DEVELOPMENT OF BLOCK HYBRID BACKWARD DIFFERENTIATION FORMULAE FOR SOLVING SOME CLASSES OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

The need for numerical solutions of second order ordinary differential equations cannot be over emphasized, as mathematical problems of it kind results from numerous filed such as science, engineering among others. In this research, we focus on the derivation of a k-step Block Hybrid Backward Differentiation Formulae of order (k+1) for the general solution of second order ordinary differential equations. The new methods were derived using the procedure of collocation and interpolation of power series at some selected grid and off-grid points. The methods were used to compute the solution of linear and nonlinear systems in a block by some discrete schemes obtained from the continuous schemes which are combined and implemented. The methods were tested on some classes of second order ordinary differential equations, the results indicated that all the methods has a maximum error of 10^{-9} and as step number increases methods give more accurate results with small errors.

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CHAPTER ONE

2.0 INTRODUCTION

1.1 Background to the Study

Numerical analysis has always been used in different fields such as in Sciences, Medicines, Engineering, and in diverse facet of life. In recent times, the integration of Ordinary Differential Equations (ODEs) are investigated using some kind of block methods. Development of Linear Multi-Step Method (LMM) for solving ODE can be generated using methods such as Taylor's series, numerical interpolation, numerical integration and collocation method, which are restricted by an assumed order of convergence. Milne (1953) proposed block methods for solving ODEs. The Milne's idea of proceeding in blocks was developed by Rosser (1967).

The general second-order ordinary differential equation of the form

$$y'' = f\left(x, y, y'\right),\ \ y\left(a\right) = y_o,\ \ y'\left(a\right) = \delta_o \tag{1.1}$$

where f satisfies a Lipschitz condition as given in Henrici (1962), is encountered in several areas of engineering and science, such as circuit theory, control theory, chemical kinetics, and biology. The approach in providing solution to (1.1) is to first convert the problem to a system of first order ODE and then solves using numerical method like the R-K method and liner multistep methods (Lambert, 1973 and 1991).

The studies on direct method for higher order ODEs reveal the advantages in speed and accuracy. The objective of numerical analysis is to solve complex numerical problems like stiff equation using only the simple operations of arithmetic, to develop and evaluate methods for computing numerical results from given data. Mehrkanoon (2011) and Ramos (2017) investigate the complicated computational work and lengthy finishing time of numerical simulations. The backward differentiation formula (BDF) is a component of the family of implicit linear multistep methods for the numerical integration of ordinary differential equations. They are characterized by a single function evaluation point and region of absolute stability. They are linear multistep method that, for a given function and point, approximate the derivative of that function using information from already computed points, thereby increasing the accuracy of the approximation. These methods can be recommended for the general solution of second order ordinary differential equations.

Linear multi-step methods constitute a power class of numerical procedures for solving first order ordinary differential equations. The multi-step methods employ idea that is of different from that of one- step methods.

Solving ordinary differential equation arise as a result of the fact that certain differential equation does not have a theoretical solution or the theoretical solution is very complex to obtain. Numerical method provides approximate solution to such differential equation. Numerical Solution of Ordinary Differential Equations (ODEs) is the most important technique ever developed since the advent of computers. Especially methods for the numerical solutions of the special order ordinary differential equations.

1.2 Statement of the Research Problem

Numerical analyst are usually faced with the challenge of obtaining starting or initial values for Linear Multistep Methods (LMM) at step number $k \geq 2$. Prior before now, one step methods like Runge-Kutta and trapezoidal methods were used to obtain the starting values for such methods. The hybrid method is not exempted from this problem as it shares the same standard methods. In order to solve ordinary differential equations using the generalized Adams method. Adamu *et al.* (2019) introduced some additional schemes which do not have direct link to the main discrete schemes. Such a combination may lead to stability problem and poor results. since these additional schemes introduced were not from the same continuous schemes.

1.3 Scope and Limitation of the Study

This research focuses majorly on the developing k-step of block hybrid backward differentiation formulae(BHBDF) for solving some classes of second order ordinary differential equations (ODEs) by incorporating two off-grid collocation points for the solution. The performance of these schemes in the solution of differential equations shall be verified and it is limited to numerical solutions of $k = 2, 3, 4$ with the two off grid points using the numerical scheme obtained.

1.4 Aim and Objectives of the Study

The aim of this study is to develop block hybrid backward differentiation formulae for solving some classes of second order ODEs. Hence the following objectives:

1. Construct block hybrid BDF for step numbers k=2,3,4 incorporating 2-off-grid points at interpolation.
2. Perform convergence analysis on the proposed methods in terms of order, error constant, zero stability, consistency and convergence.
3. The applications of the developed methods to solve some numerical problems.

1.5 Significance of the Study

Certain numerical problems arise in science and engineering in the form of differential equations which may not be solvable analytically. Hence, BDF is recommended for approximate numerical solutions of such problems, because of their infinite region of absolute stability which allows them to take larger step sizes that would be with explicit methods.

1.6 Justification of the Study

The research would contribute to numerical analysis through the formulation of new classes of efficient consistent block hybrid backward differentiation formulae for the direct solution of ordinary differential equations.

1.7 Definition of Terms

1.7.1 Linear Multistep Method (LMM): Let y_n be an approximation to $y(x_n)$ and let $f_n = (x_n, y_n)$ .If a computational method or determining the sequence $\{y_n\}$ takes the form of the linear relationship between y_{n+j}, f_{n+j} i.e $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$ (1.2)Then (1.2) is a linear multistep method (LMM) of step number k.

1.7.2 Order of LMM: The differential operator $L[y(x):h] = \sum_{j=0}^{k} [\alpha_j y(x_n + jh) - h\beta_j y'(x_n + jh)]$ and the associated LMM are said to be of order p if $C_O = C_1 = C_2 = \ldots C_p = 0,\ C_{p+1} \neq 0$

1.7.3 Error Constant: The term C_{p+1} is called error constant and it implies that the local truncation error is given by $E_{n+k} = C_{p+1} h^{p+1} y^{p+1}(x_n) + 0(h^{p+2})$

1.7.4 Consistency of LMM: A Linear Multistep Method is said to be consistent if it has order $p \geq 1$

1.7.5 Zero stability of Hybrid Block Method: The Hybrid Block Method is said to be zero stable if the roots of R of the characteristic polynomial $\overline{p}(R) = \det[RA^o - A']$ satisfies $|R| \leq 1$ and every root with $|R_o| \leq 1$ has multiplicity not exceeding two in the limit as $h \to 0$

1.7.6 Absolute Stability: The linear multi-step method is said to be absolutely stable for a given h and for all h the root r_s of

$\pi(r,h) = p(r) - h\delta(r)$ satisfy $|r_s| < 1$ $s = 1,2,3,...,k$ and it is absolutely unstable for that h otherwise

1.7.7 Convergence of LMM: The linear Multistep Method is said to be convergent if and only if it is both consistent and zero stable.

