IMPLICIT ADAMS TYPE LINEAR MULTISTEP METHOD FOR SOLVING STIFF DIFFERENTIAL EQUATIONS

BY

DAVID, Ineke Jacob
MTech/SPS/2018/8263

DEPARTMENT OF MATHEMATICS

FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA

NOVEMBER, 2021

IMPLICIT ADAMS TYPE LINEAR MULTISTEP METHOD FOR SOLVING STIFF DIFFERENTIAL EQUATIONS

BY

DAVID, Ineke Jacob
MTech/SPS/2018/8263

A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA, NIGERIA IN PARTIAL FULFILLMENT OF THE REQIUREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF TECHNOLOGY (MTech) IN MATHEMATICS

NOVEMBER, 2021

ABSTRACT

In this thesis 3-step implicit Adam's type method for solving linear, non-linear ordinary differential equations and stiff differential system of ODEs were derived through the application of power series expansion collocating at 6 and 9 off-grid points respectively. The Schemes were shown to be consistent and zero-stable, thereby establishing their convergence. Numerical examples solved attested the efficiency and reliability of the Schemes. The block method with 6 off-grid points $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{7}{3}, and\ \frac{8}{3}$ at collocation has order (11, 11, 11, 11, 11, 11, 11, 11) and the error constants are $\frac{-11899}{349192166400}$, $\frac{-179}{5456127600}, \ldots \frac{5609}{1325839006800}$. Numerical results obtained show that the methods are competitive in terms of accuracy.

TABLE OF CONTENTS

Content **Page**

LIST OF TABLES

CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

In mathematics, a differential equation is an equation that relates one or more functions and their derivatives Dennis (2012). In applications, the functions generally represent physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between the two. Mainly the study of differential equations consists of the study of their solutions (the set of functions that satisfy each equation), and of the properties of their solutions. Only the simplest differential equations are solvable by explicit formulas; however, many properties of solutions of a given differential equation may be determined without computing them exactly.

Often when a closed-form expression for the solutions is not available, solutions may be approximated numerically using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

Numerical methods are personally connected to mathematical models of the most various sorts of wonders. Generally, most of marvel procedures of hobby have been of physical character. Liquid flow, a branch of continuum mechanics has for a considerable length of time been a rich wellspring of scientific models, which take the type of partial equations, integral equations and ordinary differential equations (Beck, 2009).

Numerical approximation to the exact solution is the study of kind and quantity of uncertainty, that may be present in the solution of problems, this issue is particularly prominent in applied areas such as numerical analysis, when using numerical methods or algorithms and computing with finite precision, errors of approximation or rounding and truncation are introduced. A Newly developed method is worthless without an error analysis; neither does it make sense to use methods which introduce errors with magnitude larger than the effects to be measured or simulated. On the other hand, using a method with very high accuracy might be computationally too expensive to justify the gain in accuracy. Numerical method that gathers a wide range of materials from floating point arithmetic, Numerical Linear Algebra, Polynomials, functions evaluation and root finding, Interpolation, Numerical differentiation and integration all make use of the idea of error analysis, "The theory of constructive methods in numerical analysis". In Numerical analysis error analysis comprises both forward error analysis and backward error analysis.

Measurements and calculations can be characterized with regards to their accuracy and precision. Accuracy refers to how closely a value agrees with the true value, Precision refers to how closely true values agrees with each other. The term error represents the imprecision and inaccuracy of a numerical computation.

The formal academic area of numerical analysis changes from very theoretical mathematics studies to computer science issues including the impacts of computer hardware on the execution of specific algorithms (Gallopoulos *et al.* 1994). Conte and Boor (1965) did an excellent work on numerical analysis and programming aspect of it when the make use of Fortran IV programming they make a comparison between analytical method of solving numerical iterations and programming method of solving numerical iterations and came up with a conclusion that programming method of solving numerical iterations using computer is faster than using analytical method and safe time with a very negligible error or no error incurred at all (Hassan *et al.*, 2006).

Analytical method produces when possible, exact analytical solutions in the form of general mathematical expression. Solutions of differential equation will give expressions or functions, which are distinct from numerical values. Numerical method on the other hand produces approximate solutions in the form of discrete values or numbers; hence the concentration of this project, the simplest of the equations mentioned above is a linear algebraic equation; the exact solution of this immediate and consists of a single value. (Hansen, 2015)

There are numerous issues in mathematics for which no analytic solution is known, there are additionally others for which analytic solutions are repetitive and the answer may be in type of a boundless solution that must be deciphered after much computational exertion. (Hilbert, 1998). The coming of fast advanced computers has made numerical method more attractive for tackling pragmatic issue by evacuating the tedious dreary redundant manual mathematics calculations. This is as a consequence of the orderly approach numerical method receives in critical thinking which are like calculation. These calculations are effectively changed over to computer justifiable code. Accordingly, numerical analysis is a go-between in the middle of mathematics and computer.

There are several different ways to discretize a PDE. The Finite difference strategy is one basic technique that includes supplanting the PDE by a distinction condition which must be fulfilled by the estimations of the obscure capacity x at a limited set focuses in the space Ω, of the free factor. The Finite component technique replaces the first capacity by a capacity which as some level of smoothness over the worldwide area, yet which is piece insightful polynomial on straightforward cells, for example, little triangles or square shape. This strategy likely the most broad and surely knew discretion system accessible. In the middle of these two strategies, there are a couple of preservationist plans called Finite volume techniques, which endeavor to copy nonstop protection laws of material science.

Numerical solution of ordinary differential equation is assembled into:

i. Single step method.
ii. Multiple step method.

Single step Method

In this method one and only solution point is included in discovering the following solution point. Example is the Runge Kutta method. The method utilize a weighted whole of the estimation of $f(x, y)$ evaluated at the beginning stage of every stride and at different point over the combination step.

Multiple step method

This method makes utilization of more than one previous solution point to locating the following solution point. The corrector and predictor method fall into this classification.

1.2 Statement of the Research Problem

It has been recognized that in the numerous fields of study such as sciences i.e. physics, chemistry etc. The idea of numerical analysis is very crucial and cannot be over emphasized because there is high closeness of errors in the various fields mentioned above. Hence, the need to model a solution that will reduce this error has necessitated the research.

1.3 Aim of Study and Objectives of Study

The aim of this research is to construct an implicit Adams type linear multi-step scheme for solving stiff differential equations.

The objectives of the study are to:

i. Construct two different linear multi-step implicit block method using the interpolation and collocation techniques.

ii. Compute the order and error constant and convergence analysis of the schemes.

iii. Test for the consistency and stability of the schemes.

iv. Test for level of accuracy of the schemes using maple 18 software.

1.4 Significance of Study

Numerical Analysis is commonly used to identify the cost of student's errors when he/she consistently make mistakes, it is a process of reviewing a students work and looking for patterns of understanding. The application of mathematics is everywhere not just in the traditional sciences of physics and chemistry but in biology, Medicine, Agriculture and many more areas.

1.5 Definition of Terms

The following terms are what we come in contact with in the study of error analysis.

i. The order of $\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j}$ is p if $c_0 = c_1 = c_2 = ... = c_p = 0$ but $c_{p+1} \neq 0$ and c_{p+1} is the error constants of the method where

$$c_0 = c_1 = c_2 = ... = c_{p+1} \neq 0$$

$$
\begin{aligned}
&c_0 = a_0 + a_1 + \ldots + a_k \\
&c_1 = (a_1 + 2a_2 + \ldots + ka_k) - (b_0 + b_1 + \ldots + b_k) \\
&. \\
&. \\
&. \\
&c_p = \frac{1}{p!}\left(a_1 + 2^p a_2 + \ldots k^p a_k\right) - \frac{1}{(p-1)!}\left(b_1 + 2^{(p-1)} b_2 + \ldots + k^{p-1} b_k\right),
\end{aligned}
$$

Error Constant: : A scheme is said to be of order p if $c_0 = c_1 = c_2 = ... = c_p = 0$ and $c_{p+1} \neq 0$, c_{p+1} is the error constant.

ii. Accuracy: An algorithm is accurate if $f'(x) = f(x)$ for all inputs x when $f'(x)$ is computed.

iii. Stability: An algorithm is stable if its outputs in finite precision (floating point arithmetic) is always near it output in exact precision.

iv. Linear Multistep Method: Multistep methods attempt to gain efficiency by keeping and using the information from previous steps rather than discarding it. Multistep method uses information from the previous steps to compute the next value. In particular, a linear multistep uses a linear combination of y_j and $f(t_i, y_j)$ to calculate the value of y for the desired current step. Thus, a linear multistep method is a method of the form: Linear multistep method of step number k can be written as $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$ Sometimes an explicit multistep method is used to predict the value of y_{n+s}. That value is then used in an implicit formula to correct the value. The result is predictor-corrector method.

v. Error Terms: We can write numerical scheme in form of $\sum_{j=0}^{n}\alpha_j y_{n+1-j} = h\sum_{i=0}^{n}\beta_i f_{n+1-i}$ then the error $e_r = \sum_{j=0}^{n}\alpha_j y_{n+1-j} - h\sum_{i=0}^{n}\beta_i f_{n+1-i} = 0(h^k)$ we obtain the error term by expanding the equilibrium in power of h.

vi. One-step Method : These are numerical method that determine the solution at the suppose time through the repulsive formula $y_{n+1} = y_n + h\phi(t_n, y_n, h) n \in N$

i.e. k=1 $\sum_{j=0}^{k}\alpha_j y_{n+j} = h\phi f(y_{n+k}, y_{n+k-1}, ... y_n, t_n, h)$.

vii. Hybrid Method: K-step general linear multistep method can be written as

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h\beta_u f_{n+u}$$

Where $\alpha_k \neq 0, a_0^2 + \beta_0^2 > 0, u \in \{0,1,2,...,k\}$ and $f_{n+u} = f(x_{n+u}, y_{n+v})$ and $\alpha_j and \beta_j$ are constants.

viii. k – block, r point Methods: The k – block, r point methods for initial value problems are given by the matrix finite difference equation

$$y_m = \sum_{i=0}^{k} A^{(1)} y_{m-1} + h^2 \sum_{i=0}^{k} B^{(1)} Z_{m-1} \text{ where } y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{n+y} \end{pmatrix}, \quad f_m = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ \vdots \\ f_{n+y} \end{pmatrix}$$

Where $A^{(1)}$, $B^{(1)}$, i=0,1,2,3, …, k are (r x r) matrices respectively with element a_{ij}^i, b_{ij}^i for $i, j = 1,2,3,...,r$

ix. Definition of Collocation Points and Interpolation Points: A collocation point is the point at which the derivative of a function is evaluated while an Interpolation point is the point at which the solution is evaluated.

x. Definition of Collocation Method: This is a method which involves the determination of an approximate solution in a suitable set of functions called trial or basis functions. The approximate solution is required to satisfy the differential equation and its supplementary condition at certain points in the range of interest, called the collocation points. The collocation methods by their very nature yield continuous solutions and the principle behind multistep collocation is to allow the collocation polynomial use information from previous points in the integration.

xi. Differential equations: Numerical analysis is also concerned with computing (in an approximate way) the solution of differential equations, both ordinary differential equations and partial differential equations. Partial differential equations are solved by first discretizing the equation, bringing it into a finite-dimensional subspace. This can be done by a finite element method, a finite difference method, or (particularly in engineering) a finite volume method.

xii. Stiff Problem: In mathematics, a stiff equation is a differential equation for which certain numerical methods for solving the equation are numerically unstable, unless the step size is taken to be extremely small.

xiii. The Adams–Bashforth methods: This is a method which allows numerical analysts to explicitly compute the approximate solution at an instant time from the solutions in previous instants. That is, it is not self-starting.

xiv. Adams Method: Adams methods are based on the idea of approximating the integrand with a polynomial within the interval (t_n, t_{n+1}). ... There are two types of Adams methods, the explicit and the implicit types. The explicit type is called the Adams-Bashforth (AB) methods and the implicit type is called the Adams-Moulton (AM) methods.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Numerical Schemes for Solving Ordinary Differential Equation

Numerical analysis is the region of mathematics and computer science that makes, dissects and executes calculations for approximating the issue continuous mathematics (Kincaid. 2000).The four steps Backward Differentiation Formulae (BDF) were reformulated for applications in the continuous form. The process produces some schemes which are combined in order to form an accurate and efficient block method for parallel or sequential solution of ordinary differential equations (ODE's). The suggested approach eliminates requirement for a starting value and its speed proved to be up when computations with the Block Discrete schemes were used(Yahaya & Mohammed 2010). Yahaya (2004), The Continous Multistep Method (CMM) method produces piece-wise polynomial solutions over K-steps [Xn, Xn+k] for the first order systems/scalar ODEs. Of note, is that the implicit [CMM] interpolant is not to be directly use as the numerical integrator, but the resulting discrete multistep Schemes which is derived from it, which will now be self-starting and can be applied for parallel or sequential solutions of both the initial and boundary value problems. In this paper is part of research effort to reformulate for efficient and accurate use, the linear multistep methods and four step block differentiation formulae (BDF) is considered here. Yahaya and Mohammed (2010) presented a one parameter family of modified collocation method with large region of absolute stability for accurate and efficient solution of the ordinary differential equations. The process produce some hybrid schemes which are combined together to form the block method for parallel or sequential solution of ODEs. The suggested approach eliminates requirement for a starting value and special predictor for off- grid values in the discrete schemes.Implicit Runge- Kutta methods are used for solving stiff problems which mostly arise in real life situations. Analysis of the order, error constant, consistency and convergence will help in determining an effective Runge-Kutta Method (RKM) to use. Due to the loss of linearity in Runge –Kutta Methods and the fact that the general Runge –Kutta Method makes no mention of the differential equation makes it impossible to define the order of the method independently of the differential equation Muhammad (2020).

Yahaya and Mohammed (2009) carried out a research titled a reformation of implicit 5-step backward differentiation formulae in continuous form for solution of first order initial value problems. Their presents a reformation of 5-step backward differentiation formulae (BDF) for accurate and efficient use and with large region of absolute stability into the continuous form. The process produced some schemes which are combined together to form the block method for parallel or sequential solution of ODE's. The method was tested on simple ODE's and showed to perform satisfactorily, without recourse to any other method.

A one point implicit code in the form of Adams-Moulton method was developed for solving a system of ordinary differential equations (ODEs) with variable step size. The multistep method involves the computations of the divided differences and integration coefficients in the code when using the variable step size or variable step size and order. The idea of the developed method is to store all the coefficients involved in the code. Therefore, this strategy can avoid the lengthy computation of the coefficients during the implementation of the code as well as improve the execution time. Numerical results are given to compare the efficiency of the developed method to the 1nitial Value Problem (IVPs) method (Mohamed and Zanariah 2006).

The derivation of the new method can be easily implemented. We established the proposed method's characteristics, including order, zero-stability, and convergence. Numerical experiments are used to confirm the superiority of the method. Applications to problems in physics and engineering are given to assess the significance of the method (Reem and Fudziah, 2020).

Butcher (2000) Numerical methods for the solution of initial value problems in ordinary differential equations made enormous progress during the 20th century for several reasons. The first reason lie in the impetus that was given to the subject in the concluding years of the previous century by the seminal papers of Bashforth and Adams for linear multistep methods and Runge for Runge–Kutta methods. Other reasons, which of course apply to numerical analysis in general, are in the invention of electronic computers half way through the century and the needs in mathematical modelling of efficient numerical algorithms as an alternative to classical methods of applied mathematics. This survey paper follows many of the main strands in the developments of these methods, both for general problems, stiff systems, and for many of the special problem types that have been gaining significance as the century draws to an end.

Ehiemua (2019) presented a new approach to the Radau method of solving stiff problems in Ordinary Differential Equation (ODE). This new implicit Runge-Kutta Scheme is derived for order 5, and the formular so derived was implemented, using Maple-18 package, and the results were compared with existing Radau Method. The performance of the method has improved results over those of Radau, on comparison for consistency convergence and stability.

2.2 Linear Multi-Step Method for Solving Stiff Differential Equations

Mohammed and Adeniyi (2014) developed a three step implicit hybrid linear multistep method for the solution of third order ordinary differential equations. In their work, a linear multistep hybrid method (lmhm) with continuous coefficients is considered and directly applied to solve third order initial value problems (ivps). the continuous method is used to obtain multiple finite difference methods (mfdms) each of order 5 which are combined as simultaneous numerical integrators to provide a direct solution to ivps over sub-intervals which do not overlap. the convergence of the mfdms is discussed by conveniently representing the mfdms as a block method and verifying that the block method is zero-stable and consistent. the superiority of the mfdms over the existing methods is established numerically.

Nwachukwu and Okor (2018) Second derivative generalized backward differentiation formulae (SDGBDF) are developed herein and applied as boundary value methods (BVMs) to solve stiff initial value problems (IVPs) in ordinary differential equations (ODEs). The order, error constant, zero stability and the region of absolute stability for the SDGBDF are discussed. The methods are stable.

Karol *et al.* (2020) in their research, a comprehensive comparison of ode solvers for biochemical problems. The article is focused on a deep and detailed study on available Ordinary Differential Equations (ODEs) numerical solvers for biochemical and bioprocesses purposes, which are an important part of the renewable energy sector. A wide selection of algorithms is tested - starting from simple, single-step explicit methods, ending with implicit multi-step techniques. These include MATLAB, Python, C++, and C# implementations. The test configuration is an ODEs based model that simulates a biogas production reactor. The research shows that most of the tested solvers pass the accuracy-test (the difference didn’t exceed 0,07%), however only selected are efficient. Most of Runge-Kutta based methods are slow and require an enormous number of steps (more than 2.5×10^8). Only multi-step implicit methods are long term solutions - they provide great accuracy while dealing well with stiff, non-smooth ODEs sets. The best from tested solutions were two MATLAB solvers - ode23s and ode15s, as well as a python solver - the LSODA. The first needed averagely 84,051s of calculation time, and 96465 steps, while ode15s required just 11,529s, performing over 20-times fewer steps. The LSODA is ranked somewhere between them with 18,806s of calculation time and the total number of 23730 steps for tested ODEs set.

This research reviewed what others have and improved on by formulating a better performance scheme for approximating stiff differential equations. The fully implicit 3-point Block Extended Backward Differentiation Formula for solving stiff initial value problems. The iterative block method is proven to be convergent by establishing zero stability and consistency conditions. Numerical results are given to show the effect of zero stability and consistency. The accuracy is seen to improve as the step length tends to zero. The order of the method is also shown to be 6. Extended backward differentiation formula and proved that the method is convergent (Musa *et al.*, 2012). A two-step Lo-stable second derivative hybrid block method of order eight for the direct solution of stiff Initial Value Problems (IVPs). The main method and additional methods are obtained from the same continuous scheme derived using interpolation and collocation technique to form the block method.

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Derivation of the Three-Step Implicit, Adams Type Method

This chapter gives a step by step description of the formulation of the proposed 3-implicit, Adams type method for the solution of Stiff systems. Off-grid points were incorporated at collocation points.

The author seeks to derive a numerical scheme defined, according to Lambert's (1991), as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k f_{n+k} \tag{3.1}$$

Where $\alpha_j, j = 0, \left(\frac{1}{3}\right), \left(\frac{2}{3}\right), 1, \left(\frac{4}{3}\right), ...k$ and $\beta_k s$ are unknown constants to be uniquely determined and h is the step length. It is worthy of note that $\alpha_k = 1, \beta_k \neq 0, \alpha_0 \neq 0, k$ is the number of steps for the proposed method.

We assumed that the solution of the differential problem of interest can be approximated by a power series of the form;

$$y(x) = \sum_{j=0}^{i+c-1} \alpha_j p_j(x) \tag{3.2}$$

Where i and c are the number of interpolation and collocation points respectively, α_j's are coefficients to be determined and $p_j(x)$ can be any orthogonal polynomial. In this case power series is used.

3.2 Derivation of the three step Adam's type method with six off-grid points

The basis function for the proposed scheme is given in equation

$$Y(x) = x^{10}a_{10} + x^9 a_9 + x^8 a_8 + x^7 a_7 + x^6 a_6 + x^5 a_5 + x^4 a_4 + x^3 a_3$$
$$x^2 a_2 + x a_1 + a_0 \qquad (3.3)$$

Differentiating (3.3)

$$YI(x) = 10x^9 a_{10} + 9x^8 a_9 + 8x^7 a_8 + 7x^6 a_7 + 6x^5 a_6 + 5x^4 a_5 + 4x^3 a_4 + 3x^2 a_3$$
$$2xa_2 + a_1 \qquad (3.4)$$

Interpolating at 2h and collocating at mesh point 1/3, then

The square matrix A was generated as shown in (3.5);

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2h & h & h & h & h & h & h & h & h & h & h \\
(2h)^2 & 0 & 2h(\frac{1}{3}h) & 2h(\frac{2}{3}h) & 2h(h) & 2h(\frac{4}{3}h) & 2h(\frac{5}{3}h) & 2h(2h) & 2h(\frac{7}{3}h) & 2h(\frac{8}{3}h) & 2h(3h) \\
(2h)^3 & 0 & 3h(\frac{1}{3}h)^2 & 3h(\frac{2}{3}h)^2 & 3h(h)^2 & 3h(\frac{4}{3}h)^2 & 3h(\frac{5}{3}h)^2 & 3h(2h)^2 & 3h(\frac{7}{3}h)^2 & 3h(\frac{8}{3}h)^2 & 3h(3h)^2 \\
(2h)^4 & 0 & 4h(\frac{1}{3}h)^3 & 4h(\frac{2}{3}h)^3 & 4h(h)^3 & 4h(\frac{4}{3}h)^3 & 4h(\frac{5}{3}h)^3 & 4h(2h)^3 & 4h(\frac{7}{3}h)^3 & 4h(\frac{8}{3}h)^3 & 4h(3h)^3 \\
(2h)^5 & 0 & 5h(\frac{1}{3}h)^4 & 5h(\frac{2}{3}h)^4 & 5h(h)^4 & 5h(\frac{4}{3}h)^4 & 5h(\frac{5}{3}h)^4 & 5h(2h)^4 & 5h(\frac{7}{3}h)^4 & 5h(\frac{8}{3}h)^4 & 5h(3h)^4 \\
(2h)^6 & 0 & 6h(\frac{1}{3}h)^5 & 6h(\frac{2}{3}h)^5 & 6h(h)^5 & 6h(\frac{4}{3}h)^5 & 6h(\frac{5}{3}h)^5 & 6h(2h)^5 & 6h(\frac{7}{3}h)^5 & 6h(\frac{8}{3}h)^5 & 6h(3h)^5 \\
(2h)^7 & 0 & 7h(\frac{1}{3}h)^6 & 7h(\frac{2}{3}h)^6 & 7h(h)^6 & 7h(\frac{4}{3}h)^6 & 7h(\frac{5}{3}h)^6 & 7h(2h)^6 & 7h(\frac{7}{3}h)^6 & 7h(\frac{8}{3}h)^6 & 7h(3h)^6 \\
(2h)^8 & 0 & 8h(\frac{1}{3}h)^7 & 8h(\frac{2}{3}h)^7 & 8h(h)^7 & 8h(\frac{4}{3}h)^7 & 8h(\frac{5}{3}h)^7 & 8h(2h)^7 & 8h(\frac{7}{3}h)^7 & 8h(\frac{8}{3}h)^7 & 8h(3h)^7 \\
(2h)^9 & 0 & 9h(\frac{1}{3}h)^8 & 9h(\frac{2}{3}h)^8 & 9h(h)^8 & 9h(\frac{4}{3}h)^8 & 9h(\frac{5}{3}h)^8 & 9h(2h)^8 & 9h(\frac{7}{3}h)^8 & 9h(\frac{8}{3}h)^8 & 9h(3h)^8 \\
(2h)^{10} & 0 & 10h(\frac{1}{3}h)^9 & 10h(\frac{2}{3}h)^9 & 10h(h)^9 & 10h(\frac{4}{3}h)^9 & 10h(\frac{5}{3}h)^9 & 10h(2h)^9 & 10h(\frac{7}{3}h)^9 & 10h(\frac{8}{3}h)^9 & 10h(3h)^9
\end{pmatrix}$$

(3.5)

Which gives;

$$
A=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2h & h & h & h & h & h & h & h & h & h & h \\
4h^2 & 0 & \frac{2}{3}h^2 & \frac{4}{3}h^2 & 2h^2 & \frac{8}{3}h^2 & \frac{10}{3}h^2 & 4h^2 & \frac{14}{3}h^2 & \frac{16}{3}h^2 & 6h^2 \\
8h^3 & 0 & \frac{1}{3}h^3 & \frac{4}{3}h^3 & 3h^3 & \frac{16}{3}h^3 & \frac{25}{3}h^3 & 12h^3 & \frac{49}{3}h^3 & \frac{64}{3}h^3 & 27h^3 \\
16h^4 & 0 & \frac{4}{27}h^4 & \frac{32}{27}h^4 & 4h^4 & \frac{256}{27}h^4 & \frac{500}{27}h^4 & 32h^4 & \frac{1327}{27}h^4 & \frac{2048}{27}h^4 & 108h^4 \\
23h^5 & 0 & \frac{5}{81}h^5 & \frac{80}{81}h^5 & 5h^5 & \frac{1289}{81}h^5 & \frac{3125}{81}h^5 & 80h^5 & \frac{12005}{81}h^5 & \frac{20480}{81}h^5 & 405h^5 \\
64h^6 & 0 & \frac{2}{81}h^6 & \frac{64}{81}h^6 & 6h^6 & \frac{2048}{81}h^6 & \frac{6250}{81}h^6 & 192h^6 & \frac{33614}{81}h^6 & \frac{65536}{81}h^6 & 1458h^6 \\
128h^7 & 0 & \frac{7}{729}h^7 & \frac{448}{729}h^7 & 7h^7 & \frac{28672}{729}h^7 & \frac{109375}{729}h^7 & 448h^7 & \frac{823543}{729}h^7 & \frac{1835008}{729}h^7 & 5103h^7 \\
256h^8 & 0 & \frac{8}{2187}h^8 & \frac{1024}{2187}h^8 & 8h^8 & \frac{131072}{2187}h^8 & \frac{625000}{2187}h^8 & 1024h^8 & \frac{6588344}{2187}h^8 & \frac{16777216}{2187}h^8 & 17496h^8 \\
512h^9 & 0 & \frac{1}{729}h^9 & \frac{256}{729}h^9 & 9h^9 & \frac{65536}{729}h^9 & \frac{390625}{729}h^9 & 2304h^9 & \frac{5764801}{729}h^9 & \frac{16777216}{729}h^9 & 59049h^9 \\
1024h^{10} & 0 & \frac{10}{19683}h^{10} & \frac{5120}{19683}h^{10} & 10h^{10} & \frac{2621440}{19683}h^{10} & \frac{19531250}{19683}h^{10} & 5120h^{10} & \frac{403536070}{19683}h^{10} & \frac{1342177280}{19683}h^{10} & 196830h^{10}
\end{pmatrix}
$$

