MATHEMATICAL MODELS FOR THE KINETICS AND OPTIMISATION OF BANANA TRUNK BIOMASS HYDROLYSIS FOR ETHANOL PRODUCTION

BY

EVANS, Patience Ogheneofejiro

PhD/SPS/2017/909

DEPARTMENT OF MATHEMATICS

FEDERAL UNIVERSITY OF TECHNOLOGY,

MINNA

OCTOBER, 2021

ABSTRACT

Ethanol is a renewable energy which can be used for domestics, automobile and industrial fuel. There is the need to make ethanol cheap from agro wastes. In this Study, Mathematical models were formulated, for the production of ethanol from unpretreated and pretreated banana trunk biomass using the consecutive reaction models where the product of a reaction will be a substrate of another reaction. The kinetics of the different processes was determined and the optimum rate for the production of ethanol from banana trunk biomass was obtained analytically. The optimum revenue was obtained using the formulated revenue model obtained by modifying the Cobb-Douglass model. Necessary and Sufficient conditions for the optimality of the revenue generated were determined using the Karush-Kuhn-Tucker (KKT) condition and the Enigmatic Bordered Hessian respectively. The results from the initial value problem obtained, shows that there is significant effect of kinetic variables on the optimum ethanol yield from banana trunk biomass. The results for the batch and continuous process of pretreated and unpretreated banana trunk biomass shows that continuous process gives higher yield of ethanol for both pretreated and unpretreated, however the continuous process proves more efficient. The formulated revenue model gives optimum revenue of ₦ 19,082 with a profit of ₦ 15,082. The KKT condition resulted to $\lambda \geq 0$, proving the solution to be an optimal point. The Enigmatic Bordered Hessian matrix with principal minor determinants of – 193.20, 1.399740440×10^6, – 4.7241, – 1.404622276×10^6, $1.130920568 \times 10^{11}$ shows an indefiniteness, which means that the formulated model has a saddle point. The work concludes that increasing hydrolysis rate will increase the glucose yield thereby increasing the ethanol yield. Increase in the rate of fermentation reduces the time taken to obtain optimum ethanol yield of 3800kg/L Pilot plants for the production of ethanol from other agro waste should be developed using these models

TABLE OF CONTENTS

CHAPTER ONE

CHAPTER TWO

CHAPTER THREE

CHAPTER FOUR

CHAPTER FIVE

LIST OF FIGURES

Figure **Page**

LIST OF TABLES

Table **Page**

LIST OF PLATES

Plates **Page**

Abbreviation, Glossaries and Symbols

K_1 = maximum specific growth rate of cells in time (t), with respect to the fermentation conditions,

A = biomass concentration (g/l) and

t = the time.

A_m = maximum biomass concentration in g/L and,

A_o = minimum or initial biomass concentration in g/L.

The kinetic parameters are:

y= the ethanol mass concentration (g/L)

k_2 = the potential maximum ethanol mass concentration (g/L)

k_1 = the maximum ethanol production rate ($gl^{-1}h^{-1}$)

λ = the lag phase or the lag phase or the time to exponential ethanol production (h)

t = fermentation time

Q = the ethanol mass concentration (g/L)

Q_m = the potential maximum ethanol mass concentration (g/L)

Q_p =the maximum ethanol production rate (g/L/h), or productivity

A = output

B = capital

D = Labour

F_b = partial derivative of F with respect to B

F_d = partial derivative of F with respect to D

g(D) = constant of integration that may depend on D

h(B) = constant of integration that may depend on B

P = Total production (monetary value of all goods produced in a year)

K = Total factor productivity

L = Labour input

C = Capital input

a = labour share of output

b = capital share of output

If $a + b = 1$, return to scale are constant

If $a + b < 1$, return to scale are decreasing

If $a + b > 1$, return to scale are increasing

h = hours of labour

S = tons of steel

C_B = Banana trunk biomass

C_G = Glucose

C_E = Ethanol

β = rate kinetics constant of the formation of C_G from C_B

γ = rate kinetics constant of the formation of C_E from C_G

C_B = Biomass Concentration

C_G = Glucose Concentration

C_E = Ethanol Concentration

B_c = Pretreated banana biomass for continuous process

R_c = Residue for continuous process

G_c = Glucose for continuous process

E_c = Ethanol for continuous process
R = revenue

K = constant

h = hour of labor
b = banana trunk biomass
r = residue
g = glucose

CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

Ethanol, a volatile, colorless, flammable chemical produced from petroleum via chemical transformation of ethylene and also by anaerobic fermentation of glucose (known as simple sugar) gotten from starch using yeast or other microorganisms. The primary source of ethanol is grain containing sugar, waste from woods and organic substances (Egwim *et al.,* 2015). Bioethanol (C_2H_5OH) is derived from hydrolysis and fermentation of glucose from cellulose biomass. It can be produced from the distillation of ethanolic waste gotten from fermentation of cellulosic biomass (Graeme, 2010, Egwim *et al.,* 2015). Understanding the dynamics and inert behavior in the production of ethanol helps to optimisation the product. Fathoming the dynamic and static behavior of ethanol formed in order to control and improve the production of ethanol made from lignocellulosic resources such as agricultural residues (wheat straw, corncob, and paddy straw) (Par and Cardona, 2011). Lignocellulose, contain 10 – 20% lignin, 25 – 30% hemicellulose and 30% - 35% cellulose. They also contain water, protein and lipids (Achinas and Euverink, 2016). Biomass contains approximately 70% of cellulosic and hemicellulosic polymers and are linked to the lignin component through a variety of covalent bonds that give the lignocellulosic biomass significant strength and resistance to biochemical or physical treatment (Achinas and Euverink, 2016).

Ethanol from sugar cane biomass, bagasse, straw and other lignocellulosic materials offer unique environmentally sustainable energy sources, economic strategic benefits and are considered as a safe and cleanest liquid fuel substitute to fossil fuels. lignocellulose biomass can be a cheap raw material with constant supply as a substrate for bioconversion to fuel ethanol (Chin *et. al.*, 2011).

Energy crops such as switch grass and fast-grow trees, and forest resources have been recognized as renewable for industrial applications to produce ethanol and other biofuels (Chin *et. al*., 2011)

Banana trunk biomass is useful as a raw material for ethanol production because it contains sugar with high level of glucose or precursors to glucose (Badger, 2002; Egwim *et al.*, 2015). Historically, fermentation products were mainly food products, but in recent years an increased interest has been observed in the production of bulk chemicals, such as, ethanol and other solvents, specialty chemicals (Pharmaceuticals, industrial enzymes), biofuels and food additives (flavor modifiers). Ethanol from biomass reduces the dependency of the government and industry on fossil fuels; it is cost effective, a cheap alternative for domestic cooking and reduces greenhouse emission (Evans *et. al.,* 2021a).

Computational Engineers apply mathematical optimization in a systematic and efficient way to obtain the best solution for a given problem. In biochemical systems, optimization models are used to simulate results (Evans *et. al.,* 2021a). The two important applications of computational optimization are parameter estimation and design problem (Mendes and Kell, 1998).

1.1.3 Parameter Estimation Method

This is the determination of the optimal parameter as well as calculating the real values of unidentified parameters in a dynamic system centered on experiential input and output data of a system. The experimental results for a set of data are replicated by adjusting the model to obtain the best ways to achieve the set objectives

1.1.4 Experimental Design Problem

Useful in metabolic engineering and biochemical evolution studies, to maximize the variability of interesting products. Experimental design improved metabolic pathways in order to optimize the production of unwanted bye-products.

1.2 Statement of The Research Problem

The rising demand for ethanol either as fuel or for industrial purpose calls for higher production and more efficient bioprocesses. One of the challenges facing human development is sustainable energy crisis. The over dependency on crude oil as source of energy in Nigeria has put the country at a disadvantaged in harnessing or exploring other energy sources. Currently, there are trends in hydrolyzing banana trunk polymers, using enzyme processes to produce fermentable sugars and the fermentable sugar is further converted into ethanol making it a cheap renewable fuel. However, the mathematical model for the optimization of the various parameters leading to the production of cheap ethanol from this abundant and renewable polymer has not been achieved.

1.3 Aim and Objectives

The research aims at determining the rate at which the optimal ethanol production can be achieved for revenue generation.

The objectives are to:

ii. Formulate mathematical models for the kinetics of ethanol production.
iii. Analytically determine the optimal rate constant of ethanol produced from banana trunk biomass.
iv. Optimize the cost of ethanol production.
v. Apply Lagrange multiplier technique to obtain the extreme cost of the revenue subject to a constraint.

1.4 Justification of the Study

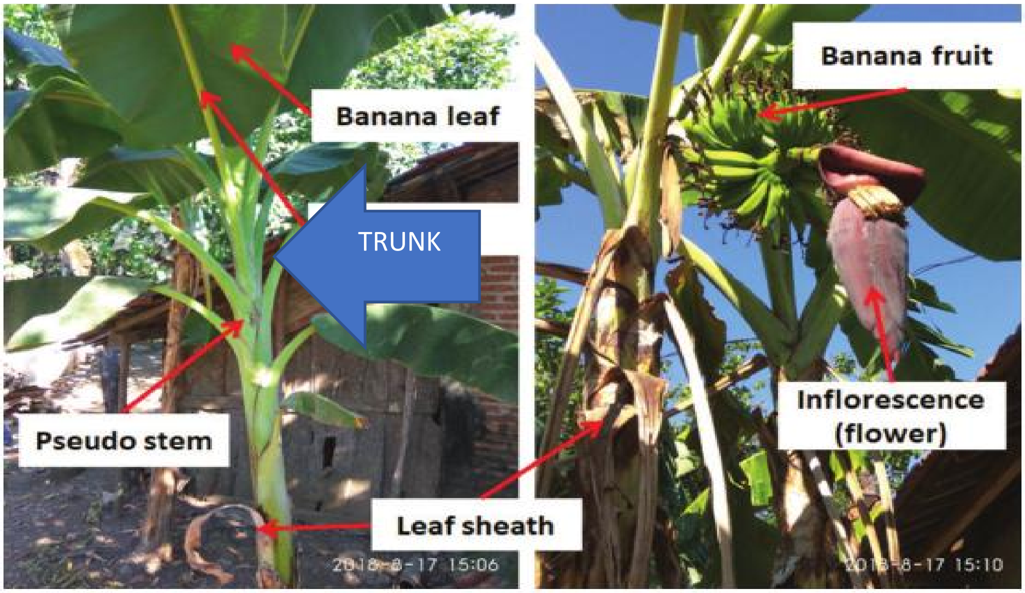
Banana trunk biomass is a renewable polymer abundant in nature particularly in Nigeria, as Nigeria is ranked among the highest producers of banana in West Africa Egwim *et a*l., 2015). Banana trunk is a lignocellulosic biomass that is not directly linked to food production (thus avoids food insecurity) and contains high level of cellulose that can be hydrolyzed to simple sugars, so can serve as a potential source of ethanol. The increasing demand in consumption of energy and the attending constituents of price fluctuation necessitates the need to look for alternative source of energy especially from Biomass sources. The development of an applicable mathematical model for ethanol production from cheap agro wastes will encourage greatly the production of ethanol from renewable energy which can complement or substitute the production of ethanol from other sources with comparative cost.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Banana Trunk

Banana trunk well-known as banana pseudo stems gotten from banana plants (one of the world's most beneficial plants), becomes biomass waste after the banana fruit is harvested (Asmanto and Adchmad, 2018). The features of banana trunk fiber, such as morphological, physical and mechanical, durability, degradability, thermal, chemical, and antibacterial properties was determined by Asmanto and Adchmad, (2018). It was concluded that the fiber can be recycled to fabricate rope, mats, paper cardboard, string thread, tea bags, high-quality textile materials, absorbent, and polymer/fiber composites.



(Asmanto and Adchmad, 2018)

Plate I: Several Parts of Banana Tree.

2.2 Biomass

Biomass resources include; solid wastes from metropolitan, industrial, forestry residues, terrestrial, agricultural and aquatic crops grown solely for energy purposes. Biomass is an attractive petroleum substitute that can be transformed to other forms of energy useful to human. Forestry and Agriculture residues, particularly paper mills residues, are biomass resources commonly used to generate electricity and power (Hossain *et al.*, 2008). Renewable liquid transportation fuel called cellulosic ethanol are gotten from biomass such as wood, grasses, non-edible parts of plants, and municipal wastes (Alex *et al.*, 2012). Bio-ethanol energy from cellulosic constituents is gaining ground more than other energy because;

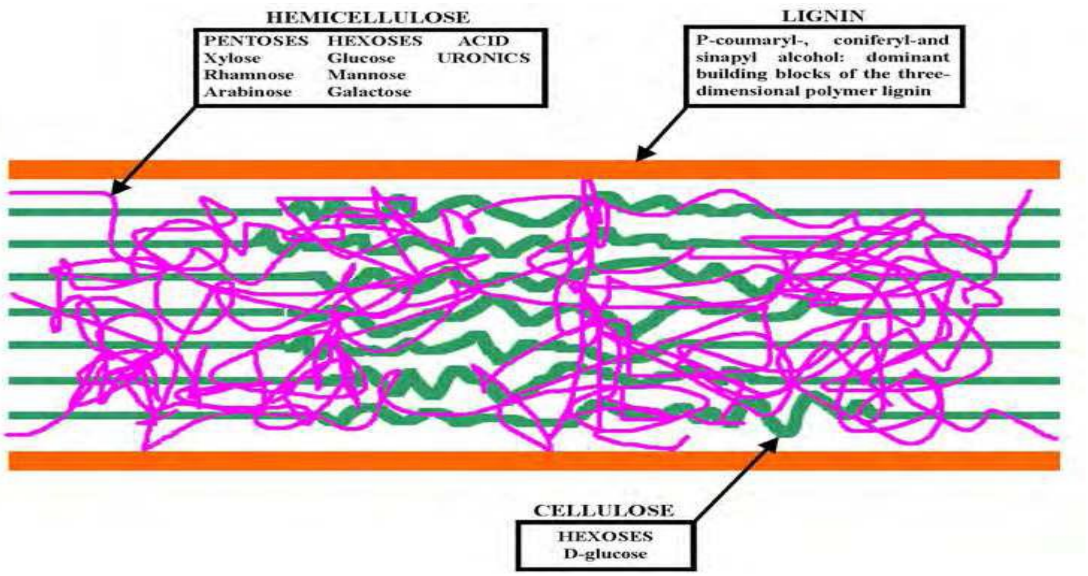
i. The raw materials are available throughout the year.
ii. Less greenhouse gas emission.
iii. It is biodegradable and less toxicity (Saha *et al.*, 2014).

Development of a biomass-to-biofuel manufacturing industry has the prospective to decrease oil importations, sustain agriculture and forestry advancement, foster a domestic bio refinery industry and reduce greenhouse gas emissions by an estimated 86% over gasoline (Alex *et al.*, 2012).

2.2.2 Lignocellulosic Sources and Composition

i. **Lignocellulosic Biomass** is used in the production of biofuel depending on the makeup, structure and process-ability of the raw materials such as; Forest sources, comprises of woody biomass containing mostly of dregs or by-products from industrial processes, biomass farmsteads, agronomic deposits such as, branches and trees. Metropolitan and manufacturing wastes are also cellulose constituents these comprise food residues and mashing slurry (Achinas and Euverink, 2016). Pectin, proteins, extractives such as non-structural sugars, nitrogenous material, chlorophyll and waxes and ash are also found in small quantity in Lignocellulosic materials (Kumar, 2009)

The structure of the biomass components can differ significantly amongst several sources. Precise quantities of the biomass components, mostly lignin and celluloses, are of primary significance since their effect assist process designs for the maximum recovery of energy and products from the raw materials.



(Kumar, 2009)

Plate II: Lignocellulosic Materials: Composition of Major Compounds.

ii. **Marine Algae:** These are appropriate raw material for numerous biochemical methods particularly owing to bio-refineries enlargement intended to produce biofuels, such as, bio-ethanol, bio-diesel, bio-gasoline and other value-added chemicals (Bonvicini, 2015). The quantity of ethanol produced from Algae per growing area is ten-fold compare to maize. Algae are also very pretty eco-friendly feedstock because they ingest high quantities of CO_2 throughout their growth (Achinas and Euverink, 2016).

iii. **Lignin:** A multifaceted polymer joined by covalent bonds to xylans translation vastness and firmness to the plant-cell-wall. Lignin comprises of three chief monomers, sinapyl-alcohol, coumaryl-alcohol, and coniferyl-alcohol. It is an abundant natural polymer and a principal component of timber at least 30–55% for hardwoods and 30–60% for softwoods, whereas agronomic deposits and grasslands cover 3–15% and 10–30% distinctly. Wheat and rice straws comprise principally hemicellulose unlike maize stalk. To upsurge the bio-ethanol yield and also to become possible basis to provide bio-refineries financial solvency, Lignin is modified through genetic engineering methods (Chen *et al*., 2006; Achinas and Euverink, 2016)

iv. **Hemicellulose:** Contains unclear and unstable configuration of hetero-polymers comprising, pentoses (arabinose, xylose) hexoses (mannose, galactose, glucose) as well as sugar/uronic acids (glucuronic, galacturonic, methyl-galacturonic). The hemi-cellulosic sequence contains of 10% arabinose and 90% xylose. Xylan is the main constituent of hemi-cellulose and its structure differs in each feedstock. This is why, hemi-cellulose needs an extensive diversity of enzymes to be fully hydrolyzed into free monomers (Girio *et al*., 2010).

v. **Forest Biomass:** These consist of cherished feedstock comprising of more lignin and less ash than pastoral deposits. Saw dusts, wood chips, and branches are forestry wastes used as bioethanol feedstock. Furthermore, metropolitan and manufacturing solid wastes are also a potential path for biofuels production.

Li (2008), proposed unified bioconversion of cellulose-enriched metropolitan solid waste offers favorable substitutions though this process is not cost effective. Nevertheless, their application related with the discarding of refuse, biological leftover and domestic by-products should be well-thought-out to avoid the effects on the ecosystem. Although increasing crop management is useful to obtain better harvests and yield, the high cost of biomass is still a critical restraint (Achinas and Euverink, 2016).

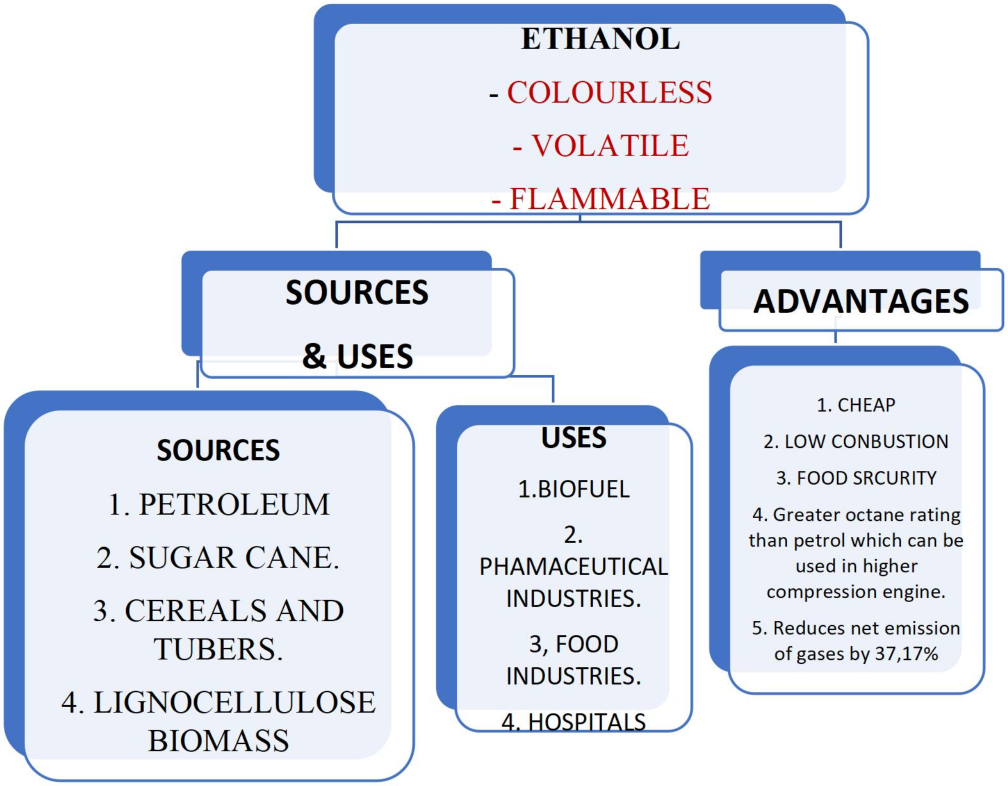
vi. **Cellulose:** This is a linear polymer consisting numerous thousands of 1,4-β-glucosidic bonds linking thousands of glucose entities. It is crystal-like in structure since it contains hydrogen bonds within the polymers. Cellulose molecule is tough and compact because of huge quantity of hydrogen bonds. Temperature as high as 320°C and pressure of 25 MPa is needed to transform cellulosic crystalline to a fluid (amorphous) form. Cellulose is the commonest carbon-based polymer on earth and make-up 30% of plant-biomass. Nevertheless, cotton comprises of virtually 100% cellulose (Achinas and Euverink, 2016).

2.3 Banana Trunk Biomass Hydrolysis

Banana trunk biomass is available in immense amount in Nigeria and a potential lignocellulosic biomass resource for ethanol creation (Evans *et al.*, 2021a). One of the advances for changing over the biomass into ethanol is the sugar stage. Specifically, the carbs (cellulose and hemicelluloses) in the biomass are first hydrolyzed into sugars by proteins, and thereafter the sugars are naturally developed or thermo falsely changed to ethanol. Considering the unmanageability of the biomass achieved by the outrageous real plan and the presence of hemicelluloses and lignin incorporating cellulose, the biomass should be pretreated truly and misleadingly to kill or reduce the obstinacy before cellulases can capably get to and pleasingly hydrolyze cellulose into glucose. Numerous methods such as, dilute and alkaline pretreatment steaming have been established and assessed for biomass pretreatment (Mosier *et al.,* (2005); Obama *et al*., 2012).

2.4 Ethanol Production

The preparation of ethanol from biomass is clearly a fascinating choice rather than standard oil subsidiary and industrial compound, which can be used as a sole fuel in vehicles with devoted motors or in pharmacological and food industries. Ethanol is at present passed on from sugars, starches and cellulosic materials. Notwithstanding, by virtue of the growing demand for human feed, for example, starches and sugars there is an earnest essential for potentially less tireless and maybe less luxurious feedstock, for instance, lignocellulose materials as vital assets for ethanol creation soon (Taherzadeh and Karimi, 2007). The substituted energies are anticipated to satisfy numerous requirements, including; most prominently considerable reduction of greenhouse gas emission, universal accessibility of raw materials and ability of being produced from renewable feedstocks (Hahn-Hagerdal *et al*., 2007). Subsequently, this mission of changing lignocellulosic materials, for example, biomasses into valuable products through chemical and enzymatic cycles has happened to be intressing (Klinke *et al*., 2004).



(Egwim *et al.,* 2015)

Figure 2.1: Ethanol: Sources, Uses and Advantages.

Ethanol is an innocuous to the environment power source made through improvement of sugars not in any manner like the oil based merchandise (Balusu *et al*., 2005; Konstantinos *et al*., 2019). Bioethanol is aliquor made by developing the sugar sections of biomass. It is made by and large from sugar and starch crops. Ethanol can be utilized as a fuel for vehicles in its unadulterated state, regardless it is routinely utilized as a gas added substance to broaden octane and further encourage vehicle transmissions. Ethanol can be mixed in with gas in moving totals to decrease the use of oil powers, likewise as to reduce air pollution (Hossain *et al*., 2008).

Interest in the bioconversion of bountiful and boundless cellulosic biomass into fuel ethanol as a decision as opposed to oil is rising all through the planet inferable from the assertion of lessening regular oil and gas assets (Balusu *et al.*, 2005; Konstantinos *et al.*, 2019). Human advancement and imaginative advancement of the cutting edge world has actuated an increase in energy utilization and the non-innocuous to the environment power sources are in like way declining and are restricted to fulfill the making need of the world (Saha *et al.*, 2014). Consequently, there is need to discover reasonable reaction for the making interest of the fuel that is eco-obliging. Possible cellulosic materials can offer reaction for this moving toward issue. It is a sensible, savvy and harmless to the climate fuel source (Shah and Rehan, 2014).

Bioethanol is one of the significant items which can be made from biomass. It has an awesome potential as a future fuel. Bioethanol has higher octane number and comparative with that of fuel alone (Shah and Rehan, 2014). The utilization of ethanol shows high squeezing factor degree and expanded energy creation in turn over motor (Shah and Rehan, 2014).

Bioenergy is potentially the principle portions to mitigate ozone hurting substance outpourings and substitute of oil based goods. Bioethanol can be conveyed from (I) sugar or starch crops (as sugar stick, sugar beet, corn and wheat), and from (ii) lignocellulosic biomass (Hossain *et al.*, 2008). Different systems for bioethanol creation have been described by specific investigators. The basic steps in the formation of bioethanol from biomass comprises; raw materials selection, pretreatment procedure, saccharification and proper choice of yeast and bacteriological strains (Shah and Rehan, 2014). Standard techniques to achieve this bioconversion consolidate destructive or protein hydrolysis of cellulose followed by fermentation of the resulting dissolvable sugars into ethanol (Balusu *et al.*, 2005).

The two main processes involve in the conversion of Biomass to bioethanol are:

2.4.1 Hydrolysis:

This is the biochemical reaction that changes the composite polysaccharides in the fresh feedstock to simple sugars. In the biomass-to-bioethanol connection, acids and enzymes are used to catalyze this reaction (Saha *et al.*, 2014) as shown in the following reactions;

$$2nC_6H_{12}O_{11} + nH_2O \longrightarrow nC_{12}H_{22}O_{11}$$

Starch (*amylase)* maltose

$$C_{12}H_{22}O_{11} + H_2O \longrightarrow 2C_6H_{12}O_6$$

Maltose (*glucoamylase*) glucose

$$\underset{Saccharose}{C_{12}H_{12}O_{11}} + \underset{Water}{H_2O} \xrightarrow[(invertase)]{} \underset{Glu\cos e}{C_6H_{12}O_6} + \underset{Fructose}{C_6H_{12}O_6}$$

2.4.2 Fermentation:

This is a sequence of biochemical reactions that convert sugars to ethanol. The developing response is accomplished by yeast (for example *Saccharomyces cerevisiae*) or bacteriological feeding on the sugars. Ethanol and carbon dioxide are made as the sugar is burned-through as shown in the following reaction;

$$C_6H_{12}O_6 \longrightarrow 2CH_3CH_2OH + 2CO_2$$

Glucose ethanol carbon dioxide

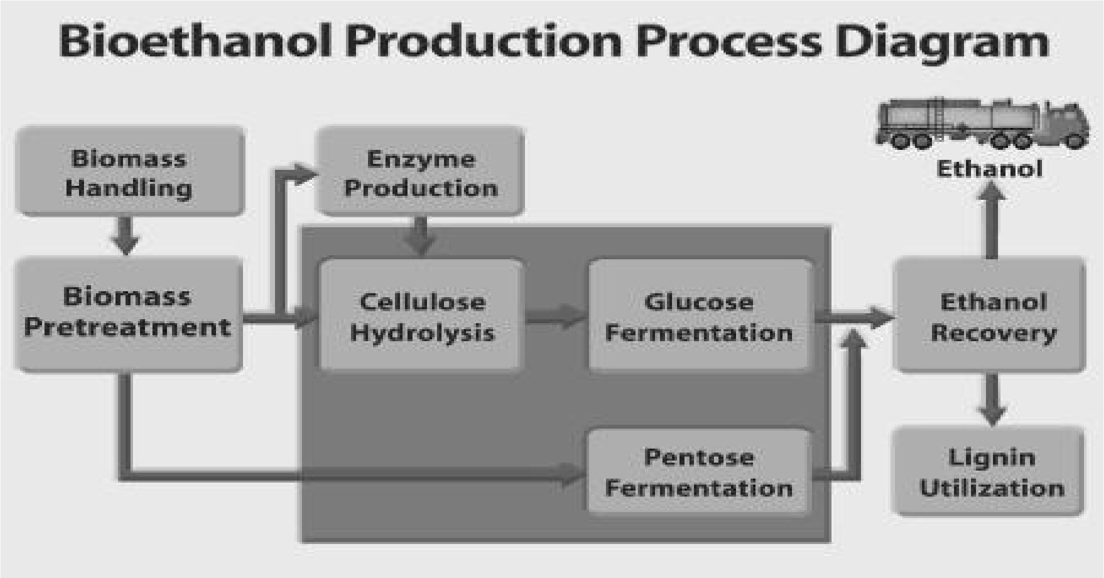
The determination of yeast strain for bioethanol creation is made by thinking about their output, usefulness, resistance to ethanol, inhibitors of fermentation, extreme pH and temperature conditions. In the greater part of the fermentation measures, *S. Cerevisiae* is utilized. *S. Cerevisiae* is a proficient bio-ethanol maker because of its high resilience to ethanol, low optimal pH range and anaerobic conditions prerequisite. In any case, *S.*

Cerevisae isn't much reasonable for ethanol creation from xylose as it needs altered strains or need the pretreatment of xylose by bacterial chemicals (Shah and Rehan, 2014).

2.5 The Basic processes in the Conversion of Biomass to Ethanol

Hydrolysis and downstream treatment can be streamlined by productive pretreatment. The central treatment systems, merge, physical and thermochemical measures which upset the difficult materials and empower the cellulose to go through hydrolysis with higher value and lower energy utilization (Achinas and Euverink, 2016). Progress of an optimal pretreatment measure is difficult, given that 'biomasses merge sources such as hardwood and softwood trees, agronomic deposits, for example; corn curbs and non-recyclable paper squander. These different feedstocks have made analysts test diverse pretreatment measures going from gurgling water and steam blast handling, to corrosive neutralizer (with alkaline) and dissolvable pretreatments, to different significant adaptations of caustic pretreatment (Mielenz, 2001).

Pretreatment philosophies can be appointed physical, compound or the mix of both and regular. Veritable pretreatment frameworks can besides be depicted into mechanical and non-mechanical, for example, criticizing and steam pretreatment autonomously. Mechanical pretreatment diminishes the biomass molecule size that makes the lignocellulosic material more powerless to biochemical hydrolysis. Non-mechanical corporal treatment decays the substrate by acquainting them with tough exterior powers. When physical and thermo-mechanical cycles joins, the energy need get broadened that can make it uneconomical (Singh *et al*., 2014). Biochemical pretreatments are solely applied for the discharge of lignin joining the cellulose and for obliterating its crystal-like arrangement. Conservatively, the paper and tissue industry has used delignification of wood to pass on high strength, long fiber paper things. Though, these procedures are rather luxurious for the pretreatment of the lignocellulosics for the formation of ethanol. Enhancements in pretreatment, as hydrolysis using dilute acids, will help in organizing feedstocks for enzymatic hydrolysis and developing without making enormous centralizations of developing inhibitors (Singh *et al*., 2014). The process of converting biomass to ethanol is show in plate III below;



(Hossian *et. al.,* 2008)

Plate III: Production of Bioethanol from biomass

In spite of the numerous benefits of creation of bioethanol from biomass, there are not many limits too. The bioethanol creation from feed stocks isn't appropriate as it influences the food holds. The utilization of non-consumable lignocellulosic biomass requires pretreatment and saccharification before transformation to bioethanol. Likewise, the utilization of mechanical squanders containing sugars is additionally not efficient because of the presence of strong deposits and other contaminations (Shah and Rehan, 2014). The focal point of most analysts is to work on the productivity and financial aspects of the ethanol creation measure by creating compounds that cooperate to productively separate cellulose. Additionally, the utilization of hereditary designing to plan microorganisms that can effectively mature both five-and six-carbon sugars to ethanol simultaneously (Hossain et al., 2008).

2.7.1 Factors that Affect the Production of Ethanol from Biomass

A few variables influence the creation of ethanol from biomass. They incorporate; Incubation period, Incubation temperature, pH, grouping of carbon source, kind of nitrogen source, inoculum size (Mohamed *et al.,* 2018; Lin *et al.,* 2014). As per the examination completed by Mohamed *et al.,* (2018), the brooding season of 12hours shows the best bioethanol creation. The ideal temperature for the maturation response utilizing the yeast isolate MHY, is 130oC. The following are factors affecting ethanol production;

i. **Effect of pH:** Fermentation response is sensitive to changes in pH. Every microorganism has its particular pH that improves explicit compounds to catalyze certain necessary responses. It is for the most part realized that yeasts favor marginally acidic climate (Mohamed et al.,2018).

ii. **Effect of Carbon Source Concentration** In the exploration completed by (Mohamed et al, 2018), on the grouping of carbon source influences, the creation of ethanol utilizing the yeast disengage MHY1 and the ethanol creation was displayed to consistently increment with expanding carbon source fixation until a centralization of 75% (Mohamed et al., 2018).

2.5.2 Optimization of Ethanol Production from Biomass

Affirmation of sensible carbon, nitrogen and different upgrades is possibly the most fundamental stages in the improvement of a proficient and monetary bioprocess. Regular and quantifiable structures are open for screening supplements in bioprocess streamlining considers. Bona fide procedures consolidate use of numerical models for organizing advancement gauges and isolating the cycle results. There are different benefits in utilizing quantifiable methods to the degree quick and dependable short posting of upgrades, understanding relationship among the improvements at different fixations and an immense decrease in complete number of assessments, accomplishing saving time, dish sets, counterfeit materials and work. Moreover, the quantifiable framework for streamlining of media adequately handles the issue of bulkiness of traditional plans. In their research,(Balusu *et al*., 2005) utilized the movement of congregations of five essential upgrades utilizing reaction surface approach (RSM), a focal composite rotatable plan (CCRD), for the creation of ethanol from cellulosic biomass by *C. thermocellum* SS19 in anaerobic cut down fermentation (Balusu *et al*., 2005).

India is among the best banana (Musa acuminata) making nations and subsequently banana pseudo stem is all around accessible nation waste to be utilized as lignocellulosic substrate. Present evaluation bases on exploitation of banana pseudo stem as a point of convergence for bioethanol creation from the sugars passed on considering diverse compound and typical pretreatments. Two irresistible strains Aspergillus ellipticus and Aspergillus fumigatus offered a clarification to make cellulolytic proteins on sugarcane bagasse were utilized under co-culture developing on banana pseudo stem to debase holocellulose and work with most incredible appearance of lessening sugars. (Snehal *et. al*., 2014). The hydrolysate got after salt and microbial meds was developed by Saccharomyces cerevisiae NCIM 3570 to pass on ethanol. Advancement of cellulosic hydrolysate (4.1 g%) gave most noticeable ethanol (17.1 g/L) with yield (84%) and capability (0.024 g%/h) after 72 h. Some crucial bits of parasitic pretreatment for saccharification of cellulosic substrate utilizing *A. ellipticus* and *A. fumigatus* for ethanol creation by S. cerevisiae NCIM 3570 have been investigated in this evaluation. It was seen that pretreated banana pseudo stem can be financially used as a more reasonable substrate for ethanol creation. (Snehal *et. al.,* 2014).

2.6 Kinetics of Biomass Hydrolysis

As indicated by Olaoye and Kolawole, (2013), logistic model can be utilized to show the rate of reaction of biomass transformation with respect to time while the improved (modified) Gomperta model can be utilized to test the kinetics of ethanol creation at a stable temperature. It has likewise been seen that outcomes from numerical models were not essentially unprecedented when separated and the exploratory outcomes. Forming keeps up with that the usage of numerical model will add to a transcendent awareness of impacts of different segments, influencing creation of ethanol.

Farah *et al*., (2011), used CPU replication of four dissimilar rate of biochemical response models which are: monod, contois, modified monod and teisser to Investigate *S. cerevisiae* improvement energy and ethanol yield. It was seen that Teisser model gave indistinguishably best fit over different models endeavored as it acquired the most basic affiliation coefficient of 0.96299. They inferred that ethanol pack advancement is a non-improvement related cycle subject to Leudking-Piret model. Alfa et *al.,* (2014), examined the possible results of frothed Cymbopogon citratus (Lemon grass) for biogas creation as a sensible elective fuel source in Nigeria.

Sheetal and Patil (2014) analyzed some literatures and saw that Kinetic models depicting the behaviour of microbiological frameworks are significant tools and decreases testing for evacuation of anticipated outcomes. They saw that, various procedure choices and dynamic models embraced towards settling the mechanical difficulties to develop an inconsequential expense business measure. To decrease the expense of ethanol creation per gallon, improvement of yield and worth should be thought of. Energy of biomass creation with respect to time could be tested by fundamental models. Energy of ethanol mass fixation creation at a working temperature and could be tested by improved kinetic models. They pondered that, usage of numerical model could add to an overwhelming enthusiasm for impacts of different elements influencing the creation of ethanol. As needs be, models empower us to get, plan and control the advancement cycle better and could be besides be utilized for additional correspondence improvement. Dynamic execution of ethanol advancement in various centralizations of foaming water sugar maple wood take out hydrolyzate was assessed in pack tests by utilizing a recombinant strain of E. coli FBWHR. Higher social affair of inside and out sugar favors the biomass improvement. The most raised ethanol centralization of 24.05 g/L was gotten utilizing an essential complete sugar concentration of 70.30 g/L. Unstructured models were made to portray cell improvement, sugar use and ethanol creation and supported by looking at the guesses of model and test information. The outcomes from this appraisal could be expected to give experiences into the participation execution smooth out the cycle and help in the game plan of cycles for huge degree making of ethanol improvement from woody biomass (Yang and Shijie, 2014).

The impact of inoculum medium composition on the production of fructose and ethanol from dates by careful fermentation was assessed. Fructose content is improved by adding minerals in the produced syrup and ethanol productivity. Adding of malt concentrate to the medium stretched out fructose yield from 94% to 97%, yet diminished ethanol yield from 78% to 69%. The presence of peptone broadened ethanol capability and fructose package in sugar, while, iron increases fructose yield to 100% in syrups with 371 g initial sugar/L. The astonishing fit by the new extended model measure to the test information makes them critical tools for additional process improvement or industrail applications (Meilana *et al*., 2014).

Oliveira *et al*., (2016), proposed a fundamental numerical model as in equation 2.1, considering limited substrate and ethanol inhibition by the substrate. The transient profiles of sugar, cell and ethanol concentration in the cultured medium were displayed by a ton of standard differential equations, which were made mathematically by the fourth order Runge-Kutta-Gill technique.

$$\frac{dS}{dt} = kS \tag{2.1}$$

Where, S is the substrate concentration, t is the time and k is the rate constant.

The model had the option to duplicate agreeably the conduct of the primary factors of the bioprocess. It was discovered that it's anything but conceivable to improve the ethanol usefulness without disabling the substrate change, since an expansion in efficiency infers a decrease in transformation if a solitary target streamlining system is utilized.

The technique for differential change was utilized to get the arrangement administering the maturation interaction; the arrangement of condition was changed utilizing differential change strategy. The outcome got from the model was seen to show that temperature assumes a significant part in the aging of ethanol, with an expansion in temperature the creation of ethanol tends to diminishes while with a low temperature the creation of ethanol increases (Shehu *et al*, 2017).

