HYBRID BACKWARD DIFFERENTIATION FORMULAS FOR THE SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

This study focuses on formulation of hybrid backward differentiation methods with power series as basis function through interpolation and collocation approach for solving initial value problems of first order ordinary differential equations. The step numbers for the derived hybrid methods are $k = 5$ and 6. The schemes are analysed using appropriate theorems to investigate their consistency, stability, convergence and the investigation shows that the developed schemes are consistent, zero stable and hence convergent. The stability property of the methods was also investigated and findings reveal that the methods are A-stable which make them suitable for solving the class of problems considered in this project such as linear and non-linear problems, oscillatory problems and stiff system. The implementation results on these problems show that the methods are of higher accuracy and have superiority over some other existing methods considered in the literature.

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

In many technological knowhow disciplines, problems are normally modeled mathematically. These models give rise to differential equations which requires answers that inadvertently resolve the problems.

The norm turned to remedy these problems, was the use of analytical or exact methods which give rise to solutions that provide insight into the behaviour of some of these problems. However, there are certain differential equations that are tough to solve by the use of analytical methods, apart from the application of numerical methods. (Abdullahi, 2017).

This research aims to derive some backward differentiation formulas that could be used to solve ordinary first order differential equations. The off grid points used in their formulation give them the name hybrid. A backward differentiation formula is termed extended when there is the presence of more than one function evaluation at the collocation point. Primarily numerical methods are used to find approximate solutions to problems. Software innovation has made numerical solutions for Ordinary Differential Equations (ODEs) the focus of numerical researchers. A differential equation is an equation involving a relationship between an unknown function and one or more of its derivatives. With y as the dependent variable, and as a function of the independent variable x, the x and y differential equation is expressed as

$$y' = f(x, y) \qquad (1.1)$$

In present times, numerical methods have been efficient tools used in the solution of first order ordinary differential equations

$$y' = f(x, y), \qquad y(a) = y_0, \quad a \le x \le b \tag{1.2}$$

Such differential equations occur in many fields of engineering science and in particular, they appear in electrical circuit, vibrations, chemical reactions, kinetics etc.

Developing methods for solving (1.2) still remains a challenge in modern numerical analysis. Many authors like Lambert (1973) and Musa *et al.* (2012) have written on the block extended backward differentiation formula that approximates the solution of (1.2) and is given by

$$\sum_{j=0}^{5} \alpha_{j,i} y_{n+j-2} = h\beta_{k,j}, f_{n+k} + h\beta_{k+1}, f_{n+k+1}, k = i = 1,2,3. \tag{1.3}$$

It was developed for higher order A–stable block methods for stiff initial value problems. An acceptable linear multistep method (LMM) must be convergent. Consistency and zero stability are the necessary and sufficient conditions for convergence of a LMM.

Akinfenwa *et al.* (2020) stated clearly that the numerical solutions of stiff systems have been one of the major worries for numerical analysts. A numerical method that is potentially good for solving systems of stiff ODEs must have some reliability in terms of its region of absolute stability and good accuracy.

According to (Gear, 1967), consistency controls the magnitude of the local truncation error while zero stability controls the manner in which the error is propagated at each step of the calculations. A method which is not both consistent and zero stable is rejected outright and has no practical interest. In this research work, some hybrid backward differentiation formulas are developed and applied to solve first order ordinary differential equations.

1.3 Statement of the Research Problem

In this research work, some five and six step hybrid and extended hybrid backward differentiation formulas with one off grid point will be developed to solve first order ordinary differential equations of the form (1.1). The order, error constant, zero stability and convergence will be analyzed. Numerical experiments will also be shown with all the newly derived schemes.

1.4 Aim and Objectives

The aim of this research work is to construct some k – step *(k=5, 6)* hybrid backward differentiation formulas and an accompanying extended hybrid backward differentiation formula for solving first order ordinary differential equations.

The objectives are to:

i. construct the continuous formulation of five and six step hybrid and six step extended hybrid backward differentiation formulas with one off step point at interpolation using power series as basis function;

ii. derive some new schemes that can be used to solve first order stiff ordinary differential equations of the form $y' = f(x, y)$;

iii. analyze the order, error constant, consistency and zero stability of all the proposed schemes;

iv. put the derived schemes to use in numerical experiments and

v. compare the results of the proposed methods with some existing methods found in the literature.

1.5 Significance of the Study

Real life phenomena such as earth sciences, saturation and diffusion problems, demographic models, fluid mechanics, and budgeting could lead to the study of First Order Ordinary Differential Equations (ODEs). The developed methods are useful to solve these problems and yield results with decreased computing time.

1.6 Scope of the study

The study is restricted to solving first order initial value problems in ordinary differential equations (ODEs) of the form (1.1). The schemes are derived to be effective in terms of accuracy and stability.

1.7 Limitations of the Study

In this study, five and six step hybrid backward differentiation formulas and also six step extended hybrid backward differentiation formulas with one off grid point shall only be considered to derive implicit schemes. These schemes can only be used to solve problems of first order ordinary differential equation.

1.8 Basic Definitions

i. Linear Multistep Method

The general form of linear multistep method (LMM) for first order differential equation that can solve equation (1.1) as presented by Oyelami (2018) is given as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (1.4)$$

The equation is explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$

ii. Hybrid Linear Multistep Method

A k – step hybrid linear multistep method is defined by Lambert (1973) as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \beta_v f_{n+v}$$

$$f_{n+v} = f(x_{n+v}, y_{n+v}) \tag{1.5}$$

α_0 and β_0 are both not zeros, $v \notin \{0,1,\ldots,k\}$. (Lambert, 1973)

iii. Collocation Method

The collocation method can be defined simply as a method involving the determination of an approximate solution with a suitable set of functions, sometimes referred to as the trial or basic function called power series in this research. It is a form of projection to solve integral and differential equations, in which the approximate solution is calculated on condition that the equation is satisfied at certain points. Fairweather (1989)

iv. Stability

One practical prerequisite for a good method to be efficient is that it has fulfilled the stability condition. If no theoretical solution to a problem is known, numerical solutions can only be sought for, given initial or limit values. But for all initial values of a certain equilibrium point in the neighborhood one needs information on the stability behaviour of the solution. The equilibrium points are again shifted to the center and established by it (Frazer *et al.*, 1937).

v. Ordinary Differential Equations (ODE)

An Ordinary Differential Equation (ODE) is a differential equation that includes one or more independent variable functions and the derivatives of those functions. It is an equality which contains a function and its derivatives.

vi. Partial Differential Equation (PDE)

A Partial Differential Equation (PDE) is a mathematical equation containing two or more independent variables, an unknown function (depending on those variables), and partial unknown function derivatives in addition to the independent variables.

vii. Linear Ordinary Differential Equation (Linear ODE)

A Linear Ordinary Differential Equation is a differential equation, described in the unknown function and its derivatives by a linear polynomial. It is a first degree equation, when the expression equated to zero is assumed to be a component of the dependent variable and its differential coefficients. If it can be written as a linear combination, then the differential equation is linear.

viii. Nonlinear Ordinary Differential Equation (Nonlinear ODE)

A Nonlinear Differential Equation is a differential equation which is not a linear equation between the unknown function and its derivatives. There are very few methods of specifically solving nonlinear differential equations, those typically known rely on the equation which has unique symmetries. A nonlinear ordinary differential equation is that which cannot be written as a linear combination.

ix. Order of an Ordinary Differential Equation.

Given F a function of x, y and derivatives of y. Then an equation of the form

$$F\left(x, y, y', \ldots, y^{(n-1)}\right) = y^{(n)} \tag{1.6}$$

Is called an explicit ordinary differential equation of order *n.*

x. Initial Value Problem

In the field of differential equations, an Initial Value Problem (also named by some writers as Cauchy Problem) is an Ordinary Differential Equation along with a defined value, called the initial condition, of the unknown function at a given point in the solution domain.

xi. Boundary Value Problem

A boundary Value Problem is a differential equation with a number of external restrictions, called boundary conditions. A solution to a Boundary Value Problem is a differential equation solution which also satisfies the boundary conditions.

xii. Backward Differentiation Formula

Backward Differentiation Formulas are especially useful in solving stiff differential equations and algebraic differential equations. These are formulas that provide a derivative estimate of a vector at a time *t* in terms of its function values at *t* and earlier periods. (Hence the word "Backward"). It is a family of implicit methods for the integration of ordinary differential equations in numerical form. They are Linear Multistep Methods which estimate the derivative of that function for a given function and time using information from already determined time points, thereby improving the approximation accuracy.

xiii. Maple Software Package

Maple is a graphical and numerical computation framework, and is also a tool of multi-paradigm. Maple also includes many areas of scientific computing, including simulation, data analysis, matrix computation, and networking, developed by Maplesoft. It manipulates algebraic sets, unbounded variables, exact rational numbers, mathematical formulas, real numbers (with accuracy), polynomials, list tables, matrices and vectors. It can solve structures of equations and can discern expressions and combine them.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Linear Multistep Methods

Linear Multistep Methods (LMMs) numerical methods for solving Initial Value Problems (IVPs) of first order. Conceptually, from an initial point, a computational process starts and then takes a step forward in time to find the next solution point. The process continues with subsequent steps to visualize the solution. Through retaining and using the knowledge from previous points and derivative values, multistep methods attempt to achieve efficiency. A linear combination of the previous points and derivative values is used for linear multistep methods (Abioyar *et al.*, 2015).

The methods are also used to solve higher order ODEs. LMMs are not self-starting hence, need starting values from single step methods like Euler's method and Runge Kutta family of methods. According to Collatz (1960), the existing methods of deriving the LMMs in discrete form include the interpolation approach, numerical integration, Taylor series expansion and through the determination of the order of LMMs be consistent.

Continuous technique of collocation and interpolation is now commonly used for LMMs derivation, block methods and hybrid methods. Several continuous LMMs were derived using different techniques and approaches: for second order ODEs; Abdullahi, (2017) derived three step continuous hybrid implicit linear multistep block methods, Oyelami (2018) derived a collocation technique based on orthogonal polynomial, Jator *et al.,* (2014) worked on blocked hybrid backward differentiation formula for solution of large stiff systems, Mohammed and Yahaya*,* (2010) worked on fully implicit four point block backward difference formulae for solving first order initial value problems; also Muhammad and Yahaya, (2012) worked on fully A sixth order implicit hybrid backward differentiation formulae for block solution of ordinary differential equations, Musa *et al.* (2012) worked on the convergence and order of the 3 point block extended backward differentiation formula, Odekunle *et al*. (2012) developed a continuous linear multistep method using interpolation and collocation for the solution of first order ODE with constant step size.

2.2 Discrete Method

Despite the fact that the discrete integration algorithm achieves satisfactory results, in comparison with their continuous counterparts, they suffer from their limitations and drawbacks. So there is an attempt to enlighten the reader about the importance of continuous formulations of discrete methods by distinguishing between the discrete methods and their continuous counterparts.

According to Adeyefa *et al.* (2014), discrete methods are methods that are used at each grid point to achieve numerical approximation, and are not represented as independent variable. Notwithstanding the simplicity and broad applicability, they involve some setback inherent in discrete numerical integration algorithm. As such recent researchers have provided interesting results for continuous integration through different approaches in IVP's numerical solution – Awoyemi, *et al.* (2015). It was also found that, for example, the continuous formulations of the discrete systems appear to possess some features over their continuous equivalents. The continuous schemes can be used for more analytical work such as ease in differentiation than the discrete schemes.

2.3 Hybrid Method

Over the years, numerical research has concentrated on solution at grid points, with little or no exposure to point other than the grid points. Chollom (2004) was upset about this finding. Though more effective and efficient than the Runge Kutta methods, these methods have the weak stability properties for a number of functions evaluation per step. Notwithstanding the benefits of these approaches, they suffer the drawback of requiring starting values and a special process to adjust the step length. Such difficulties can be minimized by reducing the linear multistep method's step number without automatically reducing its size, but in so doing, the critical zero stability condition becomes difficult to fulfill. It's also a documented fact that if the zero stability condition is to be met, an order *(k+2)* is even for k and *(k+1)* is odd. Gragg and Stetter (1964) worked together with Butcher (1964) and Gear (1965) to overcome these difficulties by introducing a modified linear multistep formula that incorporates a function evaluation at an off grid point. The approach was called Hybrid because it maintains the properties of linear multistep and Runge Kutta methods and lies between the method of extrapolation and substitution. The work of Bryne and Lambert (1966) proposed a supposition of Runge Kutta methods in which stage derivatives computed in earlier stages are used alongside stage derivatives found in the current phase in the next step to determine the output value. The iterations are examined in the same manner as the methods of Runge Kutta, derivatives evaluated in the previous step, defined as

$T_i^{[n-1]}, \quad ,i = 1,2,...,s$ and the current step derivative by $T_i^{[n]}, \quad ,i = 1,2,...,s$.

Akinfewa *et al.* (2011).

Equations for a single step of the method were represented as

$$\left.\begin{aligned} Y_i &= y_{n-1} + h\sum_{j=0}^{s} a_{ij} T_j^n T_i^n = f(x_{n-1} + hc_i Y_i) y_n \\ &= y_{n-1} + h\left(\sum_{j=0}^{s} b_i T_i^{[n]} + \sum_{j=0}^{s} \hat{b}_i \, T_i^{[n-1]} \right) \end{aligned}\right\} \tag{2.1}$$

Such methods have been called methods of Pseudo Runge Kutta. Therefore a k-step hybrid method is established as

$$\sum_{j=0}^{k} \mu_j y_{n+j} = h\sum_{j=0}^{k} \lambda_j f_{n+j} + \lambda_v f_{n+v} \tag{2.2}$$

Where $\mu_k = +1,$ and λ_0 are both not zero, $v \in [0,1,\ldots,k]$ and $f_{n+v} \in f(x_{n+v,} y_{n+v})$

2.4 Continuous Multi – Step Collocation Method (CMCM)

The continuous finite difference (CFD) approximation method by the idea of interpolation and collocation which Lie and Norsett (1989) and Oyelami (2018) referred to as the Multistep collocation (MC) is presented below. Adopting the notation

$$\underset{-}{a} = (\theta_0, \theta_1, \ldots, \theta_{(t+m-1)})^T, \; \varphi(x) = (\varphi_0(x), \varphi_1(x), \ldots, \varphi_{(t+m-1)})^T \tag{2.3}$$

Where $\theta_r, r = 0,1,\ldots, t+m-1$ are undetermined constants, $\varphi_r(x)$ are specified basis functions, T denotes transpose of, t denotes the number of interpolation point and m denotes the number of distinct collocation points. A continuous approximation (interpolant) $Y(x)$ to $y(x)$as expressed by Zarina *et al.*(2008)was considered

$$Y(x) = \sum_{r=0}^{t+m-1} \theta_r \varphi_r(x) = \theta^T \varphi(x) \tag{2.4}$$

Which is valid in the sub – intervals $x_n \leq x \leq x_{n+k}$,where $n = 0, k,..., N - K$. The quantities $x_0 = \theta$, $x_N = b, k, m, n, t$ and $\varphi_r(x)$, $r = 0,1,..., t + m - 1$ are specified values. The constant co – efficient a_r of (2.4) can be determined using the conditions

$$y'(\bar{x}_j) = f_{n+j}, \quad j = 0,1,..., m-1 \tag{2.5}$$

where

$$f_{n+j} = f(x_{n+j}, x_{n+j}) \tag{2.6}$$

The distinct collocation points $x_0,...,x_{m-1}$ can be chosen freely from the set $[x_n, x_{n+k}]$. Equations (2.4), (2.5) and (2.6) are denoted by a single set of algebraic equations of the form

$$D\underline{\theta} = \underline{F} \tag{2.7}$$

$$\left.\begin{aligned} \underline{F} &= (y_n, y_{n+1},..., y_{n+t-1}, f_n, f_{n+1}, f_{n+m-1})^T \\ \underline{\theta} &= D^{-1}\underline{F} \end{aligned}\right\} \tag{2.8}$$

where $\underline{D}$ is the non – singular matrix of dimension $(t + m)$. Mohammed and Adeniyi (2014).

$$\underset{-}{D} = \begin{pmatrix} \varphi_0(x_n) & \cdots & \varphi_{t+m-1}(x_n) \\ \vdots & \vdots & \vdots \\ \varphi_0(x_{n+t-1}) & \cdots & \varphi_{t+m-1}(x_{n+t-1}) \\ \\ \varphi_0(\bar{x}_0) & \cdots & \varphi_{t+m-1}(\bar{x}_0) \\ \vdots & \vdots & \vdots \\ \varphi_0(\bar{x}_{m-1}) & \cdots & \varphi_{t+m-1}(\bar{x}_{m-1}) \end{pmatrix} \tag{2.9}$$

By substituting (2.9) into (2.4), The MC formula was obtained

$$y(x) = F^T C^T \varphi(x), \qquad x_n \le x \le x_{n+k} \qquad n = 0, k, \ldots, N-k \tag{2.10}$$

Where

$$\underset{-}{C} \equiv \underset{-}{D}^{-1} = (c_{ij}), \qquad i, j = 1, \ldots, t+m-1 \tag{2.11}$$

$$\underset{-}{C} = \begin{pmatrix} c_{11} & \cdots & c_{1t} & c_{1t+1} & \cdots & c_{1t+m} \\ c_{21} & \cdots & c_{2t} & c_{2t+1} & \cdots & c_{2t+m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{t+m1} & \cdots & c_{t+mt} & c_{t+mt+1} & \cdots & c_{t+mt+m} \end{pmatrix} \tag{2.12}$$

with the numerical elements denoted by $c_{ij,} i, j = 1, \ldots, k+m$. By expanding $C^T \varphi(x)$ in (2.12) yields the following

$$y(x) = \sum_{j=0}^{t-1} \left(\sum_{r=0}^{t+m-1} C_{r+2,j+2} \varphi_r(x) \right) + \sum_{j=0}^{m-1} h \left(\sum_{r=0}^{k+m-1} \frac{C_{r+j+1}}{h} \varphi_r(x) \right) f_{n+j} \tag{2.13}$$

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \beta_j(x) f_{n+j} \tag{2.14}$$

θ_r can be determined as follows:

$$y(x) = \left\{ \sum_{r=0}^{t-1} \alpha_{j,r+1} y_{n+j} + h \sum_{j=0}^{m-1} \beta_{j,r+1} f_{n+j} \right\} \varphi_r(x) \tag{2.15}$$

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Construction of Proposed Hybrid Backward Differentiation Methods

A step by step description of the formulation of the proposed $k-$ step Hybrid Backward Differentiation Formula for the solution of first order ordinary differentiation equations is hereby given.

