightarrow$

$$I^* = \frac{\left((1-\theta_e)\rho_e - (\theta_e\rho_e + \mu)\right)E^*}{(1-\theta)\beta S^*} - \varepsilon_H H^* \tag{3.74}$$

Putting equation (3.72) into equation (3.74)

$$I^* = \frac{\left[(\rho_e + \mu)(\gamma_H + \delta_I + \mu)I^*\right]}{(1-\theta)\beta S^*(1-\theta_e)\rho_e} - \varepsilon_H H^* \tag{3.75}$$

$$I^* = \frac{\left((1-\theta_e)\rho_e - (\theta_e\rho_e + \mu)\right)(\gamma_H + \delta_I + \mu)I^*}{\beta S^*\rho_e(1-\theta)(1-\theta)} - \varepsilon_H H^* \tag{3.76}$$

For Simplicity

$$\lambda = \frac{\left((1-\theta_e)\rho_e - (\theta_e\rho_e + \mu)\right)(\gamma_H + \delta_I + \mu)}{\beta S^*\rho_e(1-\theta)(1-\theta)} \tag{3.77}$$

Then, we have

$$(\lambda - 1)I^* = \varepsilon_H H^* \tag{3.78}$$

$$I^* = \frac{\varepsilon_H H}{(\lambda - 1)}^{*} \tag{3.79}$$

Also

Let $A = \dfrac{\varepsilon_H}{\lambda - 1}$ (3.81)

$$I^* = AH^* \tag{3.82}$$

From equation (3.65)

$$\theta\beta S^*\left(AH^* + \varepsilon_H H^*\right) - \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q\rho_q + \mu\right)\right)\left(\frac{\left(\gamma_I + \delta_H + \mu\right)H^* + \gamma_H AH^*}{\rho_q(1-\theta_q)}\right) = 0 \quad (3.83)$$

$\Rightarrow$

$$\left[\theta\beta S^*\left(A^* + \varepsilon_H\right) - \frac{\left(\left(1-\theta_q\right)\rho_q + \left(\theta_q\rho_q + \mu\right)\right)\left(\gamma_I + \delta_H + \mu\right) + \gamma_H A}{\rho_q(1-\theta_q)}\right]H^* = 0 \quad (3.84)$$

Thus, there exists two equilibrium points from (3.84)

3.5.1 Disease-free equilibrium

At this equilibrium state, there is a nonexistence of infection. Hence, the total population will contain the susceptible individuals only, since the infected classes will be zero.

Lemma 1: A DFE of the model exists at the point:

$$\left(S^0, Q^0, E^0, I^0, \mathrm{H}^0, R^0\right) = \left[\frac{\Lambda}{\mu_1}, 0, 0, 0, 0, 0\right] \quad (3.85)$$

Proof

Let $\left(S, \mathrm{Q}, E, I, \mathrm{H}, R\right) = \left(S^0, \mathrm{Q}^0, E^0, I^0, \mathrm{H}^0, R^0\right)$ at disease free equilibrium

Suppose $H^0 = 0$ (3.86)

Substitute equation (3.86) into equation (3.82)

$$I^0 = A(0) \quad (3.87)$$

$$I^0 = 0$$

(3.88)

Substitute equation (3.88) into equation (3.72)

$$E^0 = \frac{(\delta_H + \gamma_I + \mu)(0)}{(1-\theta_e)\rho_e} \tag{3.89}$$

$$E^0 = 0$$

(3.90)

Substitute equation (3.85) into equation (3.73)

$$R^0 = \frac{\gamma_I(0)}{\mu} \tag{3.91}$$

$$R^0 = 0$$

(3.92)

From equation (3.71) we have

$$Q^0 = 0$$

(3.93)

From equation (3.67) we have

$$\Lambda - \mu S^0 = 0 \tag{3.94}$$

$$\Lambda = \mu S^0 \tag{3.95}$$

$$S^0 = \frac{\Lambda}{\mu} \tag{3.96}$$

Thus, the lemma is proved.

3 .5.2 The Endemic Equilibrium State

At the endemic equilibrium state, the virus cannot be totally eradicated but it must remain in the population of interest. For the virus to persist in the population all the compartments are greater than zero.

Let $E'=\left(S,Q,E,I,H,R\right)=\left(S^{**},Q^{**},E^{**},I^{**},H^{**},R^{**}\right)$ be the Endemic Equilibrium point,

Suppose $H\neq 0$ from (3.84)

$$\left[\theta\beta S^{**}\left(A+\varepsilon_H\right)-\frac{\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)\left(\gamma_I+\delta_H+\mu\right)+\gamma_H A}{\rho_q(1-\theta_q)}\right]=0 \tag{3.97}$$

$$\theta\beta S^{**}\left(A+\varepsilon_H\right)=\frac{\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)\left(\gamma_I+\delta_H+\mu\right)+\gamma_H A}{\rho_q(1-\theta_q)} \tag{3.98}$$

$$S^{**}=\frac{\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)\left(\gamma_I+\delta_H+\mu\right)+\gamma_H A}{\theta\beta\left(A+\varepsilon_H\right)\rho_q(1-\theta_q)} \tag{3.99}$$

Let

$$B^{*}=\left(I+\varepsilon_H H\right) \tag{3.100}$$

From equation (3.65)

$$\theta\beta S^{**}B^{*}-\left(\left(1-\theta_q\right)\rho_q-\left(\theta_q\rho_q+\mu\right)\right)Q^{**}=0 \tag{3.101}$$

Substitute equation (3.99) into equation (3.101)

$$\theta\beta B^{*}\frac{\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)\left(\gamma_I+\delta_H+\mu\right)+\gamma_H A}{\theta\beta\left(A+\varepsilon_H\right)\rho_q(1-\theta_q)}-\left(\left(1-\theta_q\right)\rho_q-\left(\theta_q\rho_q+\mu\right)\right)Q^{**}=0 \tag{3.102}$$

$$Q^{**}=\frac{\theta\beta B^{*}\left[\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)\left(\gamma_I+\delta_H+\mu\right)+\gamma_H A\right]}{\theta\beta\left(A+\varepsilon_H\right)\rho_q(1-\theta_q)\left(\left(1-\theta_q\right)\rho_q-\left(\theta_q\rho_q+\mu\right)\right)}$$

(3.103)

Substitute equation (3.103) into equation (3.66) we have

$$(1-\theta)\beta S^{**}(I^{**}+\varepsilon_H H^{**})-\left(1-\theta_e\right)\rho_e E^{**}-\left(\theta_e\rho_e+\mu\right)E^{**}=0 \qquad (3.104)$$

$$\frac{\left(1-\theta\right)\beta B^{*}\left[\left(1-\theta_e\right)\rho_e-\left(\theta_e\rho_e+\mu\right)\left(\gamma_I+\delta_H+\mu\right)+\gamma_H A\right]}{\theta\beta\left(A+\varepsilon_H\right)\rho_q(1-\theta_q)}=\left(1-\theta_e\right)\rho_e E^{**}-\left(\theta_e\rho_e+\mu\right)E^{**} \qquad (3.105)$$

$$E^{**}=\frac{\left(1-\theta\right)\beta B^{*}\left[\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\left(\gamma_I+\delta_H+\mu\right)+\gamma_H A\right]}{\theta\beta\left(A+\varepsilon_H\right)\rho_q(1-\theta_q)\left(1-\theta_e\right)\rho_e-\left(\theta_e\rho_e+\mu\right)} \qquad (3.106)$$

$E^{**}=C^{*}$ and $Q^{**}=D^{*}$ (3.107)

Substitute equation (3.107) into equation (3.72)

$$I^{**}=\frac{\left(1-\theta_e\right)\rho_e C^{*}}{\left(\gamma_I+\delta_H+\mu\right)} \qquad (3.107)$$

Substitute equation (3.106) and (3.105) into equation (3.68)

$$\left(1-\theta_q\right)\rho_q D^{*}-\left(\gamma_I+\delta_H+\mu\right)H^{**}+\frac{\gamma_H\left(1-\theta_e\right)\rho_e C^{*}}{\left(\gamma_I+\delta_H+\mu\right)}=0 \qquad (3.108)$$

$$\begin{aligned}&\left(\gamma_I+\delta_H+\mu\right)\left(1-\theta_q\right)\rho_q D^{*}-\left(\gamma_I+\delta_H+\mu\right)\left(\gamma_H+\delta_I+\mu\right)H^{**}\\&+\left(1-\theta_e\right)\rho_e C^{*}=0\end{aligned} \qquad (3.109)$$

$$H^{**} = \frac{(\gamma_H + \delta_I + \mu)(1-\theta_q)\rho_q D^* - \gamma_H (1-\theta_e)\rho_e C^*}{(\gamma_H + \delta_I + \mu)(\gamma_I + \delta_H + \mu)} \tag{3.110}$$

Substitute equation (3.110) into equation (3.72) we have

$$R^{**} = \gamma_I \frac{(\gamma_H + \delta_I + \mu)(1-\theta_q)\rho_q D^* - \gamma_H (1-\theta_e)\rho_e C^*}{\mu(\gamma_H + \delta_I + \mu)(\gamma_I + \delta_H + \mu)} \tag{3.111}$$

3.6 Effective Reproduction Number

One of the most imperative concerns about every infectious disease is its ability to spread across a population. The basic reproduction number, R_0 is the extent of the probable for disease spread in a population, and is inarguably one of the foremost and most valuable ideas that mathematical thinking has brought to epidemic theory (Heesterbeek & Dietz, 1996). It represents the average number of secondary cases generated by an infected individual if introduced into a susceptible population with no immunity to the disease in the absence of interventions to control the infection. If $R_0 < 1$, then on average, an infected individual produces less than one newly infected individual over the course of its infection period. In this case, the infection may die out in the long run. Conversely, if $R_0 > 1$, each infected individual produces, on average more than one new infection, the infection will be able to spread in a population. A large value of R_0 may indicate the possibility of a major epidemic.

Similarly, the effective reproduction number, R_C represent the average number of secondary cases generated by an infected individual if introduced into a susceptible population where control strategies are used.

Using the next generation operator technique described by Diekmann and Heesterbek (2000) and subsequently analysed by Van de Driessche and Watmough (2002), we obtained the effective reproduction number, R_C of our model which is the spectral radius of the next generation matrix FV^{-1}.

i.e

$$R_C = \rho FV^{-1} \tag{3.112}$$

Where

ρ is the spectral radius

F is matrix of infection term at disease free equilibrium

V is matrix of transmission term at disease free equilibrium

$$FV^{-1} = \left(\frac{\partial F_i(E^0)}{\partial x_i}\right)\left(\frac{\partial V_i(E^0)}{\partial x_i}\right)^{-1} \tag{3.113}$$

Thus

From equation (3.1) to equation (3.6), the infection term matrix of the jacobian at disease free equilibrium is given as

$$f_i = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} \theta\beta S(I + \varepsilon_H H) \\ (1-\theta)\beta S(I + \varepsilon_H H) \\ 0 \\ 0 \end{pmatrix} \tag{3.114}$$

$$F = \begin{pmatrix} \frac{\partial f_1}{\partial Q} & \frac{\partial f_1}{\partial E} & \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial H} \\ \frac{\partial f_2}{\partial Q} & \frac{\partial f_2}{\partial E} & \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial H} \\ \frac{\partial f_3}{\partial Q} & \frac{\partial f_3}{\partial E} & \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial H} \\ \frac{\partial f_4}{\partial Q} & \frac{\partial f_4}{\partial E} & \frac{\partial f_4}{\partial I} & \frac{\partial f_4}{\partial H} \end{pmatrix} \tag{3.115}$$

$$F_{DFE} = \begin{pmatrix} 0 & 0 & \theta\beta S & \theta\beta\varepsilon_H S \\ 0 & 0 & \beta S(1-\theta) & \beta\varepsilon_H S(1-\theta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.116}$$

$$v_i = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} A_1 Q \\ A_2 E \\ A_3 I - (1-\theta_e)\rho_e \\ A_4 H - \gamma_H I \end{pmatrix} \tag{3.117}$$

$$V_{DFE} = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & -(1-\theta_e)\rho_e & A_3 & 0 \\ 0 & 0 & -\gamma_H & A_4 \end{pmatrix} \tag{3.118}$$

where

$$\begin{aligned} &A_1 = (1-\theta_q)\rho_q + (\theta_q\rho_q + \mu), A_2 = (1-\theta_e)\rho_e + (\theta_e\rho_e + \mu), \\ &A_3 = (\gamma_H + \delta_1 + \mu), A_4 = (\gamma_1 + \delta_H + \mu) \end{aligned} \tag{3.119}$$

$$V^{-1} = \frac{adjoint}{determinant} \tag{3.120}$$

$$V^{-1}=\begin{pmatrix} \frac{1}{A_1} & 0 & 0 & 0 \\ 0 & \frac{1}{A_2} & 0 & 0 \\ 0 & \frac{(1-\theta_e)\rho_e}{A_2A_3} & \frac{1}{A_3} & 0 \\ 0 & \frac{\gamma_H(1-\theta_e)\rho_e}{A_2A_3A_4} & \frac{\gamma_H}{A_3A_4} & \frac{1}{A_4} \end{pmatrix} \tag{3.121}$$

$$FV^{-1}=\begin{pmatrix} \frac{1}{A_1} & 0 & 0 & 0 \\ 0 & \frac{1}{A_2} & 0 & 0 \\ 0 & \frac{(1-\theta_e)\rho_e}{A_2A_3} & \frac{1}{A_3} & 0 \\ 0 & \frac{\gamma_H(1-\theta_e)\rho_e}{A_2A_3A_4} & \frac{\gamma_H}{A_3A_4} & \frac{1}{A_4} \end{pmatrix}\begin{pmatrix} 0 & 0 & \theta\beta S & \theta\beta\varepsilon_H S \\ 0 & 0 & \beta S(1-\theta) & \beta\varepsilon_H S(1-\theta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.122}$$

$$FV^{-1}=\begin{pmatrix} 0 & \frac{\theta\beta S(1-\theta_e)\rho_e}{A_2A_3}+\frac{\theta\beta S\varepsilon_H\gamma_H(1-\theta_e)\rho_e}{A_2A_3} & \frac{\theta\beta S}{A_3}+\frac{\theta\beta S\varepsilon_H\gamma_H}{A_3A_4} & \frac{\theta\beta\varepsilon_H S}{A_4} \\ 0 & \frac{(1-\theta)\beta S(1-\theta_e)\rho_e}{A_2A_3} & \frac{\beta S(1-\theta)(1-\theta_e)\rho_e\varepsilon_H\gamma_H}{A_2A_3A_4} & \frac{\beta S(1-\theta)}{A_3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.123}$$

$$FV^{-1}=\begin{pmatrix} 0 & \frac{\theta\beta S(1-\theta_e)\rho_e}{A_2A_3}+\frac{\theta\beta S\varepsilon_H\gamma_H(1-\theta_e)\rho_e}{A_2A_3} & \frac{\theta\beta S}{A_3}+\frac{\theta\beta S\varepsilon_H\gamma_H}{A_3A_4} & \frac{\theta\beta\varepsilon_H S}{A_4} \\ 0 & \frac{(1-\theta)\beta S(1-\theta_e)\rho_e}{A_2A_3} & \frac{\beta S(1-\theta)(1-\theta_e)\rho_e\varepsilon_H\gamma_H}{A_2A_3A_4} & \frac{\beta S(1-\theta)}{A_3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.124}$$

solving the eigen values using maple 18.

$$FV^{-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{(1-\theta_e)\rho_e(1-\theta)\beta S\varepsilon_H\gamma_H + (1-\theta)\beta S A_4}{A_1A_2A_3A_4} \end{pmatrix} \tag{3.125}$$

thus

$$\lambda_1 = \lambda_2 = \lambda_3 = 0 \tag{3.126}$$

$$\lambda_4 = \frac{(1-\theta_e)\rho_e(1-\theta)\beta S\varepsilon_H\gamma_H + (1-\theta)\beta S A_4}{A_1A_2A_3A_4} \tag{3.127}$$

Therefore

$$R_C = \frac{(1-\theta_e)\rho_e(1-\theta)\beta S\varepsilon_H\gamma_H + (1-\theta)\beta S A_4}{A_1A_2A_3A_4} \tag{3.128}$$

At DFE

$$R_C = \frac{(1-\theta_e)\rho_e(1-\theta)\beta\Lambda\varepsilon_H\gamma_H + (1-\theta)\beta\Lambda A_4}{A_1A_2A_3A_4\mu} \tag{3.129}$$

3.7 Local Stability of Disease-Free Equilibrium

According to Deikmann and Heesterbeek (1990) theorem, the DFE is LAS if there exist R_C and $R_C < 1$. We want to further justify the theorem using Jacobian techniques for stability.

Lemma 3.3: The Disease Free Equilibrium of the model is locally asymptotically stable (LAS) if $R_C < 1$.