(3.6)

Obtaining the inverse of the transposed matrix A gives;

$$
\left[A^T\right]^{-1}=\begin{bmatrix} y_{n+2}-\frac{7}{75}hf_n-\frac{771}{1400}hf_{n+\frac{1}{3}}+\frac{51}{700}hf_{n+\frac{2}{3}}-\frac{199}{210}hf_{n+1}+\frac{249}{700}hf_{n+\frac{4}{3}}-\frac{633}{700}hf_{n+\frac{5}{3}} \\ +\frac{299}{2100}hf_{n+2}-\frac{33}{350}hf_{n+\frac{7}{3}}+\frac{3}{140}hf_{n+\frac{8}{3}}-\frac{3}{1400}hf_{n+3} \end{bmatrix},
$$

$$
[f_n],
$$

$$
\begin{bmatrix} -\frac{7129}{1680}\frac{f_n}{h}+\frac{27}{2}\frac{f_{n+\frac{1}{3}}}{h}-\frac{27f_{n+\frac{2}{3}}}{h}+\frac{42f_{n+1}}{h}-\frac{189}{4}\frac{f_{n+\frac{4}{3}}}{h}+\frac{189}{5}\frac{f_{n+\frac{5}{3}}}{h}-\frac{21f_{n+2}}{h} \\ +\frac{54}{7}\frac{f_{n+\frac{7}{3}}}{h}-\frac{27}{16}\frac{f_{n+\frac{8}{3}}}{h}+\frac{1}{6}\frac{f_{n+3}}{h} \end{bmatrix},
$$

$$
\begin{bmatrix} \frac{6515}{672}\frac{f_n}{h^2}-\frac{13827}{280}\frac{f_{n+\frac{1}{3}}}{h^2}+\frac{17607}{140}\frac{f_{n+\frac{2}{3}}}{h^2}-\frac{6289}{30}\frac{f_{n+1}}{h^2}+\frac{19497}{80}\frac{f_{n+\frac{4}{3}}}{h^2}-\frac{795}{4}\frac{f_{n+\frac{5}{3}}}{h^2} \\ +\frac{6709}{60}\frac{f_{n+2}}{h^2}-\frac{2901}{70}\frac{f_{n+\frac{7}{3}}}{h^2}+\frac{10221}{1120}\frac{f_{n+\frac{8}{3}}}{h^2}-\frac{761}{840}\frac{f_{n+3}}{h^2} \end{bmatrix},
$$

(3.7)

$$\left[\begin{array}{l}-\frac{4523}{336}\frac{f_n}{h^3}+\frac{381753}{4480}\frac{f_{n+\frac{1}{3}}}{h^3}-\frac{562599}{2240}\frac{f_{n+\frac{2}{3}}}{h^3}+\frac{72569}{160}\frac{f_{n+1}}{h^3}-\frac{176013}{320}\frac{f_{n+\frac{4}{3}}}{h^3}+\frac{29457}{64}\frac{f_{n+\frac{5}{3}}}{h^3}\\-\frac{84307}{320}\frac{f_{n+2}}{h^3}+\frac{110727}{1120}\frac{f_{n+\frac{7}{3}}}{h^3}-\frac{49221}{2240}\frac{f_{n+\frac{8}{3}}}{h^3}+\frac{29531}{13440}\frac{f_{n+3}}{h^3}\end{array}\right],$$

$$\left[\begin{array}{l}\frac{1539}{128}\frac{f_n}{h^4}-\frac{69003}{800}\frac{f_{n+\frac{1}{3}}}{h^4}+\frac{224109}{800}\frac{f_{n+\frac{2}{3}}}{h^4}-\frac{108351}{200}\frac{f_{n+1}}{h^4}+\frac{1100241}{1600}\frac{f_{n+\frac{4}{3}}}{h^4}-\frac{47457}{80}\frac{f_{n+\frac{5}{3}}}{h^4}\\+\frac{277533}{800}\frac{f_{n+2}}{h^4}-\frac{26451}{200}\frac{f_{n+\frac{7}{3}}}{h^4}+\frac{95211}{3200}\frac{f_{n+\frac{8}{3}}}{h^4}-\frac{2403}{800}\frac{f_{n+3}}{h^4}\end{array}\right],$$

$$\left[\begin{array}{l}\frac{9039}{1280}\frac{f_n}{h^5}+\frac{70263}{1280}\frac{f_{n+\frac{1}{3}}}{h^5}-\frac{6108}{320}\frac{f_{n+\frac{2}{3}}}{h^5}+\frac{124857}{320}\frac{f_{n+1}}{h^5}-\frac{330921}{640}\frac{f_{n+\frac{4}{3}}}{h^5}+\frac{294957}{640}\frac{f_{n+\frac{5}{3}}}{h^5}\\-\frac{88407}{320}\frac{f_{n+2}}{h^5}+\frac{34353}{320}\frac{f_{n+\frac{7}{3}}}{h^5}-\frac{31383}{1280}\frac{f_{n+\frac{8}{3}}}{h^5}+\frac{3207}{1280}\frac{f_{n+3}}{h^5}\end{array}\right],$$

$$\left[\begin{array}{l}\frac{1215}{448}\frac{f_n}{h^6}-\frac{891}{40}\frac{f_{n+\frac{1}{3}}}{h^6}+\frac{45603}{560}\frac{f_{n+\frac{2}{3}}}{h^6}-\frac{97443}{560}\frac{f_{n+1}}{h^6}+\frac{268353}{1120}\frac{f_{n+\frac{4}{3}}}{h^6}-\frac{24705}{112}\frac{f_{n+\frac{5}{3}}}{h^6}\\+\frac{76059}{560}\frac{f_{n+2}}{h^6}-\frac{30213}{560}\frac{f_{n+\frac{7}{3}}}{h^6}+\frac{28107}{2240}\frac{f_{n+\frac{8}{3}}}{h^6}-\frac{729}{560}\frac{f_{n+3}}{h^6}\end{array}\right],$$

$$\left[\begin{array}{l}-\frac{2349}{3584}\frac{f_n}{h^7}+\frac{14337}{2560}\frac{f_{n+\frac{1}{3}}}{h^7}-\frac{1701}{80}\frac{f_{n+\frac{2}{3}}}{h^7}+\frac{7533}{160}\frac{f_{n+1}}{h^7}-\frac{85779}{1280}\frac{f_{n+\frac{4}{3}}}{h^7}+\frac{16281}{256}\frac{f_{n+\frac{5}{3}}}{h^7}\\-\frac{12879}{320}\frac{f_{n+2}}{h^7}+\frac{36693}{2240}\frac{f_{n+\frac{7}{3}}}{h^7}-\frac{9963}{2560}\frac{f_{n+\frac{8}{3}}}{h^7}+\frac{1053}{2560}\frac{f_{n+3}}{h^7}\end{array}\right],$$

$$\left[\begin{array}{l}\frac{81}{896}\frac{f_n}{h^8}-\frac{891}{1120}\frac{f_{n+\frac{1}{3}}}{h^8}+\frac{3483}{1120}\frac{f_{n+\frac{2}{3}}}{h^8}-\frac{567}{80}\frac{f_{n+1}}{h^8}+\frac{3321}{320}\frac{f_{n+\frac{4}{3}}}{h^8}-\frac{81}{8}\frac{f_{n+\frac{5}{3}}}{h^8}\\+\frac{1053}{160}\frac{f_{n+2}}{h^8}-\frac{1539}{560}\frac{f_{n+\frac{7}{3}}}{h^8}+\frac{2997}{4480}\frac{f_{n+\frac{8}{3}}}{h^8}-\frac{81}{1120}\frac{f_{n+3}}{h^8}\end{array}\right],$$

$$\left[\begin{array}{l}-\frac{243}{44800}\frac{f_n}{h^9}+\frac{2187}{44800}\frac{f_{n+\frac{1}{3}}}{h^9}-\frac{2187}{11200}\frac{f_{n+\frac{2}{3}}}{h^9}+\frac{729}{1600}\frac{f_{n+1}}{h^9}-\frac{2187}{3200}\frac{f_{n+\frac{4}{3}}}{h^9}+\frac{2187}{3200}\frac{f_{n+\frac{5}{3}}}{h^9}\\-\frac{729}{1600}\frac{f_{n+2}}{h^9}+\frac{2187}{11200}\frac{f_{n+\frac{7}{3}}}{h^9}-\frac{2187}{44800}\frac{f_{n+\frac{8}{3}}}{h^9}+\frac{243}{44800}\frac{f_{n+3}}{h^9}\end{array}\right]$$

$$B = \begin{pmatrix} y_{n+2} \\ hf_n \\ hf_{n+\frac{1}{3}} \\ hf_{n+\frac{2}{3}} \\ hf_{n+1} \\ hf_{n+\frac{4}{3}} \\ hf_{n+\frac{5}{3}} \\ hf_{n+2} \\ hf_{n+\frac{7}{3}} \\ hf_{n+\frac{8}{3}} \\ hf_{n+3} \end{pmatrix}$$
(3.8)

Evaluating the product of $\left[A^T\right]^{-1}$ and B then simplifying,

$$a_0 = y_{n+2} - \frac{7}{75}hf_n - \frac{771}{1400}hf_{n+\frac{1}{3}} + \frac{51}{700}hf_{n+\frac{2}{3}} - \frac{199}{210}hf_{n+1} + \frac{249}{700}hf_{n+\frac{4}{3}} - \frac{633}{700}hf_{n+\frac{5}{3}}$$
$$+ \frac{299}{2100}hf_{n+2} - \frac{33}{350}hf_{n+\frac{7}{3}} + \frac{3}{140}hf_{n+\frac{8}{3}} - \frac{3}{1400}hf_{n+3}$$
(3.9)

$$a_1 = f_n$$
(3.10)

$$a_2 = -\frac{7129f_n}{1680h} + \frac{27f_{n+\frac{1}{3}}}{2h} - \frac{27f_{n+\frac{2}{3}}}{h} + \frac{42f_{n+1}}{h} - \frac{189f_{n+\frac{4}{3}}}{4h} + \frac{189f_{n+\frac{5}{3}}}{5h} - \frac{21f_{n+2}}{h} + \frac{54f_{n+\frac{7}{3}}}{7h} - \frac{27f_{n+\frac{8}{3}}}{16h} + \frac{f_{n+3}}{6h}$$
(3.11)

$$a_3 = -\frac{6515f_n}{672h^2} - \frac{13827f_{n+\frac{1}{3}}}{280h^2} + \frac{17607f_{n+\frac{2}{3}}}{140h^2} - \frac{6289f_{n+1}}{30h^2} + \frac{19497f_{n+\frac{4}{3}}}{80h^2}$$
$$-\frac{795f_{n+\frac{5}{3}}}{4h^2} + \frac{6709f_{n+2}}{60h^2} - \frac{2901f_{n+\frac{7}{3}}}{70h^2} + \frac{10221f_{n+\frac{8}{3}}}{1120h^2} - \frac{761f_{n+3}}{840h^2}$$
(3.12)

$$a_4 = -\frac{4523 f_n}{336h^3} + \frac{381753 f_{n+\frac{1}{3}}}{4480h^3} - \frac{562559 f_{n+\frac{2}{3}}}{2240h^3} + \frac{72569 f_{n+1}}{160h^3} - \frac{176013 f_{n+\frac{4}{3}}}{320h^3}$$
$$+\frac{29457 f_{n+\frac{5}{3}}}{64h^3} - \frac{84307 f_{n+2}}{320h^3} + \frac{110727 f_{n+\frac{7}{3}}}{1120h^3} - \frac{49221 f_{n+\frac{8}{3}}}{2240h^3} + \frac{29531 f_{n+3}}{13440h^3}$$
(3.13)

$$a_5 = \frac{1539 f_n}{128h^4} - \frac{69003 f_{n+\frac{1}{3}}}{800h^4} + \frac{224109 f_{n+\frac{2}{3}}}{800h^4} - \frac{108351 f_{n+1}}{200h^4} + \frac{1100241 f_{n+\frac{4}{3}}}{1600h^4}$$
$$-\frac{47457 f_{n+\frac{5}{3}}}{80h^4} + \frac{277533 f_{n+2}}{800h^4} - \frac{26451 f_{n+\frac{7}{3}}}{200h^4} + \frac{95211 f_{n+\frac{8}{3}}}{3200h^4} - \frac{2403 f_{n+3}}{800h^4}$$
(3.14)

$$a_6 = -\frac{9039 f_n}{1280h^5} + \frac{70263 f_{n+\frac{1}{3}}}{1280h^5} - \frac{61083 f_{n+\frac{2}{3}}}{320h^5} + \frac{124857 f_{n+1}}{320h^5} - \frac{330921 f_{n+\frac{4}{3}}}{640h^5}$$
$$+\frac{294957 f_{n+\frac{5}{3}}}{640h^5} - \frac{88407 f_{n+2}}{320h^5} + \frac{34353 f_{n+\frac{7}{3}}}{320h^5} - \frac{31383 f_{n+\frac{8}{3}}}{1280h^5} + \frac{3207 f_{n+3}}{1280h^5}$$
(3.15)

$$a_7 = \frac{1215 f_n}{448h^6} - \frac{891 f_{n+\frac{1}{3}}}{40h^6} + \frac{45603 f_{n+\frac{2}{3}}}{560h^6} - \frac{97443 f_{n+1}}{560h^6} + \frac{268353 f_{n+\frac{4}{3}}}{1120h^6}$$
$$-\frac{24705 f_{n+\frac{5}{3}}}{112h^6} + \frac{76059 f_{n+2}}{560h^6} - \frac{30213 f_{n+\frac{7}{3}}}{560h^6} + \frac{28107 f_{n+\frac{8}{3}}}{2240h^6} - \frac{729 f_{n+3}}{560h^6}$$
(3.16)

$$a_8 = -\frac{2349 f_n}{3584h^7} + \frac{14337 f_{n+\frac{1}{3}}}{2560h^7} - \frac{1701 f_{n+\frac{2}{3}}}{80h^7} + \frac{7533 f_{n+1}}{320h^7} - \frac{85779 f_{n+\frac{4}{3}}}{1280h^7}$$
$$+\frac{16281 f_{n+\frac{5}{3}}}{256h^7} - \frac{12879 f_{n+2}}{320h^7} + \frac{36693 f_{n+\frac{7}{3}}}{2240h^7} - \frac{9963 f_{n+\frac{8}{3}}}{2560h^7} + \frac{1053 f_{n+3}}{2560h^7}$$
(3.17)

$$a_9 = \frac{81 f_n}{896h^8} - \frac{891 f_{n+\frac{1}{3}}}{1120h^8} + \frac{3483 f_{n+\frac{2}{3}}}{1120h^6} - \frac{567 f_{n+1}}{80h^8} + \frac{3321 f_{n+\frac{4}{3}}}{320h^8}$$
$$-\frac{81 f_{n+\frac{5}{3}}}{8h^8} + \frac{1053 f_{n+2}}{160h^8} - \frac{1539 f_{n+\frac{7}{3}}}{560h^8} + \frac{2997 f_{n+\frac{8}{3}}}{4480h^8} - \frac{81 f_{n+3}}{1120h^8}$$
(3.18)

$$a_{10} = -\frac{243 f_n}{44800h^9} + \frac{2187 f_{n+\frac{1}{3}}}{44800h^9} - \frac{2187 f_{n+\frac{2}{3}}}{11200h^9} + \frac{729 f_{n+1}}{1600h^9} - \frac{2187 f_{n+\frac{4}{3}}}{3200h^9}$$
$$+\frac{2187 f_{n+\frac{5}{3}}}{3200h^9} - \frac{729 f_{n+2}}{1600h^9} + \frac{2187 f_{n+\frac{7}{3}}}{11200h^9} - \frac{2187 f_{n+\frac{8}{3}}}{44800h^9} + \frac{243 f_{n+3}}{44800h^9}$$
(3.19)

Let

$$Y(Cx) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$$
$$+a_7 x^7 + a_8 x^8 + a_9 x^9 + a_{10} x^{10}$$
(3.20)

$$
Y(cx)=x^{10}\left[\begin{array}{l}-\frac{243}{44800}\frac{f_n}{h^9}+\frac{2187}{44800}\frac{f_{n+\frac{1}{3}}}{h^9}-\frac{2187}{11200}\frac{f_{n+\frac{2}{3}}}{h^9}+\frac{729}{1600}\frac{f_{n+1}}{h^9}-\frac{2187}{3200}\frac{f_{n+\frac{4}{3}}}{h^9}+\frac{2187}{3200}\frac{f_{n+\frac{5}{3}}}{h^9}\\-\frac{729}{1600}\frac{f_{n+2}}{h^9}+\frac{2187}{11200}\frac{f_{n+\frac{7}{3}}}{h^9}-\frac{2187}{44800}\frac{f_{n+\frac{8}{3}}}{h^9}+\frac{243}{44800}\frac{f_{n+3}}{h^9}\end{array}\right]
$$

$$
+x^{9}\left[\begin{array}{l}\frac{81}{896}\frac{f_n}{h^8}-\frac{891}{1120}\frac{f_{n+\frac{1}{3}}}{h^8}+\frac{3483}{1120}\frac{f_{n+\frac{2}{3}}}{h^8}-\frac{567}{80}\frac{f_{n+1}}{h^8}+\frac{3321}{320}\frac{f_{n+\frac{4}{3}}}{h^8}-\frac{81}{8}\frac{f_{n+\frac{5}{3}}}{h^8}\\+\frac{1053}{160}\frac{f_{n+2}}{h^8}-\frac{1539}{560}\frac{f_{n+\frac{7}{3}}}{h^8}+\frac{2997}{4480}\frac{f_{n+\frac{8}{3}}}{h^8}-\frac{81}{1120}\frac{f_{n+3}}{h^8}\end{array}\right]+
$$

$$
x^{8}\left[\begin{array}{l}-\frac{2349}{3584}\frac{f_n}{h^7}+\frac{14337}{2560}\frac{f_{n+\frac{1}{3}}}{h^7}-\frac{1701}{80}\frac{f_{n+\frac{2}{3}}}{h^7}+\frac{7533}{160}\frac{f_{n+1}}{h^7}-\frac{85779}{1280}\frac{f_{n+\frac{4}{3}}}{h^7}+\frac{16281}{256}\frac{f_{n+\frac{5}{3}}}{h^7}\\-\frac{12879}{320}\frac{f_{n+2}}{h^7}+\frac{36693}{2240}\frac{f_{n+\frac{7}{3}}}{h^7}-\frac{9963}{2560}\frac{f_{n+\frac{8}{3}}}{h^7}+\frac{1053}{2560}\frac{f_{n+3}}{h^7}\end{array}\right]
$$

$$
+x^{7}\left[\begin{array}{l}\frac{1215}{448}\frac{f_n}{h^6}-\frac{891}{40}\frac{f_{n+\frac{1}{3}}}{h^6}+\frac{45603}{560}\frac{f_{n+\frac{2}{3}}}{h^6}-\frac{97443}{560}\frac{f_{n+1}}{h^6}+\frac{268353}{1120}\frac{f_{n+\frac{4}{3}}}{h^6}-\frac{24705}{112}\frac{f_{n+\frac{5}{3}}}{h^6}\\+\frac{76059}{560}\frac{f_{n+2}}{h^6}-\frac{30213}{560}\frac{f_{n+\frac{7}{3}}}{h^6}+\frac{28107}{2240}\frac{f_{n+\frac{8}{3}}}{h^6}-\frac{729}{560}\frac{f_{n+3}}{h^6}\end{array}\right]
$$

$$
+x^{6}\left[\begin{array}{l}\frac{9039}{1280}\frac{f_n}{h^5}+\frac{70263}{1280}\frac{f_{n+\frac{1}{3}}}{h^5}-\frac{6108}{320}\frac{f_{n+\frac{2}{3}}}{h^5}+\frac{124857}{320}\frac{f_{n+1}}{h^5}-\frac{330921}{640}\frac{f_{n+\frac{4}{3}}}{h^5}+\frac{294957}{640}\frac{f_{n+\frac{5}{3}}}{h^5}\\-\frac{88407}{320}\frac{f_{n+2}}{h^5}+\frac{34353}{320}\frac{f_{n+\frac{7}{3}}}{h^5}-\frac{31383}{1280}\frac{f_{n+\frac{8}{3}}}{h^5}+\frac{3207}{1280}\frac{f_{n+3}}{h^5}\end{array}\right],
$$

$$
+x^{5}\left[\begin{array}{l}\frac{1539}{128}\frac{f_n}{h^4}-\frac{69003}{800}\frac{f_{n+\frac{1}{3}}}{h^4}+\frac{224109}{800}\frac{f_{n+\frac{2}{3}}}{h^4}-\frac{108351}{200}\frac{f_{n+1}}{h^4}+\frac{1100241}{1600}\frac{f_{n+\frac{4}{3}}}{h^4}-\frac{47457}{80}\frac{f_{n+\frac{5}{3}}}{h^4}\\+\frac{277533}{800}\frac{f_{n+2}}{h^4}-\frac{26451}{200}\frac{f_{n+\frac{7}{3}}}{h^4}+\frac{95211}{3200}\frac{f_{n+\frac{8}{3}}}{h^4}-\frac{2403}{800}\frac{f_{n+3}}{h^4}\end{array}\right]
$$

$$
+x^{4}\left[\begin{array}{l}-\frac{4523}{336}\frac{f_n}{h^3}+\frac{381753}{4480}\frac{f_{n+\frac{1}{3}}}{h^3}-\frac{562599}{2240}\frac{f_{n+\frac{2}{3}}}{h^3}+\frac{72569}{160}\frac{f_{n+1}}{h^3}-\frac{176013}{320}\frac{f_{n+\frac{4}{3}}}{h^3}+\frac{29457}{64}\frac{f_{n+\frac{5}{3}}}{h^3}\\-\frac{84307}{320}\frac{f_{n+2}}{h^3}+\frac{110727}{1120}\frac{f_{n+\frac{7}{3}}}{h^3}-\frac{49221}{2240}\frac{f_{n+\frac{8}{3}}}{h^3}+\frac{29531}{13440}\frac{f_{n+3}}{h^3}\end{array}\right]
$$

$$+x^3\left[\begin{array}{l}\frac{6515}{672}\frac{f_n}{h^2}-\frac{13827}{280}\frac{f_{n+\frac{1}{3}}}{h^2}+\frac{17607}{140}\frac{f_{n+\frac{2}{3}}}{h^2}-\frac{6289}{30}\frac{f_{n+1}}{h^2}+\frac{19497}{80}\frac{f_{n+\frac{4}{3}}}{h^2}-\frac{795}{4}\frac{f_{n+\frac{5}{3}}}{h^2}\\+\frac{6709}{60}\frac{f_{n+2}}{h^2}-\frac{2901}{70}\frac{f_{n+\frac{7}{3}}}{h^2}+\frac{10221}{1120}\frac{f_{n+\frac{8}{3}}}{h^2}-\frac{761}{840}\frac{f_{n+3}}{h^2}\end{array}\right]$$

$$+x^2\left[\begin{array}{l}-\frac{7129}{1680}\frac{f_n}{h}+\frac{27}{2}\frac{f_{n+\frac{1}{3}}}{h}-\frac{27f_{n+\frac{2}{3}}}{h}+\frac{42f_{n+1}}{h}-\frac{189}{4}\frac{f_{n+\frac{4}{3}}}{h}+\frac{189}{5}\frac{f_{n+\frac{5}{3}}}{h}-\frac{21f_{n+2}}{h}\\+\frac{54}{7}\frac{f_{n+\frac{7}{3}}}{h}-\frac{27}{16}\frac{f_{n+\frac{8}{3}}}{h}+\frac{1}{6}\frac{f_{n+3}}{h}\end{array}\right]$$

$$+xf_n$$

$$+\left[\begin{array}{l}y_{n+2}-\frac{7}{75}hf_n-\frac{771}{1400}hf_{n+\frac{1}{3}}+\frac{51}{700}hf_{n+\frac{2}{3}}-\frac{199}{210}hf_{n+1}+\frac{249}{700}hf_{n+\frac{4}{3}}-\frac{633}{700}hf_{n+\frac{5}{3}}\\+\frac{299}{2100}hf_{n+2}-\frac{33}{350}hf_{n+\frac{7}{3}}+\frac{3}{140}hf_{n+\frac{8}{3}}-\frac{3}{1400}hf_{n+3}\end{array}\right]$$

(3.21)

Hence the scheme is simplified to:

$$y_{n+3}=y_{n+2}+\frac{25}{10752}hf_n-\frac{303}{12800}hf_{n+\frac{1}{3}}+\frac{1221}{11200}hf_{n+\frac{2}{3}}-\frac{5039}{16800}hf_{n+1}+\frac{24603}{44800}hf_{n+\frac{4}{3}}-\frac{6369}{8960}hf_{n+\frac{5}{3}}$$
$$+\frac{13273}{16800}hf_{n+2}-\frac{93}{1600}hf_{n+\frac{7}{3}}+\frac{49143}{89600}hf_{n+\frac{8}{3}}+\frac{1197}{12800}hf_{n+3}$$