An approximate solution of the form (3.1) is sought for

$$Y(x)=\sum_{j=0}^{r+s-1}\gamma_j x^j \tag{3.1}$$

where γ_j are unknown coefficients to be determined, $k<r$ and $s>0$ are the number of interpolation and collocation points respectively.

The continuous form of the numerical schemes is expressed as:

$$Y(x)=\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+\alpha_\mu(x)y_{n+\mu}+h\beta_k(x)f_{n+k} \tag{3.2}$$

where $\alpha_j(x)$, $\beta_k(x)$ and $\alpha_\mu(x)$ are continuous coefficients.

3.2 Five Step Hybrid Backward Differentiation Method (5SHBDM) with One Off Step Point at Interpolation.

To derive this method, (3.2) is used to obtain a continuous five step hybrid backward differentiation method with the following specification: $r=6, s=1, k=5$as follows:

$$y(x)=\alpha_0 y_n+\alpha_1 y_{n+1}+\alpha_2 y_{n+2}+\alpha_3 y_{n+3}+\alpha_4 y_{n+4}+\alpha_{\frac{9}{2}} y_{n+\frac{9}{2}}+h\beta_5 f_{n+5} \tag{3.3}$$

Using power series as the basis function,

$$Y(x) = \sum_{i=0}^{r+s-1} a_i x^j , \quad x_n \le x \le x_{n+p} \tag{3.4}$$

Where n = grid point , $n+p$ = interval of integration, $p = r+s-1$

Approximating the exact solution $Y(x)$ by a polynomial of degree 6 of the form:

$$Y(x) = \sum_{i=0}^{6} a_i x^j , \quad x_n \le x \le x_{n+p} \tag{3.5}$$

$$Y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \tag{3.6}$$

Taking the first derivative of (3.6) yields

$$Y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 \tag{3.7}$$

Interpolating equation (3.6) at $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+\frac{9}{2}}$, and collocating equation (3.7) at $x = x_{n+5}$ gives the system of equation in which the coefficients $a_j's$ are found

$$Y'(h) = a_1 + 2a_2 x_{n+5} + 3a_3 x_{n+5}^2 + 4a_4 x_{n+5}^3 + 5a_5 x_{n+5}^4 + 6a_6 x_{n+5}^5 = f_{n+5} \tag{3.8}$$

$$Y(x_n) = a_0 \tag{3.9}$$

$$Y(x_{n+1}) = a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4 x_{n+1}^4 + a_5 x_{n+1}^5 + a_6 x_{n+1}^6 \tag{3.10}$$

$$Y(x_{n+2}) = a_0 + a_1 x_{n+2} + a_2 x_{n+2}^2 + a_3 x_{n+2}^3 + a_4 x_{n+2}^4 + a_5 x_{n+2}^5 + a_6 x_{n+2}^6 \tag{3.11}$$

$$Y(x_{n+3}) = a_0 + a_1 x_{n+3} + a_2 x_{n+3}^2 + a_3 x_{n+3}^3 + a_4 x_{n+3}^4 + a_5 x_{n+3}^5 + a_6 x_{n+2}^6 \tag{3.12}$$

$$Y(x_{n+4}) = a_1 + a_1 x_{n+4} + a_2 x_{n+4}^2 + a_3 x_{n+4}^3 + a_4 x_{n+4}^4 + a_5 x_{n+4}^5 + a_6 x_{n+2}^6 \tag{3.13}$$

$$Y(x_{n+\frac{9}{2}}) = a_0 + a_1 x_{n+\frac{9}{2}} + a_2 x^2_{n+\frac{9}{2}} + a_3 x^3_{n+\frac{9}{2}} + a_4 x^4_{n+\frac{9}{2}} + a_5 x^5_{n+\frac{9}{2}} + a_6 x^6_{n+\frac{9}{2}} \quad (3.14)$$

The matrix D of the proposed method is expressed

as $$D = \begin{pmatrix} 1 & x_n & x_n & x_n & x_n & x_n & x_n \\ 1 & x_{n+1} & x^2_{n+1} & x^3_{n+1} & x^4_{n+1} & x^5_{n+1} & x^6_{n+1} \\ 1 & x_{n+2} & x^2_{n+2} & x^3_{n+2} & x^4_{n+2} & x^5_{n+2} & x^6_{n+2} \\ 1 & x_{n+3} & x^2_{n+3} & x^3_{n+3} & x^4_{n+3} & x^5_{n+3} & x^6_{n+3} \\ 1 & x_{n+4} & x^2_{n+4} & x^3_{n+4} & x^4_{n+4} & x^5_{n+4} & x^6_{n+4} \\ 1 & x_{n+\frac{9}{2}} & x^2_{n+\frac{9}{2}} & x^3_{n+\frac{9}{2}} & x^4_{n+\frac{9}{2}} & x^5_{n+\frac{9}{2}} & x^6_{n+\frac{9}{2}} \\ 0 & x_{n+5} & x^2_{n+5} & x^3_{n+5} & x^4_{n+5} & x^5_{n+5} & x^6_{n+5} \end{pmatrix} \quad (3.15)$$

The Column matrix B of the proposed method is given as:

$$B = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+\frac{9}{2}} \\ f_{n+5} \end{pmatrix} \quad (3.16)$$

Multiplying the inverse of the matrix D with the column matrix B the values of the coefficients in (3.3) is obtained as follows

$$\alpha_0(x) = 1 - \frac{23095}{9252}\frac{x}{h} + \frac{131057}{55512}\frac{x^2}{h^2} - \frac{20485}{18504}\frac{x^3}{h^3} + \frac{15293}{55512}\frac{x^4}{h^4} - \frac{215}{6168}\frac{x^5}{h^5} + \frac{49}{27756}\frac{x^6}{h^6} \quad (3.17)$$

$$\alpha_1(x) = \frac{11430}{1799}\frac{x}{h} - \frac{34201}{3598}\frac{x^2}{h^2} + \frac{118577}{21588}\frac{x^3}{h^3} - \frac{16619}{10794}\frac{x^4}{h^4} + \frac{4537}{21588}\frac{x^5}{h^5} - \frac{121}{10794}\frac{x^6}{h^6} \quad (3.18)$$

$$\alpha_2(x) = -\frac{2241}{257}\frac{x}{h} + \frac{89457}{5140}\frac{x^2}{h^2} - \frac{12211}{1028}\frac{x^3}{h^3} + \frac{3813}{1028}\frac{x^4}{h^4} - \frac{561}{1028}\frac{x^5}{h^5} + \frac{79}{2570}\frac{x^6}{h^6}$$

(3.19)

$$\alpha_3(x) = \frac{2390}{257}\frac{x}{h} - \frac{93017}{4626}\frac{x^2}{h^2} + \frac{46993}{3084}\frac{x^3}{h^3} - \frac{24071}{4626}\frac{x^4}{h^4} + \frac{2537}{3084}\frac{x^5}{h^5} - \frac{227}{4626}\frac{x^6}{h^6} \quad (3.20)$$

$$\alpha_4(x) = -\frac{9405}{1028}\frac{x}{h} + \frac{42211}{2056}\frac{x^2}{h^2} - \frac{101287}{6168}\frac{x^3}{h^3} + \frac{36917}{6168}\frac{x^4}{h^4} - \frac{6227}{6168}\frac{x^5}{h^5} + \frac{197}{3084}\frac{x^6}{h^6} \quad (3.21)$$

$$\alpha_{\frac{9}{2}}(x) = \frac{76288}{16191}\frac{x}{h} - \frac{2594432}{242865}\frac{x^2}{h^2} + \frac{140480}{16191}\frac{x^3}{h^3} - \frac{156736}{48573}\frac{x^4}{h^4} + \frac{3008}{5397}\frac{x^5}{h^5} - \frac{8768}{242865}\frac{x^6}{h^6} \quad (3.22)$$

$$\beta_5(x) = -\frac{108}{257}x + \frac{249}{257}\frac{x^2}{h} - \frac{415}{514}\frac{x^3}{h^2} + \frac{80}{257}\frac{x^4}{h^3} - \frac{29}{514}\frac{x^5}{h^4} + \frac{1}{257}\frac{x^6}{h^5} \quad (3.23)$$

Substituting (3.17) – (3.23) into (3.3) gives the continuous form of the five step implicit method expressed as

$$\begin{aligned}
\bar{y}(x) = {} & \left(1 - \frac{23095}{9252}\frac{x}{h} + \frac{131057}{55512}\frac{x^2}{h^2} - \frac{20485}{18504}\frac{x^3}{h^3} + \frac{15293}{55512}\frac{x^4}{h^4} - \frac{215}{6168}\frac{x^5}{h^5} + \frac{49}{27756}\frac{x^6}{h^6}\right) y_n \\
& + \left(\frac{11430}{1799}\frac{x}{h} - \frac{34201}{3598}\frac{x^2}{h^2} + \frac{118577}{21588}\frac{x^3}{h^3} - \frac{16619}{10794}\frac{x^4}{h^4} + \frac{4537}{21588}\frac{x^5}{h^5} - \frac{121}{10794}\frac{x^6}{h^6}\right) y_{n+1} \\
& + \left(-\frac{2241}{257}\frac{x}{h} + \frac{89457}{5140}\frac{x^2}{h^2} - \frac{12211}{1028}\frac{x^3}{h^3} + \frac{3813}{1028}\frac{x^4}{h^4} - \frac{561}{1028}\frac{x^5}{h^5} + \frac{79}{2570}\frac{x^6}{h^6}\right) y_{n+2} \\
& + \left(\frac{2390}{257}\frac{x}{h} - \frac{93017}{4626}\frac{x^2}{h^2} + \frac{46993}{3084}\frac{x^3}{h^3} - \frac{24071}{4626}\frac{x^4}{h^4} + \frac{2537}{3084}\frac{x^5}{h^5} - \frac{227}{4626}\frac{x^6}{h^6}\right) y_{n+3} \\
& + \left(-\frac{9405}{1028}\frac{x}{h} + \frac{42211}{2056}\frac{x^2}{h^2} - \frac{101287}{6168}\frac{x^3}{h^3} + \frac{36917}{6168}\frac{x^4}{h^4} - \frac{6227}{6168}\frac{x^5}{h^5} + \frac{197}{3084}\frac{x^6}{h^6}\right) y_{n+4} + \\
& \left(\frac{76288}{16191}\frac{x}{h} - \frac{2594432}{242865}\frac{x^2}{h^2} + \frac{140480}{16191}\frac{x^3}{h^3} - \frac{156736}{48573}\frac{x^4}{h^4} + \frac{3008}{5397}\frac{x^5}{h^5} - \frac{8768}{242865}\frac{x^6}{h^6}\right) y_{n+\frac{9}{2}} \\
& + \left(-\frac{108}{257}x + \frac{249}{257}\frac{x^2}{h} - \frac{415}{514}\frac{x^3}{h^2} + \frac{80}{257}\frac{x^4}{h^3} - \frac{29}{514}\frac{x^5}{h^4} + \frac{1}{257}\frac{x^6}{h^5}\right) hf_{n+5}
\end{aligned} \quad (3.24)$$

Evaluating (3.24) at $x = x_{n+5}$ yields the hybrid five step implicit method

$$y_{n+5} = -\frac{4}{771}y_n + \frac{75}{1799}y_{n+1} - \frac{40}{257}y_{n+2} + \frac{100}{257}y_{n+3} - \frac{300}{257}y_{n+4}$$

$$+\frac{10240}{5397}y_{n+\frac{9}{2}} - \frac{60}{257}hf_{n+5} \tag{3.25}$$

Taking the first derivative of (3.24), thereafter evaluating the resulting continuous polynomial solution at $x = x_{n+1}, x = x_{n+2}, x = x_{n+3}, x = x_{n+4}\ and\ x = x_{n+\frac{9}{2}}$ and rearranging give five additional methods generated as follows

$$y_{n+1} = -\frac{1274}{10965}y_n + \frac{37044}{18275}y_{n+2} - \frac{5929}{3655}y_{n+3} + \frac{5194}{3655}y_{n+4} - \frac{38912}{54825}y_{n+\frac{9}{2}} - \frac{5397}{7310}hf_{n+1} + \frac{441}{7310}hf_{n+5} \tag{3.26}$$

$$y_{n+2} = \frac{1325}{32724}y_n - \frac{3275}{6363}y_{n+1} + \frac{6175}{2727}y_{n+3} - \frac{5425}{3636}y_{n+4} - \frac{40192}{57267}y_{n+\frac{9}{2}} - \frac{1285}{909}hf_{n+2} - \frac{50}{909}hf_{n+5} \tag{3.27}$$

$$y_{n+3} = -\frac{55}{1272}y_n + \frac{153}{371}y_{n+1} - \frac{2403}{1060}y_{n+2} + \frac{2043}{424}y_{n+4} - \frac{10688}{5565}y_{n+\frac{9}{2}} - \frac{771}{212}hf_{n+3} - \frac{27}{212}hf_{n+5} \tag{3.28}$$

$$y_{n+4} = \frac{61}{6321}y_n - \frac{1208}{14749}y_{n+1} + \frac{3564}{10535}y_{n+2} - \frac{328}{301}y_{n+3} + \frac{403456}{221235}y_{n+\frac{9}{2}} - \frac{3084}{2107}hf_{n+4} - \frac{144}{2107}hf_{n+5} \tag{3.29}$$

$$y_{n+\frac{9}{2}} = \frac{89425}{25993216}y_n - \frac{366525}{12996608}y_{n+1} + \frac{1416933}{12996608}y_{n+2} - \frac{3825675}{12996608}y_{n+3} + \frac{31454325}{25993216}y_{n+4}$$

$$+\frac{80955}{203072}hf_{n+\frac{9}{2}} - \frac{297675}{6498304}hf_{n+5} \tag{3.30}$$

3.3 Six Step Hybrid Backward Differentiation Method (6SHBDM) with One-off Step Point at Interpolation.

To derive this method, (3.2) was used to obtain a continuous six step hybrid backward differentiation method with the following specification:

$r = 7$, $\left(y_n, y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+\frac{9}{4}} \right)$ $s = 1$, $k = 6$ as follows:

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + \alpha_3 y_{n+3} + \alpha_4 y_{n+4} + \alpha_5 y_{n+5}$$

$$+ \alpha_{\frac{9}{4}} y_{n+\frac{9}{4}} + h\beta_6 f_{n+6} \tag{3.31}$$

Using the power series function

$$Y(x) = \sum_{i=0}^{r+s-1} a_i x^j, \quad x_n \le x \le x_{n+p} \tag{3.32}$$

The exact solution $Y(x)$ is approximated by a polynomial of degree 7 of the form

$$Y(x) = \sum_{i=0}^{7} a_i x^j, \quad x_n \le x \le x_{n+p} \tag{3.33}$$

$$Y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \tag{3.34}$$

Taking the first derivative of (3.34) gives

$$Y^{'}(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + 7a_6 x^6 \tag{3.35}$$

Interpolating equation (3.34) at $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+\frac{9}{4}}, x_{n+5}$ and collocating

equation (3.35) at $x = x_{n+6}$ gives

$$Y^{'}(x) = a_1 + a_2 x_{n+5} + a_3 x_{n+5}^2 + a_4 x_{n+5}^3 + a_5 x_{n+5}^4 + a_6 x_{n+5}^5 + a_6 x_{n+5}^6 = f_{n+6} \tag{3.36}$$

$$Y(x_n) = a_0 \tag{3.37}$$

$$Y(x_{n+1}) = a_0 + ax_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4 x_{n+1}^4 + a_5 x_{n+1}^5 + a_6 x_{n+1}^6 + a_7 x_{n+1}^7 \tag{3.38}$$

$$Y(x_{n+2}) = a_0 + a_1 x_{n+2} + a_2 x_{n+2}^2 + a_3 x_{n+2}^3 + a_4 x_{n+2}^4 + a_5 x_{n+2}^5 + a_6 x_{n+2}^6 + a_7 x_{n+1}^7 \tag{3.39}$$

$$Y(x_{n+3}) = a_{01} + a_1 x_{n+3} + a_2 x_{n+3}^2 + a_3 x_{n+3}^3 + a_4 x_{n+3}^4 + a_5 x_{n+3}^5 + a_6 x_{n+2}^6 + a_7 x_{n+1}^7 \tag{3.40}$$

$$Y(x_{n+4}) = a_1 + a_1 x_{n+4} + a_2 x_{n+4}^2 + a_3 x_{n+4}^3 + a_4 x_{n+4}^4 + a_5 x_{n+4}^5 + a_6 x_{n+2}^6 + a_7 x_{n+1}^7$$

(3.41)

$$Y(x_{n+\frac{9}{4}}) = a_0 + a_1 x_{n+\frac{9}{4}} + a_2 x^2_{n+\frac{9}{2}} + a_3 x^3_{n+\frac{9}{4}} + a_4 x^4_{n+\frac{9}{4}} + a_5 x^5_{n+\frac{9}{4}} + a_6 x^6_{n+\frac{9}{4}} + a_7 x^7_{n+\frac{9}{4}} \quad (3.42)$$

$$Y(x_{n+5}) = a_1 + a_1 x_{n+4} + a_2 x_{n+4}^2 + a_3 x_{n+4}^3 + a_4 x_{n+4}^4 + a_5 x_{n+4}^5 + a_6 x_{n+2}^6 + a_7 x_{n+1}^7 \quad (3.43)$$

The D matrix of the proposed method is expressed as

$$D = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} & D_{17} & D_{18} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} & D_{27} & D_{28} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} & D_{37} & D_{38} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} & D_{47} & D_{48} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} & D_{57} & D_{58} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} & D_{67} & D_{68} \\ D_{71} & D_{72} & D_{73} & D_{74} & D_{75} & D_{76} & D_{77} & D_{78} \\ D_{81} & D_{82} & D_{83} & D_{84} & D_{85} & D_{86} & D_{87} & 3D_{88} \end{pmatrix} \quad (3.44)$$

where the elements of D's are expressed in appendix B. The Column matrix of the proposed method is given as:

$$B = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+\frac{9}{4}} \\ f_{n+6} \end{pmatrix} \quad (3.45)$$