Proof:

$$J=\begin{bmatrix} -\mu & \theta_q\rho_q & \theta_e\rho_e & -\beta S & -\beta\varepsilon_H S & 0 \\ 0 & -A_1 & 0 & \beta\theta S & \beta\varepsilon_H\theta S & 0 \\ 0 & 0 & -A_2 & \beta\left(1-\theta\right)S & \beta\varepsilon_H\left(1-\theta\right)S & 0 \\ 0 & 0 & \left(1-\theta_e\right)\rho_e & -A_3 & 0 & 0 \\ 0 & \left(1-\theta_q\right)\rho_q & 0 & \gamma_H & -A_4 & 0 \\ 0 & 0 & 0 & 0 & \gamma_H & -\mu \end{bmatrix}=0 \tag{3.130}$$

Reducing to upper triangular matrix

$$J=\begin{bmatrix} -\mu & \theta_q\rho_q & \theta_e\rho_e & -\beta S & -\beta\varepsilon_H S & 0 \\ 0 & -A_1 & 0 & \beta S & \beta\varepsilon_H\theta S & 0 \\ 0 & 0 & -A_2 & \beta\left(1-\theta\right)S & \beta\varepsilon_H\left(1-\theta\right)S & 0 \\ 0 & 0 & 0 & \frac{(1-\theta_e)\rho_e(1-\theta)\beta S}{A_2}-A_2A_3 & \frac{(1-\theta_e)\rho_e(1-\theta)\beta\varepsilon_H S}{A_2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{M}\left((1-\theta_e)\rho_e(1-\theta)\beta\Lambda\varepsilon_H\gamma_H+(1-\theta)\beta\Lambda A_4-A_1A_2A_3A_4\mu\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu \end{bmatrix}=0 \tag{3.131}$$

Where

$$M=A_1((1-\theta_e)\rho_e(1-\theta)\beta S-A_2A_3) \tag{3.132}$$

$$J=\begin{bmatrix} -\mu-\lambda_1 & \theta_q\rho_q & \theta_e\rho_e & -\beta S & \beta\varepsilon_H S & 0 \\ 0 & -A_1-\lambda_2 & 0 & \beta S & \beta\varepsilon_H\theta S & 0 \\ 0 & 0 & -A_2-\lambda_3 & \beta\left(1-\theta\right)S & \beta\varepsilon_H\left(1-\theta\right)S & 0 \\ 0 & 0 & 0 & \frac{(1-\theta_e)\rho_e(1-\theta)\beta S}{A_2}-A_2A_3-\lambda_4 & \frac{(1-\theta_e)\rho_e(1-\theta)\beta\varepsilon_H S}{A_2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{M}\left((1-\theta_e)\rho_e(1-\theta)\beta\Lambda\varepsilon_H\gamma_H+(1-\theta)\beta\Lambda A_4-A_1A_2A_3A_4\mu\right)-\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu-\lambda_6 \end{bmatrix}=0 \tag{3.133}$$

Therefore

$$\lambda_1 = -\mu, \lambda_2 = -A_1, \lambda_3 = -A_2, \lambda_4 = \frac{(1-\theta_e)\rho_e(1-\theta)\beta S}{A_2} - A_2 A_3, \lambda_6 = -\mu \tag{3.134}$$

All negative since

$$A_2 A_3 > \frac{(1-\theta_e)\rho_e(1-\theta)\beta S}{A_2} \tag{3.135}$$

$\lambda_5 < 0$ if $R_C < 1$ (3.136)

$$\lambda_5 = \frac{1}{M}\left((1-\theta_e)\rho_e(1-\theta)\beta\Lambda\varepsilon_H\gamma_H + (1-\theta)\beta\Lambda A_4 - A_1A_2A_3A_4\mu\right) < 0 \tag{3.137}$$

$$\lambda_5 = \frac{1}{M}\left(\frac{(1-\theta_e)\rho_e(1-\theta)\beta\Lambda\varepsilon_H\gamma_H + (1-\theta)\beta\Lambda A_4}{A_1A_2A_3A_4\mu} - 1\right) < 0 \tag{3.138}$$

$$\frac{1}{M}(R_C - 1) < 0 \tag{3.139}$$

$$(R_C - 1) < 0 \tag{3.140}$$

$$R_C < 1$$

(3.141)

Hence, the disease free equilibrium is locally assymptotically stable since $R_C < 1$.

3.8 Global Stability of the Disease-Free Equilibrium Point

Theorem: The D.F.E (E^0) of the model system is globally asymptotically stable (GAS) in the feasible region Ω if $R_C < 1$ and unstable if $R_C > 1$.

Proof: To establish the global stability of the D.F.E, the two conditions for the global stability of D.F.E as in (Castillo-Chavez and Feng, 1997) for $R_C < 1$ was used for the model system.

We can write the model system as:

$$\frac{dX_S}{dt} = A(X_S - X_{DFE_S}) + A_1 X_i \tag{3.142}$$

$$\frac{dX_i}{dt} = A_2 X_i \tag{3.143}$$

Where

$$X_S = (S^0, \mathrm{R}^0)^T \tag{3.144}$$

denote the non-infectious compartments,

$$X_i = (\mathrm{Q}^0, E^0, I^0, H^0)^T \tag{3.145}$$

denote the infectious compartments. The disease-free equilibrium is denoted as

$$E^0 = (X_s^*, 0) \tag{3.146}$$

Where

$$X_s^* = (N^0, 0) \tag{3.147}$$

$$\frac{dX_S}{dt} = F(X_S, 0) \tag{3.148}$$

$$\Lambda - \mu S \tag{3.149}$$

$$-\mu R \tag{3.150}$$

Solving the differential equations

$$\mathrm{R}^0(t) = R^0(0)e^{-\mu t} \tag{3.151}$$

Hence

$$S^0(t) + R^0(t) \to N^0(t) \ \ as\ t \to 0 \tag{3.152}$$

Irrespective of the value of

$$S^0(0), \mathrm{R}^0(0). \tag{3.153}$$

Thus

$$X_s^* = (N^0, 0) \tag{3.154}$$

Is globally asymptotically stable.

Next,

$$\bar{G}(X_s, X_i) = AX_i - G(X_s, X_i) \tag{3.155}$$

$$AX_i = \begin{pmatrix} -A_1 & 0 & \theta\beta S & \theta\beta\varepsilon_H S \\ 0 & -A_2 & (1-\theta)\beta S & (1-\theta)\beta\varepsilon_H S \\ 0 & (1-\theta_e)\rho_e & -A_3 & 0 \\ (1-\theta)\rho_q & 0 & -\gamma_H & -A_4 \end{pmatrix} \tag{3.156}$$

Where

It is obvious that this is an M-matrix (Metzler also called quasi positive matrix in which all the off diagonal elements are non-negative i.e greater than or equal to zero

$$G(X_s, X_i) = AX_i = \begin{pmatrix} -A_1Q^* & 0 & \theta\beta SI^* & \theta\beta\varepsilon_H SH^* \\ 0 & -A_2E^* & (1-\theta)\beta SI^* & (1-\theta)\beta\varepsilon_H SH^* \\ 0 & (1-\theta_e)\rho_e E^* & -A_3I^* & 0 \\ (1-\theta)\rho_q Q^* & 0 & -\gamma_H I^* & -A_4H^* \end{pmatrix}$$

(3.157)

Then

$$\bar{G}(X_s, X_i) = AX_i - G(X_s, X_i) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \quad (3.158)$$

That is,

$$\bar{G}(X_s, X_i) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.159)$$

Thus,

$$\bar{G}(X_s, X_i) = 0 \quad (3.160)$$

The theorem is complete.

3.9 Stability analysis of the Endemic Equilibrium (EES)

The characteristics equation becomes

$$\begin{bmatrix} a_1-\lambda & \theta_q\rho_q & \theta_e\rho_e & -\beta S^* & -\beta\varepsilon_H S^* & 0 \\ a_2 & -a_3-\lambda & 0 & a_4 & a_5 & 0 \\ a_6 & 0 & a_7-\lambda & a_8 & a_9 & 0 \\ 0 & 0 & a_{10} & a_{11}-\lambda & 0 & 0 \\ 0 & a_{12} & 0 & \gamma_H & a_{13}-\lambda & 0 \\ 0 & 0 & 0 & 0 & \gamma_I & -\mu-\lambda \end{bmatrix} \quad (3.161)$$

The characteristics equation obtained from the Jacobian determinant with the Eigen value of λ

Where

$$
\left.
\begin{aligned}
& l^6 + (a_{11} + a_{13} + a_1 - a_3 + a_7 + \mu) l^5 + (- r_e a_6 q_e - r_q a_2 q_q - a_1 a_3 + a_1 a_7 \\
& + a_1 a_{11} + a_{13} a_1 - a_3 a_7 - a_{13} a_3 - a_5 a_{12} + a_{11} a_7 + a_7 a_{13} - a_8 a_{10} + a_{11} a_{13} \\
& - (- a_{11} - a_{13} - a_1 + a_3 - a_7) \mu) l^4 + (- r_q a_2 a_7 q_q + r_e a_3 a_6 q_e - r_q a_2 a_{13} q_q \\
& + S^* b a_6 a_{10} - r_e a_6 a_{11} q_e - r_e a_6 a_{13} q_e - r_q a_2 a_{11} q_q - (r_e a_6 q_e + r_q a_2 q_q + a_1 a_3 \\
& - a_1 a_7 - a_1 a_{11} - a_1 a_{13} + a_3 a_7 + a_3 a_{11} + a_3 a_{13} + a_5 a_{12} - a_7 a_{11} - a_7 a_{13} + \\
& a_8 a_{10} - a_{11} a_{13}) \mu + S^* b a_2 a_{12} e_H - a_3 a_7 a_{13} + a_3 a_8 a_{10} - a_3 a_{11} a_{13} - a_5 a_7 a_{12} \\
& - a_5 a_{11} a_{12} - a_{13} a_1 a_3 - a_5 a_{12} a_1 - a_1 a_3 a_{11} + a_1 a_7 a_{11} + a_1 a_7 a_{13} - a_1 a_8 a_{10} + \\
& a_1 a_{11} a_{13} - a_3 a_7 a_{11} + a_7 a_{11} a_{13} - g_H a_{10} a_9 - a_8 a_{10} a_{13} - a_1 a_3 a_7) l^3 + (- a_{11} a_1 \\
& a_3 a_7 + a_{10} a_1 a_3 a_8 - a_1 a_3 a_7 a_{13} - a_1 a_3 a_{11} a_{13} - a_1 a_5 a_7 a_{12} - a_1 a_5 a_{11} a_{12} + a_1 a_7 \\
& a_{11} a_{13} - a_1 a_8 a_{10} a_{13} - a_1 a_9 a_{10} g_H - (r_q a_2 a_7 q_q - r_e a_3 a_6 q_e + r_q a_2 a_{13} q_q - S^* b a_6 a_{10} \\
& + r_e a_6 a_{11} q_e + r_e a_6 a_{13} q_e + r_q a_2 a_{11} q_q - S^* b a_2 a_{12} e_H + a_3 a_7 a_{13} - a_3 a_8 a_{10} + a_3 a_{11} \\
& a_{13} + a_5 a_7 a_{12} + a_5 a_{11} a_{12} + a_{13} a_1 a_3 + a_5 a_{12} a_1 + a_1 a_3 a_{11} - a_1 a_7 a_{11} - a_1 a_7 a_{13} + \\
& a_1 a_8 a_{10} - a_1 a_{11} a_{13} + a_3 a_7 a_{11} - a_7 a_{11} a_{13} + g_H a_{10} a_9 + a_8 a_{10} a_{13} + a_1 a_3 a_7) \mu - \\
& a_3 a_7 a_{11} a_{13} + a_3 a_8 a_{10} a_{13} + a_3 a_8 a_{10} a_{13} + a_3 a_9 a_{10} g_H - a_4 a_9 a_{10} a_{12} - a_5 a_7 a_{11} a_{12} + \\
& a_5 a_8 a_{10} a_{12} + S^* b a_6 a_{10} a_{13} - r_e a_2 a_9 a_{12} q_e + r_e a_3 a_6 a_{13} q_e + r_e a_5 a_6 a_{12} q_e - r_e a_6 a_{11} a_{13} q_e \\
& - r_q a_2 a_7 a_{13} q_q - r_q a_2 a_{11} a_{13} q_q + a_{11} r_e a_3 a_6 q_e - a_{11} r_q a_2 a_7 q_q + a_{10} r_q a_2 a_8 q_q - a_{10} r_q a_4 \\
& a_6 q_q - a_{10} S b a_3 a_6 + S^* b a_2 a_7 a_{12} e_H + S^* b a_2 a_{11} a_{12} e_H + S^* b a_6 a_{10} e_H g_H) l^2 + (- a_1 a_3 a_7 \\
& a_{11} a_{13} + a_1 a_3 a_8 a_{10} a_{13} + a_1 a_3 a_9 a_{10} g_H - a_1 a_5 a_7 a_{11} a_{12} + a_1 a_5 a_8 a_{10} a_{12} - a_1 a_4 a_9 a_{10} a_{12} \\
& - r_e a_2 a_9 a_{11} a_{12} q_e - r_q a_2 a_7 a_{11} a_{13} q_q + r_q a_2 a_8 a_{10} a_{13} q_q - S^* b a_2 a_8 a_{10} a_{12} e_H + r_q a_2 a_9 a_{10} g_H q_q \\
& + S^* b a_2 a_9 a_{10} a_{12} + S^* b a_2 a_7 a_{11} a_{12} e_H + r_e a_3 a_6 a_{11} a_{13} q_e + r_e a_5 a_6 a_{11} a_{12} q_e - S^* b a_3 a_6 a_{10} e_H g_H \\
& - S^* b a_3 a_6 a_{10} a_{13} - r_q a_4 a_6 a_{10} a_{13} q_q + S^* b a_4 a_6 a_{10} a_{12} e_H - r_q a_5 a_6 a_{10} g_H q_q - S^* b a_5 a_6 a_{10} a_{12} - \\
& (a_{11} a_1 a_3 a_7 - a_{10} a_1 a_3 a_8 + a_1 a_3 a_7 a_{13} + a_1 a_3 a_{11} a_{13} + a_1 a_5 a_7 a_{12} + a_1 a_5 a_{11} a_{12} - a_1 a_7 a_{11} a_{13} \\
& + a_1 a_8 a_{10} a_{13} + a_1 a_9 a_{10} g_H + a_3 a_7 a_{11} a_{13} - a_3 a_8 a_{10} a_{13} - a_3 a_9 a_{10} g_H + a_4 a_9 a_{10} a_{12} + a_5 a_7 a_{11} a_{12} \\
& - a_5 a_8 a_{10} a_{12} - S^* b a_6 a_{10} a_{13} + r_e a_2 a_9 a_{12} q_e - r_e a_3 a_6 a_{13} q_e - r_e a_5 a_6 a_{12} q_e + r_e a_6 a_{11} a_{13} q_e + \\
& r_q a_2 a_7 a_{13} q_q + r_q a_2 a_{11} a_{13} q_q - a_{11} r_e a_3 a_6 q_e + a_{11} r_q a_2 a_7 q_q - a_{10} r_q a_2 a_8 q_q + a_{10} r_q a_4 a_6 q_q + \\
& a_{10} S^* b a_3 a_6 - S^* b a_2 a_7 a_{12} e_H - S^* b a_2 a_{11} a_{12} e_H - S^* b a_6 a_{10} e_H g_H) \mu) l - (a_1 a_3 a_7 a_{11} a_{13} - \\
& a_1 a_3 a_8 a_{10} a_{13} - a_1 a_3 a_9 a_{10} g_H + a_1 a_5 a_7 a_{11} a_{12} - a_1 a_5 a_8 a_{10} a_{12} + a_1 a_4 a_9 a_{10} a_{12} + r_e a_2 a_9 a_{11} a_{12} q_e \\
& + r_q a_2 a_7 a_{11} a_{13} q_q - r_q a_2 a_8 a_{10} a_{13} q_q + S^* b a_2 a_8 a_{10} a_{12} e_H - r_q a_2 a_9 a_{10} g_H q_q - S^* b a_2 a_9 a_{10} a_{12} - \\
& S^* b a_2 a_7 a_{11} a_{12} e_H - r_e a_3 a_6 a_{11} a_{13} q_e - r_e a_5 a_6 a_{11} a_{12} q_e + S^* b a_3 a_6 a_{10} e_H g_H + S^* b a_3 a_6 a_{10} a_{13} \\
& + r_q a_4 a_6 a_{10} a_{13} q_q - S^* b a_4 a_6 a_{10} a_{12} e_H + r_q a_5 a_6 a_{10} g_H q_q + S^* b a_5 a_6 a_{10} a_{12}) \mu
\end{aligned}
\right\} \tag{3.162}
$$

We can now apply the result of Bellman and Cooke's theorem of stability,

$H(Z)=P(z,e^{2})$ where $P(z,\mathrm{w})$ is a polynomial with principal term.

Suppose $H(y)=F(y)+iG(y)$ (3.163)

If all zero of $H(y)$ have negatives real parts, then zeros of $H(y)$ and $G(y)$ are real, simple and alternate and

$F(0)G(0)-F(0)G(0)>0$ for all y belongs to real numbers (3.164)

Conversely, all zeros of $H(z)$ will be in the left hand plane provided that it is in either of the following conditions.