(3.22)

$$y_n=y_{n+2}-\frac{7}{75}hf_n-\frac{771}{1400}hf_{n+\frac{1}{3}}+\frac{51}{700}hf_{n+\frac{2}{3}}-\frac{199}{210}hf_{n+1}+\frac{249}{700}hf_{n+\frac{4}{3}}-\frac{633}{700}hf_{n+\frac{5}{3}}$$
$$+\frac{299}{2100}hf_{n+2}-\frac{33}{350}hf_{n+\frac{7}{3}}+\frac{3}{140}hf_{n+\frac{8}{3}}-\frac{3}{1400}hf_{n+3}$$

(3.23)

$$y_{n+\frac{1}{3}}=y_{n+2}+\frac{25}{10752}hf_n-\frac{101635}{870912}hf_{n+\frac{1}{3}}-\frac{48425}{108864}hf_{n+\frac{2}{3}}-\frac{11575}{54432}hf_{n+1}$$
$$+\frac{190775}{435456}hf_{n+\frac{4}{3}}-\frac{123575}{435456}hf_{n+\frac{5}{3}}-\frac{10735}{54432}hf_{n+2}-\frac{3175}{108864}hf_{n+\frac{7}{3}} \qquad (3.24)$$
$$-\frac{4675}{870912}hf_{n+\frac{8}{3}}+\frac{425}{870912}hf_{n+3}$$

$$y_{n+\frac{2}{3}} = y_{n+2} - \frac{13}{42525}hf_n + \frac{32}{6075}hf_{n+\frac{1}{3}} - \frac{5494}{42525}hf_{n+\frac{2}{3}} - \frac{17632}{42525}hf_{n+1}$$
$$-\frac{2174}{8505}hf_{n+\frac{4}{3}} - \frac{17632}{42525}hf_{n+\frac{5}{3}} - \frac{5494}{42525}hf_{n+2} - \frac{32}{6075}hf_{n+\frac{7}{3}} - \frac{13}{42525}hf_{n+\frac{8}{3}}$$

(3.25)

$$y_{n+1} = y_{n+2} + \frac{7}{38400}hf_n - \frac{201}{89600}hf_{n+\frac{1}{3}} + \frac{33}{22400}hf_{n+\frac{2}{3}} - \frac{2647}{16800}hf_{n+1}$$
$$+\frac{15909}{44800}hf_{n+\frac{4}{3}} - \frac{15909}{44800}hf_{n+\frac{5}{3}} + \frac{2647}{16800}hf_{n+2} + \frac{33}{2240}hf_{n+\frac{7}{3}} - \frac{201}{89600}hf_{n+\frac{8}{3}} + \frac{7}{38400}hf_{n+3}$$

(3.26)

$$y_{n+\frac{4}{3}} = y_{n+2} + \frac{23}{340200}hf_{n+\frac{1}{3}} - \frac{167}{170100}hf_{n+\frac{2}{3}} - \frac{701}{85050}hf_{n+1} - \frac{23189}{34020}hf_{n+\frac{4}{3}} - \frac{13903}{34020}hf_{n+\frac{5}{3}}$$
$$-\frac{23189}{170100}hf_{n+2} + \frac{701}{85050}hf_{n+\frac{7}{3}} - \frac{167}{170100}hf_{n+\frac{8}{3}} + \frac{23}{340200}hf_{n+3}$$

(3.27)

$$y_{n+\frac{5}{3}} = y_{n+2} + \frac{2497}{21772800}hf_n - \frac{27467}{21772800}hf_{n+\frac{1}{3}} + \frac{17663}{2721600}hf_{n+\frac{2}{3}}$$
$$-\frac{5779}{272160}hf_{n+1} + \frac{583073}{10886400}hf_{n+\frac{4}{3}} - \frac{2381791}{10886400}hf_{n+\frac{5}{3}} - \frac{225623}{1360800}hf_{n+2}$$
$$+\frac{42767}{2721600}hf_{n+\frac{7}{3}} - \frac{10063}{4354560}hf_{n+\frac{8}{3}} + \frac{7}{38400}hf_{n+3}$$

(3.28)

$$y_{n+\frac{7}{3}} = y_{n+2} + \frac{7}{38400}hf_n - \frac{42187}{21772800}hf_{n+\frac{1}{3}} + \frac{25759}{2721600}hf_{n+\frac{2}{3}}$$
$$-\frac{38599}{1360800}hf_{n+1} + \frac{25759}{2721600}hf_{n+\frac{4}{3}} - \frac{1083167}{10886400}hf_{n+\frac{5}{3}} + \frac{349817}{1360800}hf_{n+2}$$
$$+\frac{391711}{2721600}hf_{n+\frac{7}{3}} - \frac{163531}{21772800}hf_{n+\frac{8}{3}} + \frac{425}{870912}hf_{n+3}$$

(3.29)

$$y_{n+\frac{8}{3}} = y_{n+2} - \frac{13}{42525}hf_n + \frac{1063}{340200}hf_{n+\frac{1}{3}} - \frac{491}{34020}hf_{n+\frac{2}{3}} + \frac{3373}{85050}hf_{n+1} + \frac{12133}{170100}hf_{n+\frac{4}{3}}$$
$$+\frac{14117}{170100}hf_{n+\frac{5}{3}} + \frac{9371}{170100}hf_{n+2} + \frac{7817}{17010}hf_{n+\frac{7}{3}} + \frac{19469}{170100}hf_{n+\frac{8}{3}} - \frac{3}{1400}hf_{n+3}$$

(3.30)

3.3 Derivation of the three step Adam's type method with nine off-grid points

Consider the power series:

$$Y(x) = x^{13}a_{13} + x^{12}a_{12} + x^{11}a_{11} + x^{10}a_{10} + x^9a_9 + x^8a_8 + x^7a_7 + x^6a_6$$
$$+x^5a_5 + x^4a_4 + x^3a_3 + x^2a_2 + xa_1 + a_0 \quad (3.31)$$

Differentiating (3.31)

$$YI(x) = 13x^{12}a_{13} + 12x^{11}a_{12} + 11x^{10}a_{11} + 10x^9a_{10} + 9x^8a_9 + 8x^7a_8 + 7x^6a_7 + 6x^5a_6$$
$$+5x^4a_5 + 4x^3a_4 + 3x^2a_3 + 2xa_2 + a_1 \quad (3.32)$$

Interpolating at 2h and Collocating at mesh point 1/4

The square matrix A was generated as shown in (3.10);

$$A=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2h & h & h & h & h & h & h & h & 0 & h & h & h & h & h \\
(2h)^2 & 0 & 2h(\frac{h}{4}) & 2h(\frac{h}{2}) & 2h(\frac{3h}{4}) & 2h(h) & 2h(\frac{5h}{4}) & 2h(\frac{3h}{2}) & 2h(\frac{7h}{4}) & 2h(2h) & 2h(\frac{9h}{4}) & 2h(\frac{5h}{2}) & 2h(\frac{11h}{4}) & 2h(3h) \\
(2h)^3 & 0 & 3h(\frac{h}{4})^2 & 3h(\frac{h}{2})^2 & 3h(\frac{3h}{4})^2 & 3h(h)^2 & 3h(\frac{5h}{4})^2 & 3h(\frac{3h}{2})^2 & 3h(\frac{7h}{4})^2 & 3h(2h)^2 & 3h(\frac{9h}{4})^2 & 3h(\frac{5h}{2})^2 & 3h(\frac{11h}{4})^2 & 3h(3h)^2 \\
(2h)^4 & 0 & 4h(\frac{h}{4})^3 & 4h(\frac{h}{2})^3 & 4h(\frac{3h}{4})^3 & 4h(h)^3 & 4h(\frac{5h}{4})^3 & 4h(\frac{3h}{2})^3 & 4h(\frac{7h}{4})^3 & 4h(2h)^3 & 4h(\frac{9h}{4})^3 & 4h(\frac{5h}{2})^3 & 4h(\frac{11h}{4})^3 & 4h(3h)^3 \\
(2h)^5 & 0 & 5h(\frac{h}{4})^4 & 5h(\frac{h}{2})^4 & 5h(\frac{3h}{4})^4 & 5h(h)^4 & 5h(\frac{5h}{4})^4 & 5h(\frac{3h}{2})^4 & 5h(\frac{7h}{4})^4 & 5h(2h)^4 & 5h(\frac{9h}{4})^4 & 5h(\frac{5h}{2})^4 & 5h(\frac{11h}{4})^4 & 5h(3h)^4 \\
(2h)^6 & 0 & 6h(\frac{h}{4})^5 & 6h(\frac{h}{2})^5 & 6h(\frac{3h}{4})^5 & 6h(h)^5 & 6h(\frac{5h}{4})^5 & 6h(\frac{3h}{2})^5 & 6h(\frac{7h}{4})^5 & 6h(2h)^5 & 6h(\frac{9h}{4})^5 & 6h(\frac{5h}{2})^5 & 6h(\frac{11h}{4})^5 & 6h(3h)^5 \\
(2h)^7 & 0 & 7h(\frac{h}{4})^6 & 7h(\frac{h}{2})^6 & 7h(\frac{3h}{4})^6 & 7h(h)^6 & 7h(\frac{5h}{4})^6 & 7h(\frac{3h}{2})^6 & 7h(\frac{7h}{4})^6 & 7h(2h)^6 & 7h(\frac{9h}{4})^6 & 7h(\frac{5h}{2})^6 & 7h(\frac{11h}{4})^6 & 7h(3h)^6 \\
(2h)^8 & 0 & 8h(\frac{h}{4})^7 & 8h(\frac{h}{2})^7 & 8h(\frac{3h}{4})^7 & 8h(h)^7 & 8h(\frac{5h}{4})^7 & 8h(\frac{3h}{2})^7 & 8h(\frac{7h}{4})^7 & 8h(2h)^7 & 8h(\frac{9h}{4})^7 & 8h(\frac{5h}{2})^7 & 8h(\frac{11h}{4})^7 & 8h(3h)^7 \\
(2h)^9 & 0 & 9h(\frac{h}{4})^8 & 9h(\frac{h}{2})^8 & 9h(\frac{3h}{4})^8 & 9h(h)^8 & 9h(\frac{5h}{4})^8 & 9h(\frac{3h}{2})^8 & 9h(\frac{7h}{4})^8 & 9h(2h)^8 & 9h(\frac{9h}{4})^8 & 9h(\frac{5h}{2})^8 & 9h(\frac{11h}{4})^8 & 9h(3h)^8 \\
(2h)^{10} & 0 & 10h(\frac{h}{4})^9 & 10h(\frac{h}{2})^9 & 10h(\frac{3h}{4})^9 & 10h(h)^9 & 10h(\frac{5h}{4})^9 & 10h(\frac{3h}{2})^9 & 10h(\frac{7h}{4})^9 & 10h(2h)^9 & 10h(\frac{9h}{4})^9 & 10h(\frac{5h}{2})^9 & 10h(\frac{11h}{4})^9 & 10h(3h)^9 \\
(2h)^{11} & 0 & 11h(\frac{h}{4})^{10} & 11h(\frac{h}{2})^{10} & 11h(\frac{3h}{4})^{10} & 11h(h)^{10} & 11h(\frac{5h}{4})^{10} & 11h(\frac{3h}{2})^{10} & 11h(\frac{7h}{4})^{10} & 11h(2h)^{10} & 11h(\frac{9h}{4})^{10} & 11h(\frac{5h}{2})^{10} & 11h(\frac{11h}{4})^{10} & 11h(3h)^{10} \\
(2h)^{12} & 0 & 12h(\frac{h}{4})^{11} & 12h(\frac{h}{2})^{11} & 12h(\frac{3h}{4})^{11} & 12h(h)^{11} & 12h(\frac{5h}{4})^{11} & 12h(\frac{3h}{2})^{11} & 12h(\frac{7h}{4})^{11} & 12h(2h)^{11} & 12h(\frac{9h}{4})^{11} & 12h(\frac{5h}{2})^{11} & 12h(\frac{11h}{4})^{11} & 12h(3h)^{11} \\
(2h)^{13} & 0 & 13h(\frac{h}{4})^{12} & 13h(\frac{h}{2})^{12} & 13h(\frac{3h}{4})^{12} & 13h(h)^{12} & 13h(\frac{5h}{4})^{12} & 13h(\frac{3h}{2})^{12} & 13h(\frac{7h}{4})^{12} & 13h(2h)^{12} & 13h(\frac{9h}{4})^{12} & 13h(\frac{5h}{2})^{12} & 13h(\frac{11h}{4})^{12} & 13h(3h)^{12}
\end{pmatrix}$$

(3.33)

$$B=\begin{pmatrix} y_{n+2} \\ hf_n \\ hf_{n+\frac{1}{4}} \\ hf_{n+\frac{1}{2}} \\ hf_{n+\frac{3}{4}} \\ hf_{n+1} \\ hf_{n+\frac{5}{4}} \\ hf_{n+\frac{3}{2}} \\ hf_{n+\frac{7}{4}} \\ hf_{n+2} \\ hf_{n+\frac{9}{4}} \\ hf_{n+\frac{5}{2}} \\ hf_{n+\frac{11}{4}} \\ hf_{n+3} \end{pmatrix} \tag{3.34}$$

Evaluating the product of $\left[A^T\right]^{-1}$ and B, then simplifying gives;

$$(A^T)^{-1}B=\begin{bmatrix} y_{n+2}-\frac{4219406}{638512875}hf_n-\frac{97021984}{212837625}hf_{n+\frac{1}{4}}+\frac{19044664}{70945875}hf_{n+\frac{1}{2}}-\frac{173147168}{127702575}hf_{n+\frac{3}{4}}+\frac{22373536}{14189175}hf_{n+1} \\ -\frac{181696448}{70945875}hf_{n+\frac{5}{4}}+\frac{60377264}{30405375}hf_{n+\frac{3}{2}}-\frac{14992192}{7882875}hf_{n+\frac{7}{4}}+\frac{10913087}{14189175}hf_{n+2}-\frac{43033184}{12770275}hf_{n+\frac{9}{4}} \\ +\frac{19496392}{212837625}hf_{n+\frac{5}{2}}-\frac{358496}{2348625}hf_{n+\frac{11}{4}}+\frac{739276}{638512875}hf_{n+3} \end{bmatrix},$$

(3.35)

$[f_n],$

$$\left[\begin{array}{l} -\frac{86021}{13680}\frac{f_n}{h}+\frac{24 f_{n+\frac{1}{4}}}{h}-\frac{66 f_{n+\frac{1}{2}}}{h}+\frac{440}{3}\frac{f_{n+\frac{3}{4}}}{h}-\frac{495}{2}\frac{f_{n+1}}{h}+\frac{189}{4}\frac{f_{n+\frac{4}{3}}}{h}+\frac{1584}{5}\frac{f_{n+\frac{5}{4}}}{h}-\frac{308 f_{n+\frac{3}{2}}}{h} \\ +\frac{1584}{7}\frac{f_{n+\frac{7}{4}}}{h}-\frac{495}{4}\frac{f_{n+2}}{h}+\frac{440}{9}\frac{f_{n+\frac{9}{4}}}{h}-\frac{66}{5}\frac{f_{n+\frac{5}{2}}}{h}+\frac{24}{11}\frac{f_{n+\frac{11}{4}}}{h}-\frac{1}{6}\frac{f_{n+3}}{h} \end{array}\right],$$

$$\left[\begin{array}{l} \frac{3353402}{155925}\frac{f_n}{h^2}-\frac{466408}{3465}\frac{f_{n+\frac{1}{4}}}{h^2}+\frac{144322}{315}\frac{f_{n+\frac{1}{2}}}{h^2}-\frac{614248}{567}\frac{f_{n+\frac{3}{4}}}{h^2}+\frac{79091}{42}\frac{f_{n+1}}{h^2}-\frac{1287632}{525}\frac{f_{n+\frac{5}{4}}}{h^2} \\ -\frac{325604}{135}\frac{f_{n+\frac{3}{2}}}{h^2}-\frac{187568}{105}\frac{f_{n+\frac{7}{4}}}{h^2}+\frac{20639}{21}\frac{f_{n+2}}{h^2}-\frac{221176}{567}\frac{f_{n+\frac{9}{4}}}{h^2}+\frac{166498}{1575}\frac{f_{n+\frac{5}{2}}}{h^2} \\ -\frac{60728}{3465}\frac{f_{n+\frac{11}{4}}}{h^2}-\frac{83711}{62370}\frac{f_{n+3}}{h^2} \end{array}\right],$$

$$\left[\begin{array}{l} -\frac{5356117}{113400}\frac{f_n}{h^3}+\frac{83344}{225}\frac{f_{n+\frac{1}{4}}}{h^3}-\frac{2270987}{1575}\frac{f_{n+\frac{1}{2}}}{h^3}+\frac{1149152}{315}\frac{f_{n+\frac{3}{4}}}{h^3}-\frac{5520439}{840}\frac{f_{n+1}}{h^3}+\frac{4592864}{525}\frac{f_{n+\frac{5}{4}}}{h^3} \\ -\frac{5892794}{675}\frac{f_{n+\frac{3}{2}}}{h^3}+\frac{3430528}{525}\frac{f_{n+\frac{7}{4}}}{h^3}-\frac{434831}{120}\frac{f_{n+2}}{h^3}+\frac{4102576}{2835}\frac{f_{n+\frac{9}{4}}}{h^3}-\frac{620731}{1575}\frac{f_{n+\frac{5}{2}}}{h^3} \\ +\frac{103328}{1575}\frac{f_{n+\frac{11}{4}}}{h^3}-\frac{29531}{13440}\frac{f_{n+3}}{h^3} \end{array}\right],$$

$$\left[\begin{array}{l} \frac{14936519}{212625}\frac{f_n}{h^4}-\frac{14845352}{23625}\frac{f_{n+\frac{1}{4}}}{h^4}+\frac{63330886}{23625}\frac{f_{n+\frac{1}{2}}}{h^4}-\frac{43694344}{6075}\frac{f_{n+\frac{3}{4}}}{h^4}+\frac{4235597}{315}\frac{f_{n+1}}{h^4} \\ -\frac{20634832}{1125}\frac{f_{n+\frac{5}{4}}}{h^4}+\frac{188526796}{10125}\frac{f_{n+\frac{3}{2}}}{h^4}-\frac{111144464}{7875}\frac{f_{n+\frac{7}{4}}}{h^4}+\frac{12446459}{1575}\frac{f_{n+2}}{h^4}-\frac{27046472}{8505}\frac{f_{n+\frac{9}{4}}}{h^4} \\ +\frac{2941058}{3375}\frac{f_{n+\frac{5}{2}}}{h^4}-\frac{3444472}{23625}\frac{f_{n+\frac{11}{4}}}{h^4}+\frac{341747}{30375}\frac{f_{n+3}}{h^4} \end{array}\right],$$

$$\left[\begin{array}{l} -\frac{124891}{1701}\frac{f_n}{h^5}+\frac{869144}{1215}\frac{f_{n+\frac{1}{4}}}{h^5}-\frac{27722152}{8505}\frac{f_{n+\frac{1}{2}}}{h^5}+\frac{78076984}{8505}\frac{f_{n+\frac{3}{4}}}{h^5}-\frac{50383877}{2835}\frac{f_{n+1}}{h^5} \\ +\frac{70486576}{2835}\frac{f_{n+\frac{5}{4}}}{h^5}-\frac{3471704}{135}\frac{f_{n+\frac{3}{2}}}{h^5}+\frac{16360084}{2835}\frac{f_{n+\frac{7}{4}}}{h^5}-\frac{4534369}{405}\frac{f_{n+2}}{h^5}+\frac{38664776}{8505}\frac{f_{n+\frac{9}{4}}}{h^5} \\ -\frac{10673648}{8505}\frac{f_{n+\frac{5}{2}}}{h^5}+\frac{1796824}{8505}\frac{f_{n+\frac{11}{4}}}{h^5}-\frac{139381}{8505}\frac{f_{n+3}}{h^5} \end{array}\right],$$

$$\left[\begin{array}{l} \frac{16360084}{297675}\frac{f_n}{h^6}-\frac{11275616}{19845}\frac{f_{n+\frac{1}{4}}}{h^6}+\frac{53967512}{19845}\frac{f_{n+\frac{1}{2}}}{h^4}-\frac{474424928}{59535}\frac{f_{n+\frac{3}{4}}}{h^6}+\frac{105302396}{6615}\frac{f_{n+1}}{h^6} \\ -\frac{754956352}{33075}\frac{f_{n+\frac{5}{4}}}{h^6}+\frac{86243632}{2835}\frac{f_{n+\frac{3}{2}}}{h^6}-\frac{124342976}{6615}\frac{f_{n+\frac{7}{4}}}{h^6}+\frac{71294492}{6615}\frac{f_{n+2}}{h^6}-\frac{263240864}{59535}\frac{f_{n+\frac{9}{4}}}{h^6} \\ +\frac{1221611528}{99225}\frac{f_{n+\frac{5}{2}}}{h^6}-\frac{4143008}{19845}\frac{f_{n+\frac{11}{4}}}{h^6}+\frac{970148}{59535}\frac{f_{n+3}}{h^6} \end{array}\right],$$

$$\left[\begin{array}{l} -\frac{15548}{525}\frac{f_n}{h^7}+\frac{215768}{675}\frac{f_{n+\frac{1}{4}}}{h^7}-\frac{1072432}{675}\frac{f_{n+\frac{1}{2}}}{h^7}+\frac{649256}{153}\frac{f_{n+\frac{3}{4}}}{h^7}-\frac{148132}{15}\frac{f_{n+1}}{h^7} \\ +\frac{3260624}{225}\frac{f_{n+\frac{5}{4}}}{h^7}-\frac{3505792}{225}\frac{f_{n+\frac{3}{2}}}{h^7}+\frac{19475536}{1575}\frac{f_{n+\frac{7}{4}}}{h^7}-\frac{107804}{15}\frac{f_{n+2}}{h^7}+\frac{134216}{45}\frac{f_{n+\frac{9}{4}}}{h^7} \\ -\frac{566096}{675}\frac{f_{n+\frac{5}{2}}}{h^7}+\frac{96808}{675}\frac{f_{n+\frac{11}{4}}}{h^7}-\frac{7612}{675}\frac{f_{n+3}}{h^7} \end{array}\right],$$

$$\left[\begin{array}{l}\frac{1453504}{127575}\frac{f_n}{h^8}-\frac{540856}{42525}\frac{f_{n+\frac{1}{4}}}{h^8}+\frac{27655808}{42525}\frac{f_{n+\frac{1}{2}}}{h^8}-\frac{51590912}{25515}\frac{f_{n+\frac{3}{4}}}{h^8}+\frac{4018624}{945}\frac{f_{n+1}}{h^8}\\-\frac{90353152}{14175}\frac{f_{n+\frac{5}{4}}}{h^8}+\frac{42419456}{6075}\frac{f_{n+\frac{3}{2}}}{h^8}-\frac{79846912}{14175}\frac{f_{n+\frac{7}{4}}}{h^8}+\frac{3138496}{945}\frac{f_{n+2}}{h^8}-\frac{35610368}{25515}\frac{f_{n+\frac{9}{4}}}{h^8}\\+\frac{16873088}{42525}\frac{f_{n+\frac{5}{2}}}{h^8}-\frac{2913536}{42525}\frac{f_{n+\frac{11}{4}}}{h^8}+\frac{693184}{127575}\frac{f_{n+3}}{h^8}\end{array}\right],$$

$$\left[\begin{array}{l}-\frac{43264}{14175}\frac{f_n}{h^9}+\frac{23552}{675}\frac{f_{n+\frac{1}{4}}}{h^9}-\frac{864256}{4725}\frac{f_{n+\frac{1}{2}}}{h^9}+\frac{2747392}{4725}\frac{f_{n+\frac{3}{4}}}{h^9}-\frac{1966336}{1575}\frac{f_{n+1}}{h^9}\\+\frac{333824}{175}\frac{f_{n+\frac{5}{4}}}{h^9}-\frac{1435648}{657}\frac{f_{n+\frac{3}{2}}}{h^9}+\frac{305152}{175}\frac{f_{n+\frac{7}{4}}}{h^9}-\frac{234752}{225}\frac{f_{n+2}}{h^9}+\frac{6298624}{14175}\frac{f_{n+\frac{9}{4}}}{h^9}\\-\frac{120832}{945}\frac{f_{n+\frac{5}{2}}}{h^9}+\frac{105472}{4725}\frac{f_{n+\frac{11}{4}}}{h^9}-\frac{2816}{1575}\frac{f_{n+3}}{h^9}\end{array}\right],$$

$$\left[\begin{array}{l}\frac{252928}{467775}\frac{f_n}{h^{10}}-\frac{65536}{10395}\frac{f_{n+\frac{1}{4}}}{h^{10}}+\frac{38912}{1155}\frac{f_{n+\frac{1}{2}}}{h^{10}}-\frac{1458176}{13365}\frac{f_{n+\frac{3}{4}}}{h^{10}}+\frac{275456}{115}\frac{f_{n+1}}{h^{10}}\\-\frac{917504}{2475}\frac{f_{n+\frac{5}{4}}}{h^{10}}+\frac{1871872}{4455}\frac{f_{n+\frac{3}{2}}}{h^{10}}-\frac{1212416}{3465}\frac{f_{n+\frac{7}{4}}}{h^{10}}+\frac{736256}{3465}\frac{f_{n+2}}{h^{10}}-\frac{8585216}{93555}\frac{f_{n+\frac{9}{4}}}{h^{10}}\\+\frac{198656}{7425}\frac{f_{n+\frac{5}{2}}}{h^{10}}-\frac{16384}{3465}\frac{f_{n+\frac{11}{4}}}{h^{10}}+\frac{1024}{2673}\frac{f_{n+3}}{h^{10}}\end{array}\right],$$