Multiplying the inverse of the matrix (3.44) with the column matrix (3.45) the values of the coefficients in (3.33) were obtained as

$$\alpha_0(x) = 1 - \frac{84623}{29340}\frac{x}{h} + \frac{291923}{88020}\frac{x^2}{h^2} - \frac{233987}{117360}\frac{x^3}{h^3} + \frac{47953}{70416}\frac{x^4}{h^4} - \frac{5191}{39120}\frac{x^5}{h^5} + \frac{4823}{352080}\frac{x^6}{h^6} - \frac{17}{29340}\frac{x^7}{h^7} \quad (3.46)$$

$$\alpha_1(x) = \frac{8694x}{815\,h} - \frac{163803x^2}{8150\,h^2} + \frac{62254x^3}{4075\,h^3} - \frac{117113x^4}{19560\,h^4} + \frac{1247x^5}{978\,h^5} - \frac{13679x^6}{97800h^6} + \frac{151}{24450}\frac{x^7}{h^7} \quad (3.47)$$

$$\alpha_2(x) = -\frac{10665x}{164\,h} + \frac{101697x^2}{652\,h^2} - \frac{45299x^3}{326\,h^3} + \frac{29786x^4}{489\,h^4} - \frac{13759x^5}{978\,h^5} + \frac{3205x^6}{1956h^6} - \frac{37}{489}\frac{x^7}{h^7} \quad (3.48)$$

$$\alpha_3(x) = -\frac{3060x}{163\,h} + \frac{23411x^2}{489\,h^2} - \frac{203630x^3}{4401\,h^3} + \frac{128917x^4}{5868\,h^4} - \frac{47899x^5}{8802\,h^5} + \frac{3941x^6}{5868h^6} - \frac{143}{4401}\frac{x^7}{h^7} \quad (3.49)$$

$$\alpha_4(x) = \frac{19305x}{4564h} - \frac{25413x^2}{2282h^2} + \frac{204977x^3}{18256h^3} - \frac{307183x^4}{54768h^4} + \frac{80453x^5}{54768h^5} - \frac{10481x^6}{54768h^6} + \frac{19}{1956}\frac{x^7}{h^7} \quad (3.50)$$

$$\alpha_5(x) = -\frac{6102x}{8965h} + \frac{2973x^2}{1630h^2} - \frac{16892x^3}{8965\,h^3} + \frac{41797x^4}{43032h^4} - \frac{7114}{26895}\frac{x^5}{h^5} + \frac{7759}{215160}\frac{x^6}{h^6} - \frac{103}{53790}\frac{x^7}{h^7} \quad (3.51)$$

$$\alpha_{\frac{9}{4}}(x) = \frac{41156608x}{564795\,h} - \frac{136904704x^2}{770175\,h^2} + \frac{1377501184x^3}{8471925\,h^3} - \frac{123584512x^4}{1694385\,h^4} + \frac{5816320x^5}{338877\,h^5} - \frac{17186816x^6}{8471925\,h^6} + \frac{114688}{1210275}\frac{x^7}{h^7} \quad (3.52)$$

$$\beta_6(x) = \frac{6}{163}x - \frac{491}{4890}\frac{x^2}{h} + \frac{3121}{29340}\frac{x^3}{h^2} - \frac{37}{652}\frac{x^4}{h^3} + \frac{95}{5868}\frac{x^5}{h^4} - \frac{23}{9780}\frac{x^6}{h^5} + \frac{1}{7335}\frac{x^7}{h^6} \quad (3.53)$$

Substituting (3.46) – (3.53) into (3.33) gives the continuous form of the six step implicit method expressed as

$$
\begin{aligned}
\bar{y}(x) = {} & \left(1-\frac{84623}{29340}\frac{x}{h}+\frac{291923}{88020}\frac{x^2}{h^2}-\frac{233987}{117360}\frac{x^3}{h^3}+\frac{47953}{70416}\frac{x^4}{h^4}-\frac{5191}{39120}\frac{x^5}{h^5}+\frac{4823}{352080}\frac{x^6}{h^6}-\frac{17}{29340}\frac{x^7}{h^7}\right)y_n \\
& +\left(\frac{8694}{815}\frac{x}{h}-\frac{163803}{8150}\frac{x^2}{h^2}+\frac{62254}{4075}\frac{x^3}{h^3}-\frac{117113}{19560}\frac{x^4}{h^4}+\frac{1247}{978}\frac{x^5}{h^5}-\frac{13679}{97800}\frac{x^6}{h^6}+\frac{151}{24450}\frac{x^7}{h^7}\right)y_{n+1} \\
& +\left(-\frac{10665}{164}\frac{x}{h}+\frac{101697}{652}\frac{x^2}{h^2}-\frac{45299}{326}\frac{x^3}{h^3}+\frac{29786}{489}\frac{x^4}{h^4}-\frac{13759}{978}\frac{x^5}{h^5}+\frac{3205}{1956}\frac{x^6}{h^6}-\frac{37}{489}\frac{x^7}{h^7}\right)y_{n+2} \\
& +\left(-\frac{3060}{163}\frac{x}{h}+\frac{23411}{489}\frac{x^2}{h^2}-\frac{203630}{4401}\frac{x^3}{h^3}+\frac{128917}{5868}\frac{x^4}{h^4}-\frac{47899}{8802}\frac{x^5}{h^5}+\frac{3941}{5868}\frac{x^6}{h^6}-\frac{143}{4401}\frac{x^7}{h^7}\right)y_{n+3} \\
& +\left(\frac{19305}{4564}\frac{x}{h}-\frac{25413}{2282}\frac{x^2}{h^2}+\frac{204977}{18256}\frac{x^3}{h^3}-\frac{307183}{54768}\frac{x^4}{h^4}+\frac{80453}{54768}\frac{x^5}{h^5}-\frac{10481}{54768}\frac{x^6}{h^6}+\frac{19}{1956}\frac{x^7}{h^7}\right)y_{n+4} \\
& +\left(-\frac{6102}{8965}\frac{x}{h}+\frac{2973}{1630}\frac{x^2}{h^2}-\frac{16892}{8965}\frac{x^3}{h^3}+\frac{41797}{43032}\frac{x^4}{h^4}-\frac{7114}{26895}\frac{x^5}{h^5}+\frac{7759}{215160}\frac{x^6}{h^6}-\frac{103}{53790}\frac{x^7}{h^7}\right)y_{n+5} \\
& +\left(\frac{41156608}{564795}\frac{x}{h}-\frac{136904704}{770175}\frac{x^2}{h^2}+\frac{1377501184}{8471925}\frac{x^3}{h^3}-\frac{123584512}{1694385}\frac{x^4}{h^4}+\frac{5816320}{338877}\frac{x^5}{h^5}-\frac{17186816}{8471925}\frac{x^6}{h^6}+\frac{114688}{1210275}\frac{x^7}{h^7}\right)y_{n+\frac{9}{4}} \\
& +\left(\frac{6}{163}x-\frac{491}{4890}\frac{x^2}{h}+\frac{3121}{29340}\frac{x^3}{h^2}-\frac{37}{652}\frac{x^4}{h^3}+\frac{95}{5868}\frac{x^5}{h^4}-\frac{23}{9780}\frac{x^6}{h^5}+\frac{1}{7335}\frac{x^7}{h^6}\right)f_{n+6}
\end{aligned}
\qquad (3.54)
$$

Evaluating (3.54) at $x = x_{n+6}$ yields the hybrid six step implicit method

$$y_{n+6} = \frac{50}{489}y_n - \frac{216}{163}y_{n+1} + \frac{3375}{163}y_{n+2} + \frac{2000}{163}y_{n+3} - \frac{6750}{1141}y_{n+4} + \frac{5400}{1793}y_{n+5} - \frac{1048576}{37653}y_{n+\frac{9}{4}} + \frac{60}{163}hf_{n+6} \quad (3.55)$$

Taking the first derivative of equation (3.54), thereafter, evaluating the resulting continuous polynomial solution at

$$x = x_{n+1}, x = x_{n+2}, x = x_{n+\frac{9}{4}}, x = x_{n+3}, x = x_{n+4} \quad and \quad x = x_{n+5}$$

gives six additional methods generated as follows

$$y_{n+1} = -\frac{2750}{60693}y_n + \frac{120000}{20231}y_{n+2} - \frac{27787264}{4673361}y_{n+\frac{9}{4}} + \frac{77500}{60693}y_{n+3} - \frac{36250}{141617}y_{n+4} + \frac{8625}{222541}y_{n+5} - \frac{9780}{20231}hf_{n+1} - \frac{40}{20231}hf_{n+6} \quad (3.56)$$

$$y_{n+2} = \frac{14}{16725}y_n - \frac{498}{27875}y_{n+1} + \frac{2359296}{2146375}y_{n+\frac{9}{4}} - \frac{316}{3345}y_{n+3} + \frac{111}{7805}y_{n+4} + \frac{118}{61325}y_{n+5} - \frac{489}{2230}hf_{n+2} + \frac{1}{11150}hf_{n+6} \quad (3.57)$$

$$y_{n+\frac{9}{4}} = \frac{25050025}{25313738752}y_n - \frac{240604749}{12656869376}y_{n+1} + \frac{7383828375}{6328434688}y_{n+2} - \frac{1060549875}{6328434688}y_{n+3} + \frac{547630875}{25313738752}y_{n+4} - \frac{2083725}{744521728}y_{n+5} + \frac{564795}{1545028}hf_{n+\frac{9}{4}} + \frac{800415}{6328434688}hf_{n+6} \quad (3.58)$$

$$y_{n+3} = -\frac{173}{40320}y_n + \frac{1539}{22400}y_{n+1} - \frac{27}{16}y_{n+2} + \frac{342016}{121275}y_{n+\frac{9}{4}} - \frac{1377}{6272}y_{n+4} + \frac{1107}{49280}y_{n+5} + \frac{163}{224}hf_{n+3} - \frac{1}{1120}hf_{n+6} \quad (3.59)$$

$$y_{n+4}=\frac{2989}{256065}y_n-\frac{70952}{426775}y_{n+1}+\frac{52332}{17071}y_{n+2}-\frac{61865984}{14083575}y_{n+\frac{9}{4}}+\frac{135632}{51213}y_{n+3}-\frac{156408}{938905}y_{n+5}$$
$$+\frac{13692}{17071}hf_{n+4}+\frac{392}{85355}hf_{n+6} \tag{3.60}$$

$$y_{n+5}=-\frac{17182}{650343}y_n+\frac{76593}{216781}y_{n+1}-\frac{1258400}{216781}y_{n+2}+\frac{12058624}{1517467}y_{n+\frac{9}{4}}-\frac{2456300}{650343}y_{n+3}+\frac{3502950}{1517467}y_{n+4}$$
$$-\frac{4840}{216781}hf_{n+6} \tag{3.61}$$

3.4 Six Step Extended Hybrid Backward Differentiation Method (6SEHBDM) with One Off Step Point at Interpolation.

To derive this method, (3.1) is used to obtain a continuous six step extended hybrid backward differentiation method with the following specification:

$$r=7,\quad \left(y_n, y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+\frac{1}{3}}\right),\quad s=2,\quad \left(f_{n+6}, f_{n+7}\right),\quad k=6$$

Expressed as follows:

$$y(x)=\alpha_0 y_n+\alpha_1 y_{n+1}+\alpha_2 y_{n+2}+\alpha_3 y_{n+3}+\alpha_4 y_{n+4}+\alpha_5 y_{n+5}+\alpha_{n+\frac{1}{3}} y_{n+\frac{1}{3}}$$
$$+\beta_1 hf_{n+6}+\beta_2 hf_{n+7} \tag{3.62}$$

The D matrix of the proposed method is expressed as

$$D=\begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} & D_{17} & D_{18} & D_{19} \\ D_{21} & D_{22\backslash} & D_{23} & D_{24} & D_{25} & D_{26} & D_{27} & D_{28} & D_{29} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} & D_{37} & D_{38} & D_{39} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} & D_{47} & D_{48} & D_{49} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} & D_{57} & D_{58} & D_{59} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} & D_{67} & D_{68} & D_{69} \\ D_{71} & D_{72} & D_{73} & D_{74} & D_{75} & D_{76} & D_{77} & D_{78} & D_{79} \\ D_{81} & D_{82} & D_{83} & D_{84} & D_{85} & D_{86} & D_{87} & D_{88} & D_{89} \\ D_{91} & D_{92} & D_{93} & D_{94} & D_{95} & D_{96} & D_{97} & D_{98} & D_{99} \end{pmatrix} \tag{3.63}$$

Where the elements of D's are expressed in appendix C.

The Column matrix of the proposed method is given as:

$$B=\begin{pmatrix} y_n \\ y_{n+\frac{1}{3}} \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ hf_{n+6} \\ hf_{n+7} \end{pmatrix} \tag{3.64}$$

Multiplying the inverse of the matrix in (3.63) with the column matrix (3.64) the values of the coefficients in (3.62) as follows:

$$\alpha_0(x)=1-\frac{17638}{3165}\frac{x}{h}+\frac{120200021}{11697840}\frac{x^2}{h^2}-\frac{62955197}{7018704}\frac{x^3}{h^3}+\frac{199606373}{46791360}\frac{x^4}{h^4}-\frac{40957951}{35093520}\frac{x^5}{h^5}$$

$$+\frac{4281743}{23395680}\frac{x^6}{h^6}-\frac{13381}{877338}\frac{x^7}{h^7}+-\frac{24457}{46791360}\frac{x^8}{h^8} \tag{3.65}$$

$$\alpha_{\frac{1}{3}}(x)=\frac{81507303}{10398080}\frac{x}{h}-\frac{322935956379}{16013043200}\frac{x^2}{h^2}+\frac{10028967453}{500407600}\frac{x^3}{h^3}-\frac{652069991043}{64052172800}\frac{x^4}{h^4}$$

$$+\frac{9268337601}{3202608640}\frac{x^5}{h^5}-\frac{14890432257}{32026086400}\frac{x^6}{h^6}+\frac{630636759}{16013043200}\frac{x^7}{h^7}-\frac{87438447}{64052172800}\frac{x^8}{h^8} \tag{3.66}$$

$$\alpha_1(x)=-\frac{103825}{30384}\frac{x}{h}+\frac{438668555}{28074816}\frac{x^2}{h^2}-\frac{341858299}{1754676}\frac{x^3}{h^3}+\frac{1253240435}{112299264}\frac{x^4}{h^4}-\frac{957655211}{280748166}\frac{x^5}{h^5}$$

$$+\frac{322611133}{561496322}\frac{x^6}{h^6}-\frac{1411387}{28074816}\frac{x^7}{h^7}+\frac{66757}{374330888}\frac{x^8}{h^8} \tag{3.67}$$

$$\alpha_2(x)=\frac{2087}{1055}\frac{x}{h}-\frac{48906173}{4874100}\frac{x^2}{h^2}+\frac{37295351}{2437050}\frac{x^3}{h^3}-\frac{49135279}{4874100}\frac{x^4}{h^4}+\frac{275346}{81235}\frac{x^5}{h^5}-\frac{991689}{1624700}\frac{x^6}{h^6}$$

$$+\frac{136579}{2437050}\frac{x^7}{h^7}-\frac{3347}{1624700}\frac{x^8}{h^8} \qquad (3.68)$$

$$\alpha_3(x)=-\frac{104885}{81024}\frac{x}{h}+\frac{507660823}{74866176}\frac{x^2}{h^2}-\frac{77434435}{7018704}\frac{x^3}{h^3}+\frac{2371375135}{299464704}\frac{x^4}{h^4}-\frac{644765531}{224598528}\frac{x^5}{h^5}$$

$$+\frac{82204981}{149732352}\frac{x^6}{h^6}-\frac{11864969}{224598528}\frac{x^7}{h^7}+\frac{603131}{299464704}\frac{x^8}{h^8} \qquad (3.69)$$

$$\alpha_4(x)=\frac{13085}{20889}\frac{x}{h}-\frac{257298037}{77205744}\frac{x^2}{h^2}+\frac{431936951}{77205744}\frac{x^3}{h^3}-\frac{1298497969}{308822976}\frac{x^4}{h^4}+\frac{124211729}{77205744}\frac{x^5}{h^5}$$

$$-\frac{50006939}{154411488}\frac{x^6}{h^6}+\frac{157163}{4825359}\frac{x^7}{h^7}-\frac{132599}{102940992}\frac{x^8}{h^8} \qquad (3.70)$$

$$\alpha_5(x)=-\frac{18773}{118160}\frac{x}{h}+\frac{93051319}{109179840}\frac{x^2}{h^2}-\frac{1985413}{13647480}\frac{x^3}{h^3}+\frac{489751487}{436719360}\frac{x^4}{h^4}-\frac{16136711}{36393280}\frac{x^5}{h^5}$$

$$+\frac{6740159}{72786560}\frac{x^6}{h^6}-\frac{210587}{21835968}\frac{x^7}{h^7}+\frac{57241}{145573120}\frac{x^8}{h^8} \qquad (3.71)$$

$$\beta_6(x)=\frac{5}{211}x-\frac{25019}{194964}\frac{x^2}{h}+\frac{130597}{584892}\frac{x^3}{h^2}-\frac{138329}{779856}\frac{x^4}{h^3}+\frac{42775}{584892}\frac{x^5}{h^4}-\frac{6287}{3899288}\frac{x^6}{h^5}+$$

$$\frac{262}{146223}\frac{x^7}{h^6}-\frac{61}{779856}\frac{x^8}{h^7} \qquad (3.72)$$

$$\beta_7(x)=-\frac{37}{10128}x+\frac{929923}{46791360}\frac{x^2}{h}-\frac{51031}{1462230}\frac{x^3}{h^2}+\frac{5265451}{187165440}\frac{x^4}{h^3}-\frac{110857}{9358272}\frac{x^5}{h^4}+$$

$$\frac{251609}{93582720}\frac{x^6}{h^5}-\frac{14543}{46791360}\frac{x^7}{h^6}+\frac{893}{62388480}\frac{x^8}{h^7} \qquad (3.73)$$