1.7.8 Collocation Point: Collocation point is a point at which a derivative function is evaluated.

1.7.9 Interpolation Point: Interpolation point is a point at which the solution function is evaluated.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Existing Methods

Differential equations first came in to existence with the invention of calculus by Newton and Leibniz. Isaac Newton (1671) listed three kinds of differential equations:

$$\frac{dy}{dx} = f(x) \tag{2.1}$$

$$\frac{dy}{dx} = f(x, y) \tag{2.2}$$

$$x_1 \frac{\partial y}{\partial x_1} + x_2 \frac{\partial y}{\partial x_2} = y \tag{2.3}$$

In all the three classes, y is an unknown of x (or of x_1 and x_2),and f is a given function. He solves these equations using infinite series and discusses the non-uniqueness of solutions.

The first two classes contained only ordinary derivatives of one or more dependent variables, and are called ordinary differential equations (ODE). The third class involved the partial derivatives of one dependent variable which is known as system of partial differential equations (PDEs). Jacob Bernoulli proposed the Bernoulli differential equation in 1695. This is an ordinary differentiation of the form $y^1 + p(x)y = Q(x)y^n$.

In 17th century, Newton, Leibinitz and Bernoulli solved simple differential equations of first and second order arising from geometry and mechanics, Newton (1967).

Differential equations are among the most important mathematical tools used in producing models in physical sciences, Biological sciences, and Engineering. Over the years, several researchers developed methods in finding analytical solutions of IVP in ODEs of the form.

$$y'' = f\left(x, y, y'\right), y(x_0) = y_0, y'(x_0) = y'_0 \tag{2.4}$$

The improvement of numerical methods for the solution of Initial Value Problems (lVPs) in ODEs of the form (2.4) gave mount to two major discrete variable methods namely: single step (one step) methods and multistep methods, most especially the linear multistep method. The single step methods are very low order of accuracy and they are suitable for first order IVPs of ODEs. Such as Euler's methods, Runge-kutta methods etc.

The numerical solution of higher order single step methods such as Runge-kutta, in terms of the number of function evaluation per step, is sacrificed since more function evaluations are required. Hence, solving (2.4) using any single step methods, means reducing it to an equivalent system of first order IVPs in ODEs which increase the scale of the problem, thus increasing its size, reducing to first order is ineffective due to computational burden and also uneconomical arising from computer time wastage and gives results of low accuracy.

However, Linear Multistep Methods include methods such as Numerov method, Adams-Bashforth method, Adam-Multon method. These methods give more accuracy and are appropriate for the direct solution of (2.4) without necessarily reducing it to an equivalent system of first order IVPs of ODES.

Ordinary Differential Equations of the form (2.4) are examined by some authors including Awoyemi (1992), Lambert and Wastson (1976), Lambert (1973, 1991), Fatunla (1988), Areo and Adeniyi (2014), Adamu *et al*. (2019) and Ra'ft *et al.*(2020) among others, by first reducing them to an equivalent system of first order ordinary differential equations and then using any appropriate numerical method to solve the resultant system. The disadvantage of this is that it consumes more time, human efforts and computer program to check the accuracy of these methods are usually complicated (Adamu*et al*., 2019).

More also, in consideration of these setbacks, we considered a method that can solve LMM without reduction. Some prominent scholars have made efforts to solve higher order initial value problems of second order ordinary differential equations by a number of different methods, these includes the work Adesanya *et al.* (2008), Adeyefa *et al.* (2014), Abdelrahim *et al.* (2016), Adamu *et al.* (2019) and Ibrahim, Sunday and Pius (2020) among others. The direct methods are self-starting methods which are formulated in terms of LMMs called block methods. The block method offers the traditional advantage single step methods e.g. Rung-Kutta methods of been self starting and allow easy change of step length. Another important attribute of the block method is that all the discrete schemes are of uniform order and are obtained from a single continuous formula unlike the non-starting predictor corrector technique.

Ibrahim *et al*. (2020) construct two-step second derivative hybrid block backward Differentiation formula. The newly proposed scheme was derived based on interpolation and collocation approach. The discrete schemes were obtained from the continuous schemes. The derived method is applied to solve non-linear systems of stiff ordinary differential equations. Numerical experiments show that the method is suitable for stiff differential equations. In this research, we shall adopt the block method approach to formulate a second order numerical scheme using power series approximation as basis function.

Other numerical methods that are useful while solving ODEs are the collation methods and hybrid methods. In mathematics, collocation method for ordinary differential equation is a method for the numerical solution of ordinary differential equations, partial differential equations and integral equations. Collocation methods were used over the past decades in search of solution to a wide class of ordinary differential equations, partial differential equations, Integro-differential equations and functional equations. The attractiveness of such methods is owing to their abstract simplicity and also large applicability. The collocation is dates back to 1930s. According to Popov *et al.* (2017), the method was first proposed by Frazer, Jones and Skan (1938). The work of Frazer *et al.* was dedicated to the solution of PDEs. Collocation at the family of orthogonal polynomials is often called orthogonal collocation. Orthogonal collocation is the method for the numerical solution of partial differential equations. It uses collocation at the zeros of some orthogonal polynomials to transform the partial differential equation (PDE) to a set of ordinary differential equations (ODEs). The ODE can then be solved by any method (Ramos, 2017).

Chebyshev orthogonal collocation methods are described by Fox and Parker (1968). Special collocation methods are very much related to this form of collocation, Henrici (1962). There is a quick improvement as reported in the literature on the use of collocation methods on the use of numerical solutions of first order ODEs. The multistep collocation techniques involve obtaining solution of a set of function of a linear combination of a function known as the trial function. The analytical solution of an IVP is assumed to be approximated by the basis function. The linear combination of this basis is required to satisfy the approximation at some certain grid points called the collocation points.

The hybrid method has been anticipated in the literature. The methods share the property of utilizing data at other points other than the points the step points $\left\{x_{n+j;}x_{n+j=}x_{n+jh}\right\}$ while retraining uniqueness of the continuous linear multistep methods.

The Linear Multistep method me is helpful in reducing the step number of a method and still remains zero stable. According to Lambert (1973), hybrid method was first introduced by (Gragg and Shetter, 1964, as cited in Shchelchkov and Skrypnik, 2017).