$$\left[\begin{array}{l} -\frac{26624}{467775}\frac{f_n}{h^{11}}+\frac{4096}{6075}\frac{f_{n+\frac{1}{4}}}{h^{11}}-\frac{155648}{42525}\frac{f_{n+\frac{1}{2}}}{h^{11}}+\frac{20480}{1701}\frac{f_{n+\frac{3}{4}}}{h^{11}}-\frac{75776}{2835}\frac{f_{n+1}}{h^{11}} \\ +\frac{598016}{14175}\frac{f_{n+\frac{5}{4}}}{h^{11}}-\frac{32768}{675}\frac{f_{n+\frac{3}{2}}}{h^{11}}+\frac{581632}{14175}\frac{f_{n+\frac{7}{4}}}{h^{11}}-\frac{2048}{81}\frac{f_{n+2}}{h^{11}}+\frac{94208}{8575}\frac{f_{n+\frac{9}{4}}}{h^{11}} \\ -\frac{139264}{42525}\frac{f_{n+\frac{5}{2}}}{h^{11}}+\frac{274432}{427775}\frac{f_{n+\frac{11}{4}}}{h^{11}}-\frac{2048}{42525}\frac{f_{n+3}}{h^{11}} \end{array}\right],$$

$$\left[\begin{array}{l} \frac{16384}{6081075}\frac{f_n}{h^{12}}-\frac{65536}{2027025}\frac{f_{n+\frac{1}{4}}}{h^{12}}+\frac{32768}{184275}\frac{f_{n+\frac{1}{2}}}{h^{12}}-\frac{65536}{110505}\frac{f_{n+\frac{3}{4}}}{h^{12}}+\frac{16384}{12285}\frac{f_{n+1}}{h^{12}} \\ -\frac{131072}{61425}\frac{f_{n+\frac{5}{4}}}{h^{12}}+\frac{65536}{26325}\frac{f_{n+\frac{3}{2}}}{h^{12}}-\frac{131072}{61425}\frac{f_{n+\frac{7}{4}}}{h^{12}}+\frac{16384}{12285}\frac{f_{n+2}}{h^{12}}-\frac{65536}{2027025}\frac{f_{n+\frac{9}{4}}}{h^{12}} \\ +\frac{32768}{184275}\frac{f_{n+\frac{5}{2}}}{h^{12}}-\frac{65536}{2027025}\frac{f_{n+\frac{11}{4}}}{h^{12}}+\frac{16384}{6081075}\frac{f_{n+3}}{h^{12}} \end{array}\right],$$

$$\begin{aligned} Y(\mathrm{c}\,x) &= x^{13}a_{13}+x^{12}a_{12}+x^{11}a_{11}+x^{10}a_{10}+x^{9}a_{9}+x^{8}a_{8}+x^{7}a_{7}+x^{6}a_{6} \\ &+x^{5}a_{5}+x^{4}a_{4}+x^{3}a_{3}+x^{2}a_{2}+xa_{1}+a_{0} \end{aligned} \tag{3.36}$$

$$\begin{aligned} a_0 &= y_{n+2}-\frac{4219406}{638512875}hf_n-\frac{97021984}{212837625}hf_{n+\frac{1}{4}}+\frac{19044664}{70945875}hf_{n+\frac{1}{2}}-\frac{173147168}{127702575}hf_{n+\frac{3}{4}}+\frac{22373536}{14189175}hf_{n+1} \\ &-\frac{181696448}{70945875}hf_{n+\frac{5}{4}}+\frac{60377264}{30405375}hf_{n+\frac{3}{2}}-\frac{14992192}{7882875}hf_{n+\frac{7}{4}}+\frac{10913087}{14189175}hf_{n+2}-\frac{43033184}{12770275}hf_{n+\frac{9}{4}} \\ &+\frac{19496392}{212837625}hf_{n+\frac{5}{2}}-\frac{358496}{2348625}hf_{n+\frac{11}{4}}+\frac{739276}{638512875}hf_{n+3} \end{aligned} \tag{3.37}$$

$$a_1 = f_n \tag{3.38}$$

$$\begin{aligned} a_2 &= -\frac{86021}{13680}\frac{f_n}{h}+\frac{24f_{n+\frac{1}{4}}}{h}-\frac{66f_{n+\frac{1}{2}}}{h}+\frac{440}{3}\frac{f_{n+\frac{3}{4}}}{h}-\frac{495}{2}\frac{f_{n+1}}{h}+\frac{189}{4}\frac{f_{n+\frac{4}{3}}}{h}+\frac{1584}{5}\frac{f_{n+\frac{5}{4}}}{h}-\frac{308f_{n+\frac{3}{2}}}{h} \\ &+\frac{1584}{7}\frac{f_{n+\frac{7}{4}}}{h}-\frac{495}{4}\frac{f_{n+2}}{h}+\frac{440}{9}\frac{f_{n+\frac{9}{4}}}{h}-\frac{66}{5}\frac{f_{n+\frac{5}{2}}}{h}+\frac{24}{11}\frac{f_{n+\frac{11}{4}}}{h}-\frac{1}{6}\frac{f_{n+3}}{h} \end{aligned} \tag{3.39}$$

$$a_3 = \frac{3353402}{155925}\frac{f_n}{h^2} - \frac{466408}{3465}\frac{f_{n+\frac{1}{4}}}{h^2} + \frac{144322}{315}\frac{f_{n+\frac{1}{2}}}{h^2} - \frac{614248}{567}\frac{f_{n+\frac{3}{4}}}{h^2} + \frac{79091}{42}\frac{f_{n+1}}{h^2} - \frac{1287632}{525}\frac{f_{n+\frac{5}{4}}}{h^2}$$

$$-\frac{325604}{135}\frac{f_{n+\frac{3}{2}}}{h^2} - \frac{187568}{105}\frac{f_{n+\frac{7}{4}}}{h^2} + \frac{20639}{21}\frac{f_{n+2}}{h^2} - \frac{221176}{567}\frac{f_{n+\frac{9}{4}}}{h^2} + \frac{166498}{1575}\frac{f_{n+\frac{5}{2}}}{h^2} \qquad (3.39)$$

$$-\frac{60728}{3465}\frac{f_{n+\frac{11}{4}}}{h^2} - \frac{83711}{62370}\frac{f_{n+3}}{h^2}$$

$$a_4 = -\frac{5356117}{113400}\frac{f_n}{h^3} + \frac{83344}{225}\frac{f_{n+\frac{1}{4}}}{h^3} - \frac{2270987}{1575}\frac{f_{n+\frac{1}{2}}}{h^3} + \frac{1149152}{315}\frac{f_{n+\frac{3}{4}}}{h^3} - \frac{5520439}{840}\frac{f_{n+1}}{h^3} + \frac{4592864}{525}\frac{f_{n+\frac{5}{4}}}{h^3} \qquad (3.40)$$

$$-\frac{5892794}{675}\frac{f_{n+\frac{3}{2}}}{h^3} + \frac{3430528}{525}\frac{f_{n+\frac{7}{4}}}{h^3} - \frac{434831}{120}\frac{f_{n+2}}{h^3} + \frac{4102576}{2835}\frac{f_{n+\frac{9}{4}}}{h^3} - \frac{620731}{1575}\frac{f_{n+\frac{5}{2}}}{h^3}$$

$$+\frac{103328}{1575}\frac{f_{n+\frac{11}{4}}}{h^3} - \frac{29531}{13440}\frac{f_{n+3}}{h^3}$$

$$a_5 = \frac{14936519}{212625}\frac{f_n}{h^4} - \frac{14845352}{23625}\frac{f_{n+\frac{1}{4}}}{h^4} + \frac{63330886}{23625}\frac{f_{n+\frac{1}{2}}}{h^4} - \frac{43694344}{6075}\frac{f_{n+\frac{3}{4}}}{h^4} + \frac{4235597}{315}\frac{f_{n+1}}{h^4}$$

$$-\frac{20634832}{1125}\frac{f_{n+\frac{5}{4}}}{h^4} + \frac{188526796}{10125}\frac{f_{n+\frac{3}{2}}}{h^4} - \frac{111144464}{7875}\frac{f_{n+\frac{7}{4}}}{h^4} + \frac{12446459}{1575}\frac{f_{n+2}}{h^4} - \frac{27046472}{8505}\frac{f_{n+\frac{9}{4}}}{h^4} \qquad (3.41)$$

$$+\frac{2941058}{3375}\frac{f_{n+\frac{5}{2}}}{h^4} - \frac{3444472}{23625}\frac{f_{n+\frac{11}{4}}}{h^4} + \frac{341747}{30375}\frac{f_{n+3}}{h^4}$$

$$a_6 = -\frac{124891}{1701}\frac{f_n}{h^5} + \frac{869144}{1215}\frac{f_{n+\frac{1}{4}}}{h^5} - \frac{27722152}{8505}\frac{f_{n+\frac{1}{2}}}{h^5} + \frac{78076984}{8505}\frac{f_{n+\frac{3}{4}}}{h^5} - \frac{50383877}{2835}\frac{f_{n+1}}{h^5}$$

$$+\frac{70486576}{2835}\frac{f_{n+\frac{5}{4}}}{h^5} - \frac{3471704}{135}\frac{f_{n+\frac{3}{2}}}{h^5} + \frac{16360084}{2835}\frac{f_{n+\frac{7}{4}}}{h^5} - \frac{4534369}{405}\frac{f_{n+2}}{h^5} + \frac{38664776}{8505}\frac{f_{n+\frac{9}{4}}}{h^5} \qquad (3.42)$$

$$-\frac{10673648}{8505}\frac{f_{n+\frac{5}{2}}}{h^5} + \frac{1796824}{8505}\frac{f_{n+\frac{11}{4}}}{h^5} - \frac{139381}{8505}\frac{f_{n+3}}{h^5}$$

$$a_7 = \frac{16360084}{297675}\frac{f_n}{h^6} - \frac{11275616}{19845}\frac{f_{n+\frac{1}{4}}}{h^6} + \frac{53967512}{19845}\frac{f_{n+\frac{1}{2}}}{h^4} - \frac{474424928}{59535}\frac{f_{n+\frac{3}{4}}}{h^6} + \frac{105302396}{6615}\frac{f_{n+1}}{h^6}$$

$$-\frac{754956352}{33075}\frac{f_{n+\frac{5}{4}}}{h^6} + \frac{86243632}{2835}\frac{f_{n+\frac{3}{2}}}{h^6} - \frac{124342976}{6615}\frac{f_{n+\frac{7}{4}}}{h^6} + \frac{71294492}{6615}\frac{f_{n+2}}{h^6} - \frac{263240864}{59535}\frac{f_{n+\frac{9}{4}}}{h^6} \qquad (3.43)$$

$$+\frac{1221611528}{99225}\frac{f_{n+\frac{5}{2}}}{h^6} - \frac{4143008}{19845}\frac{f_{n+\frac{11}{4}}}{h^6} + \frac{970148}{59535}\frac{f_{n+3}}{h^6}$$

$$a_8 = -\frac{15548}{525}\frac{f_n}{h^7} + \frac{215768}{675}\frac{f_{n+\frac{1}{4}}}{h^7} - \frac{1072432}{675}\frac{f_{n+\frac{1}{2}}}{h^7} + \frac{649256}{153}\frac{f_{n+\frac{3}{4}}}{h^7} - \frac{148132}{15}\frac{f_{n+1}}{h^7}$$
$$+\frac{3260624}{225}\frac{f_{n+\frac{5}{4}}}{h^7} - \frac{3505792}{225}\frac{f_{n+\frac{3}{2}}}{h^7} + \frac{19475536}{1575}\frac{f_{n+\frac{7}{4}}}{h^7} - \frac{107804}{15}\frac{f_{n+2}}{h^7} + \frac{134216}{45}\frac{f_{n+\frac{9}{4}}}{h^7} \quad (3.44)$$
$$-\frac{566096}{675}\frac{f_{n+\frac{5}{2}}}{h^7} + \frac{96808}{675}\frac{f_{n+\frac{11}{4}}}{h^7} - \frac{7612}{675}\frac{f_{n+3}}{h^7}$$

$$a_9 = \frac{1453504}{127575}\frac{f_n}{h^8} - \frac{540856}{42525}\frac{f_{n+\frac{1}{4}}}{h^8} + \frac{27655808}{42525}\frac{f_{n+\frac{1}{2}}}{h^8} - \frac{51590912}{25515}\frac{f_{n+\frac{3}{4}}}{h^8} + \frac{4018624}{945}\frac{f_{n+1}}{h^8}$$
$$-\frac{90353152}{14175}\frac{f_{n+\frac{5}{4}}}{h^8} + \frac{42419456}{6075}\frac{f_{n+\frac{3}{2}}}{h^8} - \frac{79846912}{14175}\frac{f_{n+\frac{7}{4}}}{h^8} + \frac{3138496}{945}\frac{f_{n+2}}{h^8} - \frac{35610368}{25515}\frac{f_{n+\frac{9}{4}}}{h^8} \quad (3.45)$$
$$+\frac{16873088}{42525}\frac{f_{n+\frac{5}{2}}}{h^8} - \frac{2913536}{42525}\frac{f_{n+\frac{11}{4}}}{h^8} + \frac{693184}{127575}\frac{f_{n+3}}{h^8}$$

$$a_{10} = -\frac{43264}{14175}\frac{f_n}{h^9} + \frac{23552}{675}\frac{f_{n+\frac{1}{4}}}{h^9} - \frac{864256}{4725}\frac{f_{n+\frac{1}{2}}}{h^9} + \frac{2747392}{4725}\frac{f_{n+\frac{3}{4}}}{h^9} - \frac{1966336}{1575}\frac{f_{n+1}}{h^9}$$
$$+\frac{333824}{175}\frac{f_{n+\frac{5}{4}}}{h^9} - \frac{1435648}{657}\frac{f_{n+\frac{3}{2}}}{h^9} + \frac{305152}{175}\frac{f_{n+\frac{7}{4}}}{h^9} - \frac{234752}{225}\frac{f_{n+2}}{h^9} + \frac{6298624}{14175}\frac{f_{n+\frac{9}{4}}}{h^9} \quad (3.46)$$
$$-\frac{120832}{945}\frac{f_{n+\frac{5}{2}}}{h^9} + \frac{105472}{4725}\frac{f_{n+\frac{11}{4}}}{h^9} - \frac{2816}{1575}\frac{f_{n+3}}{h^9}$$

$$a_{11} = \frac{252928}{467775}\frac{f_n}{h^{10}} - \frac{65536}{10395}\frac{f_{n+\frac{1}{4}}}{h^{10}} + \frac{38912}{1155}\frac{f_{n+\frac{1}{2}}}{h^{10}} - \frac{1458176}{13365}\frac{f_{n+\frac{3}{4}}}{h^{10}} + \frac{275456}{115}\frac{f_{n+1}}{h^{10}}$$
$$-\frac{917504}{2475}\frac{f_{n+\frac{5}{4}}}{h^{10}} + \frac{1871872}{4455}\frac{f_{n+\frac{3}{2}}}{h^{10}} - \frac{1212416}{3465}\frac{f_{n+\frac{7}{4}}}{h^{10}} + \frac{736256}{3465}\frac{f_{n+2}}{h^{10}} - \frac{8585216}{93555}\frac{f_{n+\frac{9}{4}}}{h^{10}} \quad (3.47)$$
$$+\frac{198656}{7425}\frac{f_{n+\frac{5}{2}}}{h^{10}} - \frac{16384}{3465}\frac{f_{n+\frac{11}{4}}}{h^{10}} + \frac{1024}{2673}\frac{f_{n+3}}{h^{10}}$$

$$a_{12} = -\frac{26624}{467775}\frac{f_n}{h^{11}} + \frac{4096}{6075}\frac{f_{n+\frac{1}{4}}}{h^{11}} - \frac{155648}{42525}\frac{f_{n+\frac{1}{2}}}{h^{11}} + \frac{20480}{1701}\frac{f_{n+\frac{3}{4}}}{h^{11}} - \frac{75776}{2835}\frac{f_{n+1}}{h^{11}} + \frac{598016}{14175}\frac{f_{n+\frac{5}{4}}}{h^{11}} - \frac{32768}{675}\frac{f_{n+\frac{3}{2}}}{h^{11}} + \frac{581632}{14175}\frac{f_{n+\frac{7}{4}}}{h^{11}} - \frac{2048}{81}\frac{f_{n+2}}{h^{11}} + \frac{94208}{8575}\frac{f_{n+\frac{9}{4}}}{h^{11}} - \frac{139264}{42525}\frac{f_{n+\frac{5}{2}}}{h^{11}} + \frac{274432}{427775}\frac{f_{n+\frac{11}{4}}}{h^{11}} - \frac{2048}{42525}\frac{f_{n+3}}{h^{11}} \quad (3.48)$$

$$a_{13} = \frac{16384}{6081075}\frac{f_n}{h^{12}} - \frac{65536}{2027025}\frac{f_{n+\frac{1}{4}}}{h^{12}} + \frac{32768}{184275}\frac{f_{n+\frac{1}{2}}}{h^{12}} - \frac{65536}{110505}\frac{f_{n+\frac{3}{4}}}{h^{12}} + \frac{16384}{12285}\frac{f_{n+1}}{h^{12}} - \frac{131072}{61425}\frac{f_{n+\frac{5}{4}}}{h^{12}} + \frac{65536}{26325}\frac{f_{n+\frac{3}{2}}}{h^{12}} - \frac{131072}{61425}\frac{f_{n+\frac{7}{4}}}{h^{12}} + \frac{16384}{12285}\frac{f_{n+2}}{h^{12}} - \frac{65536}{2027025}\frac{f_{n+\frac{9}{4}}}{h^{12}} + \frac{32768}{184275}\frac{f_{n+\frac{5}{2}}}{h^{12}} - \frac{65536}{2027025}\frac{f_{n+\frac{11}{4}}}{h^{12}} + \frac{16384}{6081075}\frac{f_{n+3}}{h^{12}} \quad (3.49)$$

Thus;

$$Y(Cx) = x^{13}\left[\frac{16384}{6081075}\frac{f_n}{h^{12}} - \frac{65536}{2027025}\frac{f_{n+\frac{1}{4}}}{h^{12}} + \frac{32768}{184275}\frac{f_{n+\frac{1}{2}}}{h^{12}} - \frac{65536}{110505}\frac{f_{n+\frac{3}{4}}}{h^{12}} + \frac{16384}{12285}\frac{f_{n+1}}{h^{12}} - \frac{131072}{61425}\frac{f_{n+\frac{5}{4}}}{h^{12}} + \frac{65536}{26325}\frac{f_{n+\frac{3}{2}}}{h^{12}} - \frac{131072}{61425}\frac{f_{n+\frac{7}{4}}}{h^{12}} + \frac{16384}{12285}\frac{f_{n+2}}{h^{12}} - \frac{65536}{2027025}\frac{f_{n+\frac{9}{4}}}{h^{12}} + \frac{32768}{184275}\frac{f_{n+\frac{5}{2}}}{h^{12}} - \frac{65536}{2027025}\frac{f_{n+\frac{11}{4}}}{h^{12}} + \frac{16384}{6081075}\frac{f_{n+3}}{h^{12}}\right] \quad (3.50)$$

$$+x^{12}\left[-\frac{26624}{467775}\frac{f_n}{h^{11}} + \frac{4096}{6075}\frac{f_{n+\frac{1}{4}}}{h^{11}} - \frac{155648}{42525}\frac{f_{n+\frac{1}{2}}}{h^{11}} + \frac{20480}{1701}\frac{f_{n+\frac{3}{4}}}{h^{11}} - \frac{75776}{2835}\frac{f_{n+1}}{h^{11}} + \frac{598016}{14175}\frac{f_{n+\frac{5}{4}}}{h^{11}} - \frac{32768}{675}\frac{f_{n+\frac{3}{2}}}{h^{11}} + \frac{581632}{14175}\frac{f_{n+\frac{7}{4}}}{h^{11}} - \frac{2048}{81}\frac{f_{n+2}}{h^{11}} + \frac{94208}{8575}\frac{f_{n+\frac{9}{4}}}{h^{11}} - \frac{139264}{42525}\frac{f_{n+\frac{5}{2}}}{h^{11}} + \frac{274432}{427775}\frac{f_{n+\frac{11}{4}}}{h^{11}} - \frac{2048}{42525}\frac{f_{n+3}}{h^{11}}\right]$$

$$+x^{11}\left[\begin{array}{l}\frac{252928}{467775}\frac{f_n}{h^{10}}-\frac{65536}{10395}\frac{f_{n+\frac{1}{4}}}{h^{10}}+\frac{38912}{1155}\frac{f_{n+\frac{1}{2}}}{h^{10}}-\frac{1458176}{13365}\frac{f_{n+\frac{3}{4}}}{h^{10}}+\frac{275456}{115}\frac{f_{n+1}}{h^{10}}\\-\frac{917504}{2475}\frac{f_{n+\frac{5}{4}}}{h^{10}}+\frac{1871872}{4455}\frac{f_{n+\frac{3}{2}}}{h^{10}}-\frac{1212416}{3465}\frac{f_{n+\frac{7}{4}}}{h^{10}}+\frac{736256}{3465}\frac{f_{n+2}}{h^{10}}-\frac{8585216}{93555}\frac{f_{n+\frac{9}{4}}}{h^{10}}\\+\frac{198656}{7425}\frac{f_{n+\frac{5}{2}}}{h^{10}}-\frac{16384}{3465}\frac{f_{n+\frac{11}{4}}}{h^{10}}+\frac{1024}{2673}\frac{f_{n+3}}{h^{10}}\end{array}\right]$$

$$+x^{10}\left[\begin{array}{l}-\frac{43264}{14175}\frac{f_n}{h^{9}}+\frac{23552}{675}\frac{f_{n+\frac{1}{4}}}{h^{9}}-\frac{864256}{4725}\frac{f_{n+\frac{1}{2}}}{h^{9}}+\frac{2747392}{4725}\frac{f_{n+\frac{3}{4}}}{h^{9}}-\frac{1966336}{1575}\frac{f_{n+1}}{h^{9}}\\+\frac{333824}{175}\frac{f_{n+\frac{5}{4}}}{h^{9}}-\frac{1435648}{657}\frac{f_{n+\frac{3}{2}}}{h^{9}}+\frac{305152}{175}\frac{f_{n+\frac{7}{4}}}{h^{9}}-\frac{234752}{225}\frac{f_{n+2}}{h^{9}}+\frac{6298624}{14175}\frac{f_{n+\frac{9}{4}}}{h^{9}}\\-\frac{120832}{945}\frac{f_{n+\frac{5}{2}}}{h^{9}}+\frac{105472}{4725}\frac{f_{n+\frac{11}{4}}}{h^{9}}-\frac{2816}{1575}\frac{f_{n+3}}{h^{9}}\end{array}\right]$$

$$+x^{9}\left[\begin{array}{l}\frac{1453504}{127575}\frac{f_n}{h^{8}}-\frac{540856}{42525}\frac{f_{n+\frac{1}{4}}}{h^{8}}+\frac{27655808}{42525}\frac{f_{n+\frac{1}{2}}}{h^{8}}-\frac{51590912}{25515}\frac{f_{n+\frac{3}{4}}}{h^{8}}+\frac{4018624}{945}\frac{f_{n+1}}{h^{8}}\\-\frac{90353152}{14175}\frac{f_{n+\frac{5}{4}}}{h^{8}}+\frac{42419456}{6075}\frac{f_{n+\frac{3}{2}}}{h^{8}}-\frac{79846912}{14175}\frac{f_{n+\frac{7}{4}}}{h^{8}}+\frac{3138496}{945}\frac{f_{n+2}}{h^{8}}-\frac{35610368}{25515}\frac{f_{n+\frac{9}{4}}}{h^{8}}\\+\frac{16873088}{42525}\frac{f_{n+\frac{5}{2}}}{h^{8}}-\frac{2913536}{42525}\frac{f_{n+\frac{11}{4}}}{h^{8}}+\frac{693184}{127575}\frac{f_{n+3}}{h^{8}}\end{array}\right]$$

$$+x^{8}\left[\begin{array}{l}-\frac{15548}{525}\frac{f_n}{h^{7}}+\frac{215768}{675}\frac{f_{n+\frac{1}{4}}}{h^{7}}-\frac{1072432}{675}\frac{f_{n+\frac{1}{2}}}{h^{7}}+\frac{649256}{153}\frac{f_{n+\frac{3}{4}}}{h^{7}}-\frac{148132}{15}\frac{f_{n+1}}{h^{7}}\\+\frac{3260624}{225}\frac{f_{n+\frac{5}{4}}}{h^{7}}-\frac{3505792}{225}\frac{f_{n+\frac{3}{2}}}{h^{7}}+\frac{19475536}{1575}\frac{f_{n+\frac{7}{4}}}{h^{7}}-\frac{107804}{15}\frac{f_{n+2}}{h^{7}}+\frac{134216}{45}\frac{f_{n+\frac{9}{4}}}{h^{7}}\\-\frac{566096}{675}\frac{f_{n+\frac{5}{2}}}{h^{7}}+\frac{96808}{675}\frac{f_{n+\frac{11}{4}}}{h^{7}}-\frac{7612}{675}\frac{f_{n+3}}{h^{7}}\end{array}\right]$$

$$
+x^7\left[\begin{array}{l}
\frac{16360084}{297675}\frac{f_n}{h^6}-\frac{11275616}{19845}\frac{f_{n+\frac{1}{4}}}{h^6}+\frac{53967512}{19845}\frac{f_{n+\frac{1}{2}}}{h^4}-\frac{474424928}{59535}\frac{f_{n+\frac{3}{4}}}{h^6}+\frac{105302396}{6615}\frac{f_{n+1}}{h^6}\\
-\frac{754956352}{33075}\frac{f_{n+\frac{5}{4}}}{h^6}+\frac{86243632}{2835}\frac{f_{n+\frac{3}{2}}}{h^6}-\frac{124342976}{6615}\frac{f_{n+\frac{7}{4}}}{h^6}+\frac{71294492}{6615}\frac{f_{n+2}}{h^6}-\frac{263240864}{59535}\frac{f_{n+\frac{9}{4}}}{h^6}\\
+\frac{1221611528}{99225}\frac{f_{n+\frac{5}{2}}}{h^6}-\frac{4143008}{19845}\frac{f_{n+\frac{11}{4}}}{h^6}+\frac{970148}{59535}\frac{f_{n+3}}{h^6}
\end{array}\right]
$$