Substituting (3.65) – (3.73) into (3.62) gives the continuous form of the six step extended hybrid method expressed as

$$
\left.
\begin{aligned}
\bar{y}(x) = & \begin{pmatrix} 1-\frac{17638}{3165}\frac{x}{h}+\frac{120200021}{11697840}\frac{x^2}{h^2}-\frac{62955197}{7018704}\frac{x^3}{h^3}+\frac{199606373}{46791360}\frac{x^4}{h^4}-\frac{40957951}{35093520}\frac{x^5}{h^5} \\ +\frac{4281743}{23395680}\frac{x^6}{h^6}-\frac{13381}{877338}\frac{x^7}{h^7}+-\frac{24457}{46791360}\frac{x^8}{h^8} \end{pmatrix} y_n \\
& +\begin{pmatrix} -\frac{103825}{30384}\frac{x}{h}+\frac{438668555}{28074816}\frac{x^2}{h^2}-\frac{34185829}{1754676}\frac{x^3}{h^3}+\frac{1253240435}{112299264}\frac{x^4}{h^4}-\frac{95765521}{28074816}\frac{x^5}{h^5} \\ +\frac{32261113}{56149632}\frac{x^6}{h^6}-\frac{1411387}{28074816}\frac{x^7}{h^7}+\frac{66757}{37433088}\frac{x^8}{h^8} \end{pmatrix} y_{n+1} \\
& +\begin{pmatrix} \frac{81507303}{10398080}\frac{x}{h}-\frac{322935956379}{16013043200}\frac{x^2}{h^2}+\frac{10028967453}{500407600}\frac{x^3}{h^3}-\frac{652069991043}{64052172800}\frac{x^4}{h^4} \\ +\frac{9268337601}{3202608640}\frac{x^5}{h^5}-\frac{14890432257}{32026086400}\frac{x^6}{h^6}+\frac{630636759}{16013043200}\frac{x^7}{h^7}-\frac{87438447}{64052172800}\frac{x^8}{h^8} \end{pmatrix} y_{n+\frac{1}{3}} \\
& +\begin{pmatrix} \frac{2087}{1055}\frac{x}{h}-\frac{48906173}{4874100}\frac{x^2}{h^2}+\frac{37295351}{2437050}\frac{x^3}{h^3}-\frac{49135279}{4874100}\frac{x^4}{h^4}+\frac{275346}{81235}\frac{x^5}{h^5}-\frac{991689}{1624700}\frac{x^6}{h^6} \\ +\frac{136579}{2437050}\frac{x^7}{h^7}-\frac{3347}{1624700}\frac{x^8}{h^8} \end{pmatrix} y_{n+2} \\
& +\begin{pmatrix} -\frac{104885}{81024}\frac{x}{h}+\frac{507660823}{74866176}\frac{x^2}{h^2}-\frac{77434435}{7018704}\frac{x^3}{h^3}+\frac{2371375135}{299464704}\frac{x^4}{h^4}-\frac{644765531}{224598528}\frac{x^5}{h^5} \\ +\frac{82204981}{14973235}\frac{x^6}{h^6}-\frac{11864969}{224598528}\frac{x^7}{h^7}+\frac{603131}{299464704}\frac{x^8}{h^8} \end{pmatrix} y_{n+3} \\
& +\begin{pmatrix} \frac{13085}{20889}\frac{x}{h}-\frac{257298037}{77205744}\frac{x^2}{h^2}+\frac{431936951}{77205744}\frac{x^3}{h^3}-\frac{1298497969}{308822976}\frac{x^4}{h^4}+\frac{124211729}{77205744}\frac{x^5}{h^5} \\ -\frac{50006939}{154411488}\frac{x^6}{h^6}+\frac{157163}{4825359}\frac{x^7}{h^7}-\frac{132599}{102940992}\frac{x^8}{h^8} \end{pmatrix} y_{n+4} \\
& +\begin{pmatrix} -\frac{18773}{118160}\frac{x}{h}+\frac{93051319}{109179840}\frac{x^2}{h^2}-\frac{1985413}{1364748}\frac{x^3}{h^3}+\frac{489751487}{436719360}\frac{x^4}{h^4}-\frac{16136711}{36393280}\frac{x^5}{h^5} \\ +\frac{6740159}{72786560}\frac{x^6}{h^6}-\frac{210587}{21835968}\frac{x^7}{h^7}+\frac{57241}{145573120}\frac{x^8}{h^8} \end{pmatrix} y_{n+5} \\
& +\begin{pmatrix} \frac{5}{211}x-\frac{25019}{194964}\frac{x^2}{h}+\frac{130597}{584892}\frac{x^3}{h^2}-\frac{138329}{779856}\frac{x^4}{h^3}+\frac{42775}{584892}\frac{x^5}{h^4}-\frac{6287}{3899288}\frac{x^6}{h^5}+ \\ \frac{262}{146223}\frac{x^7}{h^6}-\frac{61}{779856}\frac{x^8}{h^7} \end{pmatrix} f_{n+6} \\
& +\begin{pmatrix} -\frac{37}{10128}x+\frac{929923}{46791360}\frac{x^2}{h}-\frac{51031}{1462230}\frac{x^3}{h^2}+\frac{5265451}{187165440}\frac{x^4}{h^3}-\frac{110857}{9358272}\frac{x^5}{h^4}+ \\ \frac{251609}{93582720}\frac{x^6}{h^5}-\frac{14543}{46791360}\frac{x^7}{h^6}+\frac{893}{62388480}\frac{x^8}{h^7} \end{pmatrix} f_{n+7}
\end{aligned}
\right\} \quad (3.74)
$$

Evaluating (3.74) at $x = x_{n+6}$ and $x = x_{n+7}$ yields the hybrid six step extended method

$$y_{n+6} = \frac{4335}{16247} y_n - \frac{29701647}{40032608} y_{n+\frac{1}{3}} + \frac{217039}{194964} y_{n+1} - \frac{26877}{16247} y_{n+2} + \frac{1177675}{519904} y_{n+3} - \frac{1293275}{536151} y_{n+4} + \frac{984045}{454916} y_{n+5} + \frac{8160}{16247} hf_{n+6} - \frac{1445}{64988} hf_{n+7}$$

(3.75)

$$y_{n+7} = -\frac{1675}{6963} y_n + \frac{531441}{816992} y_{n+\frac{1}{3}} - \frac{25375}{27852} y_{n+1} + \frac{2688}{2321} y_{n+2} - \frac{261625}{222816} y_{n+3} + \frac{37625}{76593} y_{n+4} + \frac{9525}{9284} y_{n+5} + \frac{3500}{2321} hf_{n+6} + \frac{2765}{9284} hf_{n+7} \quad (3.76)$$

Taking the first derivative of equation (3.74), thereafter, evaluating the resulting continuous polynomial solution at

$$x = x_{n+\frac{1}{3}}, x = x_{n+1}, x = x_{n+2}, x = x_{n+\frac{9}{4}}, x = x_{n+3}, x = x_{n+4} \quad and \quad x = x_{n+5}$$

gives seven additional methods generated as follows

$$y_{n+\frac{1}{3}} = -\frac{6911212000}{1602058419} y_n + \frac{11809105000}{1602058419} y_{n+1} - \frac{607760384}{178006491} y_{n+2} + \frac{3356381875}{1602058419} y_{n+3} - \frac{1575700000}{1602058419} y_{n+4} + \frac{43703000}{178006491} y_{n+5} - \frac{5199040}{1383681} hf_{n+\frac{1}{3}} - \frac{326480000}{9078331041} hf_{n+6} + \frac{2935240}{534019473} hf_{n+7} \quad (3.77)$$

$$y_{n+1} = -\frac{684476}{183335} y_n + \frac{8263376109}{564671800} y_{n+\frac{1}{3}} - \frac{13549824}{916675} y_{n+2} + \frac{2129591}{293336} y_{n+3} - \frac{1261564}{403337} y_{n+4} + \frac{961137}{1283345} y_{n+5} + \frac{2339568}{183335} hf_{n+1} - \frac{19248}{183335} hf_{n+6} + \frac{2883}{183335} hf_{n+7} \quad (3.78)$$

$$y_{n+2} = -\frac{184475}{204756} y_n + \frac{475167303}{168172928} y_{n+\frac{1}{3}} - \frac{1682125}{273008} y_{n+1} + \frac{46012375}{6552192} y_{n+3} - \frac{1707625}{750772} y_{n+4} + \frac{934475}{1911056} y_{n+5} - \frac{243705}{34126} hf_{n+2} - \frac{2125}{34126} hf_{n+6} + \frac{2445}{273008} hf_{n+7} \quad (3.79)$$

$$y_{n+3}=-\frac{1329632}{3110365}y_n+\frac{1505336157}{1197490525}y_{n+\frac{1}{3}}-\frac{4094504}{1866219}y_{n+1}+\frac{79171584}{15551825}y_{n+2}-\frac{66931616}{20528409}y_{n+4}$$
$$+\frac{11627304}{21772555}y_{n+5}+\frac{3119424}{622073}hf_{n+3}-\frac{35712}{622073}hf_{n+6}+\frac{24152}{3110365}hf_{n+7} \quad (3.80)$$

$$y_{n+4}=\frac{141977}{996335}y_n-\frac{907169787}{2231790400}y_{n+\frac{1}{3}}+\frac{1037531}{1594136}y_{n+1}-\frac{5652306}{4981675}y_{n+2}+\frac{28593653}{12753088}y_{n+3}$$
$$-\frac{27581697}{55794760}y_{n+5}+\frac{292446}{199267}hf_{n+4}+\frac{7194}{199267}hf_{n+6}-\frac{35079}{7970680}hf_{n+7} \quad (3.81)$$

$$y_{n+5}=-\frac{2463412}{18416163}y_n+\frac{202597119}{540207448}y_{n+\frac{1}{3}}-\frac{3525375}{6138721}y_{n+1}+\frac{5485312}{6138721}y_{n+2}-\frac{196497175}{147329304}y_{n+4}$$
$$+\frac{3899280}{6138721}hf_{n+5}-\frac{439600}{6138721}hf_{n+6}+\frac{43155}{6138721}hf_{n+7} \quad (3.82)$$

3.5 Order and Error Constant

Following Skwame *et al*. (2018), Let $y(x_{n+j})$, the solution to $y'(x_{n+j})$ be sufficiently differentiable, then $y(x_{n+j})$ and $y'(x_{n+j})$ can be expanded into Taylor's about point x_n to obtain

$$T_n=\frac{1}{h\sigma(1)}\left[c_0(x_n)+c_1hy'(x_n)+c_2h^2y''(x_n)+\cdots\right]$$

where

$$\left.\begin{aligned} c_0&=\sum_{j=0}^{k}\alpha_j\\ c_1&=\sum_{j=0}^{k}j\alpha_j-\sum_{j=0}^{k}\beta_j\\ &\vdots\\ c_q&=\frac{1}{q!}\sum_{j=0}^{k}j^q\alpha_j-\frac{1}{(q-1)!}\sum_{j=0}^{k}j^{q-1}\beta_j \end{aligned}\right\} \quad (3.83)$$

Definition 3.1: A linear multistep method is said to be of order of accuracy p if

$c_0 = c_1 = \cdots = c_p = 0, \qquad c_{p+1} \neq 0.$ c_{p+1} is called the error constant.

3.5.1 Order and error constant of the proposed five-step hybrid backward differentiation method (5SHBDM).

For the method in (3.25) $\alpha_0 = \frac{4}{771}$, $\alpha_1 = -\frac{75}{1799}$, $\alpha_2 = \frac{40}{257}$,

$$\alpha_3 = -\frac{100}{257}, \alpha_4 = \frac{300}{257} \alpha_5 = 1, \qquad \alpha_{\frac{9}{2}} = -\frac{10240}{5397}, \beta_{\frac{9}{2}} = 0, \beta_5 = -\frac{60}{257}$$

$$
\left.\begin{aligned}
c_0 &= \frac{4}{771} - \frac{75}{1799} + \frac{40}{257} - \frac{100}{257} + \frac{300}{257} + 1 - \frac{10240}{5397} = 0 \\
c_1 &= \left(-\frac{75}{1799}\right) + 2\left(\frac{40}{257}\right) - 3\left(\frac{100}{257}\right) + 4\left(\frac{300}{257}\right) + 5(1) \\
&\quad -\left(\frac{9}{2}\right)\left(\frac{10240}{5397}\right) - \frac{60}{257} = 0 \\
c_2 &= \frac{1}{2!}\left(\begin{aligned}&\left(-\frac{75}{1799}\right) + (2)^2\left(\frac{40}{257}\right) - (3)^2\left(\frac{100}{257}\right) \\ &+(4)^2\left(\frac{300}{257}\right) + (5)^2(1) - \left(\frac{9}{2}\right)^2\left(\frac{10240}{5397}\right)\end{aligned}\right) - 5\left(\frac{60}{257}\right) = 0 \\
c_3 &= \frac{1}{3!}\left[\begin{aligned}&\left(-\frac{75}{1799}\right) + 2^3\left(\frac{40}{257}\right) - 3^3\left(\frac{100}{257}\right) \\ &+4^3\left(\frac{300}{257}\right) + 5^3(1) - \left(\frac{9}{2}\right)^3\left(\frac{10240}{5397}\right)\end{aligned}\right] - \frac{1}{2!}\left(5^2\left(\frac{60}{257}\right)\right) = 0 \\
c_4 &= \frac{1}{4!}\left[\begin{aligned}&\left(-\frac{75}{1799}\right) + 2^4\left(\frac{40}{257}\right) - 3^4\left(\frac{100}{257}\right) \\ &+4^4\left(\frac{300}{257}\right) + 5^4(1) - \left(\frac{9}{2}\right)^4\left(\frac{10240}{5397}\right)\end{aligned}\right] - \frac{1}{3!}\left(5^3\left(\frac{60}{257}\right)\right) = 0 \\
c_5 &= \frac{1}{5!}\left[\begin{aligned}&\left(-\frac{75}{1799}\right) + 2^5\left(\frac{40}{257}\right) - 3^5\left(\frac{100}{257}\right) \\ &+4^5\left(\frac{300}{257}\right) + 5^5(1) - \left(\frac{9}{2}\right)^5\left(\frac{10240}{5397}\right)\end{aligned}\right] - \frac{1}{4!}\left(5^4\left(\frac{60}{257}\right)\right) = 0 \\
c_6 &= \frac{1}{6!}\left[\begin{aligned}&\left(-\frac{75}{1799}\right) + 2^6\left(\frac{40}{257}\right) - 3^6\left(\frac{100}{257}\right) \\ &+4^6\left(\frac{300}{257}\right) + 5^6(1) - \left(\frac{9}{2}\right)^6\left(\frac{10240}{5397}\right)\end{aligned}\right] - \frac{1}{5!}\left(5^5\left(\frac{60}{257}\right)\right) = 0 \\
c_7 &= \frac{1}{7!}\left[\begin{aligned}&\left(-\frac{75}{1799}\right) + 2^7\left(\frac{40}{257}\right) - 3^7\left(\frac{100}{257}\right) \\ &+4^7\left(\frac{300}{257}\right) + 5^7(1) - \left(\frac{9}{2}\right)^7\left(\frac{10240}{5397}\right)\end{aligned}\right] - \frac{1}{6!}\left(5^6\left(\frac{60}{257}\right)\right) = -\frac{5}{1799}
\end{aligned}\right\} \quad (3.84)
$$

Hence the method is of order $p = 6$ with error constant $c_{p+1} = -\frac{5}{1799}$

For the method in (3.26) $\alpha_0 = \frac{1274}{10965}$, $\alpha_1 = 1, \alpha_2 = \frac{37044}{18275}$, $\alpha_3 = \frac{5929}{3655}$, $\alpha_4 = -\frac{5194}{3655}$,

$\alpha_5 = 0$, $\alpha_{\frac{9}{2}} = \frac{38912}{54825}$, $\beta_1 = -\frac{5397}{7310}$, $\beta_5 = \frac{441}{7310}$

$$
\left.
\begin{array}{l}
c_0 = \frac{1274}{10965} + 1 - \frac{37044}{18275} + \frac{5929}{3655} - \frac{5194}{3655} + \frac{38912}{54825} = 0 \\[2ex]
c_1 = 1 + 2\left(-\frac{37044}{18275}\right) + 3\left(\frac{5929}{3655}\right) + 4\left(-\frac{5194}{3655}\right) + \frac{9}{2}\left(\frac{38912}{54825}\right) + 5(0)) \\
\quad -\left(-\frac{5397}{7310} + \frac{441}{7310}\right) = 0 \\[2ex]
c_2 = \frac{1}{2!}\left[\begin{array}{l} 1 - 2^2\left(\frac{37044}{18275}\right) + 3^2\left(\frac{5929}{3655}\right) \\ -4^2\left(-\frac{5194}{3655}\right) - \left(\frac{9}{2}\right)^2\left(\frac{38912}{54825}\right) - 5^2(0) \end{array}\right] - \left(-\frac{5397}{7310} - 5^1\left(\frac{441}{7310}\right)\right) = 0 \\[2ex]
c_3 = \frac{1}{3!}\left[\begin{array}{l} 1 - 2^3\left(\frac{37044}{18275}\right) + 3^3\left(\frac{5929}{3655}\right) \\ -4^3\left(-\frac{5194}{3655}\right) - \left(\frac{9}{2}\right)^3\left(\frac{38912}{54825}\right) - 5^3(0) \end{array}\right] - \frac{1}{2!}\left(-\frac{5397}{7310} - 5^2\left(\frac{441}{7310}\right)\right) = 0 \\[2ex]
c_4 = \frac{1}{4!}\left[\begin{array}{l} 1 - 2^4\left(\frac{37044}{18275}\right) + 3^4\left(\frac{5929}{3655}\right) \\ -4^4\left(-\frac{5194}{3655}\right) - \left(\frac{9}{2}\right)^4\left(\frac{38912}{54825}\right) - 5^4(0) \end{array}\right] - \frac{1}{3!}\left(-\frac{5397}{7310} - 5^3\left(\frac{441}{7310}\right)\right) = 0 \\[2ex]
c_5 = \frac{1}{5!}\left[\begin{array}{l} 1 - 2^5\left(\frac{37044}{18275}\right) + 3^5\left(\frac{5929}{3655}\right) \\ -4^5\left(-\frac{5194}{3655}\right) - \left(\frac{9}{2}\right)^5\left(\frac{38912}{54825}\right) - 5^5(0) \end{array}\right] - \frac{1}{4!}\left(-\frac{5397}{7310} - 5^4\left(\frac{441}{7310}\right)\right) = 0 \\[2ex]
c_6 = \frac{1}{6!}\left[\begin{array}{l} 1 - 2^6\left(\frac{37044}{18275}\right) + 3^6\left(\frac{5929}{3655}\right) \\ -4^6\left(-\frac{5194}{3655}\right) - \left(\frac{9}{2}\right)^6\left(\frac{38912}{54825}\right) - 5^6(0) \end{array}\right] - \frac{1}{5!}\left(-\frac{5397}{7310} - 5^5\left(\frac{441}{7310}\right)\right) = 0 \\[2ex]
c_7 = \frac{1}{7!}\left[\begin{array}{l} 1 - 2^7\left(\frac{37044}{18275}\right) + 3^7\left(\frac{5929}{3655}\right) \\ -4^7\left(-\frac{5194}{3655}\right) - \left(\frac{9}{2}\right)^7\left(\frac{38912}{54825}\right) - 5^7(0) \end{array}\right] - \frac{1}{6!}\left(-\frac{5397}{7310} - 5^6\left(\frac{441}{7310}\right)\right) = -\frac{14}{1075}
\end{array}
\right\}
\quad (3.85)
$$