Mounira *et al.,* (2017), examined the development energy and displaying of ethanol creation from inulin by *Pichia caribbica* (KC977491) in a cluster framework. Unstructured models were proposed of ethanol was related with *P. caribbica* cell development, a decent understanding between model forecasts and test information was gotten. R2 upsides of 0.91, 0.96, and 0.95 were noticed for biomass, ethanol creation and substrate utilization, individually. To approve the proposed models, analysis of variance (ANOVA) was likewise utilized. As per the got results, the anticipated active qualities and test information concurred well. It is feasible to anticipate the advancement of *P. caribbica* utilizing these models.

The study and discussion of chemical reactions with respect to reaction rates, formation of intermediates effect of various variables and re-arrangement of atoms is called Chemical kinetics. Comparatively, simple rate laws exist for zero order reactions (for which reaction rates are independent of concentration), first order reactions and second order reactions, can be derived for others. elementary reactions follow the law of mass action, but the rate law of stepwise reactions has to be derived by combining the rate laws of the various elementary steps, and can become rather complex. In consecutive reactions, the rate-determining step often determines the kinetics. In consecutive first order reactions, a steady state approximation can simplify the rate law. The main factors that influence the reaction rate include: the physical state of the reactants, the concentrations of the reactants, the temperature at which the reaction occurs, and whether or not any catalysts are present in the reaction (Gorban and Yablonsky, 2015).

Gorban and Yablonsky (2015) suggested that the history of chemical dynamics can be divided into three eras. The first is the Van't Hoff wave searching for the general laws of chemical reactions and relating kinetics to thermodynamics. The second may be called the Semenov--Hinshelwood wave with emphasis on reaction mechanisms, especially for chain reactions. The third is associated with the detailed mathematical description of chemical reaction networks..

The experimental determination of reaction rates involves measuring how the concentrations of reactants or products change over time. For instance, spectrophotometry method is used to measure, the concentration of a reactant at a wavelength where no other reactant or product in the system absorbs light. For reactions which take at least several minutes, it is possible to start the observations after the reactants have been mixed at the temperature of interest. (Atkins and de Paula, 2006).

In chemical kinetics, the overall rate of a reaction is often approximately determined by the slowest step, known as the rate-limiting step or rate-determining step (RDS). The prediction of the corresponding rate equation for a given reaction mechanism (in comparison with the experimental rate law) is frequently simplified by using this approximation of the rate-determining step. (Murdoch, 1981).

In principle, the set of simultaneous rate equations for the individual steps of the mechanism can be used to determine the time evolution of the reactant and product concentrations from, one for each step. However, the analytical solution of these differential equations is not always easy, and in some cases numerical integration may even be required. The hypothesis of a single rate-determining step can greatly simplify the mathematics. In the simplest case the initial step is the slowest, and the overall rate is just the rate of the first step. Also, the rate equations for mechanisms with a single rate-determining step are usually in a simple mathematical form, whose relation to the mechanism and choice of rate-determining step is clear. The correct rate-determining step can be identified by predicting the rate law for each possible choice and comparing the different predictions with the experimental law. The concept of the rate-determining step is very important to the optimization and understanding of many chemical processes such as catalysis and combustion. (Murdoch, 1981).

The concentration of a reactive intermediate such as $[NO_3]$ remains low and almost constant. It may therefore be estimated by the steady-state approximation, which specifies that the rate at which it is formed equals the (total) rate at which it is consumed.

The statement that the first step is the slow step actually means that the first step *in the backward reverse direction* is slower than the second step in the forward direction, so that almost all NO_3 is consumed by reaction with CO and not with "NO". That is, $r_{-1} \ll r_2$, so that $r_1 - r_2 \approx 0$. But the overall rate of reaction is the rate of formation of final product (here CO_2), so that $r = r_2 \approx r_1$. That is, the overall rate is determined by the rate of the first step, and (almost) all molecules that react at the first step continue to the fast second step (Zumdahl, 2005).

2.7 Mathematical Models

The numerical models which portray chemical kinetics reaction provides chemists and compound designers, such as, chemical engineers with devices to even more immediately comprehend and depict substance cycles (chemical process) like food disintegrating, microbial growth, stratospheric ozone decay, and the investigation of normal frameworks. These models can also be utilized in the course of action or change of designed reactors to streamline product yield, all of the more profitably separate products, and dispose of normally harming results. When performing synergist breaking of significant hydrocarbons into gas and light gas, for instance, dynamic models can be utilized to discover the temperature and pressure variable at which the best return of liberal hydrocarbons into gas will happen. Substance Kinetics is at times embraced and investigated through showing in express packs as a fragment of ordinary differential equation-solving (ODE) and curve fitting (Espenson, 2002).

Some existing models for the kinetics of ethanol production are as follows;

i.	Logistic Model	$y = \frac{a}{1 + exp(b - cx)}$	(Sheetal and Patil, 2014; Olaoye and Kolawole, 2013)
ii.	Gompertz Model	$y = a.exp\{-exp(b - cx)\}$	(Olaoye and Kolawole, 2013)
iii.	Modified Gompertz Model	$y = a.exp\left\{-exp\left[\begin{matrix}\frac{\mu_m exp(1)}{a}\\ (\lambda - t) + 1\end{matrix}\right]\right\}$	(Olaoye and Kolawole, 2013)
iv.	Monod Model	$y = \frac{y_{max}[S]}{k + [S]}$	(Farah *et al.*, 2011)
v.	Defferential Transform Method	$F(k) = \left[\frac{1}{k}\ \frac{d^k f(x)}{dx^k}\right]_{x-x_o}$	(Shehu *et al.,* 2017)

2.7.1 Biomass Kinetic Growth Model

As indicated by Olaoye and Kolawole (2013), a non-structured, sigmoidal-shaped model, particularly logistic model is broadly used to depict microbial growth. A lot of polysaccharide developing cycles and biomass progression have been depicted by fundamental condition. Under ideal progression condition and when the; inhibitory impacts of substrate and products were neglected; the speed of cell development follows the familiar exponential relationship as conferred in equation (2.2)

$$\frac{dA}{dt} = k_1 A \tag{2.2}$$

where,

k_1 = greatest explicit development pace of cells on schedule (t), as for the aging conditions

A = biomass fixation (g/l)

t = the time.

The condition above implies that A increases with time paying little heed to substrate accessibility. Equation (2.3), shows the real life situation of the given hyperbolic relationship governing cell growths;

$$\frac{dA}{dt} = k_1 \left[1 - \frac{A}{A_m} \right] \tag{2.3}$$

where,

Am = the greatest biomass focus in g/l,

A_0 = the base or beginning biomass focus in g/l.

This condition is known as the Riccati condition, utilizing the limit condition at t=0 then A = Ao, gives a sigmoidal variety of An as a component of time.

Integrating equation (2.3), to obtain, equation (2.4).

$$A = \frac{A_m A_0 e^{k_1}}{A_m - A_0 + A_0 e^{k_1 t}} \tag{2.4}$$

Further simplification gives equation (2.5) and (2.6):

$$A = \frac{A_m A_0 e^{k_1 t}}{A_m - A_0 \left[1 - e^{k_1 t}\right]} \tag{2.5}$$

Dividing both the numerator and the denominator by A_m, we have;

$$A = \frac{A_0 e^{k_1 t}}{1 - \frac{A_0}{A_m}\left[1 - e^{k_1 t}\right]} \tag{2.6}$$

Equation (2.6) known as logistic equation which can precisely portray the rate of change of biomass with time. To determine the kinetics of biomass growth, experimental data of the dynamic of the biomass growth was adopted and solved using equation (2.6).

2.7.2 Ethanol Concentration Model

The adjusted Gompertz model can effectively depict the information of the fermentative formation of ethanol from glucose biomass utilizing a thermo tolerant strain of *Kluveromyces maxianus*. The twofold exponential Gompertz function Model is by equation (2.7);

$$y = k_2 \ \exp \ \left[- \exp \left(b - c\ x\right)\right] \tag{2.7}$$

$$y = k_2 \exp\left\{ -\exp\left[\frac{k_1 \exp(1)}{k_2} (\lambda - t) + 1 \right] \right\} \tag{2.8}$$

where,

y= the ethanol mass fixation (g/L)

k2 = the potential most conspicuous ethanol mass fixation (g/L)

k1 = the best ethanol creation rate (g/l/h)

λ = the slack stage or the room stage or the chance to enthusiastic ethanol creation (h)

t = fermentation time.

Equation (2.8) can be framed as shown in equation (2.9),

$$Q = Q_m \, exp\left\{ - \, exp\left[\frac{Q_p \, exp\,(1)}{Q_m} (\lambda - t) + 1 \right] \right\} \tag{2.9}$$

where,

Q = the ethanol mass focus (g/L),

Q(m) = the potential most incredible ethanol mass fixation (g/L),

Q(p)=the most important ethanol creation rate (gl-1h-1), or viability

, the motor furthest reaches of the ethanol mass social event of the exploratory information was gotten and introduced to changed Gompertz model and the outcome will plotted close by with the genuine test information. Olaoye and Kolawole (2013).

2.7.3 Lagrange Multiplier Method:

Lagrange multiplier strategy is a method for tracking down a most extreme or least of a capacity F(x; y; z) subject to an imperative (likewise called side state) of the structure G(x; y; z) = 0. Mathematical premise of Lagrange multiplier technique can be clarified if the capacities are of two factors. So we start by attempting to track down the outrageous upsides of F (x; y) subject to a requirement of the structure G(x; y) = 0. At the end of the day, we look for the outrageous upsides of F (x; y) when the point (x; y) is limited to lie on the level curve G (x; y) = 0 (Salih, 2013).

$$\nabla F\left(x_0, y_0\right) = -\lambda \nabla G\left(x_0, y_0\right) \qquad (2.10)$$

λ is a scalar parameter called Lagrange multiplier. The procedure for solving the above equation F(x,y) is based on chain rule,

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0 \qquad (2.11)$$

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy = 0 \qquad (2.12)$$

Multiply equation (2.12) by λ and add to equation (2.11) yields

$$\left(\frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x}\right) dx + \left(\frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y}\right) dy = 0 \qquad (2.13)$$

By choosing λ to satisfy

$$\left(\frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x}\right) = 0 \qquad (2.14)$$

So that,

$$\left(\frac{\partial F}{\partial x}+\lambda\frac{\partial G}{\partial x}\right)=0 \tag{2.15}$$

Equation (2.14) and (2.15) are component of $\nabla F+\lambda\nabla G=0$, can be calculated by solving the following set of equations simultaneously;

$$\begin{aligned}&\frac{\partial F}{\partial x}+\lambda\frac{\partial G}{\partial x}=0\\&\frac{\partial F}{\partial y}+\lambda\frac{\partial G}{\partial y}=0\\&G(x,y)=0\end{aligned} \tag{2.16}$$

This is a system of three equation with three unknowns x, y and λ.

In the event that the capacity to be extremized F and the side condition, G are capacity of three autonomous factors x, y, and z, the accompanying arrangement of condition is addressed to get the minimum or maximum of F.

$$\begin{aligned}&\frac{\partial F}{\partial x}+\lambda\frac{\partial G}{\partial x}=0\\&\frac{\partial F}{\partial y}+\lambda\frac{\partial G}{\partial y}=0\\&\frac{\partial F}{\partial z}+\lambda\frac{\partial G}{\partial z}=0\\&G(x,y.z)=0\end{aligned} \tag{2.17}$$

Assume we have two limitations, So that λ and μ are known as the Lagrange multipliers to such an extent that;

$$\nabla F\left(x_0, y_0, z_0\right)=\lambda\nabla G\left(x_0, y_0, z_0\right) \qquad 2.18$$

The extreme values are gotten by solving for five unknowns x, y, z, λ and μ. This is done by composing the above condition as far as the parts and utilizing the limitation conditions:

$$
\begin{aligned}
&Fx = \lambda Gx + \mu Hx \\
&Fy = \lambda Gy + \mu Hy \\
&Fz = \lambda Gz + \mu Hz \\
&G(x, y, z) = k \\
&H(x, y, z) = c
\end{aligned}
\tag{2.19}
$$

2.7.4 Cobb – Douglas Production Model

Theorem 2.1:

Let X, Y and Z be Banach space, and let y_1≤ k be the direct fractional request in Y prompted by a shut, nonempty, curved cone K in Y: in y1 ≤ k y2 iff y2 – y1 ϵ K. We mean the polar cone of K by K+ := {y^* ϵ Y^* :⟨y^*,y⟩≥0,∀ y∈K}. Think about the accompanying class of compelled streamlining issues, for (y; z) ϵ Y × Z,

P(y; z) : min f(x) subject to: g(x) ≤ K y, h(x) = z; x ϵ C;

where C is a shut subset of X, f : X → IR is lower semi-ceaseless, g : X →Y is lower semi-ceaseless as for ≤K and h : X → Z is persistent.

2.7.5 Steps to Optimize the Multivariate Function

To upgrade or confine a multivariable cutoff f(x, y,…) subject to the essential that another multivariable limit rises to a predictable, g(x,y,…) = C. The going with pushes are appropriate:

i. Introduce another variable λ, and portray another limit L

The limit L is known as the "Lagrangian", and the new factor λ, is implied as "Lagrangian multiplier".

By the day's end, find the essential characteristics of L.

iii. Consider each game plan, which will take after (X0, Y0, … , λ0). Connection each one into f, whichever one gives the best or tiniest worth is the best or least point we are searching for. According to Cobb and Douglas, (1928), creation work is given by Q = AKaLb where 'a' and 'b' are positive bits. Q = $AK^aL^{(1-a)}$.

Re-write the Cobb – Douglas production model as:

$$A = k\ B^{1-a}\ D^{a} = F\ (B\ ,D) \qquad (2.20)$$

where,

$$_{p}F\ (B\ ,D) -_{r} B -_{q} D \qquad (2.21)$$

The first order conditions for an interior maximum are;

$$_{p}F_{b}\ (B^{*}\ ,D^{*}) = r \qquad (2.22)$$

$$_{p}F_{d}\ (B^{*}\ ,D^{*}) = q \qquad (2.23)$$

Where F_b and F_d is the partial derivatives of F with respect B and D respectively.

Accepting the small amount of yield paid to Labor is a steady 'a'. for Cobb and Douglas they picked . This is composed as:

$$(1-a)_{p}F(B^{*},D^{*}) = rB \qquad (2.24)$$

$$ {}_{ap}F\left(B^*,D^*\right)=qD \tag{2.25} $$

$$ \frac{1}{B^*}=\frac{F_b\left(B^*,D^*\right)}{(1-a)F\left(B^*,D^*\right)_p} \tag{2.26} $$

Given that,

$$ \frac{d}{d\ x}\ \ln\left(f(x)\right)=\frac{f'\ (x)}{f\ (x)} \tag{2.27} $$

Using chain rule equation (2.26) becomes,

$$ \frac{\partial}{\partial B}\ln F=\frac{F_b}{F}=\frac{1-a}{B^*} \tag{2.28} $$

Similarly,

$$ \frac{\partial}{\partial D}\ln F=\frac{a}{D^*} \tag{2.29} $$

Thus, we have eliminated p, q and r. The above equations holds for every (B*,D*) that can result as a profit maximum in R^2. We treat equation (2.28) and (2.29)

$$ \int\frac{1}{x}=ln\,(x)+c, \tag{2.30} $$

$$ \ln F\ (B,D)=(1-\alpha)\ln B+g\ (D)\ +c \tag{2.31} $$

where g(D) is a constant of integration that may depend on D: and

$$ \ln F\ (B,D)=\ \alpha\ \ln\ D+h\ \ (B)\ +c \tag{2.32} $$

$$\ln F(B,D) = (1-\alpha)\ln B + \alpha \ln D + C \qquad (2.33)$$

taking the exponential of equation (2.33) and letting

$$k = e^{c}$$

$$F(B,D) = kB^{1-a}D^{a} \qquad (2.34a)$$

The Cobb – Douglas Production function can also be written as

$$P = kL^{a}C^{b} \qquad (2.34b)$$

where,

P = Total creation (that is the financial worth of all products delivered in a year)

K = Total factor efficiency

L = Labor input

C = Capital information

a and b are the work and a lot of yield individually. These qualities are constants and dictated by the avalaible innovation.

In the event that a + b = 1, get back to scale are consistent

In the event that a + b < 1, get back to scale are diminishing

In the event that a + b > 1, get back to scale are expanding.

Get back to scale is a specialized property of production that check changes in yield resulting to a corresponding change in all data sources (Mervin *et, al*., 2011).

2.7.6 Budgetary Constraints Optimization Problem

As per Khan Academy (2016), we can manage the budgetary objectives issue for a mechanical office passing on gadget that requires steel as a crude material. The expense are human work, which is \$20 consistently for their laborers and the steel for production is \$170 per ton. The compensation (Revenue) R is displayed by the condition (2.35) under,

$$R(h, s) = 200\, h^{\frac{2}{3}}\, s^{\frac{1}{3}} \tag{2.35}$$

Where, h = hours of labour and S = tons of steel.

The \$20 each hour work and the \$170 per ton steel cost disclose to us that the absolute expense of creation as far as h and S, is given as,

$$20\,h + 170\,s \tag{2.36}$$

$$20\,h + 170\,s = 20{,}000 \tag{2.37}$$

The Lagrangian function for the optimisation problem above is as follows;

$$L(h,S,\lambda) = 200h^{2/3}S^{1/3} - \lambda(20h + 170S - 20{,}000) \tag{2.38}$$

2.7.7 Maximum Theoretical Ethanol Yield.

Konstantinos *et. al*., (2019) used Cotton stalks (CS) in the formation of fuel-ethanol due to its abundance and high carbon content, it was observed that, direct transformation without pretreatment, reliably achieves incredibly low yields due to the obstinate nature of lignocelluloses. They contemplated the impact of different substance and physicochemical pretreatment techniques, i.e., salt, microwave-helped ruinous, organosolv, liquid treatment, and ceaselessly organosolv and watery pretreatment, on compound piece of cotton tail and following ethanol creation applying pre-hydrolysis and concurrent saccharification and developing (PSSF) at high strong stacking. It was viewed as that to be best outcomes the degree that ethanol creation were developed by the reformist mix of organosolv and watery pretreatment (32.3 g/L, utilizing 15% w/v substrate fixation and 6 h pre-hydrolysis) with an improvement of 32% to half in ethanol creation separated from the other pretreatments. Slackening up pre-hydrolysis time to 14 h and stretching out substrate place to 20% w/v, ethanol creation displayed at 47.0 g/L (standing out from an ethanol yield of 52%) after 30 h of advancement. They analyzed the Parameters that impacted the presentation of PSSF which were substrate fixation and length of pre-hydrolysis time.

The following conditions were used for the assessment (Konstantinos *et. al*., 2019).

1. Glucose potential ,

$$\lfloor G_{Potential} \rfloor = f \times (S_0) \times 1.111 \tag{2.39}$$

2. Maximum theoretical ethanol

$$(\lfloor EtOH_{theoretical} \rfloor) = 0.511 \times f \times (S_0) \times .1.111 \tag{2.40}$$

3. Hydrolysis yield

$$HY\ (\%) = \frac{[Glucose_1 - Glucose_0]}{f \times (S_0) \times 1.111} \qquad (2.41)$$

4. Ethanol yield

$$TEY(\%) = \frac{[EtOH_t - EtOH_0]}{0.511.\times f \times (S_0) \times 1.111} \qquad (2.42)$$

2.8 Karush-Kuhn-Tucker (KKT) Necessary and Sufficient Conditions

The KKT condition is a numerical streamlining first request essential condition (first order condition) otherwise called Kuhn-Tuker conditions for the ideal arrangement of a nonlinear programming issue. In nonlinear programming KKT hypothesis likewise alluded to as seat point hypothesis utilizes the strategy for Lagrange multipliers on the balance compelled issue to change the obliged enhancement issue over to an unconstrained issue to acquire a nearby as well as worldwide greatest (least) in the space. As indicated by Taha (2010), the KKT condition give the most binding together hypothesis to all nonlinear programing issues.

KKT condition can be applied by acquiring the second request halfway subsidiary of the inside greatest for every one of the variable which is addressed as a framework. The arrangement will be a most extreme, least or a seat point utilizing the strategy for head minor determinant which is the essential and adequate conditions for optimality (Taha, 2010). To foster the overall Karush-Kuhn Tucker (KKT) important conditions for deciding the fixed focuses which are additionally adequate under specific guidelines, Taha, (2010) thinks about the issue of the structure,

$$Optimize: Z = f(U) \tag{2.43}$$

$$Subject\ to: g(U) \leq 0 \tag{2.44}$$

The inequality constraint may be converted into equality constraint equations by using nonnegative slack variables let S_i^2 (≥ 0) be the slack quantity added to the ith constraint $g_i(U) \leq 0$ and define.

$$S = (S_1, S_2 \ \ldots, Sm)^T, S^2 = \left(S_1{}^2, S_2{}^2 \ \ldots, Sm^2\right)^T \tag{4.45}$$

Where m is the number of inequality constraints. The Lagrangean function is thus given by,

$$L\ (S, U, \lambda) = f(U) - \lambda\left(g(U) + S^2\right) \tag{2.46}$$

Given the constraints,

$$g(U) \leq 0 \tag{2.47}$$

A necessary condition for optimality is that λ be nonnegative (nonpositive) for maximization (minimization) problems. This result is justified by noting that the vector λ measures the rate of variation of with respect to g…that is,

$$\lambda = \frac{\partial f}{\partial g} \tag{2.48}$$

In maximization case, as the right-hand side of the constraint $\mathrm{g(U)} \leq 0$ increases from 0 to the vector ∂g, the solution space becomes less constraint and hence f cannot decrease, meaning that $\lambda \geq 0$. Similarly, for minimization, as the right-hand side of the constraints increases, f cannot increase, which implies that $\lambda \leq 0$. If the care equalities, that is, $g(U) = 0$, then λ becomes unrestricted in sign.

The restrictions on λ holds as part of the KKT necessary conditions. The other conditions are developed as follows,

Taking the partial derivatives of L with respect to U, S, and λ, we obtain,

$$\frac{\partial L}{\partial X} = \nabla f(U) - \lambda \qquad \forall \nabla g(U) = 0 \tag{2.49}$$

$$\frac{\partial L}{\partial S_i} = -2 \lambda_i S_i = 0, \; i = 1,2,3, \ldots, m \tag{2.50}$$

$$\frac{\partial L}{\partial \lambda} = -\left(g(U) + S^2\right) = 0 \tag{2.51}$$

These sets of equations reveal the results as follows (Taha, 2010; Rardin, 1998.):

1. If $\lambda_i = 0$, then S_i^2 this means that the corresponding resource is scares, and hence it is consumed completely (inequality constraint).
2. If $S_i^2 > 0$, then $\lambda_i = 0$. This means resources, i is not scarce and, consequently it has no effect on the value of f $\left(i.e.\, \lambda_i = \frac{\partial y}{\partial u} = 0\right)$

   From equations (2.49) and (2.51), we obtain,

$$\lambda_i \, g(U) = 0 \qquad i = 1, 2, ,\ldots, m \tag{2.52}$$

This new condition essentially repeats the forgoing arguments, because if

$\lambda > 0$, then $g(U) = 0$, or $S_i^2=0$ and if $\lambda < 0$ then $g(U) < 0$, and $S_i^2>0$ and $\lambda = 0$

$\lambda > 0$ then $g(U) = 0$ or $S_i^2 = 0$ and if then $g(U) < 0$ and $S_i^2 > 0$ and $\lambda = 0,$

According to Taha (2003) the KKT necessary conditions for maximization problem are summarized as:

$$\lambda \geq 0 \qquad (2.53)$$

$$\nabla f(U) - \lambda \nabla g(U) = 0 \qquad (2.54)$$

$$\lambda_i \; g_i = 0, \; i = 1, 2, ,\ldots, m \qquad (2.55)$$

$$g(U) \leq 0 \qquad (2.56)$$

These conditions apply to minimization case as well, except that λ must be non-positive. In both maximization and minimization, the Lagrange multipliers corresponding to equality constraints are restricted in sign.

On the Sufficiency of the KKT Conditions, the Kuhn-Tucker necessary conditions are also sufficient if the objective function and the solution space satisfy specific conditions (Taha, 2010). These conditions are summarized in Table 2.1.

It is simpler to verify that a function is convex or concave than to prove that a solution space is a convex set. For this reason, we provide a list of conditions that are easier to apply in practice in the sense that the convexity of the solution space can be established by checking the convexity or concavity of the constraint functions. To provide these conditions, we define the generalized nonlinear problems as,

$$Maximize\ or\ minimize\ \ Z = f\,(U) \qquad (2.57)$$

$$Subject\ to \qquad g_i\,(U) \le\ 0, \qquad i = 1, 2, ..., r \qquad (2.58)$$

$$g_i\,(U) \ge\ 0, \qquad i = 1, 2, ..., p \qquad (2.59)$$

$$g_i\,(U) =\ 0, \qquad i = 1, 2, ..., m \qquad (2.60)$$

$$L(U, S, \lambda) = f\,(U) - \sum_{i=1}^{t} \lambda_i \left(g_i\,(U) + S_i^2 \right) - \sum_{i=r+1}^{p} \lambda_i \left(g_i\,(U) - S_i^2 \right) - \sum_{i=p+1}^{m} \lambda_i \left(g_i (U) \right) \quad (2.61)$$

where λi is the Lagrangean multiplier associated with constraint i (Bazarra *et al.* (1993; Taha, 2010). The conditions for establishing the sufficiency of the KKT conditions are summarized in Table 2.2.

The conditions in Table 2.2 represent only a subset of the conditions in Table2.1

because a solution space may be convex without satisfying the conditions in Table 2.2.

Table 2.1: Required Conditions for Concave and Convex Function

Sense of Optimization	Objective Function	Solution Space
Minimization	Concave	Convex Set
Maximization	Convex	Convex Set

Table 2.2: Required Conditions for Establishing the Sufficiency of the KKT Conditions are summarized

Sense of Optimisation	Objective Function	Solution Space	Constraints	average
maximization	Concave	Convex	≥ 0	(1 ≤ i ≤0)
		concave	≤ 0	(r +1 ≤ i ≤p)
		Linear	Unrestricted	(p + 1 ≤ i ≤ m)
minimization	Convex	Convex	≤ 0	(1 ≤ i ≤0)
		concave	≥ 0	(r +1 ≤ i ≤p)
		Linear	Unrestricted	(p + 1 ≤ i ≤ m)

Table 2.2 is valid because the given conditions yield a concave Langrangean function L(U, S, λ) in case of maximization and a Convex L(U, S, λ) in case of minimization. This result is verified by noticing that if g(U) is convex, then λg(U) is convex, if $\lambda_i \geq 0$ and concave if $\lambda_i \leq 0$. Similar interpretations can be established for all the remaining conditions. A linear function is both convex and concave. If a function f is concave, then (-f) is convex, and vice versa (Beightler *et al.*1979; Taha, 2010).

Given a problem of the form

$$Min: \ f\ (U) \tag{2.62}$$

$$s.t = \ g_i - b_i \geq 0 \qquad i = 1, \ldots, k \tag{2.63}$$

$$g_i - b_i = 0 \qquad i = k + 1, \ldots, m \tag{2.64}$$

There are four KKT necessary conditions model optimality

1. Feasibility

   $$g_i\ (U) - b_i \tag{2.65}$$

   Equation (2.65) is feasible (applies to 2.63 and 2.64)

2. No direction which improves objective and feasibility,

   $$\nabla\ f\ (U)^* - \sum_{i=1}^{m} \lambda_i^*\ g_i\ (U^*) = 0 \tag{2.66}$$

   Equation (2.66) applies to 2.62, 2.63 and 2.64

3. Complimentary slackness,

   $$\lambda_i^*\ (g_i(U^*) - b_i\ ) = 0 \qquad i = 1, \ldots.\ k \tag{2.67}$$

   Equation (2.67) applies to 2.63,

4. Positive Langrange multipliers

   $$\lambda_i \geq 0 \qquad i = 1,\ \ldots,\ k \tag{2.68}$$

   Equation (2.68) applies to 2.63,

CHAPTER THREE

3.0 MATERIAL AND METHODS

3.1 Research Methodology

The present work improved on the work of Olaoye and Kolawole (2013), Konstantinos *et. al.,* (2019) and Khan Academy (2016) that optimized the various parameters such as time, effect of pretreatment and hydrolysis rate on the concentration of biomass as well as the parameters in the fermentation of the resulting fermentable sugars, such parameters include fermentation time and fermentation rate on ethanol yield, substrate concentration. Overall, the work is expected to bring out parameters that could leads to batch and continuous reactors for commercialization of bioethanol production from banana trunk biomass.

3.2 Kinetics of Biomass Hydrolysis

Figure 3.1 is a flow Chart showing step by step procedure of producing ethanol from biomass.

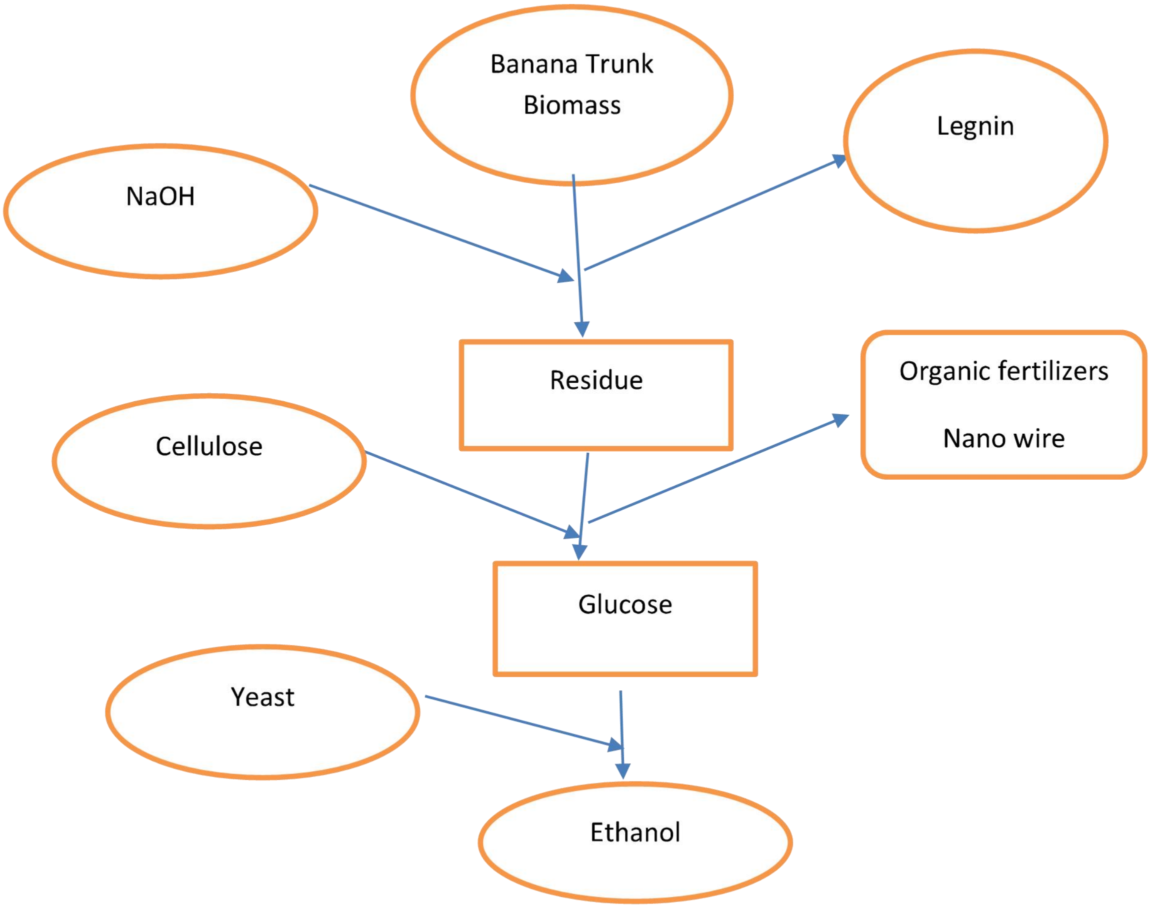


Figure 3.1: A Flow Chart for Ethanol Production from Banana Trunk Biomass

3.3 Mathematical Model Formulation

The following models were developed in order to achieve the set objections:

From the flow diagram, particular mathematical models were used to study the kinetics of ethanol production. At the end, the mathematical model that best portrays the transformation of banana trunk biomass to ethanol was established. Optimisation model was formulated for the production of ethanol from banana trunk biomass, to evaluate the financial, economic, business and commercial relevance of the production process.

In the making of ethanol from banana trunk biomass a pattern of consecutive reactions was consider where the consequence of one reaction transforms into the reactant for various reactions (Olaoye and Kolawole, 2013).

3.3.1 CASE 1:

Conversion of banana trunk biomass to ethanol without pretreatment

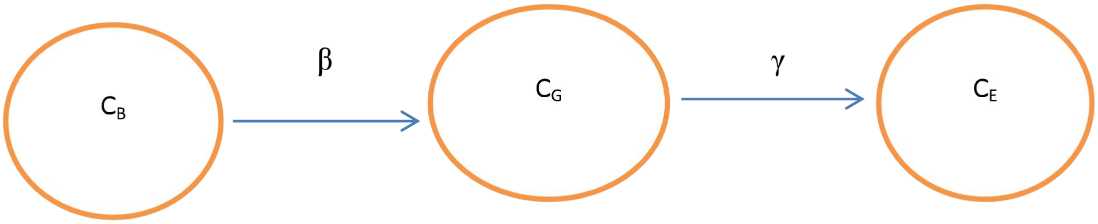


Figure 3.2. Model of Conversion of Banana Trunk Biomass to Ethanol without Pretreatment

C_B is the main reactant (Banana trunk Biomass) which gives the product C_G (Glucose) at the rate β , the product C_G formed becomes the reactant for the next reactions which undergoes fermentation to give the substance C_E (ethanol) at the rate γ. The kinetics of such a reaction is determined by the slowest step to the above reaction.

If the first reaction is faster than the second, the kinetics will be determined by the second reaction. In general, if a reaction occurs in series, the slowest step will be the rate determining reaction.

Considering the reaction in Figure 3.2, where;

C_B = Concentration of Banana trunk biomass

C_G = Concentration of Glucose

C_E = Concentration of Ethanol

β = rate kinetics constant of the formation of C_G from C_B

γ = rate kinetics constant of the formation of C_E from C_G

The ODE$_s$ that describe the rate of change of each reactant with time is an initial value problem written as,

$$\frac{d\,C_G}{d\,t} = \beta\, C_B - \gamma\, C_G \qquad C_G(0) = 0 \tag{3.2}$$

$$\frac{d\,C_B}{d\,t} = -\,\beta\, C_B \qquad C_B(0) = C_B \tag{3.1}$$

$$\frac{d\,C_E}{d\,t} = -\,\gamma\, C_G \qquad ,\quad C_E(0) = 0 \tag{3.3}$$

where,

C_B = Concentration of Banana trunk biomass

C_G = Concentration of Glucose

C_E = Concentration of Ethanol

From equation (3.1), we have

$$\frac{d\,C_B}{C_B} = -\beta\, d\, t \tag{3.4}$$

Solving equation (3.4) we obtain;

$$C_B\,(t) \;= C_{B_0}\, e^{-\beta\, d\, t} \tag{3.5}$$

Substituting equation (3.5) into equation (3.2), we have,

$$\frac{d\,C_G}{d\,t} + \gamma\; C_G = \;\beta\, C_{B_0}\, e^{-\beta t} \tag{3.6}$$

Solving equation (3.6) we obtain;

$$C_G(t) = \frac{\beta\, C_{B_0}}{(\gamma-\beta)}\left(e^{-\beta t} \;-\; e^{-\gamma t}\right) \;+ C\, e^{-\gamma\, t} \tag{3.7}$$

$at\ t=0,\ C_G(0)=C=0 \Rightarrow C=0\ \ therefore,$

$$C_G(t) = \frac{\beta\, C_{B_0}}{(\gamma-\beta)}\;\left(e^{-\beta t} - e^{-\gamma t}\right) \qquad\qquad \forall\ \beta\ \neq\ \gamma \tag{3.8}$$

Substitute Equation (3.8) into Equation (3.3), we have,

$$\frac{dC_E}{dt} = \frac{\beta \gamma C_{B_0}}{(\gamma-\beta)} \left(e^{-\beta t} - e^{-\gamma t}\right) \tag{3.9}$$

Solving equation (3.9), we obtain,

$$C_E(t) = \frac{\beta \gamma C_{B_0}}{(\gamma-\beta)} \left(\frac{1}{\gamma} e^{-\gamma t} - \frac{1}{\beta} e^{-\beta t}\right) + C_1 \tag{3.10}$$

at t=0,

$$\Rightarrow C_1 = \frac{\beta C_{B_0}}{(\gamma-\beta)} \left(\frac{1}{\beta} - \frac{1}{\gamma}\right) \tag{3.11}$$

therefore,

$$C_E(t) = \frac{\beta \gamma C_{B_0}}{(\gamma-\beta)} \left(\frac{1}{\gamma} e^{-\gamma t} - \frac{1}{\beta} e^{-\beta t} + \left(\frac{1}{\beta} - \frac{1}{\gamma}\right)\right) \quad \forall\, \beta \neq \gamma \tag{3.12}$$

Boundary assessment technique (parameter estimation method) was utilized to decide the convergence of the biomass, glucose yield and ethanol created at various time utilizing Maple 17 softwaer.

3.3.2 CASE 2:

Model of conversion of banana trunk biomass to ethanol with NaOH pretreatment.

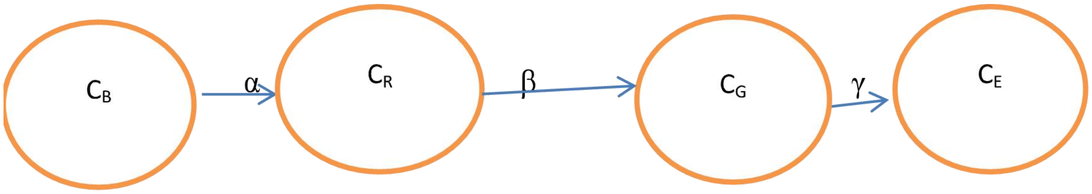


Figure 3.3 Model of Conversion of Banana Trunk Biomass to Ethanol with NaOH Pretreatment.

C_B is the main reactant (banana trunk biomass) which gives the product C_R (Residue) at the rate constant α, the product "C_R" formed becomes the reactant for the next reactions which react to give C_G (glucose) at a rate constant β, the glucose formed reacts in the present of yeast to give the substance C_E (ethanol) with rate constant γ, the kinetics of such reaction is determined by the slowest step to the above reaction.

If the first and second reactions are faster than the third, then the kinetics will be determined by the third reaction. In general, if a reaction occurs in series the slowest step will be the rate determining reaction.