$$
+x^6\left[\begin{array}{l}
-\frac{124891}{1701}\frac{f_n}{h^5}+\frac{869144}{1215}\frac{f_{n+\frac{1}{4}}}{h^5}-\frac{27722152}{8505}\frac{f_{n+\frac{1}{2}}}{h^5}+\frac{78076984}{8505}\frac{f_{n+\frac{3}{4}}}{h^5}-\frac{50383877}{2835}\frac{f_{n+1}}{h^5}\\
+\frac{70486576}{2835}\frac{f_{n+\frac{5}{4}}}{h^5}-\frac{3471704}{135}\frac{f_{n+\frac{3}{2}}}{h^5}+\frac{16360084}{2835}\frac{f_{n+\frac{7}{4}}}{h^5}-\frac{4534369}{405}\frac{f_{n+2}}{h^5}+\frac{38664776}{8505}\frac{f_{n+\frac{9}{4}}}{h^5}\\
-\frac{10673648}{8505}\frac{f_{n+\frac{5}{2}}}{h^5}+\frac{1796824}{8505}\frac{f_{n+\frac{11}{4}}}{h^5}-\frac{139381}{8505}\frac{f_{n+3}}{h^5}
\end{array}\right]
$$

$$
+x^5\left[\begin{array}{l}
\frac{14936519}{212625}\frac{f_n}{h^4}-\frac{14845352}{23625}\frac{f_{n+\frac{1}{4}}}{h^4}+\frac{63330886}{23625}\frac{f_{n+\frac{1}{2}}}{h^4}-\frac{43694344}{6075}\frac{f_{n+\frac{3}{4}}}{h^4}+\frac{4235597}{315}\frac{f_{n+1}}{h^4}\\
-\frac{20634832}{1125}\frac{f_{n+\frac{5}{4}}}{h^4}+\frac{188526796}{10125}\frac{f_{n+\frac{3}{2}}}{h^4}-\frac{111144464}{7875}\frac{f_{n+\frac{7}{4}}}{h^4}+\frac{12446459}{1575}\frac{f_{n+2}}{h^4}-\frac{27046472}{8505}\frac{f_{n+\frac{9}{4}}}{h^4}\\
+\frac{2941058}{3375}\frac{f_{n+\frac{5}{2}}}{h^4}-\frac{3444472}{23625}\frac{f_{n+\frac{11}{4}}}{h^4}+\frac{341747}{30375}\frac{f_{n+3}}{h^4}
\end{array}\right]
$$

$$
+x^4\left[\begin{array}{l}
-\frac{5356117}{113400}\frac{f_n}{h^3}+\frac{83344}{225}\frac{f_{n+\frac{1}{4}}}{h^3}-\frac{2270987}{1575}\frac{f_{n+\frac{1}{2}}}{h^3}+\frac{1149152}{315}\frac{f_{n+\frac{3}{4}}}{h^3}-\frac{5520439}{840}\frac{f_{n+1}}{h^3}+\frac{4592864}{525}\frac{f_{n+\frac{5}{4}}}{h^3}\\
-\frac{5892794}{675}\frac{f_{n+\frac{3}{2}}}{h^3}+\frac{3430528}{525}\frac{f_{n+\frac{7}{4}}}{h^3}-\frac{434831}{120}\frac{f_{n+2}}{h^3}+\frac{4102576}{2835}\frac{f_{n+\frac{9}{4}}}{h^3}-\frac{620731}{1575}\frac{f_{n+\frac{5}{2}}}{h^3}\\
+\frac{103328}{1575}\frac{f_{n+\frac{11}{4}}}{h^3}-\frac{29531}{13440}\frac{f_{n+3}}{h^3}
\end{array}\right]
$$

$$+x^3\left[\begin{array}{l}\frac{3353402}{155925}\frac{f_n}{h^2}-\frac{466408}{3465}\frac{f_{n+\frac{1}{4}}}{h^2}+\frac{144322}{315}\frac{f_{n+\frac{1}{2}}}{h^2}-\frac{614248}{567}\frac{f_{n+\frac{3}{4}}}{h^2}+\frac{79091}{42}\frac{f_{n+1}}{h^2}-\frac{1287632}{525}\frac{f_{n+\frac{5}{4}}}{h^2}\\-\frac{325604}{135}\frac{f_{n+\frac{3}{2}}}{h^2}-\frac{187568}{105}\frac{f_{n+\frac{7}{4}}}{h^2}+\frac{20639}{21}\frac{f_{n+2}}{h^2}-\frac{221176}{567}\frac{f_{n+\frac{9}{4}}}{h^2}+\frac{166498}{1575}\frac{f_{n+\frac{5}{2}}}{h^2}\\-\frac{60728}{3465}\frac{f_{n+\frac{11}{4}}}{h^2}-\frac{83711}{62370}\frac{f_{n+3}}{h^2}\end{array}\right]$$

$$+x^2\left[\begin{array}{l}-\frac{86021}{13680}\frac{f_n}{h}+\frac{24f_{n+\frac{1}{4}}}{h}-\frac{66f_{n+\frac{1}{2}}}{h}+\frac{440}{3}\frac{f_{n+\frac{3}{4}}}{h}-\frac{495}{2}\frac{f_{n+1}}{h}+\frac{189}{4}\frac{f_{n+\frac{4}{3}}}{h}+\frac{1584}{5}\frac{f_{n+\frac{5}{4}}}{h}-\frac{308f_{n+\frac{3}{2}}}{h}\\+\frac{1584}{7}\frac{f_{n+\frac{7}{4}}}{h}-\frac{495}{4}\frac{f_{n+2}}{h}+\frac{440}{9}\frac{f_{n+\frac{9}{4}}}{h}-\frac{66}{5}\frac{f_{n+\frac{5}{2}}}{h}+\frac{24}{11}\frac{f_{n+\frac{11}{4}}}{h}-\frac{1}{6}\frac{f_{n+3}}{h}\end{array}\right]$$

$$+x\left[f_n\right]$$

$$+\left[\begin{array}{l}y_{n+2}-\frac{4219406}{638512875}hf_n-\frac{97021984}{212837625}hf_{n+\frac{1}{4}}+\frac{190044664}{70945875}hf_{n+\frac{1}{2}}-\frac{173147168}{127702575}hf_{n+\frac{3}{4}}+\frac{22373536}{14189175}hf_{n+1}\\-\frac{181696448}{70945875}hf_{n+\frac{5}{4}}+\frac{60377264}{30405375}hf_{n+\frac{3}{2}}-\frac{14992192}{7882875}hf_{n+\frac{7}{4}}+\frac{10913087}{14189175}hf_{n+2}-\frac{43033184}{12770275}hf_{n+\frac{9}{4}}\\+\frac{19496392}{212837625}hf_{n+\frac{5}{2}}-\frac{358496}{2348625}hf_{n+\frac{11}{4}}+\frac{739276}{638512875}hf_{n+3}\end{array}\right]$$

$$y_{n+3}=y_{n+2}-\frac{5942359}{5108103000}hf_n+\frac{3247592}{212837625}hf_{n+\frac{1}{4}}-\frac{6564377}{7094875}hf_{n+\frac{1}{2}}+\frac{43882936}{127702575}hf_{n+\frac{3}{4}}$$
$$-\frac{19812941}{22702680}hf_{n+1}+\frac{113671024}{70945875}hf_{n+\frac{5}{4}}-\frac{66615022}{30405375}hf_{n+\frac{3}{2}}+\frac{17826416}{7882875}hf_{n+\frac{7}{4}}-\frac{190748297}{113513400}hf_{n+2}$$
$$+\frac{34799384}{25540515}hf_{n+\frac{9}{4}}-\frac{57330731}{212837625}hf_{n+\frac{5}{2}}+\frac{10782568}{23648625}hf_{n+\frac{11}{4}}-\frac{337524401}{5108103000}hf_{n+3}$$

(3.51)

$$y_n = y_{n+2} - \frac{42194069}{638512875} hf_n - \frac{97021984}{212837625} hf_{n+\frac{1}{4}} + \frac{19044664}{70945875} hf_{n+\frac{1}{2}} - \frac{173147168}{127702575} hf_{n+\frac{3}{4}}$$
$$+ \frac{22373536}{14189175} hf_{n+1} - \frac{181696448}{70945875} hf_{n+\frac{5}{4}} + \frac{60377264}{30405375} hf_{n+\frac{3}{2}} - \frac{14992192}{7882875} hf_{n+\frac{7}{4}} + \frac{10913087}{14189175} hf_{n+2}$$
$$- \frac{43033184}{127702575} hf_{n+\frac{9}{4}} + \frac{19496392}{212837625} hf_{n+\frac{5}{2}} - \frac{358496}{23648625} hf_{n+\frac{11}{4}} + \frac{739276}{638512875} hf_{n+3}$$

(3.52)

$$y_{n+\frac{1}{4}} = y_{n+2} + \frac{250951589}{213497856000} hf_n - \frac{2895487553}{35582976000} hf_{n+\frac{1}{4}} - \frac{6476481263}{17791488000} hf_{n+\frac{1}{2}} - \frac{293241529}{4269957120} hf_{n+\frac{3}{4}}$$
$$- \frac{809066881}{158145620} hf_{n+1} + \frac{313854457}{5930496000} hf_{n+\frac{5}{4}} - \frac{144310733}{277992000} hf_{n+\frac{3}{2}} - \frac{475063673}{5930496000} hf_{n+\frac{7}{4}} - \frac{138491}{63258624} hf_{n+2}$$
$$+ \frac{1098060061}{21249785600} hf_{n+\frac{9}{4}} - \frac{227568593}{17791488000} hf_{n+\frac{5}{2}} + \frac{72298177}{3582976000} hf_{n+\frac{11}{4}} - \frac{32315941}{213496780000} hf_{n+3}$$

(3.53)

$$y_{n+\frac{1}{2}} = y_{n+2} - \frac{44983}{336336000} hf_n + \frac{81383}{28028000} hf_{n+\frac{1}{4}} - \frac{642261}{7007000} hf_{n+\frac{1}{2}} - \frac{5494261}{16816800} hf_{n+\frac{3}{4}}$$
$$- \frac{719601}{4484480} hf_{n+1} - \frac{4912861}{14014000} hf_{n+\frac{5}{4}} - \frac{18979}{125125} hf_{n+\frac{3}{2}} - \frac{4742691}{14014000} hf_{n+\frac{7}{4}} - \frac{1858267}{22422400} hf_{n+2} \quad (3.54)$$
$$- \frac{4559}{3363360} hf_{n+\frac{9}{4}} + \frac{8389}{7007000} hf_{n+\frac{5}{2}} - \frac{6887}{28028000} hf_{n+\frac{11}{4}} + \frac{6887}{336336000} hf_{n+3}$$

$$y_{n+\frac{3}{4}} = y_{n+2} + \frac{318845}{83691159552} hf_n - \frac{8919685}{13948526592} hf_{n+\frac{1}{4}} + \frac{13995335}{2324754432} hf_{n+\frac{1}{2}} - \frac{4330920085}{41845579776} hf_{n+\frac{3}{4}} \quad (3.55)$$
$$- \frac{2749121675}{9299017728} hf_{n+1} - \frac{512838115}{2324754432} hf_{n+\frac{5}{4}} - \frac{4052105}{15567552} hf_{n+\frac{3}{2}} - \frac{68674805}{258306048} hf_{n+\frac{7}{4}} - \frac{1116220405}{9299017728} hf_{n+2}$$
$$+ \frac{528744725}{41845579776} hf_{n+\frac{9}{4}} - \frac{17042965}{6974263296} hf_{n+\frac{5}{2}} + \frac{47735}{140894208} hf_{n+\frac{11}{4}} - \frac{1935865}{83691159552} hf_{n+3}$$
$$y_{n+1} = y_{n+2} - \frac{28151}{5108103000} hf_n + \frac{21128}{212837625} hf_{n+\frac{1}{4}} - \frac{196739}{212837625} hf_{n+\frac{1}{2}} + \frac{849752}{127702575} hf_{n+\frac{3}{4}}$$
$$- \frac{3920003}{37837800} hf_{n+1} - \frac{21258704}{70945875} hf_{n+\frac{5}{4}} - \frac{6237758}{30405375} hf_{n+\frac{3}{2}} - \frac{21258704}{70945875} hf_{n+\frac{7}{4}} - \frac{3920003}{37837800} hf_{n+2}$$
$$+ \frac{849752}{127702575} hf_{n+\frac{9}{4}} - \frac{196739}{212837625} hf_{n+\frac{5}{2}} + \frac{21128}{212837625} hf_{n+\frac{11}{4}} - \frac{28151}{5108103000} hf_{n+3}$$

(3.56)

$$y_{n+\frac{5}{4}} = y_{n+2} + \frac{521303}{43051008000}hf_n - \frac{1244171}{7175168000}hf_{n+\frac{1}{4}} + \frac{4264699}{3587584000}hf_{n+\frac{1}{2}} - \frac{22944767}{4305100800}hf_{n+\frac{3}{4}} + \frac{54961629}{287067200}hf_{n+1} - \frac{647419943}{3587584000}hf_{n+\frac{5}{4}} - \frac{2033753}{8008000}hf_{n+\frac{3}{2}} - \frac{985667673}{3587584000}hf_{n+\frac{7}{4}} - \frac{329208157}{2870067200}hf_{n+2} + \frac{45169279}{4305100800}hf_{n+\frac{9}{4}} - \frac{6696251}{3587584000}hf_{n+\frac{5}{2}} + \frac{152459}{7175168000}hf_{n+\frac{11}{4}} - \frac{688087}{4305100800}hf_{n+3}$$

(3.57)

$$y_{n+\frac{3}{2}} = y_{n+2} + \frac{133787}{81729648000}hf_n - \frac{133787}{6810804000}hf_{n+\frac{1}{4}} + \frac{56333}{567567000}hf_{n+\frac{1}{2}} - \frac{181091}{817296480}hf_{n+\frac{3}{4}} - \frac{583669}{1816214400}hf_{n+1} + \frac{6742003}{1135134000}hf_{n+\frac{5}{4}} - \frac{3118879}{30405375}hf_{n+\frac{3}{2}} - \frac{38542363}{126126000}hf_{n+\frac{7}{4}} - \frac{7503059}{72648576}hf_{n+2} + \frac{28097519}{4068482400}hf_{n+\frac{9}{4}} - \frac{1742911}{1702701000}hf_{n+\frac{5}{2}} + \frac{89987}{756756000}hf_{n+\frac{11}{4}} - \frac{584203}{81729648000}hf_{n+3} \quad (3.58)$$

$$y_{n+\frac{7}{4}} = y_{n+2} + \frac{9959263}{951035904000}hf_n - \frac{252766961}{1743565824000}hf_{n+\frac{1}{4}} + \frac{821346049}{871782912000}hf_{n+\frac{1}{2}} - \frac{4014966413}{1046139494400}hf_{n+\frac{3}{4}} + \frac{172090819}{15498362887}hf_{n+1} - \frac{7236570071}{29059430400}hf_{n+\frac{5}{4}} + \frac{94985467}{1945944000}hf_{n+\frac{3}{2}} - \frac{49214636201}{290594304000}hf_{n+\frac{7}{4}} - \frac{54121130127}{7749184400}hf_{n+2} + \frac{2507349353}{209227898880}hf_{n+\frac{9}{4}} - \frac{184216480}{871782912000}hf_{n+\frac{5}{2}} + \frac{475414129}{1743565824000}hf_{n+\frac{11}{4}} - \frac{184329877}{1046139494400}hf_{n+3} \quad (3.59)$$

$$y_{n+\frac{9}{4}} = y_{n+2} + \frac{184329877}{10461394944000}hf_n - \frac{417640049}{1743565824000}hf_{n+\frac{1}{4}} + \frac{441509227}{290594304000}hf_{n+\frac{1}{2}} - \frac{6257449741}{1046139494400}hf_{n+\frac{3}{4}} + \frac{3821011693}{232475443200}hf_{n+1} - \frac{9816495959}{29059430400}hf_{n+\frac{5}{4}} + \frac{107296613}{1945944000}hf_{n+\frac{3}{2}} - \frac{2552320801}{32288256000}hf_{n+\frac{7}{4}} + \frac{44643543443}{232475443200}hf_{n+2} + \frac{115234170509}{10461394944000}hf_{n+\frac{9}{4}} - \frac{6054093569}{871782912000}hf_{n+\frac{5}{2}} + \frac{143115689}{193729536000}hf_{n+\frac{11}{4}} - \frac{456196373}{10461394944000}hf_{n+3} \quad (3.60)$$

$$y_{n+\frac{5}{2}} = y_{n+2} - \frac{193087}{7429968000}hf_n + \frac{2349637}{6810804000}hf_{n+\frac{1}{4}} - \frac{3612439}{1702701000}hf_{n+\frac{1}{2}} + \frac{32731249}{4086482400}hf_{n+\frac{3}{4}} - \frac{12546839}{605404800}hf_{n+1} + \frac{44018707}{1135134000}hf_{n+\frac{5}{4}} - \frac{1625861}{30405375}hf_{n+\frac{3}{2}} + \frac{57802477}{1135134000}hf_{n+\frac{7}{4}} + \frac{34426087}{605404800}hf_{n+2} + \frac{1362297487}{4086482400}hf_{n+\frac{9}{4}} + \frac{154495511}{1702701000}hf_{n+\frac{5}{2}} - \frac{19099973}{6810804000}hf_{n+\frac{11}{4}} + \frac{10480453}{81729648000}hf_{n+3} \quad (3.61)$$

$$y_{n+\frac{11}{4}} = y_{n+2} + \frac{6279127}{43051008000}hf_n - \frac{13866379}{7175168000}hf_{n+\frac{1}{4}} + \frac{42570291}{3587584000}hf_{n+\frac{1}{2}} - \frac{38554547}{861020160}hf_{n+\frac{3}{4}}$$
$$+\frac{333308253}{82700067200}hf_{n+1} - \frac{787623847}{3587584000}hf_{n+\frac{5}{4}} + \frac{2514233}{8008000}hf_{n+\frac{3}{2}} - \frac{1264870617}{3587584000}hf_{n+\frac{7}{4}} +$$
$$\frac{46838683}{114802688}hf_{n+2} + \frac{324301823}{43051008000}hf_{n+\frac{9}{4}} + \frac{1302640901}{3587584000}hf_{n+\frac{5}{2}} + \frac{584576011}{7175168000}hf_{n+\frac{11}{4}} - \frac{50840663}{43051008000}hf_{n+3} \quad (3.62)$$

3.4 Analysis of the methods

This section addresses order of accuracy and error constant, consistency, zero stability, convergence and region of absolute stability of the newly formulated methods.

3.4.1 Order of accuracy and error constant

Following Mohammed & Adeniyi (2014) and Fudziah et al. (2020), let y_{n+j} be the solution to y'_{n+j} be sufficiently differentiable, then y_{n+j} and y'_{n+j} can be expanded into a Taylor's series about point x_n to obtain

$$T_n = \frac{1}{h\sigma(1)}[c_0\, y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + ...]$$

(3.63)

Where

$$c_0 = \sum_{j=0}^{k} \alpha_j$$

(3.64)

$$c_1 = \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j \,,$$

(3.65)

$$c_q = \frac{1}{q!}\sum_{j=0}^{k} j^q \alpha_j - \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{q-1}\beta_j$$

(3.66)

Definition: A linear multistep method is said to be of order p if

$c_0 = c_1 = ... = c_p = 0, c_{p+1} \neq 0$, c_{p+1} is the error constant.

3.4.2 Order of accuracy and Error constants of three step Adam's type method with six off-grid points.

For the method,

$$y_{n+3} = y_{n+2} + \frac{25}{10752}hf_n - \frac{303}{12800}hf_{n+\frac{1}{3}} + \frac{1221}{11200}hf_{n+\frac{2}{3}} - \frac{5039}{16800}hf_{n+1} + \frac{24603}{44800}hf_{n+\frac{4}{3}} - \frac{6369}{8960}hf_{n+\frac{5}{3}}$$
$$+\frac{13273}{16800}hf_{n+2} - \frac{93}{1600}hf_{n+\frac{7}{3}} + \frac{49143}{89600}hf_{n+\frac{8}{3}} + \frac{1197}{12800}hf_{n+3}$$

(3.67)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 1, \alpha_{\frac{1}{3}} = 0, \beta_0 = \frac{25}{10752}, \beta_{\frac{1}{3}} = -\frac{303}{12800} = \beta_{\frac{2}{3}} = \frac{1221}{11200}, \beta_1 = -\frac{5039}{16800},$$
$$\beta_{\frac{4}{3}} = \frac{24603}{44800}, \beta_{\frac{5}{3}} = -\frac{6369}{18960}, \beta_2 = \frac{13273}{16800}, \beta_{\frac{7}{3}} = -\frac{93}{1600}, \beta_{\frac{8}{3}} = \frac{49143}{89600}, \beta_3 = \frac{1197}{12800}$$

(3.68)

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_{\frac{1}{3}} = 0$$

(3.69)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{3}\alpha_{\frac{1}{3}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{3}} + \beta_{\frac{2}{3}} + \beta_1 + \beta_{\frac{4}{3}} + \beta_{\frac{5}{3}} + \beta_2 + \beta_{\frac{7}{3}} + \beta_{\frac{8}{3}} + \beta_3) = 0$$

(3.70)

$$C_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{3})^2\alpha_{\frac{1}{3}} + 3^2\alpha_3) - (\frac{1}{3}\beta_{\frac{1}{3}} + \frac{2}{3}\beta_{\frac{2}{3}} + \beta_1 + \frac{4}{3}\beta_{\frac{4}{3}} + \frac{5}{3}\beta_{\frac{5}{3}} + 2\beta_2 + \frac{7}{3}\beta_{\frac{7}{3}} + \frac{8}{3}\beta_{\frac{8}{3}} + 3\beta_3) = 0$$

(3.71)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{3})^p\alpha_{\frac{1}{3}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{3})^{p-1}\beta_{\frac{1}{3}} + (\frac{2}{3})^{p-1}\beta_{\frac{2}{3}} + \beta_1 + (\frac{4}{3})^{p-1}\beta_{\frac{4}{3}}$$
$$+(\frac{5}{3})^{p-1}\beta_{\frac{5}{3}} + (2)^{p-1}\beta_2 + (\frac{7}{3})^{p-1}\beta_{\frac{7}{3}} + (\frac{8}{3})^{p-1}\beta_{\frac{8}{3}} + (3)^{p-1}\beta_3)$$

(3.72)

$$c_3 = 0 \ldots$$

(3.73)

$$c_{10} = 0$$

(3.74)

$$c_{11} = -\frac{11899}{349192166400}$$

(3.75)

$$y_n = y_{n+2} - \frac{7}{75}hf_n - \frac{771}{1400}hf_{n+\frac{1}{3}} + \frac{51}{700}hf_{n+\frac{2}{3}} - \frac{199}{210}hf_{n+1} + \frac{249}{700}hf_{n+\frac{4}{3}} - \frac{633}{700}hf_{n+\frac{5}{3}}$$
$$+\frac{299}{2100}hf_{n+2} - \frac{33}{350}hf_{n+\frac{7}{3}} + \frac{3}{140}hf_{n+\frac{8}{3}} - \frac{3}{1400}hf_{n+3}$$

(3.76)

$$\alpha_0 = 1, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{1}{3}} = 0, \beta_0 = -\frac{7}{75}, \beta_{\frac{1}{3}} = -\frac{771}{1400} = \beta_{\frac{2}{3}} = \frac{51}{700}, \beta_1 = -\frac{199}{210},$$
$$\beta_{\frac{4}{3}} = \frac{249}{700}, \beta_{\frac{5}{3}} = -\frac{633}{700}, \beta_2 = \frac{299}{2100}, \beta_{\frac{7}{3}} = -\frac{33}{350}, \beta_{\frac{8}{3}} = \frac{3}{140}, \beta_3 = -\frac{3}{1400}$$

(3.77)

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_{\frac{1}{3}} = 0$$

(3.78) $c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{3}\alpha_{\frac{1}{3}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{3}} + \beta_{\frac{2}{3}} + \beta_1 + \beta_{\frac{4}{3}} + \beta_{\frac{5}{3}} + \beta_2 + \beta_{\frac{7}{3}} + \beta_{\frac{8}{3}} + \beta_3) = 0$

(3.79)

$$C_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{3})^2\alpha_{\frac{1}{3}} + 3^2\alpha_3) - (\frac{1}{3}\beta_{\frac{1}{3}} + \frac{2}{3}\beta_{\frac{2}{3}} + \beta_1 + \frac{4}{3}\beta_{\frac{4}{3}} + \frac{5}{3}\beta_{\frac{5}{3}} + 2\beta_2 + \frac{7}{3}\beta_{\frac{7}{3}} + \frac{8}{3}\beta_{\frac{8}{3}} + 3\beta_3) = 0$$

(3.80)

$c_2 = 0$

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{3})^p\alpha_{\frac{1}{3}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{3})^{p-1}\beta_{\frac{1}{3}} + (\frac{2}{3})^{p-1}\beta_{\frac{2}{3}} + \beta_1 + (\frac{4}{3})^{p-1}\beta_{\frac{4}{3}} + (\frac{5}{3})^{p-1}\beta_{\frac{5}{3}} + (2)^{p-1}\beta_2 + (\frac{7}{3})^{p-1}\beta_{\frac{7}{3}} + (\frac{8}{3})^{p-1}\beta_{\frac{8}{3}} + (3)^{p-1}\beta_3)$$