The Method involves the determination of an approximate solution in a suitable set of functions, sometimes called basis function. According to Lambert (1973), hybrid method is not a method in its own accurate since particular predictors were needed to estimate the solution of the off-step point and the derivative function as well. In view of the disadvantage mentioned above, many researchers focused on efforts in improving the numerical solution of Initial Value Problems (lVPs) of ODEs. One of the outcomes is the development of a class of methods called Block method. The contribution of Bolaji (2017) also proposed for a family of Hybrid Backward Differentiation Formulae and a three step Hybrid Linear Multistep method for a direct solution of second and third order ODEs and the solution of second order IVPs.

2.2 Collocation method

A collocation method can simply be described as a method, which involves the determination of an approximate solution in a suitable set of functions, sometimes called basis function. The approximate solution is required to satisfy the initial or boundary conditions along with the differential equations (2.4) at certain points called the collocation points.

Collocation methods have been used to solve integral equation for more than sixty years. More recently, the so-called h-,p- and hp versions of the standard finite Element Method have attracted the interest of many investigators (Yahaya and Tijjani, 2015) The accuracy of the h- inversion is achieved by refining the mesh size; and the p-version improves its accuracy by increasing the polynomial degree.

Obviously, over the past years, collocation methods evolved as valuable methods for the solution of abroad class of problems covering ordinary and partial differential equations, functional equations and Butcher (2008) first proposed the collocation method, specifically intended for the solution of partial differential equations in two variables, with collocation being applied in two variables, with collocation being applied in one variable for each fixed value of the second. This actually is a method of lines procedure. The work of Kayode and Obarhua (2017) was dedicated to the solution of ODEs. While the applicability of collocation method to the solution of partial differential equations was mentioned in (Kayode and Obarhua, 2017), not only discussed collocation for both ordinary and partial differential equations, but also provided some numerical examples. These methods have in common the option of polynomial for the basis function.

2.3 Block Methods

The narrative property of the method that can be briefly discuss this chapter is that of simultaneously producing approximations to the solution of initial value problem at k points $x_{n+1,}x_{n+2,}x_{n+N}$. Although these methods will be formulated in terms of linear multistep methods, it can be observed that they are equivalent to certain Runge- Kutta method and preserve the traditional Runge – Kutta advantage of being self –starting and permitting easy change of step length (Lambert,1973).Their advantage over conventional Runge – Kutta method lies in the fact that they are less expensive in terms of function evaluations forgiven order blocks method appear to have been first proposed by Miln (1953),who advocated their use only as a means of obtaining starting values of corrector of block method consists of a set of all new functions values which are evaluated during each application of the relative formula to producek new set of values of solution in each computational step. (Akinfenwa, 2011). Although these methods is formulated in terms of linear multi-step methods, it can be observed that they are equivalent to certain Runge- Kutta advantage of being self- starting and permitting easy change of step length Lambert (1973) their advantage over conventional Runge – Kutta method lies in the fact that they are less expensive in terms of function evaluations forgiven order method, appear to have been first proposed by Miln (1953), who advocated their use only as a means of obtaining starting values for predictor – corrector algorithms (Areo and Adeniyi, 2014).

2.4 Hybrid Methods

According to Kayode and Obarhua(2017) numerical analysis has over the years been determined on solution at the grid points ignoring what happens at other points than the grid points. Searching for higher order numerical methods has led to researchers throwing in additional off-step points in the process of formulation. Methods formulated using this approach are called hybrid methods, they preserve the self-starting property of Runge-Kutta methods as well as being able to provide more solution at a single application. They are also said to capable of overcoming Dalquist barrier theorem which states that a linear multistep method cannot have order greater than k+1 for k odd and k + 2 for k even. There have been successful methods developed in this area too. Like the methods in (Areo and Adeniyi, 2014; Badmus, Yahaya and Subair, 2014; Kuboye and Omar, 2015 and Kayode and Obarhua, 2017).

2.7 TheBackward Differentiation Formula (BDF)

Backward differentiation formula (BDF) is a linear multistep method suitable for solving differential equations and stiff initial value problems. The Backward Differentiation Formula is an example implicit multistep method with a strange uniqueness of function evaluation at a single point. Method was first introduced by Curtiss and Hirschfelder (1952). Generally written as:

$$\sum_{j=0}^{v} \alpha_j y_{n+j} = h\beta_v f(x_{n+v}, y_{n+v}) \quad (2.5)$$

$\alpha_v = 1, \alpha_{j,} \left(j = 0,1,\ldots \text{v}-1 \right)$ and β_v are unknown coefficient to be exclusively determined and h is the step sizes.

There are other modifications of this method such as the blended backward differentiation formula and the extended backward differentiation formula. The backward differentiation formula of order k is said to be A-stable up to order 2.

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Derivation of the Numerical Schemes

We present the derivation of some Hybrid Backward Differentiation Formula (HBDF) for solving some classes of second-order ordinary differential equations of the form

$$\frac{d^2y(x)}{dx^2} = f\left(x, y, \frac{dy(x)}{dx}\right) \tag{3.1}$$

coupled with appropriate initial conditions

$$y(x_0) = \varphi_1, \frac{dy(x_0)}{dx} = \varphi_2 \tag{3.2}$$

where f is a continuous function $f: \mathbb{R}^{n+1} \to \mathbb{R}^n$, x_0 is the initial point, $y \in \mathbb{R}$ is an n −dimensional vector, x is a scalar variable, φ_1 and φ_2 are the initial values.

In this research, we seek to develop numerical schemes in the form of HBDF as:

$$Y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_v(x) y_{n+v} + h^2 \beta_k(x) f_{n+k} \tag{3.3}$$

where h is the chosen step size and $\alpha_j(x): j = 0,1,2, \dots, k$, $\alpha_v(x), \beta_k(x)$ are unknown continuous coefficients to be determined, $\alpha_k = 1$ and $\beta_k \neq 0$. In this study, we will derive HBDF for the step numbers $k = 2,3,4$ step numbers of the proposed method using power series function as the basis function.