1. All zeros of $F(y)$ and $G(y)$ are real, simple and the inequality (3.164) is satisfied for at least one y
2. All the zeros of $F(y)$ are real and for each zero the relation (3.163) is satisfied
3. All the zeros of $G(y)$ are real and for each zero the relation (3.163) is satisfied

Let the equation (3.162) take the form

$$
H(l)=\left.\begin{aligned}
&l^{6}+(a_{11}+a_{13}+a_{1}-a_{3}+a_{7}+\mu)l^{5}+(-r_{e}a_{6}q_{e}-r_{q}a_{2}q_{q}-a_{1}a_{3}+a_{1}a_{7}\\
&+a_{1}a_{11}+a_{13}a_{1}-a_{3}a_{7}-a_{13}a_{3}-a_{5}a_{12}+a_{11}a_{7}+a_{7}a_{13}-a_{8}a_{10}+a_{11}a_{13}\\
&-(-a_{11}-a_{13}-a_{1}+a_{3}-a_{7})\mu)l^{4}+(-r_{q}a_{2}a_{7}q_{q}+r_{e}a_{3}a_{6}q_{e}-r_{q}a_{2}a_{13}q_{q}\\
&+Sba_{6}a_{10}-r_{e}a_{6}a_{11}q_{e}-r_{e}a_{6}a_{13}q_{e}-r_{q}a_{2}a_{11}q_{q}-(r_{e}a_{6}q_{e}+r_{q}a_{2}q_{q}+a_{1}a_{3}\\
&-a_{1}a_{7}-a_{1}a_{11}-a_{1}a_{13}+a_{3}a_{7}+a_{3}a_{11}+a_{3}a_{13}+a_{5}a_{12}-a_{7}a_{11}-a_{7}a_{13}+\\
&a_{8}a_{10}-a_{11}a_{13})\mu+Sba_{2}a_{12}e_{H}-a_{3}a_{7}a_{13}+a_{3}a_{8}a_{10}-a_{3}a_{11}a_{13}-a_{5}a_{7}a_{12}\\
&-a_{5}a_{11}a_{12}-a_{13}a_{1}a_{3}-a_{5}a_{12}a_{1}-a_{1}a_{3}a_{11}+a_{1}a_{7}a_{11}+a_{1}a_{7}a_{13}-a_{1}a_{8}a_{10}+\\
&a_{1}a_{11}a_{13}-a_{3}a_{7}a_{11}+a_{7}a_{11}a_{13}-g_{H}a_{10}a_{9}-a_{8}a_{10}a_{13}-a_{1}a_{3}a_{7})l^{3}+(-a_{11}a_{1}\\
&a_{3}a_{7}+a_{10}a_{1}a_{3}a_{8}-a_{1}a_{3}a_{7}a_{13}-a_{1}a_{3}a_{11}a_{13}-a_{1}a_{5}a_{7}a_{12}-a_{1}a_{5}a_{11}a_{12}+a_{1}a_{7}\\
&a_{11}a_{13}-a_{1}a_{8}a_{10}a_{13}-a_{1}a_{9}a_{10}g_{H}-(r_{q}a_{2}a_{7}q_{q}-r_{e}a_{3}a_{6}q_{e}+r_{q}a_{2}a_{13}q_{q}-Sba_{6}a_{10}\\
&+r_{e}a_{6}a_{11}q_{e}+r_{e}a_{6}a_{13}q_{e}+r_{q}a_{2}a_{11}q_{q}-Sba_{2}a_{12}e_{H}+a_{3}a_{7}a_{13}-a_{3}a_{8}a_{10}+a_{3}a_{11}\\
&a_{13}+a_{5}a_{7}a_{12}+a_{5}a_{11}a_{12}+a_{13}a_{1}a_{3}+a_{5}a_{12}a_{1}+a_{1}a_{3}a_{11}-a_{1}a_{7}a_{11}-a_{1}a_{7}a_{13}+\\
&a_{1}a_{8}a_{10}-a_{1}a_{11}a_{13}+a_{3}a_{7}a_{11}-a_{7}a_{11}a_{13}+g_{H}a_{10}a_{9}+a_{8}a_{10}a_{13}+a_{1}a_{3}a_{7})\mu-\\
&a_{3}a_{7}a_{11}a_{13}+a_{3}a_{8}a_{10}a_{13}+a_{3}a_{8}a_{10}a_{13}+a_{3}a_{9}a_{10}g_{H}-a_{4}a_{9}a_{10}a_{12}-a_{5}a_{7}a_{11}a_{12}+\\
&a_{5}a_{8}a_{10}a_{12}+Sba_{6}a_{10}a_{13}-r_{e}a_{2}a_{9}a_{12}q_{e}+r_{e}a_{3}a_{6}a_{13}q_{e}+r_{e}a_{5}a_{6}a_{12}q_{e}-r_{e}a_{6}a_{11}a_{13}q_{e}\\
&-r_{q}a_{2}a_{7}a_{13}q_{q}-r_{q}a_{2}a_{11}a_{13}q_{q}+a_{11}r_{e}a_{3}a_{6}q_{e}-a_{11}r_{q}a_{2}a_{7}q_{q}+a_{10}r_{q}a_{2}a_{8}q_{q}-a_{10}r_{q}a_{4}\\
&a_{6}q_{q}-a_{10}Sba_{3}a_{6}+Sba_{2}a_{7}a_{12}e_{H}+Sba_{2}a_{11}a_{12}e_{H}+Sba_{6}a_{10}e_{H}g_{H})l^{2}+(-a_{1}a_{3}a_{7}\\
&a_{11}a_{13}+a_{1}a_{3}a_{8}a_{10}a_{13}+a_{1}a_{3}a_{9}a_{10}g_{H}-a_{1}a_{5}a_{7}a_{11}a_{12}+a_{1}a_{5}a_{8}a_{10}a_{12}-a_{1}a_{4}a_{9}a_{10}a_{12}\\
&-r_{e}a_{2}a_{9}a_{11}a_{12}q_{e}-r_{q}a_{2}a_{7}a_{11}a_{13}q_{q}+r_{q}a_{2}a_{8}a_{10}a_{13}q_{q}-Sba_{2}a_{8}a_{10}a_{12}e_{H}+r_{q}a_{2}a_{9}a_{10}g_{H}q_{q}\\
&+Sba_{2}a_{9}a_{10}a_{12}+Sba_{2}a_{7}a_{11}a_{12}e_{H}+r_{e}a_{3}a_{6}a_{11}a_{13}q_{e}+r_{e}a_{5}a_{6}a_{11}a_{12}q_{e}-Sba_{3}a_{6}a_{10}e_{H}g_{H}\\
&-Sba_{3}a_{6}a_{10}a_{13}-r_{q}a_{4}a_{6}a_{10}a_{13}q_{q}+Sba_{4}a_{6}a_{10}a_{12}e_{H}-r_{q}a_{5}a_{6}a_{10}g_{H}q_{q}-Sba_{5}a_{6}a_{10}a_{12}-\\
&(a_{11}a_{1}a_{3}a_{7}-a_{10}a_{1}a_{3}a_{8}+a_{1}a_{3}a_{7}a_{13}+a_{1}a_{3}a_{11}a_{13}+a_{1}a_{5}a_{7}a_{12}+a_{1}a_{5}a_{11}a_{12}-a_{1}a_{7}a_{11}a_{13}\\
&+a_{1}a_{8}a_{10}a_{13}+a_{1}a_{9}a_{10}g_{H}+a_{3}a_{7}a_{11}a_{13}-a_{3}a_{8}a_{10}a_{13}-a_{3}a_{9}a_{10}g_{H}+a_{4}a_{9}a_{10}a_{12}+a_{5}a_{7}a_{11}a_{12}\\
&-a_{5}a_{8}a_{10}a_{12}-Sba_{6}a_{10}a_{13}+r_{e}a_{2}a_{9}a_{12}q_{e}-r_{e}a_{3}a_{6}a_{13}q_{e}-r_{e}a_{5}a_{6}a_{12}q_{e}+r_{e}a_{6}a_{11}a_{13}q_{e}+\\
&r_{q}a_{2}a_{7}a_{13}q_{q}+r_{q}a_{2}a_{11}a_{13}q_{q}-a_{11}r_{e}a_{3}a_{6}q_{e}+a_{11}r_{q}a_{2}a_{7}q_{q}-a_{10}r_{q}a_{2}a_{8}q_{q}+a_{10}r_{q}a_{4}a_{6}q_{q}+\\
&a_{10}Sba_{3}a_{6}-Sba_{2}a_{7}a_{12}e_{H}-Sba_{2}a_{11}a_{12}e_{H}-Sba_{6}a_{10}e_{H}g_{H})\mu)l-(a_{1}a_{3}a_{7}a_{11}a_{13}-\\
&a_{1}a_{3}a_{8}a_{10}a_{13}-a_{1}a_{3}a_{9}a_{10}g_{H}+a_{1}a_{5}a_{7}a_{11}a_{12}-a_{1}a_{5}a_{8}a_{10}a_{12}+a_{1}a_{4}a_{9}a_{10}a_{12}+r_{e}a_{2}a_{9}a_{11}a_{12}q_{e}\\
&+r_{q}a_{2}a_{7}a_{11}a_{13}q_{q}-r_{q}a_{2}a_{8}a_{10}a_{13}q_{q}+Sba_{2}a_{8}a_{10}a_{12}e_{H}-r_{q}a_{2}a_{9}a_{10}g_{H}q_{q}-Sba_{2}a_{9}a_{10}a_{12}-\\
&Sba_{2}a_{7}a_{11}a_{12}e_{H}-r_{e}a_{3}a_{6}a_{11}a_{13}q_{e}-r_{e}a_{5}a_{6}a_{11}a_{12}q_{e}+Sba_{3}a_{6}a_{10}e_{H}g_{H}+Sba_{3}a_{6}a_{10}a_{13}\\
&+r_{q}a_{4}a_{6}a_{10}a_{13}q_{q}-Sba_{4}a_{6}a_{10}a_{12}e_{H}+r_{q}a_{5}a_{6}a_{10}g_{H}q_{q}+Sba_{5}a_{6}a_{10}a_{12})\mu
\end{aligned}\right\} \tag{3.165}
$$

Now set $\lambda = iw$ into equation (3.165) and apply the of Bellman and cooke's theorem (1963)

$$
H(iw) = \left\{ \begin{aligned}
& w^6 + (a_{11} + a_{13} + a_1 - a_3 + a_7 + \mu) w^5 + (- r_e a_6 q_e - r_q a_2 q_q - a_1 a_3 + a_1 a_7 \\
& + a_1 a_{11} + a_{13} a_1 - a_3 a_7 - a_{13} a_3 - a_5 a_{12} + a_{11} a_7 + a_7 a_{13} - a_8 a_{10} + a_{11} a_{13} \\
& - (- a_{11} - a_{13} - a_1 + a_3 - a_7)\mu) w^4 + i(- r_q a_2 a_7 q_q + r_e a_3 a_6 q_e - r_q a_2 a_{13} q_q \\
& + Sba_6 a_{10} - r_e a_6 a_{11} q_e - r_e a_6 a_{13} q_e - r_q a_2 a_{11} q_q - (r_e a_6 q_e + r_q a_2 q_q + a_1 a_3 \\
& - a_1 a_7 - a_1 a_{11} - a_1 a_{13} + a_3 a_7 + a_3 a_{11} + a_3 a_{13} + a_5 a_{12} - a_7 a_{11} - a_7 a_{13} + \\
& a_8 a_{10} - a_{11} a_{13})\mu + Sba_2 a_{12} e_H - a_3 a_7 a_{13} + a_3 a_8 a_{10} - a_3 a_{11} a_{13} - a_5 a_7 a_{12} \\
& - a_5 a_{11} a_{12} - a_{13} a_1 a_3 - a_5 a_{12} a_1 - a_1 a_3 a_{11} + a_1 a_7 a_{11} + a_1 a_7 a_{13} - a_1 a_8 a_{10} + \\
& a_1 a_{11} a_{13} - a_3 a_7 a_{11} + a_7 a_{11} a_{13} - g_H a_{10} a_9 - a_8 a_{10} a_{13} - a_1 a_3 a_7) w^3 + (- a_{11} a_1 \\
& a_3 a_7 + a_{10} a_1 a_3 a_8 - a_1 a_3 a_7 a_{13} - a_1 a_3 a_{11} a_{13} - a_1 a_5 a_7 a_{12} - a_1 a_5 a_{11} a_{12} + a_1 a_7 \\
& a_{11} a_{13} - a_1 a_8 a_{10} a_{13} - a_1 a_9 a_{10} g_H - (r_q a_2 a_7 q_q - r_e a_3 a_6 q_e + r_q a_2 a_{13} q_q - Sba_6 a_{10} \\
& + r_e a_6 a_{11} q_e + r_e a_6 a_{13} q_e + r_q a_2 a_{11} q_q - Sba_2 a_{12} e_H + a_3 a_7 a_{13} - a_3 a_8 a_{10} + a_3 a_{11} \\
& a_{13} + a_5 a_7 a_{12} + a_5 a_{11} a_{12} + a_{13} a_1 a_3 + a_5 a_{12} a_1 + a_1 a_3 a_{11} - a_1 a_7 a_{11} - a_1 a_7 a_{13} + \\
& a_1 a_8 a_{10} - a_1 a_{11} a_{13} + a_3 a_7 a_{11} - a_7 a_{11} a_{13} + g_H a_{10} a_9 + a_8 a_{10} a_{13} + a_1 a_3 a_7)\mu - \\
& a_3 a_7 a_{11} a_{13} + a_3 a_8 a_{10} a_{13} + a_3 a_8 a_{10} a_{13} + a_3 a_9 a_{10} g_H - a_4 a_9 a_{10} a_{12} - a_5 a_7 a_{11} a_{12} + \\
& a_5 a_8 a_{10} a_{12} + Sba_6 a_{10} a_{13} - r_e a_2 a_9 a_{12} q_e + r_e a_3 a_6 a_{13} q_e + r_e a_5 a_6 a_{12} q_e - r_e a_6 a_{11} a_{13} q_e \\
& - r_q a_2 a_7 a_{13} q_q - r_q a_2 a_{11} a_{13} q_q + a_{11} r_e a_3 a_6 q_e - a_{11} r_q a_2 a_7 q_q + a_{10} r_q a_2 a_8 q_q - a_{10} r_q a_4 \\
& a_6 q_q - a_{10} Sba_3 a_6 + Sba_2 a_7 a_{12} e_H + Sba_2 a_{11} a_{12} e_H + Sba_6 a_{10} e_H g_H) w^2 + i(- a_1 a_3 a_7 \\
& a_{11} a_{13} + a_1 a_3 a_8 a_{10} a_{13} + a_1 a_3 a_9 a_{10} g_H - a_1 a_5 a_7 a_{11} a_{12} + a_1 a_5 a_8 a_{10} a_{12} - a_1 a_4 a_9 a_{10} a_{12} \\
& - r_e a_2 a_9 a_{11} a_{12} q_e - r_q a_2 a_7 a_{11} a_{13} q_q + r_q a_2 a_8 a_{10} a_{13} q_q - Sba_2 a_8 a_{10} a_{12} e_H + r_q a_2 a_9 a_{10} g_H q_q \\
& + Sba_2 a_9 a_{10} a_{12} + Sba_2 a_7 a_{11} a_{12} e_H + r_e a_3 a_6 a_{11} a_{13} q_e + r_e a_5 a_6 a_{11} a_{12} q_e - Sba_3 a_6 a_{10} e_H g_H \\
& - Sba_3 a_6 a_{10} a_{13} - r_q a_4 a_6 a_{10} a_{13} q_q + Sba_4 a_6 a_{10} a_{12} e_H - r_q a_5 a_6 a_{10} g_H q_q - Sba_5 a_6 a_{10} a_{12} - \\
& (a_{11} a_1 a_3 a_7 - a_{10} a_1 a_3 a_8 + a_1 a_3 a_7 a_{13} + a_1 a_3 a_{11} a_{13} + a_1 a_5 a_7 a_{12} + a_1 a_5 a_{11} a_{12} - a_1 a_7 a_{11} a_{13} \\
& + a_1 a_8 a_{10} a_{13} + a_1 a_9 a_{10} g_H + a_3 a_7 a_{11} a_{13} - a_3 a_8 a_{10} a_{13} - a_3 a_9 a_{10} g_H + a_4 a_9 a_{10} a_{12} + a_5 a_7 a_{11} a_{12} \\
& - a_5 a_8 a_{10} a_{12} - Sba_6 a_{10} a_{13} + r_e a_2 a_9 a_{12} q_e - r_e a_3 a_6 a_{13} q_e - r_e a_5 a_6 a_{12} q_e + r_e a_6 a_{11} a_{13} q_e + \\
& r_q a_2 a_7 a_{13} q_q + r_q a_2 a_{11} a_{13} q_q - a_{11} r_e a_3 a_6 q_e + a_{11} r_q a_2 a_7 q_q - a_{10} r_q a_2 a_8 q_q + a_{10} r_q a_4 a_6 q_q + \\
& a_{10} Sba_3 a_6 - Sba_2 a_7 a_{12} e_H - Sba_2 a_{11} a_{12} e_H - Sba_6 a_{10} e_H g_H)\mu) w - (a_1 a_3 a_7 a_{11} a_{13} - \\
& a_1 a_3 a_8 a_{10} a_{13} - a_1 a_3 a_9 a_{10} g_H + a_1 a_5 a_7 a_{11} a_{12} - a_1 a_5 a_8 a_{10} a_{12} + a_1 a_4 a_9 a_{10} a_{12} + r_e a_2 a_9 a_{11} a_{12} q_e \\
& + r_q a_2 a_7 a_{11} a_{13} q_q - r_q a_2 a_8 a_{10} a_{13} q_q + Sba_2 a_8 a_{10} a_{12} e_H - r_q a_2 a_9 a_{10} g_H q_q - Sba_2 a_9 a_{10} a_{12} - \\
& Sba_2 a_7 a_{11} a_{12} e_H - r_e a_3 a_6 a_{11} a_{13} q_e - r_e a_5 a_6 a_{11} a_{12} q_e + Sba_3 a_6 a_{10} e_H g_H + Sba_3 a_6 a_{10} a_{13} \\
& + r_q a_4 a_6 a_{10} a_{13} q_q - Sba_4 a_6 a_{10} a_{12} e_H + r_q a_5 a_6 a_{10} g_H q_q + Sba_5 a_6 a_{10} a_{12})\mu = 0
\end{aligned} \right\} \tag{3.166}
$$