(3.81) $c_3 = 0 \ldots$

(3.82)

$c_{10} = 0$

(3.83)

$$c_{11} = -\frac{179}{5456127600}$$

(3.84)

$$y_{n+\frac{1}{3}} = y_{n+2} + \frac{25}{10752}hf_n - \frac{101365}{870912}hf_{n+\frac{1}{3}} - \frac{48425}{108864}hf_{n+\frac{2}{3}} - \frac{11575}{54432}hf_{n+1} - \frac{190775}{435456}hf_{n+\frac{4}{3}} - \frac{123575}{435456}hf_{n+\frac{5}{3}} - \frac{10735}{54432}hf_{n+2} + \frac{3175}{108864}hf_{n+\frac{7}{3}} - \frac{4675}{870912}hf_{n+\frac{8}{3}} + \frac{425}{870912}hf_{n+3}$$

(3.85)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{1}{3}} = 1, \beta_0 = \frac{25}{10752}, \beta_{\frac{1}{3}} = -\frac{101635}{870912} = \beta_{\frac{2}{3}} = -\frac{48425}{108864}, \beta_1 = -\frac{11575}{54432}, \text{(3.86)}$$

$$\beta_{\frac{4}{3}} = -\frac{190775}{435456}, \beta_{\frac{5}{3}} = -\frac{123575}{435456}, \beta_2 = -\frac{10735}{54432}, \beta_{\frac{7}{3}} = -\frac{3175}{108864}, \beta_{\frac{8}{3}} = -\frac{4675}{870912}, \beta_3 = \frac{425}{870912}$$

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_{\frac{1}{3}} = 0$$

(3.87) $c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{3}\alpha_{\frac{1}{3}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{3}} + \beta_{\frac{2}{3}} + \beta_1 + \beta_{\frac{4}{3}} + \beta_{\frac{5}{3}} + \beta_2 + \beta_{\frac{7}{3}} + \beta_{\frac{8}{3}} + \beta_3) = 0$

(3.88)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{3})^2\alpha_{\frac{1}{3}} + 3^2\alpha_3) - (\frac{1}{3}\beta_{\frac{1}{3}} + \frac{2}{3}\beta_{\frac{2}{3}} + \beta_1 + \frac{4}{3}\beta_{\frac{4}{3}} + \frac{5}{3}\beta_{\frac{5}{3}} + 2\beta_2 + \frac{7}{3}\beta_{\frac{7}{3}} + \frac{8}{3}\beta_{\frac{8}{3}} + 3\beta_3) = 0 \quad \text{(3.89)}$$

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{3})^p\alpha_{\frac{1}{3}} + 3^p\alpha_3) - \frac{1}{(p-1)!}(((\frac{1}{3})^{p-1}\beta_{\frac{1}{3}} + (\frac{2}{3})^{p-1}\beta_{\frac{2}{3}} + \beta_1 + (\frac{4}{3})^{p-1}\beta_{\frac{4}{3}}$$
$$+(\frac{5}{3})^{p-1}\beta_{\frac{5}{3}} + (2)^{p-1}\beta_2 + (\frac{7}{3})^{p-1}\beta_{\frac{7}{3}} + (\frac{8}{3})^{p-1}\beta_{\frac{8}{3}} + (3)^{p-1}\beta_3)$$

(3.90)

$c_3 = 0 \ldots$

(3.91)

$c_{10} = 0$ (3.92)

$$c_{11} = -\frac{18665}{3394147857408}$$

(3.93)

For the scheme

$$y_{n+\frac{2}{3}} = y_{n+2} - \frac{13}{42525}hf_n + \frac{32}{6075}hf_{n+\frac{1}{3}} - \frac{5494}{42525}hf_{n+\frac{2}{3}} - \frac{17632}{42525}hf_{n+1} - \frac{2174}{8505}hf_{n+\frac{4}{3}}$$
$$-\frac{17632}{42525}hf_{n+\frac{5}{3}} - \frac{5495}{54432}hf_{n+2} + \frac{32}{6075}hf_{n+\frac{7}{3}} - \frac{13}{42525}hf_{n+\frac{8}{3}}$$

(3.94)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{2}{3}} = 1, \beta_0 = -\frac{13}{42525}, \beta_{\frac{1}{3}} = \frac{32}{6075} = \beta_{\frac{2}{3}} = -\frac{5494}{42525}, \beta_1 = -\frac{17632}{42525}, \text{(3.95)}$$
$$\beta_{\frac{4}{3}} = -\frac{2174}{8505}, \beta_{\frac{5}{3}} = -\frac{17632}{42525}, \beta_2 = -\frac{5494}{42525}, \beta_{\frac{7}{3}} = \frac{32}{6075}, \beta_{\frac{8}{3}} = -\frac{13}{42525}, \beta_3 = 0$$

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_{\frac{1}{3}} = 0$$

(3.96) $c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{3}\alpha_{\frac{1}{3}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{3}} + \beta_{\frac{2}{3}} + \beta_1 + \beta_{\frac{4}{3}} + \beta_{\frac{5}{3}} + \beta_2 + \beta_{\frac{7}{3}} + \beta_{\frac{8}{3}} + \beta_3)$

(3.97)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{3})^2\alpha_{\frac{1}{3}} + 3^2\alpha_3) - (\frac{1}{3}\beta_{\frac{1}{3}} + \frac{2}{3}\beta_{\frac{2}{3}} + \beta_1 + \frac{4}{3}\beta_{\frac{4}{3}} + \frac{5}{3}\beta_{\frac{5}{3}} + 2\beta_2 + \frac{7}{3}\beta_{\frac{7}{3}} + \frac{8}{3}\beta_{\frac{8}{3}} + 3\beta_3) = 0$$

(3.98)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{3})^p\alpha_{\frac{1}{3}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{3})^{p-1}\beta_{\frac{1}{3}} + (\frac{2}{3})^{p-1}\beta_{\frac{2}{3}} + \beta_1 + (\frac{4}{3})^{p-1}\beta_{\frac{4}{3}}$$
$$+(\frac{5}{3})^{p-1}\beta_{\frac{5}{3}} + (2)^{p-1}\beta_2 + (\frac{7}{3})^{p-1}\beta_{\frac{7}{3}} + (\frac{8}{3})^{p-1}\beta_{\frac{8}{3}} + (3)^{p-1}\beta_3)$$

(3.99)

$$c_3 = 0 \ldots$$

(3.100)

$$c_{10} = 0$$

(3.101)

$$c_{11} = \frac{62}{82864937925}$$

(3.102)

$$y_{n+\frac{4}{3}} = y_{n+2} + \frac{23}{340200}hf_{n+\frac{1}{3}} - \frac{167}{170100}hf_{n+\frac{2}{3}} + \frac{701}{85050}hf_{n+1} - \frac{23189}{170100}hf_{n+\frac{4}{3}}$$
$$-\frac{13903}{34020}hf_{n+\frac{5}{3}} - \frac{23189}{170100}hf_{n+2} + \frac{701}{85050}hf_{n+\frac{7}{3}} - \frac{167}{170100}hf_{n+\frac{8}{3}} + \frac{23}{340200}hf_{n+3}$$

(3.103)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{4}{3}} = 1, \beta_0 = 0, \beta_{\frac{1}{3}} = \frac{23}{340200} = \beta_{\frac{2}{3}} = -\frac{167}{170100}, \beta_1 = \frac{701}{85050},$$
$$\beta_{\frac{4}{3}} = -\frac{23189}{170100}, \beta_{\frac{5}{3}} = -\frac{13903}{34020}, \beta_2 = -\frac{23189}{170100}, \beta_{\frac{7}{3}} = \frac{701}{85050}, \beta_{\frac{8}{3}} = -\frac{167}{170100}, \beta_3 = \frac{23}{340200}$$

(3.104)

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_{\frac{1}{3}} = 0$$

(3.105)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{3}\alpha_{\frac{1}{3}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{3}} + \beta_{\frac{2}{3}} + \beta_1 + \beta_{\frac{4}{3}} + \beta_{\frac{5}{3}} + \beta_2 + \beta_{\frac{7}{3}} + \beta_{\frac{8}{3}} + \beta_3) = 0$$

(3.106)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{3})^2\alpha_{\frac{1}{3}} + 3^2\alpha_3) - (\frac{1}{3}\beta_{\frac{1}{3}} + \frac{2}{3}\beta_{\frac{2}{3}} + \beta_1 + \frac{4}{3}\beta_{\frac{4}{3}} + \frac{5}{3}\beta_{\frac{5}{3}} + 2\beta_2 + \frac{7}{3}\beta_{\frac{7}{3}} + \frac{8}{3}\beta_{\frac{8}{3}} + 3\beta_3) = 0$$

(3.107)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{3})^p\alpha_{\frac{1}{3}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{3})^{p-1}\beta_{\frac{1}{3}} + (\frac{2}{3})^{p-1}\beta_{\frac{2}{3}} + \beta_1 + (\frac{4}{3})^{p-1}\beta_{\frac{4}{3}}$$
$$+(\frac{5}{3})^{p-1}\beta_{\frac{5}{3}} + (2)^{p-1}\beta_2 + (\frac{7}{3})^{p-1}\beta_{\frac{7}{3}} + (\frac{8}{3})^{p-1}\beta_{\frac{8}{3}} + (3)^{p-1}\beta_3)$$

(3.108)

$$c_3 = 0 \ldots$$

(3.109)

$$c_{10} = 0$$

(3.110)

$$c_{11} = -\frac{263}{1325839006800}$$

(3.111)

$$y_{n+\frac{5}{3}} = y_{n+2} + \frac{2497}{21772800}hf_n - \frac{27467}{21772800}hf_{n+\frac{1}{3}} + \frac{17663}{2721600}hf_{n+\frac{2}{3}} - \frac{5779}{272160}hf_{n+1} + \frac{583073}{10886400}hf_{n+\frac{4}{3}}$$
$$-\frac{2381791}{10886400}hf_{n+\frac{5}{3}} - \frac{225623}{1360800}hf_{n+2} + \frac{42767}{2721600}hf_{n+\frac{7}{3}} - \frac{10063}{4354560}hf_{n+\frac{8}{3}} + \frac{7}{38400}hf_{n+3}$$

(3.112)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{5}{3}} = 1, \beta_0 = \frac{2497}{21772800}, \beta_{\frac{1}{3}} = -\frac{27467}{21772800} = \beta_{\frac{2}{3}} = \frac{17663}{272160}, \beta_1 = -\frac{5779}{272160},$$
$$\beta_{\frac{4}{3}} = \frac{5831791}{10886400}, \beta_{\frac{5}{3}} = -\frac{2381791}{10886400}, \beta_2 = -\frac{225623}{1360800}, \beta_{\frac{7}{3}} = \frac{42767}{2721600}, \beta_{\frac{8}{3}} = -\frac{10063}{4354560}, \beta_3 = \frac{7}{38400}$$

(3.113)

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_{\frac{1}{3}} = 0$$

(3.114)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{3}\alpha_{\frac{1}{3}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{3}} + \beta_{\frac{2}{3}} + \beta_1 + \beta_{\frac{4}{3}} + \beta_{\frac{5}{3}} + \beta_2 + \beta_{\frac{7}{3}} + \beta_{\frac{8}{3}} + \beta_3) = 0$$

(3.115)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{3})^2\alpha_{\frac{1}{3}} + 3^2\alpha_3) - (\frac{1}{3}\beta_{\frac{1}{3}} + \frac{2}{3}\beta_{\frac{2}{3}} + \beta_1 + \frac{4}{3}\beta_{\frac{4}{3}} + \frac{5}{3}\beta_{\frac{5}{3}} + 2\beta_2 + \frac{7}{3}\beta_{\frac{7}{3}} + \frac{8}{3}\beta_{\frac{8}{3}} + 3\beta_3) = 0$$

(3.116)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{3})^p\alpha_{\frac{1}{3}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{3})^{p-1}\beta_{\frac{1}{3}} + (\frac{2}{3})^{p-1}\beta_{\frac{2}{3}} + \beta_1 + (\frac{4}{3})^{p-1}\beta_{\frac{4}{3}}$$
$$+(\frac{5}{3})^{p-1}\beta_{\frac{5}{3}} + (2)^{p-1}\beta_2 + (\frac{7}{3})^{p-1}\beta_{\frac{7}{3}} + (\frac{8}{3})^{p-1}\beta_{\frac{8}{3}} + (3)^{p-1}\beta_3)$$

(3.117)

$$c_3 = 0 \ldots$$

(3.118)

$$c_{10} = 0$$

(3.119)

$$c_{11}=-\frac{90817}{84853696435200}$$

(3.120)

$$y_{n+\frac{7}{3}}=y_{n+2}+\frac{7}{38400}hf_n-\frac{42187}{21772800}hf_{n+\frac{1}{3}}+\frac{25759}{2721600}hf_{n+\frac{2}{3}}-\frac{38599}{1360800}hf_{n+1}+\frac{129581}{2177280}hf_{n+\frac{4}{3}}$$
$$-\frac{1083167}{10886400}hf_{n+\frac{5}{3}}+\frac{349817}{1360800}hf_{n+2}+\frac{391711}{2721600}hf_{n+\frac{7}{3}}-\frac{163531}{21772800}hf_{n+\frac{8}{3}}+\frac{425}{870912}hf_{n+3}$$

(3.121)

$$\alpha_0=0,\alpha_1=0,\alpha_2=-1,\alpha_3=0,\alpha_{\frac{7}{3}}=1,\beta_0=\frac{7}{38400},\beta_{\frac{1}{3}}=-\frac{42187}{21772800}=\beta_{\frac{2}{3}}=\frac{25759}{2721600},\beta_1=-\frac{35899}{1360800},$$
$$\beta_{\frac{4}{3}}=\frac{129581}{2177280},\beta_{\frac{5}{3}}=-\frac{1083167}{10886400},\beta_2=\frac{349817}{1360800},\beta_{\frac{7}{3}}=\frac{391711}{2721600},\beta_{\frac{8}{3}}=-\frac{163531}{21772800},\beta_3=\frac{425}{370912}$$

(3.122)

$$c_0=\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_{\frac{1}{3}}=0$$

(3.123)

$$c_1=\alpha_1+2\alpha_2+\frac{1}{3}\alpha_{\frac{1}{3}}+3\alpha_3-(\beta_0+\beta_{\frac{1}{3}}+\beta_{\frac{2}{3}}+\beta_1+\beta_{\frac{4}{3}}+\beta_{\frac{5}{3}}+\beta_2+\beta_{\frac{7}{3}}+\beta_{\frac{8}{3}}+\beta_3)=0$$

(3.124)

$$c_2=\frac{1}{2}(\alpha_1+2^2\alpha_2+(\frac{1}{3})^2\alpha_{\frac{1}{3}}+3^2\alpha_3)-(\frac{1}{3}\beta_{\frac{1}{3}}+\frac{2}{3}\beta_{\frac{2}{3}}+\beta_1+\frac{4}{3}\beta_{\frac{4}{3}}+\frac{5}{3}\beta_{\frac{5}{3}}+2\beta_2+\frac{7}{3}\beta_{\frac{7}{3}}+\frac{8}{3}\beta_{\frac{8}{3}}+3\beta_3)=0$$

(3.125)

$$c_p=\frac{1}{(p)!}(\alpha_1+2^p\alpha_2+(\frac{1}{3})^p\alpha_{\frac{1}{3}}+3^p\alpha_3)-\frac{1}{(p-1)!}((\frac{1}{3})^{p-1}\beta_{\frac{1}{3}}+(\frac{2}{3})^{p-1}\beta_{\frac{2}{3}}+\beta_1+(\frac{4}{3})^{p-1}\beta_{\frac{4}{3}}$$
$$+(\frac{5}{3})^{p-1}\beta_{\frac{5}{3}}+(2)^{p-1}\beta_2+(\frac{7}{3})^{p-1}\beta_{\frac{7}{3}}+(\frac{8}{3})^{p-1}\beta_{\frac{8}{3}}+(3)^{p-1}\beta_3)$$

(3.126)

$$c_3=0\ \ldots$$

(3.127)

$c_{10} = 0$

(3.128)

$c_{11} = -\frac{171137}{84853696435200}$

(3.129)

$$y_{n+\frac{8}{3}} = y_{n+2} - \frac{13}{42525}hf_n + \frac{1063}{340200}hf_{n+\frac{1}{3}} - \frac{491}{34020}hf_{n+\frac{2}{3}} + \frac{3373}{85050}hf_{n+1} - \frac{12133}{170100}hf_{n+\frac{4}{3}}$$
$$+\frac{14117}{170100}hf_{n+\frac{5}{3}} + \frac{9371}{170100}hf_{n+2} + \frac{7817}{17010}hf_{n+\frac{7}{3}} + \frac{19469}{170100}hf_{n+\frac{8}{3}} - \frac{3}{1400}hf_{n+3}$$

(3.130)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{8}{3}} = 1, \beta_0 = -\frac{13}{42525}, \beta_{\frac{1}{3}} = \frac{1063}{340200} = \beta_{\frac{2}{3}} = -\frac{491}{34020}, \beta_1 = \frac{3373}{35050},$$
$$\beta_{\frac{4}{3}} = -\frac{12133}{170100}, \beta_{\frac{5}{3}} = \frac{14117}{170100}, \beta_2 = \frac{9371}{170100}, \beta_{\frac{7}{3}} = \frac{9371}{170100}, \beta_{\frac{8}{3}} = \frac{19469}{170100}, \beta_3 = -\frac{3}{1400}$$

(3.131)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{3}\alpha_{\frac{1}{3}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{3}} + \beta_{\frac{2}{3}} + \beta_1 + \beta_{\frac{4}{3}} + \beta_{\frac{5}{3}} + \beta_2 + \beta_{\frac{7}{3}} + \beta_{\frac{8}{3}} + \beta_3) = 0$$

(3.133)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{3})^2\alpha_{\frac{1}{3}} + 3^2\alpha_3) - (\frac{1}{3}\beta_{\frac{1}{3}} + \frac{2}{3}\beta_{\frac{2}{3}} + \beta_1 + \frac{4}{3}\beta_{\frac{4}{3}} + \frac{5}{3}\beta_{\frac{5}{3}} + 2\beta_2 + \frac{7}{3}\beta_{\frac{7}{3}} + \frac{8}{3}\beta_{\frac{8}{3}} + 3\beta_3) = 0$$

(3.134)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{3})^p\alpha_{\frac{1}{3}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{3})^{p-1}\beta_{\frac{1}{3}} + (\frac{2}{3})^{p-1}\beta_{\frac{2}{3}} + \beta_1 + (\frac{4}{3})^{p-1}\beta_{\frac{4}{3}}$$
$$+(\frac{5}{3})^{p-1}\beta_{\frac{5}{3}} + (2)^{p-1}\beta_2 + (\frac{7}{3})^{p-1}\beta_{\frac{7}{3}} + (\frac{8}{3})^{p-1}\beta_{\frac{8}{3}} + (3)^{p-1}\beta_3)$$

(3.135)

$c_3 = 0 \ldots$

(3.136)

$c_{10} = 0$

(3.137)

$$c_{11} = \frac{5609}{1325839006800} \qquad (3.138)$$

Table 3.1: Order of accuracy and error constants of three step Adam's type method with six off-grid points

Scheme	Order	Error Constant
y_{n+3}	11	$\frac{-11899}{349192166400}$
y_n	11	$\frac{-179}{5456127600}$
$y_{n+\frac{1}{3}}$	11	$\frac{-18665}{3394147857408}$
$y_{n+\frac{2}{3}}$	11	$\frac{62}{82864937925}$

$y_{n+\frac{4}{3}}$	11	$\frac{-263}{1325839006800}$
$y_{n+\frac{5}{3}}$	11	- $\frac{-90817}{84853696435200}$
$y_{n+\frac{7}{3}}$	11	- $\frac{-171137}{84853696435200}$
$y_{n+\frac{8}{3}}$	11	$\frac{5609}{1325839006800}$

3.4.3 The Stability of three step Adam's type method with six off-grid points

$$y_{n+3} = y_{n+2} + \frac{25}{10752}hf_n - \frac{303}{12800}hf_{n+\frac{1}{3}} + \frac{1221}{11200}hf_{n+\frac{2}{3}} - \frac{5039}{16800}hf_{n+1} + \frac{24603}{44800}hf_{n+\frac{4}{3}} - \frac{6369}{8960}hf_{n+\frac{5}{3}}$$
$$+\frac{13273}{16800}hf_{n+2} - \frac{93}{1600}hf_{n+\frac{7}{3}} + \frac{49143}{89600}hf_{n+\frac{8}{3}} + \frac{1197}{12800}hf_{n+3}$$

(3.139)

$$\rho(r) = r^3 - r^2$$

(3.140)

$$\rho(r) = 1, 0, 0$$

(3.141)

3.4.4 The Consistence of three step Adam's type method with six off-grid points

$$\sum \alpha = 1 - 1$$

(3.142)

$$\sum \alpha = 0$$

(3.143)

$$\rho(r) = r^3 - r^2$$

(3.144)

$$\frac{\rho(r)}{dr} = 3r^2 - 2r$$

(3.145)

$$\frac{\rho(1)}{dr} = 1$$

(3.146)

$$\sigma(1) = \frac{25}{10752} - \frac{303}{12800} + \frac{1221}{11200} - \frac{5039}{16800} + \frac{24603}{44800} - \frac{6369}{8960} + \frac{13273}{16800} - \frac{93}{1600} + \frac{49143}{89600} + \frac{1197}{12800}$$

(3.147)

$$\sigma(1) = 1$$

(3.148)

3.4.5 Order of accuracy and error constant of three step Adam's type method with nine off-grid points

$$y_{n+3}=y_{n+2}-\frac{5942359}{5108103000}hf_n+\frac{3247592}{212837625}hf_{n+\frac{1}{4}}-\frac{6564377}{70945875}hf_{n+\frac{1}{2}}+\frac{43882936}{127702575}hf_{n+\frac{3}{4}}$$
$$-\frac{19812941}{22702680}hf_{n+1}+\frac{113671024}{70945875}hf_{n+\frac{5}{4}}-\frac{66615022}{30405375}hf_{n+\frac{3}{2}}+\frac{17826416}{7882875}hf_{n+\frac{7}{4}}-\frac{190748297}{113513400}hf_{n+2}$$
$$+\frac{34799384}{25540515}hf_{n+\frac{9}{4}}-\frac{57330731}{212837625}hf_{n+\frac{5}{2}}+\frac{10782568}{23648625}hf_{n+\frac{11}{4}}+\frac{337524401}{5108103000}hf_{n+3}$$

(3.149)

$$\alpha_0=0,\alpha_1=0,\alpha_2=-1,\alpha_3=1,\alpha_{\frac{1}{4}}=0,\beta_0=-\frac{5942359}{5108103000},\beta_{\frac{1}{4}}=\frac{3247592}{212837625}=\beta_{\frac{1}{2}}=-\frac{6564377}{70945875},$$
$$\beta_{\frac{3}{4}}=\frac{43882936}{127702575},\beta_1=-\frac{19812941}{22702680},\beta_{\frac{5}{4}}=\frac{113671024}{70945875},\beta_{\frac{3}{2}}=-\frac{66615022}{30405375},\beta_{\frac{7}{4}}=\frac{17826416}{7882875},$$
$$\beta_2=-\frac{190748297}{113513400},\beta_{\frac{9}{4}}=\frac{34799384}{25540515},\beta_{\frac{5}{2}}=-\frac{57330731}{212837625},\beta_{\frac{11}{4}}=\frac{10782568}{23648625},\beta_3=\frac{337524401}{5108103000}$$

(3.150)

$$c_0=0$$

(3.151)

$$c_1=\alpha_1+2\alpha_2+\frac{1}{4}\alpha_{\frac{1}{4}}+3\alpha_3-(\beta_0+\beta_{\frac{1}{4}}+\beta_{\frac{1}{2}}+\beta_{\frac{3}{4}}+\beta_1+\beta_{\frac{5}{4}}+\beta_{\frac{3}{2}}+\beta_{\frac{7}{4}}+\beta_2+\beta_{\frac{9}{4}}+\beta_{\frac{5}{2}}+\beta_{\frac{11}{4}}+\beta_3)=0$$

(3.152)

$$c_2=\frac{1}{2}(\alpha_1+2^2\alpha_2+(\frac{1}{4})^2\alpha_{\frac{1}{4}}+3^2\alpha_3)-(\frac{1}{4}\beta_{\frac{1}{4}}+\frac{1}{2}\beta_{\frac{1}{2}}+\frac{3}{4}\beta_{\frac{3}{4}}+\beta_1+\frac{5}{4}\beta_{\frac{5}{4}}+\frac{3}{2}\beta_{\frac{3}{2}}$$
$$+\frac{7}{4}\beta_{\frac{7}{4}}+2\beta_2+\frac{9}{4}\beta_{\frac{9}{4}}+\frac{5}{2}\beta_{\frac{5}{2}}+\frac{11}{4}\beta_{\frac{11}{4}}+3\beta_3)=0 \quad (3.153)$$

$$c_p=\frac{1}{(p)!}(\alpha_1+2^p\alpha_2+(\frac{1}{4})^p\alpha_{\frac{1}{4}}+3^p\alpha_3)-\frac{1}{(p-1)!}((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}}+(\frac{1}{2})^{p-1}\beta_{\frac{1}{2}}+(\frac{3}{4})^{p-1}\beta_{\frac{3}{4}}+\beta_1+(\frac{5}{4})^{p-1}\beta_{\frac{5}{4}}$$
$$+(\frac{3}{2})^{p-1}\beta_{\frac{3}{2}}+(\frac{7}{4})^{p-1}\beta_{\frac{7}{4}}+(2)^{p-1}\beta_2+(\frac{9}{4})^{p-1}\beta_{\frac{9}{4}}+(\frac{5}{2})^{p-1}\beta_{\frac{5}{2}}+(\frac{11}{4})^{p-1}\beta_{\frac{11}{4}}+(3)^{p-1}\beta_3)$$