3.2 Specifications of the Method

3.2.1 2-Step block hybrid backward differentiation formulae (2SBHBDF)

We seek an approximation of the form;

$$Y(x) = \sum_{j=0}^{t+c-1} p_j x^j \tag{3.4}$$

where t is the interpolation points, c is the collocation points and p_j are unknown coefficients to be determined. Then, we take

$$Y(x) = y_{n+j}, j = 0,1,2, \ldots, k-1 \tag{3.5}$$

$$Y''(x_{n+k}) = f_{n+k} \tag{3.6}$$

To derive 2SBHBDF, we take $t = 4$, $c = 1$and $x \in [x_n, x_{n+2}]$. Therefore, (3.4) becomes;

$$Y(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 \tag{3.7}$$

Interpolating (3.5) at $x_{n+i}; i = 0, \frac{1}{2}, 1, \frac{3}{2}$ and collocate (3.6) at $x_{n+i}; i = 2$. This results in a system of equations;

$$D\psi = Y \tag{3.8}$$

where

$\psi = \left(\alpha_0, \alpha_{\frac{1}{2}}, \alpha_1, \alpha_{\frac{3}{2}}, \beta_2\right)^T, Y = \left(y_n, y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, f_{n+2}\right)^T$ and the matrix D of the proposed method is expressed as:

$$D=\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 1 & x_{n+\frac{3}{2}} & x^2_{n+\frac{3}{2}} & x^3_{n+\frac{3}{2}} & x^4_{n+\frac{3}{2}} \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2} \end{pmatrix}$$

Solving (3.8) using matrix inversion approach with the aid of Maple 2017 software package to obtain the values of the continuous coefficients as;

$$\left.\begin{aligned} \alpha_0 &= \frac{16}{35}\frac{x^4}{h^4}-\frac{284}{105}\frac{x^3}{h^3}+\frac{184}{35}\frac{x^2}{h^2}-\frac{421}{105}\frac{x}{h}+1 \\ \alpha_{\frac{1}{2}} &=-\frac{8}{5}\frac{x^4}{h^4}+\frac{44}{5}\frac{x^3}{h^3}-\frac{72}{5}\frac{x^2}{h^2}+\frac{36}{5}\frac{x}{h} \\ \alpha_1 &= \frac{64}{35}\frac{x^4}{h^4}-\frac{332}{35}\frac{x^3}{h^3}+\frac{456}{35}\frac{x^2}{h^2}-\frac{153}{35}\frac{x}{h} \\ \alpha_{\frac{3}{2}} &=-\frac{24}{35}\frac{x^4}{h^4}+\frac{356}{105}\frac{x^3}{h^3}-\frac{136}{35}\frac{x^2}{h^2}+\frac{124}{105}\frac{x}{h} \\ \beta_2 &= \frac{2}{35}\frac{x^4}{h^2}-\frac{6}{35}\frac{x^3}{h}+\frac{11}{70}x^2-\frac{3}{70}xh \end{aligned}\right\} \tag{3.9}$$

The values of the continuous coefficients are then substituted into the proposed method in (3.3) to obtain;

$$y(x) = \alpha_0(x)y_n + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + \alpha_1(x)y_{n+1} + \alpha_{\frac{3}{2}}(x)y_{n+\frac{3}{2}} + \beta_2(x)f_{n+2} \tag{3.10}$$

Expressing (3.10) further gives the continuous form of the 2SHBDF with 2-offstep interpolation points as;

$$y(x)=\left(\frac{16}{35}\frac{x^4}{h^4}-\frac{284}{105}\frac{x^3}{h^3}+\frac{184}{35}\frac{x^2}{h^2}-\frac{421}{105}\frac{x}{h}+1\right)y_n$$
$$+\left(-\frac{8}{5}\frac{x^4}{h^4}+\frac{44}{5}\frac{x^3}{h^3}-\frac{72}{5}\frac{x^2}{h^2}+\frac{36}{5}\frac{x}{h}\right)y_{n+\frac{1}{2}}$$
$$+\left(\frac{64}{35}\frac{x^4}{h^4}-\frac{332}{35}\frac{x^3}{h^3}+\frac{456}{35}\frac{x^2}{h^2}-\frac{153}{35}\frac{x}{h}\right)y_{n+1}$$
$$+\left(-\frac{24}{35}\frac{x^4}{h^4}+\frac{356}{105}\frac{x^3}{h^3}-\frac{136}{35}\frac{x^2}{h^2}+\frac{124}{105}\frac{x}{h}\right)y_{n+\frac{3}{2}}$$
$$+\left(\frac{2}{35}\frac{x^4}{h^2}-\frac{6}{35}\frac{x^3}{h}+\frac{11}{70}x^2-\frac{3}{70}xh\right)f_{n+2} \quad (3.11)$$

Evaluating (3.11) at $x=x_{n+2}$, gives the discrete scheme as

$$y_{n+2}=-\frac{11}{35}y_n+\frac{8}{5}y_{n+\frac{1}{2}}-\frac{114}{35}y_{n+1}+\frac{104}{35}y_{n+\frac{3}{2}}+\frac{3}{35}h^2f_{n+2} \quad (3.12)$$

To obtain the sufficient schemes required, we obtain the first derivative of (3.11) and evaluate the continuous function at $x=x_n, x=x_{n+\frac{1}{2}}, x=x_{n+1}, x=x_{n+\frac{3}{2}}, x=x_{n+2}$ to obtain;

$$\left.\begin{aligned}
hz_n &=-\frac{421}{105}y_n+\frac{36}{5}y_{n+\frac{1}{2}}-\frac{153}{35}y_{n+1}+\frac{124}{105}y_{n+\frac{3}{2}}-\frac{3}{70}h^2f_{n+2}\\
hz_{n+\frac{1}{2}} &= -\frac{58}{105}y_n-\frac{7}{5}y_{n+\frac{1}{2}}+\frac{86}{35}y_{n+1}-\frac{53}{105}y_{n+\frac{3}{2}}+\frac{1}{70}h^2f_{n+2}\\
hz_{n+1} &=\frac{23}{105}y_n-\frac{8}{5}y_{n+\frac{1}{2}}+\frac{19}{35}y_{n+1}+\frac{88}{105}y_{n+\frac{3}{2}}-\frac{1}{70}h^2f_{n+2}\\
hz_{n+\frac{3}{2}} &=-\frac{34}{105}y_n+\frac{9}{5}y_{n+\frac{1}{2}}-\frac{162}{35}y_{n+1}+\frac{331}{105}y_{n+\frac{3}{2}}+\frac{3}{70}h^2f_{n+2}\\
hz_{n+2} &=-\frac{17}{21}y_n+4y_{n+\frac{1}{2}}-\frac{53}{7}y_{n+1}+\frac{92}{21}y_{n+\frac{3}{2}}+\frac{5}{14}h^2f_{n+2}
\end{aligned}\right\} \quad (3.13)$$

where$z$ is the first derivative of y.