Resolving into real and imaginary parts we have

$$H(iw)=F(w)+iG(w) \tag{3.167}$$

$$F(w)=\left.\begin{aligned}
&w^6+(-r_e a_6 q_e-r_q a_2 q_q-a_1a_3+a_1a_7+a_1a_{11}+a_{13}a_1-a_3a_7-a_{13}a_3-a_5a_{12}\\
&+a_{11}a_7+a_7a_{13}-a_8a_{10}+a_{11}a_{13}-(-a_{11}-a_{13}-a_1+a_3-a_7)\mu)w^4+(-a_{11}a_1\\
&a_3a_7+a_{10}a_1a_3a_8-a_1a_3a_7a_{13}-a_1a_3a_{11}a_{13}-a_1a_5a_7a_{12}-a_1a_5a_{11}a_{12}+a_1a_7\\
&a_{11}a_{13}-a_1a_8a_{10}a_{13}-a_1a_9a_{10}g_H-(r_q a_2a_7q_q-r_e a_3a_6q_e+r_q a_2a_{13}q_q-Sba_6a_{10}\\
&+r_e a_6a_{11}q_e+r_e a_6a_{13}q_e+r_q a_2a_{11}q_q-Sba_2a_{12}e_H+a_3a_7a_{13}-a_3a_8a_{10}+a_3a_{11}\\
&a_{13}+a_5a_7a_{12}+a_5a_{11}a_{12}+a_{13}a_1a_3+a_5a_{12}a_1+a_1a_3a_{11}-a_1a_7a_{11}-a_1a_7a_{13}+\\
&a_1a_8a_{10}-a_1a_{11}a_{13}+a_3a_7a_{11}-a_7a_{11}a_{13}+g_Ha_{10}a_9+a_8a_{10}a_{13}+a_1a_3a_7)\mu-\\
&(a_3a_7a_{11}a_{13}+a_3a_8a_{10}a_{13}+a_3a_8a_{10}a_{13}+a_3a_9a_{10}g_H-a_4a_9a_{10}a_{12}-a_5a_7a_{11}a_{12}+\\
&a_5a_8a_{10}a_{12}+Sba_6a_{10}a_{13}-r_e a_2a_9a_{12}q_e+r_e a_3a_6a_{13}q_e+r_e a_5a_6a_{12}q_e-r_e a_6a_{11}a_{13}q_e\\
&-r_q a_2a_7a_{13}q_q-r_q a_2a_{11}a_{13}q_q+a_{11}r_e a_3a_6q_e-a_{11}r_q a_2a_7q_q+a_{10}r_q a_2a_8q_q-a_{10}r_q a_4\\
&a_6q_q-a_{10}Sba_3a_6+Sba_2a_7a_{12}e_H+Sba_2a_{11}a_{12}e_H+Sba_6a_{10}e_Hg_H)w^2-(a_1a_3a_7a_{11}a_{13}-\\
&a_1a_3a_8a_{10}a_{13}-a_1a_3a_9a_{10}g_H+a_1a_5a_7a_{11}a_{12}-a_1a_5a_8a_{10}a_{12}+a_1a_4a_9a_{10}a_{12}+r_e a_2a_9a_{11}a_{12}q_e\\
&+r_q a_2a_7a_{11}a_{13}q_q-r_q a_2a_8a_{10}a_{13}q_q+Sba_2a_8a_{10}a_{12}e_H-r_q a_2a_9a_{10}g_Hq_q-Sba_2a_9a_{10}a_{12}-\\
&Sba_2a_7a_{11}a_{12}e_H-r_e a_3a_6a_{11}a_{13}q_e-r_e a_5a_6a_{11}a_{12}q_e+Sba_3a_6a_{10}e_Hg_H+Sba_3a_6a_{10}a_{13}\\
&+r_q a_4a_6a_{10}a_{13}q_q-Sba_4a_6a_{10}a_{12}e_H+r_q a_5a_6a_{10}g_Hq_q+Sba_5a_6a_{10}a_{12})\mu=0
\end{aligned}\right\} \tag{3.168}$$

$$G(w)=\left.\begin{aligned}
&(a_{11}+a_{13}+a_1-a_3+a_7+\mu)w^5-(r_e a_6q_e+r_q a_2q_q+a_1a_3\\
&-a_1a_7-a_1a_{11}-a_1a_{13}+a_3a_7+a_3a_{11}+a_3a_{13}+a_5a_{12}-a_7a_{11}-a_7a_{13}+\\
&a_8a_{10}-a_{11}a_{13})\mu+Sba_2a_{12}e_H-a_3a_7a_{13}+a_3a_8a_{10}-a_3a_{11}a_{13}-a_5a_7a_{12}\\
&-a_5a_{11}a_{12}-a_{13}a_1a_3-a_5a_{12}a_1-a_1a_3a_{11}+a_1a_7a_{11}+a_1a_7a_{13}-a_1a_8a_{10}+\\
&a_1a_{11}a_{13}-a_3a_7a_{11}+a_7a_{11}a_{13}-g_Ha_{10}a_9-a_8a_{10}a_{13}-a_1a_3a_7)w^3-\\
&(a_{11}a_1a_3a_7-a_{10}a_1a_3a_8+a_1a_3a_7a_{13}+a_1a_3a_{11}a_{13}+a_1a_5a_7a_{12}+a_1a_5a_{11}a_{12}-a_1a_7a_{11}a_{13}\\
&+a_1a_8a_{10}a_{13}+a_1a_9a_{10}g_H+a_3a_7a_{11}a_{13}-a_3a_8a_{10}a_{13}-a_3a_9a_{10}g_H+a_4a_9a_{10}a_{12}+a_5a_7a_{11}a_{12}\\
&-a_5a_8a_{10}a_{12}-(Sba_6a_{10}a_{13}+r_e a_2a_9a_{12}q_e-r_e a_3a_6a_{13}q_e-r_e a_5a_6a_{12}q_e+r_e a_6a_{11}a_{13}q_e+\\
&r_q a_2a_7a_{13}q_q+r_q a_2a_{11}a_{13}q_q-a_{11}r_e a_3a_6q_e+a_{11}r_q a_2a_7q_q-a_{10}r_q a_2a_8q_q+a_{10}r_q a_4a_6q_q+\\
&a_{10}Sba_3a_6-Sba_2a_7a_{12}e_H-Sba_2a_{11}a_{12}e_H-Sba_6a_{10}e_Hg_H)\mu)w
\end{aligned}\right\} \tag{3.169}$$

Differentiating equation (3.168) and (3.169) with respect to w and setting $w=0$ we have $F'(0)=0$

$$F'(0)=\left.\begin{matrix} 6w^5+4(-r_e a_6 q_e - r_q a_2 q_q - a_1 a_3 + a_1 a_7 + a_1 a_{11} + a_{13} a_1 - a_3 a_7 - a_{13} a_3 - a_5 a_{12} \\ + a_{11} a_7 + a_7 a_{13} - a_8 a_{10} + a_{11} a_{13} - (-a_{11} - a_{13} - a_1 + a_3 - a_7)\mu) w^3 - 2(a_3 a_7 a_{11} a_{13} + \\ a_3 a_8 a_{10} a_{13} + a_3 a_8 a_{10} a_{13} + a_3 a_9 a_{10} g_H - a_4 a_9 a_{10} a_{12} - a_5 a_7 a_{11} a_{12} + \\ a_5 a_8 a_{10} a_{12} + Sba_6 a_{10} a_{13} - r_e a_2 a_9 a_{12} q_e + r_e a_3 a_6 a_{13} q_e + r_e a_5 a_6 a_{12} q_e - r_e a_6 a_{11} a_{13} q_e \\ - r_q a_2 a_7 a_{13} q_q - r_q a_2 a_{11} a_{13} q_q + a_{11} r_e a_3 a_6 q_e - a_{11} r_q a_2 a_7 q_q + a_{10} r_q a_2 a_8 q_q - a_{10} r_q a_4 \\ a_6 q_q - a_{10} Sba_3 a_6 + Sba_2 a_7 a_{12} e_H + Sba_2 a_{11} a_{12} e_H + Sba_6 a_{10} e_H g_H) w \end{matrix}\right\} \quad (3.170)$$

Also set $w=0$ into equation (3.168) and (3.169)

$$G'(0)=\left.\begin{matrix} -(a_{11} a_1 a_3 a_7 - a_{10} a_1 a_3 a_8 + a_1 a_3 a_7 a_{13} + a_1 a_3 a_{11} a_{13} + a_1 a_5 a_7 a_{12} + a_1 a_5 a_{11} a_{12} - a_1 a_7 a_{11} a_{13} \\ + a_1 a_8 a_{10} a_{13} + a_1 a_9 a_{10} g_H + a_3 a_7 a_{11} a_{13} - a_3 a_8 a_{10} a_{13} - a_3 a_9 a_{10} g_H + a_4 a_9 a_{10} a_{12} + a_5 a_7 a_{11} a_{12} \\ - a_5 a_8 a_{10} a_{12} - Sba_6 a_{10} a_{13} + r_e a_2 a_9 a_{12} q_e - r_e a_3 a_6 a_{13} q_e - r_e a_5 a_6 a_{12} q_e + r_e a_6 a_{11} a_{13} q_e + \\ r_q a_2 a_7 a_{13} q_q + r_q a_2 a_{11} a_{13} q_q - a_{11} r_e a_3 a_6 q_e + a_{11} r_q a_2 a_7 q_q - a_{10} r_q a_2 a_8 q_q + a_{10} r_q a_4 a_6 q_q + \\ a_{10} Sba_3 a_6 - Sba_2 a_7 a_{12} e_H - Sba_2 a_{11} a_{12} e_H - Sba_6 a_{10} e_H g_H)\mu) \end{matrix}\right\}$$

(3.171)

$$G(0)=0 \quad (3.172)$$

Since

$$F'(0)=0, F(0)\neq 0, G'(0)\neq 0 \text{ and } G(0)=0 \quad (3.173)$$

Hence

$$F(0)G'(0)-F'(0)G(0)\neq 0 \quad (3.174)$$

Therefore, the nonzero equilibrium state is stable

3.10 Analytical Solution of the Model

3.10.1 Semi- Analytical solution of the model using homotopy perturbation method

The fundamental of Homotopy Perturbation Method (HPM) was first proposed by Ji-Haun (2000). The Homotopy Perturbation Method (HPM), which provides analytical approximate solution, is applied to various linear and non-linear equations (Abubakar *et al.*, 2013). The homotopy perturbation method (HPM) is a series expansion method used in the solution of nonlinear full differential equations (Jiya, 2010).

To show the simple concepts of this method, he considered the following non-linear differential equation given as Equation (3.132) to Equation (3.133) (Somma *et al.* 2017):

$$A_3(U) - f(r) = 0, \; r \in \Omega \tag{3.175}$$

Subject to the boundary condition

$$B_3\left(U, \frac{\partial U}{\partial n}\right) = 0, \qquad r \in \Gamma \tag{3.176}$$

Where A_3 is a general differential operator, B_3 a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A_3 can be divided into two parts L and N, where L is the linear part, and N is the nonlinear part. Equation (3.119) can be written as:

$$L(U) + N(U) - f(r) = 0, \; r \in \Omega \tag{3.177}$$

The Homotopy Perturbation structure is shown as follows

$$H(V,h) = (1-h)[L(V) - L(U_0)] + h[A(V) - f(r)] = 0 \tag{3.178}$$

Where $V(r,P) : \Omega \in [0,1] \rightarrow R$ (3.179)

In Equation (3.117) $P \in [0,1]$ is an embedding parameter and U_0 is the approximation that satisfies the boundary condition. It can be assumed that the solution of the equation (3.177) can be written as power series in h given as Equation (3.180) to Equation (3.181):

$$V = V_0 + hV_1 + h^2V_2 + \ldots \tag{3.180}$$

And the best approximation for the solution is:

$$U = \lim_{h \to 1} v = v_0 + hv_1 + h^2 v_2 + \ldots \tag{3.181}$$

The series (3.181) is convergent for most cases. However, the convergent rate depends on the nonlinear operator A (V)

3.10.2 Solution of the model equations

From differentiation Equation given as Equation (3.1) to Equation (3.5)

$$\frac{dS}{dt} + \beta SI + \beta S\varepsilon_H H + \mu S - \theta_e \rho_e E - \theta_q \rho_q Q - \Lambda = 0 \tag{3.182}$$

$$\frac{dQ}{dt} + \left(1 - \theta_q\right)\rho_q Q + \left(\theta_q \rho_q + \mu\right)Q - \theta\beta S(I + \varepsilon_H H) = 0 \tag{3.183}$$

$$\frac{dE}{dt} + \left(1 - \theta_e\right)\rho_e E + \left(\theta_e \rho_e + \mu\right)E - (1-\theta)\beta S(I + \varepsilon_H H) = 0 \tag{3.184}$$

$$\frac{dI}{dt} + (\gamma_H + \delta_I + \mu)I - (1-\theta_e)\rho_e E = 0 \tag{3.185}$$

$$\frac{dH}{dt} + (\gamma_1 + \delta_H + \mu)H - (1-\theta_q)\rho_q Q - \gamma_H I = 0 \tag{3.186}$$

$$\frac{dR}{dt} + \mu R - \gamma_1 H = 0 \tag{3.187}$$

With the initial condition given as equation

Let

$$S(t) = u_0 + hu_1 + h^2u_1 + ... \tag{3.188}$$

$$Q(t) = v_0 + hv_1 + h^2v_1 + ... \tag{3.189}$$

$$E(t) = w_0 + hw_1 + h^2w_1 + ... \tag{3.190}$$

$$I(t) = x_0 + hx_1 + h^2x_1 + ... \tag{3.191}$$

$$H(t) = y_0 + hy_1 + h^2y_1 + ... \tag{3.192}$$

$$R(t) = z_0 + hz_1 + h^2z_1 + ... \tag{3.193}$$

Applying HPM into equation (3.139)

$$(1-p)\frac{dS}{dt} + p\left[\frac{dS}{dt} + \beta SI + \beta\varepsilon_H SH + \mu S - \theta_q\rho_q Q - \theta_e\rho_e E\right] = 0 \tag{3.194}$$

Substitute equation (3.145), (3.148), (3.149), (3.146), (3.147) and into (3.151)

$$\left(u_0^1 + hu_1^1 + h^2u_1^1 + ...\right)p\begin{bmatrix}\beta\left(u_0 + hu_1 + h^2u_1 + ...\right)\left(x_0 + hx_1 + h^2x_1 + ...\right) + \beta\varepsilon_H \\ \left(u_0 + hu_1 + h^2u_1 + ...\right)\left(y_0 + hy_1 + h^2y_1 + ...\right) + \mu \\ \left(u_0 + hu_1 + h^2u_1 + ...\right) - \theta_q\rho_q\left(v_0 + hv_1 + h^2v_1 + ...\right) - \\ \theta_e\rho_e\left(w_0 + hw_1 + h^2w_1 + ...\right)\end{bmatrix} = 0 \tag{3.195}$$

Collecting the coefficient of power p given as equation

$$p^0 : \mathrm{u}_0^1 = 0 \tag{3.196}$$

$$p^1 : u_1^1 + \beta u_0x_0 + \beta\varepsilon_H u_0y_0 + \mu u_0 - \theta_q\rho_q v_0 - \theta_e\rho_e w_0 - \Lambda = 0 \tag{3.197}$$

$$p^2 : u_2^1 + \beta\left(u_1 x_0 + u_0 x_1\right) + \beta\varepsilon_H\left(u_1 y_0 + u_0 y_1\right) + \mu u_1 - \theta_q \rho_q v_1 - \theta_e \rho_e w_1 = 0 \tag{3.198}$$

Applying HPM to equation (3.183)

$$\left(1-p\right)\frac{dQ}{dt} + p\left[\frac{dQ}{dt} + \left(1-\theta_q\right)\rho_q Q + \left(\theta_q \rho_q + \mu\right)Q - \theta\beta S(I + \varepsilon_H H)\right] = 0 \tag{3.199}$$

Substitute equation (3.188), (3.189), (3.190), (3.191), (3.192) and (3.193) into (3.199)

$$\left(v_0^1 + hv_1^1 + h^2 v_1^1 + \ldots\right) p \left[\begin{array}{l} \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q\rho_q + \mu\right)\right)\left(v_0 + hv_1 + h^2 v_1 + \ldots\right) - \theta\beta\left(u_0 + hu_1 + h^2 u_1 + \ldots\right) \\ \left(x_0 + hx_1 + h^2 x_1 + \ldots\right) - \theta\beta\varepsilon_H\left(u_0 + hu_1 + h^2 u_1 + \ldots\right) \\ \left(y_0 + hy_1 + h^2 y_1 + \ldots\right) \end{array}\right] = 0 \tag{3.200}$$

Collecting the coefficient of power p given as equation

$$p^0 : \mathrm{v}_0^1 = 0 \tag{3.201}$$

$$p^1 : \mathrm{v}_1^1 + \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q\rho_q + \mu\right)\right)v_0 - \theta\beta u_0 x_0 - \theta\beta\varepsilon_H u_0 y_0 = 0$$

(3.202)

$$p^2 : \mathrm{v}_2^1 + \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q\rho_q + \mu\right)\right)v_1 - \theta\beta\left(u_1 x_0 + u_0 x_1\right) - \theta\beta\varepsilon_H\left(u_1 y_0 + u_0 y_1\right) = 0 \tag{3.203}$$

Applying HPM to equation (3.184)

$$\left(1-p\right)\frac{dE}{dt} + p\left[\frac{dE}{dt} + \left(1-\theta_e\right)\rho_e E + \left(\theta_e \rho_e + \mu\right)E - (1-\theta)\beta S(I + \varepsilon_H H)\right] = 0 \tag{3.204}$$

Substitute equation (3.188), (3.189), (3.190), (3.191), (3.192) and (3.193) into (3.204)

$$\left(w_0^1+hw_1^1+h^2w_1^1+\ldots\right)p\begin{bmatrix}\left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)\left(w_0+hw_1+h^2w_2+\ldots\right)-\left(1-\theta\right)\beta\left(u_0+hu_1+h^2u_1+\ldots\right)\\\left(x_0+hx_1+h^2x_1+\ldots\right)-\left(1-\theta\right)\beta\varepsilon_H\left(u_0+hu_1+h^2u_1+\ldots\right)\\\left(y_0+hy_1+h^2y_1+\ldots\right)\end{bmatrix}=0 \quad (3.205)$$

Collecting the coefficient of power p given as equation

$$p^0:\mathrm{w}_0^1=0 \quad (3.206)$$

$$p^1:\mathrm{w}_1^1+\left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)w_0-\left(1-\theta\right)\beta u_0x_0-\left(1-\theta\right)\beta\varepsilon_H u_0y_0=0 \quad (3.207)$$

$$p^2:\mathrm{w}_2^1+\left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)w_1-\left(1-\theta\right)\beta\left(u_1x_0+u_0x_1\right)-\left(1-\theta\right)\beta\varepsilon_H\left(u_1y_0+u_0y_1\right)=0 \quad (3.208)$$

Applying HPM to equation (3.185)

$$\left(1-p\right)\frac{dI}{dt}+p\left[\frac{dI}{dt}+(\gamma_H+\delta_I+\mu)I-(1-\theta_e)\rho_eE\right]=0 \quad (3.209)$$

Substitute equation (3.188), (3.189), (3.190), (3.191), (3.192) and (3.193) into (3.209)

$$\left(x_0^1+hx_1^1+h^2x_2^1+\ldots\right)p\left[(\gamma_H+\delta_I+\mu)\left(x_0+hx_1+h^2x_2+\ldots\right)-\left(1-\theta_e\right)\rho_e\left(w_0+hw_1+h^2w_2+\ldots\right)\right]=0 \quad (3.210)$$