(3.154)

$$c_3=0\ldots$$

(3.155)

$c_{13} = 0$

(3.156)

$$c_{14} = -\frac{7619}{486930382848000}$$

(3.157)

$$y_n = y_{n+2} - \frac{42194069}{638512875}hf_n - \frac{97021984}{212837625}hf_{n+\frac{1}{4}} + \frac{19044664}{70945875}hf_{n+\frac{1}{2}} - \frac{173147168}{127702575}hf_{n+\frac{3}{4}}$$
$$+\frac{22373536}{14189175}hf_{n+1} - \frac{181696448}{70945875}hf_{n+\frac{5}{4}} + \frac{60377264}{30405375}hf_{n+\frac{3}{2}} - \frac{14992192}{7882875}hf_{n+\frac{7}{4}} + \frac{10913087}{14189175}hf_{n+2}$$
$$-\frac{43033184}{127702575}hf_{n+\frac{9}{4}} + \frac{19496392}{212837625}hf_{n+\frac{5}{2}} - \frac{358496}{23648625}hf_{n+\frac{11}{4}} + \frac{739276}{638512875}hf_{n+3}$$

(3.158)

$$\alpha_0 = 1, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{1}{4}} = 0, \beta_0 = -\frac{42194069}{638512875}, \beta_{\frac{1}{4}} = -\frac{97021984}{212837625} = \beta_{\frac{1}{2}} = \frac{19044664}{70945875},$$
$$\beta_{\frac{3}{4}} = -\frac{173147168}{127702575}, \beta_1 = \frac{22373536}{14189175}, \beta_{\frac{5}{4}} = -\frac{181696448}{70945875}, \beta_{\frac{3}{2}} = \frac{60377264}{30405375}, \beta_{\frac{7}{4}} = -\frac{14992192}{7882875},$$
$$\beta_2 = \frac{10913087}{14189175}, \beta_{\frac{9}{4}} = -\frac{43033184}{127702575}, \beta_{\frac{5}{2}} = \frac{19496392}{212837625}, \beta_{\frac{11}{4}} = -\frac{358496}{23648625}, \beta_3 = \frac{739276}{638512875}$$

(3.159)

$c_0 = 0$

(3.160)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{4}\alpha_{\frac{1}{4}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{1}{2}} + \beta_{\frac{3}{4}} + \beta_1 + \beta_{\frac{5}{4}} + \beta_{\frac{3}{2}} + \beta_{\frac{7}{4}} + \beta_2 + \beta_{\frac{9}{4}} + \beta_{\frac{5}{2}} + \beta_{\frac{11}{4}} + \beta_3) = 0$$

(3.161)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{4})^2\alpha_{\frac{1}{4}} + 3^2\alpha_3) - (\frac{1}{4}\beta_{\frac{1}{4}} + \frac{1}{2}\beta_{\frac{1}{2}} + \frac{3}{4}\beta_{\frac{3}{4}} + \beta_1 + \frac{5}{4}\beta_{\frac{5}{4}} + \frac{3}{2}\beta_{\frac{3}{2}}$$
$$+\frac{7}{4}\beta_{\frac{7}{4}} + 2\beta_2 + \frac{9}{4}\beta_{\frac{9}{4}} + \frac{5}{2}\beta_{\frac{5}{2}} + \frac{11}{4}\beta_{\frac{11}{4}} + 3\beta_3) = 0$$

(3.162)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{4})^p\alpha_{\frac{1}{4}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}} + (\frac{1}{2})^{p-1}\beta_{\frac{1}{2}} + (\frac{3}{4})^{p-1}\beta_{\frac{3}{4}} + \beta_1 + (\frac{5}{4})^{p-1}\beta_{\frac{5}{4}}$$
$$+(\frac{3}{2})^{p-1}\beta_{\frac{3}{2}} + (\frac{7}{4})^{p-1}\beta_{\frac{7}{4}} + (2)^{p-1}\beta_2 + (\frac{9}{4})^{p-1}\beta_{\frac{9}{4}} + (\frac{5}{2})^{p-1}\beta_{\frac{5}{2}} + (\frac{11}{4})^{p-1}\beta_{\frac{11}{4}} + (3)^{p-1}\beta_3)$$

(3.163)

$c_3 = 0 \ldots$

(3.164)

$C_{13} = 0$

(3.165)

$$c_{14} = -\frac{7619}{486930382848000}$$

(3.166)

$$y_{n+\frac{1}{4}} = y_{n+2} + \frac{250951589}{213497856000}hf_n - \frac{2895487553}{35582976000}hf_{n+\frac{1}{4}} - \frac{6476481263}{17791488000}hf_{n+\frac{1}{2}} - \frac{293241529}{4269957120}hf_{n+\frac{3}{4}} \quad (3.167)$$
$$-\frac{809066881}{1581465600}hf_{n+1} + \frac{313854457}{5930496000}hf_{n+\frac{5}{4}} - \frac{144310733}{277992000}hf_{n+\frac{3}{2}} - \frac{47506373}{5930496000}hf_{n+\frac{7}{4}} - \frac{138999991}{63258624}hf_{n+2}$$
$$-\frac{1098060061}{213497856000}hf_{n+\frac{9}{4}} - \frac{227568593}{17791488000}hf_{n+\frac{5}{2}} + \frac{72298177}{35582976000}hf_{n+\frac{11}{4}} - \frac{32315941}{213497856000}hf_{n+3}$$

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{1}{4}} = 1, \beta_0 = \frac{250951589}{213497856000}, \beta_{\frac{1}{4}} = -\frac{2895487553}{3558276000} = \beta_{\frac{1}{2}} = -\frac{6476481263}{17791488000}, \quad (3.168)$$
$$\beta_{\frac{3}{4}} = -\frac{293241529}{4269957120}, \beta_1 = -\frac{809066881}{15814656000}, \beta_{\frac{5}{4}} = \frac{313854457}{5930496000}, \beta_{\frac{3}{2}} = -\frac{144310733}{277992000}, \beta_{\frac{7}{4}} = -\frac{475063673}{5930496000},$$
$$\beta_2 = -\frac{13899991}{63258624}, \beta_{\frac{9}{4}} = \frac{1098060061}{21349785600}, \beta_{\frac{5}{2}} = -\frac{227568593}{17791488000}, \beta_{\frac{11}{4}} = \frac{72298177}{35582976000}, \beta_3 = -\frac{32315941}{21349785600}$$

$c_0 = 0$

(3.169)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{4}\alpha_{\frac{1}{4}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{1}{2}} + \beta_{\frac{3}{4}} + \beta_1 + \beta_{\frac{5}{4}} + \beta_{\frac{3}{2}} + \beta_{\frac{7}{4}} + \beta_2 + \beta_{\frac{9}{4}} + \beta_{\frac{5}{2}} + \beta_{\frac{11}{4}} + \beta_3) = 0$$

(3.170)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{4})^2\alpha_{\frac{1}{4}} + 3^2\alpha_3) - (\frac{1}{4}\beta_{\frac{1}{4}} + \frac{1}{2}\beta_{\frac{1}{2}} + \frac{3}{4}\beta_{\frac{3}{4}} + \beta_1 + \frac{5}{4}\beta_{\frac{5}{4}} + \frac{3}{2}\beta_{\frac{3}{2}}$$
$$+\frac{7}{4}\beta_{\frac{7}{4}} + 2\beta_2 + \frac{9}{4}\beta_{\frac{9}{4}} + \frac{5}{2}\beta_{\frac{5}{2}} + \frac{11}{4}\beta_{\frac{11}{4}} + 3\beta_3) = 0$$

(3.171)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{4})^p\alpha_{\frac{1}{4}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}} + (\frac{1}{2})^{p-1}\beta_{\frac{1}{2}} + (\frac{3}{4})^{p-1}\beta_{\frac{3}{4}} + \beta_1 + (\frac{5}{4})^{p-1}\beta_{\frac{5}{4}}$$
$$+(\frac{3}{2})^{p-1}\beta_{\frac{3}{2}} + (\frac{7}{4})^{p-1}\beta_{\frac{7}{4}} + (2)^{p-1}\beta_2 + (\frac{9}{4})^{p-1}\beta_{\frac{9}{4}} + (\frac{5}{2})^{p-1}\beta_{\frac{5}{2}} + (\frac{11}{4})^{p-1}\beta_{\frac{11}{4}} + (3)^{p-1}\beta_3)$$

(3.172)

$$c_3 = 0 \ldots$$

(3.173)

$$c_{13} = 0$$

(3.174)

$$c_{14} = \frac{19061}{1072038370816000}$$

(3.175)

$$y_{n+\frac{1}{2}} = y_{n+2} - \frac{44983}{336336000}hf_n + \frac{81383}{28028000}hf_{n+\frac{1}{4}} - \frac{642261}{7007000}hf_{n+\frac{1}{2}} - \frac{5494261}{16816800}hf_{n+\frac{3}{4}}$$
$$-\frac{719601}{4484480}hf_{n+1} - \frac{4912861}{14014000}hf_{n+\frac{5}{4}} - \frac{18979}{125125}hf_{n+\frac{3}{2}} - \frac{4742691}{14014000}hf_{n+\frac{7}{4}} - \frac{1858267}{22422400}hf_{n+2}$$
$$-\frac{4559}{3363360}hf_{n+\frac{9}{4}} + \frac{8389}{7007000}hf_{n+\frac{5}{2}} - \frac{6887}{28028000}hf_{n+\frac{11}{4}} + \frac{6887}{336336000}hf_{n+3}$$

(3.176)

$$\alpha_0=0,\alpha_1=0,\alpha_2=-1,\alpha_3=0,\alpha_{\frac{1}{2}}=1,\beta_0=-\frac{44983}{336336000},\beta_{\frac{1}{4}}=\frac{81383}{28028000}=\beta_{\frac{1}{2}}=-\frac{642261}{7007000},$$
$$\beta_{\frac{3}{4}}=-\frac{5494261}{16816800},\beta_1=-\frac{719601}{4484480},\beta_{\frac{5}{4}}=-\frac{4912861}{4484480},\beta_{\frac{3}{2}}=-\frac{18979}{125125},\beta_{\frac{7}{4}}=-\frac{4742691}{14014000},$$
$$\beta_2=-\frac{1858267}{22422400},\beta_{\frac{9}{4}}=-\frac{4559}{3363360},\beta_{\frac{5}{2}}=\frac{8389}{7007000},\beta_{\frac{11}{4}}=-\frac{6887}{28028000},\beta_3=-\frac{6887}{336336000}$$

(3.177)

$$c_0=0$$

(3.178)

$$c_1=\alpha_1+2\alpha_2+\frac{1}{4}\alpha_{\frac{1}{4}}+3\alpha_3-(\beta_0+\beta_{\frac{1}{4}}+\beta_{\frac{1}{2}}+\beta_{\frac{3}{4}}+\beta_1+\beta_{\frac{5}{4}}+\beta_{\frac{3}{2}}+\beta_{\frac{7}{4}}+\beta_2+\beta_{\frac{9}{4}}+\beta_{\frac{5}{2}}+\beta_{\frac{11}{4}}+\beta_3)=0$$

(3.179)

$$c_2=\frac{1}{2}(\alpha_1+2^2\alpha_2+(\frac{1}{4})^2\alpha_{\frac{1}{4}}+3^2\alpha_3)-(\frac{1}{4}\beta_{\frac{1}{4}}+\frac{1}{2}\beta_{\frac{1}{2}}+\frac{3}{4}\beta_{\frac{3}{4}}+\beta_1+\frac{5}{4}\beta_{\frac{5}{4}}+\frac{3}{2}\beta_{\frac{3}{2}}$$
$$+\frac{7}{4}\beta_{\frac{7}{4}}+2\beta_2+\frac{9}{4}\beta_{\frac{9}{4}}+\frac{5}{2}\beta_{\frac{5}{2}}+\frac{11}{4}\beta_{\frac{11}{4}}+3\beta_3)=0$$

(3.180)

$$c_p=\frac{1}{(p)!}(\alpha_1+2^p\alpha_2+(\frac{1}{4})^p\alpha_{\frac{1}{4}}+3^p\alpha_3)-\frac{1}{(p-1)!}((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}}+(\frac{1}{2})^{p-1}\beta_{\frac{1}{2}}+(\frac{3}{4})^{p-1}\beta_{\frac{3}{4}}+\beta_1+(\frac{5}{4})^{p-1}\beta_{\frac{5}{4}}$$
$$+(\frac{3}{2})^{p-1}\beta_{\frac{3}{2}}+(\frac{7}{4})^{p-1}\beta_{\frac{7}{4}}+(2)^{p-1}\beta_2+(\frac{9}{4})^{p-1}\beta_{\frac{9}{4}}+(\frac{5}{2})^{p-1}\beta_{\frac{5}{2}}+(\frac{11}{4})^{p-1}\beta_{\frac{11}{4}}+(3)^{p-1}\beta_3)$$

(3.181)

$$c_3=0\ldots$$

(3.182)

$$c_{13}=0$$

(3.183)

$$c_{14} = \frac{6887}{22571126882304000}$$

(3.184)

$$y_{n+\frac{3}{4}} = y_{n+2} + \frac{3188345}{83691159552}hf_n - \frac{8919685}{13948526592}hf_{n+\frac{1}{4}} + \frac{13995335}{2324754432}hf_{n+\frac{1}{2}} - \frac{4330920085}{41845579776}hf_{n+\frac{3}{4}}$$
$$-\frac{2749121675}{9299017728}hf_{n+1} - \frac{512838115}{2324754432}hf_{n+\frac{5}{4}} - \frac{4052105}{15567552}hf_{n+\frac{3}{2}} - \frac{68674805}{258306048}hf_{n+\frac{7}{4}} - \frac{1116220405}{9299017728}hf_{n+2}$$
$$+\frac{5285744725}{41845579776}hf_{n+\frac{9}{4}} - \frac{17042965}{6974263296}hf_{n+\frac{5}{2}} + \frac{47735}{140894208}hf_{n+\frac{11}{4}} - \frac{1935865}{83691159552}hf_{n+3}$$

(3.185)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{3}{4}} = 1, \beta_0 = -\frac{3188345}{83691159552}, \beta_{\frac{1}{4}} = -\frac{8919685}{13948526592} = \beta_{\frac{1}{2}} = -\frac{13995335}{2324754432},$$
$$\beta_{\frac{3}{4}} = -\frac{4330920085}{41845579776}, \beta_1 = -\frac{2749121675}{9299017728}, \beta_{\frac{5}{4}} = -\frac{512838115}{2324754432}, \beta_{\frac{3}{2}} = -\frac{4052105}{15567552}, \beta_{\frac{7}{4}} = -\frac{68674805}{258306048},$$
$$\beta_2 = -\frac{11162220405}{9299017728}, \beta_{\frac{9}{4}} = -\frac{528744725}{41845579776}, \beta_{\frac{5}{2}} = \frac{17042965}{6974263296}, \beta_{\frac{11}{4}} = -\frac{47735}{140894208}, \beta_3 = -\frac{1935865}{83691159552}$$

(3.186)

$$c_0 = 0$$

(3.187)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{4}\alpha_{\frac{1}{4}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{1}{2}} + \beta_{\frac{3}{4}} + \beta_1 + \beta_{\frac{5}{4}} + \beta_{\frac{3}{2}} + \beta_{\frac{7}{4}} + \beta_2 + \beta_{\frac{9}{4}} + \beta_{\frac{5}{2}} + \beta_{\frac{11}{4}} + \beta_3) = 0$$

(3.188)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{4})^2\alpha_{\frac{1}{4}} + 3^2\alpha_3) - (\frac{1}{4}\beta_{\frac{1}{4}} + \frac{1}{2}\beta_{\frac{1}{2}} + \frac{3}{4}\beta_{\frac{3}{4}} + \beta_1 + \frac{5}{4}\beta_{\frac{5}{4}} + \frac{3}{2}\beta_{\frac{3}{2}}$$
$$+\frac{7}{4}\beta_{\frac{7}{4}} + 2\beta_2 + \frac{9}{4}\beta_{\frac{9}{4}} + \frac{5}{2}\beta_{\frac{5}{2}} + \frac{11}{4}\beta_{\frac{11}{4}} + 3\beta_3) = 0$$

(3.189)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{4})^p\alpha_{\frac{1}{4}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}} + (\frac{1}{2})^{p-1}\beta_{\frac{1}{2}} + (\frac{3}{4})^{p-1}\beta_{\frac{3}{4}} + \beta_1 + (\frac{5}{4})^{p-1}\beta_{\frac{5}{4}}$$
$$+(\frac{3}{2})^{p-1}\beta_{\frac{3}{2}} + (\frac{7}{4})^{p-1}\beta_{\frac{7}{4}} + (2)^{p-1}\beta_2 + (\frac{9}{4})^{p-1}\beta_{\frac{9}{4}} + (\frac{5}{2})^{p-1}\beta_{\frac{5}{2}} + (\frac{11}{4})^{p-1}\beta_{\frac{11}{4}} + (3)^{p-1}\beta_3)$$

(3.190)

$c_3 = 0 \ldots$

(3.191)

$c_{13} = 0$

(3.192)

$c_{14} = \frac{1935865}{1123283728875493785 6}$

(3.193)

$$y_{n+1} = y_{n+2} - \frac{28151}{5108103000} hf_n - \frac{21128}{212837625} hf_{n+\frac{1}{4}} - \frac{196739}{212837625} hf_{n+\frac{1}{2}} + \frac{849752}{127702575} hf_{n+\frac{3}{4}}$$
$$- \frac{3920003}{37837800} hf_{n+1} - \frac{21258704}{70945875} hf_{n+\frac{5}{4}} - \frac{6237758}{30405375} hf_{n+\frac{3}{2}} - \frac{21258704}{70945875} hf_{n+\frac{7}{4}} - \frac{3920003}{37837800} hf_{n+2}$$
$$+ \frac{849752}{127702575} hf_{n+\frac{9}{4}} - \frac{196739}{212837625} hf_{n+\frac{5}{2}} + \frac{21128}{212837625} hf_{n+\frac{11}{4}} - \frac{28151}{5108103000} hf_{n+3}$$

(3.194)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{3}{4}} = 1, \beta_0 = -\frac{28151}{5108103000}, \beta_{\frac{1}{4}} = \frac{21128}{212837625} = \beta_{\frac{1}{2}} = -\frac{196739}{212837625},$$
$$\beta_{\frac{3}{4}} = \frac{849752}{127702575}, \beta_1 - \frac{3920003}{37837800}, \beta_{\frac{5}{4}} = -\frac{21258704}{70945875}, \beta_{\frac{3}{2}} = -\frac{6237758}{30405375}, \beta_{\frac{7}{4}} = -\frac{21258704}{70945875},$$
$$\beta_2 = -\frac{3920003}{37837800}, \beta_{\frac{9}{4}} = -\frac{849752}{127702575}, \beta_{\frac{5}{2}} = \frac{196739}{212837625}, \beta_{\frac{11}{4}} = -\frac{21128}{212837625}, \beta_3 = -\frac{28151}{5108103000}$$

(3.195)

$c_0 = 0$

(3.196)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{4}\alpha_{\frac{1}{4}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{1}{2}} + \beta_{\frac{3}{4}} + \beta_1 + \beta_{\frac{5}{4}} + \beta_{\frac{3}{2}} + \beta_{\frac{7}{4}} + \beta_2 + \beta_{\frac{9}{4}} + \beta_{\frac{5}{2}} + \beta_{\frac{11}{4}} + \beta_3) = 0$$

(3.197)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{4})^2\alpha_{\frac{1}{4}} + 3^2\alpha_3) - (\frac{1}{4}\beta_{\frac{1}{4}} + \frac{1}{2}\beta_{\frac{1}{2}} + \frac{3}{4}\beta_{\frac{3}{4}} + \beta_1 + \frac{5}{4}\beta_{\frac{5}{4}} + \frac{3}{2}\beta_{\frac{3}{2}}$$
$$+\frac{7}{4}\beta_{\frac{7}{4}} + 2\beta_2 + \frac{9}{4}\beta_{\frac{9}{4}} + \frac{5}{2}\beta_{\frac{5}{2}} + \frac{11}{4}\beta_{\frac{11}{4}} + 3\beta_3) = 0$$

(3.198)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{4})^p\alpha_{\frac{1}{4}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}} + (\frac{1}{2})^{p-1}\beta_{\frac{1}{2}} + (\frac{3}{4})^{p-1}\beta_{\frac{3}{4}} + \beta_1 + (\frac{5}{4})^{p-1}\beta_{\frac{5}{4}}$$
$$+(\frac{3}{2})^{p-1}\beta_{\frac{3}{2}} + (\frac{7}{4})^{p-1}\beta_{\frac{7}{4}} + (2)^{p-1}\beta_2 + (\frac{9}{4})^{p-1}\beta_{\frac{9}{4}} + (\frac{5}{2})^{p-1}\beta_{\frac{5}{2}} + (\frac{11}{4})^{p-1}\beta_{\frac{11}{4}} + (3)^{p-1}\beta_3)$$

(3.199)

$$c_3 = 0 \ldots$$

(3.200)

$$c_{13} = 0$$

(3.201)

$$c_{14} = \frac{1909}{514198484287488000}$$

(3.202)

$$y_{n+\frac{5}{4}} = y_{n+2} + \frac{521303}{4305100800}hf_n - \frac{1244171}{7175168000}hf_{n+\frac{1}{4}} + \frac{4264699}{3587584000}hf_{n+\frac{1}{2}} - \frac{22944767}{4305100800}hf_{n+\frac{3}{4}}$$
$$+\frac{54961629}{2870067200}hf_{n+1} - \frac{467419943}{3587584000}hf_{n+\frac{5}{4}} - \frac{2033753}{8008000}hf_{n+\frac{3}{2}} - \frac{985667673}{3587584000}hf_{n+\frac{7}{4}} - \frac{329208157}{2870067200}hf_{n+2}$$
$$+\frac{45169279}{4305100800}hf_{n+\frac{9}{4}} - \frac{6696251}{3587584000}hf_{n+\frac{5}{2}} + \frac{1752459}{7175168000}hf_{n+\frac{11}{4}} - \frac{688087}{43051008000}hf_{n+3}$$

(3.203)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{3}{4}} = 1, \beta_0 = \frac{521303}{43051008000}, \beta_{\frac{1}{4}} = -\frac{1244171}{7175168000} = \beta_{\frac{1}{2}} = \frac{4264699}{3587584000},$$
$$\beta_{\frac{3}{4}} = -\frac{22944767}{43051008000}\beta_1 = \frac{54961629}{2870067200}, \beta_{\frac{5}{4}} = -\frac{467419943}{3587584000}, \beta_{\frac{3}{2}} = -\frac{2033753}{80080000}, \beta_{\frac{7}{4}} = -\frac{958667673}{3587584000},$$
$$\beta_2 = -\frac{329208157}{2870067200}, \beta_{\frac{9}{4}} = \frac{45169279}{4305100800}, \beta_{\frac{5}{2}} = \frac{6696251}{3587584000}, \beta_{\frac{11}{4}} = \frac{1752459}{7175168000}, \beta_3 = -\frac{688087}{43051008000}$$

(3.204)

$$c_0 = 0$$

(3.205)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{4}\alpha_{\frac{1}{4}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{1}{2}} + \beta_{\frac{3}{4}} + \beta_1 + \beta_{\frac{5}{4}} + \beta_{\frac{3}{2}} + \beta_{\frac{7}{4}} + \beta_2 + \beta_{\frac{9}{4}} + \beta_{\frac{5}{2}} + \beta_{\frac{11}{4}} + \beta_3) = 0$$

(3.206)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{4})^2\alpha_{\frac{1}{4}} + 3^2\alpha_3) - (\frac{1}{4}\beta_{\frac{1}{4}} + \frac{1}{2}\beta_{\frac{1}{2}} + \frac{3}{4}\beta_{\frac{3}{4}} + \beta_1 + \frac{5}{4}\beta_{\frac{5}{4}} + \frac{3}{2}\beta_{\frac{3}{2}}$$
$$+ \frac{7}{4}\beta_{\frac{7}{4}} + 2\beta_2 + \frac{9}{4}\beta_{\frac{9}{4}} + \frac{5}{2}\beta_{\frac{5}{2}} + \frac{11}{4}\beta_{\frac{11}{4}} + 3\beta_3) = 0$$

(3.207)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{4})^p\alpha_{\frac{1}{4}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}} + (\frac{1}{2})^{p-1}\beta_{\frac{1}{2}} + (\frac{3}{4})^{p-1}\beta_{\frac{3}{4}} + \beta_1 + (\frac{5}{4})^{p-1}\beta_{\frac{5}{4}}$$
$$+ (\frac{3}{2})^{p-1}\beta_{\frac{3}{2}} + (\frac{7}{4})^{p-1}\beta_{\frac{7}{4}} + (2)^{p-1}\beta_2 + (\frac{9}{4})^{p-1}\beta_{\frac{9}{4}} + (\frac{5}{2})^{p-1}\beta_{\frac{5}{2}} + (\frac{11}{4})^{p-1}\beta_{\frac{11}{4}} + (3)^{p-1}\beta_3)$$

(3.208)

$$c_3 = 0 \ldots$$

(3.209)

$$c_{13} = 0$$

(3.210)

$$c_{14} = \frac{521303}{5778208481869824000}$$

(3.211)

$$y_{n+\frac{3}{2}} = y_{n+2} + \frac{133787}{81729648000}hf_n - \frac{133787}{6810804000}hf_{n+\frac{1}{4}} + \frac{56333}{567567000}hf_{n+\frac{1}{2}} - \frac{181091}{817296480}hf_{n+\frac{3}{4}}$$
$$-\frac{583669}{181621440}hf_{n+1} + \frac{6742003}{1135134000}hf_{n+\frac{5}{4}} - \frac{3118879}{30405375}hf_{n+\frac{3}{2}} - \frac{38542363}{126126000}hf_{n+\frac{7}{4}} - \frac{7503059}{72648576}hf_{n+2}$$
$$+\frac{28097519}{4086482400}hf_{n+\frac{9}{4}} - \frac{1742911}{1702701000}hf_{n+\frac{5}{2}} + \frac{89987}{756756000}hf_{n+\frac{11}{4}} - \frac{584203}{81729648000}hf_{n+3}$$