Likewise, we further obtain the second derivatives of (3.11), thereafter, evaluating at $x=x_{n+\frac{1}{2}}, x=x_{n+\frac{3}{2}}$ to obtain;

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= \frac{11}{21}y_n + \frac{3}{7}y_{n+1} + \frac{1}{21}y_{n+\frac{3}{2}} - \frac{5}{36}h^2 f_{n+\frac{1}{2}} - \frac{1}{252}h^2 f_{n+2} \\ y_{n+\frac{3}{2}} &= \frac{13}{37}y_n - \frac{63}{37}y_{n+\frac{1}{2}} + \frac{13}{37}y_{n+1} - \frac{35}{148}h^2 f_{n+\frac{3}{2}} - \frac{11}{148}h^2 f_{n+2} \end{aligned}\right\} \qquad (3.14)$$

The equations (3.12) – (3.14) are the proposed2SBHBDF for solving second order ordinary differential equations.

3.2.2 3-Step block hybrid backward differentiation formulae (3SBHBDF)

To derive a 3SBHBDF, we take $t = 5, c = 1$ and $x \in [x_n, x_{n+3}]$. Therefore, (3.4) becomes;

$$Y(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + p_5 x^5 \qquad (3.15)$$

Interpolating (3.5) at $x_{n+i}; i = 0, 1, 2, \frac{5}{2}, \frac{11}{4}$ and collocate (3.6) at $x_{n+i}; i = 3$. This results in a system of equations;

$$D\psi = Y \qquad (3.16)$$

where

$\psi = \left(\alpha_0, \alpha_1, \alpha_2, \alpha_{\frac{5}{2}}, \alpha_{\frac{11}{4}}, \beta_3\right)^T$, $Y = \left(y_n, y_{n+1}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+\frac{11}{4}}, f_{n+3}\right)^T$ andthe matrix D of the proposed method is expressed as

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 \\ 1 & x_{n+\frac{5}{2}} & x_{n+\frac{5}{2}}^2 & x_{n+\frac{5}{2}}^3 & x_{n+\frac{5}{2}}^4 & x_{n+\frac{5}{2}}^5 \\ 1 & x_{n+\frac{11}{4}} & x_{n+\frac{11}{4}}^2 & x_{n+\frac{11}{4}}^3 & x_{n+\frac{11}{4}}^4 & x_{n+\frac{11}{4}}^5 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2} & 20x_{n+2} \end{pmatrix}$$

We proceed by solving (3.16) using matrix inversion method with the aid of Maple 2017 software to obtain the continuous coefficients;

$$\left.\begin{aligned}
\alpha_0 &= -\frac{7}{330}\frac{x^5}{h^5}+\frac{109}{440}\frac{x^4}{h^4}-\frac{2963}{2640}\frac{x^3}{h^3}+\frac{2157}{880}\frac{x^2}{h^2}-\frac{3373}{1320}\frac{x}{h}+1 \\
\alpha_1 &= \frac{7}{45}\frac{x^5}{h^5}-\frac{233}{140}\frac{x^4}{h^4}+\frac{16613}{2520}\frac{x^3}{h^3}-\frac{3209}{280}\frac{x^2}{h^2}+\frac{1859}{252}\frac{x}{h} \\
\alpha_2 &= -\frac{79}{90}\frac{x^5}{h^5}+\frac{343}{40}\frac{x^4}{h^4}-\frac{21563}{720}\frac{x^3}{h^3}+\frac{3479}{80}\frac{x^2}{h^2}-\frac{1529}{72}\frac{x}{h} \\
\alpha_{\frac{5}{2}} &= \frac{16}{9}\frac{x^5}{h^5}-\frac{84}{5}\frac{x^4}{h^4}+\frac{2522}{45}\frac{x^3}{h^3}-\frac{386}{5}\frac{x^2}{h^2}+\frac{1628}{45}\frac{x}{h} \\
\alpha_{\frac{11}{4}} &= -\frac{512}{495}\frac{x^5}{h^5}+\frac{3712}{385}\frac{x^4}{h^4}-\frac{109376}{3465}\frac{x^3}{h^3}+\frac{16448}{385}\frac{x^2}{h^2}-\frac{13696}{693}\frac{x}{h} \\
\beta_3 &= \frac{1}{30}\frac{x^5}{h^3}-\frac{11}{40}\frac{x^4}{h^2}+\frac{197}{240}\frac{x^3}{h}-\frac{83}{80}x^2+\frac{11}{24}xh
\end{aligned}\right\} \quad (3.17)$$

The values of the continuous coefficients are then substituted in to proposed method in (3.3) to obtain

$$\begin{aligned} y(x) = {} & \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_{\frac{5}{2}}(x)y_{n+\frac{5}{2}} + \alpha_{\frac{11}{4}}(x)y_{n+\frac{11}{4}} \\ & + \beta_3(x)f_{n+3} \end{aligned} \quad (3.18)$$

The values of the continuous coefficients (3.17) are then substituted in to the proposed method (3.3) to obtain (3.18).

Expressing (3.18) further gives the continuous form of the 3SHBDF with 2-step off-step interpolation point as

$$y(x) = \left(-\frac{7}{330}\frac{x^5}{h^5} + \frac{109}{440}\frac{x^4}{h^4} - \frac{2963}{2640}\frac{x^3}{h^3} + \frac{2157}{880}\frac{x^2}{h^2} - \frac{3373}{1320}\frac{x}{h} + 1\right) y_n$$
$$+ \left(\frac{7}{45}\frac{x^5}{h^5} - \frac{233}{140}\frac{x^4}{h^4} + \frac{16613}{2520}\frac{x^3}{h^3} - \frac{3209}{280}\frac{x^2}{h^2} + \frac{1859}{252}\frac{x}{h}\right) y_{n+1}$$
$$+ \left(-\frac{79}{90}\frac{x^5}{h^5} + \frac{343}{40}\frac{x^4}{h^4} - \frac{21563}{720}\frac{x^3}{h^3} + \frac{3479}{80}\frac{x^2}{h^2} - \frac{1529}{72}\frac{x}{h}\right) y_{n+2}$$
$$+ \left(\frac{16}{9}\frac{x^5}{h^5} - \frac{84}{5}\frac{x^4}{h^4} + \frac{2522}{45}\frac{x^3}{h^3} - \frac{386}{5}\frac{x^2}{h^2} + \frac{1628}{45}\frac{x}{h}\right) y_{n+\frac{5}{2}}$$
$$+ \left(-\frac{512}{495}\frac{x^5}{h^5} + \frac{3712}{385}\frac{x^4}{h^4} - \frac{109376}{3465}\frac{x^3}{h^3} + \frac{16448}{385}\frac{x^2}{h^2} - \frac{13696}{693}\frac{x}{h}\right) y_{n+\frac{11}{4}}$$
$$+ \left(\frac{1}{30}\frac{x^5}{h^3} - \frac{11}{40}\frac{x^4}{h^2} + \frac{197}{240}\frac{x^3}{h} - \frac{83}{80}x^2 + \frac{11}{24}xh\right) f_{n+3} \qquad (3.19)$$