The ODEs that describe the above process is given by,

$$\frac{dC_B}{dt} = -\alpha C_B \qquad C_B(0) = C_{B_0} \tag{3.13}$$

$$\frac{dC_R}{dt} = \alpha C_B - \beta C_R \qquad C_R(0) = 0 \tag{3.14}$$

$$\frac{dC_G}{dt} = \beta C_R - \gamma C_G \qquad C_G(0) = 0, \tag{3.15}$$

$$\frac{d\,C_E}{d\,t} = -\gamma\,C_G \qquad\qquad C_E(0) = 0, \tag{3.16}$$

From equation (3.13), the rate of change of biomass with time is given by,

$$C_B(t) = C_{B_0}\,e^{\alpha\,t} \tag{3.17}$$

Equation (3.17) is the concentration of biomass at time t.

From equation (3.14), the concentration of residue at time t, becomes;

$$\frac{d\,C_R}{d\,t} + \beta\,C_R = \alpha\,C_{B_0}\,e^{-\alpha t} \tag{3.18}$$

Solving equation (3.18), we have,

$$C_R(t) = \frac{\alpha\,C_{B_0}}{(\beta-\alpha)}\left(e^{-\alpha t} - e^{-\beta t}\right) + C\,e^{-\beta t} \tag{3.19}$$

$$C_R(0) = C_2 = 0 \Rightarrow C_2 = 0 \tag{3.20}$$

Therefore,

$$C_R(t) = \frac{\alpha\,C_{B_0}}{(\beta-\alpha)}\left(e^{-\alpha t} - e^{-\beta t}\right) \qquad\qquad \forall\,\alpha \neq \beta \tag{3.21}$$

Equation (3.21) is the concentration of residue at time t.

From equation (3.15) the rate of change of glucose with time becomes;

$$\frac{d\,C_G}{d\,t} + \gamma\, C_G = \frac{\alpha\,\beta\,C_{B_0}}{(\beta-\alpha)}\left(e^{-\alpha t} - e^{-\beta t}\right) \qquad (3.22)$$

Solving equation (3.22), we have,

$$C_G(t) = \int_0^t \frac{\alpha\,\beta\,C_{B_0}}{(\beta-\alpha)}\left(e^{(\gamma-\alpha)x} - e^{(\gamma-\beta)x}\right)d\,x + C_3\,e^{-\gamma t} \qquad (3.23)$$

$$C_G(t) = \frac{\alpha\,\beta\,C_{B_0}}{(\beta-\alpha)}\left(\frac{\left(e^{-\alpha t} - e^{-\gamma t}\right)}{(\gamma-\alpha)} - \frac{\left(e^{-\beta t} - e^{-\gamma t}\right)}{(\gamma-\beta)}\right) + C_3\,e^{-\gamma t} \qquad (3.24)$$

$$C_G\,(0) = C_3 = 0 \;\Rightarrow\; C_3 = 0 \qquad (3.25)$$

Therefore,

$$C_G\,(t) = \frac{\alpha\,\beta\,C_{B_0}}{(\beta-\alpha)}\left(\frac{\left(e^{-\alpha t} - e^{-\gamma t}\right)}{(\gamma-\alpha)} - \frac{\left(e^{-\beta t} - e^{-\gamma t}\right)}{(\gamma-\beta)}\right) \quad \forall\ \alpha \neq \beta \neq \gamma \quad (3.26)$$

Equation (3.26) is the concentration of glucose at time t.

From (3.16), the rate of production of ethanol with time becomes,

$$\frac{d\,C_E}{d\,t} = \frac{\alpha\,\beta\,\gamma\,C_{B_0}}{(\beta-\alpha)}\left(\frac{e^{-\alpha t} - e^{-\gamma t}}{(\gamma-\alpha)} - \frac{e^{-\beta t} - e^{-\gamma t}}{(\gamma-\beta)}\right) \qquad (3.27)$$

Solving equation (3.27), we have,

$$C_E(t)=\frac{\alpha\beta\gamma C_{B_0}}{(\beta-\alpha)}\left(\frac{e^{-\gamma t}}{\gamma(\gamma-\alpha)}-\frac{e^{-\alpha t}}{\alpha(\gamma-\alpha)}+\frac{e^{-\beta t}}{\beta(\gamma-\beta)}-\frac{e^{-\gamma t}}{\gamma(\gamma-\beta)}\right)+C_4 \qquad (3.28)$$

At t=0

$$\Rightarrow \quad C_4=\frac{\alpha\beta\gamma C_{B_0}}{(\beta-\alpha)}\left(\frac{1}{\alpha(\gamma-\alpha)}-\frac{1}{\gamma(\gamma-\alpha)}+\frac{1}{\gamma(\gamma-\beta)}-\frac{1}{\beta(\gamma-\beta)}\right) \qquad (3.29)$$

Therefore,

$$C_E(t)=\frac{\alpha\beta\gamma C_{B_0}}{(\beta-\alpha)}\left(\frac{\left(e^{-\gamma t}-1\right)}{\gamma(\gamma-\alpha)}-\frac{\left(e^{-\alpha t}-1\right)}{\alpha(\gamma-\alpha)}+\frac{\left(e^{-\beta t}-1\right)}{\beta(\gamma-\beta)}-\frac{\left(e^{-\gamma t}-1\right)}{\gamma(\gamma-\beta)}\right)$$

$$\forall\, \alpha \neq \beta \neq \gamma \qquad (3.30)$$

Equation (3.30) is the concentration of ethanol at time t.

From the process in equation (3.17), (3.21), (3.26) and (3.30), we obtain the total differential. Appropriate differentiation and substitution of the particular solution gives: α, β and γ. Simulations and graphs for the batch process are shown in chapter 4.

3.3.3 CASE 3:

Model for the conversion of banana trunk biomass to ethanol without pretreatment Continuous Process

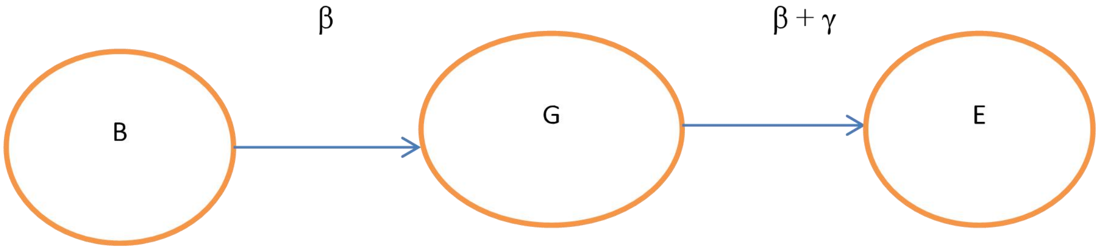


Figure 3.4. Model of Conversion of Banana Trunk Biomass to Ethanol without Pretreatment Continuous process

1. Unpretreated

$$\frac{dB}{dt} = -\beta B \qquad B(0) = B_0 \qquad (3.31)$$

$$\frac{dG}{dt} = \beta B - (\beta + \gamma) G \qquad G(0) = 0 \qquad (3.32)$$

$$\frac{dE}{dt} = (\beta + \gamma) G \qquad E(0) = 0 \qquad (3.33)$$

Concentration of biomass at time t time the continuous process unpretreated biomass is given in equation (3.34).

$$B(t) = B_0 e^{-\beta t} \tag{3.34}$$

Concentration of glucose at time t for the continuous process unpretreated biomass is given in equation (3.35).

$$G(t) = \left(\frac{\beta B_0 e^{-\gamma t}}{\gamma} - \frac{\beta B_0}{\gamma} \right) e^{-(\beta+\gamma)t} \tag{3.35}$$

Concentration of ethanol at time t for the continuous process unpretreated biomass is given in equation (3.36),

$$E(t) = -\frac{\beta B_0 (\beta+\gamma)\left(\frac{e^{-\beta t}}{\beta} + \frac{e^{(\beta+\gamma)t}}{-\beta-\gamma} \right)}{\gamma} + \frac{\beta B_0 (\beta+\gamma)\left(\frac{1}{\beta} + \frac{1}{-\beta-\gamma} \right)}{\gamma} \tag{3.36}$$

3.3.4 CASE 4:

Model of Conversion of banana trunk biomass to ethanol with NaOH pretreatment continuous process.

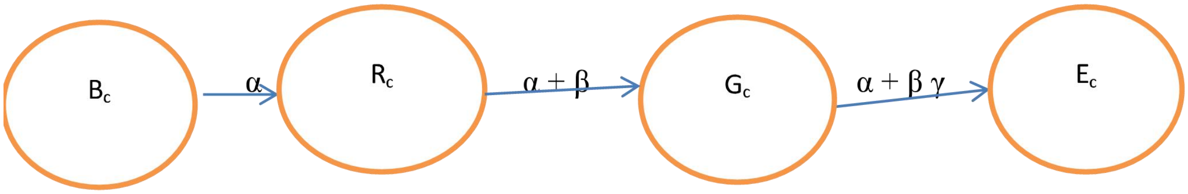


Figure 3.5 Model of Conversion of Banana Trunk Biomass to Ethanol with NaOH Pretreatment Continuous process

2. Pretreated

$$\frac{dB_c}{dt} = -\alpha B_c \qquad B_c(0) = B_0 \qquad (3.37)$$

$$\frac{dR_c}{dt} = \alpha B_c - (\alpha + \beta) R \qquad R_c(0) = 0 \qquad (3.38)$$

$$\frac{dG_c}{dt} = (\alpha + \beta) R_c - (\alpha + \beta + \gamma) G_c \qquad G_c(0) = 0 \qquad (3.39)$$

$$\frac{dE_c}{dt} = (\alpha + \beta + \gamma) G_c \qquad E_c(0) = 0 \qquad (3.40)$$

Concentration of biomass at time t for the continuous process pretreated biomass is given in equation (3.41)

$$B_c(t) = B_0 e^{-\alpha t} \qquad (3.41)$$

Concentration of residue at time t for the continuous process pretreated biomass is given in equation (3.42),

$$R_c(t) = \left(\frac{\alpha B_0 e^{-\beta t}}{\beta} - \frac{\alpha B_0}{\beta} \right) e^{-(\alpha + \beta) t} \qquad (3.42)$$

Concentration of glucose at time t for the continuous process pretreated biomass is given in equation (3.43),

$$G_c(t) = \left[\frac{\alpha B_0(\alpha+\beta)\left(\frac{e^{(\gamma+\beta)t}}{(\gamma+\beta)} - \frac{e^{\gamma t}}{\gamma}\right)}{\beta} - \frac{\alpha B_0(\alpha+\beta)\left(\frac{1}{(\gamma+\beta)} + \frac{1}{\gamma}\right)}{\beta} \right] e^{-(\alpha+\beta+\gamma)t} \qquad (3.43)$$

Concentration of ethanol at time t for the continuous process pretreated biomass is given in equation (3.44),

$$E_c(t) = -\frac{1}{\beta(\gamma+\beta)\gamma}\left[B_0(\alpha+\beta)(\alpha+\beta+\gamma)\left(\begin{array}{c} \frac{2\gamma e^{-(\alpha+\beta+\gamma)t}}{-\alpha-\beta-\gamma} + \frac{\beta e^{-(\alpha+\beta+\gamma)t}}{-\alpha-\beta-\gamma} \\ -\frac{\gamma e^{-(\alpha+\beta)t}}{-\alpha-\beta} + \frac{\gamma e^{-\alpha t}}{\alpha} - \frac{\beta e^{-(\alpha+\beta)t}}{-\alpha-\beta} \end{array} \right) \right]$$

$$+\frac{1}{\beta(\gamma+\beta)\gamma}\left[B_0(\alpha+\beta)(\alpha+\beta+\gamma)\left(\begin{array}{c} \frac{2\gamma}{-\alpha-\beta-\gamma} + \frac{\beta}{-\alpha-\beta-\gamma} \\ -\frac{\gamma}{-\alpha-\beta} + \frac{\gamma}{\alpha} - \frac{\beta}{-\alpha-\beta} \end{array} \right) \right] \qquad (3.44)$$

3.4 Revenue Model Formulation

Appling theorem 2.1, to alter an income (revenue) model to deal with the costing with a heuristic methodology for the creation of ethanol from biomass. The procedure of heuristic joined Lagrange multiplier that includes local search with budgetary-constraints was utilized to formulate Lagrange cost model as multi – level, multi – item capacitated income generation. We simplified the Lagrange multiplier by utilizing proxy sub inclination (sub-gradient) technique that guarantees the intermingling of the estimated solution. An achievable result of the first question is developed from the solution of the Lagrange multiplier problem at every cycle which is subsequently improved by neighborhood search that changes the value of at least one of the factors at each time (Khan Academy, 2016).

Our budgetary limitations for running a plant producing ethanol that requires Banana Trunk Biomass as a crude material; our expense is overwhelmingly human work which is ₦ 120 every hour for laborers, cost of social occasion introductory Biomass is ₦ 500, price of Sodium Hydroxide for the pretreatment is ₦ 40 to acquire the filtrate (Residue) and worth of yeast for fermentation of glucose is ₦ 70, delivered from the biomass per 2.5L of ethanol. We need to boost (maximize) the income R, by adjusting the Cobb Douglass model, to have,

$$R(h,b,r,g) = k\,h^{m}\,b^{n}\,r^{p}\,g^{q}$$
$$\forall\, m,n,p,q \in x, \forall\, x \in Y, \mathit{such\ that}\ (0 < Y < 1) \tag{3.45}$$

Assuming our budget is ₦ 4000 per 2.5 L, what will be our revenue? This budget can be translated to a constraint give in (3.46),

$$120\,h + 500b + 40\,r + 70\,g = 4{,}000\ \ \forall\ \ h, b, r, g > 0 \tag{3.46}$$

where,

R is revenue,

K is a constant

h is hour of labor,

b is banana trunk biomass / Cost of obtaining the Biomass.

r is residue, / Cost of Sodium Hydroxide for the pretreatment process, (residue to glucose)

g is glucose, / Cost of Yeast for the Fermentation Process (glucose to ethanol)

i. Maximizing the Function

In order to demonstrate our model, by heuristics; we let

$m = \frac{3}{4}$, $n = \frac{1}{2}$, p= $\frac{1}{6}$, $q = \frac{1}{12}$. $k = 200$

$$\left.\begin{array}{ll} Max: & R(h,b,r,g) = 200h^{\frac{3}{4}} b^{\frac{1}{2}} r^{\frac{1}{6}} g^{\frac{1}{12}} \\ Subject\ to: & 120h + 500b + 40r + 70g = 4{,}000 \qquad \forall\, h,b,r,g > 0 \end{array}\right\} \qquad (3.47)$$

Taking the Lagrange function of the formulated model in (3.47), we have an unconstrained problem as,

$$L\ (h,b,r,g) = 200h^{\frac{3}{4}} b^{\frac{1}{2}} r^{\frac{1}{6}} g^{\frac{1}{12}} - \lambda\ (120h + 500b + 40r + 70g - 4{,}000) = 0 \quad \forall\, h,b,r,g > 0 \quad (3.48)$$

ii. Model Solution

We set the gradient L equal to 0. This is the same as setting each partial derivative equal to 0 as follows

Differentiating with respect to h, we have:

$$\frac{\partial}{\partial h}\left(200h^{\frac{3}{4}} b^{\frac{1}{2}} r^{\frac{1}{6}} g^{\frac{1}{12}} - \lambda\ (120h + 500b + 40r + 70g - 4{,}000) \right) = 0 \quad (3.49)$$

We obtained,

$$\left(\frac{600}{4}\right) h^{\left(\frac{-1}{4}\right)} b^{\frac{1}{2}} r^{\frac{1}{6}} g^{\frac{1}{12}} - \lambda 120 = 0 \qquad (3.50)$$

Differentiating with respect to b, we have:

$$\frac{\partial}{\partial b}\left(200h^{\frac{3}{4}}b^{\frac{1}{2}}r^{\frac{1}{6}}g^{\frac{1}{12}} - \lambda\left(120h + 500b + 40r + 70g - 4{,}000\right)\right) = 0 \qquad (3.51)$$

We obtained,

$$\left(\frac{200}{2}\right)h^{\frac{3}{4}}b^{\left(-\frac{1}{2}\right)}r^{\frac{1}{6}}g^{\frac{1}{12}} - \lambda 500 = 0 \qquad (3.52)$$

Differentiating with respect to r, we have:

$$\frac{\partial}{\partial r}\left(200h^{\frac{3}{4}}b^{\frac{1}{2}}r^{\frac{1}{6}}g^{\frac{1}{12}} - \lambda\left(120h + 500b + 40r + 70g - 4{,}000\right)\right) = 0 \qquad (3.53)$$

We obtained,

$$\left(\frac{200}{6}\right)h^{\frac{3}{4}}b^{\frac{1}{2}}r^{\left(\frac{-5}{6}\right)}g^{\frac{1}{12}} - \lambda 40 = 0 \qquad (3.54)$$

Differentiating with respect to g, we have:

$$\frac{\partial}{\partial g}\left(200h^{\frac{3}{4}}b^{\frac{1}{2}}r^{\frac{1}{6}}g^{\frac{1}{12}} - \lambda\left(120h + 500b + 40r + 70g - 4{,}000\right)\right) = 0 \qquad (3.55)$$

We obtained,

$$\left(\frac{200}{12}\right)h^{\frac{3}{4}}b^{\frac{1}{2}}r^{\frac{1}{6}}g^{\left(\frac{-11}{12}\right)} - \lambda 70 = 0 \qquad (3.56)$$

Differentiating with respect to λ, we have:

$$\frac{\partial}{\partial \lambda}\left(200h^{\frac{3}{4}}b^{\frac{1}{2}}r^{\frac{1}{6}}g^{\frac{1}{12}}-\lambda\left(120h+500b+40r+70g-4{,}000\right)\right)=0 \qquad (3.57)$$

We obtained,

$$-120-500b-40r-70g+4000=0 \qquad (3.58)$$

The maximum revenue can be obtained, using the relationship below,

$$R(h,b,r,g)=200h^{\frac{3}{4}}b^{\frac{1}{2}}r^{\frac{1}{6}}g^{\frac{1}{12}} \qquad (3.59)$$

Solving equations3.50, 3.52, 3.54, 3.56 and 3.58 simultaneously, we have,

$$\left.\begin{array}{l}
\frac{600}{4}h^{\frac{-1}{4}}b^{\frac{1}{2}}r^{\frac{1}{6}}g^{\frac{1}{12}}-120\lambda=0\\
\frac{200}{2}h^{\frac{3}{4}}b^{\frac{-1}{2}}r^{\frac{1}{6}}g^{\frac{1}{12}}-500\lambda=0\\
\frac{200}{6}h^{\frac{3}{4}}b^{\frac{1}{2}}r^{\frac{-5}{6}}g^{\frac{1}{12}}-40\lambda=0\\
\frac{200}{12}h^{\frac{3}{4}}b^{\frac{1}{2}}r^{\frac{1}{6}}g^{\frac{-11}{12}}-70\lambda=0\\
-120h-500b-40r-70g+4000=0
\end{array}\right) \qquad (3.60)$$

$$prob\,1 = \begin{pmatrix} \frac{600}{4} h^{-\frac{1}{4}} b^{\frac{1}{2}} r^{\frac{1}{6}} g^{\frac{1}{12}} - 120, \frac{200}{2} h^{\frac{3}{4}} b^{-\frac{1}{2}} r^{\frac{1}{6}} g^{\frac{1}{12}} - 500, \frac{200}{6} h^{\frac{3}{4}} b^{\frac{1}{2}} r^{-\frac{5}{6}} g^{\frac{1}{4}} - 40, \\ \frac{200}{12} h^{\frac{3}{4}} b^{\frac{1}{2}} r^{\frac{1}{6}} g^{-\frac{11}{12}} - 70 = 0, -120\,h - 500\,b - 40\,r - 79\,g + 4000 = 0 \end{pmatrix} \qquad (3.61)$$

$$h = \frac{50}{3}, b = \frac{8}{3}, r = \frac{100}{9}, g = \frac{200}{63}, \lambda = 1.1667$$

$$R(h,b,r,g) = 200\left(\frac{50}{3}\right)^{\frac{3}{4}}\left(\frac{8}{3}\right)^{\frac{1}{2}}\left(\frac{100}{9}\right)^{\frac{1}{6}}\left(\frac{200}{93}\right)^{\frac{1}{12}} = 4.430 \qquad (3.62)$$

When

$$m^* = \frac{1}{6}, n^* = \frac{1}{12}, p^* = \frac{3}{4}, q^* = \frac{1}{2}, \ and\, k = 200$$

$$Max\,R^*(h,b,r,g) = K\,h^{\frac{1}{6}} b^{\frac{1}{12}} r^{\frac{3}{4}} g^{\frac{1}{2}}$$

$$subject\ to: 120h + 500b + 40r + 70g = 4{,}000 \ \ \forall h,b,r,g > 0 \qquad (3.63)$$

Taking the Lagrange function of the formulated model in (3.67), we have an unconstrained problem asl

$$L^*(h,b,r,g) = 200h^{\frac{1}{6}} b^{\frac{1}{12}} r^{\frac{3}{4}} g^{\frac{1}{2}} - \lambda\left(120h + 500\,b + 40r + 70g - 4000\right) = 0 \ \ \forall h,b,r,g > 0 \quad (3.64)$$

iii. Model Solution

We set the gradient L^* equal to 0. This is the same as setting each partial derivative equal to 0 as follows

Differentiating with respect to h, we have:

$$\frac{\partial}{\partial h}\left(200h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}}-\lambda(120h+500b+40r+70g-4000)\right)=0 \tag{3.65}$$

we obtained,

$$\frac{dL^{*}}{dh}=\left(\frac{200}{6}\right)h^{\frac{-5}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}}-120\lambda=0 \tag{3.66}$$

Differentiating with respect to b, we have:,

$$\frac{\partial}{\partial b}\left(200h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}}-\lambda(120h+500b+40r+70g-4000)\right)=0 \tag{3.67}$$

We obtained,

$$\frac{dL^{*}}{db}=\left(\frac{200}{12}\right)h^{\frac{1}{6}}b^{\frac{-11}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}}-500\lambda=0 \tag{3.68}$$

Differentiating with respect to r, we have:

$$\frac{\partial}{\partial r}\left(200h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}}-\lambda(120h+500b+40r+70g-4000)\right)=0 \qquad (3.69)$$

we obtained,

$$\frac{dL^*}{dr}=\left(\frac{600}{4}\right)h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{-1}{4}}g^{\frac{1}{2}}-40\lambda=0 \qquad (3.70)$$

Differentiating with respect to g, we have:

$$\frac{\partial}{\partial g}\left(200h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}}-\lambda(120h+500b+40r+70g-4000)\right)=0 \qquad (3.71)$$

we have,,

$$\frac{dL^*}{dg}=\left(\frac{200}{2}\right)h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{-1}{2}}-70\lambda=0 \qquad (3.72)$$

Differentiating with respect to λ, we have:

$$\frac{\partial}{\partial \lambda}\left(200h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}}-\lambda(120h+500b+40r+70g-4000)\right)=0 \qquad (3.73)$$

we have,

$$\frac{dL^*}{d\lambda}=-120\,h-500b-40r-70g+4000=0 \tag{3.74}$$

The maximum revenue can be obtained, using the relationship below,

$$R^*(h,b,r,g)=K\,h^{\frac{1}{6}}\,b^{\frac{1}{12}}\,r^{\frac{3}{4}}\,g^{\frac{1}{2}} \tag{3.75}$$

Solving equations 3.66, 3.68, 3.70, 3.72, and 3.74 simultaneously, we have,

$$\left.\begin{array}{l}
\frac{200}{6}h^{\frac{-5}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}}-120\,\lambda=0\\
\frac{200}{12}h^{\frac{1}{6}}b^{\frac{-11}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}}-500\,\lambda=0\\
\frac{600}{4}h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{-1}{4}}g^{\frac{1}{2}}-40\,\lambda=0\\
\frac{200}{2}h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{-1}{2}}-70\,\lambda=0\\
\\
-120h-500b-40r-70g+4000=0
\end{array}\right) \tag{3.76}$$

$$prob2=\left(\begin{array}{l}
\frac{200}{6}\,h^{-\frac{5}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}}-120,\ \frac{200}{12}\,h^{\frac{1}{6}}b^{-\frac{11}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}}-500,\\
\frac{600}{4}\,h^{\frac{1}{6}}b^{\frac{1}{12}}r^{-\frac{1}{4}}g^{\frac{1}{2}}-40,\ \frac{200}{2}\,h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{-\frac{1}{2}}-70=0,\\
-120\,h-500b-40r-70g+4000=0\\
\\
\end{array}\right) \tag{3.77}$$

$$h=\frac{100}{27},b=\frac{4}{9},\ r=\frac{100}{2},g=\frac{400}{21},\lambda=7.1548$$

$$R^*(h,b,r,g)=200\left(\frac{100}{27}\right)^{\frac{1}{6}}\left(\frac{4}{9}\right)^{\frac{1}{12}}\left(\frac{100}{2}\right)^{\frac{3}{4}}\left(\frac{400}{21}\right)^{\frac{1}{2}}\ =\ 19{,}082 \tag{3.78}$$

3.5 Optimality and Extrema Point of Our Formulated Model

To test the optimality of our model using the KKT conditions and the extrema point using the Hessian approach (Evans *et. al.*, 2021b).

The formulated model is given as:

$$Max: R^*(h,b,r,g) = 200\, h^{\frac{1}{6}} b^{\frac{1}{12}} r^{\frac{3}{4}} g^{\frac{1}{2}} \tag{3.79}$$

$$Subject\,to: 120h+500b+40r+70g=4000 \tag{3.80}$$

1st KKT condition requires the equality constraints to be in residing form,

$$120h+500b+40r+70g-4000=0 \tag{3.81}$$

2nd KKT condition requires the application of Lagrange multiplier

$$L^*(h,b,r,g) = 200h^{\frac{1}{6}} b^{\frac{1}{12}} r^{\frac{3}{4}} g^{\frac{1}{2}} - \lambda\left(120h+500b+40r+70g-4000\right)=0 \;\; \forall h,b,r,g>0 \tag{3.82}$$

Differentiating equation (3.82) partially with respect to h, b, r, g and λ, we obtain,

$$L_h = \frac{200}{6} h^{\frac{-5}{6}} b^{\frac{1}{12}} r^{\frac{3}{4}} g^{\frac{1}{2}} - 120\lambda = 0 \tag{3.83}$$

$$L_b = \frac{200}{12} h^{\frac{1}{6}} b^{\frac{-11}{12}} r^{\frac{3}{4}} g^{\frac{1}{2}} - 500\lambda = 0 \tag{3.84}$$

$$L_r = \frac{600}{4} h^{\frac{1}{6}} b^{\frac{1}{12}} r^{\frac{-1}{4}} g^{\frac{1}{2}} - 40\lambda = 0 \qquad (3.85)$$

$$L_g = \frac{200}{2} h^{\frac{1}{6}} b^{\frac{1}{12}} r^{\frac{3}{4}} g^{\frac{-1}{2}} - 70\lambda = 0 \qquad (3.86)$$

$$L(\lambda) = -120h - 500b - 40r - 70g + 4000 = 0 \qquad (3.87)$$

The 3rd KKT condition does not apply to the problem, since there is no inequality constraint.

Solving equations (3.83), (3.84), (3.85), (3.86) and (3.87), simultaneously, we obtain,

$h = 3.7037, b = 0.4444, r = 50, g = 19.0480 \; and \; \lambda = 7.1548$

Upon substitution we have (3.88) our revenue model,

$$R^*(h,b,r,g) = 200\,(3.7037)^{\frac{1}{6}}\,(0.4444)^{\frac{1}{12}}\,(50)^{\frac{3}{4}}\,(19.0480)^{\frac{1}{2}} = 19{,}082 \qquad (3.88)$$

KKT condition of $\lambda^* \geq 0$ $(\lambda = 7.1548)$ shows that the solution is optimal. If λ^* is negative it means it is not optimum and needs further investigation (Taha 2010).

The 4th KKT condition is satisfied since λ^* is positive, this means the solution is optimum.

Again, it is required to know if the optimum solution is a maximum or minimum which leads to the sufficient (second order condition) which put restrictions on the Enigmatic Bordered Hessian denoted by H$_L$(*)

The Enigmatic Bordered Hessian matrix for $L(h,b,r,g)$ is given by (3.89)

$$H_L = \begin{pmatrix} \frac{\partial^2 L}{\partial h^2} & \frac{\partial^2 L}{\partial h \partial b} & \frac{\partial^2 L}{\partial h \partial r} & \frac{\partial^2 L}{\partial h \partial g} & \frac{\partial^2 L}{\partial h \partial \lambda} \\ \frac{\partial^2 L}{\partial b \partial h} & \frac{\partial^2 L}{\partial b^2} & \frac{\partial^2 L}{\partial b \partial r} & \frac{\partial^2 L}{\partial b \partial g} & \frac{\partial^2 L}{\partial b \partial \lambda} \\ \frac{\partial^2 L}{\partial r \partial h} & \frac{\partial^2 L}{\partial r \partial b} & \frac{\partial^2 L}{\partial r^2} & \frac{\partial^2 L}{\partial r \partial g} & \frac{\partial^2 L}{\partial g \partial \lambda} \\ \frac{\partial^2 L}{\partial g \partial h} & \frac{\partial^2 L}{\partial g \partial b} & \frac{\partial^2 L}{\partial g \partial r} & \frac{\partial^2 L}{\partial g^2} & \frac{\partial^2 L}{\partial r \partial \lambda} \\ \frac{\partial^2 L}{\partial h \partial \lambda} & \frac{\partial^2 L}{\partial b \partial \lambda} & \frac{\partial^2 L}{\partial g \partial \lambda} & \frac{\partial^2 L}{\partial r \partial \lambda} & \frac{\partial^2 L}{\partial \lambda^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial^2 L}{\partial h^2} & \frac{\partial^2 L}{\partial h \partial b} & \frac{\partial^2 L}{\partial h \partial r} & \frac{\partial^2 L}{\partial h \partial g} & -ph \\ \frac{\partial^2 L}{\partial b \partial h} & \frac{\partial^2 L}{\partial b^2} & \frac{\partial^2 L}{\partial b \partial r} & \frac{\partial^2 L}{\partial b \partial g} & -pb \\ \frac{\partial^2 L}{\partial r \partial h} & \frac{\partial^2 L}{\partial r \partial b} & \frac{\partial^2 L}{\partial r^2} & \frac{\partial^2 L}{\partial r \partial g} & -pr \\ \frac{\partial^2 L}{\partial g \partial h} & \frac{\partial^2 L}{\partial g \partial b} & \frac{\partial^2 L}{\partial g \partial r} & \frac{\partial^2 L}{\partial g^2} & -pg \\ -ph & -pb & -pr & -pg & 0 \end{pmatrix} \qquad (3.89$$

Differentiating equation 3.83 partially with respect to h, b, r, g and λ, to obtain the second partial derivatives of equation (3.82), we have,

$$\frac{\partial^2 L}{\partial h^2} = -\frac{1000}{36} h^{\frac{-11}{6}} b^{\frac{1}{12}} r^{\frac{3}{4}} g^{\frac{1}{2}} = 0 \qquad (3.90)$$

$$\frac{\partial^2 L}{\partial h \partial b} = \frac{200}{72} h^{\frac{-5}{6}} b^{\frac{-11}{12}} r^{\frac{3}{4}} g^{\frac{1}{2}} = 0 \qquad (3.91)$$

$$\frac{\partial^2 L}{\partial h \partial r} = \frac{600}{24} h^{\frac{-5}{6}} b^{\frac{1}{12}} r^{\frac{-1}{4}} g^{\frac{1}{2}} = 0 \qquad (3.92)$$

$$\frac{\partial^2 L}{\partial h \partial g} = \frac{200}{12} h^{\frac{-5}{6}} b^{\frac{1}{12}} r^{\frac{3}{4}} g^{\frac{-1}{2}} = 0 \qquad (3.93)$$

$$\frac{\partial^2 L}{\partial h \partial \lambda} = -120 \qquad (3.94)$$

Differentiating equation 3.84 partially with respect to h, b, r, g and λ, we obtain,

$$\frac{\partial^2 L}{\partial b \partial h} = \frac{200}{72} h^{\frac{-5}{6}} b^{\frac{-11}{12}} r^{\frac{3}{4}} g^{\frac{1}{2}} = 0 \qquad (3.95)$$

$$\frac{\partial^2 L}{\partial b^2} = -\frac{2200}{144} h^{\frac{1}{6}} b^{\frac{-23}{12}} r^{\frac{3}{4}} g^{\frac{1}{2}} = 0 \qquad (3.96)$$

$$\frac{\partial^2 L}{\partial b \partial r} = \frac{600}{48} h^{\frac{1}{6}} b^{\frac{-11}{12}} r^{\frac{-1}{4}} g^{\frac{1}{2}} = 0 \qquad (3.97)$$

$$\frac{\partial^2 L}{\partial b \partial g} = \frac{200}{24} h^{\frac{1}{6}} b^{\frac{-11}{12}} r^{\frac{3}{4}} g^{\frac{-1}{2}} = 0 \qquad (3.98)$$

$$\frac{\partial^2 L}{\partial b \partial \lambda} = -500 \qquad (3.99)$$

Differentiating equation 3.85 partially with respect to h, b, r, g and λ, we obtain

$$\frac{\partial^2 L}{\partial r \partial h} = \frac{600}{24} h^{\frac{-5}{6}} b^{\frac{1}{12}} r^{\frac{-1}{4}} g^{\frac{1}{2}} = 0 \qquad (3.100)$$

$$\frac{\partial^2 L}{\partial r \partial b} = \frac{600}{48} h^{\frac{1}{6}} b^{\frac{-11}{12}} r^{\frac{-1}{4}} g^{\frac{1}{2}} = 0 \qquad (3.101)$$

$$\frac{\partial^2 L}{\partial r^2} = -\frac{600}{16} h^{\frac{1}{6}} b^{\frac{1}{12}} r^{\frac{-5}{4}} g^{\frac{1}{2}} = 0 \qquad (3.102)$$

$$\frac{\partial^2 L}{\partial r \partial g} = \frac{600}{8} h^{\frac{1}{6}} b^{\frac{1}{12}} r^{\frac{-1}{4}} g^{\frac{-1}{2}} = 0 \qquad (3.103)$$

$$\frac{\partial^2 L}{\partial r \partial \lambda} = -40 \qquad (3.104)$$

Differentiating equation (3.86) partially with respect to h, b, r, g and λ, we obtain,

$$\frac{\partial^2 L}{\partial g \partial h} = \frac{200}{12} h^{\frac{-5}{6}} b^{\frac{1}{12}} r^{\frac{3}{4}} g^{\frac{-1}{2}} = 0 \qquad (3.105)$$

$$\frac{\partial^2 L}{\partial g \partial b} = \frac{200}{24} h^{\frac{1}{6}} b^{\frac{-11}{12}} r^{\frac{3}{4}} g^{\frac{-1}{2}} = 0 \qquad (3.106)$$

$$\frac{\partial^2 L}{\partial g \partial r} = \frac{600}{8} h^{\frac{1}{6}} b^{\frac{1}{12}} r^{\frac{-1}{4}} g^{\frac{-1}{2}} = 0 \tag{3.107}$$

$$\frac{\partial^2 L}{\partial g^2} = -\frac{200}{4} h^{\frac{1}{6}} b^{\frac{1}{12}} r^{\frac{3}{4}} g^{\frac{-3}{2}} = 0 \tag{3.108}$$

$$\frac{\partial^2 L}{\partial g \partial \lambda} = -70 \tag{3.109}$$

Differentiating equation (3.87) partially with respect to h, b, r, g and λ, we obtain,

$$\frac{\partial^2 L}{\partial \lambda \partial h} = -120 \tag{3.110}$$

$$\frac{\partial^2 L}{\partial \lambda \partial b} = -500 \tag{3.111}$$

$$\frac{\partial^2 L}{\partial \lambda \partial r} = -40 \tag{3.112}$$

$$\frac{\partial^2 L}{\partial \lambda \partial g} = -70 \tag{3.113}$$

$$\frac{\partial^2 L}{\partial \lambda^2} = 0 \tag{3.114}$$

Equation (3.90) to (3.114) can be written in matrix form as shown in (3.115)

$$H_L = \begin{pmatrix} -\frac{1000}{36}h^{\frac{-11}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}} & \frac{200}{72}h^{\frac{-5}{6}}b^{\frac{-11}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}} & \frac{600}{24}h^{\frac{-5}{6}}b^{\frac{1}{12}}r^{\frac{-1}{4}}g^{\frac{1}{2}} & \frac{200}{12}h^{\frac{-5}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{-1}{2}} & -120 \\ \frac{200}{72}h^{\frac{-5}{6}}b^{\frac{-11}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}} & -\frac{2200}{144}h^{\frac{1}{6}}b^{\frac{-23}{12}}r^{\frac{3}{4}}g^{\frac{1}{2}} & \frac{600}{48}h^{\frac{1}{6}}b^{\frac{-11}{12}}r^{\frac{-1}{4}}g^{\frac{1}{2}} & \frac{200}{24}h^{\frac{1}{6}}b^{\frac{-11}{12}}r^{\frac{3}{4}}g^{\frac{-1}{2}} & -500 \\ \frac{600}{24}h^{\frac{-5}{6}}b^{\frac{1}{12}}r^{\frac{-1}{4}}g^{\frac{1}{2}} & \frac{600}{48}h^{\frac{1}{6}}b^{\frac{-11}{12}}r^{\frac{-1}{4}}g^{\frac{1}{2}} & -\frac{600}{16}h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{-5}{4}}g^{\frac{1}{2}} & \frac{600}{8}h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{-1}{4}}g^{\frac{-1}{2}} & -40 \\ \frac{200}{12}h^{\frac{-5}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{-1}{2}} & \frac{200}{24}h^{\frac{1}{6}}b^{\frac{-11}{12}}r^{\frac{3}{4}}g^{\frac{-1}{2}} & \frac{600}{8}h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{-1}{4}}g^{\frac{-1}{2}} & -\frac{200}{4}h^{\frac{1}{6}}b^{\frac{1}{12}}r^{\frac{3}{4}}g^{\frac{-3}{2}} & -70 \\ -120 & -500 & -40 & -70 & 0 \end{pmatrix}$$

(3.115)

Solving Equation (3.90) to (3.114) simultaneously we obtain the result in (3.116) (Evans *et. al*., 2021b).

$$H_L = \begin{pmatrix} -193.20 & 161.00 & 12.880 & 22.539 & -120 \\ 161.00 & -7379.2 & 53.666 & 93.913 & -500 \\ 12.880 & 53.666 & -1.4311 & 7.5131 & -40 \\ 22.539 & 93.913 & 7.5131 & -2.1088 & -70 \\ -120 & -500 & -40 & -70 & 0 \end{pmatrix} \qquad (3.116)$$

This is the Enigmatic Bordered Hessian Matrix.