Collecting the coefficient of power p given as equation

$$p^0:\mathrm{x}_0^1=0 \quad (3.211)$$

$$p^1:\mathrm{x}_1^1+(\gamma_H+\delta_I+\mu)\,\mathrm{x}_0-\left(1-\theta_e\right)\rho_e w_0=0 \quad (3.212)$$

$$p^2:\mathrm{x}_2^1+(\gamma_H+\delta_I+\mu)\,\mathrm{x}_1-\left(1-\theta_e\right)\rho_e w_1=0 \quad (3.213)$$

Applying HPM to equation (3.186)

$$\left(1-p\right)\frac{dH}{dt}+p\left[\frac{dH}{dt}+(\gamma_1+\delta_H+\mu)H-(1-\theta_q)\rho_q Q-\gamma_H I\right]=0 \quad (3.214)$$

Substitute equation (3.188), (3.189), (3.190), (3.191), (3.192) and (3.193) into (3.214)

$$\left(y_0^1+hy_1^1+h^2y_2^1+...\right)p\begin{bmatrix}(\gamma_I+\delta_H+\mu)\left(y_0+hy_1+h^2y_2+...\right)-\gamma_H\left(x_0+hx_1+h^2x_2+...\right)-\\ \left(1-\theta_q\right)\rho_q\left(v_0+hv_1+h^2v_2+...\right)\end{bmatrix}=0 \quad (3.215)$$

Collecting the coefficient of power p given as equation

$$p^0 : \mathrm{y}_0^1=0 \quad (3.216)$$

$$p^1 : \mathrm{y}_1^1+(\gamma_I+\delta_H+\mu)\,\mathrm{y}_0-\gamma_H x_0-\left(1-\theta_q\right)\rho_q v_0=0 \quad (3.217)$$

$$p^2 : \mathrm{y}_2^1+(\gamma_I+\delta_H+\mu)\,\mathrm{y}_1-\gamma_H x_1-\left(1-\theta_q\right)\rho_q v_1=0 \quad (3.218)$$

Applying HPM to equation (3.187)

$$\left(1-p\right)\frac{dR}{dt}+p\left[\frac{dR}{dt}+\mu R-\gamma_1 H\right]=0 \quad (3.219)$$

Substitute equation (3.188), (3.189), (3.190), (3.191), (3.192) and (3.193) into (3.219)

$$\left(z_0^1+hz_1^1+h^2z_2^1+...\right)p\left[\mu\left(z_0+hz_1+h^2z_2+...\right)-\gamma_I\left(y_0+hy_1+h^2y_2+...\right)\right]=0 \quad (3.220)$$

Collecting the coefficient of power p given as equation

$$p^0 : \mathrm{z}_0^1=0 \quad (3.221)$$

$$p^1 : \mathrm{z}_1^1+\mu\mathrm{z}_0-\gamma_I y_0=0 \quad (3.222)$$

$$p^2 : \mathrm{z}_2^1+\mu z_1-\gamma_I y_1=0 \quad (3.223)$$

From equation (3.196)

$$p^0 : \mathrm{u}_0^1 = 0 \tag{3.224}$$

Integration both sides

$$u_0 = T_1 \tag{3.225}$$

Applying the initial condition

$$u_0(0) = S_0 = u_0 \tag{3.226}$$

$$T_1 = S_0$$

(3.227)

$$u_0 = S_0 \tag{3.228}$$

From equation (3.201)

$$p^0 : \mathrm{v}_0^1 = 0 \tag{3.229}$$

Integration both sides

$$v_0 = T_2$$

(3.230)

Applying the initial condition

$$v_0(0) = Q_0 = v_0 \tag{3.231}$$

$$T_2 = Q_0 \tag{3.232}$$

$$v_0 = Q_0 \tag{3.233}$$

From equation (3.206)

$$p^0 : \mathrm{w}_0^1 = 0 \tag{3.234}$$

Integration both sides

$$w_0 = T_3 \tag{3.235}$$

Applying the initial condition

$$w_0(0) = E_0 = w_0 \tag{3.236}$$

$$T_3 = E_0 \tag{3.237}$$

$$w_0 = E_0 \tag{3.238}$$

From equation (3.211)

$$p^0 : \mathrm{x}_0^1 = 0 \tag{3.239}$$

Integration both sides

$$x_0 = T_4 \tag{3.240}$$

Applying the initial condition

$$x_0(0) = I_0 = x_0 \quad (3.241)$$

$$T_4 = I_0 \quad (3.242)$$

$$x_0 = I_0 \quad (3.243)$$

From equation (3.216)

$$p^0 : \mathrm{y}_0^1 = 0 \quad (3.244)$$

Integration both sides

$$y_0 = T_5 \quad (3.245)$$

Applying the initial condition

$$y_0(0) = H_0 = y_0 \quad (3.246)$$

$$T_5 = H_0 \quad (3.247)$$

$$y_0 = H_0 \quad (3.248)$$

From equation (3.221)

$$p^0 : \mathrm{z}_0^1 = 0 \quad (3.249)$$

Integration both sides

$$z_0 = T_6 \tag{3.250}$$

Applying the initial condition

$$z_0(0) = R_0 = z_0 \tag{3.251}$$

$$T_6 = R_0 \tag{3.252}$$

$$z_0 = R_0 \tag{3.253}$$

From equation (3.197)

$$: u_1^1 + \beta u_0 x_0 + \beta \varepsilon_H u_0 y_0 + \mu u_0 - \theta_q \rho_q v_0 - \theta_e \rho_e w_0 - \Lambda = 0 \tag{3.254}$$

$$u_1^1 = \Lambda + \theta_q \rho_q v_0 + \theta_e \rho_e w_0 - \beta u_0 x_0 - \beta \varepsilon_H u_0 y_0 - \mu u_0 \tag{3.255}$$

Integrating both sides

$$\int du_1^1 = \int \left(\Lambda + \theta_q \rho_q v_0 + \theta_e \rho_e w_0 - \beta u_0 x_0 - \beta \varepsilon_H u_0 y_0 - \mu u_0 \right) dt \tag{3.256}$$

$$u_1 = \left(\Lambda + \theta_q \rho_q v_0 + \theta_e \rho_e w_0 - \beta u_0 x_0 - \beta \varepsilon_H u_0 y_0 - \mu u_0 \right) t + T_7 \tag{3.257}$$

Applying the initial condition

$$T_7 = 0 \tag{3.258}$$

$$u_1 = \left(\Lambda + \theta_q \rho_q v_0 + \theta_e \rho_e w_0 - \beta u_0 x_0 - \beta \varepsilon_H u_0 y_0 - \mu u_0 \right) t \tag{3.259}$$

Substitute equation (3.228), (3.233), (3.238), (3.243) and (3.248) into (3.259)

$$u_1 = \left(\Lambda + \theta_q \rho_q Q_0 + \theta_e \rho_e E_0 - \beta S_0 I_0 - \beta \varepsilon_H S_0 H_0 - \mu S_0\right) t \quad (3.217)$$

From equation (3.202)

$$v_1^1 + \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q \rho_q + \mu\right)\right) v_0 - \theta\beta u_0 x_0 - \theta\beta\varepsilon_H u_0 y_0 = 0 \quad (3.261)$$

$$v_1^1 = \theta\beta u_0 x_0 + \theta\beta\varepsilon_H u_0 y_0 - \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q \rho_q + \mu\right)\right) v_0 \quad (3.262)$$

Integrating both sides

$$\int dv_1^1 = \int \left(\theta\beta u_0 x_0 + \theta\beta\varepsilon_H u_0 y_0 - \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q \rho_q + \mu\right)\right) v_0\right) dt \quad (3.263)$$

$$v_1(t) = \left(\theta\beta u_0 x_0 + \theta\beta\varepsilon_H u_0 y_0 - \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q \rho_q + \mu\right)\right) v_0\right) t + T_8 \quad (3.264)$$

Applying the initial condition

$$T_8 = 0 \quad (3.265)$$

$$v_1(t) = \left(\theta\beta u_0 x_0 + \theta\beta\varepsilon_H u_0 y_0 - \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q \rho_q + \mu\right)\right) v_0\right) t \quad (3.266)$$

Substitute equation (3.228), (3.233), (3.238), (3.243), (3.248) and (3.253) into (3.266)

$$v_1(t) = \left(\theta\beta S_0 I_0 + \theta\beta\varepsilon_H S_0 H_0 - \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q \rho_q + \mu\right)\right) Q_0\right) t \quad (3.267)$$

From equation (3.207)

$$w_1^1 + \left(\left(1-\theta_e\right)\rho_e + \left(\theta_e \rho_e + \mu\right)\right) w_0 - \left(1-\theta\right)\beta u_0 x_0 - \left(1-\theta\right)\beta\varepsilon_H u_0 y_0 = 0 \quad (3.268)$$

$$w_1^1 = \left(1-\theta\right)\beta u_0 x_0 + \left(1-\theta\right)\beta\varepsilon_H u_0 y_0 - \left(\left(1-\theta_e\right)\rho_e + \left(\theta_e \rho_e + \mu\right)\right) w_0 \quad (3.269)$$

Integrating both sides

$$\int dw_1^1 = \int \left(\left(1-\theta\right)\beta u_0 x_0 + \left(1-\theta\right)\beta\varepsilon_H u_0 y_0 - \left(\left(1-\theta_e\right)\rho_e + \left(\theta_e\rho_e + \mu\right)\right) w_0 \right) dt \qquad (3.270)$$

$$w_1\left(t\right) = \left(\left(1-\theta\right)\beta u_0 x_0 + \left(1-\theta\right)\beta\varepsilon_H u_0 y_0 - \left(\left(1-\theta_e\right)\rho_e + \left(\theta_e\rho_e + \mu\right)\right) w_0 \right) t + T_9$$

(3.271)

Applying the initial condition

$$T_9 = 0 \qquad (3.272)$$

$$w_1\left(t\right) = \left(\left(1-\theta\right)\beta u_0 x_0 + \left(1-\theta\right)\beta\varepsilon_H u_0 y_0 - \left(\left(1-\theta_e\right)\rho_e + \left(\theta_e\rho_e + \mu\right)\right) w_0 \right) t \qquad (3.273)$$

Substitute equation (3.228), (3.233), (3.238), (3.243), (3.248) and (3.253) into (3.273)

$$w_1\left(t\right) = \left(\left(1-\theta\right)\beta S_0 I_0 + \left(1-\theta\right)\beta\varepsilon_H S_0 H_0 - \left(\left(1-\theta_e\right)\rho_e + \left(\theta_e\rho_e + \mu\right)\right) E_0 \right) t \qquad (3.274)$$

From equation (3.212)

$$\mathrm{x}_1^1 + (\gamma_H + \delta_I + \mu)\,\mathrm{x}_0 - \left(1-\theta_e\right)\rho_e w_0 = 0 \qquad (3.275)$$

$$\mathrm{x}_1^1 = \left(1-\theta_e\right)\rho_e w_0 - (\gamma_H + \delta_I + \mu)\,\mathrm{x}_0 \qquad (3.276)$$

Integrating both sides

$$\int dx_1^1 = \int \left(\left(1-\theta_e\right)\rho_e w_0 - (\gamma_H + \delta_I + \mu)\,\mathrm{x}_0 \right) dt \qquad (3.277)$$

$$x_1\left(t\right) = \left(\left(1-\theta_e\right)\rho_e w_0 - (\gamma_H + \delta_I + \mu)\,\mathrm{x}_0 \right) t + T_{10} \qquad (3.278)$$

Applying the initial condition

$$T_{10} = 0 \tag{3.279}$$

$$x_1(t) = \left(\left(1-\theta_e\right)\rho_e w_0 - (\gamma_H + \delta_I + \mu)\,\mathrm{x}_0\right)t \tag{3.280}$$

Substitute equation (3.228), (3.233), (3.238), (3.243), (3.248) and (3.253) into (3.280)

$$x_1(t) = \left(\left(1-\theta_e\right)\rho_e E_0 - (\gamma_H + \delta_I + \mu)\,\mathrm{I}_0\right)t \tag{3.281}$$

From equation (3.217)

$$y_1^1 + (\gamma_I + \delta_H + \mu)\,\mathrm{y}_0 - \gamma_H x_0 - \left(1-\theta_q\right)\rho_q v_0 = 0 \tag{3.282}$$

$$y_1^1 = \left(1-\theta_q\right)\rho_q v_0 + \gamma_H x_0 - (\gamma_I + \delta_H + \mu)\,\mathrm{y}_0 \tag{3.283}$$

Integrating both sides

$$\int dy_1^1 = \int \left(\left(1-\theta_q\right)\rho_q v_0 + \gamma_H x_0 - (\gamma_I + \delta_H + \mu)\,\mathrm{y}_0\right)dt \tag{3.284}$$

$$y_1(t) = \left(\left(1-\theta_q\right)\rho_q v_0 + \gamma_H x_0 - (\gamma_I + \delta_H + \mu)\,\mathrm{y}_0\right)t + T_{11} \tag{3.285}$$

Applying the initial condition

$$T_{11} = 0 \tag{3.286}$$

$$y_1(t) = \left(\left(1-\theta_q\right)\rho_q v_0 + \gamma_H x_0 - (\gamma_I + \delta_H + \mu)\,\mathrm{y}_0\right)t \tag{3.287}$$

Substitute equation (3.228), (3.233), (3.238), (3.243), (3.248) and (3.253) into (3.287)

$$y_1(t) = \left(\left(1-\theta_q\right)\rho_q \mathrm{Q}_0 + \gamma_H I_0 - (\gamma_I + \delta_H + \mu)\,\mathrm{H}_0\right)t \tag{3.288}$$

From equation (3.222)

$$z_1^1 + \mu z_0 - \gamma_I y_0 = 0 \tag{3.289}$$

$$z_1^1 = \gamma_I y_0 - \mu z_0 \tag{3.290}$$

Integrating both sides

$$\int dz_1^1 = \int (\gamma_I y_0 - \mu z_0) dt \tag{3.291}$$

$$z_1(t) = (\gamma_I y_0 - \mu z_0)t + T_{12} \tag{3.292}$$

Applying the initial condition

$$T_{12} = 0 \tag{3.293}$$

$$z_1(t) = (\gamma_I y_0 - \mu z_0)t \tag{3.294}$$

Substitute equation (3.228), (3.233), (3.238), (3.243), (3.248) and (3.253) into (3.294)

$$z_1(t) = (\gamma_I H_0 - \mu R_0)t \tag{3.295}$$

From equation (3.198)

From equation (3.155)

$$u_2^1 + \beta(u_1 x_0 + u_0 x_1) + \beta\varepsilon_H (u_1 y_0 + u_0 y_1) + \mu u_1 - \theta_q \rho_q v_1 - \theta_e \rho_e w_1 = 0 \tag{3.296}$$

$$u_2^1 = \theta_q \rho_q v_1 + \theta_e \rho_e w_1 - \beta(u_1 x_0 + u_0 x_1) - \beta\varepsilon_H (u_1 y_0 + u_0 y_1) - \mu u_1 \tag{3.297}$$

Substitute equation (3.260), (3.267), (3.274), (3.281),(3.288) and (3.295) into (3.297)

$$u_2^1=\theta_q\rho_q\left(\theta\beta S_0I_0+\theta\beta\varepsilon_H S_0H_0-\left(\rho_q+\mu\right)Q_0\right)t+\theta_e\rho_e\begin{pmatrix}\left(1-\theta\right)\beta\varepsilon_H S_0H_0\\+\left(1-\theta\right)\beta S_0I_0-\left(\rho_e+\mu\right)E_0\end{pmatrix}t$$
$$-\beta\begin{pmatrix}\left(\Lambda+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\beta S_0I_0-\beta\varepsilon_H S_0H_0-\mu S_0\right)tI_0\\+S_0\left(\left(1-\theta_e\right)\rho_eE_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)t\end{pmatrix}\tag{3.298}$$
$$-\beta\varepsilon_H\begin{pmatrix}\left(\Lambda+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\beta S_0I_0-\beta\varepsilon_H S_0H_0-\mu S_0\right)tH_0\\+S_0\left(\left(1-\theta_q\right)\rho_qQ_0+\gamma_HI_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\right)t\end{pmatrix}$$
$$-\mu\left(\Lambda+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\beta S_0I_0-\beta\varepsilon_H S_0H_0-\mu S_0\right)t$$

$$u_2^1=\begin{pmatrix}\theta_q\rho_q\left(\theta\beta S_0I_0+\theta\beta\varepsilon_H S_0H_0-\left(\rho_q+\mu\right)Q_0\right)+\theta_e\rho_e\begin{pmatrix}\left(1-\theta\right)\beta\varepsilon_H S_0H_0\\+\left(1-\theta\right)\beta S_0I_0-\left(\rho_e+\mu\right)E_0\end{pmatrix}\\-\beta\begin{pmatrix}\left(\Lambda+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\beta S_0I_0-\beta\varepsilon_H S_0H_0-\mu S_0\right)I_0\\+S_0\left(\left(1-\theta_e\right)\rho_eE_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)\end{pmatrix}\\-\beta\varepsilon_H\begin{pmatrix}\left(\Lambda+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\beta S_0I_0-\beta\varepsilon_H S_0H_0-\mu S_0\right)tH_0\\+S_0\left(\left(1-\theta_q\right)\rho_qQ_0+\gamma_HI_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\right)\end{pmatrix}\\-\mu\left(\Lambda+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\beta S_0I_0-\beta\varepsilon_H S_0H_0-\mu S_0\right)\end{pmatrix}t\tag{3.299}$$