Evaluating (3.19) at $x = x_{n+3}$ gives the discrete scheme as

$$y_{n+3} = \frac{1}{440}y_n - \frac{11}{420}y_{n+1} + \frac{41}{120}y_{n+2} - \frac{28}{15}y_{n+\frac{5}{2}} + \frac{2944}{1155}y_{n+\frac{11}{4}} + \frac{1}{40}h^2 f_{n+3} \quad (3.20)$$

To obtain the sufficient schemes required, we obtain the first derivative of (3.19) and evaluate the continuous function at $x = x_n, x = x_{n+1}, x = x_{n+2}, x = x_{n+\frac{5}{2}}, x = x_{n+\frac{11}{4}} x = x_{n+3}$ to obtain;

$$
\left.
\begin{aligned}
hz_n &=-\frac{3373}{1320}y_n+\frac{1859}{252}y_{n+1}-\frac{1529}{72}y_{n+2}+\frac{1628}{45}y_{n+\frac{5}{2}}\\
&\quad-\frac{13696}{693}y_{n+\frac{11}{4}}+\frac{11}{24}h^2f_{n+3}\\
hz_{n+1} &=-\frac{119}{880}y_n-\frac{461}{280}y_{n+1}+\frac{1393}{240}y_{n+2}-\frac{42}{5}y_{n+\frac{5}{2}}\\
&\quad+\frac{5056}{1155}y_{n+\frac{11}{4}}-\frac{7}{80}h^2f_{n+3}\\
hz_{n+2} &=\frac{1}{88}y_n-\frac{71}{420}y_{n+1}-\frac{299}{120}y_{n+2}+\frac{68}{15}y_{n+\frac{5}{2}}\\
&\quad-\frac{2176}{1155}y_{n+\frac{11}{4}}+\frac{1}{40}h^2f_{n+3}\\
hz_{n+\frac{5}{2}} &=-\frac{13}{3520}y_n+\frac{31}{672}y_{n+1}-\frac{161}{192}y_{n+2}-\frac{53}{30}y_{n+\frac{5}{2}}\\
&\quad+\frac{2054}{385}y_{n+\frac{11}{4}}+\frac{77}{2560}h^2f_{n+3}\\
hz_{n+\frac{11}{4}} &=\frac{133}{28160}y_n-\frac{1507}{26880}y_{n+1}+\frac{2079}{2560}y_{n+2}-\frac{1463}{240}y_{n+\frac{5}{2}}\\
&\quad+\frac{2054}{385}y_{n+\frac{11}{4}}+\frac{77}{2560}h^2f_{n+3}\\
hz_{n+3} &=\frac{31}{2640}y_n-\frac{337}{2520}y_{n+1}+\frac{1207}{720}y_{n+2}-\frac{74}{9}y_{n+\frac{5}{2}}\\
&\quad+\frac{23104}{3465}y_{n+\frac{11}{4}}+\frac{47}{240}h^2f_{n+3}
\end{aligned}
\right\}
\quad (3.21)
$$

where$z$ is the first derivative of y.

We further obtain the second derivatives of (3.19), thereafter, evaluating at $x = x_{n+\frac{5}{2}}, x = x_{n+\frac{11}{4}}, x = x_{n+2}$ to obtain;

$$
\left.
\begin{aligned}
y_{n+\frac{5}{2}} &=\frac{141}{143968}y_n-\frac{617}{45808}y_{n+1}+\frac{4727}{13088}y_{n+2}+\frac{20512}{31493}y_{n+\frac{11}{4}}\\
&\quad-\frac{45}{818}h^2f_{n+\frac{5}{2}}+\frac{21}{13088}h^2f_{n+3}\\
y_{n+\frac{11}{4}} &=-\frac{4641}{1037312}y_n+\frac{26521}{518656}y_{n+1}-\frac{654577}{1037312}y_{n+2}+\frac{51359}{32416}y_{n+\frac{5}{2}}\\
&\quad+\frac{3465}{32416}h^2f_{n+\frac{11}{4}}-\frac{41811}{1037312}h^2f_{n+3}\\
y_{n+2} &=-\frac{273}{4961}y_n+\frac{3002}{3157}y_{n+1}-\frac{1376}{451}y_{n+\frac{5}{2}}+\frac{109568}{34727}y_{n+\frac{11}{4}}\\
&\quad-\frac{360}{451}h^2f_{n+2}-\frac{3}{41}h^2f_{n+3}
\end{aligned}
\right\}
\quad (3.22)
$$

The equations (3.20) – (3.22) are the proposed3SBHBDFfor solving second order ordinary differential equations.

3.2.3 4-Step block hybrid backward differentiation formulae (4SBHBDF)

To derive 4SBHBDF, we take $t = 6$, $c = 1$and $x \in [x_n, x_{n+3}]$. Therefore, (3.4) becomes;

$$Y(x) = p_0 + p_1x + p_2x^2 + p_3x^3 + p_4x^4 + p_5x^5 + p_6x^6 \quad (3.23)$$

Interpolating (3.5) at $x_{n+i}; i = 0, 1, 2, 3, \frac{15}{4}, \frac{31}{8}$ and collocate (3.6) at $x_{n+i}; i = 4$. This results in a system of equations;

$$D\psi = Y \quad (3.24)$$

where

$\psi = \left(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_{\frac{15}{4}}, \alpha_{\frac{31}{8}}, \beta_4\right)^T$, $Y = \left(y_n, y_{n+1}, y_{n+2}, y_{n+3}, y_{n+\frac{15}{4}}, y_{n+\frac{31}{8}}, f_{n+4}\right)^T$ andthe matrix D of the proposed method is expressed as