First Principal Minor Determinant

$$A = [-193.20]$$

Determinant of A is – 193.20

Second Principal Minor Determinant

$$B=\begin{bmatrix} -193.20 & 161.00 \\ 161.00 & -7379.0 \end{bmatrix}$$

Determinant of B is 1.399740440 x 10^6

Third Principal Minor Determinant

$$C=\begin{bmatrix} -193.20 & 161.00 & 12.880 \\ 161.00 & -7379.2 & 53.666 \\ 12.880 & 53.666 & -1.4311 \end{bmatrix}$$

Determinant of C is – 4.7241

Fourth Principal Minor Determinant

$$D=\begin{bmatrix} -193.20 & 161.00 & 12.880 & 22.539 \\ 161.00 & -7379.2 & 53.666 & 93.913 \\ 12.880 & 53.666 & -1.4311 & 7.5131 \\ 22.539 & 93.913 & 7.5131 & -2.1088 \end{bmatrix}$$

Determinant of D is – 1.404622276 x 10^6

Fifth Principal Minor Determinant

$$E=\begin{pmatrix} -193.20 & 161.00 & 12.880 & 22.539 & -120 \\ 161.00 & -7379.2 & 53.666 & 93.913 & -500 \\ 12.880 & 53.666 & -1.4311 & 7.5131 & -40 \\ 22.539 & 93.913 & 7.5131 & -2.1088 & -70 \\ -120 & -500 & -40 & -70 & 0 \end{pmatrix}$$

Determinant of E is 1.130920568 x 10^{11}

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Results

The graph for equations 3.7, 3.11, 3.15, 3.20, 3.24, 3.29, 3.33, 3.37, 3.38, 3.39, 3.44, 3.45, 3.46 and 3.47 for batch and continuous process in Figures 4.1 to 4.31.

4.2 Effect of rate of reaction on the biomass, residue glucose and ethanol yield are shown in Figures 4.1 to 4.18.

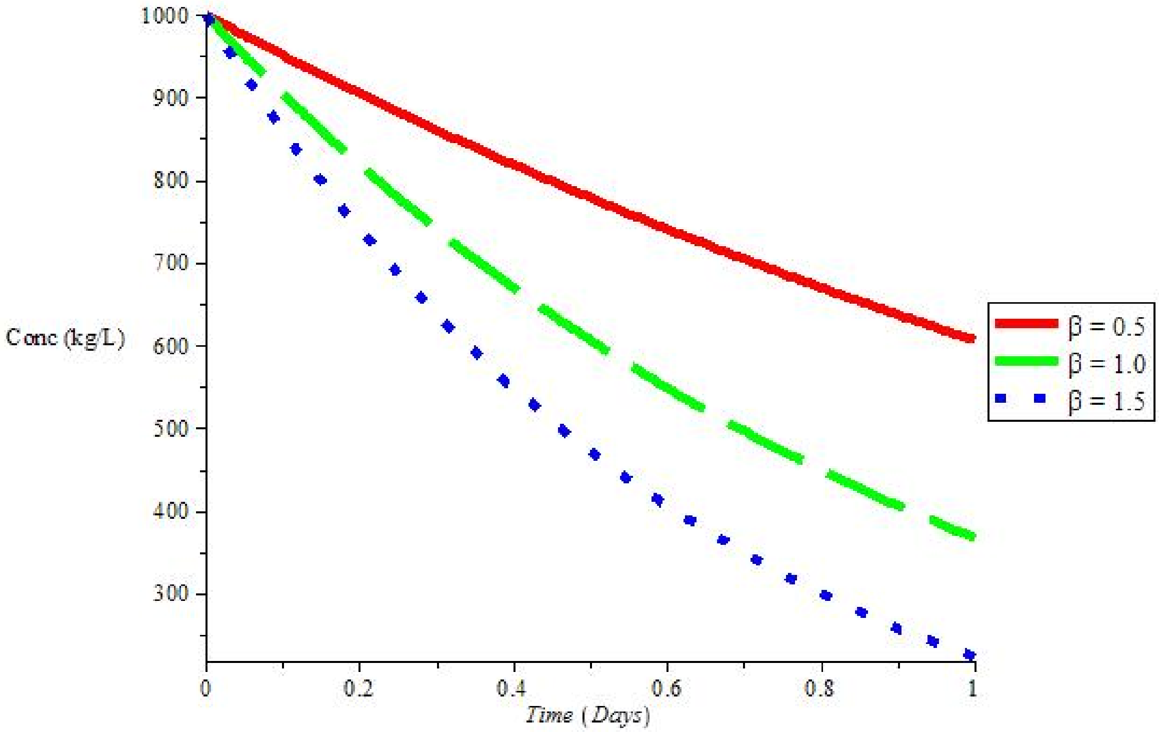


Figure 4.1: Biomass – Time profile for different β (without pretreatment biomass).

Figure 4.1 shows that at high hydrolysis rate β, more of the unpretreated (without pretreatment) biomass was converted to glucose. This implies that increase in β decreases the biomass.

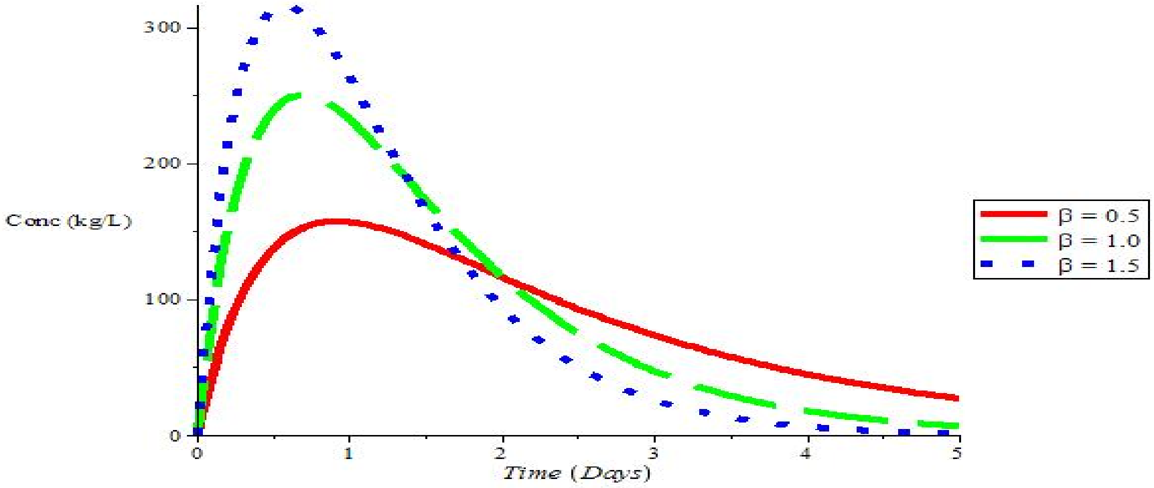


Figure 4.2: Glucose – Time profile for different β (Unpretreated biomass).

Figure 4.2 shows the effect of increases in the rate of hydrolysis β on the glucose, where the optimum glucose concentration is 330 kg/L at day one. This implies that an increase in β increases the glucose concentration.

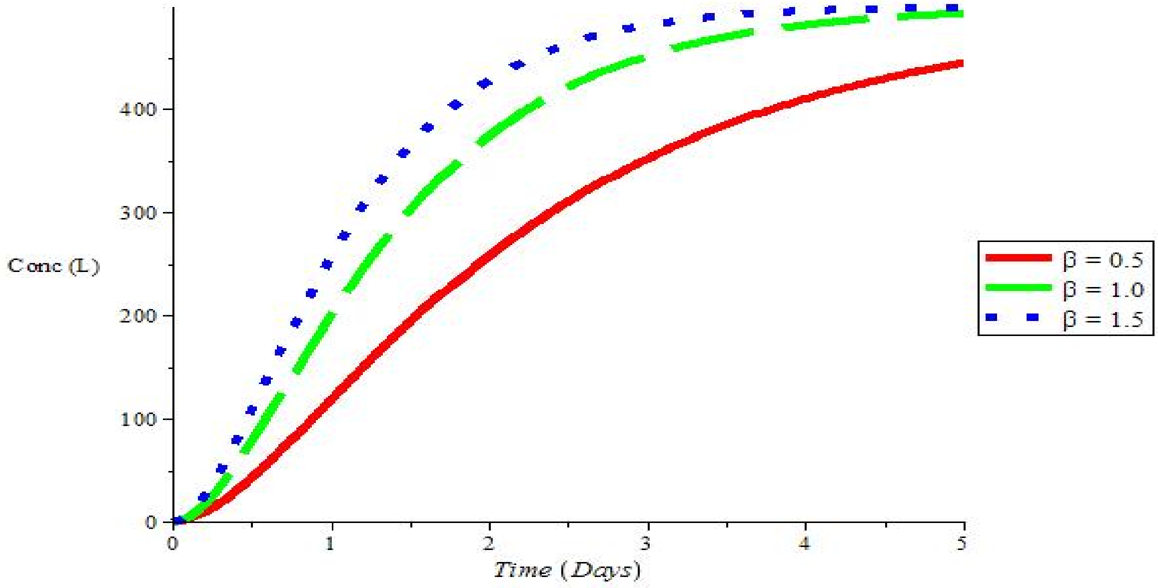


Figure 4.3: Ethanol – Time profile for different β (Unpretreated biomass).

Figure 4.3 depicts the effect of rate of hydrolysis, β on the ethanol yield, where the optimum ethanol yield was 499 L on day two when β = 1.5. this implies that an increase β increases the ethanol yield.

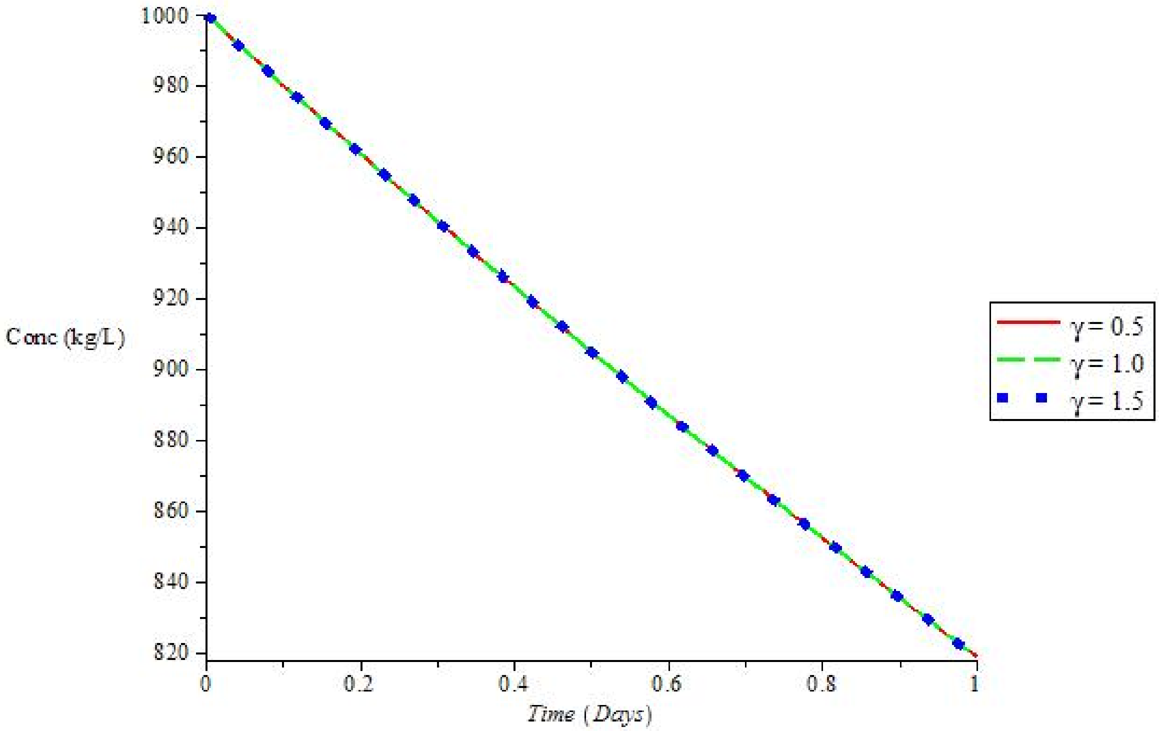


Figure 4.4 Biomass – Time profile for different γ (Unpretreated Biomass)

Figure 4.4 shows the effect of fermentation rate γ on the unpretreated biomass. This implies that a decrease or an increase in fermentation rate γ does not have significant effect on the biomass. This means that γ does not affect biomass depletion and as such, the overall reaction may not be a reversible reaction.

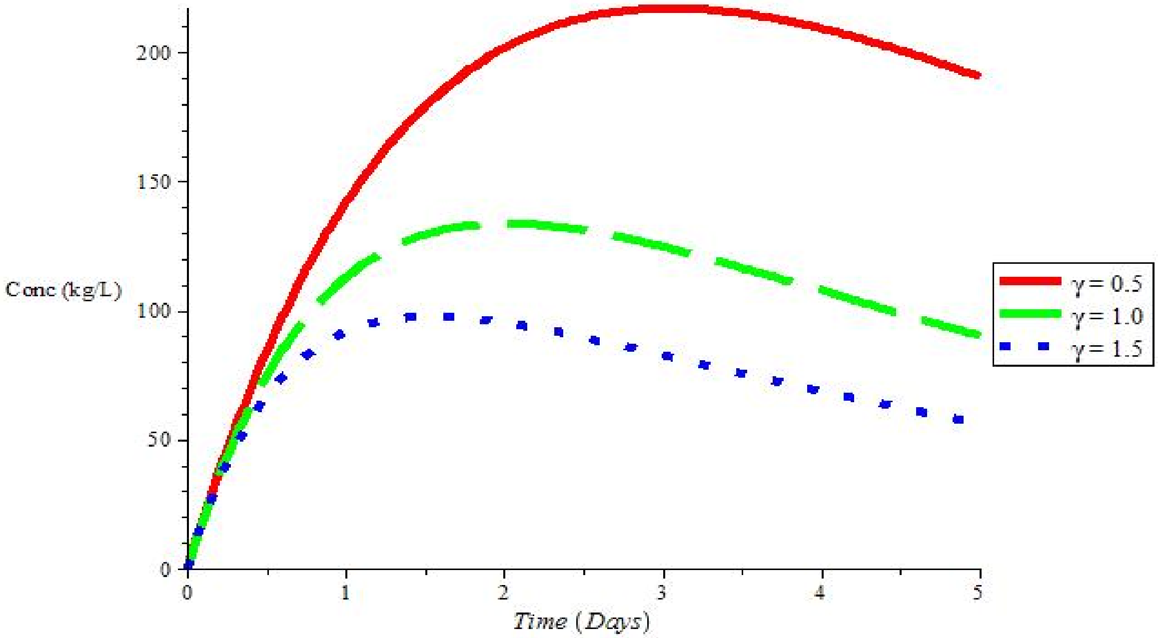


Figure 4.5 Glucose – Time profile for different γ (Unpretreated Biomass)

Figure 4.5 illustrates the effect of fermentation rate γ on glucose, where the optimum is at day three with a glucose yield of 215 kg/L, when γ = 0.5. This implies that decrease fermentation rate γ increases the glucose yield. This is understandable, because the more the glucose the more the ethanol, therefore, the higher the fermentation rate the less the glucose yield.

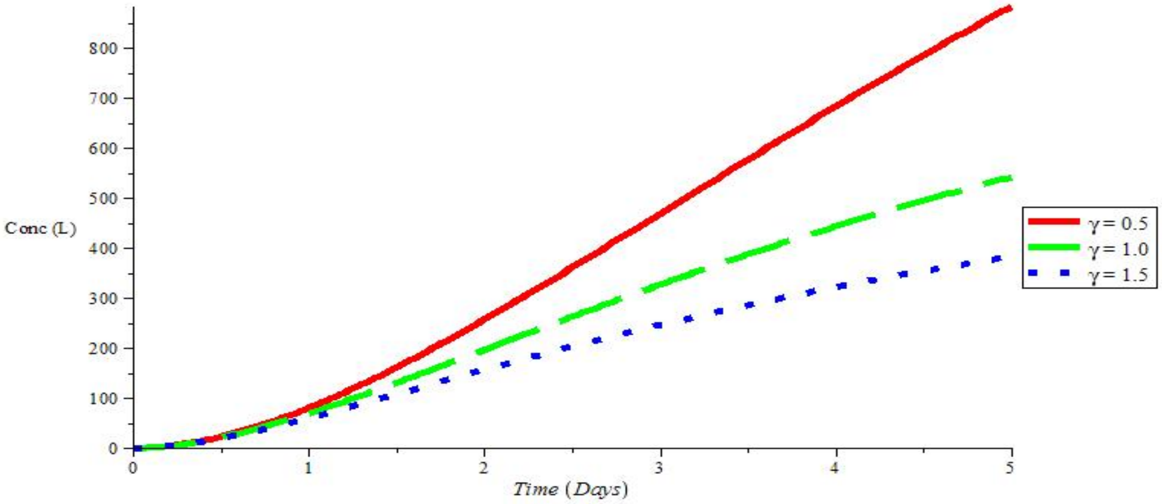


Figure 4.6 Ethanol – Time profile for different γ (Unpretreated Biomass).

Figure 4.6 depicts the effect of fermentation rate γ on ethanol, yield where the optimum is at day five with a yield of 901 L when γ = 0.5. This implies that decrease in fermentation rate γ increases the ethanol yield.

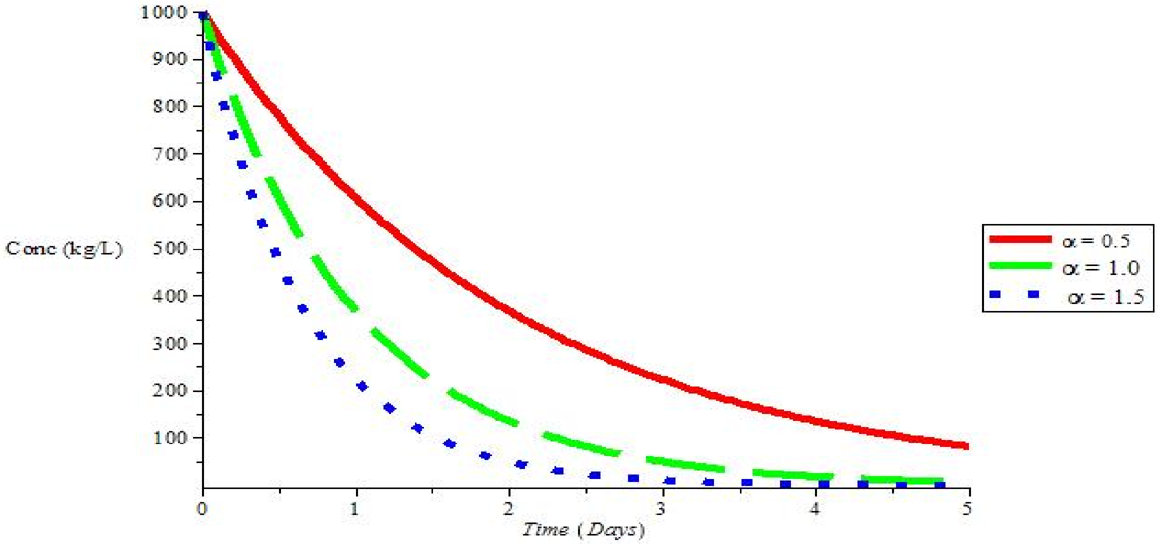


Figure 4.7: Biomass – Time Profile for different α (Pretreated Biomass).

Figure 4.7 shows the effect of pretreatment rate, α on the pretreated biomass. This implies that higher pretreatment rate α, increases the biomass depletion.

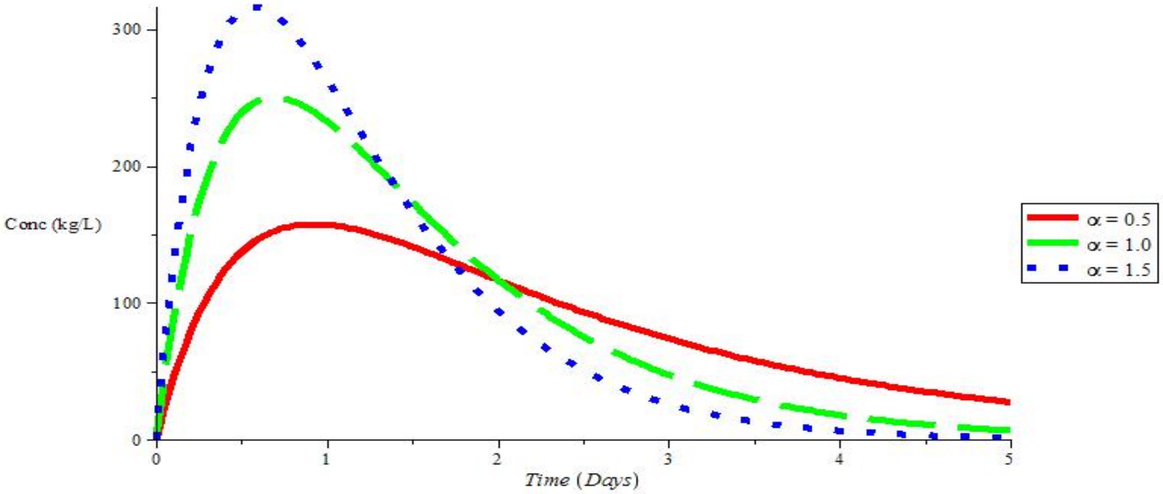


Figure 4.8: Residue – Time Profile for different α (Pretreated. Biomass).

Figure 4.8 shows the effect of pretreatment rate, α on the residue yield, where the optimum is at day one with a residue yield of 330 kg/L, when α =1.5. This implies that increase in increases the residue yield.

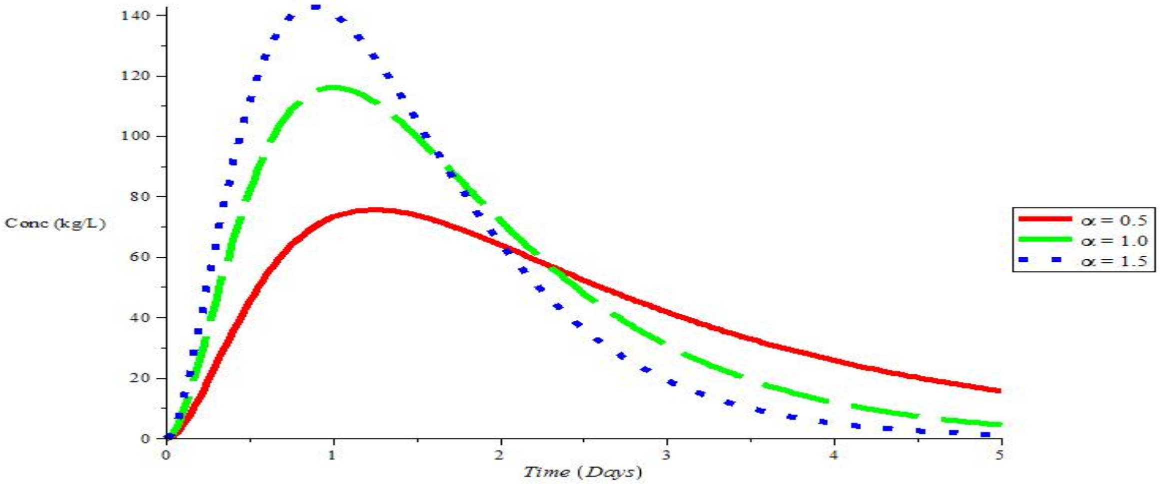


Figure 4.9: Glucose – Time Profile for different α (Pretreated Biomass).

Figure 4.9 illustrates the effect of pretreatment rate α on the glucose, where the optimum glucose yield is 140 kg/L on day one when α =1.5. This implies that increase in α increases the glucose yield.

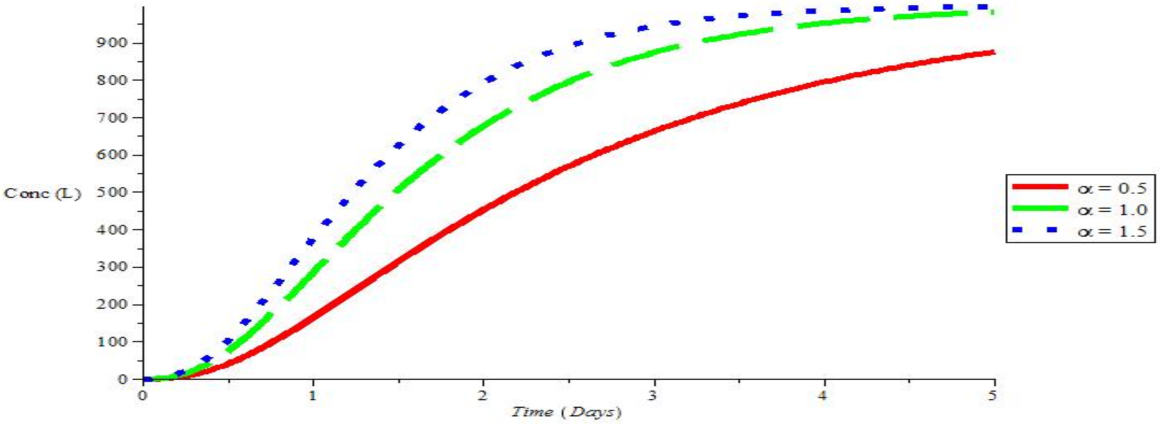


Figure 4.10: Ethanol – Time Profile for different α (Pretreated Biomass).

Figure 4.10 depicts the effect of pretreatment rate α on ethanol yield, where the optimum is at day three with a yield of 930 L when α = 1.5. This implies that increase in α, increases the glucose yield.

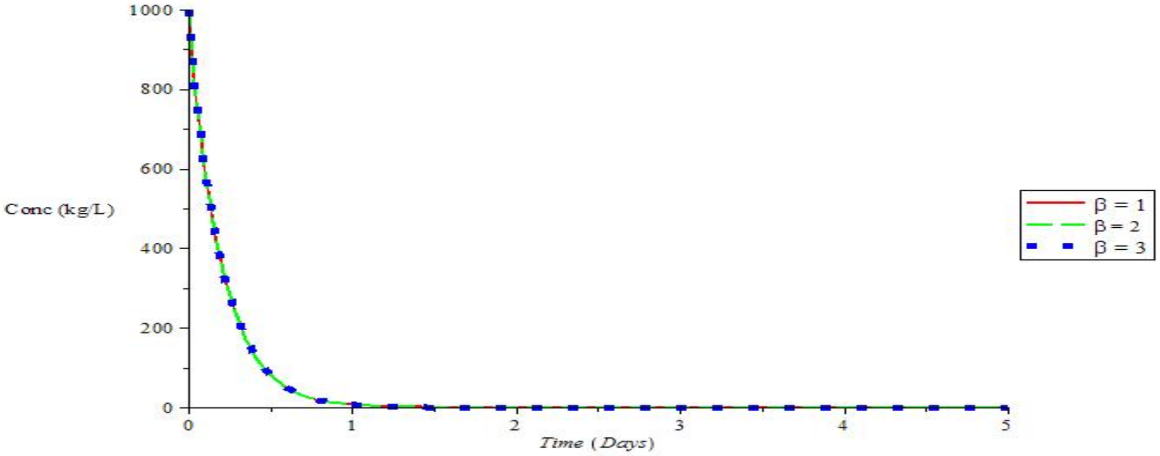


Figure 4.11: Biomass – Time Profile for different β (Pretreated Biomass).

Figure 4.11 shows the effect of hydrolysis rate, β, on the pretreated biomass the plot overlapped on each other. This implies that decrease or increase of β does not have significant effect on the biomass depletion.

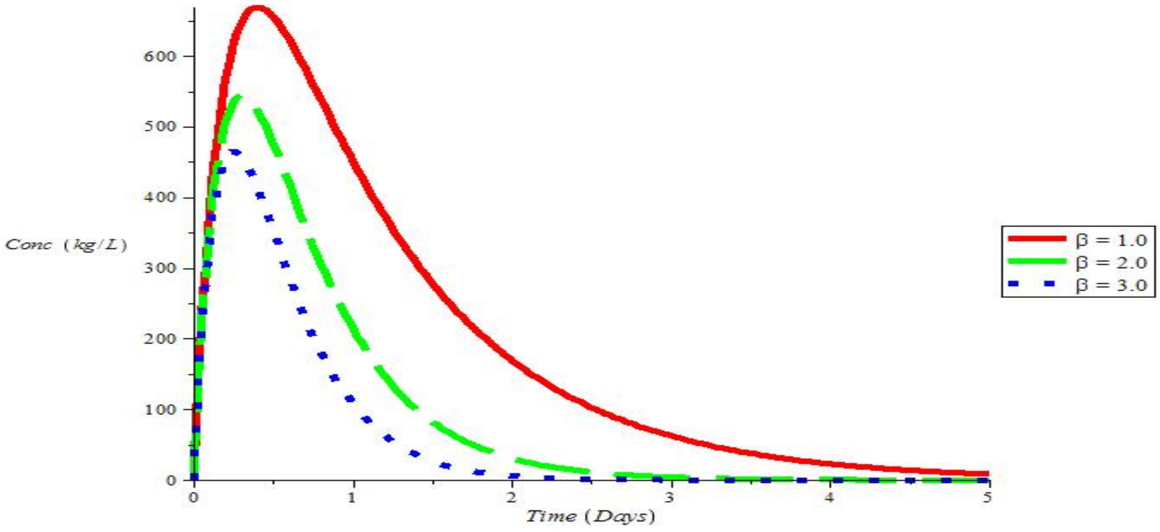


Figure 4.12: Residue – Time Profile for different β (Pretreated Biomass).

Figure 4.12 illustrates the effect of hydrolysis rate, β, on residue, where the optimum is at day one with a residue yield of 655 kg/L when β = 1.0. This implies that low hydrolysis rate β gives high the residue yield.

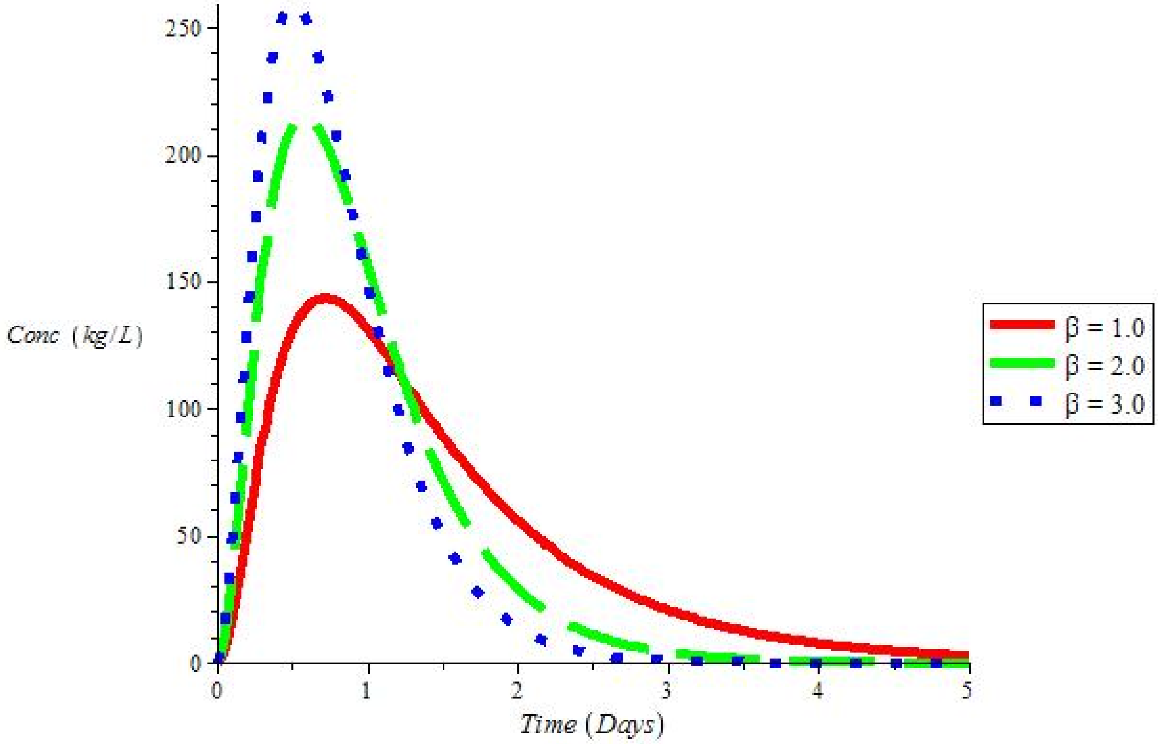


Figure 4.13: Glucose – Time Profile for different β (Pretreated Biomass).

Figure 4.13 depicts the effect of hydrolysis rate, β, on glucose concentration, where the optimum yield on day one was 260 kg/L when β = 3.0. This implies that increase in β increases the glucose yield.

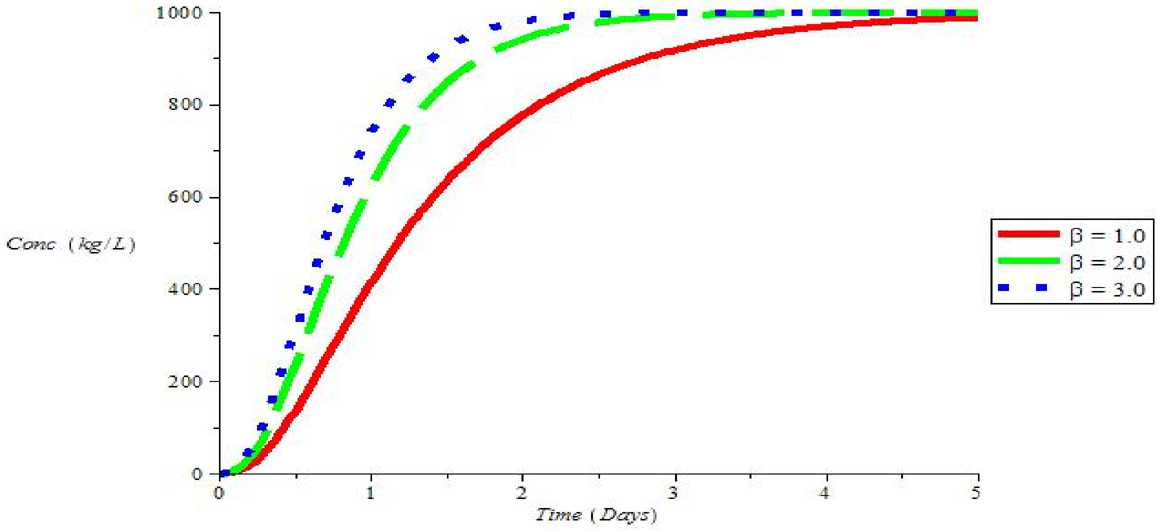


Figure 4.14: Ethanol – Time profile for different β (pretreated Biomass).

The result in Figure 4.14 depicts the effect of hydrolysis rate β on ethanol yield, where the optimum is at day one with a yield of 1000 L when β = 3.0. This implies that increase β increases the ethanol yield. It takes a shorter time to obtain optimum ethanol yield using pretreated biomass, in comparism with figures 4.3, 4.5 and 4.10

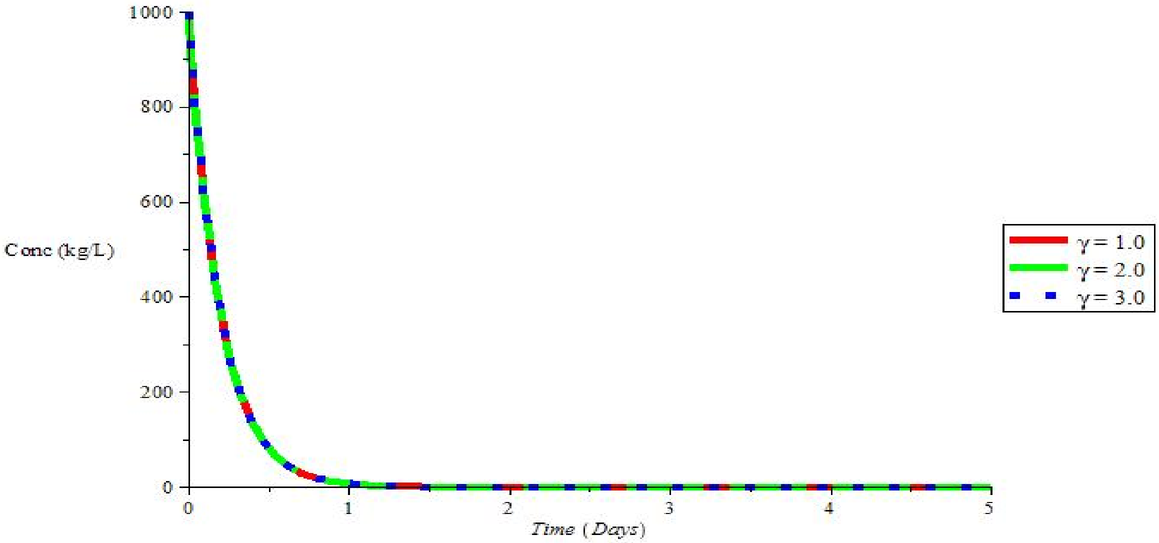


Figure 4.15: Biomass – Time Profile for different γ (Pretreated Biomass)

Figure 4.15 shows the effect of fermentation rate γ on the biomass. All plots overlapped with each other and show an exponential decay curve. This implies that decrease or increase of γ does not have significant effect on the biomass.

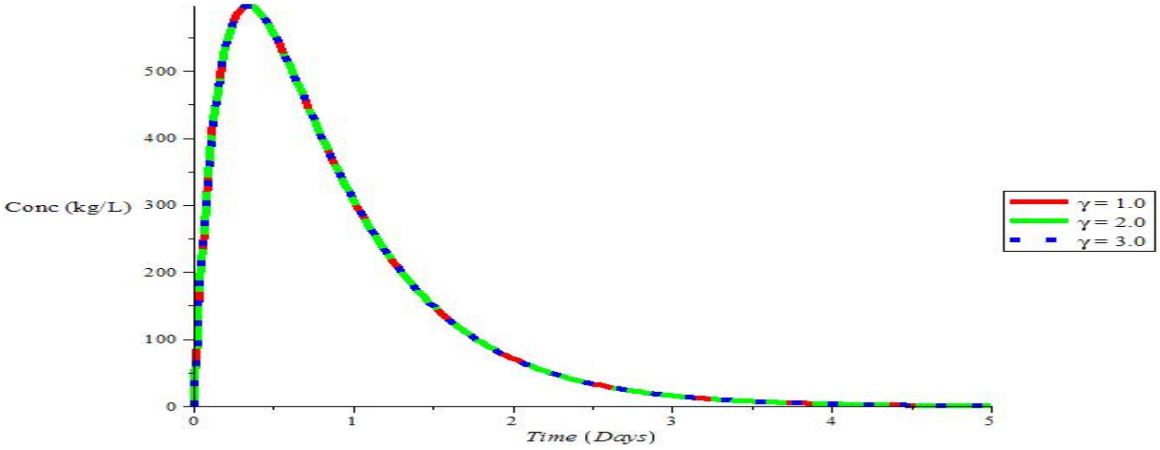


Figure 4.16: Residue – Time Profile for different γ (Pretreated Biomass).