Hence the method is of order $p = 6$ with error constant $c_{p+1} = -\frac{14}{1075}$

For the method in (3.27) $\alpha_0 = -\frac{1325}{32724}$ $\alpha_1 = \frac{3275}{6363}$ $\alpha_2 = 1$

$\alpha_3 = -\frac{6175}{2727}$ $\alpha_4 = \frac{5425}{3636}$, $\alpha_5 = 0$, $\alpha_{\frac{9}{2}} = -\frac{40192}{57267}$, $\beta_2 = -\frac{1285}{909}$,

$\beta_5 = -\frac{50}{909}$

$$\left.\begin{aligned}
c_0 &= -\frac{1325}{32724} + \frac{3275}{6363} + 1 - \frac{6175}{2727} + \frac{5425}{3636} - \frac{40192}{57267} = 0 \\
c_1 &= \begin{bmatrix} \frac{3275}{6363} + 2(1) - 3\left(\frac{6175}{2727}\right) + 4\left(\frac{5425}{3636}\right) \\ -\left(\frac{9}{2}\right)\left(\frac{40192}{57267}\right) - \left(-\frac{1285}{909} - \frac{50}{909}\right) \end{bmatrix} = 0 \\
c_2 &= \frac{1}{2!}\begin{bmatrix} \frac{3275}{6363} + 2^2(1) - 3^2\left(\frac{6175}{2727}\right) \\ +4^2\left(\frac{5425}{3636}\right) - \left(\frac{9}{2}\right)^2\left(\frac{40192}{57267}\right) \end{bmatrix} - \left(-2\left(\frac{1285}{909}\right) - 5\left(\frac{50}{909}\right)\right) = 0 \\
c_3 &= \frac{1}{3!}\begin{bmatrix} \frac{3275}{6363} + 2^3(1) - 3^3\left(\frac{6175}{2727}\right) \\ +4^3\left(\frac{5425}{3636}\right) - \left(\frac{9}{2}\right)^3\left(\frac{40192}{57267}\right) \end{bmatrix} - \frac{1}{2!}\left(-2^2\left(\frac{1285}{909}\right) - 5^2\left(\frac{50}{909}\right)\right) = 0 \\
c_4 &= \frac{1}{4!}\begin{bmatrix} \frac{3275}{6363} + 2^4(1) - 3^4\left(\frac{6175}{2727}\right) \\ +4^4\left(\frac{5425}{3636}\right) - \left(\frac{9}{2}\right)^4\left(\frac{40192}{57267}\right) \end{bmatrix} - \frac{1}{3!}\left(-2^3\left(\frac{1285}{909}\right) - 5^3\left(\frac{50}{909}\right)\right) = 0 \\
c_5 &= \frac{1}{5!}\begin{bmatrix} \frac{3275}{6363} + 2^5(1) - 3^5\left(\frac{6175}{2727}\right) \\ +4^5\left(\frac{5425}{3636}\right) - \left(\frac{9}{2}\right)^5\left(\frac{40192}{57267}\right) \end{bmatrix} - \frac{1}{4!}\left(-2^4\left(\frac{1285}{909}\right) - 5^4\left(\frac{50}{909}\right)\right) = 0 \\
c_6 &= \frac{1}{6!}\begin{bmatrix} \frac{3275}{6363} + 2^6(1) - 3^6\left(\frac{6175}{2727}\right) \\ +4^6\left(\frac{5425}{3636}\right) - \left(\frac{9}{2}\right)^6\left(\frac{40192}{57267}\right) \end{bmatrix} - \frac{1}{5!}\left(-2^5\left(\frac{1285}{909}\right) - 5^5\left(\frac{50}{909}\right)\right) = 0 \\
c_7 &= \frac{1}{7!}\begin{bmatrix} \frac{3275}{6363} + 2^7(1) - 3^7\left(\frac{6175}{2727}\right) \\ +4^7\left(\frac{5425}{3636}\right) - \left(\frac{9}{2}\right)^7\left(\frac{40192}{57267}\right) \end{bmatrix} - \frac{1}{6!}\left(-2^6\left(\frac{1285}{909}\right) - 5^6\left(\frac{50}{909}\right)\right) = \frac{1385}{152712}
\end{aligned}\right\} \quad (3.86)$$

Hence the method is of order $p = 6$ with error constant $c_{p+1} = \frac{1385}{152712}$

For the method in (3.28) $\alpha_0 = \frac{55}{1272}$ $\alpha_1 = -\frac{153}{371}$ $\alpha_2 = \frac{2403}{1060}$ $\alpha_3 = 1$ $\alpha_4 = -\frac{2043}{424}$

$$\alpha_5 = 0 \ \alpha_{\frac{9}{2}} = \frac{10688}{5565} \ \beta_3 = -\frac{771}{212} \qquad \beta_5 = -\frac{27}{212}$$

$$\left.\begin{aligned}
c_0 &= \frac{55}{1272} - \frac{153}{371} + \frac{2403}{1060} + 1 - \frac{2403}{424} + \frac{10688}{5565} = 0 \\
c_1 &= \begin{bmatrix} -\frac{153}{371} + 2\left(\frac{2403}{1060}\right) + 3(1) \\ -4\left(\frac{2403}{424}\right) + \left(\frac{9}{2}\right)\left(\frac{10688}{5565}\right) \end{bmatrix} - \left(-\frac{771}{212} + \frac{27}{212}\right) = 0 \\
c_2 &= \frac{1}{2!}\begin{bmatrix} -\frac{153}{371} + 2^2\left(\frac{2403}{1060}\right) + 3^2(1) \\ -4^2\left(\frac{2403}{424}\right) + \left(\frac{9}{2}\right)^2\left(\frac{10688}{5565}\right) \end{bmatrix} - \left(-3\left(\frac{771}{212}\right) + 5\left(\frac{27}{212}\right)\right) = 0 \\
c_3 &= \frac{1}{3!}\begin{bmatrix} -\frac{153}{371} + 2^3\left(\frac{2403}{1060}\right) + 3^3(1) \\ -4^3\left(\frac{2403}{424}\right) + \left(\frac{9}{2}\right)^3\left(\frac{10688}{5565}\right) \end{bmatrix} - \frac{1}{2!}\left(-3^2\left(\frac{771}{212}\right) + 5^2\left(\frac{27}{212}\right)\right) = 0 \\
c_4 &= \frac{1}{4!}\begin{bmatrix} -\frac{153}{371} + 2^4\left(\frac{2403}{1060}\right) + 3^4(1) \\ -4^4\left(\frac{2403}{424}\right) + \left(\frac{9}{2}\right)^4\left(\frac{10688}{5565}\right) \end{bmatrix} - \frac{1}{3!}\left(-3^3\left(\frac{771}{212}\right) + 5^3\left(\frac{27}{212}\right)\right) = 0 \\
c_5 &= \frac{1}{5!}\begin{bmatrix} -\frac{153}{371} + 2^5\left(\frac{2403}{1060}\right) + 3^5(1) \\ -4^5\left(\frac{2403}{424}\right) + \left(\frac{9}{2}\right)^5\left(\frac{10688}{5565}\right) \end{bmatrix} - \frac{1}{4!}\left(-3^4\left(\frac{771}{212}\right) + 5^4\left(\frac{27}{212}\right)\right) = 0 \\
c_6 &= \frac{1}{6!}\begin{bmatrix} -\frac{153}{371} + 2^6\left(\frac{2403}{1060}\right) + 3^6(1) \\ -4^6\left(\frac{2403}{424}\right) + \left(\frac{9}{2}\right)^6\left(\frac{10688}{5565}\right) \end{bmatrix} - \frac{1}{5!}\left(-3^5\left(\frac{771}{212}\right) + 5^5\left(\frac{27}{212}\right)\right) = 0 \\
c_7 &= \frac{1}{7!}\begin{bmatrix} -\frac{153}{371} + 2^7\left(\frac{2403}{1060}\right) + 3^7(1) \\ -4^7\left(\frac{2403}{424}\right) + \left(\frac{9}{2}\right)^7\left(\frac{10688}{5565}\right) \end{bmatrix} - \frac{1}{6!}\left(-3^6\left(\frac{771}{212}\right) + 5^6\left(\frac{27}{212}\right)\right) = -\frac{123}{8480}
\end{aligned}\right\} \quad (3.87)$$

Hence the method is of order $p=6$ with error constant $c_{p+1}=-\frac{123}{8480}$

For the method in (3.29) $\alpha_0=-\frac{61}{6321}$ $\alpha_1=\frac{1208}{14749}$

$$\alpha_2=-\frac{3564}{10535}\ \alpha_3=\frac{328}{301},\alpha_4=1\ \alpha_5=0\ \alpha_{\frac{9}{2}}=-\frac{403456}{221235}\ \beta_4=-\frac{3084}{2107}\ \beta_5=-\frac{144}{2107}$$

$$\left.\begin{aligned}
c_0&=-\frac{61}{6321}+\frac{1208}{14749}-\frac{3564}{10535}+\frac{328}{301}+1-\frac{403456}{221235}=0\\
c_1&=\begin{bmatrix}\frac{1208}{14749}-2\left(\frac{3564}{10535}\right)+3\left(\frac{328}{301}\right)\\+4(1)-\left(\frac{9}{2}\right)\left(\frac{403456}{221235}\right)\end{bmatrix}-\left(-\frac{3084}{2107}-\frac{144}{2107}\right)=0\\
c_2&=\frac{1}{2!}\begin{bmatrix}\frac{1208}{14749}-2^2\left(\frac{3564}{10535}\right)+3^2\left(\frac{328}{301}\right)\\+4^2(1)-\left(\frac{9}{2}\right)^2\left(\frac{403456}{221235}\right)\end{bmatrix}-\left(-4\left(\frac{3084}{2107}\right)-5\left(\frac{144}{2107}\right)\right)=0\\
c_3&=\frac{1}{3!}\begin{bmatrix}\frac{1208}{14749}-2^3\left(\frac{3564}{10535}\right)+3^3\left(\frac{328}{301}\right)\\+4^3(1)-\left(\frac{9}{2}\right)^3\left(\frac{403456}{221235}\right)\end{bmatrix}-\frac{1}{2!}\left(-4^2\left(\frac{3084}{2107}\right)-5^2\left(\frac{144}{2107}\right)\right)=0\\
c_4&=\frac{1}{4!}\begin{bmatrix}\frac{1208}{14749}-2^4\left(\frac{3564}{10535}\right)+3^4\left(\frac{328}{301}\right)\\+4^4(1)-\left(\frac{9}{2}\right)^4\left(\frac{403456}{221235}\right)\end{bmatrix}-\frac{1}{3!}\left(-4^3\left(\frac{3084}{2107}\right)-5^3\left(\frac{144}{2107}\right)\right)=0\\
c_5&=\frac{1}{5!}\begin{bmatrix}\frac{1208}{14749}-2^5\left(\frac{3564}{10535}\right)+3^5\left(\frac{328}{301}\right)\\+4^5(1)-\left(\frac{9}{2}\right)^5\left(\frac{403456}{221235}\right)\end{bmatrix}-\frac{1}{4!}\left(-4^4\left(\frac{3084}{2107}\right)-5^4\left(\frac{144}{2107}\right)\right)=0\\
c_6&=\frac{1}{6!}\begin{bmatrix}\frac{1208}{14749}-2^6\left(\frac{3564}{10535}\right)+3^6\left(\frac{328}{301}\right)\\+4^6(1)-\left(\frac{9}{2}\right)^6\left(\frac{403456}{221235}\right)\end{bmatrix}-\frac{1}{5!}\left(-4^5\left(\frac{3084}{2107}\right)-5^5\left(\frac{144}{2107}\right)\right)=0\\
c_7&=\frac{1}{7!}\begin{bmatrix}\frac{1208}{14749}-2^7\left(\frac{3564}{10535}\right)+3^7\left(\frac{328}{301}\right)\\+4^7(1)-\left(\frac{9}{2}\right)^7\left(\frac{403456}{221235}\right)\end{bmatrix}-\frac{1}{6!}\left(-4^6\left(\frac{3084}{2107}\right)-5^6\left(\frac{144}{2107}\right)\right)=\frac{317}{73745}
\end{aligned}\right\}\quad(3.88)$$

Hence the method is of order $p = 6$ with error constant $c_{p+1} = \frac{317}{73745}$

For the method in (3.30)

$$\alpha_0 = -\frac{89425}{25993216}\ \alpha_1 = \frac{366525}{12996608}\ \alpha_2 = -\frac{1416933}{12996608}\ \alpha_3 = \frac{3825675}{12996608}\ \alpha_4 = -\frac{31454325}{25993216}$$

$$\alpha_5 = 0 \; \alpha_{\frac{9}{2}} = 1 \; \beta_{\frac{9}{4}} = \frac{80955}{203072} \; \beta_5 = -\frac{297675}{6498304}$$

$$\left.\begin{aligned}
c_0 &= \left[-\frac{89425}{25993216}+\frac{366525}{12996608}-\frac{1416933}{12996608}+\frac{3825675}{12996608}-\frac{31454325}{25993216}+1\right]=0 \\
c_1 &= \left[\begin{matrix}\frac{366525}{12996608}-2\left(\frac{1416933}{12996608}\right)+3\left(\frac{3825675}{12996608}\right)\\ -4\left(\frac{31454325}{25993216}\right)+\left(\frac{9}{2}\right)\end{matrix}\right]-\left(\frac{80955}{203072}-\frac{297675}{6498304}\right)=0 \\
c_2 &= \frac{1}{2!}\left[\begin{matrix}\frac{366525}{12996608}-2^2\left(\frac{1416933}{12996608}\right)+3^2\left(\frac{3825675}{12996608}\right)\\ -4^2\left(\frac{31454325}{25993216}\right)+\left(\frac{9}{2}\right)^2\end{matrix}\right]-\left(\frac{9}{2}\left(\frac{80955}{203072}\right)-5\left(\frac{297675}{6498304}\right)\right)=0 \\
c_3 &= \frac{1}{3!}\left[\begin{matrix}\frac{366525}{12996608}-2^3\left(\frac{1416933}{12996608}\right)+3^3\left(\frac{3825675}{12996608}\right)\\ -4^3\left(\frac{31454325}{25993216}\right)+\left(\frac{9}{2}\right)^3\end{matrix}\right]-\frac{1}{2!}\left(\left(\frac{9}{2}\right)^2\left(\frac{80955}{203072}\right)-5^2\left(\frac{297675}{6498304}\right)\right)=0 \\
c_4 &= \frac{1}{4!}\left[\begin{matrix}\frac{366525}{12996608}-2^4\left(\frac{1416933}{12996608}\right)+3^4\left(\frac{3825675}{12996608}\right)\\ -4^4\left(\frac{31454325}{25993216}\right)+\left(\frac{9}{2}\right)^4\end{matrix}\right]-\frac{1}{3!}\left(\left(\frac{9}{2}\right)^3\left(\frac{80955}{203072}\right)-5^3\left(\frac{297675}{6498304}\right)\right)=0 \\
c_4 &= \frac{1}{4!}\left[\begin{matrix}\frac{366525}{12996608}-2^4\left(\frac{1416933}{12996608}\right)+3^4\left(\frac{3825675}{12996608}\right)\\ -4^4\left(\frac{31454325}{25993216}\right)+\left(\frac{9}{2}\right)^4\end{matrix}\right]-\frac{1}{3!}\left(\left(\frac{9}{2}\right)^3\left(\frac{80955}{203072}\right)-5^3\left(\frac{297675}{6498304}\right)\right)=0 \\
c_5 &= \frac{1}{5!}\left[\begin{matrix}\frac{366525}{12996608}-2^5\left(\frac{1416933}{12996608}\right)+3^5\left(\frac{3825675}{12996608}\right)\\ -4^5\left(\frac{31454325}{25993216}\right)+\left(\frac{9}{2}\right)^5\end{matrix}\right]-\frac{1}{4!}\left(\left(\frac{9}{2}\right)^4\left(\frac{80955}{203072}\right)-5^4\left(\frac{297675}{6498304}\right)\right)=0 \\
c_6 &= \frac{1}{6!}\left[\begin{matrix}\frac{366525}{12996608}-2^6\left(\frac{1416933}{12996608}\right)+3^6\left(\frac{3825675}{12996608}\right)\\ -4^6\left(\frac{31454325}{25993216}\right)+\left(\frac{9}{2}\right)^6\end{matrix}\right]-\frac{1}{5!}\left(\left(\frac{9}{2}\right)^5\left(\frac{80955}{203072}\right)-5^5\left(\frac{297675}{6498304}\right)\right)=0 \\
c_7 &= \frac{1}{7!}\left[\begin{matrix}\frac{366525}{12996608}-2^7\left(\frac{1416933}{12996608}\right)+3^7\left(\frac{3825675}{12996608}\right)\\ -4^7\left(\frac{31454325}{25993216}\right)+\left(\frac{9}{2}\right)^7\end{matrix}\right]-\frac{1}{6!}\left(\left(\frac{9}{2}\right)^6\left(\frac{80955}{203072}\right)-5^6\left(\frac{297675}{6498304}\right)\right)=\frac{356265}{207945728}
\end{aligned}\right\} \qquad (3.89)$$

Hence the method is of order $p = 6$ with error constant $c_{p+1} = \frac{356265}{207945728}$

The same method was used for the 6SHBDM and 6SEHBDM and the summary of the order and error constants is presented below in Tables 3.1, 3.2 and 3.3

Table 3.1: Order and Error Constants for the Proposed 5SHBDM

Equation	Order P	Error Constant C_{p+1}
3.25	6	$-\frac{5}{1799}$
3.26	6	$-\frac{14}{1075}$
3.27	6	$\frac{1385}{152712}$
3.28	6	$-\frac{123}{8480}$
3.29	6	$\frac{317}{73745}$
3.30	6	$\frac{356265}{207945728}$

Table 3.2: Order and Error Constants for the Proposed 6SHBDM

Equation	Order P	Error Constant C_{p+1}
3.55	7	$-\frac{225}{9128}$
3.56	7	$\frac{625}{323696}$
3.57	7	$-\frac{89}{1248800}$
3.58	7	$-\frac{307016325}{3240158560256}$
3.59	7	$\frac{549}{1003520}$
3.60	7	$-\frac{1351}{682840}$
3.61	7	$\frac{134915}{24279472}$

Table 3.3: Order and Error Constants for the Proposed 6SEHBDM

Equation	Order	Error Constant C_{p+1}
3.75	8	$-\frac{189295}{24565464}$
3.77	8	$-\frac{325451500}{43255577313}$
3.78	8	$-\frac{150277}{7700070}$
3.79	8	$-\frac{6425}{683424}$
3.80	8	$-\frac{3921524}{587858985}$
3.81	8	$\frac{987173}{334768560}$
3.76	8	$\frac{1775}{167112}$

3.6 Consistency

Definition: A linear multistep method is said to be consistent if the following conditions are satisfied, Butcher, (1964).