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 \\ 1 & x_{n+\frac{15}{4}} & x_{n+\frac{15}{4}}^2 & x_{n+\frac{15}{4}}^3 & x_{n+\frac{15}{4}}^4 & x_{n+\frac{15}{4}}^5 & x_{n+\frac{15}{4}}^6 \\ 1 & x_{n+\frac{31}{8}} & x_{n+\frac{31}{8}}^2 & x_{n+\frac{31}{8}}^3 & x_{n+\frac{31}{8}}^4 & x_{n+\frac{31}{8}}^5 & x_{n+\frac{31}{8}}^6 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2} & 20x_{n+2} & 30x_{n+2} \end{pmatrix}$$

We proceed by solving (3.24) using matrix inversion method with the aid of Maple 2017 software to obtain the continuous coefficients;

$$
\left.
\begin{aligned}
\alpha_0 &= \frac{352}{130479}\frac{x^6}{h^6} - \frac{94388}{1957185}\frac{x^5}{h^5} + \frac{682219}{1957185}\frac{x^4}{h^4} - \frac{5070023}{3914370}\frac{x^3}{h^3} \\
&\quad + \frac{5060234}{1957185}\frac{x^2}{h^2} - \frac{1127893}{434930}\frac{x}{h} + 1 \\
\alpha_1 &= -\frac{6896}{354959}\frac{x^6}{h^6} + \frac{116406}{354959}\frac{x^5}{h^5} - \frac{1549923}{709918}\frac{x^4}{h^4} + \frac{461407}{64538}\frac{x^3}{h^3} \\
&\quad - \frac{4075758}{354959}\frac{x^2}{h^2} + \frac{2558430}{354959}\frac{x}{h} \\
\alpha_2 &= \frac{1984}{29463}\frac{x^6}{h^6} - \frac{52536}{49105}\frac{x^5}{h^5} + \frac{968068}{147315}\frac{x^4}{h^4} - \frac{1889777}{98210}\frac{x^3}{h^3} \\
&\quad + \frac{1285926}{49105}\frac{x^2}{h^2} - \frac{245799}{19642}\frac{x}{h} \\
\alpha_3 &= -\frac{1936}{9821}\frac{x^6}{h^6} + \frac{259850}{88389}\frac{x^5}{h^5} - \frac{2961029}{176778}\frac{x^4}{h^4} + \frac{7939457}{176778}\frac{x^3}{h^3} \\
&\quad - \frac{4903190}{88389}\frac{x^2}{h^2} + \frac{723850}{29463}\frac{x}{h} \\
\alpha_{\frac{15}{4}} &= \frac{237568}{324093}\frac{x^6}{h^6} - \frac{51426304}{4861395}\frac{x^5}{h^5} + \frac{282386432}{4861395}\frac{x^4}{h^4} - \frac{66307072}{441945}\frac{x^3}{h^3} \\
&\quad + \frac{872353792}{4861395}\frac{x^2}{h^2} - \frac{125833216}{1620465}\frac{x}{h} \\
\alpha_{\frac{31}{8}} &= -\frac{12320768}{21007119}\frac{x^6}{h^6} + \frac{295108608}{35011865}\frac{x^5}{h^5} - \frac{4839440384}{105035595}\frac{x^4}{h^4} \\
&\quad + \frac{4149280768}{35011865}\frac{x^3}{h^3} - \frac{4945870848}{35011865}\frac{x^2}{h^2} + \frac{427032576}{7002373}\frac{x}{h} \\
\beta_4 &= \frac{16}{1403}\frac{x^6}{h^4} - \frac{218}{1403}\frac{x^5}{h^3} + \frac{2281}{2806}\frac{x^4}{h^2} - \frac{2833}{1403}\frac{x^3}{h} + \frac{6579}{2806}x^2 - \frac{1395}{1403}xh
\end{aligned}
\right\} \qquad (3.25)
$$

The values of the continuous coefficients are then substituted in to the proposed method in (3.3) to obtain

$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2} + \alpha_3(x)y_{n+3} + \alpha_{\frac{15}{4}}(x)y_{n+\frac{15}{4}} + \alpha_{\frac{31}{8}}(x)y_{n+\frac{31}{8}} + \beta_4(x)f_{n+4}$(3.26)

The values of the continuous coefficients (3.25) are then substituted in to the proposed method (3.3) to obtain (3.26).

Expressing (3.26) further gives the continuous form of the 4SHBDF with 2-off step interpolation point as;

$$y(x) = \left(\frac{352}{130479}\frac{x^6}{h^6} - \frac{94388}{1957185}\frac{x^5}{h^5} + \frac{682219}{1957185}\frac{x^4}{h^4} - \frac{5070023}{3914370}\frac{x^3}{h^3} + \frac{5060234}{1957185}\frac{x^2}{h^2} - \frac{1127893}{434930}\frac{x}{h} + 1\right)y_n$$
$$+ \left(-\frac{6896}{354959}\frac{x^6}{h^6} + \frac{116406}{354959}\frac{x^5}{h^5} - \frac{1549923}{709918}\frac{x^4}{h^4} + \frac{461407}{64538}\frac{x^3}{h^3} - \frac{4075758}{354959}\frac{x^2}{h^2} + \frac{2558430}{354959}\frac{x}{h}\right)y_{n+1} + \left(\frac{1984}{29463}\frac{x^6}{h^6} - \frac{52536}{49105}\frac{x^5}{h^5}\right.$$
$$\left. + \frac{968068}{147315}\frac{x^4}{h^4} - \frac{1889777}{98210}\frac{x^3}{h^3} + \frac{1285926}{49105}\frac{x^2}{h^2} - \frac{245799}{19642}\frac{x}{h}\right)y_{n+2} + \left(\right.$$
$$\left. -\frac{1936}{9821}\frac{x^6}{h^6} + \frac{259850}{88389}\frac{x^5}{h^5} - \frac{2961029}{176778}\frac{x^4}{h^4} + \frac{7939457}{176778}\frac{x^3}{h^3} - \frac{4903190}{88389}\frac{x^2}{h^2} + \frac{723850}{29463}\frac{x}{h}\right)y_{n+3} + \left(\frac{237568}{324093}\frac{x^6}{h^6} - \frac{51426304}{4861395}\frac{x^5}{h^5} + \frac{282386432}{4861395}\frac{x^4}{h^4}\right.$$
$$\left. - \frac{66307072}{441945}\frac{x^3}{h^3} + \frac{872353792}{4861395}\frac{x^2}{h^2} - \frac{125833216}{1620465}\frac{x}{h}\right)y_{n+\frac{15}{4}}$$
$$+ \left(-\frac{12320768}{21007119}\frac{x^6}{h^6} + \frac{295108608}{35011865}\frac{x^5}{h^5} - \frac{4839440384}{105035595}\frac{x^4}{h^4} + \frac{4149280768}{35011865}\frac{x^3}{h^3} - \frac{4945870848}{35011865}\frac{x^2}{h^2} + \frac{427032576}{7002373}\frac{x}{h}\right)y_{n+\frac{31}{8}}$$
$$+ \left(\frac{16}{1403}\frac{x^6}{h^4} - \frac{218}{1403}\frac{x^5}{h^3} + \frac{2281}{2806}\frac{x^4}{h^2} - \frac{2833}{1403}\frac{x^3}{h} + \frac{6579}{2806}x^2 - \frac{1395}{1403}xh\right)f_{n+4} \quad (3.27)$$