Figure 4.16 depicts the effect of fermentation rate γ on residue, where the optimum is less than day one with a yield of 580 kg/L. This implies that decrease or increase of γ does not have any effect on the residue yield.

n

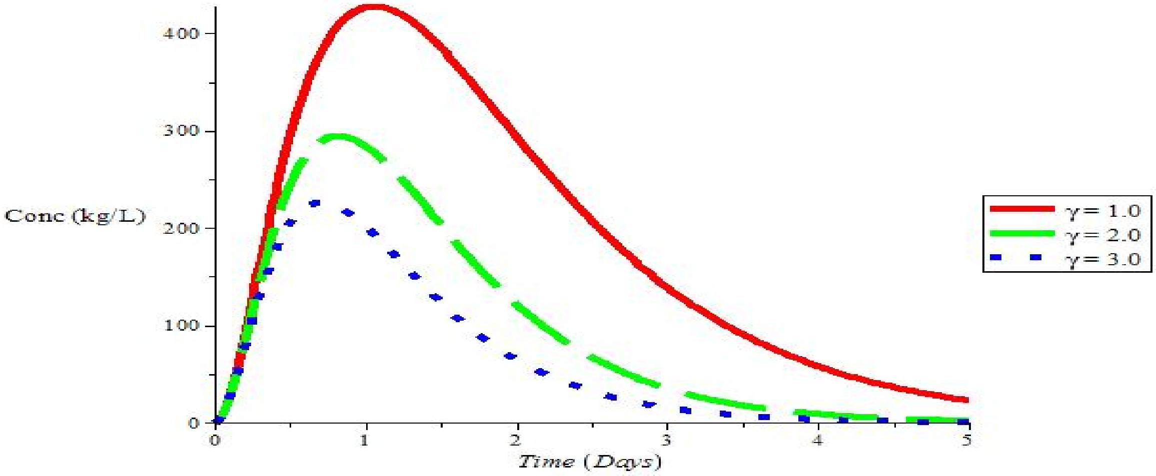


Figure 4.17: Glucose – Time Profile for different γ (Pretreated Biomass).

Figure 4.17 shows the effect of fermentation rate γ on glucose yield, where the optimum yield is at day one with a yield of 420 kg/ L when γ = 1.0. This implies that decrease γ increases glucose yield.

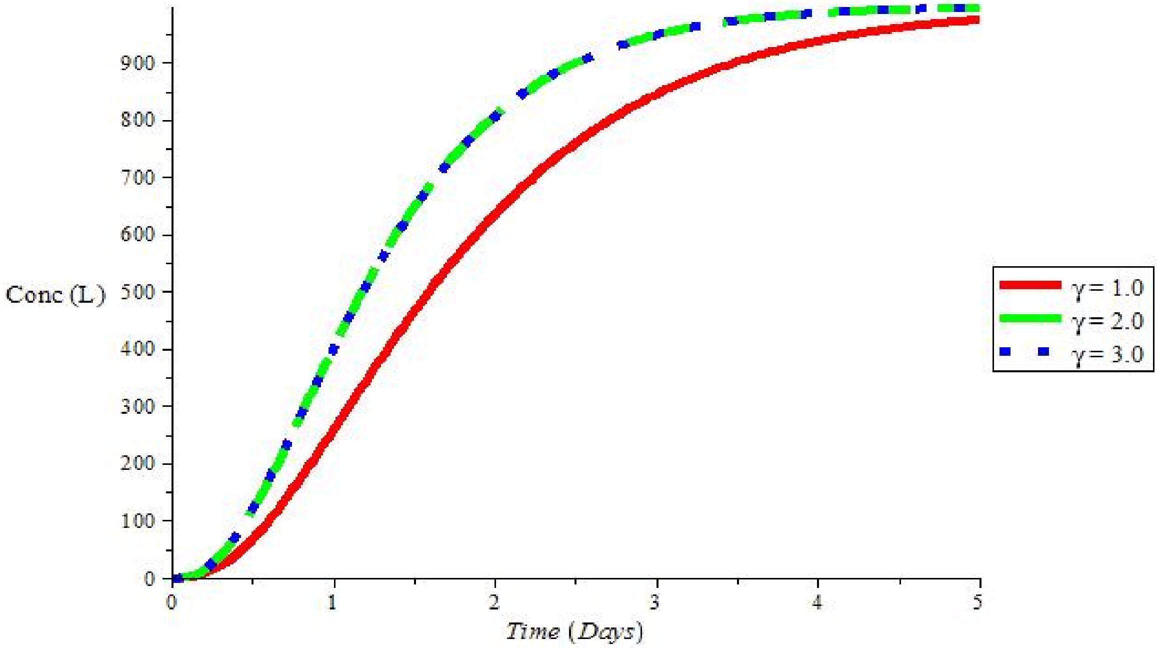


Figure 4.18: Ethanol – Time Profile for different γ (Pretreated Biomass).

The result in Figure 4.18 illustrates the effect of fermentation rate γ on ethanol yield, where the optimum yield on day two with a yield of 975 L when γ = 3.0. This implies that increase γ increases ethanol yield.

The present study has shown that the conversion of banana trunk biomass is controlled by rate kinetics of hydrolysis, β, for unpretreated biomass, while; for pretreated biomass; the whole process depended on initial kinetic step α, where the biomass is treated to remove lignin and the second step β where the residue is converted to glucose. This study, therefore, suggests that the pretreatment step is very important in the industrial scale of converting banana trunk biomass to ethanol.

4.3 Batch Process for the Unpretreated Banana Trunk Biomass

Batch Process for the unpretreated banana trunk biomass is given in Figures 4.19 to 4.22 at $\alpha = 0.5, 2.5, 5, 0.6$ and $\beta = 1.0$ and 10.0 respectively.

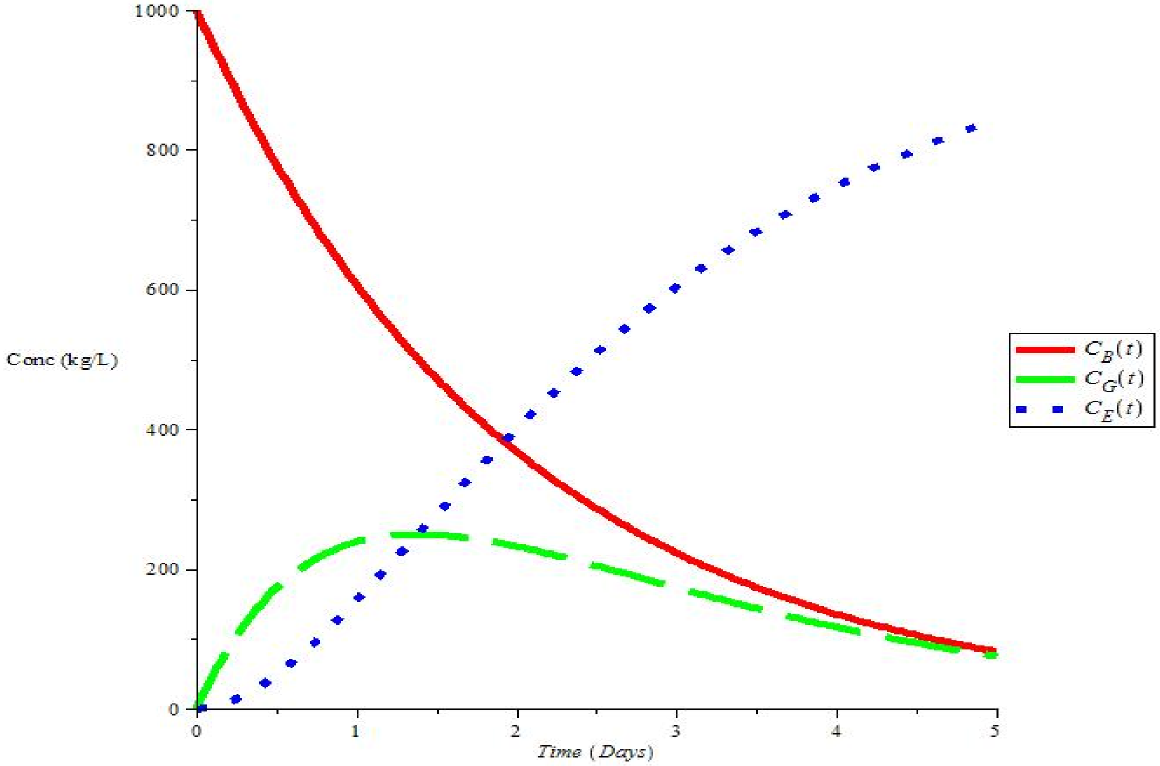


Figure 4.19: The rate of change of Unpretreated Banana Trunk Biomass, Glucose and Ethanol with time at β = 0.5 and γ = 1.0 (Batch Process)

Figure 4.19 shows the dynamics of banana trunk unpretreated biomass, glucose and ethanol, in a batch process. The banana trunk biomass decreased at the rate of $\beta = 0.5$ from the initial value of 1000 kg/L to 96 kg/L. at day 5 due to hydrolysis. The glucose yield increases from day zero at the rate of 1.0 to an optimum yield of 230 kg/L at day three, but after day three there was decrease in glucose due to the utilization of the available glucose for the ethanol production. The ethanol yield increase at the rate of 1.0 and reached optimum of 810 L on day five. The rate of change of ethanol with time was inversely proportional to that of biomass. This agrees with the studies of Olaoye and Kolawole. (2013) and Konstantinos *et. al.,* (2019). Who have shown that, using logistic model to describe microbial growth, polysaccharide fermentation processes and biomass growth, that under optimal growth condition and when the inhibitory effects of substrate and product were neglected, the rate of cell growth follows the well-known exponential relation.

Konstantinos *et. al.,* (2019) used cotton stalks in the production of fuel-ethanol due to its abundance and high carbohydrate content, it was observed that the direct conversion without pretreatment always results in extremely low yields due to the recalcitrant nature of lignocelluloses.

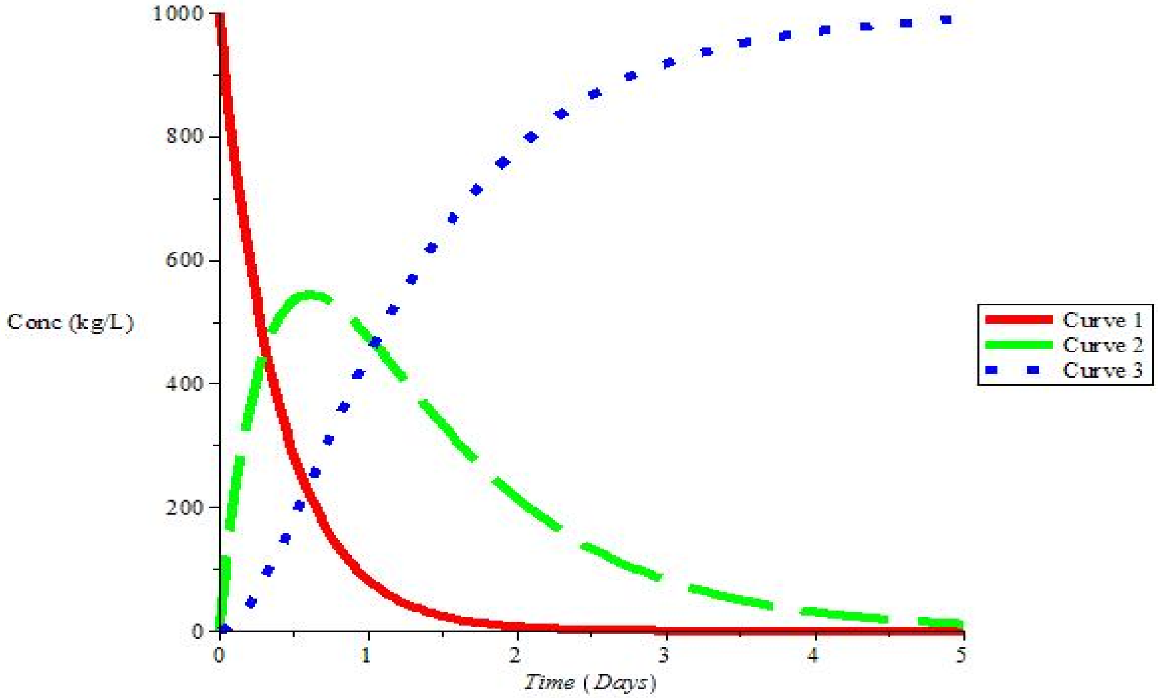


Figure 4.20: The rate of change of Unpretreated Banana Trunk Biomass, Glucose and Ethanol with time at β = 2.5 and γ = 1.0 (Batch Process)

Figure 4.20 illustrates the dynamics of banana trunk unpretreated biomass glucose and ethanol, in a batch process. The rate of depletion of banana trunk biomass was 2.5. The initial value of 1000 kg/L reduces to 0.00001 kg/L. at day 5. The glucose and ethanol yield increased at the rate of 1.0 to an optimum yield of 574 kg/L at day one, but on day two to five, there was a decrease in glucose due to the utilization of the available glucose concentration for the ethanol production. The ethanol yield increase at a rate of 1.0 and got to the optimum of 900 L on day three. The rate of change of ethanol with time is inversely proportional to that of biomass. This agrees with the works of Olaoye and Kolawole. (2013) and Konstantinos *et. al.*. (2019).

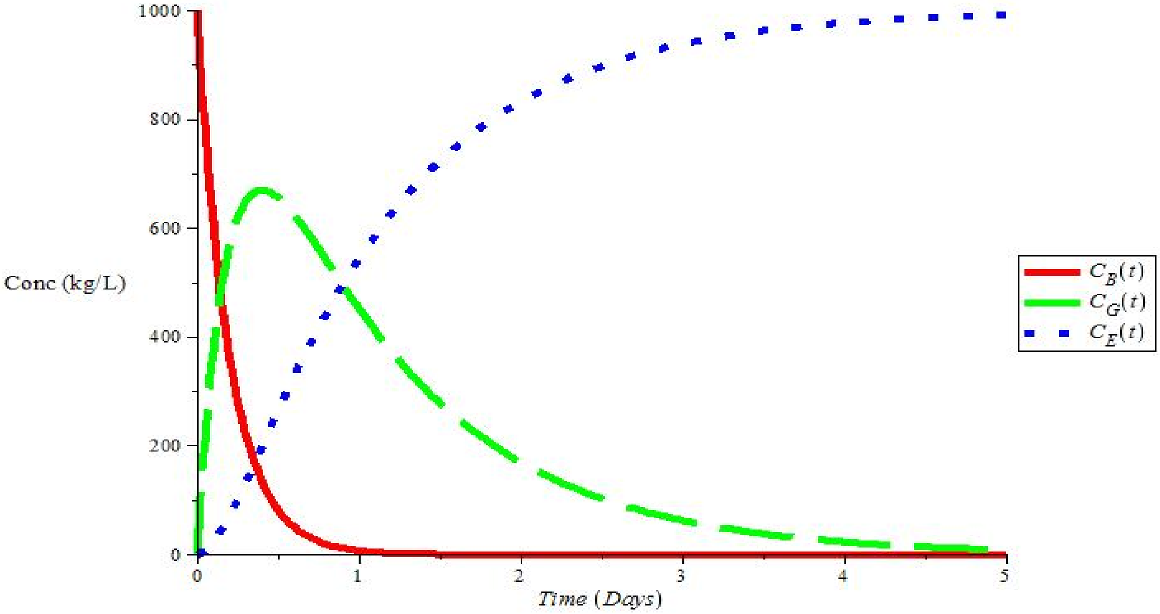


Figure 4.21: The rate of change of Unpretreated Banana Trunk Biomass, Glucose and Ethanol with time at β = 5 and γ = 1.0 (Batch Process)

Figure 4.21 illustrates the dynamics of banana trunk unpretreated biomass glucose and ethanol, in a batch process. The rate of depletion of banana trunk biomass is $\beta = 5$ given the initial value of 1000 kg/L to 0.00006 kg/L. on day 1. The glucose yield increased from zero at the rate of 1.0 to an optimum yield of 700 kg/L at half a day, but in day one to five there is decrease in glucose due to the utilization of the available glucose for the ethanol production. The ethanol yield increase at the rate of 1.0 and got to the optimum of 930 L on day two. The rate of change of ethanol with time was inversely proportional to that of biomass. This agrees with the works of Olaoye and Kolawole. (2013) and Konstantinos *et. al.*. (2019).

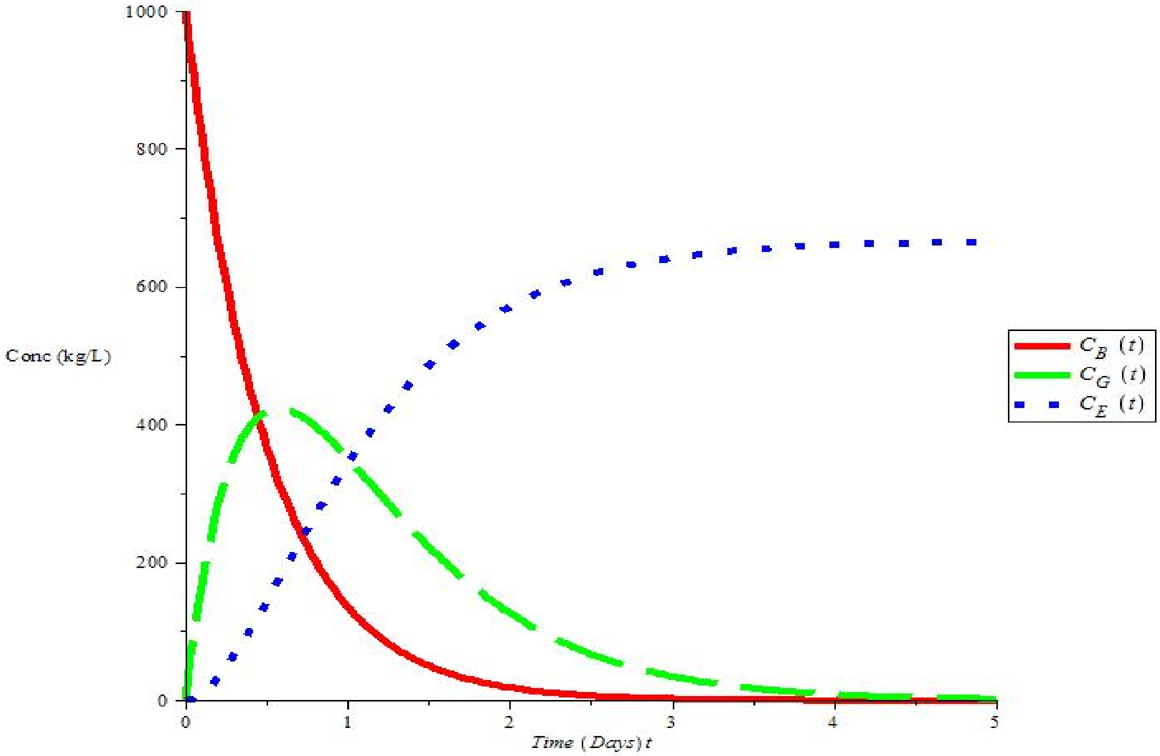


Figure 4.22: The rate of change of unpretreated banana trunk biomass, glucose and ethanol with time.

Figure 4.22 illustrates the dynamics of banana trunk unpretreated biomass, glucose and ethanol, in a batch process. The rate of depletion of banana trunk biomass is $\beta = 2.0,$ given the initial value of 1000 kg/L to 0.000096 kg/L. on day 5. The glucose yield increased from day one at the rate of 1.5 to an optimum yield of 4315 kg/L at 36 hours, but on day two to five there is decrease in glucose due to the utilization of the available glucose for the ethanol production. The ethanol yield increase at the rate of 1.5. The rate of change of ethanol with time was inversely proportional to that of biomass. This agrees with the works of Olaoye and Kolawole. (2013) and Konstantinos *et. al.*. (2019). In general, as the hydrolysis rate increases glucose and ethanol yield increases and also the time of production and optimum yield reduces for batch process of unpretreated biomass as seen in Figures 4,19 to 4.22.

This observation suggests that the use of unpretreated biomass for ethanol production would be possible; this would depend on the initial biomass and the rate of hydrolysis β.

4.4 Batch Process for the Pretreated Banana Trunk Biomass

Batch Process for the Pretreated Banana Trunk Biomass is given43 in figure (4.23) at α = 0.5, β = 0.6 and γ = 10.0 respectively.

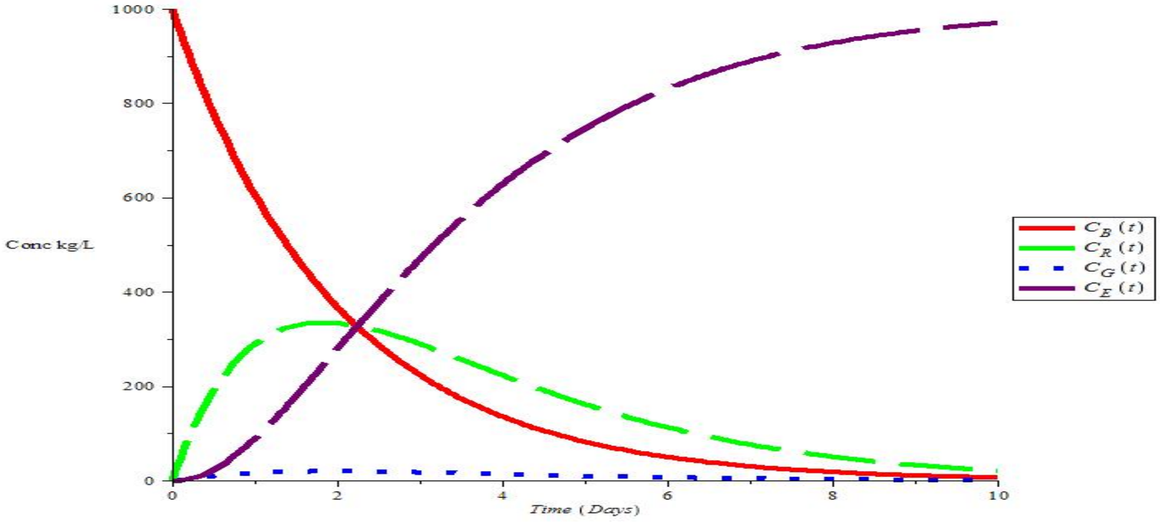


Figure 4.23: the rate of change of Pretreated Banana Trunk Biomass, Residue, Glucose and Ethanol with Time at α = 0.5, β = 0.6 and γ = 10.0 (Batch Process)

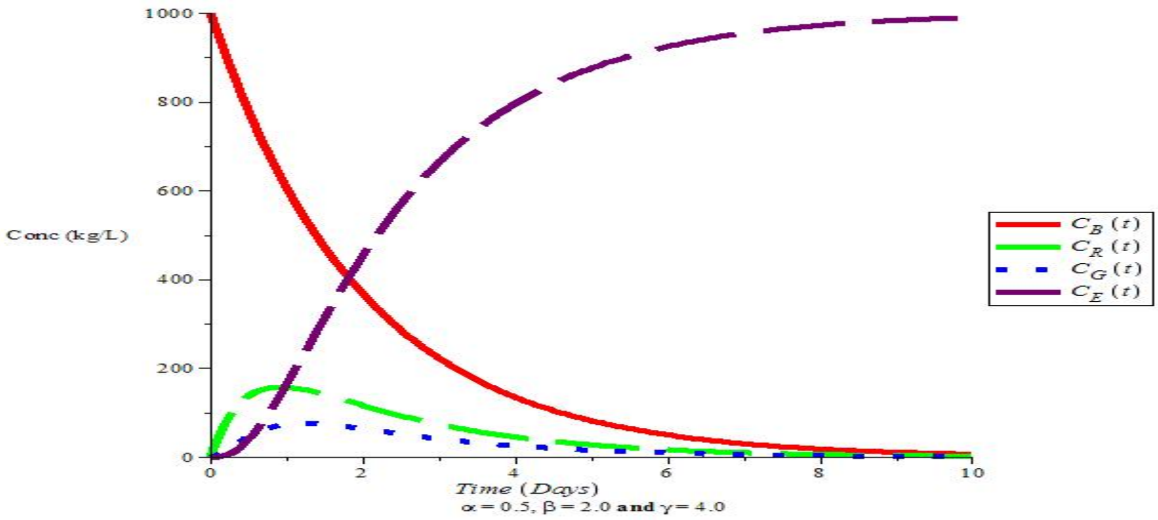


Figure 4.23a: The rate of change of Pretreated Banana Trunk Biomass, Residue, Glucose and Ethanol with time at β= 2 and γ=4.

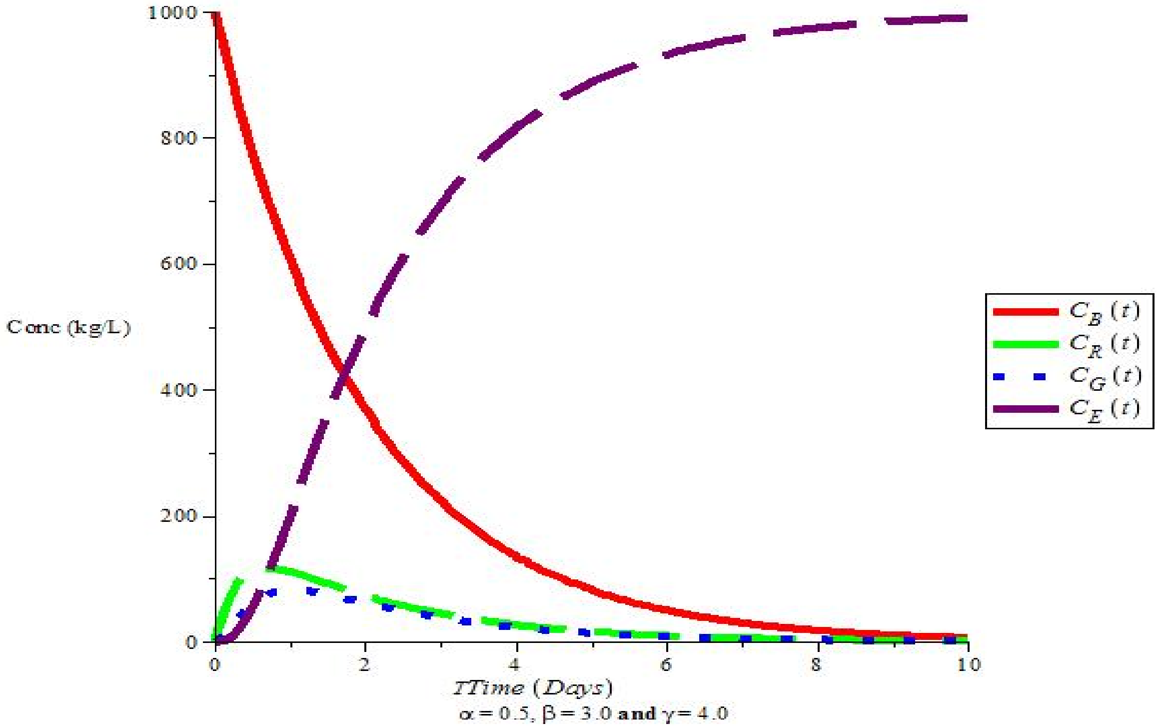


Figure 4.23b: The Rate of Change of Pretreated Banana Trunk Biomass, Residue, Glucose and Ethanol with Time at β= 3 and γ=4.

It was revealed that the rate of hydrolysis β was central in the optimized production of ethanol from pretreated biomass. By increasing β from 0.6 to 2.0, ethanol yield attained optimum on day 4, while, increasing β to 2.0 to 3.0 yielded optimum ethanol in less than 3 days. This clearly shows the equation (3.17), (3.21), (3.26) and (3.30) is useful for optimization of ethanol production from banana trunk biomass.

4.5 Continuous Process for the Unpretreated Banana Trunk Biomass

Batch Process for the Unpretreated Banana Trunk Biomass is given in figure (4.24 to 4.27) at α = 0.5, 2.5, 5, 0.6 and β = 1.0 and 10.0 respectively.

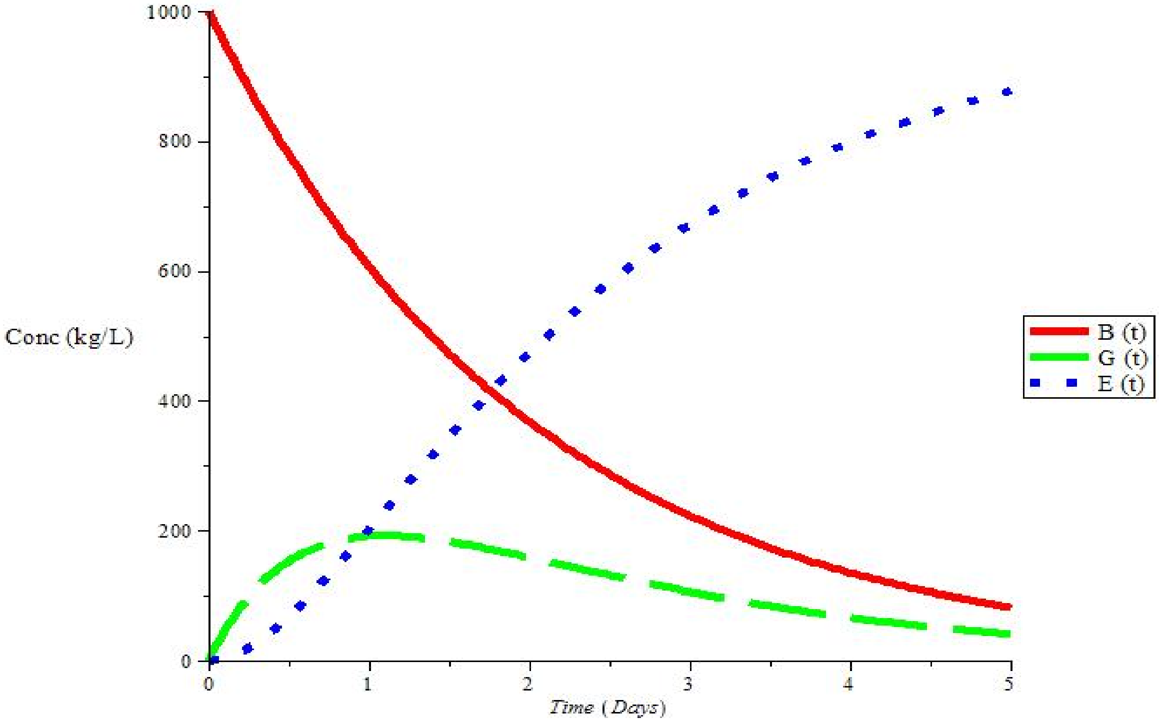


Figure 4.24: The Rate of Change Unpretreated Banana Trunk Biomass, Glucose and Ethanol with Time at β = 0.5 and γ = 1.0 (Continuous Process)

Figure 4.24 illustrates the dynamics of unpretreated banana trunk biomass, glucose and ethanol, in a continuous process. The rate of depletion of banana trunk biomass is $\beta = 0.5$, given the initial value of 1000 kg/L to 100 kg/L. on day 5. The glucose level increased from day one at the rate of 1.0 to an optimum yield of 200 kg/L on day one, but from day two to five there is decrease in glucose as the available glucose is being used for the ethanol production. The ethanol yield increase at the rate of 1.0 and gets to the optimum of 810 kg/L on day 5. The rate of change of ethanol with time is inversely proportional to that of biomass. This agrees with the works of Olaoye and Kolawole. (2013) and Konstantinos *et. al.,* (2019)

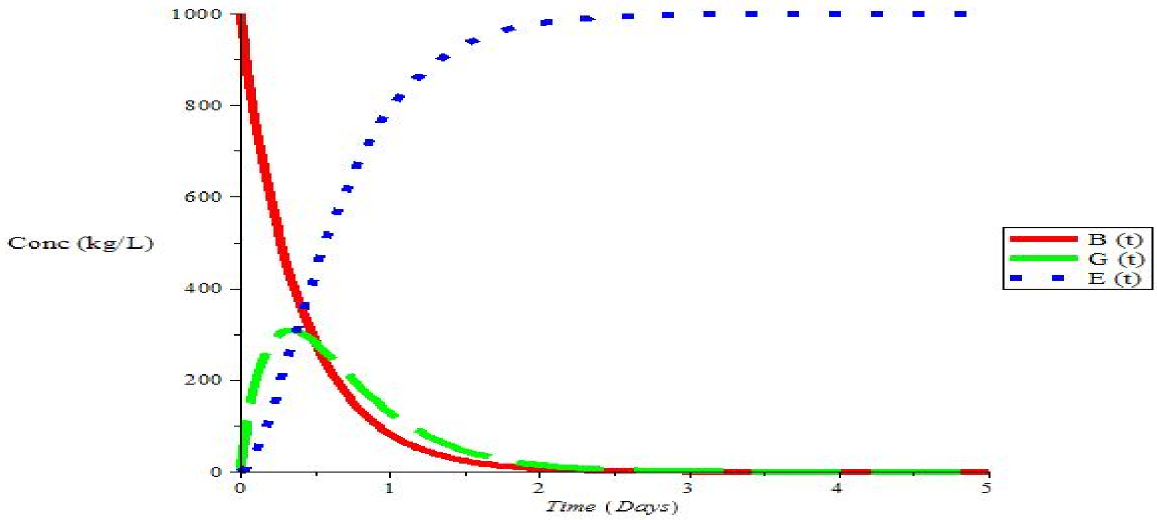


Figure 4.25: The Rate of Change Unpretreated Banana Trunk Biomass, Glucose and Ethanol with Time at β = 2.5 and γ = 1.0 (Continuous Process)

Figure 4.25 depicts the dynamics of unpretreated banana trunk biomass, glucose and ethanol, in a continuous process. The rate of depletion of banana trunk biomass is $\beta = 2.5$, given the initial value of 1000 kg/L to 0.00025 kg/L. on day 2. The glucose yield increases at the rate of 1.0 to an optimum yield of 320kg/L within 12 hours, but from one to three there is decrease in glucose as the available glucose is being used for the ethanol production. The Ethanol yield increase at the rate of 1.0 and gets to the optimum of 950 kg/L on day two. The rate of change of Ethanol with time is inversely proportional to that of biomass.

This agrees with the works of Olaoye and Kolawole. (2013) and Konstantinos *et. al.*. (2019).

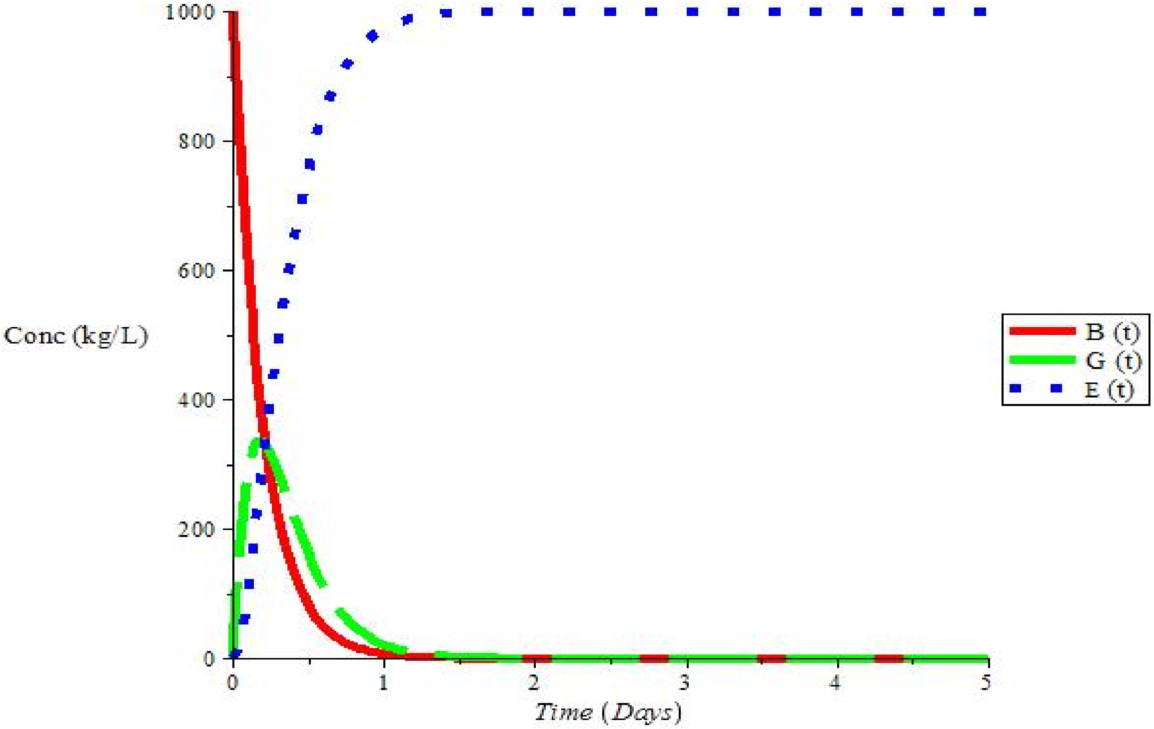


Figure 4.26: The Rate of Change Unpretreated Banana Trunk Biomass, Glucose and Ethanol with Time at β = 5 and γ = 1.0 (Continuous Process)

Figure 4.26 shows the dynamics of unpretreated banana trunk biomass, glucose and ethanol, in a continuous process. The rate of depletion of banana trunk biomass is $\beta = 5,$ given the initial value of 1000 kg/L to 0.00013 kg/L. on day 1. The glucose yield increases at the rate of 1.0 to an optimum yield of 370 g/l /l within 6 hours, but from 7hours to day one there is decrease in glucose since the available glucose is being used for the ethanol production. The ethanol yield increase at the rate of 1.0 and gets to the optimum of 1000g/l within day one. The rate of change of ethanol with time is inversely proportional to that of biomass as reported by Olaoye and Kolawole. (2013) and Konstantinos *et. al.*. (2019).