(i) the order of accuracy $p > 1$

(ii) $\sum_{j=0}^{k} \alpha_j = 0$

(iii) $\rho'(1) = \sigma(1)$, where $\rho(r)$ *and* $\sigma(r)$, are respectively first and second characteristic polynomials of the methods.

From section 3.5, conditions (i) and (ii) are satisfied for all the proposed methods, since in each case the order $p > 1$ and $C_0 = \sum_{j=0}^{k} \alpha_j = 0$.

For the third condition, the first and second characteristic polynomials are obtained and evaluated in what follows. For all the methods, conditions for consistency are satisfied. Hence they are consistent with uniform order of accuracy $p > 1$.

The summary of parameters for measuring consistency is presented in Tables 3.4, 3.5 and 3.6

Table 3.4: Parameters for determining consistency for the proposed 5SHBDM

Equation	Order P	$\sum \alpha_j$	$\rho^{'}(1)$	$\sigma(1)$
3.25	6	0	$-\frac{60}{257}$	$-\frac{60}{257}$
3.26	6	0	$\frac{236}{257}$	$\frac{236}{257}$
3.27	6	0	$\frac{267}{257}$	$\frac{267}{257}$
3.28	6	0	$\frac{248}{257}$	$\frac{248}{257}$
3.29	6	0	$\frac{269}{257}$	$\frac{269}{257}$
3.30	6	0	$\frac{7279}{8224}$	$\frac{7279}{8224}$

Table 3.5: Parameters for determining consistency for the proposed 6SHBDM

Equation	Order P	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
3.55	7	0	$-\frac{60}{163}$	$-\frac{60}{163}$
3.56	7	0	$\frac{491}{489}$	$\frac{491}{489}$
3.57	7	0	$\frac{2444}{2445}$	$\frac{2444}{2445}$
3.58	7	0	$\frac{814}{815}$	$\frac{814}{815}$
3.59	7	0	$\frac{2459}{2445}$	$\frac{2459}{2445}$
3.60	7	0	$\frac{467}{489}$	$\frac{467}{489}$
3.61	7	0	$\frac{667879}{667648}$	$\frac{667879}{667648}$

Table 3.6: Parameters for determining consistency for the proposed 6SEHBDM

Equation	Order P	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
3.75	8	0	$-\frac{31195}{64988}$	$-\frac{31195}{64988}$
3.77	8	0	$\frac{774401}{779856}$	$\frac{774401}{779856}$
3.78	8	0	$\frac{3721589}{3691656}$	$\frac{3721589}{3691656}$
3.79	8	0	$\frac{392839}{389928}$	$\frac{392839}{389928}$
3.80	8	0	$\frac{1930339}{1949640}$	$\frac{1930339}{1949640}$
3.81	8	0	$\frac{362137}{354480}$	$\frac{362137}{354480}$
3.76	8	0	$\frac{100081}{111408}$	$\frac{100081}{111408}$
3.82	8	0	$-\frac{16765}{9284}$	$-\frac{16765}{9284}$

3.7 Zero Stability

Following Akinfenwa *et al.* (2013), the derived Hybrid Backward Differentiation Formula can be written in block form as follows

$$A^{(1)}Y_{\omega+1} = A^{(0)}Y_{\omega-1} + hBF_{\omega+1} \tag{3.90}$$

whose first characteristic polynomial is given as

$$\rho(\lambda) = \det\left[\lambda A^{(1)} - A^{(0)}\right] \tag{3.91}$$

Definition 3.3: The block method (3.138) is said to be zero stable if no root of the first characteristic polynomial $\rho(\lambda)$ satisfies $\left|\lambda_j\right| \le 1, j = 1,2,3,\cdots$ and for those roots with $\left|\lambda_j\right| = 1$, the multiplicity must not exceed 1. (Anake 2011)

3.7.1 Zero stability of the proposed 5SHBDM with one off step grid point

Expressing the methods represented in (3.25), (3.26), (3.27), (3.28), (3.29) and (3.30) in the form (3.90)

$$A(1) = \begin{pmatrix} 1 & -\frac{37044}{18275} & \frac{5929}{3655} & -\frac{5194}{3655} & \frac{38912}{54825} & 0 \\ \frac{3275}{6363} & 1 & -\frac{6175}{2727} & \frac{5425}{3636} & -\frac{40192}{57267} & 0 \\ -\frac{153}{371} & \frac{2403}{1060} & 1 & -\frac{2043}{424} & \frac{10688}{5565} & 0 \\ \frac{1208}{14749} & -\frac{3564}{10535} & \frac{328}{301} & 1 & -\frac{403456}{221235} & 0 \\ \frac{366525}{12996608} & -\frac{1416933}{12996608} & \frac{3825675}{12996608} & -\frac{31454325}{25993216} & 1 & 0 \\ -\frac{75}{1799} & \frac{40}{257} & -\frac{100}{257} & \frac{300}{257} & -\frac{10240}{5397} & 1 \end{pmatrix} \tag{3.92}$$

$$A(0)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1274}{10965} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1325}{32724} \\ 0 & 0 & 0 & 0 & 0 & \frac{55}{1272} \\ 0 & 0 & 0 & 0 & 0 & -\frac{61}{6321} \\ 0 & 0 & 0 & 0 & 0 & -\frac{89425}{25993216} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{771} \end{pmatrix} \tag{3.93}$$

$$B=\begin{pmatrix} -\frac{5397}{7310} & 0 & 0 & 0 & 0 & \frac{441}{7310} \\ 0 & -\frac{1285}{909} & 0 & 0 & 0 & -\frac{50}{909} \\ 0 & 0 & -\frac{771}{212} & 0 & 0 & \frac{27}{212} \\ 0 & 0 & 0 & -\frac{3084}{2107} & 0 & -\frac{144}{2107} \\ 0 & 0 & 0 & 0 & \frac{80955}{203072} & -\frac{297675}{6498304} \\ 0 & 0 & 0 & 0 & 0 & \frac{60}{257} \end{pmatrix} \tag{3.94}$$

$$P(\lambda)=\lambda.A(1)-A(0)$$

$$\left|P(\lambda)\right|=\frac{2944667520675}{4271133391016}\lambda^{5}+\frac{2944667520675}{4271133391016}\lambda^{6} \tag{3.95}$$

$$\lambda=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Thus the method is zero stable.

3.7.2 Zero stability of the proposed 6SHBDM with one off grid point

Expressing the methods presented in equations (3.55), to (3.61) in the form (3.90)

$$A(1)=\begin{pmatrix} 1 & -\frac{120000}{20231} & \frac{27787264}{4673361} & -\frac{77500}{60693} & \frac{36250}{141617} & -\frac{8625}{222541} & 0 \\ \frac{498}{27875} & 1 & -\frac{2359296}{2146375} & \frac{316}{3345} & -\frac{111}{7805} & \frac{118}{61325} & 0 \\ -\frac{240604749}{12656869376} & -\frac{7383828375}{6328434688} & 1 & \frac{1060549875}{6328434688} & -\frac{547630875}{25313738752} & \frac{2083725}{744521728} & 0 \\ -\frac{1539}{22400} & \frac{27}{16} & -\frac{342016}{121275} & 1 & \frac{1377}{6272} & -\frac{1107}{49280} & 0 \\ \frac{70952}{426775} & -\frac{52332}{17071} & \frac{61865984}{14083575} & -\frac{135632}{51213} & 1 & \frac{156408}{938905} & 0 \\ -\frac{76593}{216781} & \frac{1258400}{216781} & -\frac{12058624}{1517467} & \frac{2456300}{650343} & -\frac{3502950}{1517467} & 1 & 0 \\ \frac{216}{163} & -\frac{2275}{163} & -\frac{2000}{163} & \frac{6750}{1141} & -\frac{5400}{1793} & \frac{1048576}{37653} & 1 \end{pmatrix} \quad (3.96)$$

$$A(0)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{2750}{60693} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{14}{16725} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{25050025}{25313738752} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{173}{40320} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2989}{256065} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{17182}{650343} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{50}{489} \end{pmatrix} \quad (3.97)$$

$$B=\begin{pmatrix} -\frac{9780}{20231} & 0 & 0 & 0 & 0 & 0 & -\frac{40}{20231} \\ 0 & -\frac{489}{2230} & 0 & 0 & 0 & 0 & \frac{1}{11150} \\ 0 & 0 & \frac{564795}{1545028} & 0 & 0 & 0 & \frac{800415}{6328434688} \\ 0 & 0 & 0 & \frac{163}{224} & 0 & 0 & -\frac{1}{1120} \\ 0 & 0 & 0 & 0 & \frac{13692}{17071} & 0 & \frac{392}{85355} \\ 0 & 0 & 0 & 0 & 0 & \frac{107580}{216781} & -\frac{4840}{216781} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{60}{163} \end{pmatrix} \tag{3.98}$$

$$P(\lambda)=\lambda.A(1)-A(0)$$

$$\left|P(\lambda)\right|=\frac{828887805319255605 00}{64487977093483978432 91}\lambda^{6}+\frac{82888780531925560500}{6448797709348397843291}\lambda^{7}$$

$$\lambda=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{3.99}$$

Thus the method is zero stable.

3.7.3 Zero Stability of 6SEHBDM with one off grid point

Expressing the method presented in equations (3.75) to (3.82) in the form (3.90)

$$A(0)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6911212000}{1602058419} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{684476}{183335} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{184475}{204756} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1329632}{3110365} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{141977}{996335} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2463412}{18416163} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1675}{16247} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1675}{6963} \end{pmatrix} \qquad (3.100)$$

$$A(1)=\begin{pmatrix} 1 & -\frac{11809105000}{1602058419} & \frac{607760384}{178006491} & -\frac{3356381875}{1602058419} & \frac{1575700000}{1602058419} & -\frac{43703000}{178006491} & 0 & 0 \\ -\frac{8263376109}{564671800} & 1 & \frac{13549824}{916675} & -\frac{2129591}{293336} & \frac{1261564}{403337} & -\frac{961137}{1283345} & 0 & 0 \\ -\frac{475167303}{168172928} & \frac{1682125}{273008} & 1 & -\frac{46012375}{6552192} & \frac{1707625}{750772} & -\frac{934475}{1911056} & 0 & 0 \\ -\frac{1505336157}{1197490525} & \frac{4094504}{1866219} & -\frac{79171584}{15551825} & 1 & \frac{66931616}{20528409} & -\frac{11627304}{21772555} & 0 & 0 \\ \frac{907169787}{2231790400} & -\frac{1037531}{1594136} & \frac{5652306}{4981675} & -\frac{28593653}{12753088} & 1 & \frac{27581697}{55794760} & 0 & 0 \\ -\frac{202597119}{540207448} & \frac{3525375}{6138721} & -\frac{5485312}{6138721} & \frac{196497175}{147329304} & -\frac{119735700}{67525931} & 1 & 0 & 0 \\ \frac{29701647}{40032608} & -\frac{217039}{194964} & \frac{26877}{16247} & -\frac{1177675}{519904} & \frac{1293275}{536151} & -\frac{984045}{454916} & 1 & 0 \\ -\frac{531441}{816992} & \frac{25375}{27852} & -\frac{2688}{2321} & \frac{261625}{222816} & -\frac{37625}{76593} & -\frac{9525}{9284} & 0 & 1 \end{pmatrix} \qquad (3.101)$$

$$
B=\begin{pmatrix}
-\frac{5199040}{1383681} & 0 & 0 & 0 & 0 & 0 & -\frac{326480000}{9078331041} & \frac{2935240}{534019473} \\
0 & \frac{2339568}{183335} & 0 & 0 & 0 & 0 & -\frac{19248}{183335} & \frac{2883}{183335} \\
0 & 0 & -\frac{243705}{34126} & 0 & 0 & 0 & -\frac{2125}{34126} & \frac{2445}{273008} \\
0 & 0 & 0 & \frac{3119424}{622073} & 0 & 0 & -\frac{35712}{622073} & \frac{24152}{3110365} \\
0 & 0 & 0 & 0 & \frac{292446}{199267} & 0 & \frac{7194}{199267} & -\frac{35079}{7970680} \\
0 & 0 & 0 & 0 & 0 & \frac{3899280}{6138721} & -\frac{439600}{6138721} & \frac{43155}{6138721} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{8160}{16247} & -\frac{1445}{64988} \\
 & & 0 & 0 & 0 & 0 & \frac{3500}{2321} & \frac{2765}{9284}
\end{pmatrix} \quad (3.102)
$$

$$
P(\lambda)=\begin{pmatrix} \frac{908620734362473661163695662694400000}{1291668892810635664552602360826 1}\lambda^7 \\ +\frac{908620734362473661163695662694400000}{1291668892810635664552602360826 1}\lambda^6 \end{pmatrix}\lambda \quad (3.103)
$$

$$
\lambda=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}
$$

Thus the method is zero stable.

3.8 Convergence

The necessary and sufficient condition for linear multistep method to be convergent is for it to be consistent and zero stable (Lambert, 1973). Following this theorem, each of the block methods developed are convergent.

3.9 Region of Absolute Stability

Following Akinfenwa *et al,* (2020) the region of absolute stability is determined by obtaining the stability polynomial of the form:

$$\sigma(z) = \left(A^{(1)} - zB^{(1)}\right)^{-1}\left(A^{(0)}\right) \tag{3.104}$$

where $z = \lambda h$

The matrix $\sigma(z)$ has Eigen values $\{0, 0, 0, ..., \lambda_k\}$, and the dominant Eigenvalue $\lambda_k : \mathbb{C} \rightarrow \mathbb{C}$ is a rational function (called the stability function) with real coefficients given by

$$\lambda_k = P(z) \tag{3.105}$$

3.9.1 Stability region of 5SHBDM with one off grid point:

The stability function is given as:

$$P(z) = -\frac{12z^5 + 98z^4 + 40z^3 + 960z^2 + 1260z + 720}{540z^6 - 1353z^5 + 2573z^4 - 3645z^3 + 3660z^2 - 2340z + 720} \qquad (3.106)$$

It is clear from the stability functions that for $\text{Re}(z) < 0, \quad |\lambda_k| \le 1$. Below is the region of absolute stability region of the proposed method.

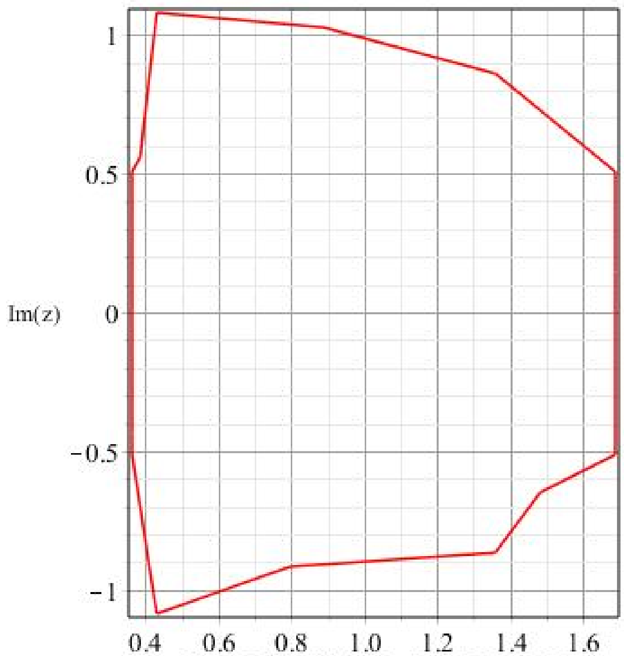

Figure 3.1: Stability Region of 5SHBDM

Figure 3.1 shows the stability region of 5SHBDM and found to be an A-stable method since its region of absolute stability contains the left half-plane $\mathbb{C}^-$.

3.9.2 Stability region of 6SHBDM with one off grid point:

The stability function is given as:

$$P(z)=\frac{300z^6+1530z^5+4471z^4+8700z^3+11300z^2+9000z+3360}{1080z^7-3126z^6+7224z^5-13111z^4+18060z^3-17780z^2+11160z-3360} \quad (3.107)$$

It is clear from the stability functions that for $\mathrm{Re}(z)<0,\quad |\lambda_k|\le 1$. below is the region of absolute stability region of the proposed method.

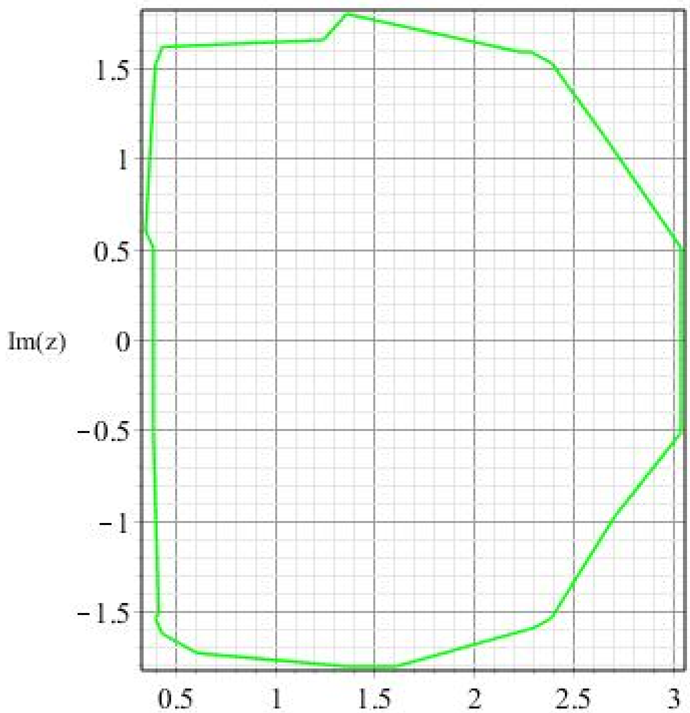

Figure 3.2: Stability Region of 6SHBDM

Figure 3.2 shows the stability region of 6SHBDM and found to be an A-stable method since its region of absolute stability contains the left half-plane $\mathbb{C}^-$.