Integrating both sides

$$\int du_2^1=\int\begin{pmatrix}\theta_q\rho_q\left(\theta\beta S_0I_0+\theta\beta\varepsilon_H S_0H_0-\left(\rho_q+\mu\right)Q_0\right)+\theta_e\rho_e\begin{pmatrix}\left(1-\theta\right)\beta\varepsilon_H S_0H_0\\+\left(1-\theta\right)\beta S_0I_0-\left(\rho_e+\mu\right)E_0\end{pmatrix}\\-\beta\begin{pmatrix}\left(\Lambda+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\beta S_0I_0-\beta\varepsilon_H S_0H_0-\mu S_0\right)I_0\\+S_0\left(\left(1-\theta_e\right)\rho_eE_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)\end{pmatrix}\\-\beta\varepsilon_H\begin{pmatrix}\left(\Lambda+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\beta S_0I_0-\beta\varepsilon_H S_0H_0-\mu S_0\right)tH_0\\+S_0\left(\left(1-\theta_q\right)\rho_qQ_0+\gamma_HI_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\right)\end{pmatrix}\\-\mu\left(\Lambda+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\beta S_0I_0-\beta\varepsilon_H S_0H_0-\mu S_0\right)\end{pmatrix}tdt\tag{3.300}$$

$$u_2(t)=\begin{pmatrix} \theta_q\rho_q\left(\theta\beta S_0 I_0+\theta\beta\varepsilon_H S_0 H_0-\left(\rho_q+\mu\right)Q_0\right)+\theta_e\rho_e\begin{pmatrix}\left(1-\theta\right)\beta\varepsilon_H S_0 H_0\\ +\left(1-\theta\right)\beta S_0 I_0-\left(\rho_e+\mu\right)E_0\end{pmatrix} \\ -\beta\begin{pmatrix}\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)I_0\\ +S_0\left(\left(1-\theta_e\right)\rho_e E_0-(\gamma_H+\delta_I+\mu)I_0\right)\end{pmatrix} \\ -\beta\varepsilon_H\begin{pmatrix}\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)tH_0\\ +S_0\left(\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-(\gamma_I+\delta_H+\mu)H_0\right)\end{pmatrix} \\ -\mu\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right) \end{pmatrix}\frac{t^2}{2}+T_{13} \quad (3.301)$$

Applying the initial condition

$$T_{13}=0 \quad (3.302)$$

$$u_2(t)=\begin{pmatrix} \theta_q\rho_q\left(\theta\beta S_0 I_0+\theta\beta\varepsilon_H S_0 H_0-\left(\rho_q+\mu\right)Q_0\right)+\theta_e\rho_e\begin{pmatrix}\left(1-\theta\right)\beta\varepsilon_H S_0 H_0\\ +\left(1-\theta\right)\beta S_0 I_0-\left(\rho_e+\mu\right)E_0\end{pmatrix} \\ -\beta\begin{pmatrix}\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)I_0\\ +S_0\left(\left(1-\theta_e\right)\rho_e E_0-(\gamma_H+\delta_I+\mu)I_0\right)\end{pmatrix} \\ -\beta\varepsilon_H\begin{pmatrix}\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)tH_0\\ +S_0\left(\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-(\gamma_I+\delta_H+\mu)H_0\right)\end{pmatrix} \\ -\mu\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right) \end{pmatrix}\frac{t^2}{2} \quad (3.303)$$

Substitute equation (3.228), (3.260) and (3.303) into (3.188)

$$S(t)=\left(u_0+hu_1+h^2u_2+\ldots\right)t \quad (3.304)$$

$$S(t)=\lim_{h\to 1}\left(u_0+hu_1+h^2u_2+\ldots\right)t \quad (3.305)$$

$$S(t)=\left(u_0+u_1+u_2+\ldots\right)t \quad (3.306)$$

Hence

$$S(t)=S_0+\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)t+$$
$$\left(\begin{array}{l}\theta_q\rho_q\left(\theta\beta S_0 I_0+\theta\beta\varepsilon_H S_0 H_0-\left(\rho_q+\mu\right)Q_0\right)+\theta_e\rho_e\left(\begin{array}{l}(1-\theta)\beta\varepsilon_H S_0 H_0\\+(1-\theta)\beta S_0 I_0-\left(\rho_e+\mu\right)E_0\end{array}\right)\\-\beta\left(\begin{array}{l}\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)I_0\\+S_0\left(\left(1-\theta_e\right)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)\end{array}\right)\\-\beta\varepsilon_H\left(\begin{array}{l}\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)tH_0\\+S_0\left(\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\right)\end{array}\right)\\-\mu\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)\end{array}\right)\frac{t^2}{2} \quad (3.307)$$

From equation (3.203)

$$p^2:\mathrm{v}_2^1+\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)v_1-\theta\beta\left(u_1x_0+u_0x_1\right)-\theta\beta\varepsilon_H\left(u_1y_0+u_0y_1\right)=0 \quad (3.308)$$

$$v_2^1=\theta\beta\left(u_1x_0+u_0x_1\right)+\theta\beta\varepsilon_H\left(u_1y_0+u_0y_1\right)-\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)v_1 \quad (3.309)$$

Substitute equation (3.260), (3.267), (3.274), (3.281),(3.288), and (3.295) into (3.309)

$$v_2^1=\theta\beta\left(\begin{array}{l}\left(\Lambda-(1-\theta)S_0I_0-(1-\theta)S_0H_0-\beta\theta S_0I_0-\beta\varepsilon_H\theta S_0H_0+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\mu S_0\right)tI_0\\+S_0\left(\left(1-\theta_e\right)\rho_eE_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)t\end{array}\right)+$$
$$\theta\beta\varepsilon_H\left(\begin{array}{l}\left(\Lambda-(1-\theta)S_0I_0-(1-\theta)S_0H_0-\beta\theta S_0I_0-\beta\varepsilon_H\theta S_0H_0+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\mu S_0\right)tH_0+\\S_0\left(\left(1-\theta_q\right)\rho_qQ_0+\gamma_HI_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\right)t\end{array}\right)- \quad (3.310)$$
$$\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)\left(\theta\beta S_0I_0+\theta\beta\varepsilon_HS_0H_0-\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)Q_0\right)t$$

$$v_2^1 = \begin{pmatrix} \theta\beta\Big(\big(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\big)I_0+S_0\big(\left(1-\theta_e\right)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\big)\Big)+ \\ \theta\beta\varepsilon_H\begin{pmatrix}\big(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\big)tH_0+ \\ S_0\big(\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\big)\end{pmatrix}- \\ \big(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\big)\Big(\theta\beta S_0 I_0+\theta\beta\varepsilon_H S_0 H_0-\big(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\big)Q_0\Big) \end{pmatrix} t \qquad (3.311)$$

Integrating both sides

$$\int dv_2^1 = \int\begin{pmatrix} \theta\beta\Big(\big(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\big)I_0+S_0\big(\left(1-\theta_e\right)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\big)\Big)+ \\ \theta\beta\varepsilon_H\begin{pmatrix}\big(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\big)vtH_0+ \\ S_0\big(\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\big)\end{pmatrix}- \\ \big(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\big)\Big(\theta\beta S_0 I_0+\theta\beta\varepsilon_H S_0 H_0-\big(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\big)Q_0\Big) \end{pmatrix} tdt \qquad (3.312)$$

$$v_2\left(t\right) = \begin{pmatrix} \theta\beta\Big(\big(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\big)I_0+S_0\big(\left(1-\theta_e\right)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\big)\Big)+ \\ \theta\beta\varepsilon_H\left(\big(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\big)vtH_0+S_0\begin{pmatrix}\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0- \\ (\gamma_I+\delta_H+\mu)\mathrm{H}_0\end{pmatrix}\right)- \\ \big(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\big)\Big(\theta\beta S_0 I_0+\theta\beta\varepsilon_H S_0 H_0-\big(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\big)Q_0\Big) \end{pmatrix}\frac{t^2}{2}+T_{14} \qquad (3.313)$$

Applying the initial condition

$$T_{14}=0 \qquad (3.314)$$

$$v_2(t)=\begin{pmatrix}\theta\beta\left(\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)I_0+S_0\left(\left(1-\theta_e\right)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)\right)+\\ \theta\beta\varepsilon_H\left(\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)tH_0+S_0\begin{pmatrix}\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-\\ (\gamma_I+\delta_H+\mu)\mathrm{H}_0\end{pmatrix}\right)-\\ \left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)\left(\theta\beta S_0 I_0+\theta\beta\varepsilon_H S_0 H_0-\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)Q_0\right)\end{pmatrix}\frac{t^2}{2} \qquad (3.315)$$

From (3.183)

$$Q(t)=\left(v_0+hv_1+h^2v_2+...\right)t \qquad (3.316)$$

Setting h=1, i.e

$$Q(t)=\lim_{h\to1}\left(v_0+hv_1+h^2v_2+...\right)t \qquad (3.317)$$

$$Q(t)=\left(v_0+v_1+v_2+...\right)t \qquad (3.318)$$

Hence

$$Q(t)=Q_0+\left(\theta\beta S_0 I_0+\theta\beta\varepsilon_H S_0 H_0-\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)Q_0\right)t+$$
$$\begin{pmatrix}\theta\beta\left(\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)I_0+S_0\left(\left(1-\theta_e\right)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)\right)+\\ \theta\beta\varepsilon_H\left(\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)tH_0+S_0\begin{pmatrix}\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-\\ (\gamma_I+\delta_H+\mu)\mathrm{H}_0\end{pmatrix}\right)-\\ \left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)\left(\theta\beta S_0 I_0+\theta\beta\varepsilon_H S_0 H_0-\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)Q_0\right)\end{pmatrix}\frac{t^2}{2} \qquad (3.319)$$

From equation (3.208)

$$p^2:\mathrm{w}_2^1+\left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)w_1-\left(1-\theta\right)\beta\left(u_1x_0+u_0x_1\right)-\left(1-\theta\right)\beta\varepsilon_H\left(u_1y_0+u_0y_1\right)=0 \qquad (3.320)$$

$$w_2^1=\left(1-\theta\right)\beta\left(u_1x_0+u_0x_1\right)+\left(1-\theta\right)\beta\varepsilon_H\left(u_1y_0+u_0y_1\right)-\left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)w_1 \qquad (3.321)$$

Substitute equation (3.260), (3.267), (3.274), (3.281),(3.288), and (3.295) into (3.321)

$$w_2^1 = (1-\theta)\beta\left(\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)tI_0+S_0\left(\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\right)t\right)+ (1-\theta)\beta\varepsilon_H\begin{pmatrix} tH_0+ \\ S_0\left(\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\right)t \end{pmatrix}- \left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)\left(\left(1-\theta\right)\beta S_0 I_0+\left(1-\theta\right)\beta\varepsilon_H S_0 H_0-\left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)E_0\right)t \quad (3.322)$$

$$w_2^1 = \begin{pmatrix} (1-\theta)\beta\left(\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)I_0+S_0\begin{pmatrix}\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0- \\ (\gamma_I+\delta_H+\mu)\mathrm{H}_0\end{pmatrix}\right)+ \\ (1-\theta)\beta\varepsilon_H\left(\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)H_0+S_0\begin{pmatrix}\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0- \\ (\gamma_I+\delta_H+\mu)\mathrm{H}_0\end{pmatrix}\right)- \\ \left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)\left(\left(1-\theta\right)\beta S_0 I_0+\left(1-\theta\right)\beta\varepsilon_H S_0 H_0-\left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)E_0\right) \end{pmatrix} t \quad (3.323)$$

Integrating both sides

$$\int dw_2^1 = \int\begin{pmatrix} (1-\theta)\beta\left(\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)I_0+S_0\begin{pmatrix}\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0- \\ (\gamma_I+\delta_H+\mu)\mathrm{H}_0\end{pmatrix}\right)+ \\ (1-\theta)\beta\varepsilon_H\left(\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0 I_0-\beta\varepsilon_H S_0 H_0-\mu S_0\right)H_0+S_0\begin{pmatrix}\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0- \\ (\gamma_I+\delta_H+\mu)\mathrm{H}_0\end{pmatrix}\right)- \\ \left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)\left(\left(1-\theta\right)\beta S_0 I_0+\left(1-\theta\right)\beta\varepsilon_H S_0 H_0-\left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)E_0\right) \end{pmatrix} tdt \quad (3.324)$$

$$w_2(t)=\begin{pmatrix} (1-\theta)\beta\begin{pmatrix}\left(\Lambda-(1-\theta)S_0I_0-(1-\theta)S_0H_0-\beta\theta S_0I_0-\beta\varepsilon_H\theta S_0H_0+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\mu S_0\right)I_0\\ +S_0\left(\left(1-\theta_q\right)\rho_qQ_0+\gamma_HI_0-(\gamma_I+\delta_H+\mu)H_0\right)\end{pmatrix}+\\ (1-\theta)\beta\varepsilon_H\begin{pmatrix}\left(\Lambda-(1-\theta)S_0I_0-(1-\theta)S_0H_0-\beta\theta S_0I_0-\beta\varepsilon_H\theta S_0H_0+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\mu S_0\right)H_0+\\ S_0\left(\left(1-\theta_q\right)\rho_qQ_0+\gamma_HI_0-(\gamma_I+\delta_H+\mu)H_0\right)\end{pmatrix}-\\ \left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)\left(\left(1-\theta\right)\beta S_0I_0+\left(1-\theta\right)\beta\varepsilon_HS_0H_0-\left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)E_0\right) \end{pmatrix}\frac{t^2}{2}+T_{15} \quad (3.325)$$

Applying the initial condition

$$T_{15}=0 \tag{3.326}$$

$$w_2(t)=\begin{pmatrix} (1-\theta)\beta\left(\left(\Lambda+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\beta S_0I_0-\beta\varepsilon_HS_0H_0-\mu S_0\right)I_0+S_0\begin{pmatrix}\left(1-\theta_q\right)\rho_qQ_0+\gamma_HI_0-\\ (\gamma_I+\delta_H+\mu)H_0\end{pmatrix}\right)+\\ (1-\theta)\beta\varepsilon_H\left(\left(\Lambda+\theta_q\rho_qQ_0+\theta_e\rho_eE_0-\beta S_0I_0-\beta\varepsilon_HS_0H_0-\mu S_0\right)H_0+S_0\begin{pmatrix}\left(1-\theta_q\right)\rho_qQ_0+\gamma_HI_0-\\ (\gamma_I+\delta_H+\mu)H_0\end{pmatrix}\right)-\\ \left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)\left(\left(1-\theta\right)\beta S_0I_0+\left(1-\theta\right)\beta\varepsilon_HS_0H_0-\left(\left(1-\theta_e\right)\rho_e+\left(\theta_e\rho_e+\mu\right)\right)E_0\right) \end{pmatrix}\frac{t^2}{2} \tag{3.327}$$

Substitute equation (3.228), (3.274) and (3.327) into (3.327)

$$E(t)=\left(w_0+hw_1+h^2w_2+...\right)t \tag{3.328}$$

$$E(t)=\lim_{h\to1}\left(w_0+hw_1+h^2w_2+...\right)t \tag{3.329}$$

$$E(t)=\left(w_0+w_1+w_2+...\right)t \tag{3.330}$$

Hence

$$E(t)=E_0+\left((1-\theta)\beta S_0I_0+(1-\theta)\beta\varepsilon_H S_0H_0-\left((1-\theta_e)\rho_e+(\theta_e\rho_e+\mu)\right)E_0\right)t+$$
$$\left(\begin{array}{l}(1-\theta)\beta\left(\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0I_0-\beta\varepsilon_H S_0H_0-\mu S_0\right)I_0+S_0\begin{pmatrix}(1-\theta_q)\rho_q Q_0+\gamma_H I_0-\\(\gamma_I+\delta_H+\mu)\mathrm{H}_0\end{pmatrix}\right)+\\(1-\theta)\beta\varepsilon_H\left(\left(\Lambda+\theta_q\rho_q Q_0+\theta_e\rho_e E_0-\beta S_0I_0-\beta\varepsilon_H S_0H_0-\mu S_0\right)H_0+S_0\begin{pmatrix}(1-\theta_q)\rho_q Q_0+\gamma_H I_0-\\(\gamma_I+\delta_H+\mu)\mathrm{H}_0\end{pmatrix}\right)-\\\left((1-\theta_e)\rho_e+(\theta_e\rho_e+\mu)\right)\left((1-\theta)\beta S_0I_0+(1-\theta)\beta\varepsilon_H S_0H_0-\left((1-\theta_e)\rho_e+(\theta_e\rho_e+\mu)\right)E_0\right)\end{array}\right)\frac{t^2}{2} \quad (3.331)$$

From equation (3.213)

$$x_2^1+(\gamma_H+\delta_I+\mu)\mathrm{x}_1-(1-\theta_e)\rho_e w_1=0 \quad (3.332)$$

$$x_2^1=(1-\theta_e)\rho_e w_1-(\gamma_H+\delta_I+\mu)\mathrm{x}_1 \quad (3.333)$$

Substitute equation (3.260), (3.267), (3.274), (3.281),(3.288), and (3.295) into (3.333)

$$x_2^1=(1-\theta_e)\rho_e\left((1-\theta)\beta S_0I_0+(1-\theta)\beta\varepsilon_H S_0H_0-\left((1-\theta_e)\rho_e+(\theta_e\rho_e+\mu)\right)E_0\right)t-$$
$$(\gamma_H+\delta_I+\mu)\left((1-\theta_e)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)t \quad (3.334)$$

$$x_2^1=\begin{pmatrix}(1-\theta_e)\rho_e\left((1-\theta)\beta S_0I_0+(1-\theta)\beta\varepsilon_H S_0H_0-\left((1-\theta_e)\rho_e+(\theta_e\rho_e+\mu)\right)E_0\right)-\\(\gamma_H+\delta_I+\mu)\left((1-\theta_e)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)\end{pmatrix}t \quad (3.335)$$

Integrating both sides

$$\int dx_2^1=\begin{pmatrix}(1-\theta_e)\rho_e\left((1-\theta)\beta S_0I_0+(1-\theta)\beta\varepsilon_H S_0H_0-\left((1-\theta_e)\rho_e+(\theta_e\rho_e+\mu)\right)E_0\right)-\\(\gamma_H+\delta_I+\mu)\left((1-\theta_e)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)\end{pmatrix}tdt \quad (3.336)$$

$$x_2(t)=\begin{pmatrix}(1-\theta_e)\rho_e\left((1-\theta)\beta S_0 I_0+(1-\theta)\beta\varepsilon_H S_0 H_0-\left((1-\theta_e)\rho_e+(\theta_e\rho_e+\mu)\right)E_0\right)-\\(\gamma_H+\delta_I+\mu)\left((1-\theta_e)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)\end{pmatrix}\frac{t^2}{2}+T_{16} \quad (3.337)$$