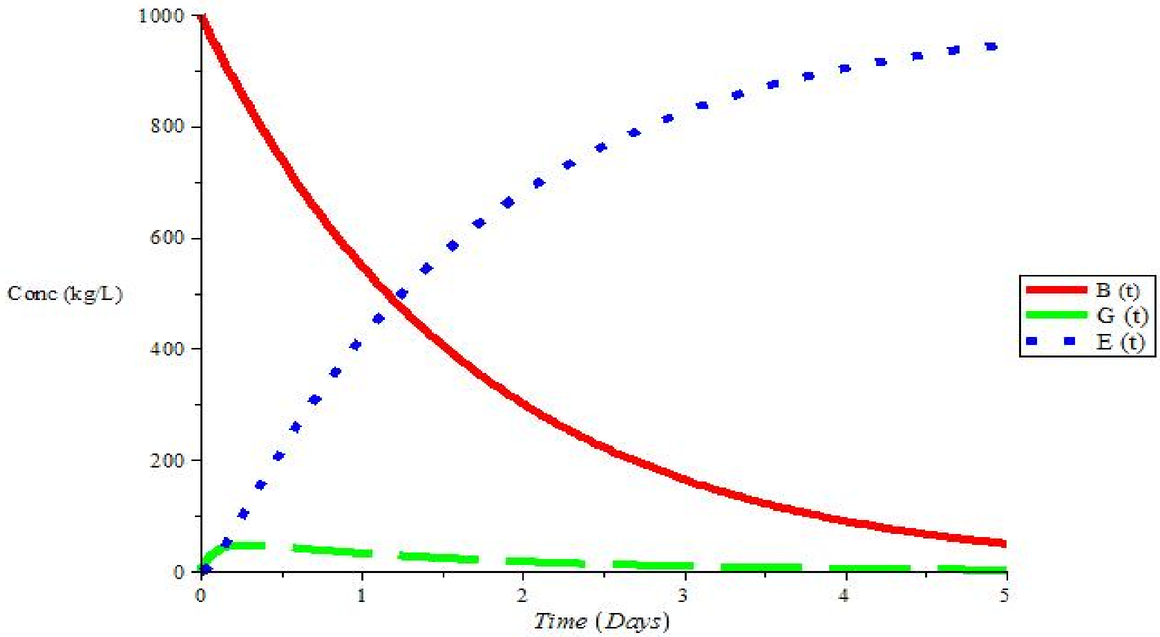


Figure 4.27: The Rate of Change Unpretreated Banana Trunk Biomass, Glucose and Ethanol with Time at β = 0.6 and γ = 10.0 (Continuous Process)

Figure 4.27 illustrates the rate of change of unpretreated banana trunk biomass, glucose and ethanol, in a continuous process. The rate of decrease of banana trunk biomass is $\beta = 0.6$, given the initial value of 1000 kg/L to 50 kg/L. on day 5. The glucose yield increases at the rate of 10.0 to an optimum yield of 60 kg/L within 2 hours, but latter, there was a decrease in glucose concentration since the available glucose is being used for the ethanol production. The ethanol yield increase at the rate of 10.0 and gets to the optimum of 800 kg/L on day 5. The rate of change of ethanol with time is inversely proportional to that of biomass. This agrees with the works of Olaoye and Kolawole. (2013) and Konstantinos *et. al.*. (2019).

In general, as the hydrolysis rate increases glucose and ethanol yield increases, the time of optimum production of ethanol reduced in continuous process of unpretreated biomass as seen in Figures 4.24 to 4.27. As the rate of fermentation increases it takes shorter time to obtain optimum yield. This finding has shown that the mathematical model in equation 3.31 to 3.33 and 3.34 to 3.36 is dependable to predicting the kinetic variable for optimum production of ethanol from unpretreated banana trunk biomass in a continuous process

4.6 Continuous Process for the Pretreated Banana Trunk Biomass

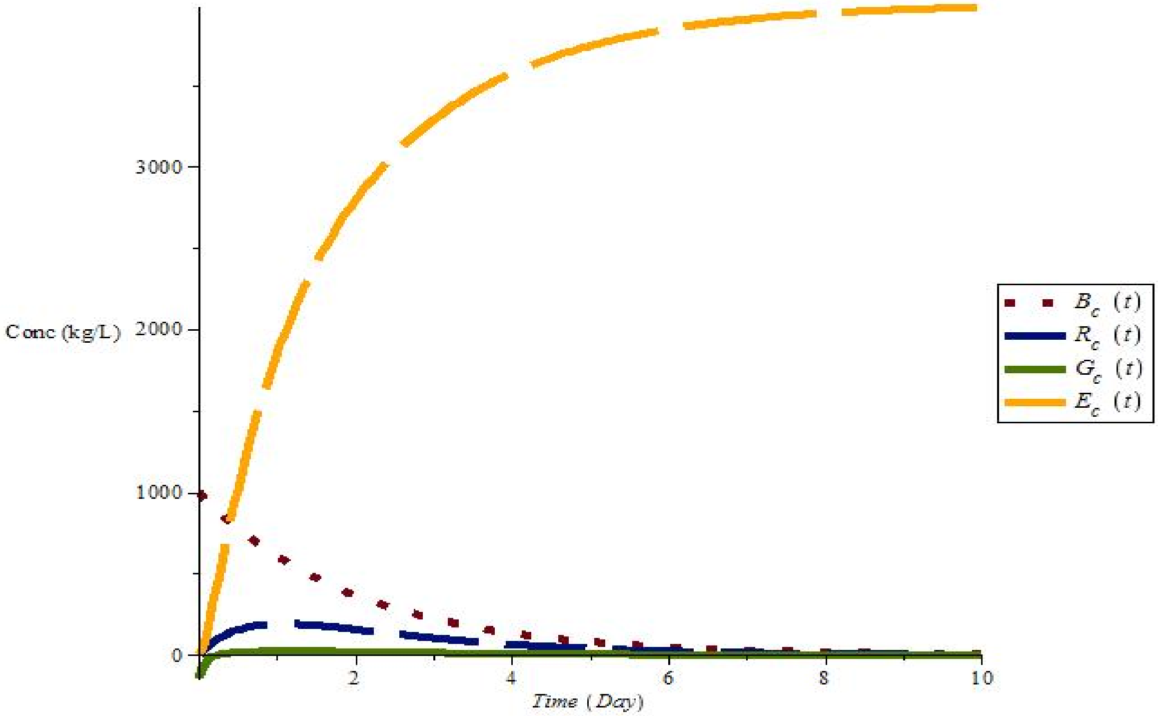


Figure 4.28: The Rate of Change Pretreated Banana Trunk Biomass, Residue, Glucose and Ethanol with Time at α = 0.5, β = 1.0 and γ = 10.0 (Continuous Process)

The kinetics of ethanol produced from pretreated banana trunk biomass in a continuous process is given in Figure 4.28. The pretreated process shows the dynamics of banana trunk biomass, residue, glucose yield and ethanol yield, banana trunk biomass decreases at the rate of $\alpha = 0.5$ from the initial value of 1000 kg/L to 0.00001 kg/L on day 5. The residue increases at the rate of 1.0 and get to the optimum of 284 kg/L, at day one it decreases at the rate of 1.0 as it is being used to form glucose. The glucose yield increases at the rate of 1.0 to an optimum yield of 3.9 kg/L on day two. It starts reducing at the rate of 10.0 due to the production of ethanol. The rate of production of ethanol is 10.0 with optimum of 3800 kg/L on day 5. The rate of change of Ethanol with time is inversely proportional to that of biomass. This agrees with the work of Konstantinos *et. al.,* (2019).

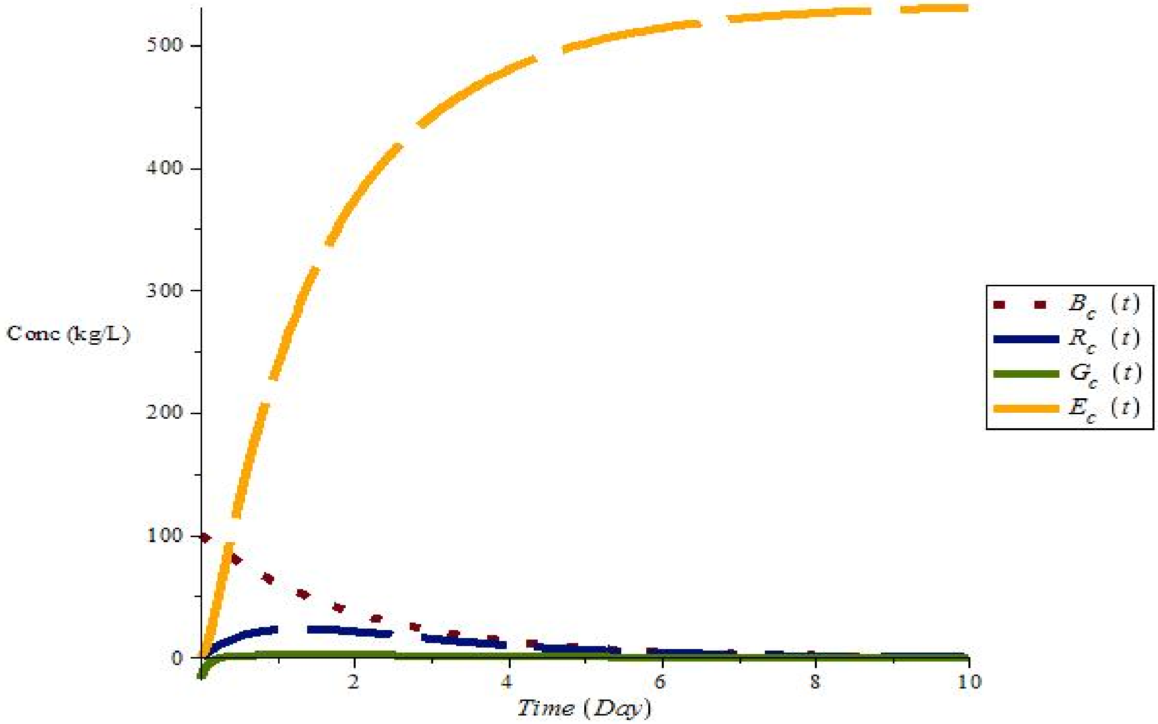


Figure 4.29: The Rate of Change Pretreated Banana Trunk Biomass, Residue, Glucose and Ethanol with Time at α = 0.5, β = 0.6 and γ = 10.0 (Continuous Process) B_c (0) = 100

Figure 4.29 illustrates the kinetics of ethanol produced from pretreated banana trunk biomass in a continuous process the pretreated process shows the dynamics of banana trunk biomass, residue, glucose yield and ethanol yield. Banana trunk biomass decreases at the rate of $\alpha = 0.5$ from the initial value of 100 kg/L to 0.00001 kg/L on day 4. The residue increases at the rate of 0.6 and get to the optimum of 26 kg/L on day one it decreases at the rate of 0.6 as it is being used to form glucose. As the glucose is formed almost all of it is converted to ethanol immediately in the continuous process at the rate of 10.0. The rate of production of ethanol was 10.0 with optimum of 500 kg/L at day 3. The rate of change of ethanol with time is inversely proportional to that of biomass. This agrees with the work of Konstantinos *et. al.,* (2019).

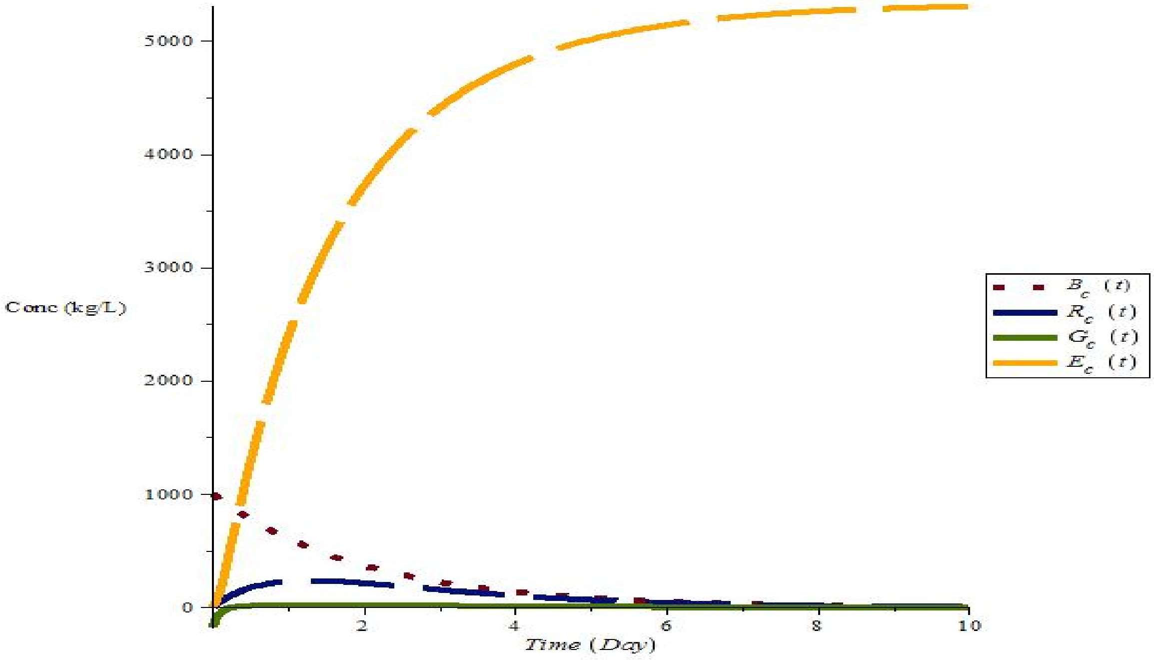


Figure 4.30: The Rate of Change Pretreated Banana Trunk Biomass, Residue, Glucose and Ethanol with Time at α = 0.5, β = 0.6 and γ = 10.0 (Continuous Process) B_c (0) = 1000

Figure 4.30 shows the kinetics of ethanol produced from pretreated banana trunk biomass in a continuous process. The pretreated process shows the dynamics of banana trunk biomass, residue, glucose yield and ethanol yield. Banana trunk biomass decreases at the rate of $\alpha = 0.5$ from the initial value of 1000 kg/L to 0.0025 kg/L on day 5. The residue increases at the rate of 0.6 and get to the optimum of 260 kg/L, on day one it decreases at the rate of 0.6 as it is being used to form glucose. As the glucose is formed almost all of it is converted to ethanol immediately in the continuous process at the rate of 10.0. The rate of production of ethanol is 10.0 with optimum of 5000 kg/L on day 3. The rate of change of

Ethanol with time is inversely proportional to that of biomass. This agrees with the work of Konstantinos *et. al.,* (2019).

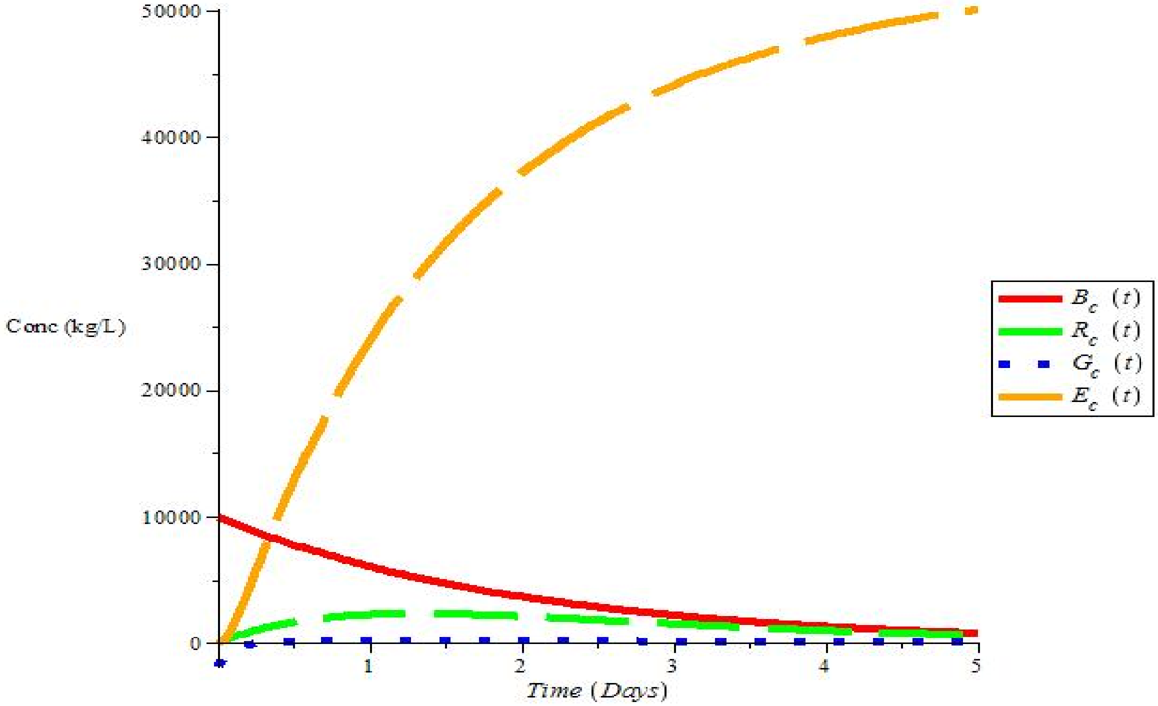


Figure 4.31: The Rate of Change Pretreated Banana Trunk Biomass, Residue, Glucose and Ethanol with Time at α = 0.5, β = 0.6 and γ = 10.0 (Continuous Process) B_c (0) = 10,000

Figure 4.31 depicts the kinetics of ethanol produced from pretreated banana trunk biomass in a continuous process. The pretreated process shows the dynamics of banana trunk biomass, residue, glucose yield and ethanol yield. banana trunk biomass decreases at the rate of $\alpha = 0.5$ from the initial value of 10000 kg/L to 0.045 kg/L on day 5. The residue increases at the rate of 0.6 and get to the optimum of 2600 kg/l, on day one it decreases at the rate of 0.6 as it is being used to form glucose. As the glucose is formed almost all of it is converted to ethanol immediately in the continuous process at the rate of 10.0. The rate of production of ethanol is 10.0 with optimum of 50000kg/L on day 3. The rate of change of ethanol with time is inversely proportional to that of biomass. This agrees with the work of Konstantinos *et. al.,* (2019).

The observation from changing initial values of biomass in continuous process with pretreated biomass shows that a 10 fold increase in initial biomass value resulted in a 10 fold increase in ethanol yield. This agrees with what is expected for any initial value problem (IVP). This further supports the fact that equations generated form equation 3.37, 3.38, 3.39 and 3.40 to equation 3.41, 3.42, 3.43 and 3.45 can be applied in the industrial scale for the optimal production of ethanol from pretreated banana trunk biomass.

In general, as the hydrolysis rate increases glucose and ethanol yield increases and also the time of production and optimum yield reduces for continuous process of pretreated biomass as seen in Figures 4,28 to 4.31. As the rate of fermentation increases it takes shorter time to obtain optimum yield. Figure 4.29 to 4.31 also shows that the higher the initial quantity biomass used the higher the ethanol yield.

4.7 Revenue Cost Function.

The results in equation (3.62) suggest that the producer of ethanol should employ 16.67 hours of human Labour, use 2.67kg/L of banana trunk biomass, 11.11 kg/L of residue and 3.37 kg/L of glucose to obtain revenue of ₦ 4,430. This result is not optimal. By varying the parameters, MAPLE17 Software has shown that, 3.7 hours of human labour, 44.44 kg/L of banana trunk biomass, 50 kg/L of residue and 19.048 kg/L of glucose will yield optimal revenue of ₦ 19,082 using similar computation as shown equation 3.82. See the variations in Table 4.1.

Table 4.1: Result of the Variation of the Parameter using MAPLE 17 Software.

S/No	*k*	*m*	*N*	*p*	*q*	*Hour* *(h)*	*Biomass* *(b)*	*Residue* *(r)*	*Glucose* *(g)*	*Lagrange* *Function* (λ)	*Revenue* *(R)*
1	200	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	16.67	2.67	1.053	0.58	0.87	2396
2.	200	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	17.78	2.13	13.33	3.81	1.0	3019
3	200	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	16.67	2.67	11.11	3.17	1.6	4430
4	200	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	14.04	3.37	10.53	3.01	1.7	5638
5	200	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{8}$	13.68	3.28	10.26	4.40	1.8	7805
6	200	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	30.90	0.27	2.58	0.74	4.0	1310
7	200	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{4}$	30.90	0.27	2.58	0.74	4.0	1646
8	200	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	8.33	2.00	25.00	14.29	0.4	1756
9	200	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	6.67	0.29	20.00	11.43	0.7	2235
10	200	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	6.67	1.60	20.00	22.86	1.1	3654
11	200	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{4}$	1.85	0.89	33.33	28.57	5.5	14,731
12	200	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{3}{4}$	$\frac{1}{2}$	3.70	0.44	50.00	19.05	7.2	19,082

The result obtained from the numerical experiment shows that by using the formulated model the producers of ethanol stand to have optimal revenue of 19,045 naira which is better than the current revenue. By this, the producers of ethanol will be able to supply to the retailers at a cost that enables retailers to make more profits.

4.8 Second Order Partial Derivative is given in the Enigmatic Bordered Hessian Matrix (3.116)

The hessian matrix obtained in (3.116) is a Symmetric matrix The Principal Minor determinates are,

The 1st principal minor determinant = – 193.20

The 2nd principal minor determinant is 1.399740440×10^6

The 3rd principal minor determinant is – 4.7241

The 4rd principal minor determinant is -1.404622276×10^6

The 5th principal minor determinant is $1.130920568 \times 10^{11}$

According to Taha (2010), Since the principal minor determinants are indefinite, this indicates a saddle point.

The indefinite result obtained shows that the optimum solution by the formulated revenue model is actually a saddle that can be obtained under that market condition (Evans *et. al*., 2021b). This implies that, to achieve optimum revenue, the producer of ethanol from banana trunk biomass should maintain a scaling of;

$m = \frac{1}{6},\ n = \frac{1}{12},\ p = \frac{3}{4},\ p = \frac{1}{2}\ and\ \ parameters\ \ h = 3.70,\ b = 0.44,\ r = 50,\ g = 19.05,$

The present study concludes that the formulated revenue model fulfills all the conditions of KKT, it is therefore a reliable model for revenue optimisation.

CHAPTER FIVE

5.0 CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

In conclusion the initial value problem for the production of ethanol from banana trunk biomass was formulated, the models obtained were used to determine glucose and ethanol yields respectively. The optimum glucose yield was obtained for the four cases considered. The ethanol yield was inversely proportional to the biomass consumption.

The results obtained from the batch and continuous process for pretreated and unpretreated biomass shows that:

i. Initial value problem formulated in this work is effective in determining the effect of kinetic variables on the optimum yield of ethanol from banana trunk biomass

ii. A revenue model was formulated to obtain the optimum revenue for ethanol production.

iii. Increase in the rate of fermentation of glucose also reduced the time taken to attain optimal yield of ethanol in the batch process of unpretreated biomass.

iv. The higher the hydrolysis rate of glucose, the higher the yield of ethanol. Increase in the rate of glucose hydrolysis also reduced the time taken to attain optimal yield of ethanol in the batch process of unpretreated biomass.

v. As the rate of fermentation increased it took shorter time to obtain optimum ethanol yield.

vi. Comparing batch and continuous process, it was observed higher yield of ethanol was obtained from continuous process both for pretreated and unpretreated, but more ethanol was attained from pretreated biomass in the continuous process.

vii. The revenue model obtained in this work shows that to get optimum revenue for ethanol from banana trunk biomass, 3.7 man hour, 44.44kg/L of banana trunk biomass, 50 kg/L of residue and 19.048L of glucose will be required to get an optimum revenue of ₦ 19,082 for a profit of ₦ 15,082. This model fulfills all the conditions of KKT and hence, establishes the reliability of the model.

5.2 Contribution to Knowledge

The consecutive reaction models were modified to obtain Initial Value Problem (IVP equation) capable of determining the effects of kinetic variables on banana trunk biomass, residue, glucose and ethanol yield.

The revenue model to determine the optimum revenue for the ethanol production were formulated.

The Karush-Kuhn Tucker (KKT) conditions were used to determine the optimality of the optimum revenue obtained.

1. The Consecutive Reaction models were modified to obtain Initial Value Problem capable of determining the effects of kinetic variables α, β and γ on banana trunk biomass, residue, glucose and ethanol yield.

2. The revenue model to determine the optimum revenue for the ethanol production were formulated. and Karush-Kuhn Tucker (KKT) conditions were used to determine the optimality of the optimum revenue obtained..
3. A modified consecutive reaction model that optimized the various parameters such as time t, effect of pretreatment α and hydrolysis rate β on the concentration of biomass as well as the parameters in the fermentation of the resulting fermentable sugars, such parameters includes fermentation time and fermentation rate γ on ethanol yield was developed. Overall, the experiment is expected to bring out parameters that could leads to batch and continuous reactors for commercialization of bio ethanol production from banana trunk biomass at $\alpha = 0.5$, $\beta= 0.6$ and $\gamma=10$. The method shows a 5-fold optimum yield of ethanol for pretreated biomass in continuous reactors as compare with the batch process. Further analysis was obtained for the optimisation of the revenue of the various parameters leading to the production of ethanol from Banana trunk biomass gives optimum revenue of ₦ 19,082 naira at $\lambda=7.2$.

5.3 Recommendations

1. These formulated models recommend for optimum yield of ethanol from banana trunk biomass to the industrialists.
2. Pilot plants for ethanol production should be developed by using these models
3. For further research other agro waste biomass can be investigated for ethanol production using similar mathematical models.

REFERENCES

Achinas, S. & Euverink, G. J. W. (2016). Consolidated Briefing of Biochemical Ethanol Production from Lignocellulosic Biomass. *Electronic Journal of Biotechnology*, *23*, 44–53. https://doi.org/10.1016/j.ejbt.2016.07.006

Alex M. W., Schmidt, L. D., Benjaafar, S., Tiffany, D. G. & Daoutidis, P. (2012). Economic Optimization of a Lignocellulosic Biomass-to-Ethanol Supply Chain. *Chemical Engineering Science*, *67*(1), 68–79. https://doi.org/10.1016/j.ces.2011.05.055

Alfa, M.. Okuofu, C. A., Adie, D. B., Dahunsi, S.O., Oranusi, U.S. & Idowu. S.A, (2012). Evaluation of Biogas Potentials of *Cymbopogon citratus* as Alternative Energy in Nigeria. *International Journal of Green Chemistry and Bioproces*, 2(4), 34 – 38.

Asmanto, S., & Adchmad, C. (2018). Banana Pseudo-Stem Fiber: Preparation, Characteristics, and Applications. Intech. *DOI*: http://dx.doi.org/10.5772/intechopen .82204

Atkins, P. & de Paula, J. (2006). *Physical Chemistry* (8th ed.) W.H. Freeman 793 ISBN 0-7167-8759-8.

Badger, P. C. (2002). *Ethanol from Cellulose a General Review*. ASHS Press, Alexandra. Pp 17 – 21.

Balusu, R., Paduru, R. R., Kuravi, S. K., Seenayya, G. & Reddy, G. (2005). Optimization of critical medium components using response surface methodology for ethanol production from cellulosic biomass by *Clostridium thermocellum* SS19. *Process Biochemistry*, *40*(9), 3025–3030. https://doi.org/10.1016/j.procbio.2005.02.003

Bazarra, M., Shrali, H. & Shetty, C. (1993). *Nonlinear Programming Theory and Algorithms,* 2nd ed.,Wiley, New York,

Beightler, C, Phillips, D. & Wilde, D. (1979). *Foundations of Optimization,* 2nd ed., Prentice Hall, New Jersey,.

Bonvicini, G. (2015). Algae Bioenergy Siting, Commercial Deployment and Development Analysis. Final Feport. Brussels: European Commission DG ENERGY no. 12–920-H3.

Chen, F, Srinivasa R. M. S., Temple S., Jackson L., Shadle G. & Dixon R. A. (2006). Multi-site Genetic Modulation of Monolignol Biosynthesis Suggests New Routes for Formation of Syringyl Lignin and Wall-bound Ferulic Acid in Alfalfa (*Medicago sativa* L.). *Plant Journal*. 48,113–124. http://dx.doi.org/10.1111/j.1365-313X.2006.02857.x.

Chin, K. L, Hing, P. S., Wong, L. J., Tey B. T. & Paridah M. T. (2011). Production of Glucose from Oil Palm Trunk and Sawdust of Rubber Wood and Mixed Hardwood. *Applied Energy*. 88:4222–4228.

Celis, K. I., Driessche, R., Mouton, G. & Vanhoylands, H. (2001). Kinetics of Consecutive Reactions in the Solid State: Thermal Decomposition of Oxalates. ResearchGate. Pp 206 – 213. DOI:10.4028/www.scientific.net/KEM.206-213.807

Cobb, C. W. & Douglas, P. H. (1928) A Theory of production. American Economic, Review. Supplement Paper and Proceedings of the Fortieth Annual Meeting of the American Economic Association, 18(1), 139 – 167.

Egwim, E.C., Shittu, K. O. & Komolafe, D. (2015). Microwave – Alkaline Assisted Pretreatment of Banana Trunk for Bioethanol Production. *Journal of Energy and Power Engineering,* 9, 705 – 713. doi: 10.17265/1934-8975/2015.08.003

Espenson, J.H. (2002). *Chemical Kinetics & Reaction Mechanisms* (2nd ed., McGraw-Hill 2002), p.254-256 ISBN 0-07-288362-6

Evans P. O., Jiya M., Nyor N. & Yisa J. (2021a). Mathematical Modeling of Enzyme Hydrolysis and Fermentation of Banana Trunk Biomass for Optimum Ethanol Production, A paper presented at the International Conference On Contemporary Developments In Mathematical Sciences **(**ICCDMS**)** In Honour Of Prof. K. R. Adeboye. Book of Proceedings, 368 - 381**.**

Evans P. O., Jiya M., Nyor N., & Yisa J. (2021b) On the Necessary and Sufficient Conditions for Optimality of Pat-Jiyor Model. *Journal of Science, Technology, Mathematics and Education (JOSTMED)*, 17(1), 110 – 115**.**

Farah, A., Ahmad, T., Jameel, M., Hider K. & Maizirwan, M. (2011). Study of Growth Kinetic and Modeling of Ethanol Production by *Saccharomyces cerevisae*. *African Journal of Biotechnology,* 16(1), 18842 – 18846.

Girio, F. M., Fonseca, C., Carvalheiro, F., Duarte, L. C., Marques, S. & Bogel-Lukasic, R. (2010). Hemicelluloses for Fuel Ethanol: A Review. *Bioresour Technol*. 101, 4775–4800. http://dx.doi.org/10.1016/j.biortech.2010.01.088.

Gorban, A. N. & Yablonsky, G. S. (2015). Three Waves of Chemical Dynamics. *Mathematical Modelling of Natural Phenomena*, 10(5), 1–5.

Graeme, M. W. (2010). Bioethanol: Science and Technology of Fuel Alcohol. GraemeM. Walker and Ventus Publishing Aps. Copenhagen. Pp 49-52.

Hahn-H, B., Karhumaa, H. B. K., Fonseca, C., Spencer-Martins, I. & Gorwa- Grauslund M. F. (2007). Toward Industrial Pentose-Fermenting Yeast Strains. *Applied Microbiology Biotechnology,* 74, 937-953.

Hossain, A. B. M. S., Saleh, A. A., Aishah, S., Boyce, A. N., Chowdhury, P. P., & Naqiuddin, M. (2008). Bioethanol Production from Agricultural Waste Biomass as a Renewable Bioenergy Resource in Biomaterials. *IFMBE Proceedings*, 21(1), 300–305. https://doi.org/10.1007/978-3-540-69139-6-77

Khan Academy (2016). Langrange Multiplier.

https://www.khanacademy.org/maths/multivariable.calculusapplication-of-multivariable-derivatives/constraints-optimizationvialagrange-multipliers.example

Klinke, H. B., Thomsen, A. B. & Ahring, B. K. (2004). Inhibition of Ethanol-producing Yeast and Bacteria by Degradation Products Produced during Pretreatment of Biomass. *Applied Microbiology Biotechnology,* 66, 10-26.

Konstantinos, D., Thomas, P., Argiro, L., Konstantinos, G. K., Angelos, A. L., Nikolaos, P., Dimitris, K. & Diomi, M. (2019). Effect of Various Pretreatment Methods on Bioethanol Production from Cotton Stalks. *Fermentation Journal,* 5(5), 1 – 12. doi:10.3390/fermentation5010005 www.mdpi.com/journal/fermentation

Kumar, M. S. (2009). Enzymes: Creating Opportunities for Greener Industries in Europe. Retrieved from Frost website: http://www.frost.com/prod/servlet/marketinsighttop.

Li, A. (2008). Bioconversion of Biodegradable Municipal Solid Waste (BMSW) to Glucose for Bio-ethanol Production. London: University College of London; [unpublished Doctoral thesis].

Lin, Y., Zhang, W., Li, C., Sakakibara, K., Tanaka, S. & Kong, H. (2014). Factors Affecting Ethanol Fermentation using *Saccharomyces cerevisiae* BY4742. *Biomass and Bioenergy*, *47*, 395–401. https://doi.org/10.1016/j.biombioe.2012.09.019

Meilana, D., Putra, A. E., Abasaeed, H. K., Ativeh, S. M., Al-Zahrani, M. H. & Ashraf, K. S. (2014). Kinetic Modeling and Enhanced Production of Fructose and Ethanol from Date Fruit Extract. *Chemical Engineering Communications,* 202(12), 16 – 18.

Mendes, P. & Kell, D. B. (1998). Non-linear Optimization of Biochemical Pathways: Applications to Metabolic Engineering and Parameter Estimation. *Bioinformatics,* 14, 869-883.

Mervin L. D., Lanka, S., & Drye, V. S. (2011). Application of Cobb – Douglas Function for Analyzing Agricultural Production: A Case Study from Sri Lanka. *Transaction,* 33(2), 251 - 263

Mielenz J. R. (2001). Ethanol production from biomass: technology and commercialization status. *Current Opinion in Microbiology* 4, 324–329.

Mohamed, H., Mervat, A., Abo-State, S. A. E.-T. & El-Sheikh, H. H. (2018). A Study on Ticks Affecting Camels (*Camelus dromedarius*) in Jigjiga District of Somali Region , Eastern Ethiopia. *International Journal of Advanced Research in Biological Sciences*, *5*, 121–130. https://doi.org/10.22192/ijarbs

Mounira, K., Ali, S., Hiligsmann, N., Outili, R. C. & Noreddine, K. C. (2017). Kinetic Models and Parameters Estimation Study of Biomass and Ethanol Production from

Inulin by *Pichia caribbica*. *African Journal of Biotechnology,* 16(3), 124-131, DOI:10.5897/AJB2016.1574

Mosier N., Hendrickson R., Brewer M., Ho N., Sedlak M. & Dreshel R. (2005). Industrial Scale-up of pH-controlled Liquid hot water pretreatment to Corn Fiber for Fuel Ethanol Production. *Applied Biochem Biotechnol*, 125, 77–97.

Murdoch, J. R., (1981). What is the Rate-Limiting Step of a Multistep Reaction? *Journal of Chemical Education*, 58(1),32 – 36. Bibcode: 1981 JChEd. 58..32M.doi:10.1021/ed058p32

Nyor N., Evans P. O., Jiya M. and Yisa J. (2018). A Lagrange Revenue Model For Ethanol Production from Biomass. *Transactions of the Nigerian Association of Mathematical Physics*. 7, 225 – 228.

Obama, P., Ricochon, G., Muniglia, L. & Brosse, N. (2012). Combination of Enzymatic Hydrolysis and Ethanol Organosolv Pretreatments: Effect on Lignin Structures, Delignification Yields and Cellulose-to-glucose Conversion. *Bioresource Technology*; 112, 156–63.

Olaoye, O. S. & Kolawole, O. S., (2013). Modeling of the Kinetics of Ethanol fermentation from Glucose Biomass in Batch Culture with a Non – Structural Model. *International Journal of Engineering Reseach and Application*, 3(4), 562 – 565.

Oliveira, S. C., Oliveira, R. C., Tacin, M. V. & Gattás, E. A. L. (2016). Kinetic Modeling and Optimization of a Batch Ethanol Fermentation Process. *Journal of Bioprocess Biotech* 6, 266. doi:10.4172/2155-9821.1000266

Par Astudila, I. C. & Cardona, A. C. A. (2011). Importance of Stability Study of Continuous Systems for Ethanol Production. *Journal Bioethanol,* 151, 43 – 55.

Rardin, R.,(1998.) *Optimization in Operations Research,* Prentice Hall, New Jessey.

Saha, P., Baishnab, A. C., Alam, F., Khan, M. R. & Islam, A. (2014). Production of bio-fuel (Bio-ethanol) from Biomass (pteris) by Fermentation Process with Yeast. *Procedia Engineering*, *90*, 504–509. https://doi.org/10.1016/j.proeng.2014.11.764

Salih, A. (2013). Department of Aerospace Engineering Indian Institute of Space Science and Technology, Thiruvananthapuram.

Sheetal, B. G. & Patil, I. D. (2014). Economic Study of Fermentation Kinetics for Production of Ethanol from damaged Sorghum and Corn grains: a Critical Review. *Pratibha: International Journal of Science, Spiritual, Business and Technology* (IJSSBT), 2(2), 32 – 37.

Shah, N. & Rehan, T. (2014). Bioethanol Production from Biomass. *Journal of Chemistry* *and Biochemistry*, *2*(2), 161–167. https://doi.org/10.15640/jcb.v2n2a8

Shehu, M. D., Jiya, M., Elebute, K. O. & Ahmed, H. O. (2017). Optimal Control of Beer Fermentation Process Using Differential Transform Method *Journal of Appl. Sci. Environ. Manage,* 21(4), 751-758. Full-text Available Online at www.ajol.info and www.bioline.org.br/ja

Singh, R., Shukla, A., Tiwari, S. & Srivastava, M. (2014). A Review on Delignification of Lignocellulosic Biomass for Enhancement of Ethanol Production Potential. *Renewable and Sustainable Energy Reviews*, *32*, 713–728. https://doi.org/10.1016/j.rser.2014.01.051

Snehal, I., Sanket J. J. & Akshaya, G. (2014). Production of Bioethanol using Agricultural Waste: Banana Pseudo Stem. *Brazilian Journal of Microbiology*, 45(3), 885-892. ISSN 1678-4405 www.sbmicrobiologia.org.br

Taha, H. A. (2010) *Operations Research: An Introduction* ~9th ed. Pearson and Prentice Hallm upper saddle River, New Jersey. ISBN 0-13-188923·0

Taherzadeh, M. J. & Karimi, K. (2007). Acid-Based Hydrolysis Processes for Ethanol from Lignocellulosic Materials: A Review. *BioResources*, 2, 472-499.

Yang, W. & Shijie, L. (2014). Kinetic Modeling of Ethanol Batch Fermentation by *Escherichia Coli* FBWHR Using Hot-Water Sugar Maple Wood Extract Hydrolyzate as Substrate. *Energies, 7*, 8411-8426; doi:10.3390/en7128411 www.mdpi.com/journal/energies

Zumdahl, S. S. (2005). *Chemical Principles* (5th ed.). Houghton Mifflin. Pp. 727–8. ISBN 0618372067.