3.7.3 Stability region of 6SEHBDM with one off grid point:

The stability function is given as:

$$P(z) = -\frac{12\left(600z^7+3120z^6+9443z^5+19572z^4+28525z^3+28350z^2+17430z+3040\right)}{2520z^8-14094z^7+52336z^6-138495z^5+267204z^4-372120z^3+357840z^2-214200z+60480} \quad (3.108)$$

It is clear from the stability functions that for $\operatorname{Re}(z) < 0$, $|\lambda_k| \leq 1$. below is the region of absolute stability region of the proposed method.

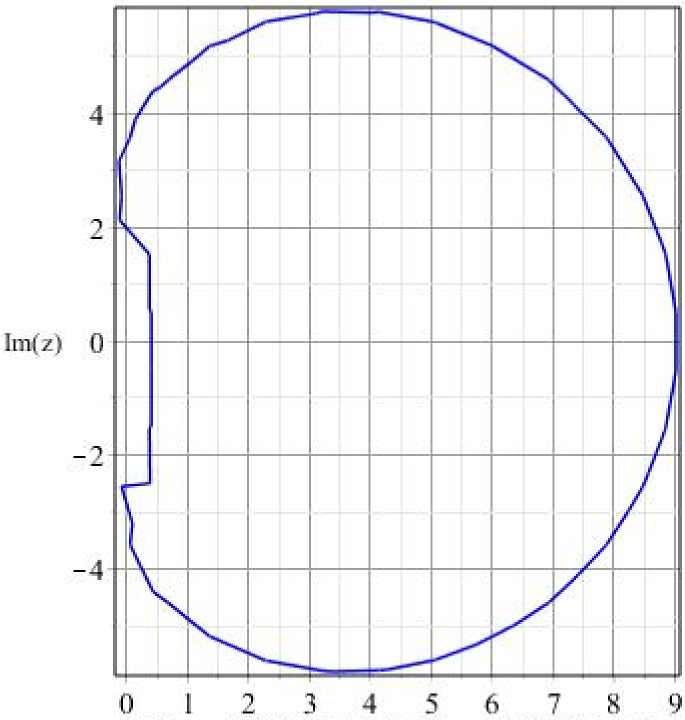

Figure 3.3: Stability Region of 6SEHBDM

Figure 3.3 shows the stability region of 6SEHBDM and found to be an A-stable method since its region of absolute stability contains the left half-plane $\mathbb{C}^-$.

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Numerical Experiments

In this section, the results of the hybrid methods developed in chapter three were presented on some problems of first order differential equations.

Example 4.1

Consider the Initial Value Problem

$$y' = -y$$

$$y(0) = 1, \qquad 0 \le x \le 1$$

Exact solution

$$y(x) = e^{-x}$$ (Mohammed *et al.* 2010)

Example 4.2

Consider the non-linear Initial Value Problem

$$y' = -\frac{y^3}{2}$$

$$y(0) = 1, \qquad 0 \le x \le 1$$

Exact solution $y(x) = \frac{1}{\sqrt{1+x}}$ Source: (Musa *et al.* 2012)

Example 4.3

Consider the nonlinear problem given by

$$y' = -10(y-1)^2 \qquad y(0) = 2$$

With the exact solution $y(x) = 1 + \dfrac{1}{1+10x}$ (Sunday *et al.* 2014).

Example 4.4

Consider the linear problem

$$y' = -20y + 20\sin x + \cos x \qquad y(0) = 1,$$

With exact solution

$y(x) = \sin x + e^{-20x}$ (Mohamad *et al.* 2018).

Example 4.5

Consider the cosine problem

$$y' = -2\pi\sin(2\pi x) - \frac{1}{\varepsilon}(y - \cos(2\pi x)) \qquad y(0) = 1,$$

The problem becomes increasingly stiff as $\varepsilon \longrightarrow 0$ we choose $\varepsilon = 10^{-3}$.

The exact solution is given by

$y = \cos(2\pi x)$. (Musa *et al.* 2013).

Example 4.6

Consider the system of nonlinear stiff initial value problem

$$y^{'} = -(\varepsilon^{-1} + 2)y_1 - \varepsilon^{-1} y_2^2 \qquad y_1(0) = 1,$$

(Musa *et al.* 2012)

$$y_2^{'} = y_1 - y_2(1 + y_2) \qquad y_2(0) = 1,$$

Table 4.1: Comparing the exact solution with the solution of the proposed methods for problem 4.1

x	Exact Solution	5SHBDM	6SHBDM	6SEHBDM
0.0	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000
0.1	0.90483741803596	0.90483743369648	0.90483741863555	0.90483741803798
0.2	0.81873075307798	0.81873076659355	0.81873075360897	0.81873075308081
0.3	0.74081822068172	0.74081823308749	0.74081822116118	0.74081822068373
0.4	0.67032004603564	0.67032005717511	0.67032004647336	0.67032004603791
0.5	0.60653065971263	0.60653066977675	0.60653066009888	0.60653065971415
0.6	0.54881163609403	0.54881165019586	0.54881163649042	0.54881163609659
0.7	0.49658530379141	0.49658531615421	0.49658532900886	0.49658530343599
0.8	0.44932896411722	0.44932897541052	0.44932898692854	0.44932896379614
0.9	0.40656965974060	0.40656966990721	0.40656968038062	0.40656965944978
1.0	0.36787944117144	0.36787945036129	0.36787945984945	0.36787944090855

Table 4.1 compares the result of the proposed methods with that of the exact solution. The 6SEHBDM is closer to the exact solution than the 6SHBDM and 5SHBDM.

Table 4.2: Comparison between the proposed methods with Mohammed *et al.* (2010) for problem 4.1

x	5SHBDM	6SHBDM	6SEHBDM	Mohammed and Yahaya, (2010)
0.0	0	0	0	0
0.1	1.56E-08	5.99E-10	2.07E-12	2.52E-06
0.2	1.35E-08	5.31E-10	2.90E-12	2.09E-06
0.3	1.24E-08	4.79E-10	2.11E-12	0.04E-06
0.4	1.11E-08	4.37E-10	2.33E-12	1.61E-06
0.5	1.01E-08	3.86E-10	1.55E-12	3.16E-06
0.6	1.86E-08	3.96E-10	2.66E-12	2.73E-06
0.7	1.64E-08	6.87E-10	3.56E-12	2.55E-06
0.8	1.49E-08	6.16E-10	2.19E-12	2.17E-06
0.9	1.35E-08	5.56E-10	1.47E-12	3.10E-06
1.0	1.22E-08	5.06E-10	1.59E-12	2.72E-06

The table shows that the proposed methods performed better than those found in the literature.

Table 4.3: Comparing results of the proposed hybrid methods and the exact solution. h=0.1 for problem 4.2

x	Exact Solution	5SHBDM	6SHBDM	6SEHBDM
0.0	1.00000000000000	1.00000000000000	1.00000000000000	1.00000000000000
0.1	0.95346258924555	0.95346607404119	0.95346333786536	0.95346260520907
0.2	0.91287092917530	0.91287385239118	0.91287157303250	0.91287095035505
0.3	0.87705801930706	0.87706064586272	0.87705858928920	0.87705803473186
0.4	0.84515425472853	0.84515658939020	0.84515476865000	0.84515427107951
0.5	0.81649658092772	0.81649868354160	0.81649703478199	0.81649659248620
0.6	0.79056941504206	0.79057162051442	0.79056986892131	0.79056943243178
0.7	0.76696498884739	0.76696699079590	0.76696544011202	0.76696497397092
0.8	0.74535599249992	0.74535783298632	0.74535640615649	0.74535597859437
0.9	0.72547625011002	0.72547794581749	0.72547663150077	0.72547623741120
1.0	0.70710678118654	0.70710835109019	0.70710713449674	0.70710676936891

It is observed from the table that the 6SEHBDM performed better than the 6SHBDM and 5SHBDM.

Table 4.4: Comparison of the proposed methods with Musa *et al* (2012) for problem 4.2

x	5SHBDM	6SHBDM	6SEHBDM	Musa *et al* (2012)
0.0	0	0	0	0
0.1	3.5E-06	7.4E-07	1.5E-8	3.2E-04
0.2	2.9E-06	6.4E-07	2.1E-8	2.6E-04
0.3	2.6E-06	5.6E-07	1.5E-8	1.3E-03
0.4	2.3E-06	5.1E-07	1.6E-8	5.6E-05
0.5	2.1E-06	4.5E-07	1.1E-8	1.5E-03
0.6	2.19E-06	4.5E-07	1.7E-8	-
0.7	2.01E-06	4.4E-07	1.5E-8	-
0.8	1.86E-06	4.1E-07	3.6E-8	-
0.9	1.7E-06	3.7E-07	0.4E-8	-
1.0	1.58E-06	3.4E-07	2.9E-8	-

Table 4.2 shows that the proposed methods produces results that are closer to the exact solution than those found in the literature.

Table 4.5: Comparing the exact solution with the solution of the proposed methods for problem 4.3

x	Exact Solution	5SHBDM	6SHBDM	6SEHBDM
0.01	1.9090909090909	1.909104965956	1.909094123383	1.909090979989
0.02	1.8333333333333	1.833344593696	1.833335977179	1.833333425922
0.03	1.7692307692308	1.769240503263	1.769233017437	1.769230832614
0.04	1.7142857142857	1.714294043570	1.714287669892	1.714285780734
0.05	1.6666666666667	1.666673911780	1.666666832836	1.666666709781
0.06	1.6250000000000	1.625007405843	1.625001644650	1.625000069938
0.07	1.5882352941176	1.588241813111	1.588236857237	1.588235212927
0.08	1.5555555555556	1.555561380916	1.555556947903	1.555555613059
0.09	1.5263157894737	1.526321012852	1.526317038961	1.526315840715
0.10	1.5000000000000	1.500004713280	1.500001128256	1.500000046530

The table shows that the 6SEHBDM is better than the 6SHBDM and 5SHBDM

Table 4.6: Comparison Between the Proposed Methods with Sunday *et al.* (2014) for Problem 4.3

x	5SHBDM	6SHBDM	6SEHBDM	Sunday *et al.* (2014)
0.01	1.4057E-05	3.21438E-06	7.0989E-08	3.41458E-06
0.02	1.12607E-05	2.64418E-06	9.2922E-08	2.7493E-06
0.03	9.73426E-06	2.24844E-06	6.3614E-08	1.34292E-05
0.04	8.32957E-06	1.95589E-06	6.6734E-08	9.09062E-05
0.05	7.24478E-06	1.65836E-07	4.2781E-08	7.96972E-05
0.06	7.40584E-06	1.64465E-06	6.9938E-08	6.99489E-05
0.07	6.51911E-06	1.56324E-06	8.1073E-08	6.27004E-05
0.08	5.82492E-06	1.3919E-06	5.7059E-08	6.01715E-05
0.09	5.22385E-06	1.24996E-06	5.1715E-08	5.41126E-05
0.10	4.71328E-06	1.12826E-06	4.653E-08	4.88098E-05

The table compares the proposed methods with those found in the literature. The proposed methods performed better than those found in the literature.

Table 4.7: Comparing the Exact Solution with the Solution of the Proposed Methods for Problem 4.4

x	Exact Solution	5SHBDM	6SHBDM	6SEHBDM
0.01	0.2351686998834	0.235148199426231	0.234912416385453	0.235180254529486
0.02	0.2169849696838	0.216916588179879	0.215763389645848	0.217031405831369
0.03	0.2979989588380	0.297879965300621	0.297325064893373	0.298034633122212
0.04	0.3897538049366	0.389584969422429	0.388096171002468	0.389842560523135
0.05	0.4794709385340	0.479253897145130	0.476164958355442	0.479610093807347
0.06	0.5646486176074	0.564385532859160	0.563132685473904	0.564732224849823
0.07	0.6442185187664	0.643912018597408	0.641276739676042	0.644379489975150
0.08	0.7173562034347	0.717009350210699	0.712256964508480	0.717576048013345
0.09	0.7833269248575	0.782943184212025	0.781103804043485	0.783451050864750
0.10	0.8414709868691	0.841054193010872	0.837507766722072	0.841689801949880

The table shows that the 6SEHBDM is better than the other methods proposed.

Table 4.8: Comparison of errors between the proposed methods for problem 4.4

x	5SHBDM	6SHBDM	6SEHBDM
0.01	0.000020500457210	0.000256283497988	0.000011554646045
0.02	0.000068381503916	0.001221580037947	0.000046436147574
0.03	0.000118993537385	0.000673893944633	0.000035674284206
0.04	0.000168835514124	0.001657633934085	0.000088755586582
0.05	0.000217041388835	0.003305980178523	0.000139155273382
0.06	0.000263084748228	0.001515932133484	0.000083607242435
0.07	0.000306500169002	0.002941779090368	0.000160971208740
0.08	0.000346853223999	0.005099238926218	0.000219844578647
0.09	0.000383740645438	0.002223120813978	0.000124126007287
0.10	0.000416793858179	0.003963220146979	0.000218815080829

The table shows that the SEHBDM has less error compared to the 6SHBDM and 5SHBDM

Table 4.9: Comparing the Maximum error (MAXE) of the Proposed Methods with Existing Methods for Problem 4.4

METHOD	MAXE
5SHBDM	1.48E-3
6SHBDM	9.56E-3
6SEHBDM	1.69E-4
Mohammad *et al.* (2018)	1.83E-2

The table shows the maximum errors in the proposed methods with that found in the literature. The proposed methods have less error compared to that found in the literature.

Table 4.10: Comparing the exact solution with the solution of the proposed methods for problem 4.5

x	Exact	5SHBDM	6SHBDM	6SEHBDM
0.01	1.00	0.999999999999753	1.0000000000000397737	1.0000000000000002795
0.02	1.00	0.999999999999753	0.9999999999999852279	1.0000000000000000381
0.03	1.00	0.999999999999753	1.0000000000009905037	0.9999999999999987331
0.04	1.00	0.999999999999756	1.0000000000000397737	0.9999999999999994421
0.05	1.00	0.999999999999756	0.9999999999999852279	0.9999999999999998686
0.06	1.00	0.999999999999756	1.0000000000009905022	1.0000000000000001603
0.07	1.00	0.999999999999753	1.0000000000000397737	1.0000000000000139048
0.08	1.00	0.999999999999756	0.9999999999999852278	1.0000000000000002796
0.09	1.00	0.999999999999756	1.0000000000009905031	1.0000000000000000381
0.10	1.00	0.999999999999756	1.0000000000000397737	0.9999999999999987332

The table compares the analytical solution with the solutions of the proposed methods. The 6SEHBDM performed better than the 6SHBDM and 5SHBDM.

Table 4.11: Comparison of errors between the proposed methods for problem 4.5

x	5SHBDM	6SHBDM	6SEHBDM
0.01	2.47E-13	3.97E-14	2.79E-16
0.02	2.47E-13	1.47E-14	3.81E-17
0.03	2.47E-13	9.91E-13	1.26E-15
0.04	2.44E-13	3.98E-14	5.58E-16
0.05	2.44E-13	1.48E-14	1.31E-16
0.06	2.44E-13	9.91E-13	1.60E-16
0.07	2.47E-13	3.98E-14	1.39E-14
0.08	2.44E-13	1.48E-14	2.79E-16
0.09	2.44E-13	9.91E-13	3.81E-17
0.10	2.44E-13	3.98E-14	1.26E-15

The table shows that the 6SEHBDM having higher order performs better than the 6SHBDM and 5SHBDM.

Table 4.12: Comparing the Maximum Error MAXE of the Proposed Methods with Existing Method for Problem 4.5

Methods	Number of Iteration	MAXE
5SHBDM	1000	9.0563E-12
6SHBDM	1000	1.00440586E-12
6SEHBDM	1000	7.925235E-14
Musa *et al* (2013)	472	3.19856E-9

The table compares results of the proposed methods with that found in the literature. The proposed methods performed better than that found in the literature.

Table 4.13: Comparison of errors between the proposed methods for problem 4.6

x	5SHBDM		6SHBDM		6SEHBDM	
	y_1	y_2	y_1	y_2	y_1	y_2
0.01	1.86E-13	3.67E-1	4.20E-17	3.67E-1	1.19E-20	3.67E-1
0.02	5.07E-14	1.35E-1	1.13E-17	1.35E-1	3.41E-21	1.35E-1
0.03	1.02E-14	4.97E-2	2.27E-18	4.97E-2	6.87E-22	4.97E-1
0.04	1.85E-15	1.83E-2	4.11E-19	1.83E-2	1.32E-22	1.83E-3
0.05	3.11E-16	6.73E-3	6.98E-20	6.73E-3	2.22E-23	6.73E-3
0.06	5.06E-17	2.47E-3	1.13E-20	2.47E-3	3.62-24	2.47E-3
0.07	8.02E-18	9.11E-4	1.78E-21	9.11E-4	6.24E-25	9.11E-4
0.08	1.24E-18	3.35E-4	2.76E-22	3.35E-4	8.87E-26	3.35E-4
0.09	1.89E-19	1.23E-4	4.19E-23	1.23E-4	1.35E-26	1.23E-4
0.10	2.84E-20	4.53E-5	6.30E-24	4.53E-5	2.01E-27	4.53E-5

The table shows that the 6SEHBDM performed better than the 6SHBDM and 5SHBDM

Table 4.14: Comparing the Maximum Error (MAXE) of the Proposed Methods with that Found in the Literature for Problem 4.6

Methods	Number of Iteration		MAXE	
	y_1	y_2	y_1	y_2
5SHBDM	20	20	2.90E-13	2.72E-13
6SHBDM	20	20	6.18E-17	5.89E-17
6SEHBDM	20	20	4.98E-20	1.84E-20
Musa *et al* (2013)	169	118	2.94E-10	8.11E-7

The table shows that the proposed method performed better than that in the literature.