Applying the initial condition

$$T_{16}=0 \quad (3.338)$$

$$x_2(t)=\begin{pmatrix}(1-\theta_e)\rho_e\left((1-\theta)\beta S_0 I_0+(1-\theta)\beta\varepsilon_H S_0 H_0-\left((1-\theta_e)\rho_e+(\theta_e\rho_e+\mu)\right)E_0\right)-\\(\gamma_H+\delta_I+\mu)\left((1-\theta_e)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)\end{pmatrix}\frac{t^2}{2} \quad (3.339)$$

From (3.185)

$$I(t)=\left(x_0+hx_1+h^2x_2+\ldots\right)t \quad (3.340)$$

$$I(t)=\lim_{h\to 1}\left(x_0+hx_1+h^2x_2+\ldots\right)t \quad (3.341)$$

$$I(t)=\left(x_0+x_1+x_2+\ldots\right)t \quad (3.342)$$

Hence

$$I(t)=I_0+\left((1-\theta_e)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)t+\begin{pmatrix}(1-\theta_e)\rho_e\begin{pmatrix}(1-\theta)\beta S_0 I_0+(1-\theta)\beta\varepsilon_H S_0 H_0-\\\left((1-\theta_e)\rho_e+(\theta_e\rho_e+\mu)\right)E_0\end{pmatrix}-\\(\gamma_H+\delta_I+\mu)\left((1-\theta_e)\rho_e E_0-(\gamma_H+\delta_I+\mu)\mathrm{I}_0\right)\end{pmatrix}\frac{t^2}{2} \quad (3.343)$$

From equation (3.218)

$$y_2^1+(\gamma_I+\delta_H+\mu)\,\mathrm{y}_1-\gamma_H x_1-\left(1-\theta_q\right)\rho_q \mathrm{v}_1=0 \quad (3.344)$$

$$y_2^1=\left(1-\theta_q\right)\rho_q \mathrm{v}_1+\gamma_H x_1-(\gamma_I+\delta_H+\mu)\,\mathrm{y}_1 \quad (3.345)$$

Substitute equation (3.260), (3.267), (3.274), (3.281),(3.288), and (3.295) into (3.345)

$$y_2^1 = \left(1-\theta_q\right)\rho_q\left(\theta\beta S_0 I_0 + \theta\beta\varepsilon_H S_0 H_0 - \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q\rho_q + \mu\right)\right)Q_0\right)t + \gamma_H\begin{pmatrix}\left(1-\theta_e\right)\rho_e E_0 - \\ (\gamma_H + \delta_I + \mu)\mathrm{I}_0\end{pmatrix}t - (\gamma_I + \delta_H + \mu)\left(\left(1-\theta_q\right)\rho_q \mathrm{Q}_0 + \gamma_H I_0 - (\gamma_I + \delta_H + \mu)\mathrm{H}_0\right)t \quad (3.346)$$

$$y_2^1 = \begin{pmatrix}\left(1-\theta_q\right)\rho_q\left(\theta\beta S_0 I_0 + \theta\beta\varepsilon_H S_0 H_0 - \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q\rho_q + \mu\right)\right)Q_0\right) + \gamma_H\begin{pmatrix}\left(1-\theta_e\right)\rho_e E_0 - \\ (\gamma_H + \delta_I + \mu)\mathrm{I}_0\end{pmatrix} - \\ (\gamma_I + \delta_H + \mu)\left(\left(1-\theta_q\right)\rho_q \mathrm{Q}_0 + \gamma_H I_0 - (\gamma_I + \delta_H + \mu)\mathrm{H}_0\right)\end{pmatrix}t \quad (3.347)$$

Integrating both sides

$$\int dy_2^1 = \int\begin{pmatrix}\left(1-\theta_q\right)\rho_q\left(\theta\beta S_0 I_0 + \theta\beta\varepsilon_H S_0 H_0 - \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q\rho_q + \mu\right)\right)Q_0\right) + \gamma_H\begin{pmatrix}\left(1-\theta_e\right)\rho_e E_0 - \\ (\gamma_H + \delta_I + \mu)\mathrm{I}_0\end{pmatrix} - \\ (\gamma_I + \delta_H + \mu)\left(\left(1-\theta_q\right)\rho_q \mathrm{Q}_0 + \gamma_H I_0 - (\gamma_I + \delta_H + \mu)\mathrm{H}_0\right)\end{pmatrix}tdt \quad (3.348)$$

$$y_2(t) = \begin{pmatrix}\left(1-\theta_q\right)\rho_q\left(\theta\beta S_0 I_0 + \theta\beta\varepsilon_H S_0 H_0 - \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q\rho_q + \mu\right)\right)Q_0\right) + \gamma_H\begin{pmatrix}\left(1-\theta_e\right)\rho_e E_0 - \\ (\gamma_H + \delta_I + \mu)\mathrm{I}_0\end{pmatrix} - \\ (\gamma_I + \delta_H + \mu)\left(\left(1-\theta_q\right)\rho_q \mathrm{Q}_0 + \gamma_H I_0 - (\gamma_I + \delta_H + \mu)\mathrm{H}_0\right)\end{pmatrix}\frac{t^2}{2} + T_{17} \quad (3.349)$$

Applying the initial condition

$$T_{17} = 0 \quad (3.350)$$

$$y_2(t) = \begin{pmatrix}\left(1-\theta_q\right)\rho_q\left(\theta\beta S_0 I_0 + \theta\beta\varepsilon_H S_0 H_0 - \left(\left(1-\theta_q\right)\rho_q + \left(\theta_q\rho_q + \mu\right)\right)Q_0\right) + \gamma_H\begin{pmatrix}\left(1-\theta_e\right)\rho_e E_0 - \\ (\gamma_H + \delta_I + \mu)\mathrm{I}_0\end{pmatrix} - \\ (\gamma_I + \delta_H + \mu)\left(\left(1-\theta_q\right)\rho_q \mathrm{Q}_0 + \gamma_H I_0 - (\gamma_I + \delta_H + \mu)\mathrm{H}_0\right)\end{pmatrix}\frac{t^2}{2} \quad (3.351)$$

From (3.186)

$$H(t)=\left(y_0+hy_1+h^2y_2+...\right)t$$

(3.352)

Setting h=1, i.e

$$H(t)=\lim_{h\to 1}\left(y_0+hy_1+h^2y_2+...\right)t \tag{3.353}$$

$$H(t)=\left(y_0+y_1+y_2+...\right)t \tag{3.354}$$

Hence

$$H(t)=H_0+\left(\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\right)t+$$
$$\begin{pmatrix}\left(1-\theta_q\right)\rho_q\left(\theta\beta S_0 I_0+\theta\beta\varepsilon_H S_0 H_0-\left(\left(1-\theta_q\right)\rho_q+\left(\theta_q\rho_q+\mu\right)\right)Q_0\right)+\gamma_H\begin{pmatrix}\left(1-\theta_e\right)\rho_e E_0-\\(\gamma_H+\delta_I+\mu)\mathrm{I}_0\end{pmatrix}-\\(\gamma_I+\delta_H+\mu)\left(\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\right)\end{pmatrix}\frac{t^2}{2} \tag{3.355}$$

From equation (3.223)

$$z_2^1+\mu z_1-\gamma_I y_1=0 \tag{3.356}$$

$$z_2^1=\gamma_I y_1-\mu z_1 \tag{3.357}$$

Substitute equation (3.260), (3.267), (3.274), (3.281),(3.288), and (3.295) into (3.357)

$$z_2^1=\gamma_I\left(\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\right)t-\mu\left(\gamma_I H_0-\mu R_0\right)t \tag{3.358}$$

$$z_2^1=\left(\left(\gamma_I\left(\left(1-\theta_q\right)\rho_q Q_0+\gamma_H I_0-(\gamma_I+\delta_H+\mu)\mathrm{H}_0\right)-\mu\left(\gamma_I H_0-\mu R_0\right)\right)\right)t \tag{3.359}$$

Integrating both sides

$$\int dz_2^1 = \int \gamma_I \left(\left(1-\theta_q\right)\rho_q Q_0 + \gamma_H I_0 - (\gamma_I + \delta_H + \mu)\mathrm{H}_0\right)t - \mu\left(\gamma_I H_0 - \mu R_0\right)tdt \tag{3.360}$$

$$z_2(t) = \left(\gamma_I \left(\left(1-\theta_q\right)\rho_q Q_0 + \gamma_H I_0 - (\gamma_I + \delta_H + \mu)\mathrm{H}_0\right) - \mu\left(\gamma_I H_0 - \mu R_0\right)\right)\frac{t^2}{2} + T_{18} \tag{3.361}$$

Applying the initial condition

$$T_{18} = 0$$

(3.362)

$$z_2(t) = \left(\gamma_I \left(\left(1-\theta_q\right)\rho_q Q_0 + \gamma_H I_0 - (\gamma_I + \delta_H + \mu)\mathrm{H}_0\right) - \mu\left(\gamma_I H_0 - \mu R_0\right)\right)\frac{t^2}{2} \tag{3.363}$$

From (3.187)

$$R(t) = \left(z_0 + hz_1 + h^2 z_2 + \ldots\right)t \tag{3.364}$$

Setting h=1, i.e

$$R(t) = \lim_{h \to 1}\left(z_0 + hz_1 + h^2 z_2 + \ldots\right)t \tag{3.365}$$

$$R(t) = \left(z_0 + z_1 + z_2 + \ldots\right)t \tag{3.366}$$

Hence

$$R(t) = R_0 + \left(\gamma_I H_0 - \mu R_0\right)t + \begin{pmatrix} \gamma_I \left(\left(1-\theta_q\right)\rho_q Q_0 + \gamma_H I_0 - (\gamma_I + \delta_H + \mu)\mathrm{H}_0\right) - \\ \mu\left(\gamma_I H_0 - \mu R_0\right) \end{pmatrix}\frac{t^2}{2} \tag{3.367}$$

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Simulations

It is difficult to get a reliable data, we estimated the parameter value based on the available data from the Nigerian Centre for Disease Control (NCDC) and reliable literature. The estimates are clearly explained in the following sub-sections as shown in Table 4.1.

Table 4.1 Initial conditions for each plot and parameters values

Parameters and State Variables	Value	Source
N_0	200000000	Assumed
S_0	199887700	NCDC (2020)
Q_0	3300	NCDC (2020)
I_0	19000	NCDC (2020)
E_0	88000	NCDC (2020)
H_0	1200	NCDC (2020)
R_0	800	NCDC (2020)
Λ	0.000300	Fitted
β	0.00000350	Estimated
μ	0.000015	NCDC (2020)
θ	0.1 to 0.9	NCDC (2020)
ε_H	0.000005	Estimated
θ_q	0.000300	Estimated
ρ_q	0.020	Fitted

θ_e	0.000400	Estimated
ρ_e	0.040	Fitted
γ_H	0.060	NCDC (2020)
δ_I	1.0×10^{-8}	NCDC (2020)
δ_H	1.0×10^{-7}	NCDC (2020)
γ_1	0.30	NCDC (2020)

4.2 Graphical Representation of Solutions of Model Equations

The graphical representation are from the analytical solutions of the model equations.

They are plotted using MAPLE software.

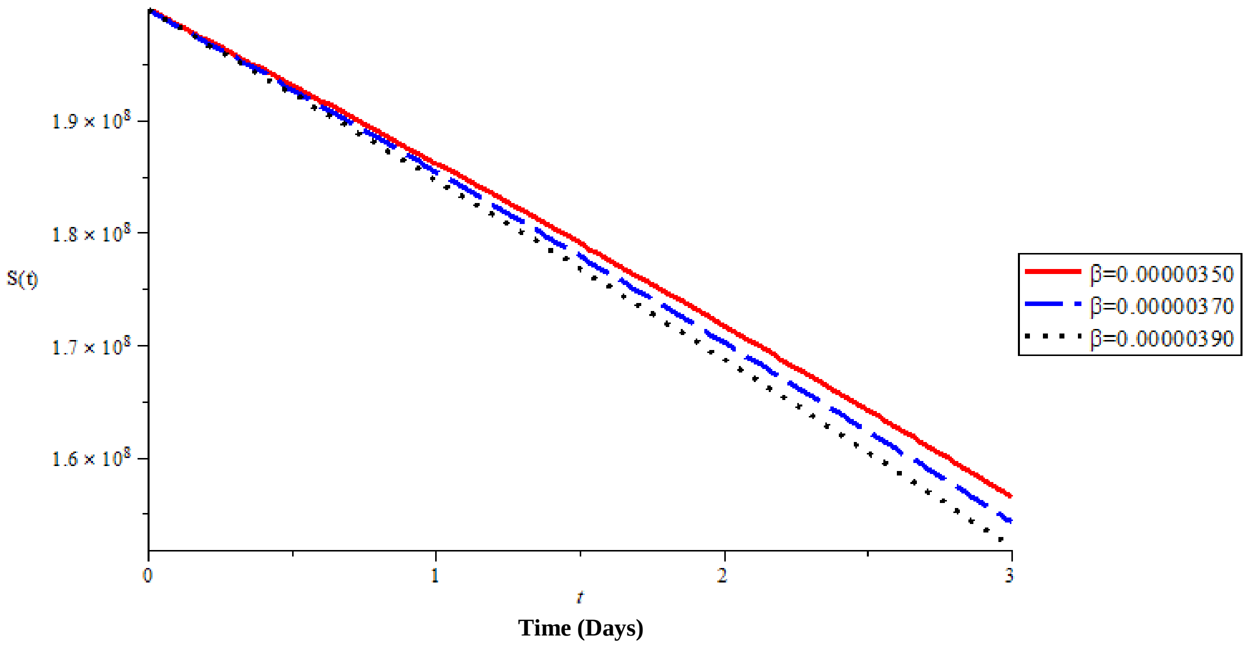
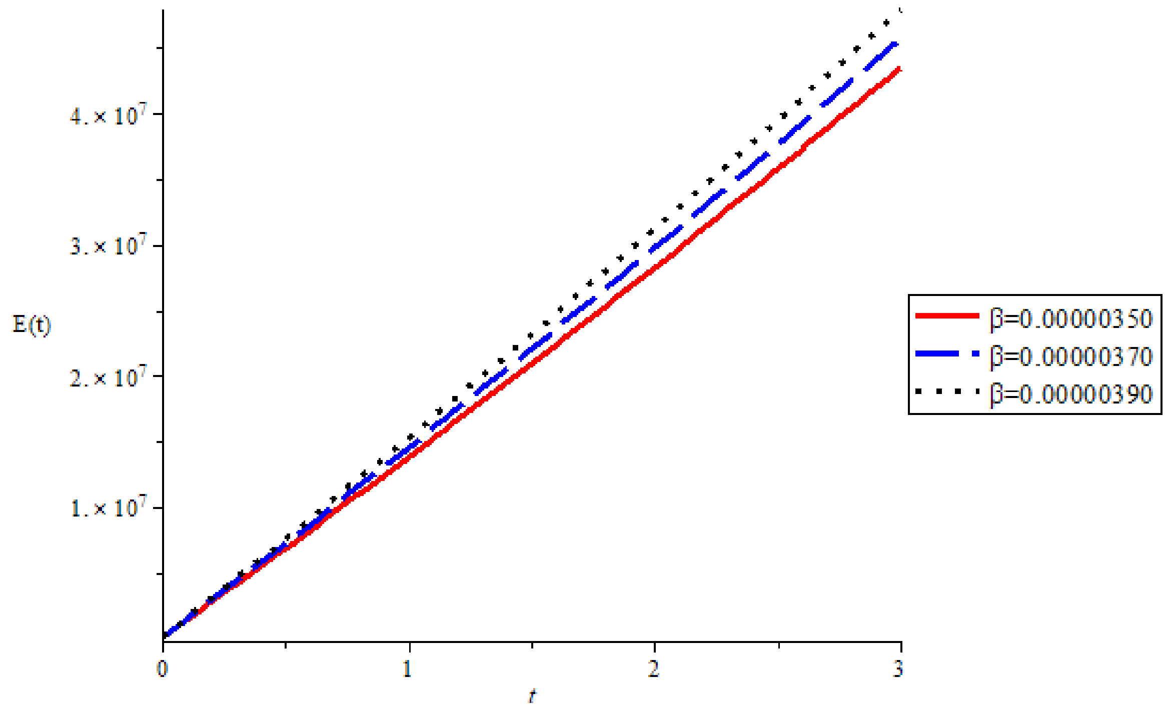


Figure 4.1: Graph of Susceptible Individual Against Time for Different Values of Contact Rate Between Susceptible Individuals and Infected Individuals.

Figure 4.1 is the graph of susceptible individuals against time for different contact rate. It is observed that the population of susceptible individuals decreases with different values of contact rate. The higher the contact rate between the susceptible and the infected, the higher the decrease in susceptible population.



Time (Days)

Figure 4.2: Graph of Untraced Individuals Who Are Exposed to COVID-19 Against Time for Different Values of Contact Rate Between Susceptible Individuals And Infected Individuals.

Figure 4.2 is the graph of untraced individuals who are exposed to COVID over time. It is observed that the population of untraced individuals who are exposed to COVID increases with different values of contact rate. The higher the contact rate between the susceptible and the infected, the higher the increases in the untraced exposed population.