APPENDIX A

FIGURES 4.1 TO 4.3

> *restart*;

> $C(t) := C[B0]\cdot\exp(-\beta\cdot t);$

$$C(t) := C_{B0}\,\mathrm{e}^{-\beta t}$$

> $C[G](t) := \dfrac{\beta\cdot C[B0]}{(\gamma-\beta)}\cdot(\exp(-\beta\cdot t)-\exp(-\gamma\cdot t));$

$$C_G(t) := \frac{\beta\,C_{B0}\left(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\gamma t}\right)}{\gamma-\beta}$$

> $C[E](t) := \dfrac{\beta\cdot C[B0]}{(\gamma-\beta)}\cdot\left(\dfrac{1}{\gamma}\cdot\exp(-\gamma\cdot t)-\dfrac{1}{\beta}\cdot\exp(-\beta\cdot t)+\left(\dfrac{1}{\beta}-\dfrac{1}{\gamma}\right)\right);$

$$C_E(t) := \frac{\beta\,C_{B0}\left(\frac{\mathrm{e}^{-\gamma t}}{\gamma}-\frac{\mathrm{e}^{-\beta t}}{\beta}+\frac{1}{\beta}-\frac{1}{\gamma}\right)}{\gamma-\beta}$$

> $B := eval\left(C(t), \left\{C_{B0}=1000, \beta=0.5, \gamma=2\right\}\right);$

$$B := 1000\,\mathrm{e}^{-0.5\,t}$$

> $B1 := eval\left(C(t), \left\{C_{B0}=1000, \beta=1.0, \gamma=2\right\}\right);$

$$B1 := 1000\,\mathrm{e}^{-1.0\,t}$$

> $B2 := eval\left(C(t), \left\{C_{B0}=1000, \beta=1.5, \gamma=2\right\}\right);$

$$B2 := 1000\,\mathrm{e}^{-1.5\,t}$$

> *plot*([*B*, *B1*, *B2*], *t* = 0 ..1, *thickness* = [1, 2, 3], *colour* = [*red*, *green*, *blue*], *linestyle* = [*solid*, *dash*, *dot*]);

> $B3 := eval\left(C_G(t), \left\{C_{B0}=1000, \beta=0.5, \gamma=2\right\}\right);$

$$B3 := 333.3333333\,\mathrm{e}^{-0.5\,t}-333.3333333\,\mathrm{e}^{-2\,t}$$

> $B4 := eval\left(C_G(t), \left\{C_{B0}=1000, \beta=1.0, \gamma=2\right\}\right);$

$$B4 := 1000.000000\,\mathrm{e}^{-1.0\,t}-1000.000000\,\mathrm{e}^{-2\,t}$$

> $B5 := eval\left(C_G(t), \left\{C_{B0}=1000, \beta=1.5, \gamma=2\right\}\right);$

$$B5 := 3000.000000\,\mathrm{e}^{-1.5\,t}-3000.000000\,\mathrm{e}^{-2\,t}$$

>

> *plot*([*B3*, *B4*, *B5*], *t* = 0 ..5, *thickness* = [1, 2, 3], *colour* = [*red*, *green*, *blue*], *linestyle* = [*solid*, *dash*, *dot*]);

> $B6 := eval\left(C_E(t), \left\{C_{B0}=1000, \beta=0.5, \gamma=2\right\}\right);$

$$B6 := 166.6666666\,\mathrm{e}^{-2\,t}-666.6666666\,\mathrm{e}^{-0.5\,t}+500.0000000$$

> $B7 := eval\left(C_E(t), \left\{C_{B0}=1000, \beta=1.0, \gamma=2\right\}\right);$

$$B7 := 500.0000000\,\mathrm{e}^{-2\,t} - 1000.000000\,\mathrm{e}^{-1.0\,t} + 500.0000000$$

> $B8 := eval\left(C_E(t), \left\{C_{B0} = 1000, \beta = 1.5, \gamma = 2\right\}\right);$

$$B8 := 1500.000000\,\mathrm{e}^{-2\,t} - 2000.000000\,\mathrm{e}^{-1.5\,t} + 500.0000001$$

> $plot([B6, B7, B8], t = 0\,..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]);$

APPENDIX B

FIGURES 4.4 TO 4.6

$>$ $\mathit{restart};$

$>$ $C(t) := C[B0]\cdot\exp(-\beta\cdot t);$

$$C(t) := C_{B0}\,\mathrm{e}^{-\beta t}$$

$>$ $C[G](t) := \dfrac{\beta\cdot C[B0]}{(\gamma-\beta)}\cdot(\exp(-\beta\cdot t)-\exp(-\gamma\cdot t));$

$$C_G(t) := \frac{\beta\, C_{B0}\left(\mathrm{e}^{-\beta t}-\mathrm{e}^{-\gamma t}\right)}{\gamma-\beta}$$

$>$ $C[E](t) := \dfrac{\beta\cdot C[B0]}{(\gamma-\beta)}\cdot\left(\dfrac{1}{\gamma}\cdot\exp(-\gamma\cdot t)-\dfrac{1}{\beta}\cdot\exp(-\beta\cdot t)+\left(\dfrac{1}{\beta}-\dfrac{1}{\gamma}\right)\right);$

$$C_E(t) := \frac{\beta\, C_{B0}\left(\frac{\mathrm{e}^{-\gamma t}}{\gamma}-\frac{\mathrm{e}^{-\beta t}}{\beta}+\frac{1}{\beta}-\frac{1}{\gamma}\right)}{\gamma-\beta}$$

$>$

$>$ $B := \mathit{eval}\left(C(t), \left\{C_{B0}=1000, \beta=0.2, \gamma=0.5\right\}\right);$

$$B := 1000\,\mathrm{e}^{-0.2\,t}$$

$>$ $B1 := \mathit{eval}\left(C(t), \left\{C_{B0}=1000, \beta=0.2, \gamma=1.0\right\}\right);$

$$B1 := 1000\,\mathrm{e}^{-0.2\,t}$$

$>$ $B2 := \mathit{eval}\left(C(t), \left\{C_{B0}=1000, \beta=0.2, \gamma=1.5\right\}\right);$

$$B2 := 1000\,\mathrm{e}^{-0.2\,t}$$

$>$ $\mathit{plot}([B, B1, B2], t=0..1, \mathit{thickness}=[1,2,3], \mathit{colour}=[\mathit{red}, \mathit{green}, \mathit{blue}], \mathit{linestyle}=[\mathit{solid}, \mathit{dash}, \mathit{dot}]);$

$>$ $B3 := \mathit{eval}\left(C_G(t), \left\{C_{B0}=1000, \beta=0.2, \gamma=0.5\right\}\right);$

$$B3 := 666.6666667\,\mathrm{e}^{-0.2\,t}-666.6666667\,\mathrm{e}^{-0.5\,t}$$

$>$ $B4 := \mathit{eval}\left(C_G(t), \left\{C_{B0}=1000, \beta=0.2, \gamma=1.0\right\}\right);$

$$B4 := 250.0000000\,\mathrm{e}^{-0.2\,t}-250.0000000\,\mathrm{e}^{-1.0\,t}$$

$>$ $B5 := \mathit{eval}\left(C_G(t), \left\{C_{B0}=1000, \beta=0.2, \gamma=1.5\right\}\right);$

$$B5 := 153.8461538\,\mathrm{e}^{-0.2\,t}-153.8461538\,\mathrm{e}^{-1.5\,t}$$

$>$

$>$ $\mathit{plot}([B3, B4, B5], t=0..5, \mathit{thickness}=[1,2,3], \mathit{colour}=[\mathit{red}, \mathit{green}, \mathit{blue}], \mathit{linestyle}=[\mathit{solid}, \mathit{dash}, \mathit{dot}]);$

$> \ B6 := eval\left(C_E(t), \left\{C_{B0} = 1000, \beta = 0.2, \gamma = 0.5\right\}\right);$

$$B6 := 1333.333333\,\mathrm{e}^{-0.5\,t} - 3333.333334\,\mathrm{e}^{-0.2\,t} + 2000.000000$$

$> \ B7 := eval\left(C_E(t), \left\{C_{B0} = 1000, \beta = 0.2, \gamma = 1.0\right\}\right);$

$$B7 := 250.0000000\,\mathrm{e}^{-1.0\,t} - 1250.000000\,\mathrm{e}^{-0.2\,t} + 1000.000000$$

$> \ B8 := eval\left(C_E(t), \left\{C_{B0} = 1000, \beta = 0.2, \gamma = 1.5\right\}\right);$

$$B8 := 102.5641025\,\mathrm{e}^{-1.5\,t} - 769.2307690\,\mathrm{e}^{-0.2\,t} + 666.6666664$$

$> \ plot([B6, B7, B8], t = 0\,..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]);$

APPENDIX C

FIGURES 7 TO 10

> $restart;$

> $C(t) := C[B0]\cdot\exp(-\alpha\cdot t);$

$$C(t) := C_{B0}\,\mathrm{e}^{-\alpha t}$$

> $C[R](t) := \dfrac{\alpha\cdot C[B0]}{(\beta-\alpha)}\cdot(\exp(-\alpha\cdot t)-\exp(-\beta\cdot t));$

$$C_R(t) := \frac{\alpha\,C_{B0}\left(\mathrm{e}^{-\alpha t}-\mathrm{e}^{-\beta t}\right)}{\beta-\alpha}$$

> $C[G](t) := \dfrac{\alpha\cdot\beta\cdot C[B0]}{(\beta-\alpha)}\cdot\left(\dfrac{\exp(-\alpha\cdot t)-\exp(-\gamma\cdot t)}{(\gamma-\alpha)}-\dfrac{\exp(-\beta\cdot t)-\exp(-\gamma\cdot t)}{(\gamma-\beta)}\right);$

$$C_G(t) := \frac{\alpha\,\beta\,C_{B0}\left(\frac{\mathrm{e}^{-\alpha t}-\mathrm{e}^{-\gamma t}}{\gamma-\alpha}-\frac{\mathrm{e}^{-\beta t}-\mathrm{e}^{-\gamma t}}{\gamma-\beta}\right)}{\beta-\alpha}$$

> $C[E](t) := \dfrac{\alpha\cdot\beta\cdot\gamma\cdot C[B0]}{(\beta-\alpha)}\cdot\left(\dfrac{(\exp(-\gamma\cdot t)-1)}{\gamma\cdot(\gamma-\alpha)}-\dfrac{(\exp(-\alpha\cdot t)-1)}{\alpha\cdot(\gamma-\alpha)}+\dfrac{(\exp(-\beta\cdot t)-1)}{\beta\cdot(\gamma-\beta)}-\dfrac{(\exp(-\gamma\cdot t)-1)}{\gamma\cdot(\gamma-\beta)}\right);$

$$C_E(t) := \frac{\alpha\,\beta\,\gamma\,C_{B0}\left(\frac{\mathrm{e}^{-\gamma t}-1}{\gamma\,(\gamma-\alpha)}-\frac{\mathrm{e}^{-\alpha t}-1}{\alpha\,(\gamma-\alpha)}+\frac{\mathrm{e}^{-\beta t}-1}{\beta\,(\gamma-\beta)}-\frac{\mathrm{e}^{-\gamma t}-1}{\gamma\,(\gamma-\beta)}\right)}{\beta-\alpha}$$

> $B := eval\left(C(t), \left\{C_{B0}=1000, \alpha=0.5, \beta=2, \gamma=4\right\}\right);$

$$B := 1000\,\mathrm{e}^{-0.5\,t}$$

> $B1 := eval\left(C(t), \left\{C_{B0}=1000, \alpha=1.0, \beta=2, \gamma=4\right\}\right);$

$$B1 := 1000\,\mathrm{e}^{-1.0\,t}$$

> $B2 := eval\left(C(t), \left\{C_{B0}=1000, \alpha=1.5, \beta=2, \gamma=4\right\}\right);$

$$B2 := 1000\,\mathrm{e}^{-1.5\,t}$$

> $plot([B, B1, B2], t=0..1, thickness=[1,2,3], colour=[red, green, blue], linestyle=[solid, dash, dot]);$

> $B3 := eval\left(C_R(t), \left\{C_{B0}=1000, \alpha=0.5, \beta=2, \gamma=4\right\}\right);$

$$B3 := 333.3333333\,\mathrm{e}^{-0.5\,t}-333.3333333\,\mathrm{e}^{-2\,t}$$

> $B4 := eval\left(C_R(t), \left\{C_{B0}=1000, \alpha=1.0, \beta=2, \gamma=4\right\}\right);$

$$B4 := 1000.000000\,\mathrm{e}^{-1.0\,t}-1000.000000\,\mathrm{e}^{-2\,t}$$

> $B5 := eval\left(C_R(t), \left\{C_{B0} = 1000, \alpha = 1.5, \beta = 2, \gamma = 4\right\}\right);$

$$B5 := 3000.000000\,\mathrm{e}^{-1.5\,t} - 3000.000000\,\mathrm{e}^{-2\,t}$$

>

> $plot([B3, B4, B5], t = 0\,..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]);$

> $B6 := eval\left(C_G(t), \left\{C_{B0} = 1000, \alpha = 0.5, \beta = 2, \gamma = 4\right\}\right);$

$$B6 := 190.4761905\,\mathrm{e}^{-0.5\,t} + 142.8571429\,\mathrm{e}^{-4\,t} - 333.3333334\,\mathrm{e}^{-2\,t}$$

> $B7 := eval\left(C_G(t), \left\{C_{B0} = 1000, \alpha = 1.0, \beta = 2, \gamma = 4\right\}\right);$

$$B7 := 666.6666666\,\mathrm{e}^{-1.0\,t} + 333.3333334\,\mathrm{e}^{-4\,t} - 1000.000000\,\mathrm{e}^{-2\,t}$$

> $B8 := eval\left(C_G(t), \left\{C_{B0} = 1000, \alpha = 1.5, \beta = 2, \gamma = 4\right\}\right);$

$$B8 := 2400.000000\,\mathrm{e}^{-1.5\,t} + 600.0000000\,\mathrm{e}^{-4\,t} - 3000.000000\,\mathrm{e}^{-2\,t}$$

>

> $plot([B6, B7, B8], t = 0\,..5, thickness = [1, 2, 3], colour = [red, green, blue,], linestyle = [solid, dash, dot]);$

> $B9 := eval\left(C_E(t), \left\{C_{B0} = 1000, \alpha = 0.5, \beta = 2.0, \gamma = 4.0\right\}\right);$

$$B9 := -142.8571429\,\mathrm{e}^{-4.0\,t} + 1000.000000 - 1523.809524\,\mathrm{e}^{-0.5\,t} + 666.6666668\,\mathrm{e}^{-2.0\,t}$$

> $B10 := eval\left(C_E(t), \left\{C_{B0} = 1000, \alpha = 1.0, \beta = 2.0, \gamma = 4.0\right\}\right);$

$$B10 := -333.3333334\,\mathrm{e}^{-4.0\,t} + 1000.000000 - 2666.666666\,\mathrm{e}^{-1.0\,t} + 2000.000000\,\mathrm{e}^{-2.0\,t}$$

> $B11 := eval\left(C_E(t), \left\{C_{B0} = 1000, \alpha = 1.5, \beta = 2.0, \gamma = 4.0\right\}\right);$

$$B11 := -600.0000000\,\mathrm{e}^{-4.0\,t} + 1000.000001 - 6400.000001\,\mathrm{e}^{-1.5\,t} + 6000.000000\,\mathrm{e}^{-2.0\,t}$$

>

> $plot([B9, B10, B11], t = 0\,..5, thickness = [1, 2, 3], colour = [red, green, blue, yellow], linestyle = [solid, dash, dot]);$

APPENDIX D

FIGURES 11 TO 14

> *restart*;

> $C(t) := C[B0]\cdot\exp(-\alpha\cdot t);$

$$C(t) := C_{B0}\,\mathrm{e}^{-\alpha t}$$

> $C[R](t) := \dfrac{\alpha\cdot C[B0]}{(\beta-\alpha)}\cdot(\exp(-\alpha\cdot t)-\exp(-\beta\cdot t));$

$$C_R(t) := \frac{\alpha\, C_{B0}\left(\mathrm{e}^{-\alpha t}-\mathrm{e}^{-\beta t}\right)}{\beta-\alpha}$$

> $C[G](t) := \dfrac{\alpha\cdot\beta\cdot C[B0]}{(\beta-\alpha)}\cdot\left(\dfrac{\exp(-\alpha\cdot t)-\exp(-\gamma\cdot t)}{(\gamma-\alpha)}-\dfrac{\exp(-\beta\cdot t)-\exp(-\gamma\cdot t)}{(\gamma-\beta)}\right);$

$$C_G(t) := \frac{\alpha\,\beta\, C_{B0}\left(\dfrac{\mathrm{e}^{-\alpha t}-\mathrm{e}^{-\gamma t}}{\gamma-\alpha}-\dfrac{\mathrm{e}^{-\beta t}-\mathrm{e}^{-\gamma t}}{\gamma-\beta}\right)}{\beta-\alpha}$$

> $C[E](t) := \dfrac{\alpha\cdot\beta\cdot\gamma\cdot C[B0]}{(\beta-\alpha)}\cdot\left(\dfrac{(\exp(-\gamma\cdot t)-1)}{\gamma\cdot(\gamma-\alpha)}-\dfrac{(\exp(-\alpha\cdot t)-1)}{\alpha\cdot(\gamma-\alpha)}+\dfrac{(\exp(-\beta\cdot t)-1)}{\beta\cdot(\gamma-\beta)}\right.$
> $\left.-\dfrac{(\exp(-\gamma\cdot t)-1)}{\gamma\cdot(\gamma-\beta)}\right);$

$$C_E(t) := \frac{\alpha\,\beta\,\gamma\, C_{B0}\left(\dfrac{\mathrm{e}^{-\gamma t}-1}{\gamma\,(\gamma-\alpha)}-\dfrac{\mathrm{e}^{-\alpha t}-1}{\alpha\,(\gamma-\alpha)}+\dfrac{\mathrm{e}^{-\beta t}-1}{\beta\,(\gamma-\beta)}-\dfrac{\mathrm{e}^{-\gamma t}-1}{\gamma\,(\gamma-\beta)}\right)}{\beta-\alpha}$$

> $B := eval\left(C(t), \left\{C_{B0}=1000, \alpha=5, \beta=1, \gamma=4\right\}\right);$

$$B := 1000\,\mathrm{e}^{-5\,t}$$

> $B1 := eval\left(C(t), \left\{C_{B0}=1000, \alpha=5, \beta=2,\ \gamma=4\right\}\right);$

$$B1 := 1000\,\mathrm{e}^{-5\,t}$$

> $B2 := eval\left(C(t), \left\{C_{B0}=1000, \alpha=5, \beta=3, \gamma=4\right\}\right);$

$$B2 := 1000\,\mathrm{e}^{-5\,t}$$

> *plot*([*B*, *B1*, *B2*], *t* = 0 ..1, *thickness* = [1, 2, 3], *colour* = [*red*, *green*, *blue*], *linestyle* = [*solid*, *dash*, *dot*]);

> $B3 := eval\left(C_R(t), \left\{C_{B0}=1000, \alpha=5, \beta=1, \gamma=4\right\}\right);$

$$B3 := -1250\,\mathrm{e}^{-5\,t}+1250\,\mathrm{e}^{-t}$$

```
> B4 := eval(C_R(t), {C_B0 = 1000, α = 5, β = 2, γ = 4});
```

$$B4 := -\frac{5000}{3}\,\mathrm{e}^{-5\,t} + \frac{5000}{3}\,\mathrm{e}^{-2\,t}$$

```
> B5 := eval(C_R(t), {C_B0 = 1000, α = 5, β = 3, γ = 4});
```

$$B5 := -2500\,\mathrm{e}^{-5\,t} + 2500\,\mathrm{e}^{-3\,t}$$

```
>
> plot([B3, B4, B5], t = 0 ..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid,
      dash, dot]);
> B6 := eval(C_G(t), {C_B0 = 1000, α = 5, β = 1, γ = 4});
```

$$B6 := 1250\,\mathrm{e}^{-5\,t} - \frac{5000}{3}\,\mathrm{e}^{-4\,t} + \frac{1250}{3}\,\mathrm{e}^{-t}$$

```
> B7 := eval(C_G(t), {C_B0 = 1000, α = 5, β = 2, γ = 4});
```

$$B7 := \frac{10000}{3}\,\mathrm{e}^{-5\,t} - 5000\,\mathrm{e}^{-4\,t} + \frac{5000}{3}\,\mathrm{e}^{-2\,t}$$

```
> B8 := eval(C_G(t), {C_B0 = 1000, α = 5, β = 3, γ = 4});
```

$$B8 := 7500\,\mathrm{e}^{-5\,t} - 15000\,\mathrm{e}^{-4\,t} + 7500\,\mathrm{e}^{-3\,t}$$

```
>
> plot([B6, B7, B8], t = 0 ..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid,
      dash, dot]);


> B9 := eval(C_E(t), {C_B0 = 1000, α = 5, β = 1, γ = 4});
```

$$B9 := \frac{5000}{3}\,\mathrm{e}^{-4\,t} + 1000 - 1000\,\mathrm{e}^{-5\,t} - \frac{5000}{3}\,\mathrm{e}^{-t}$$

```
> B10 := eval(C_E(t), {C_B0 = 1000, α = 5, β = 2, γ = 4});
```

$$B10 := 5000\,\mathrm{e}^{-4\,t} + 1000 - \frac{8000}{3}\,\mathrm{e}^{-5\,t} - \frac{10000}{3}\,\mathrm{e}^{-2\,t}$$

```
> B11 := eval(C_E(t), {C_B0 = 1000, α = 5, β = 3, γ = 4});
```

$$B11 := 15000\,\mathrm{e}^{-4\,t} + 1000 - 6000\,\mathrm{e}^{-5\,t} - 10000\,\mathrm{e}^{-3\,t}$$

```
>
> plot([B9, B10, B11], t = 0 ..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle
       = [solid, dash, dot]);
```

APPENDIX E

FIGURES 4.15 TO 4.18

> $restart;$

> $C(t) := C[B0]\cdot\exp(-\alpha\cdot t);$

$$C(t) := C_{B0}\,e^{-\alpha t}$$

> $C[R](t) := \dfrac{\alpha\cdot C[B0]}{(\beta-\alpha)}\cdot(\exp(-\alpha\cdot t)-\exp(-\beta\cdot t));$

$$C_R(t) := \frac{\alpha\,C_{B0}\left(e^{-\alpha t}-e^{-\beta t}\right)}{\beta-\alpha}$$

> $C[G](t) := \dfrac{\alpha\cdot\beta\cdot C[B0]}{(\beta-\alpha)}\cdot\left(\dfrac{\exp(-\alpha\cdot t)-\exp(-\gamma\cdot t)}{(\gamma-\alpha)}-\dfrac{\exp(-\beta\cdot t)-\exp(-\gamma\cdot t)}{(\gamma-\beta)}\right);$

$$C_G(t) := \frac{\alpha\,\beta\,C_{B0}\left(\frac{e^{-\alpha t}-e^{-\gamma t}}{\gamma-\alpha}-\frac{e^{-\beta t}-e^{-\gamma t}}{\gamma-\beta}\right)}{\beta-\alpha}$$

> $C[E](t) := \dfrac{\alpha\cdot\beta\cdot\gamma\cdot C[B0]}{(\beta-\alpha)}\cdot\left(\dfrac{(\exp(-\gamma\cdot t)-1)}{\gamma\cdot(\gamma-\alpha)}-\dfrac{(\exp(-\alpha\cdot t)-1)}{\alpha\cdot(\gamma-\alpha)}+\dfrac{(\exp(-\beta\cdot t)-1)}{\beta\cdot(\gamma-\beta)}-\dfrac{(\exp(-\gamma\cdot t)-1)}{\gamma\cdot(\gamma-\beta)}\right);$

$$C_E(t) := \frac{\alpha\,\beta\,\gamma\,C_{B0}\left(\frac{e^{-\gamma t}-1}{\gamma\,(\gamma-\alpha)}-\frac{e^{-\alpha t}-1}{\alpha\,(\gamma-\alpha)}+\frac{e^{-\beta t}-1}{\beta\,(\gamma-\beta)}-\frac{e^{-\gamma t}-1}{\gamma\,(\gamma-\beta)}\right)}{\beta-\alpha}$$

> $B := eval\left(C(t),\left\{C_{B0}=1000,\alpha=5,\beta=1.5,\gamma=1.0\right\}\right);$

$$B := 1000\,e^{-5\,t}$$

> $B1 := eval\left(C(t),\left\{C_{B0}=1000,\alpha=5,\beta=1.5,\gamma=2.0\right\}\right);$

$$B1 := 1000\,e^{-5\,t}$$

> $B2 := eval\left(C(t),\left\{C_{B0}=1000,\alpha=5,\beta=1.5,\gamma=3.0\right\}\right);$

$$B2 := 1000\,e^{-5\,t}$$

>

> $plot([B, B1, B2], t=0..1, thickness=[1,2,3], colour=[red, green, blue], linestyle=[solid, dot, dash]);$

> $B3 := eval\left(C_R(t),\left\{C_{B0}=1000,\alpha=5,\beta=1.5,\gamma=1.0\right\}\right);$

$$B3 := -1428.571429\,e^{-5\,t}+1428.571429\,e^{-1.5\,t}$$

> $B4 := eval\left(C_R(t),\left\{C_{B0}=1000,\alpha=5,\beta=1.5,\gamma=2.0\right\}\right);$

$$B4 := -1428.571429\,\mathrm{e}^{-5\,t} + 1428.571429\,\mathrm{e}^{-1.5\,t}$$

\> $B5 := eval\left(C_R(t), \left\{C_{B0} = 1000, \alpha = 5, \beta = 1.5, \gamma = 3.0\right\}\right);$

$$B5 := -1428.571429\,\mathrm{e}^{-5\,t} + 1428.571429\,\mathrm{e}^{-1.5\,t}$$

\>

\> $plot([B3, B4, B5], t = 0\,..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]);$

\> $B6 := eval\left(C_G(t), \left\{C_{B0} = 1000, \alpha = 5, \beta = 1.5, \gamma = 1.0\right\}\right);$

$$B6 := 535.7142858\,\mathrm{e}^{-5\,t} + 3750.000000\,\mathrm{e}^{-1.0\,t} - 4285.714286\,\mathrm{e}^{-1.5\,t}$$

\> $B7 := eval\left(C_G(t), \left\{C_{B0} = 1000, \alpha = 5, \beta = 1.5, \gamma = 2.0\right\}\right);$

$$B7 := 714.2857143\,\mathrm{e}^{-5\,t} - 5000.000000\,\mathrm{e}^{-2.0\,t} + 4285.714286\,\mathrm{e}^{-1.5\,t}$$

\> $B8 := eval\left(C_G(t), \left\{C_{B0} = 1000, \alpha = 5, \beta = 1.5, \gamma = 3.0\right\}\right);$

$$B8 := 1071.428572\,\mathrm{e}^{-5\,t} - 2500.000001\,\mathrm{e}^{-3.0\,t} + 1428.571429\,\mathrm{e}^{-1.5\,t}$$

\>

\> $plot([B6, B7, B8], t = 0\,..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]);$

\> $B9 := eval\left(C_E(t), \left\{C_{B0} = 1000, \alpha = 5, \beta = 1.5, \gamma = 1.0\right\}\right);$

$$B9 := -3750.000000\,\mathrm{e}^{-1.0\,t} + 1000.000001 - 107.1428572\,\mathrm{e}^{-5\,t} + 2857.142857\,\mathrm{e}^{-1.5\,t}$$

\> $B10 := eval\left(C_E(t), \left\{C_{B0} = 1000, \alpha = 5, \beta = 1.5, \gamma = 2.0\right\}\right);$

$$B10 := 5000.000002\,\mathrm{e}^{-2.0\,t} + 999.9999986 - 285.7142857\,\mathrm{e}^{-5\,t} - 5714.285713\,\mathrm{e}^{-1.5\,t}$$

\> $B11 := eval\left(C_E(t), \left\{C_{B0} = 1000, \alpha = 5, \beta = 1.5, \gamma = 3.0\right\}\right);$

$$B11 := 2500.000000\,\mathrm{e}^{-3.0\,t} + 1000.000001 - 642.8571429\,\mathrm{e}^{-5\,t} - 2857.142858\,\mathrm{e}^{-1.5\,t}$$

\>

\> $plot([B9, B10, B11], t = 0\,..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]);$

APPENDIX F

FIGURES 19 TO 22 A

> $restart;$

> $C(t) := C[B0]\cdot \exp(-\beta \cdot t);$

$$C(t) := C_{B0}\, \mathrm{e}^{-\beta\, t}$$

> $C[G](t) := \dfrac{\beta \cdot C[B0]}{(\gamma - \beta)} \cdot (\exp(-\beta \cdot t) - \exp(-\gamma \cdot t));$

$$C_G(t) := \frac{\beta\, C_{B0} \left(\mathrm{e}^{-\beta\, t} - \mathrm{e}^{-\gamma t}\right)}{\gamma - \beta}$$

> $C[E](t) := \dfrac{\beta \cdot C[B0]}{(\gamma - \beta)} \cdot \left(\dfrac{1}{\gamma} \cdot \exp(-\gamma \cdot t) - \dfrac{1}{\beta} \cdot \exp(-\beta \cdot t) + \left(\dfrac{1}{\beta} - \dfrac{1}{\gamma}\right)\right);$

$$C_E(t) := \frac{\beta\, C_{B0} \left(\frac{\mathrm{e}^{-\gamma t}}{\gamma} - \frac{\mathrm{e}^{-\beta\, t}}{\beta} + \frac{1}{\beta} - \frac{1}{\gamma}\right)}{\gamma - \beta}$$

> $p1 := eval\left(C(t), \left[\beta = 2, C_{B0} = 1000\right]\right)$

$$p1 := 1000\, \mathrm{e}^{-2\, t}$$

> $p2 := eval\left(C_G(t), \left[\beta = 2, \gamma = 0.5, C_{B0} = 1000\right]\right)$

$$p2 := -1333.333333\, \mathrm{e}^{-2\, t} + 1333.333333\, \mathrm{e}^{-0.5\, t}$$

> $p3 := eval\left(C_E(t), \left[\beta = 2, \gamma = 0.5, C_{B0} = 1000\right]\right)$

$$p3 := -2666.666666\, \mathrm{e}^{-0.5\, t} + 666.6666665\, \mathrm{e}^{-2\, t} + 2000.000000$$

>

> $plot([p1, p2, p3],\ t = 0 ..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]);$

> $p4 := eval\left(C(t), \left[\beta = 2, C_{B0} = 1000\right]\right)$

$$p4 := 1000\, \mathrm{e}^{-2\, t}$$

> $p5 := eval\left(C_G(t), \left[\beta = 2, \gamma = 1.0, C_{B0} = 1000\right]\right)$

$$p5 := -2000.000000\, \mathrm{e}^{-2\, t} + 2000.000000\, \mathrm{e}^{-1.0\, t}$$

> $p6 := eval\left(C_E(t), \left[\beta = 2, \gamma = 1.0, C_{B0} = 1000\right]\right)$

$$p6 := -2000.000000\, \mathrm{e}^{-1.0\, t} + 1000.000000\, \mathrm{e}^{-2\, t} + 1000.000000$$

>

> $plot([p4, p5, p6],\ t = 0 ..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]);$

Figure 4.20:a

> $p7 := eval\left(C(t), \left[\beta = 2, C_{B0} = 1000\right]\right)$

$$p7 := 1000\,\mathrm{e}^{-2\,t}$$

> $p8 := eval\left(C_G(t), \left[\beta = 2, \gamma = 1.5, C_{B0} = 1000\right]\right)$

$$p8 := -4000.000000\,\mathrm{e}^{-2\,t} + 4000.000000\,\mathrm{e}^{-1.5\,t}$$

> $p9 := eval\left(C_E(t), \left[\beta = 2, \gamma = 1.5, C_{B0} = 1000\right]\right)$

$$p9 := -2666.666667\,\mathrm{e}^{-1.5\,t} + 2000.000000\,\mathrm{e}^{-2\,t} + 666.6666668$$

>

> $plot([p7, p8, p9],\ t = 0\,..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]\,);$

APPENDIX G

FIGURES 23 TO 25

$$> p1 := eval\left(B(t), \left[\alpha = 0.5, B_0 = 1000\right]\right)$$

$$p1 := 1000\, e^{-0.5\, t}$$

$$> p2 := eval\left(G(t), \left[\alpha = 0.5, \beta = 1.0, B_0 = 1000\right]\right)$$

$$p2 := \left(500.0000000\, e^{1.0\, t} - 500.0000000\right)\, e^{-1.5\, t}$$

$$> p3 := eval\left(E(t), \left[\alpha = 0.5, \beta = 1.0, B_0 = 1000\right]\right)$$

$$p3 := -1500.000000\, e^{-0.5\, t} + 500.0000000\, e^{-1.5\, t} + 999.9999998$$

$$>$$

$$> plot([p1, p2, p3],\ t = 0..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]\ );$$

$$> p4 := eval\left(B(t), \left[\alpha = 2.5, B_0 = 1000\right]\right)$$

$$p4 := 1000\, e^{-2.5\, t}$$

$$> p5 := eval\left(G(t), \left[\alpha = 2.5, \beta = 1.0, B_0 = 1000\right]\right)$$

$$p5 := \left(2500.000000\, e^{1.0\, t} - 2500.000000\right)\, e^{-3.5\, t}$$

$$> p6 := eval\left(E(t), \left[\alpha = 2.5, \beta = 1.0, B_0 = 1000\right]\right)$$

$$p6 := -3500.000000\, e^{-2.5\, t} + 2500.000000\, e^{-3.5\, t} + 1000.000000$$

$$>$$

$$> plot([p4, p5, p6], t = 0..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]\ )$$

$$> p7 := eval\left(B(t), \left[\alpha = 5, B_0 = 1000\right]\right)$$

$$p7 := 1000\, e^{-5\, t}$$

$$> p8 := eval\left(G(t), \left[\alpha = 5, \beta = 1.0, B_0 = 1000\right]\right)$$

$$p8 := \left(5000.000000\, e^{1.0\, t} - 5000.000000\right)\, e^{-6.0\, t}$$

$$> p9 := eval\left(E(t), \left[\alpha = 5, \beta = 1.0, B_0 = 1000\right]\right)$$

$$p9 := -6000.000000\, e^{-5\, t} + 5000.000001\, e^{-6.0\, t} + 999.9999990$$

$$>$$

$$> plot([p7, p8, p9], t = 0..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]\ )$$

$$> p10 := eval\left(B(t), \left[\alpha = 0.6, B_0 = 1000\right]\right)$$

$$p10 := 1000\, e^{-0.6\, t}$$

$$> p11 := eval\left(G(t), \left[\alpha = 0.6, \beta = 10.0, B_0 = 1000\right]\right)$$

$p11 := \left(60.00000000\,\mathrm{e}^{10.0\,t} - 60.00000000\right)\,\mathrm{e}^{-10.6\,t}$

> $p12 \coloneqq eval\left(E(t), \left[\alpha = 0.6, \beta = 10.0, B_0 = 1000\right]\right)$

$p12 := -1060.000000\,\mathrm{e}^{-0.6\,t} + 60.00000000\,\mathrm{e}^{-10.6\,t} + 1000.000000$

>

> $plot([p10, p11, p12],\ t = 0\,..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]\ );$

> $p10 \coloneqq eval\left(B(t), \left[\alpha = 0.6, B_0 = 1000\right]\right)$

$p10 := 1000\,\mathrm{e}^{-0.6\,t}$

> $p11 \coloneqq eval\left(G(t), \left[\alpha = 0.6, \beta = 10.0, B_0 = 1000\right]\right)$

$p11 := \left(60.00000000\,\mathrm{e}^{10.0\,t} - 60.00000000\right)\,\mathrm{e}^{-10.6\,t}$

> $p12 \coloneqq eval\left(E(t), \left[\alpha = 0.6, \beta = 10.0, B_0 = 1000\right]\right)$

$p12 := -1060.000000\,\mathrm{e}^{-0.6\,t} + 60.00000000\,\mathrm{e}^{-10.6\,t} + 1000.000000$

>

> $plot([p10, p11, p12],\ t = 0\,..5, thickness = [1, 2, 3], colour = [red, green, blue], linestyle = [solid, dash, dot]\ );$

APPENDIX H

FIGURES 26 TO 31

> *restart*
> $\text{eqn1} := \mathit{diff}(B(t), t) = -\alpha \cdot B(t)$

$$\mathit{eqn1} := \frac{d}{dt}\, B(t) = -\alpha\, B(t)$$

> $\mathit{ics1} := B(0) = \mathrm{B}[0]$

$$\mathit{ics1} := B(0) = B_0$$

> $\mathit{sol1} := \mathit{dsolve}(\{\mathit{eqn1}, \mathit{ics1}\})$

$$\mathit{sol1} := B(t) = B_0\, \mathrm{e}^{-\alpha\, t}$$

> $B(t) := B_0\, \mathrm{e}^{-\alpha\, t}$

$$B(t) := B_0\, \mathrm{e}^{-\alpha\, t}$$

>
> $\mathit{eqn2} := \mathit{diff}(R(t), t) + (\alpha + \beta) \cdot R(t) = \alpha \cdot B(t)$

$$\mathit{eqn2} := \frac{d}{dt}\, R(t) + (\alpha + \beta)\, R(t) = \alpha\, B_0\, \mathrm{e}^{-\alpha\, t}$$

>
> $\mathit{ics2} := R(0) = 0$

$$\mathit{ics2} := R(0) = 0$$

> $\mathit{sol2} := \mathit{dsolve}(\{\mathit{eqn2}, \mathit{ics2}\})$

$$\mathit{sol2} := R(t) = \left(\frac{\alpha\, B_0\, \mathrm{e}^{t\, \beta}}{\beta} - \frac{B_0\, \alpha}{\beta} \right) \mathrm{e}^{-(\alpha + \beta)\, t}$$

> $R(t) := \left(\frac{\alpha \cdot B_0\, \mathrm{e}^{t\, \beta}}{\beta} - \frac{\alpha \cdot B_0}{\beta} \right) \cdot \mathrm{e}^{-(\alpha + \beta)\, t}$

$$R(t) := \left(\frac{\alpha\, B_0\, \mathrm{e}^{t\, \beta}}{\beta} - \frac{B_0\, \alpha}{\beta} \right) \mathrm{e}^{-(\alpha + \beta)\, t}$$

> $\mathit{eqn3} := \mathit{diff}(G(t), t) + (\alpha + \beta + \gamma) \cdot G(t) = (\alpha + \beta) \cdot R(t)$

>
>
> $\mathit{ics3} := G(0) = 0$

$$\mathit{ics3} := G(0) = 0$$

> $\mathit{sol3} := \mathit{dsolve}(\{\mathit{eqn3}, \mathit{ics3}\})$

> $$G(t) := \left(\frac{\alpha \cdot B_0 \cdot (\alpha+\beta) \cdot \left(\frac{e^{t(\gamma+\beta)}}{\gamma+\beta} - \frac{e^{t\gamma}}{\gamma} \right)}{\beta} - \frac{\alpha \cdot B_0 \cdot (\alpha+\beta) \cdot \left(\frac{1}{\gamma+\beta} + \frac{1}{\gamma} \right)}{\beta} \right) \cdot e^{-(\alpha+\beta+\gamma)\,t}$$

> $$eqn4 := diff(E(t), t) = (\alpha+\beta+\gamma) \cdot \left(\frac{\alpha \cdot B_0 \cdot (\alpha+\beta) \cdot \left(\frac{e^{t(\gamma+\beta)}}{\gamma+\beta} - \frac{e^{t\gamma}}{\gamma} \right)}{\beta} - \frac{\alpha \cdot B_0 \cdot (\alpha+\beta) \cdot \left(\frac{1}{\gamma+\beta} + \frac{1}{\gamma} \right)}{\beta} \right) \cdot e^{-(\alpha+\beta+\gamma)\,t}$$

$$eqn4 := \frac{d}{dt} E(t) = (\alpha+\beta+\gamma) \left(\frac{B_0\, \alpha\, (\alpha+\beta) \left(\frac{e^{t(\gamma+\beta)}}{\gamma+\beta} - \frac{e^{t\gamma}}{\gamma} \right)}{\beta} - \frac{\alpha\, B_0\, (\alpha+\beta) \left(\frac{1}{\gamma+\beta} + \frac{1}{\gamma} \right)}{\beta} \right) e^{-(\alpha+\beta+\gamma)\,t}$$