4.2 Discussion of Results

In this research, the construction of a new class of hybrid backward differential methods that are capable of solving initial value problems of first order differential equations has been the central of concern. The power series was employed to develop the schemes using the interpolation and collocation technique, with the incorporation of one off grid point in the derivation process of the methods. Five-step and six-step methods and an extended-six-step method were derived to provide approximate solutions to initial value problems of first order ODEs. The schemes were implemented as block method and therefore the continuous forms of each schemes have the capacity to generate simultaneous solutions at different points in a single application of the methods; hence, the methods are self-starting and take care of the problems associated with the predictor-corrector methods.

The convergence properties of the methods were investigated and findings from the analysis of the basic properties of the methods show that they are consistent, zero – stable and hence, convergent. The stability of the newly derived methods were obtained using the boundary locus approach which involves finding the roots of the stability functions of a rational complex function. The regions of absolute stability in the figures 3.1, 3.2 and 3.3 show that the methods are A-stable. This stability property makes the methods suitable for stiff problems in the numerical experiments such as examples 4.1, 4.4, and 4.6. Similarly, the non-linear problems in examples 4.2, 4.3 and the oscillatory problems considered in examples 4.4, and 4.5. Tables 4.1 and 4.2 show the accuracy and stability of the methods on the stiff problem in example 4.1. Results show that the 6SEHBDM, 6SHBDM and 5SHBDM respectively have greater accuracy. Furthermore, the 5SHBDM which is the least accurate of the proposed methods outperform the method of Mohammed and Yahaya (2010).

The accuracy and stability of the derived methods are also demonstrated on the non-linear initial values problems in examples 4.2 and 4.3. Results in Tables 4.3, 4.4, 4.5 and 4.6 show that the derived methods have greater accuracy that the methods of Musa *et al.* (2012) and Sunday *et al.* (2014). Similarly, the oscillatory problems considered in examples 4.4 and 4.5 are solved by the proposed methods and results in tables 4.7 – 4.8 show the superiority of our methods over that of Musa *et al.* (2013) and Mohammed *et al.* (2018) It is obvious that the results of the proposed methods perfumed better than those found in the literatures mentioned.

CHAPTER FIVE

5.0 CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

Implicit five-step, six-step and six-step extended hybrid backward differentiation methods with higher orders of accuracy have been developed using interpolation and collocation techniques with the incorporation of one off – step point in the derivation process for the approximation of the solutions of first order ordinary differential equations. For this purpose, we used the power series as basis function. The basic properties of numerical methods were carried out and found to be of higher order, zero stable and consistent, therefore the methods are convergent. Also the stability property of the methods was also obtained using the locus boundary method and the stability region of the methods show and the methods are A-stable. Six problems which include the stiff problems, stiff system, nonlinear problems and oscillatory problems were considered and results compared with existing methods to test the effectiveness and accuracy of the new methods show the superiority of our methods over some existing methods in the literature. The extended hybrid method having higher function evaluation is better than the other two. So the order of the method is proportional to the accuracy.

5.2 Contribution to Knowledge

The incorporation of off-grid points in the derivation process of the new BDFs allows the 5SHBDM to have a order of accuracy of $\left(6, 6,\ 6,\ 6,\ 6,\ 6,\ 6\right)^T$ with very small error constants: $\left(-\frac{5}{1799}, -\frac{14}{1075}, \frac{1385}{152712}, -\frac{123}{8480}, \frac{317}{73745}, \frac{356265}{207945728}\right)^T$. Also the methods are applicable to solve both linear and non – linear problems the stiff problems considered were solved with higher accuracy and maximum error of 1.84×10^{-20} compared with that of Musa *et al.* (2013) with maximum error of 8.11×10^{-7}

5.3 Recommendations

Research in numerical analysis is a continuous process. Thus, future research can be carried out in the following areas.

1. The development of software for all the classes of block methods in this thesis can be done;
2. The method may be extended to handle boundary value problems;
3. reformulation of the methods to handle higher order ODE's may also be considered;
4. The method may be reformulated using a different basis function;
5. The number of off – grid point may also be increased.

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Appendix A: Maple Software code for the solution of problem 4.1 for 5SHBDM

>

for n **from** 0 **by** 5 **to** N **do**

$h := 0.1; y[0] := 1;$

$f[n+0] := -y[n+0];$

$f[n+1] := -y[n+1];$

$f[n+2] := -y[n+2];$

$f[n+3] := -y[n+3];$

$f[n+4] := -y[n+4];$

$f\left[n+\frac{9}{2}\right] := -y\left[n+\frac{9}{2}\right];$

$f[n+5] := -y[n+5];$

$$A := \left(y_{n+5} = -\frac{4}{771} y_n + \frac{75}{1799} y_{n+1} - \frac{40}{257} y_{n+2} + \frac{100}{257} y_{n+3} - \frac{300}{257} y_{n+4} + \frac{10240}{5397} y_{n+\frac{9}{2}} + \frac{60}{257} h f_{n+5}, f_{n+\frac{9}{2}} = -\frac{2555}{296064} \frac{y_n}{h} + \frac{8145}{115136} \frac{y_{n+1}}{h} - \frac{22491}{82240} \frac{y_{n+2}}{h} + \frac{12145}{16448} \frac{y_{n+3}}{h} - \frac{99855}{32896} \frac{y_{n+4}}{h} + \frac{203072}{80955} \frac{y_{n+\frac{9}{2}}}{h} + \frac{945}{8224} f_{n+5}, f_{n+4} = \frac{61}{9252} \frac{y_n}{h} - \frac{302}{5397} \frac{y_{n+1}}{h} + \frac{297}{1285} \frac{y_{n+2}}{h} - \frac{574}{771} \frac{y_{n+3}}{h} - \frac{2107}{3084} \frac{y_{n+4}}{h} + \frac{100864}{80955} \frac{y_{n+\frac{9}{2}}}{h} - \frac{12}{257} f_{n+5}, f_{n+3} = -\frac{55}{4626} \frac{y_n}{h} + \frac{204}{1799} \frac{y_{n+1}}{h} - \frac{801}{1285} \frac{y_{n+2}}{h} - \frac{212}{771} \frac{y_{n+3}}{h} + \frac{681}{514} \frac{y_{n+4}}{h} - \frac{42752}{80955} \frac{y_{n+\frac{9}{2}}}{h} + \frac{9}{257} f_{n+5}, f_{n+2} = \frac{265}{9252} \frac{y_n}{h} - \frac{655}{1799} \frac{y_{n+1}}{h} - \frac{909}{1285} \frac{y_{n+2}}{h} + \frac{1235}{771} \frac{y_{n+3}}{h} - \frac{1085}{1028} \frac{y_{n+4}}{h} + \frac{40192}{80955} \frac{y_{n+\frac{9}{2}}}{h} - \frac{10}{257} f_{n+5}, f_{n+1} = -\frac{364}{2313} \frac{y_n}{h} - \frac{7310}{5397} \frac{y_{n+1}}{h} + \frac{3528}{1285} \frac{y_{n+2}}{h} - \frac{1694}{771} \frac{y_{n+3}}{h} + \frac{1484}{771} \frac{y_{n+4}}{h} - \frac{77824}{80955} \frac{y_{n+\frac{9}{2}}}{h} + \frac{21}{257} f_{n+5} \right);$$

$P := fsolve(\{A\}); Q := eval([y[n+1], y[n+2], y[n+3], y[n+4], y[n+5]], P); y[n+1] := Q[1]; y[n+2] := Q[2]; y[n+3] := Q[3]; y[n+4] := Q[4]; y[n+5] := Q[5];$ **end do**:

> $N := 10:$

> **for** n **from** 0 **to** N **do** $y_n := y_n$ **end do**;

$$y_0 := 1$$
$$y_1 := 0.904837433696604$$
$$y_2 := 0.818730766593666$$
$$y_3 := 0.740818233087546$$
$$y_4 := 0.670320057175161$$
$$y_5 := 0.606530669776725$$
$$y_6 := 0.548811654699055$$
$$y_7 := 0.496585320228867$$
$$y_8 := 0.449328979097400$$
$$y_9 := 0.406569673243224$$
$$y_{10} := 0.367879453379803$$

> **for** n **from** 0 **to** N **do** $x[n] := 0.1 \cdot n;\ Y[n] := \exp(-x_n)$ **end do**;

$$x_0 := 0.$$
$$Y_0 := 1.$$
$$x_1 := 0.1$$
$$Y_1 := 0.904837418035960$$
$$x_2 := 0.2$$
$$Y_2 := 0.818730753077982$$
$$x_3 := 0.3$$
$$Y_3 := 0.740818220681718$$
$$x_4 := 0.4$$
$$Y_4 := 0.670320046035639$$
$$x_5 := 0.5$$
$$Y_5 := 0.606530659712633$$
$$x_6 := 0.6$$
$$Y_6 := 0.548811636094026$$
$$x_7 := 0.7$$
$$Y_7 := 0.496585303791410$$
$$x_8 := 0.8$$
$$Y_8 := 0.449328964117222$$
$$x_9 := 0.9$$
$$Y_9 := 0.406569659740599$$
$$x_{10} := 1.0$$
$$Y_{10} := 0.367879441171442$$

> **for** n **from** 0 **to** N **do** $E[n] := \text{abs}(Y[n] - y[n])$ **end do**;

$$E_0 := 0.$$

$$E_1 := 1.5660644\ 10^{-8}$$

$$E_2 := 1.3515684\ 10^{-8}$$

$$E_3 := 1.2405828\ 10^{-8}$$

$$E_4 := 1.1139522\ 10^{-8}$$

$$E_5 := 1.0064092\ 10^{-8}$$

$$E_6 := 1.8605029\ 10^{-8}$$

$$E_7 := 1.6437457\ 10^{-8}$$

$$E_8 := 1.4980178\ 10^{-8}$$

$$E_9 := 1.3502625\ 10^{-8}$$

$$E_{10} := 1.2208361\ 10^{-8}$$

Appendix B: D's for the DMatrix of 6SHBDM

$D_{11} = 1 \quad D_{12} = x_n \quad D_{13} = x_n \quad D_{14} = x_n \quad D_{15} = x_n \quad D_{16} = x_n \quad D_{17} = x_n$

$D_{18} = x_n$

$D_{21} = 1 \quad D_{22} = x_{n+1} \quad D_{23} = x_{n+1}^2 \quad D_{24} = x_{n+1}^3 \quad D_{25} = x_{n+1}^4 \quad D_{26} = x_{n+1}^5 \quad D_{27} = x_{n+1}^6$

$D_{28} = x_{n+1}^6$

$D_{31} = 1 \quad D_{32} = 2x_{n+2} \quad D_{33} = 4x_{n+2}^2 \quad D_{34} = 8x_{n+2}^3 \quad D_{35} = 16x_{n+2}^4 \quad D_{36} = 32x_{n+2}^5$

$D_{37} = 64x_{n+2}^6 \quad D_{38} = 128x_{n+2}^7$

$D_{41} = 1 \quad D_{42} = 3x_{n+3} \quad D_{43} = 9x_{n+3}^2 \quad D_{44} = 27x_{n+3}^3 \quad D_{45} = 81x_{n+3}^4 \quad D_{46} = 243x_{n+3}^5$

$D_{47} = 729x_{n+3}^6 \quad D_{48} = 2187x_{n+3}^7$

$D_{51} = 1 \quad D_{52} = 4x_{n+4} \quad D_{53} = 16x_{n+4}^2 \quad D_{54} = 64x_{n+4}^3 \quad D_{55} = 256x_{n+4}^4 \quad D_{56} = 1024x_{n+4}^5$

$D_{57} = 4096x_{n+4}^6 \quad D_{58} = 16384x_{n+4}^7$

$D_{61} = 1 \quad D_{62} = 5x_{n+5} \quad D_{63} = 25x_{n+5}^2 \quad D_{64} = 125x_{n+5}^3 \quad D_{65} = 625x_{n+5}^4 \quad D_{66} = 3125x_{n+5}^5$

$D_{67} = 15625x_{n+5}^6 \quad D_{68} = 78125x_{n+5}^7$

$D_{71} = 1 \quad D_{72} = \frac{9}{4}x_{n+6} \quad D_{73} = \frac{81}{16}x_{n+6}^2 \quad D_{74} = \frac{729}{64}x_{n+6}^3 \quad D_{75} = \frac{6561}{256}x_{n+6}^4 \quad D_{76} = \frac{59049}{1024}x_{n+6}^5$

$D_{77} = \frac{531441}{4096}x_{n+6}^6 \quad D_{78} = \frac{4782969}{16384}x_{n+6}^7$

$D_{81} = 0 \quad D_{82} = x_{n+7} \quad D_{83} = 12x_{n+7}^2 \quad D_{84} = 108x_{n+7}^3 \quad D_{85} = 864x_{n+7}^4 \quad D_{86} = 6480x_{n+7}^5$

$D_{87} = 46656x_{n+7}^6 \quad D_{88} = 326592x_{n+7}^7$

Appendix C: D's for the D Matrix of 6SEHBDM

$D_{11} = 1 \quad D_{12} = x_n \quad D_{13} = x_n \quad D_{14} = x_n \quad D_{15} = x_n \quad D_{16} = x_n \quad D_{17} = x_n \quad D_{18} = x_n$

$D_{19} = x_n$

$D_{21} = -\frac{17638}{3165} x_{n+1} \quad D_{22} = \frac{81507303}{10398080} x_{n+1} \quad D_{23} = -\frac{103825}{30384} x_{n+1} \quad D_{24} = \frac{2087}{1055} x_{n+1}$

$D_{25} = -\frac{104885}{81024} x_{n+1} \quad D_{26} = \frac{13085}{20889} x_{n+1} \quad D_{27} = -\frac{18773}{118160} x_{n+1} \quad D_{28} = \frac{5}{211} x_{n+1}$

$D_{29} = -\frac{37}{10128} x_{n+1}$

$D_{31} = \frac{120200021}{11697840} x_{n+2} \quad D_{32} = -\frac{322935956379}{16013043200} x_{n+2} \quad D_{33} = \frac{438668555}{28074816} x_{n+2}$

$D_{34} = -\frac{48906173}{4874100} x_{n+2} \quad D_{35} = \frac{507660823}{74866176} x_{n+2} \quad D_{36} = -\frac{257298037}{77205744} x_{n+2}$

$D_{37} = \frac{93051319}{109179840} x_{n+2} \quad D_{38} = -\frac{25019}{194964} x_{n+2} \quad D_{39} = \frac{929923}{46791360} x_{n+2}$

$D_{41} = -\frac{62955197}{7018704} x_{n+3} \quad D_{42} = \frac{10028967453}{500407600} x_{n+3} \quad D_{43} = -\frac{34185829}{1754676} x_{n+3}$

$D_{44} = \frac{37295351}{2437050} x_{n+3} \quad D_{45} = -\frac{77434435}{7018704} x_{n+3} \quad D_{46} = \frac{431936951}{77205744} x_{n+3}$

$D_{47} = -\frac{1985413}{1364748} x_{n+3} \quad D_{48} = \frac{130597}{584893} x_{n+3} \quad D_{49} = -\frac{51031}{1462230} x_{n+3}$

$D_{51} = \frac{199606373}{46791360} x_{n+4} \quad D_{52} = -\frac{652069991043}{64052172800} x_{n+4} \quad D_{53} = \frac{1253240435}{112299264} x_{n+4}$

$D_{54} = -\frac{49135279}{4874100} x_{n+4} \quad D_{55} = \frac{2371375135}{299464704} x_{n+4} \quad D_{56} = -\frac{1298497969}{308822976} x_{n+4}$

$D_{57} = \frac{489751487}{436719360} x_{n+4} \quad D_{58} = -\frac{138329}{779856} x_{n+4} \quad D_{59} = \frac{5265451}{187165440} x_{n+4}$

$$D_{61} = -\frac{40957951}{35093520}x_{n+5} \quad D_{62} = \frac{9268337601}{3202608640}x_{n+5} \quad D_{63} = -\frac{95765521}{28074816}x_{n+5}$$

$$D_{64} = \frac{275346}{81235}x_{n+5} \quad D_{65} = -\frac{644765531}{224598528}x_{n+5} \quad D_{66} = \frac{124211729}{77205744}x_{n+5}$$

$$D_{67} = -\frac{16136711}{36393280}x_{n+5} \quad D_{67} = \frac{42775}{584892}x_{n+5} \quad D_{68} = -\frac{110857}{9358272}x_{n+5}$$

$$D_{71} = \frac{4281743}{23395680}x_{n+6} \quad D_{72} = -\frac{14890432257}{32026086400}x_{n+6} \quad D_{73} = \frac{32261113}{56149632}x_{n+6}$$

$$D_{74} = -\frac{991689}{1624700}x_{n+6} \quad D_{75} = \frac{82204981}{149732352}x_{n+6} \quad D_{76} = -\frac{50006939}{154411488}x_{n+6}$$

$$D_{77} = \frac{6740159}{72786560}x_{n+6} \quad D_{78} = -\frac{6287}{389928}x_{n+6} \quad D_{79} = \frac{251609}{93582720}x_{n+6}$$

$$D_{81} = -\frac{13381}{877338}x_{n+7} \quad D_{82} = \frac{630636759}{16013043200}x_{n+7} \quad D_{83} = -\frac{1411387}{28074816}x_{n+7}$$

$$D_{84} = \frac{136579}{2437050}x_{n+7} \quad D_{85} = -\frac{11864969}{224598528}x_{n+7} \quad D_{86} = \frac{157163}{4825359}x_{n+7}$$

$$D_{87} = -\frac{210587}{21835968}x_{n+7} \quad D_{88} = \frac{262}{146223}x_{n+7} \quad D_{89} = -\frac{14543}{46791360}x_{n+7}$$

$$D_{91} = \frac{2447}{46791360}x_{n+8} \quad D_{92} = -\frac{87438447}{64052172800}x_{n+8} \quad D_{93} = \frac{66757}{37433088}x_{n+8}$$

$$D_{94} = -\frac{3347}{1624700}x_{n+8} \quad D_{95} = \frac{603131}{299464704}x_{n+8} \quad D_{96} = -\frac{132599}{102940992}x_{n+8}$$

$$D_{97} = \frac{57241}{145573120}x_{n+8} \quad D_{98} = -\frac{61}{779856}x_{n+8} \quad D_{99} = \frac{893}{62388480}x_{n+8}$$