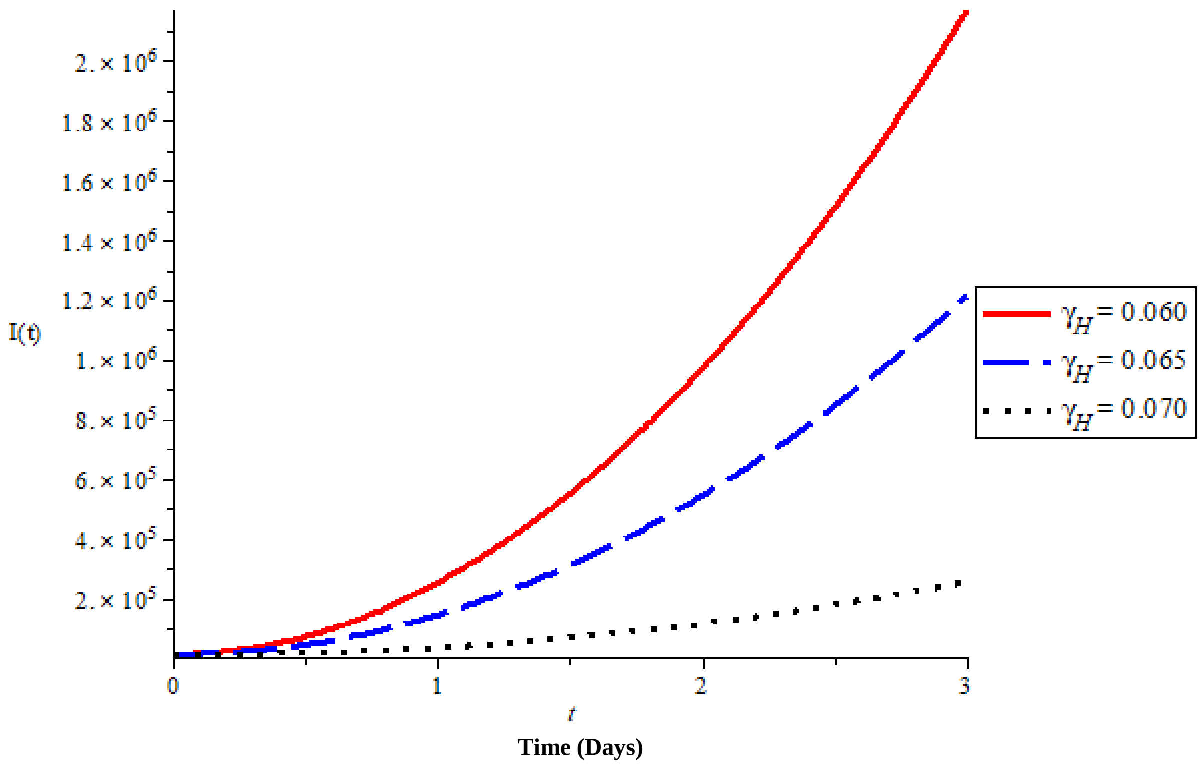


Figure 4.3: Graph of Infected Individuals against Time for Different Values of Rate at which Infected Individuals are Hospitalized.

Figure 4.3 is the graph of Infected individuals against time. That the number of infected individual increases as the **of** rate at which infected individuals are hospitalized decreases.

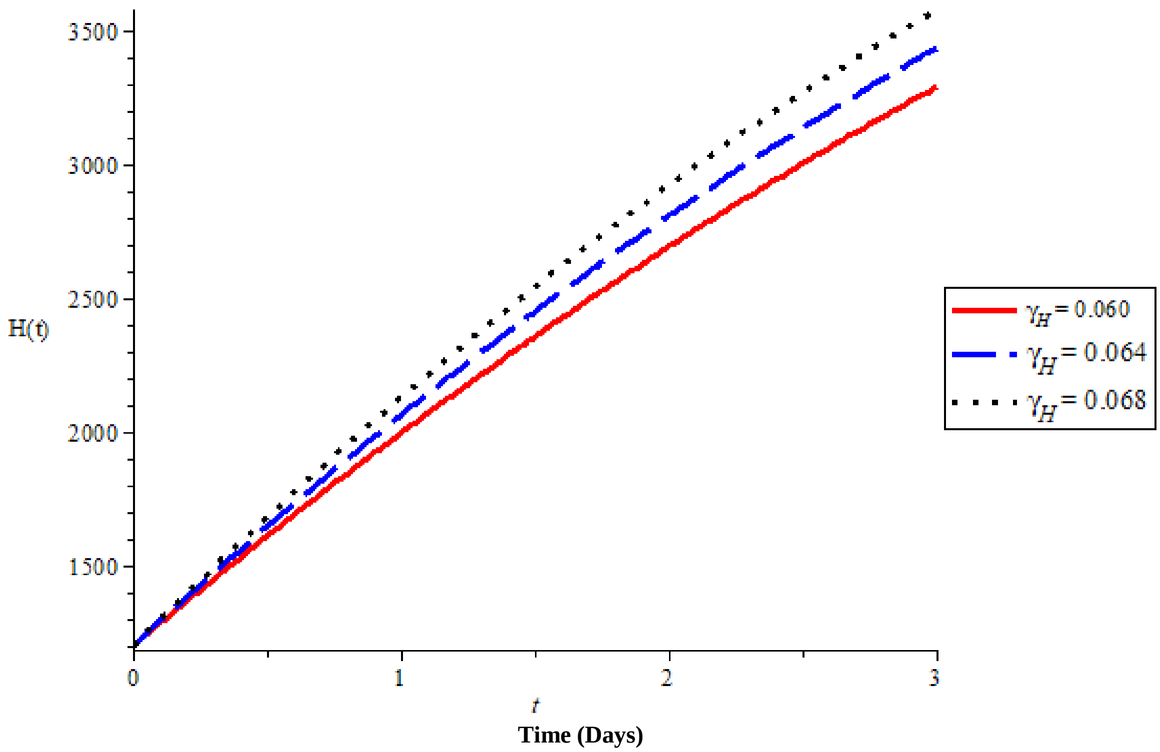


Figure 4.4: Graph of Hospitalized Individuals Against Time for Different Values of Rate at which Infected Individuals are Hospitalized.

Figure 4.4 is the graph of hospitalized individuals against time. It is observed that the population of hospitalized individuals increases as the rate of infected individuals being hospitalized increases.

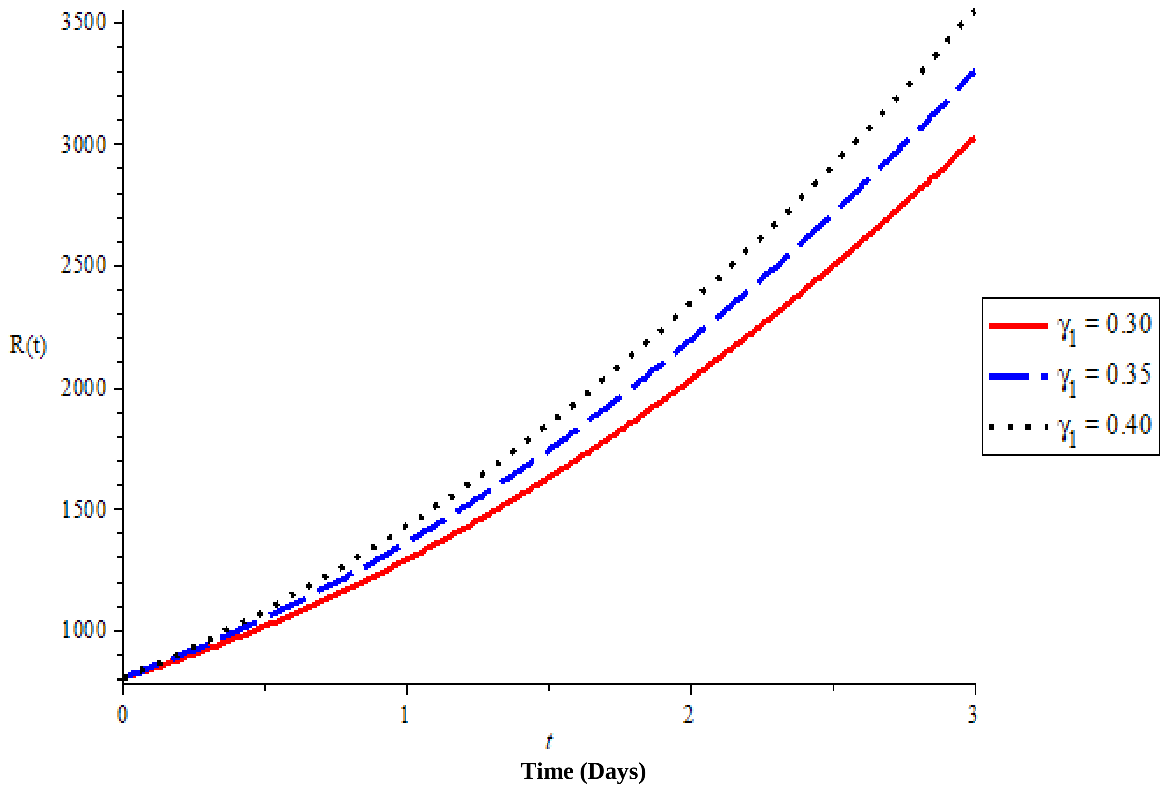


Figure 4.5: Graph of Recovered Individuals against time for Different Values of Proportion of Hospitalized Individuals who leave the Compartment to Recovered Class.

Figure 4.5 is the graph of recovered individuals against time for different at proportion of hospitalized individuals who leave the compartment to recovered class. It is observed that the population of the recovered human increases as the proportion of hospitalized individuals who leave the compartment to recovered class increases.

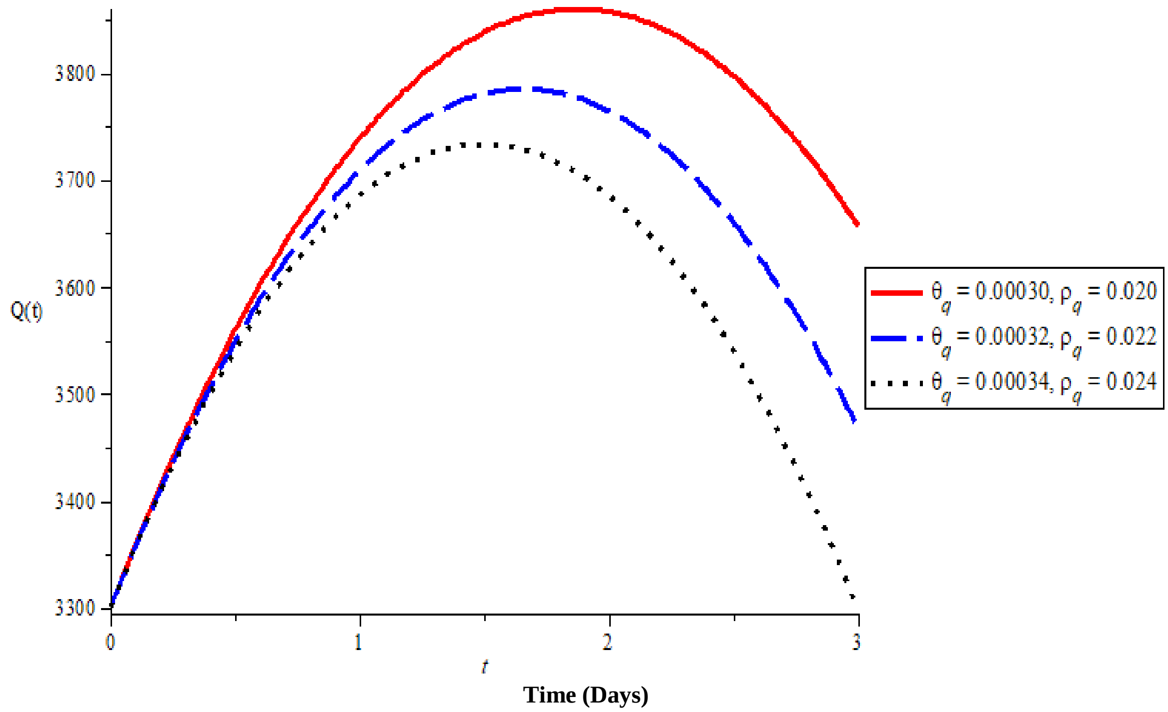


Figure 4.6: Graph of Quarantined Individuals Against time for Different Values of Individuals who Leaves Quarantine Class to Susceptible Class.

Figure 4.6 is the graph of quarantined individuals against time. it is observed that the number of quarantine individuals increases with time and letter decease as a proportion of individuals quarantined who leave the compartment to susceptible class increases.

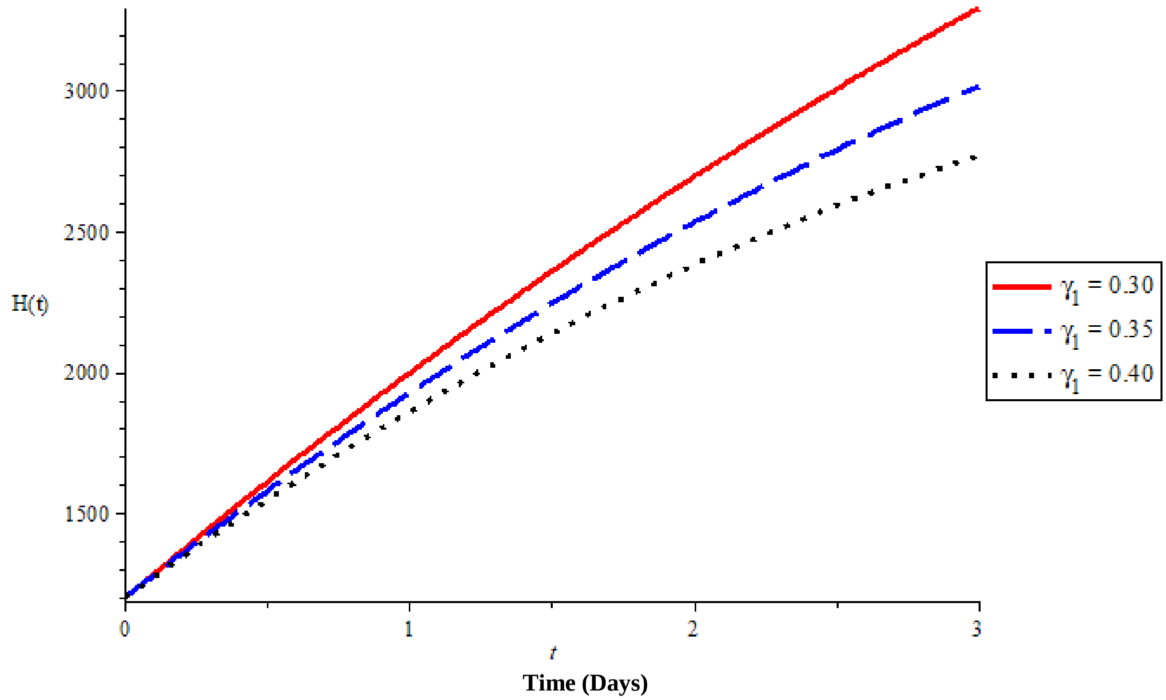


Figure 4.7: Graph of Hospitalized Individuals against Time for Different Proportion of Hospitalized Individuals who Leaves the Class to Recovery Class.

Figure 4.7 is the graph of hospitalized individuals against time. It is observed that the population of hospitalized individuals increases as the proportion of individuals who leaves to the recovered class increases.

4.8 Comparison of Results

In Figure 4.1, it was observed that the population of Susceptible individuals decreases with different values of contact rate. The higher the contact rate between the susceptible and the infected, the higher the decrease in susceptible population. It is also seen in Mustapha and Hanane (2020).

Also in Figure 4.2, it was observed that the population of untraced individuals who are exposed to COVID-19 increases with different values of contact rate. The higher the contact rate between the susceptible and the infected, the higher the increases in the untraced exposed population. It is also seen in Mustapha and Hanane (2020).

Figure 4.3, also revealed that the graph of infected individuals against time. That the number of infected individual increases as the of rate at which Infected Individuals are hospitalized decreases. In Figure 4.4, it was observed that the population of hospitalized individuals increases as the rate of infected individuals being hospitalized increases. In Figure 4.5, it was observed that the population of the recovered human increases as the proportion of hospitalized individuals who leave the compartment to recovered class increases. It is also seen in Mustapha and Hanane (2020).

Also, in Figure 4.6 it was observed that the number of quarantine individuals increases with time and letter decease as a proportion of individuals quarantined who leave the compartment to susceptible class increases. And finally, Figure 4.7, it was observed that the population of hospitalized individuals increases as the Proportion of individuals who leaves to the recovered class increases.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this study, a mathematical model for the transmission dynamics of COVID-19 pandemic with contact tracing and full recovery was developed and analyzed using system of first order ordinary differential equations. It was discovered that model has two equilibrium state; we carried analysis on the developed model. The equilibrium states were obtained and analyzed for their stability relatively to the effective reproduction number. The result shows that, the disease-free equilibrium was stable and the criteria for stability of the endemic equilibrium are established. We were able to show that the COVID-19 infectious free equilibrium is locally and globally asymptotically stable if $R_0 < 1$. The analytical solution was obtained using Homotopy Perturbation Method (HPM) and effective reproduction number was computed in order to measure the relative impact for individual or combined intervention for effective disease control. Numerical simulations of the model show that, the disease will be eradicated from both humans and the non-human primates with the proposed interventions of the model in due time.

5.2 Recommendations

(i) The model shows that the spread of COVID-19 infection depends largely on the contact rate, hence the National hospital should emphasize on the improvement in early detection of COVID-19 infection cases so that transmission can be minimized.

(ii) Infectious human’s individuals should be isolated and treated immediately.Individuals infected with COVID-19 should be given antiretroviral drugs immediately.

(iii) We also want to recommend to World Health Organization, CDC and NAFDAC that the efficacy COVID-19 drugs should be at 0.015 and above respectively to have a stable population.

(iv) One of the limitations of this study is the unavailability of records of COVID-19 case; therefore, health workers should make data available for researchers.

(v) Optimal control strategy can be incorporated into the model for greater insight into the dynamics.

5.3 Contributions to Knowledge

The study has developed:

(a) Developed and validated a mathematical model for the transmission dynamics of COVID-19 pandemic with contact tracing and full recovery.

(b) The model incorporates the mathematical model for the human to human transmission.

(c) The work has shown the positivity criteria for the endemic equilibrium state to be stable.

(d) The work has also shown that humans population will be reduced when the treatment rates and their effectiveness are high.

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