(3.212)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{3}{4}} = 1, \beta_0 = \frac{133787}{81729648000}, \beta_{\frac{1}{4}} = -\frac{133787}{6810804000} = \beta_{\frac{1}{2}} = \frac{56333}{567567000},$$
$$\beta_{\frac{3}{4}} = -\frac{181091}{817296480}\beta_1 = -\frac{583669}{1816214400}, \beta_{\frac{5}{4}} = \frac{6742003}{1135134000}, \beta_{\frac{3}{2}} = -\frac{3118879}{30405375}, \beta_{\frac{7}{4}} = -\frac{38542363}{126126000},$$
$$\beta_2 = -\frac{7503059}{72648576}, \beta_{\frac{9}{4}} = \frac{28097519}{4086482400}, \beta_{\frac{5}{2}} = \frac{1742911}{1702701000}, \beta_{\frac{11}{4}} = \frac{89987}{7567566000}, \beta_3 = -\frac{584203}{81729648000}$$

(3.213)

$$c_0 = 0$$

(3.214)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{4}\alpha_{\frac{1}{4}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{1}{2}} + \beta_{\frac{3}{4}} + \beta_1 + \beta_{\frac{5}{4}} + \beta_{\frac{3}{2}} + \beta_{\frac{7}{4}} + \beta_2 + \beta_{\frac{9}{4}} + \beta_{\frac{5}{2}} + \beta_{\frac{11}{4}} + \beta_3) = 0$$

(3.215)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{4})^2\alpha_{\frac{1}{4}} + 3^2\alpha_3) - (\frac{1}{4}\beta_{\frac{1}{4}} + \frac{1}{2}\beta_{\frac{1}{2}} + \frac{3}{4}\beta_{\frac{3}{4}} + \beta_1 + \frac{5}{4}\beta_{\frac{5}{4}} + \frac{3}{2}\beta_{\frac{3}{2}}$$
$$+\frac{7}{4}\beta_{\frac{7}{4}} + 2\beta_2 + \frac{9}{4}\beta_{\frac{9}{4}} + \frac{5}{2}\beta_{\frac{5}{2}} + \frac{11}{4}\beta_{\frac{11}{4}} + 3\beta_3) = 0$$

(3.216)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{4})^p\alpha_{\frac{1}{4}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}} + (\frac{1}{2})^{p-1}\beta_{\frac{1}{2}} + (\frac{3}{4})^{p-1}\beta_{\frac{3}{4}} + \beta_1 + (\frac{5}{4})^{p-1}\beta_{\frac{5}{4}}$$
$$+(\frac{3}{2})^{p-1}\beta_{\frac{3}{2}} + (\frac{7}{4})^{p-1}\beta_{\frac{7}{4}} + (2)^{p-1}\beta_2 + (\frac{9}{4})^{p-1}\beta_{\frac{9}{4}} + (\frac{5}{2})^{p-1}\beta_{\frac{5}{2}} + (\frac{11}{4})^{p-1}\beta_{\frac{11}{4}} + (3)^{p-1}\beta_3)$$

(3.217)

$$c_3 = 0 \ldots$$

(3.218)

$$c_{13} = 0$$

(3.219)

$$c_{14} = \frac{133787}{5484783832399872000}$$

(3.220)

$$y_{n+\frac{7}{4}} = y_{n+2} + \frac{9959263}{951035904000}hf_n - \frac{252766961}{1743565824000}hf_{n+\frac{1}{4}} + \frac{821346049}{871782912000}hf_{n+\frac{1}{2}}$$
$$-\frac{4014966413}{1046139494400}hf_{n+\frac{3}{4}} + \frac{172090819}{15498362880}hf_{n+1} - \frac{7236570071}{290594304000}hf_{n+\frac{5}{4}} + \frac{94985467}{1945944000}hf_{n+\frac{3}{2}}$$
$$-\frac{49214636201}{290594304000}hf_{n+\frac{7}{4}} - \frac{9512130127}{77491814400}hf_{n+2} + \frac{2507349353}{209227898880}hf_{n+\frac{9}{4}} - \frac{1842164801}{871782912000}hf_{n+\frac{5}{2}} +$$
$$\frac{475414129}{1743565824000}hf_{n+\frac{11}{4}} - \frac{184329877}{10461394944000}hf_{n+3}$$

(3.221)

$$-\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{7}{4}} = 1, \beta_0 = \frac{9959263}{951035904000}, \beta_{\frac{1}{4}} = -\frac{252766961}{1743565824000}$$
$$\beta_{\frac{1}{2}} = \frac{821346049}{871782912000}, \beta_{\frac{3}{4}} = -\frac{4014966413}{1046139494400}\beta_1 = \frac{172090819}{15498362880}, \beta_{\frac{5}{4}} = -\frac{7236570071}{290594304000},$$
$$\beta_{\frac{3}{2}} = \frac{94985467}{1945944000}, \beta_{\frac{7}{4}} = -\frac{49214636201}{290594304000}\beta_2 = -\frac{9512130127}{77491814400}, \beta_{\frac{9}{4}} = \frac{2507349353}{209227898880},$$
$$\beta_{\frac{5}{2}} = -\frac{1842164801}{871782912000}, \beta_{\frac{11}{4}} = \frac{475414129}{1743565824000}, \beta_3 = -\frac{184329877}{1046139494000}$$

(3.222)

$$c_0 = 0$$

(3.223)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{4}\alpha_{\frac{1}{4}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{1}{2}} + \beta_{\frac{3}{4}} + \beta_1 + \beta_{\frac{5}{4}} + \beta_{\frac{3}{2}} + \beta_{\frac{7}{4}} + \beta_2 + \beta_{\frac{9}{4}} + \beta_{\frac{5}{2}} + \beta_{\frac{11}{4}} + \beta_3) = 0$$

(3.224)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{4})^2\alpha_{\frac{1}{4}} + 3^2\alpha_3) - (\frac{1}{4}\beta_{\frac{1}{4}} + \frac{1}{2}\beta_{\frac{1}{2}} + \frac{3}{4}\beta_{\frac{3}{4}} + \beta_1 + \frac{5}{4}\beta_{\frac{5}{4}} + \frac{3}{2}\beta_{\frac{3}{2}}$$
$$+ \frac{7}{4}\beta_{\frac{7}{4}} + 2\beta_2 + \frac{9}{4}\beta_{\frac{9}{4}} + \frac{5}{2}\beta_{\frac{5}{2}} + \frac{11}{4}\beta_{\frac{11}{4}} + 3\beta_3) = 0$$

(3.225)

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{4})^p\alpha_{\frac{1}{4}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}} + (\frac{1}{2})^{p-1}\beta_{\frac{1}{2}} + (\frac{3}{4})^{p-1}\beta_{\frac{3}{4}} + \beta_1 + (\frac{5}{4})^{p-1}\beta_{\frac{5}{4}}$$
$$+ (\frac{3}{2})^{p-1}\beta_{\frac{3}{2}} + (\frac{7}{4})^{p-1}\beta_{\frac{7}{4}} + (2)^{p-1}\beta_2 + (\frac{9}{4})^{p-1}\beta_{\frac{9}{4}} + (\frac{5}{2})^{p-1}\beta_{\frac{5}{2}} + (\frac{11}{4})^{p-1}\beta_{\frac{11}{4}} + (3)^{p-1}\beta_3)$$

(3.226)

$$c_3 = 0 \ldots$$

(3.227)

$$c_{13} = 0$$

(3.228)

$$c_{14} = \frac{521303}{577208481869824000}$$

(3.229)

$$y_{n+\frac{9}{4}} = y_{n+2} + \frac{184329877}{10461394944000}hf_n - \frac{417640049}{1743565824000}hf_{n+\frac{1}{4}} + \frac{441509227}{290594304000}hf_{n+\frac{1}{2}} -$$
$$\frac{6257449741}{1046139494400}hf_{n+\frac{3}{4}} + \frac{3821011693}{232475443200}hf_{n+1} - \frac{9816495959}{290594304000}hf_{n+\frac{5}{4}} + \frac{107296613}{1945944000}hf_{n+\frac{3}{2}} -$$
$$\frac{2552320801}{32288256000}hf_{n+\frac{7}{4}} + \frac{44643543443}{232475443200}hf_{n+2} + \frac{115234170509}{1046139494400}hf_{n+\frac{9}{4}} - \frac{6054093569}{871782912000}hf_{n+\frac{5}{2}} +$$
$$\frac{143115689}{193729536000}hf_{n+\frac{11}{4}} - \frac{456196373}{10461394944000}hf_{n+3}$$

(3.230)

$$-\alpha_0=0,\alpha_1=0,\alpha_2=-1,\alpha_3=0,\alpha_{\frac{7}{4}}=1,\beta_0=\frac{184329877}{10461394944000},\beta_{\frac{1}{4}}=-\frac{417640049}{1743565824000},$$
$$\beta_{\frac{1}{2}}=\frac{441509227}{290594304000},\beta_{\frac{3}{4}}=-\frac{6257449741}{1046139494400}\beta_1=\frac{3821011693}{232475443200},\beta_{\frac{5}{4}}=-\frac{9816495959}{290594304000},$$
$$\beta_{\frac{3}{2}}=\frac{107296613}{1945944000},\beta_{\frac{7}{4}}=-\frac{2552320801}{32288256000}\beta_2=\frac{44643543443}{232475443200},\beta_{\frac{9}{4}}=\frac{115234170509}{1046139494400},$$
$$\beta_{\frac{5}{2}}=-\frac{6054093569}{871782912000},\beta_{\frac{11}{4}}=\frac{143115689}{193729536000},\beta_3=-\frac{456196373}{1046134944000}$$

(3.231)

$$c_0=0$$

(3.232)

$$c_1=\alpha_1+2\alpha_2+\frac{1}{4}\alpha_{\frac{1}{4}}+3\alpha_3-(\beta_0+\beta_{\frac{1}{4}}+\beta_{\frac{1}{2}}+\beta_{\frac{3}{4}}+\beta_1+\beta_{\frac{5}{4}}+\beta_{\frac{3}{2}}+\beta_{\frac{7}{4}}+\beta_2+\beta_{\frac{9}{4}}+\beta_{\frac{5}{2}}+\beta_{\frac{11}{4}}+\beta_3)=0$$

(3.233)

$$c_2=\frac{1}{2}(\alpha_1+2^2\alpha_2+(\frac{1}{4})^2\alpha_{\frac{1}{4}}+3^2\alpha_3)-(\frac{1}{4}\beta_{\frac{1}{4}}+\frac{1}{2}\beta_{\frac{1}{2}}+\frac{3}{4}\beta_{\frac{3}{4}}+\beta_1+\frac{5}{4}\beta_{\frac{5}{4}}+\frac{3}{2}\beta_{\frac{3}{2}}$$
$$+\frac{7}{4}\beta_{\frac{7}{4}}+2\beta_2+\frac{9}{4}\beta_{\frac{9}{4}}+\frac{5}{2}\beta_{\frac{5}{2}}+\frac{11}{4}\beta_{\frac{11}{4}}+3\beta_3)=0 \quad (3.234)$$

$$c_p=\frac{1}{(p)!}(\alpha_1+2^p\alpha_2+(\frac{1}{4})^p\alpha_{\frac{1}{4}}+3^p\alpha_3)-\frac{1}{(p-1)!}((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}}+(\frac{1}{2})^{p-1}\beta_{\frac{1}{2}}+(\frac{3}{4})^{p-1}\beta_{\frac{3}{4}}+\beta_1+(\frac{5}{4})^{p-1}\beta_{\frac{5}{4}}$$
$$+(\frac{3}{2})^{p-1}\beta_{\frac{3}{2}}+(\frac{7}{4})^{p-1}\beta_{\frac{7}{4}}+(2)^{p-1}\beta_2+(\frac{9}{4})^{p-1}\beta_{\frac{9}{4}}+(\frac{5}{2})^{p-1}\beta_{\frac{5}{2}}+(\frac{11}{4})^{p-1}\beta_{\frac{11}{4}}+(3)^{p-1}\beta_3)$$

(3.235)

$$c_3=0\ \ldots$$

(3.236)

$$c_{13}=0$$

(3.237)

$$c_{14} = \frac{1935865}{1123283728875493785 6}$$

(3.238)

$$y_{n+\frac{5}{2}} = y_{n+2} - \frac{193087}{7429968000}hf_n + \frac{2349637}{6810804000}hf_{n+\frac{1}{4}} - \frac{3612439}{1702701000}hf_{n+\frac{1}{2}} + \frac{32731249}{4086482400}hf_{n+\frac{3}{4}} - \frac{12546839}{605404800}hf_{n+1} + \frac{44018707}{1135134000}hf_{n+\frac{5}{4}} - \frac{1625861}{30405375}hf_{n+\frac{3}{2}} + \frac{57802477}{1135134000}hf_{n+\frac{7}{4}} + \frac{34426087}{605404800}hf_{n+2} + \frac{1362297487}{4086482400}hf_{n+\frac{9}{4}} + \frac{154495511}{1702701000}hf_{n+\frac{5}{2}} - \frac{19099973}{6810804000}hf_{n+\frac{11}{4}} + \frac{10480453}{81729648000}hf_{n+3}$$

(3.239)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{5}{2}} = 1, \beta_0 = -\frac{193087}{7429968000}, \beta_{\frac{1}{4}} = \frac{2349637}{6810804000}, \beta_{\frac{1}{2}} = -\frac{3612439}{1702701000}, \beta_{\frac{3}{4}} = \frac{32731249}{4086482400} \beta_1 = -\frac{12546839}{605404800}, \beta_{\frac{5}{4}} = \frac{44018707}{1135134000}, \beta_{\frac{3}{2}} = -\frac{1625861}{30405375}, \beta_{\frac{7}{4}} = \frac{57802477}{1135134000} \beta_2 = \frac{34426087}{605404800}, \beta_{\frac{9}{4}} = \frac{1362297487}{4086482400}, \beta_{\frac{5}{2}} = \frac{154495511}{1702701000}, \beta_{\frac{11}{4}} = -\frac{19099973}{6810804000}, \beta_3 = \frac{10480453}{81729648000}$$

(3.240)

$$c_0 = 0$$

(3.241)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{4}\alpha_{\frac{1}{4}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{1}{2}} + \beta_{\frac{3}{4}} + \beta_1 + \beta_{\frac{5}{4}} + \beta_{\frac{3}{2}} + \beta_{\frac{7}{4}} + \beta_2 + \beta_{\frac{9}{4}} + \beta_{\frac{5}{2}} + \beta_{\frac{11}{4}} + \beta_3) = 0$$

(3.242)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{4})^2\alpha_{\frac{1}{4}} + 3^2\alpha_3) - (\frac{1}{4}\beta_{\frac{1}{4}} + \frac{1}{2}\beta_{\frac{1}{2}} + \frac{3}{4}\beta_{\frac{3}{4}} + \beta_1 + \frac{5}{4}\beta_{\frac{5}{4}} + \frac{3}{2}\beta_{\frac{3}{2}} + \frac{7}{4}\beta_{\frac{7}{4}} + 2\beta_2 + \frac{9}{4}\beta_{\frac{9}{4}} + \frac{5}{2}\beta_{\frac{5}{2}} + \frac{11}{4}\beta_{\frac{11}{4}} + 3\beta_3) \quad (3.243)$$

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{4})^p\alpha_{\frac{1}{4}} + 3^p\alpha_3) - \frac{1}{(p-1)!}(((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}} + (\frac{1}{2})^{p-1}\beta_{\frac{1}{2}} + (\frac{3}{4})^{p-1}\beta_{\frac{3}{4}} + \beta_1 + (\frac{5}{4})^{p-1}\beta_{\frac{5}{4}}$$
$$+(\frac{3}{2})^{p-1}\beta_{\frac{3}{2}} + (\frac{7}{4})^{p-1}\beta_{\frac{7}{4}} + (2)^{p-1}\beta_2 + (\frac{9}{4})^{p-1}\beta_{\frac{9}{4}} + (\frac{5}{2})^{p-1}\beta_{\frac{5}{2}} + (\frac{11}{4})^{p-1}\beta_{\frac{11}{4}} + (3)^{p-1}\beta_3)$$

(3.244)

$c_3 = 0 \ldots$

(3.245)

$c_{13} = 0$ (3.246)

$$c_{14} = \frac{6887}{22571126882304000}$$
(3.247)

$$y_{n+\frac{11}{4}} = y_{n+2} + \frac{6279127}{43051008000}hf_n - \frac{13866379}{7175168000}hf_{n+\frac{1}{4}} + \frac{42572091}{3587584000}hf_{n+\frac{1}{2}} -$$
$$\frac{38554547}{861020160}hf_{n+\frac{3}{4}} + \frac{333308253}{2870067200}hf_{n+1} - \frac{787623847}{3587584000}hf_{n+\frac{5}{4}} + \frac{2514233}{8008000}hf_{n+\frac{3}{2}} -$$
$$\frac{1264870617}{3587584000}hf_{n+\frac{7}{4}} + \frac{46838683}{114802688}hf_{n+2} + \frac{324301823}{4305100800}hf_{n+\frac{9}{4}} + \frac{1302640901}{3587584000}hf_{n+\frac{5}{2}} +$$
$$\frac{584576011}{7175168000}hf_{n+\frac{11}{4}} - \frac{50840663}{43051008000}hf_{n+3}$$

(3.248)

$$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0, \alpha_{\frac{11}{4}} = 1, \beta_0 = \frac{6279127}{43051008000}, \beta_{\frac{1}{4}} = -\frac{13866379}{7175168000},$$
$$\beta_{\frac{1}{2}} = \frac{42572091}{3587584000}, \beta_{\frac{3}{4}} = -\frac{385554547}{861020160}\beta_1 = \frac{333308253}{2870067200}, \beta_{\frac{5}{4}} = -\frac{787623847}{3587584000},$$
$$\beta_{\frac{3}{2}} = \frac{2514233}{8008000}, \beta_{\frac{7}{4}} = -\frac{1264870617}{3587584000}\beta_2 = \frac{46838683}{114802688}, \beta_{\frac{9}{4}} = \frac{324301823}{4305100800},$$
$$\beta_{\frac{5}{2}} = \frac{1302640901}{3587584000}, \beta_{\frac{11}{4}} = \frac{584576011}{7175168000}, \beta_3 = -\frac{50840663}{43051008000}$$

(3.249)

$c_0 = 0$

(3.250)

$$c_1 = \alpha_1 + 2\alpha_2 + \frac{1}{4}\alpha_{\frac{1}{4}} + 3\alpha_3 - (\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{1}{2}} + \beta_{\frac{3}{4}} + \beta_1 + \beta_{\frac{5}{4}} + \beta_{\frac{3}{2}} + \beta_{\frac{7}{4}} + \beta_2 + \beta_{\frac{9}{4}} + \beta_{\frac{5}{2}} + \beta_{\frac{11}{4}} + \beta_3) = 0$$

(3.251)

$$c_2 = \frac{1}{2}(\alpha_1 + 2^2\alpha_2 + (\frac{1}{4})^2\alpha_{\frac{1}{4}} + 3^2\alpha_3) - (\frac{1}{4}\beta_{\frac{1}{4}} + \frac{1}{2}\beta_{\frac{1}{2}} + \frac{3}{4}\beta_{\frac{3}{4}} + \beta_1 + \frac{5}{4}\beta_{\frac{5}{4}} + \frac{3}{2}\beta_{\frac{3}{2}} + \frac{7}{4}\beta_{\frac{7}{4}} + 2\beta_2 + \frac{9}{4}\beta_{\frac{9}{4}} + \frac{5}{2}\beta_{\frac{5}{2}} + \frac{11}{4}\beta_{\frac{11}{4}} + 3\beta_3) = 0 \quad (3.252)$$

$$c_p = \frac{1}{(p)!}(\alpha_1 + 2^p\alpha_2 + (\frac{1}{4})^p\alpha_{\frac{1}{4}} + 3^p\alpha_3) - \frac{1}{(p-1)!}((\frac{1}{4})^{p-1}\beta_{\frac{1}{4}} + (\frac{1}{2})^{p-1}\beta_{\frac{1}{2}} + (\frac{3}{4})^{p-1}\beta_{\frac{3}{4}} + \beta_1 + (\frac{5}{4})^{p-1}\beta_{\frac{5}{4}} + (\frac{3}{2})^{p-1}\beta_{\frac{3}{2}} + (\frac{7}{4})^{p-1}\beta_{\frac{7}{4}} + (2)^{p-1}\beta_2 + (\frac{9}{4})^{p-1}\beta_{\frac{9}{4}} + (\frac{5}{2})^{p-1}\beta_{\frac{5}{2}} + (\frac{11}{4})^{p-1}\beta_{\frac{11}{4}} + (3)^{p-1}\beta_3)$$

(3.253)

$c_3 = 0 \ldots$

(3.254)

$c_{13} = 0$ (3.255)

$$c_{14} = \frac{19061}{107202383708160000}$$

(3.256)

3.4.6 The Stability of three step Adam's type method with nine off-grid points

$$y_{n+3} = y_{n+2} - \frac{5942359}{5108103000}hf_n + \frac{3247592}{212837625}hf_{n+\frac{1}{4}} - \frac{6564377}{70945875}hf_{n+\frac{1}{2}} + \frac{43882936}{127702575}hf_{n+\frac{3}{4}} - \frac{19812941}{22702680}hf_{n+1} + \frac{113671024}{70945875}hf_{n+\frac{5}{4}} - \frac{66615022}{30405375}hf_{n+\frac{3}{2}} + \frac{17826416}{7882875}hf_{n+\frac{7}{4}} - \frac{190748297}{113513400}hf_{n+2} + \frac{34799384}{25540515}hf_{n+\frac{9}{4}} - \frac{57330731}{212837625}hf_{n+\frac{5}{2}} + \frac{10782568}{23648625}hf_{n+\frac{11}{4}} + \frac{337524401}{5108103000}hf_{n+3}$$

(3.257)

3.4.7 The Consistence of of three step Adam's type method with nine off-grid points

$$\sum \alpha = 1 - 1$$

(3.258)

$$\sum \alpha = 0$$

(3.259)

$$\rho(r) = r^3 - r^2$$

(3.260)

$$\frac{\rho(r)}{dr} = 3r^2 - 2r$$

(3.261)

$$\frac{\rho(1)}{dr} = 1$$

(3.262)

$$\sigma(\mathrm{r}) = \frac{337524401}{5108103000}r^3 + \frac{10782568}{23648625}r^{\frac{11}{4}} - \frac{57330731}{212837625}r^{\frac{5}{2}} + \frac{34799384}{25540515}r^{\frac{9}{4}} - \frac{190748297}{113513400}r^2 + \frac{17826416}{7882875}r^{\frac{7}{4}} - \frac{66615022}{30405375}r^{\frac{3}{2}} + \frac{113671024}{70945875}r^{\frac{5}{4}} - \frac{19812941}{22702680}r + \frac{43882936}{127702575}r^{\frac{3}{4}} - \frac{6564377}{70945875}r^{\frac{1}{2}} + \frac{3247592}{212837625}r^{\frac{1}{4}} - \frac{5942359}{5108103000}$$

(3.263)

$$\sigma(1) = 1$$

(3.264)

Table 3.2: Order of accuracy and error constants of three step Adam's type method with nine off-grid points

Scheme	Order	Error Constant
y_{n+3}	13	$\frac{-7619}{486930382848000}$
y_n	13	$\frac{-7617}{486930382848000}$
$y_{n+\frac{1}{4}}$	13	$\frac{-19061}{1072038370816000}$
$y_{n+\frac{1}{2}}$	13	$\frac{-6887}{22571126882304000}$

$y_{n+\frac{3}{4}}$	13	$\frac{1935865}{11232837288754937856}$
y_{n+1}	13	$\frac{1909}{514198484287488000}$
$y_{n+\frac{5}{4}}$	13	$\frac{521303}{57782084869824000}$
$y_{n+\frac{3}{2}}$	13	$\frac{133787}{5484783832399872000}$

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Numerical Experiments

In this section, the effectiveness of the three step Adam's type method with six off-grid points and the three step Adam's type method with nine off-grid points formulated in chapter three is tested on some differential problems.

Comparison of the exact solution of problem 1 with the two methods for h = 0.1

Problem 1. Considered the following IVP on the range $0 \le x \le 1.$

$$y' = 998y + 1998z, \quad y(0) = 1$$
$$z' = -999y - 1999z, \quad z(0) = 1$$

Exact solution:

$$y(x) = 4e^{-x} - 3e^{-100x}, \qquad z(x) = -2e^{x} + 3e^{-100x}$$

Problem 2: Consider the nonlinear problem given by

$y' = 10(y-1)^2$; $y(0) = 2$

Exact solution: $y(x) = 1 + \dfrac{1}{1+10x}$

Problem 3: Consider the nonlinear problem given by

$y' = -\dfrac{y^3}{2}$; $y(0) = 1$

Exact solution: $y(x) = \dfrac{1}{\sqrt{1+x}}$

Problem 4. Considered the following nonlinear system over the range $0 \le x \le 1$.

$$y' = -1002y + 1000z^2, \quad y(0) = 1$$
$$z' = y - z(1+z), \quad y(0) = 1$$

Exact solution: $y(x) = e^{-2x}, \quad z(x) = e^{-x}$

Problem 5. Also considered the following nonlinear problem over the range $0 \le x \le 1$.

$y^1 = \frac{y(1-y)}{2y-1}$ $y(0) = \frac{5}{6}$ $\qquad 0 \le x \le 1$

Exact Solution

$$y(x) = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{5}{36}e^{-x}}$$