> $ics4 := E(0) = 0$

$$ics4 := E(0) = 0$$

> $sol4 := dsolve(\{eqn4, ics4\})$

> $$E(t) := -\frac{1}{\beta(\gamma+\beta)\gamma} \left(B_0 (\alpha+\beta)(\alpha+\beta+\gamma) \left(\frac{2\gamma e^{-(\alpha+\beta+\gamma)\,t}}{-\alpha-\beta-\gamma} + \frac{\beta e^{-(\alpha+\beta+\gamma)\,t}}{-\alpha-\beta-\gamma} - \frac{\gamma e^{-(\alpha+\beta)\,t}}{-\alpha-\beta} + \frac{\gamma e^{-\alpha t}}{\alpha} - \frac{\beta e^{-(\alpha+\beta)\,t}}{-\alpha-\beta} \right) \right) + \frac{1}{\beta(\gamma+\beta)\gamma} \left(B_0 (\alpha+\beta)(\alpha+\beta+\gamma) \left(\frac{2\gamma}{-\alpha-\beta-\gamma} + \frac{\beta}{-\alpha-\beta-\gamma} - \frac{\gamma}{-\alpha-\beta} + \frac{\gamma}{\alpha} - \frac{\beta}{-\alpha-\beta} \right) \right)$$

> $p1 := eval\left(B(t), \left[\alpha = 0.5, B_0 = 1000\right]\right)$

$$p1 := 1000\, e^{-0.5\, t}$$

> $p2 := eval\left(R(t), \left[\alpha = 0.5, \beta = 1.0, B_0 = 1000\right]\right)$

$$p2 := \left(500.0000000\, e^{1.0\, t} - 500.0000000\right) e^{-1.5\, t}$$

> $p3 := eval\left(G(t), \left[\alpha = 0.5, \beta = 1.0, B_0 = 1000, \gamma = 10.0\right]\right)$

$$p3 := \left(68.18181818\, e^{11.0\, t} - 75.00000000\, e^{10.0\, t} - 143.1818182\right) e^{-11.5\, t}$$

> $p4 := eval\left(E(t), \left[\alpha = 0.5, \beta = 1.0, B_0 = 1000, \gamma = 10.0\right]\right)$

$$p4 := 286.3636364\, e^{-11.5\, t} - 1150.000000\, e^{-1.5\, t} - 3136.363636\, e^{-0.5\, t} + 4000.000001$$

```
> plot([p1,p2,p3,p4], t = 0 ..10, linestyle = [dot, dash, solid, LongDash] )
```

```
> p5 := eval(B(t), [α = 0.5, B_0 = 100])
```

$$p5 := 100\,\mathrm{e}^{-0.5\,t}$$

```
> p6 := eval(R(t), [α = 0.5, β = 0.6, B_0 = 100])
```

$$p6 := \left(83.33333335\,\mathrm{e}^{0.6\,t} - 83.33333335\right)\mathrm{e}^{-1.1\,t}$$

```
> p7 := eval(G(t), [α = 0.5, β = 0.6, B_0 = 100, γ = 10.0])
```

$$p7 := \left(8.647798742\,\mathrm{e}^{10.6\,t} - 9.166666667\,\mathrm{e}^{10.0\,t} - 17.81446540\right)\mathrm{e}^{-11.1\,t}$$

```
> p8 := eval(E(t), [α = 0.5, β = 0.6, B_0 = 100, γ = 10.0])
```

$$p8 := 35.62893082\,\mathrm{e}^{-11.1\,t} - 185.0000000\,\mathrm{e}^{-1.1\,t} - 383.9622642\,\mathrm{e}^{-0.5\,t} + 533.3333335$$

```
> plot([p5,p6,p7,p8], t = 0 ..10, linestyle = [dot, dash, solid, LongDash] )
```

```
> p9 := eval(B(t), [α = 0.5, B_0 = 10000])
```

$$p9 := 10000\,\mathrm{e}^{-0.5\,t}$$

```
> p10 := eval(R(t), [α = 0.5, β = 0.6, B_0 = 10000])
```

$$p10 := \left(8333.333335\,\mathrm{e}^{0.6\,t} - 8333.333335\right)\mathrm{e}^{-1.1\,t}$$

```
> p11 := eval(G(t), [α = 0.5, β = 0.6, B_0 = 10000, γ = 10.0])
```

$$p11 := \left(864.7798742\,\mathrm{e}^{10.6\,t} - 916.6666667\,\mathrm{e}^{10.0\,t} - 1781.446540\right)\mathrm{e}^{-11.1\,t}$$

```
> p12 := eval(E(t), [α = 0.5, β = 0.6, B_0 = 10000, γ = 10.0])
```

$$p12 := 3562.893082\,\mathrm{e}^{-11.1\,t} - 18500.00000\,\mathrm{e}^{-1.1\,t} - 38396.22642\,\mathrm{e}^{-0.5\,t} + 53333.33335$$

```
> plot([p9,p10,p11,p12], t = 0 ..10, linestyle = [dot, dash, solid, LongDash] )
```

APPENDIX I

> *TABLE* 4.1 *NO.* 2

> $prob1 := \left(\frac{400}{3} \cdot h^{-\frac{1}{3}} \cdot b^{\frac{1}{3}} \cdot r^{\frac{1}{6}} \cdot g^{\frac{1}{12}} - 120 \cdot \lambda = 0, \frac{200}{3} \cdot h^{\frac{2}{3}} \cdot b^{-\frac{2}{3}} \cdot r^{\frac{1}{6}} \cdot g^{\frac{1}{12}} - 500 \cdot \lambda = 0, \frac{200}{6} \cdot h^{\frac{2}{3}} \cdot b^{\frac{1}{3}} \cdot r^{-\frac{5}{6}} \cdot g^{\frac{1}{12}} - 40 \cdot \lambda = 0, \frac{200}{12} \cdot h^{\frac{2}{3}} \cdot b^{\frac{1}{3}} \cdot r^{\frac{1}{6}} \cdot g^{-\frac{11}{12}} - 70 \cdot \lambda = 0, -120 \cdot h - 500 \cdot b - 40 \cdot r - 70 \cdot g + 4000 = 0 \right);$

$$prob1 := \frac{400}{3} \frac{b^{1/3} r^{1/6} g^{1/12}}{h^{1/3}} - 120\,\lambda = 0, \frac{200}{3} \frac{h^{2/3} r^{1/6} g^{1/12}}{b^{2/3}} - 500\,\lambda = 0, \frac{100}{3} \frac{h^{2/3} b^{1/3} g^{1/12}}{r^{5/6}} - 40\,\lambda = 0, \frac{50}{3} \frac{h^{2/3} b^{1/3} r^{1/6}}{g^{11/12}} - 70\,\lambda = 0, -120\,h - 500\,b - 40\,r - 70\,g + 4000 = 0$$

>

> $sol1 := solve(\{prob1\}, \{h, b, r, g, \lambda\})$

$$sol1 := \left\{ b = \frac{32}{15}, g = \frac{80}{21}, h = \frac{160}{9}, r = \frac{40}{3}, \lambda = \frac{1}{272160}\, 160^{2/3}\, 9^{1/3}\, 32^{1/3}\, 15^{2/3}\, 80^{1/12}\, 21^{11/12}\, 2560^{1/6}\, 3^{5/6} \right\}$$

> $f[0] := 200 \cdot \left(\frac{160}{9}\right)^{\frac{2}{3}} \cdot \left(\frac{32}{15}\right)^{\frac{1}{3}} \cdot \left(\frac{40}{3}\right)^{\frac{1}{6}} \cdot \left(\frac{80}{21}\right)^{\frac{1}{12}}$

$$f_0 := \frac{40}{1701}\, 160^{2/3}\, 9^{1/3}\, 32^{1/3}\, 15^{2/3}\, 40^{1/6}\, 3^{5/6}\, 80^{1/12}\, 21^{11/12}$$

> $evalf[5]\left(200 * \left(\frac{160}{9}\right) \hat{} (2/3) * \left(\frac{32}{15}\right) \hat{} (1/3) * \left(\frac{40}{3}\right) \hat{} (1/6) * \left(\frac{80}{21}\right) \hat{} ((1/12)) \right)$

$$3019.0$$

> $b := \frac{32}{15}$

$$b := \frac{32}{15}$$

> $evalf[5](\ 32/15\ )$

$$2.1333$$

>

> $h := \frac{160}{9}$

$$h := \frac{160}{9}$$

> $evalf[5](\ 160/9\ )$

$$17.778$$

>

> $g := \frac{80}{21}$

$$g := \frac{80}{21}$$

> *evalf*[5](80/21)

$$3.8095$$

> $r := \frac{40}{3}$

$$r := \frac{40}{3}$$

> *evalf*[5](40/3)

$$13.333$$

>

> $\lambda := \frac{1}{272160}\, 160^{2/3}\, 9^{1/3}\, 32^{1/3}\, 15^{2/3}\, 80^{1/12}\, 21^{11/12}\, 2560^{1/6}\, 3^{5/6}$

$$\lambda := \frac{1}{272160}\, 160^{2/3}\, 9^{1/3}\, 32^{1/3}\, 15^{2/3}\, 80^{1/12}\, 21^{11/12}\, 2560^{1/6}\, 3^{5/6}$$

> *evalf*[5]((1/272160)*160^(2/3)*9^(1/3)*32^(1/3)*15^(2/3)*80^(1/12)*21^(11/12)*2560^(1/6)*3^(5/6))

$$0.94345$$

> *TABLE* 4.1 *NO.* 4

> $prob1 := \left(\frac{400}{3} \cdot h^{-\frac{1}{3}} \cdot b^{\frac{2}{3}} \cdot r^{\frac{1}{6}} \cdot g^{\frac{1}{12}} - 120 \cdot \lambda = 0, \frac{400}{3} \cdot h^{\frac{2}{3}} \cdot b^{-\frac{1}{3}} \cdot r^{\frac{1}{6}} \cdot g^{\frac{1}{12}} - 500 \cdot \lambda = 0, \frac{200}{6} \cdot h^{\frac{2}{3}} \cdot b^{\frac{2}{3}} \cdot r^{-\frac{5}{6}} \cdot g^{\frac{1}{12}} - 40 \cdot \lambda = 0, \frac{200}{12} \cdot h^{\frac{2}{3}} \cdot b^{\frac{2}{3}} \cdot r^{\frac{1}{6}} \cdot g^{-\frac{11}{12}} - 70 \cdot \lambda = 0, -120 \cdot h - 500 \cdot b - 40 \cdot r - 70 \cdot g + 4000 = 0 \right);$

$$prob1 := \frac{400}{3}\, \frac{b^{2/3}\, r^{1/6}\, g^{1/12}}{h^{1/3}} - 120\,\lambda = 0,\ \frac{400}{3}\, \frac{h^{2/3}\, r^{1/6}\, g^{1/12}}{b^{1/3}} - 500\,\lambda = 0,\ \frac{100}{3}\, \frac{h^{2/3}\, b^{2/3}\, g^{1/12}}{r^{5/6}} - 40\,\lambda = 0,\ \frac{50}{3}\, \frac{h^{2/3}\, b^{2/3}\, r^{1/6}}{g^{11/12}} - 70\,\lambda = 0,\ -120\,h - 500\,b - 40\,r - 70\,g + 4000 = 0$$

>

> $sol1 := solve(\{prob1\}, \{h, b, r, g, \lambda\})$

$$sol1 := \left\{ b = \frac{64}{19}, g = \frac{400}{133}, h = \frac{800}{57}, r = \frac{200}{19}, \lambda = \frac{1}{3638880}\, 800^{2/3}\, 57^{1/3}\, 64^{2/3}\, 19^{1/6}\, 400^{1/12}\, 133^{11/12}\, 12800^{1/6} \right\}$$

> $f[0] := 200 \cdot \left(\frac{800}{57}\right)^{\frac{2}{3}} \cdot \left(\frac{64}{19}\right)^{\frac{2}{3}} \cdot \left(\frac{200}{19}\right)^{\frac{1}{6}} \cdot \left(\frac{200}{63}\right)^{\frac{1}{12}}$

$$f_0 := \frac{200}{68229}\,800^{2/3}\,57^{1/3}\,64^{2/3}\,19^{1/6}\,200^{1/4}\,63^{11/12}$$

> $evalf[5]\left(200*\left(\frac{800}{57}\right)\text{^}(2/3)*\left(\frac{64}{19}\right)\text{^}(2/3)*\left(\frac{100}{9}\right)\text{^}(1/6)*\left(\frac{400}{133}\right)\text{^}(4*(1/12))\right)$

$$5638.8$$

> $b := \frac{64}{19}$

$$b := \frac{64}{19}$$

> $evalf[5](\,64/19\,)$

$$3.3684$$

>

> $h := \frac{800}{57}$

$$h := \frac{800}{57}$$

> $evalf[5](\,800/57\,)$

$$14.035$$

>

> $g := \frac{400}{133}$

$$g := \frac{400}{133}$$

> $evalf[5](\,400/133)$

$$3.0075$$

> $r := \frac{200}{19}$

$$r := \frac{200}{19}$$

> $evalf[5](\,200/19)$

$$10.526$$

> $\lambda := \frac{1}{3638880}\,800^{2/3}\,57^{1/3}\,64^{2/3}\,19^{1/6}\,400^{1/12}\,133^{11/12}\,12800^{1/6}$

$$\lambda := \frac{1}{3638880}\,800^{2/3}\,57^{1/3}\,64^{2/3}\,19^{1/6}\,400^{1/12}\,133^{11/12}\,12800^{1/6}$$

>

> evalf[5]((1/3638880)*800^(2/3)*57^(1/3)*64^(2/3)*19^(1/6)*400^(1/12)*133^(11/12)*12800^(1/6))

$$1.6795$$

>

>

> *TABLE* 4.1 *NO.* 4

> $prob1 := \left(\frac{400}{3}\cdot h^{-\frac{1}{3}}\cdot b^{\frac{2}{3}}\cdot r^{\frac{1}{6}}\cdot g^{\frac{1}{12}} - 120\cdot\lambda = 0, \frac{400}{3}\cdot h^{\frac{2}{3}}\cdot b^{-\frac{1}{3}}\cdot r^{\frac{1}{6}}\cdot g^{\frac{1}{12}} - 500\cdot\lambda = 0, \frac{200}{6}\cdot h^{\frac{2}{3}}\cdot b^{\frac{2}{3}}\cdot r^{-\frac{5}{6}}\cdot g^{\frac{1}{12}} - 40\cdot\lambda = 0, \frac{200}{12}\cdot h^{\frac{2}{3}}\cdot b^{\frac{2}{3}}\cdot r^{\frac{1}{6}}\cdot g^{-\frac{11}{12}} - 70\cdot\lambda = 0, -120\cdot h - 500\cdot b - 40\cdot r - 70\cdot g + 4000 = 0\right);$

$$prob1 := \frac{400}{3}\,\frac{b^{2/3}\,r^{1/6}\,g^{1/12}}{h^{1/3}} - 120\,\lambda = 0,\ \frac{400}{3}\,\frac{h^{2/3}\,r^{1/6}\,g^{1/12}}{b^{1/3}} - 500\,\lambda = 0,\ \frac{100}{3}\,\frac{h^{2/3}\,b^{2/3}\,g^{1/12}}{r^{5/6}} - 40\,\lambda = 0,\ \frac{50}{3}\,\frac{h^{2/3}\,b^{2/3}\,r^{1/6}}{g^{11/12}} - 70\,\lambda = 0,\ -120\,h - 500\,b - 40\,r - 70\,g + 4000 = 0$$

>

> $sol1 := solve(\{prob1\}, \{h, b, r, g, \lambda\})$

$$sol1 := \left\{b = \frac{64}{19}, g = \frac{400}{133}, h = \frac{800}{57}, r = \frac{200}{19}, \lambda = \frac{1}{3638880}\,800^{2/3}\,57^{1/3}\,64^{2/3}\,19^{1/6}\,400^{1/12}\,133^{11/12}\,12800^{1/6}\right\}$$

> $f[0] := 200\cdot\left(\frac{800}{57}\right)^{\frac{2}{3}}\cdot\left(\frac{64}{19}\right)^{\frac{2}{3}}\cdot\left(\frac{200}{19}\right)^{\frac{1}{6}}\cdot\left(\frac{200}{63}\right)^{\frac{1}{12}}$

$$f_0 := \frac{200}{68229}\,800^{2/3}\,57^{1/3}\,64^{2/3}\,19^{1/6}\,200^{1/4}\,63^{11/12}$$

> $evalf[5]\left(200*\left(\frac{800}{57}\right)\hat{}\,(2/3)*\left(\frac{64}{19}\right)\hat{}\,(2/3)*\left(\frac{100}{9}\right)\hat{}\,(1/6)*\left(\frac{400}{133}\right)\hat{}\,(4*(1/12))\right)$

$$5638.8$$

> $b := \frac{64}{19}$

$$b := \frac{64}{19}$$

> $evalf[5](\ 64/19\ )$

$$3.3684$$

>

> $h := \frac{800}{57}$

$$h := \frac{800}{57}$$

> $evalf[5](\ 800/57\ )$

$$14.035$$

>

> $g := \frac{400}{133}$

$$g := \frac{400}{133}$$

> *evalf*[5](400/133)

$$3.0075$$

> $r := \frac{200}{19}$

$$r := \frac{200}{19}$$

> *evalf*[5](200/19)

$$10.526$$

> $\lambda := \frac{1}{3638880}\, 800^{2/3}\, 57^{1/3}\, 64^{2/3}\, 19^{1/6}\, 400^{1/12}\, 133^{11/12}\, 12800^{1/6}$

$$\lambda := \frac{1}{3638880}\, 800^{2/3}\, 57^{1/3}\, 64^{2/3}\, 19^{1/6}\, 400^{1/12}\, 133^{11/12}\, 12800^{1/6}$$

>

> *evalf*[5]((1/3638880)*800^(2/3)*57^(1/3)*64^(2/3)*19^(1/6)*400^(1/12)*133^(11/12)*12800^(1/6))

$$1.6795$$

> *TABLE* 4.1 *NO.* 5

> $prob1 := \left(\frac{400}{3} \cdot h^{-\frac{1}{3}} \cdot b^{\frac{2}{3}} \cdot r^{\frac{1}{6}} \cdot g^{\frac{1}{8}} - 120 \cdot \lambda = 0, \frac{400}{3} \cdot h^{\frac{2}{3}} \cdot b^{-\frac{1}{3}} \cdot r^{\frac{1}{6}} \cdot g^{\frac{1}{8}} - 500 \cdot \lambda = 0, \frac{200}{6} \cdot h^{\frac{2}{3}} \cdot b^{\frac{2}{3}} \cdot r^{-\frac{5}{6}} \cdot g^{\frac{1}{8}} - 40 \cdot \lambda = 0, \frac{200}{8} \cdot h^{\frac{2}{3}} \cdot b^{\frac{2}{3}} \cdot r^{\frac{1}{6}} \cdot g^{-\frac{7}{8}} - 70 \cdot \lambda = 0, -120 \cdot h - 500 \cdot b - 40 \cdot r - 70 \cdot g + 4000 = 0 \right);$

$$prob1 := \frac{400}{3}\,\frac{b^{2/3}\, r^{1/6}\, g^{1/8}}{h^{1/3}} - 120\,\lambda = 0,\ \frac{400}{3}\,\frac{h^{2/3}\, r^{1/6}\, g^{1/8}}{b^{1/3}} - 500\,\lambda = 0,\ \frac{100}{3}\,\frac{h^{2/3}\, b^{2/3}\, g^{1/8}}{r^{5/6}} - 40\,\lambda = 0,\ \frac{25\, h^{2/3}\, b^{2/3}\, r^{1/6}}{g^{7/8}} - 70\,\lambda = 0,\ -120\,h - 500\,b - 40\,r - 70\,g + 4000 = 0$$

>

> $sol1 := solve(\{prob1\}, \{h, b, r, g, \lambda\})$

$$sol1 := \left\{ b = \frac{128}{39}, g = \frac{400}{91}, h = \frac{1600}{117}, r = \frac{400}{39}, \lambda = \frac{1}{10221120}\, 1600^{2/3}\, 117^{1/3}\, 400^{1/8}\, 91^{7/8}\, 128^{2/3}\, 39^{1/6}\, 25600^{1/6} \right\}$$

> $f[0] := 200 \cdot \left(\frac{1600}{117}\right)^{\frac{2}{3}} \cdot \left(\frac{128}{39}\right)^{\frac{2}{3}} \cdot \left(\frac{400}{39}\right)^{\frac{1}{6}} \cdot \left(\frac{400}{91}\right)^{\frac{1}{8}}$

$$f_0 := \frac{200}{415233}\, 1600^{2/3}\, 117^{1/3}\, 128^{2/3}\, 39^{1/6}\, 400^{7/24}\, 91^{7/8}$$

> $evalf[5]\left(200*\left(\frac{1600}{117}\right)\text{^}(2/3)*\left(\frac{128}{39}\right)\text{^}(2/3)*\left(\frac{400}{39}\right)\text{^}(1/6)*\left(\frac{400}{91}\right)\text{^}(4*(1/8))\right)$

$$7805.8$$

> $b := \frac{128}{39}$

$$b := \frac{128}{39}$$

> $evalf[5](\ 128/39\ )$

$$3.2821$$

>

> $h := \frac{1600}{117}$

$$h := \frac{1600}{117}$$

> $evalf[5](\ 1600/117\ )$

$$13.675$$

>

> $g := \frac{400}{91}$

$$g := \frac{400}{91}$$

> $evalf[5](\ 400/91)$

$$4.3956$$

> $r := \frac{400}{39}$

$$r := \frac{400}{39}$$

> $evalf[5](\ 400/39)$

$$10.256$$

> $\lambda := \frac{1}{10221120}\ 1600^{2/3}\ 117^{1/3}\ 400^{1/8}\ 91^{7/8}\ 128^{2/3}\ 39^{1/6}\ 25600^{1/6}$

$$\lambda := \frac{1}{10221120}\ 1600^{2/3}\ 117^{1/3}\ 400^{1/8}\ 91^{7/8}\ 128^{2/3}\ 39^{1/6}\ 25600^{1/6}$$

>

> $evalf[5](\ (1/10221120)*1600\text{^}(2/3)*117\text{^}(1/3)*404\text{^}(1/8)*91\text{^}(7/8)*128\text{^}(2/3)*39\text{^}(1/6)*25600\text{^}(1/6)\ )$

$$1.8223$$

TABLE 4.1 *NO.* 9

> $prob1 := \left(\frac{200}{4}\cdot h^{-\frac{3}{4}}\cdot b^{\frac{1}{2}}\cdot r^{\frac{1}{4}}\cdot g^{\frac{1}{4}} - 120\cdot\lambda = 0, \frac{200}{2}\cdot h^{\frac{1}{4}}\cdot b^{-\frac{1}{2}}\cdot r^{\frac{1}{4}}\cdot g^{\frac{1}{4}} - 500\cdot\lambda = 0, \frac{200}{4}\cdot h^{\frac{1}{4}}\cdot b^{\frac{1}{2}}\cdot r^{-\frac{3}{4}}\cdot g^{\frac{1}{4}} - 40\cdot\lambda = 0, \frac{200}{4}\cdot h^{\frac{1}{4}}\cdot b^{\frac{1}{2}}\cdot r^{\frac{1}{4}}\cdot g^{-\frac{3}{4}} - 70\cdot\lambda = 0, -120\cdot h - 500\cdot b - 40\cdot r - 70\cdot g + 4000 = 0\right);$

$$prob1 := \frac{50\sqrt{b}\, r^{1/4} g^{1/4}}{h^{3/4}} - 120\,\lambda = 0,\ \frac{100\, h^{1/4} r^{1/4} g^{1/4}}{\sqrt{b}} - 500\,\lambda = 0,\ \frac{50\, h^{1/4}\sqrt{b}\, g^{1/4}}{r^{3/4}} - 40\,\lambda = 0,\ \frac{50\, h^{1/4}\sqrt{b}\, r^{1/4}}{g^{3/4}} - 70\,\lambda = 0,\ -120\,h - 500\,b - 40\,r - 70\,g + 4000 = 0$$

\>

\> $sol1 := solve(\{prob1\}, \{h, b, r, g, \lambda\})$

$$sol1 := \left\{b = \frac{16}{5}, g = \frac{80}{7}, h = \frac{20}{3}, r = 20, \lambda = \frac{1}{840}\, 20^{1/4}\, 3^{3/4}\, 80^{1/4}\, 7^{3/4}\sqrt{5}\, 320^{1/4}\right\}$$

\> $f[0] := 200 \cdot \left(\frac{20}{3}\right)^{\frac{1}{4}} \cdot \left(\frac{16}{5}\right)^{\frac{1}{2}} \cdot (20)^{\frac{1}{4}} \cdot \left(\frac{80}{7}\right)^{\frac{1}{4}}$

$$f_0 := \frac{40}{21}\sqrt{20}\, 3^{3/4}\sqrt{16}\sqrt{5}\, 80^{1/4}\, 7^{3/4}$$

\> $evalf[5]\left(200 * \left(\frac{20}{3}\right)\hat{}(1/4) * \left(\frac{16}{5}\right)\hat{}(1/2) * (20)\hat{}(1/4) * \left(\frac{80}{7}\right)\hat{}((1/4))\right)$

$$2235.3$$

\> $b := \frac{16}{5}$

$$b := \frac{16}{5}$$

\> $evalf[5](16/55)$

$$0.29091$$

\>

\> $h := \frac{20}{3}$

$$h := \frac{20}{3}$$

\> $evalf[5](20/3)$

$$6.6667$$

\>

\> $g := \frac{80}{7}$

$$g := \frac{80}{7}$$

\> $evalf[5](80/7)$

$$11.429$$

\> $r := 20$

$$r := 20$$

\> $evalf[5](20)$

$$20.$$

\>

\> $\lambda = \frac{1}{840}\, 20^{1/4}\, 3^{3/4}\, 80^{1/4}\, 7^{3/4}\sqrt{5}\, 320^{1/4}$

$$\lambda = \frac{1}{840}\, 20^{1/4}\, 3^{3/4}\, 80^{1/4}\, 7^{3/4}\, \sqrt{5}\, 320^{1/4}$$

```
> evalf[5]( (1/840)*20^(1/4)*3^(3/4)*80^(1/4)*7^(3/4)*5^(1/2)*321^(1/4) )
```

$$0.69913$$

```
>
```

TABLE 4.1 *NO.* 9

$$> \; prob1 := \left(\frac{200}{4}\cdot h^{-\frac{3}{4}}\cdot b^{\frac{1}{2}}\cdot r^{\frac{1}{4}}\cdot g^{\frac{1}{4}} - 120\cdot\lambda = 0, \frac{200}{2}\cdot h^{\frac{1}{4}}\cdot b^{-\frac{1}{2}}\cdot r^{\frac{1}{4}}\cdot g^{\frac{1}{4}} - 500\cdot\lambda = 0, \frac{200}{4}\cdot h^{\frac{1}{4}}\cdot b^{\frac{1}{2}}\cdot r^{-\frac{3}{4}}\cdot g^{\frac{1}{4}} - 40\cdot\lambda = 0, \frac{200}{4}\cdot h^{\frac{1}{4}}\cdot b^{\frac{1}{2}}\cdot r^{\frac{1}{4}}\cdot g^{-\frac{3}{4}} - 70\cdot\lambda = 0, -120\cdot h - 500\cdot b - 40\cdot r - 70\cdot g + 4000 = 0 \right);$$

$$prob1 := \frac{50\sqrt{b}\, r^{1/4}\, g^{1/4}}{h^{3/4}} - 120\,\lambda = 0, \frac{100\, h^{1/4}\, r^{1/4}\, g^{1/4}}{\sqrt{b}} - 500\,\lambda = 0, \frac{50\, h^{1/4}\sqrt{b}\, g^{1/4}}{r^{3/4}} - 40\,\lambda = 0, \frac{50\, h^{1/4}\sqrt{b}\, r^{1/4}}{g^{3/4}} - 70\,\lambda = 0, -120\, h - 500\, b - 40\, r - 70\, g + 4000 = 0$$

```
>
```

$$> \; sol1 := solve(\{prob1\}, \{h, b, r, g, \lambda\})$$

$$sol1 := \left\{ b = \frac{16}{5}, g = \frac{80}{7}, h = \frac{20}{3}, r = 20, \lambda = \frac{1}{840}\, 20^{1/4}\, 3^{3/4}\, 80^{1/4}\, 7^{3/4}\, \sqrt{5}\, 320^{1/4} \right\}$$

$$> \; f[0] := 200\cdot\left(\frac{20}{3}\right)^{\frac{1}{4}}\cdot\left(\frac{16}{5}\right)^{\frac{1}{2}}\cdot(20)^{\frac{1}{4}}\cdot\left(\frac{80}{7}\right)^{\frac{1}{4}}$$

$$f_0 := \frac{40}{21}\sqrt{20}\, 3^{3/4}\sqrt{16}\,\sqrt{5}\, 80^{1/4}\, 7^{3/4}$$

$$> \; evalf[5]\left(200 * \left(\frac{20}{3}\right)\text{^}(1/4) * \left(\frac{16}{5}\right)\text{^}(1/2) * (20)\text{^}(1/4) * \left(\frac{80}{7}\right)\text{^}((1/4)) \right)$$

$$2235.3$$

$$> \; b := \frac{16}{5}$$

$$b := \frac{16}{5}$$

```
> evalf[5]( 16/55 )
```

$$0.29091$$

```
>
```

$$> \; h := \frac{20}{3}$$

$$h := \frac{20}{3}$$

```
> evalf[5]( 20/3 )
```

$$6.6667$$

>

> $g := \frac{80}{7}$

$$g := \frac{80}{7}$$

> *evalf*[5](80/7)

11.429

> $r := 20$

$$r := 20$$

> *evalf*[5](20)

20.

>

> $\lambda = \frac{1}{840}\,20^{1/4}\,3^{3/4}\,80^{1/4}\,7^{3/4}\sqrt{5}\,320^{1/4}$

$$\lambda = \frac{1}{840}\,20^{1/4}\,3^{3/4}\,80^{1/4}\,7^{3/4}\sqrt{5}\,320^{1/4}$$

> *evalf*[5]((1/840)*20^(1/4)*3^(3/4)*80^(1/4)*7^(3/4)*5^(1/2)*321^(1/4))

0.69913

>

TABLE 4.1 *NO.* 10

> $$prob1 := \left(\frac{200}{4}\cdot h^{-\frac{3}{4}}\cdot b^{\frac{1}{4}}\cdot r^{\frac{1}{4}}\cdot g^{\frac{1}{2}} - 120\cdot\lambda = 0, \frac{200}{4}\cdot h^{\frac{1}{4}}\cdot b^{-\frac{3}{4}}\cdot r^{\frac{1}{4}}\cdot g^{\frac{1}{2}} - 500\cdot\lambda = 0, \frac{200}{4}\cdot h^{\frac{1}{4}}\cdot b^{\frac{1}{4}}\cdot r^{-\frac{3}{4}}\cdot g^{\frac{1}{2}} - 40\cdot\lambda = 0, \frac{200}{2}\cdot h^{\frac{1}{4}}\cdot b^{\frac{1}{4}}\cdot r^{\frac{1}{4}}\cdot g^{-\frac{1}{2}} - 70\cdot\lambda = 0, -120\cdot h - 500\cdot b - 40\cdot r - 70\cdot g + 4000 = 0\right);$$

$$prob1 := \frac{50\,b^{1/4}\,r^{1/4}\sqrt{g}}{h^{3/4}} - 120\,\lambda = 0, \frac{50\,h^{1/4}\,r^{1/4}\sqrt{g}}{b^{3/4}} - 500\,\lambda = 0, \frac{50\,h^{1/4}\,b^{1/4}\sqrt{g}}{r^{3/4}} - 40\,\lambda = 0, \frac{100\,h^{1/4}\,b^{1/4}\,r^{1/4}}{\sqrt{g}} - 70\,\lambda = 0, -120\,h - 500\,b - 40\,r - 70\,g + 4000 = 0$$

>

> $sol1 := solve(\{prob1\}, \{h, b, r, g, \lambda\})$

$$sol1 := \left\{b = \frac{8}{5}, g = \frac{160}{7}, h = \frac{20}{3}, r = 20, \lambda = \frac{1}{840}\,20^{1/4}\,3^{3/4}\sqrt{10}\,\sqrt{7}\,8^{1/4}\,5^{3/4}\,320^{1/4}\right\}$$

> $f[0] := 200\cdot\left(\frac{20}{3}\right)^{\frac{1}{4}}\cdot\left(\frac{8}{5}\right)^{\frac{1}{4}}\cdot(20)^{\frac{1}{4}}\cdot\left(\frac{160}{7}\right)^{\frac{1}{2}}$

$$f_0 := \frac{40}{21}\sqrt{20}\,3^{3/4}\,8^{1/4}\,5^{3/4}\sqrt{160}\,\sqrt{7}$$

> $evalf[5]\left(200*\left(\frac{20}{3}\right){}^\wedge(1/4)*\left(\frac{8}{5}\right){}^\wedge(1/4)*(20){}^\wedge(1/4)*\left(\frac{160}{7}\right){}^\wedge((1/2))\right)$

$$3654.2$$

> $b := \frac{8}{5}$

$$b := \frac{8}{5}$$

> $evalf[5](\,8/5\,)$

$$1.6000$$

>

> $h := \frac{20}{3}$

$$h := \frac{20}{3}$$

> $evalf[5](\,20/3\,)$

$$6.6667$$

>

> $g := \frac{160}{7}$

$$g := \frac{160}{7}$$

> $evalf[5](\,160/7)$

$$22.857$$

> $r := 20$

$$r := 20$$

> $evalf[5](\,20)$

$$20.$$

>

> $\lambda = \frac{1}{840}\,20^{1/4}\,3^{3/4}\,\sqrt{10}\,\sqrt{7}\,8^{1/4}\,5^{3/4}\,320^{1/4}$

$$\lambda = \frac{1}{840}\,20^{1/4}\,3^{3/4}\,\sqrt{10}\,\sqrt{7}\,8^{1/4}\,5^{3/4}\,320^{1/4}$$

> $evalf[5](\,(1/840)*20{}^\wedge(1/4)*3{}^\wedge(3/4)*10{}^\wedge(1/2)*7{}^\wedge(1/2)*8{}^\wedge(1/4)*5{}^\wedge(3/4)*320{}^\wedge(1/4)\,)$

$$1.1420$$

>

TABLE 4.1 *NO.* 12

> $$prob1 := \left(\frac{200}{6} \cdot h^{-\frac{5}{6}} \cdot b^{\frac{1}{12}} \cdot r^{\frac{3}{4}} \cdot g^{\frac{1}{2}} - 120 \cdot \lambda = 0, \frac{200}{12} \cdot h^{\frac{1}{6}} \cdot b^{-\frac{11}{12}} \cdot r^{\frac{3}{4}} \cdot g^{\frac{1}{2}} - 500 \cdot \lambda = 0, \frac{600}{4} \cdot h^{\frac{1}{6}} \cdot b^{\frac{1}{12}} \cdot r^{-\frac{1}{4}} \cdot g^{\frac{1}{2}} - 40 \cdot \lambda = 0, \frac{200}{2} \cdot h^{\frac{1}{6}} \cdot b^{\frac{1}{12}} \cdot r^{\frac{3}{4}} \cdot g^{-\frac{1}{2}} - 70 \cdot \lambda = 0, -120 \cdot h - 500 \cdot b - 40 \cdot r - 70 \cdot g + 4000 = 0 \right);$$

$$prob1 := \frac{100}{3} \frac{b^{1/12} r^{3/4} \sqrt{g}}{h^{5/6}} - 120\,\lambda = 0, \frac{50}{3} \frac{h^{1/6} r^{3/4} \sqrt{g}}{b^{11/12}} - 500\,\lambda = 0, \frac{150\, h^{1/6} b^{1/12} \sqrt{g}}{r^{1/4}} - 40\,\lambda = 0, \frac{100\, h^{1/6} b^{1/12} r^{3/4}}{\sqrt{g}} - 70\,\lambda = 0, -120\,h - 500\,b - 40\,r - 70\,g + 4000 = 0$$

>

> $$sol1 := solve(\{prob1\}, \{h, b, r, g, \lambda\})$$

$$sol1 := \left\{ b = \frac{4}{9}, g = \frac{400}{21}, h = \frac{100}{27}, r = 50, \lambda = \frac{1}{378}\, 100^{1/6}\, 27^{5/6} \sqrt{21}\, RootOf\left(3\_Z^6 - 2, index = 1\right) RootOf\left(\_Z^4 - 50, index = 1\right)^3 \right\}$$

> $$f[0] := 200 \cdot \left(\frac{100}{27}\right)^{\frac{1}{6}} \cdot \left(\frac{4}{9}\right)^{\frac{1}{12}} \cdot (50)^{\frac{3}{4}} \cdot \left(\frac{400}{21}\right)^{\frac{1}{2}}$$

$$f_0 := \frac{200}{5103}\, 100^{1/6}\, 27^{5/6}\, 4^{1/12}\, 9^{11/12}\, 50^{3/4} \sqrt{400} \sqrt{21}$$

> $$evalf[5]\left(200 * \left(\frac{100}{27}\right)\text{^}(1/6) * \left(\frac{4}{9}\right)\text{^}(1/12) * (50)\text{^}(3/4) * \left(\frac{400}{21}\right)\text{^}((1/2)) \right)$$

19082.

> $$b := \frac{4}{9}$$

$$b := \frac{4}{9}$$

> *evalf*[5](4/9)

0.44444

>

> $$h := \frac{100}{27}$$

$$h := \frac{100}{27}$$

> *evalf*[5](100/27)

3.7037

>

> $$g := \frac{400}{21}$$

$$g := \frac{400}{21}$$

```
> evalf[5]( 400/21)
```

$$19.048$$

```
> r := 50
```

$$r := 50$$

```
> evalf[5]( 50)
```

$$50.$$

```
>
```

$$> \lambda := \frac{1}{378}\, 100^{1/6}\, 27^{5/6} \sqrt{21}\, \mathit{RootOf}\left(3\_Z^6 - 2, \mathit{index} = 1\right) \mathit{RootOf}\left(\_Z^4 - 50, \mathit{index} = 1\right)^3$$

$$\lambda := \frac{1}{378}\, 100^{1/6}\, 27^{5/6} \sqrt{21}\, \mathit{RootOf}\left(3\_Z^6 - 2, \mathit{index} = 1\right) \mathit{RootOf}\left(\_Z^4 - 50, \mathit{index} = 1\right)^3$$

```
> evalf[5]( (1/378)*100^(1/6)*27^(5/6)*21^(1/2)*RootOf(3 _Z^6 - 2, index = 1)
      *RootOf(_Z^4 - 50, index = 1)^(3) )
```

$$7.1548$$

```
>
```