Evaluating (3.27) at $x = x_{n+4}$ gives the discrete scheme as

$$y_{n+4} = -\frac{83}{652395}y_n + \frac{40}{32269}y_{n+1} - \frac{326}{49105}y_{n+2} + \frac{1256}{29463}y_{n+3} - \frac{180224}{147315}y_{n+\frac{15}{4}} + \frac{76546048}{35011865}y_{n+\frac{31}{8}} + \frac{12}{1403}h^2 f_{n+4} \quad (3.28)$$

To obtain the sufficient schemes required, we obtain the first derivative of (3.27) and evaluate the continuous function at $x = x_n, x = x_{n+1}, x = x_{n+2}, x = x_{n+3}, x = x_{n+\frac{15}{4}}, x = x_{n+\frac{31}{8}} x = x_{n+4}$ to obtain;

$$
\left.\begin{aligned}
hz_n &=-\frac{1127893}{434930}y_n+\frac{2558430}{354959}y_{n+1}-\frac{245799}{19642}y_{n+2}+\frac{723850}{29463}y_{n+3}\\
&\quad-\frac{125833216}{1620465}y_{n+\frac{15}{4}}+\frac{427032576}{7002373}y_{n+\frac{31}{8}}-\frac{1395}{1403}h^2f_{n+4}\\
hz_{n+1} &=-\frac{11803}{85095}y_n-\frac{46875}{30866}y_{n+1}+\frac{3179}{915}y_{n+2}-\frac{39391}{7686}y_{n+3}\\
&\quad+\frac{452608}{30195}y_{n+\frac{15}{4}}-\frac{53346304}{4566765}y_{n+\frac{31}{8}}+\frac{11}{61}h^2f_{n+4}\\
hz_{n+2} &=\frac{5201}{260958}y_n-\frac{102060}{354959}y_{n+1}-\frac{308381}{294630}y_{n+2}+\frac{12460}{4209}y_{n+3}\\
&\quad-\frac{2277376}{324093}y_{n+\frac{15}{4}}+\frac{80740352}{15005085}y_{n+\frac{31}{8}}-\frac{105}{1403}h^2f_{n+4}\\
hz_{n+3} &=-\frac{2891}{652395}y_n+\frac{34083}{709918}y_{n+1}-\frac{2349}{7015}y_{n+2}-\frac{27873}{19642}y_{n+3}\\
&\quad+\frac{1386496}{231495}y_{n+\frac{15}{4}}-\frac{149815296}{35011865}y_{n+\frac{31}{8}}+\frac{63}{1403}h^2f_{n+4}\\
hz_{n+\frac{15}{4}} &=\frac{25487}{41753280}y_n-\frac{274365}{45434752}y_{n+1}+\frac{5247}{157136}y_{n+2}-\frac{128975}{538752}y_{n+3}\\
&\quad-\frac{3621812}{540155}y_{n+\frac{15}{4}}+\frac{6918912}{1000339}y_{n+\frac{31}{8}}-\frac{3465}{179584}h^2f_{n+4}\\
hz_{n+\frac{31}{4}} &=-\frac{3479}{5809152}y_n+\frac{185535}{31606784}y_{n+1}-\frac{14911}{468480}y_{n+2}+\frac{558155}{2623488}y_{n+3}\\
&\quad-\frac{73501}{8052}y_{n+\frac{15}{4}}+\frac{40836448}{4566765}y_{n+\frac{31}{8}}+\frac{3255}{124928}h^2f_{n+4}\\
hz_{n+4} &=-\frac{4909}{3914370}y_n+\frac{4326}{354959}y_{n+1}-\frac{19129}{294630}y_{n+2}+\frac{36230}{88389}y_{n+3}\\
&\quad-\frac{49278976}{4861395}y_{n+\frac{15}{4}}+\frac{1027342336}{105035595}y_{n+\frac{31}{8}}+\frac{169}{1403}h^2f_{n+4}
\end{aligned}\right\}\quad(3.29)
$$

where$z$ is the first derivative of y.

We further obtain the second derivatives of (3.27), thereafter, evaluating at $x = x_{n+2}, , x = x_{n+3} x = x_{n+\frac{15}{4}}, x = x_{n+\frac{31}{8}}$to obtain;

$$
\left.\begin{aligned}
y_{n+2} &= -\frac{13643}{433269}y_n + \frac{2438765}{4059891}y_{n+1} - \frac{54635}{433269}y_{n+3} + \frac{14501888}{4765959}y_{n+\frac{15}{4}} \\
&\quad - \frac{917504}{369081}y_{n+\frac{31}{8}} - \frac{49105}{96282}h^2 f_{n+2} + \frac{4445}{96282}h^2 f_{n+4} \\
y_{n+3} &= \frac{215551}{41563715}y_n - \frac{4164615}{67842709}y_{n+1} + \frac{720603}{1340765}y_{n+2} + \frac{2791424}{14748415}y_{n+\frac{15}{4}} \\
&\quad + \frac{314966016}{955965445}y_{n+\frac{31}{8}} - \frac{88389}{268153}h^2 f_{n+3} - \frac{5103}{268153}h^2 f_{n+4} \\
y_{n+\frac{15}{4}} &= -\frac{2842763}{6898264832}y_n + \frac{84349755}{20472269824}y_{n+1} - \frac{10370943}{445049344}y_{n+2} \\
&\quad + \frac{161178325}{890098688}y_{n+3} + \frac{519683472}{619765981}y_{n+\frac{31}{8}} - \frac{4861395}{111262336}h^2 f_{n+\frac{15}{4}} \\
&\quad + \frac{8201655}{890098688}h^2 f_{n+4} \\
y_{n+\frac{31}{8}} &= \frac{177976967}{249813835776}y_n - \frac{1409110115}{203552014336}y_{n+1} + \frac{170473309}{4626182144}y_{n+2} \\
&\quad - \frac{115328331095}{499627671552}y_{n+3} + \frac{12883055113}{10734188256}y_{n+3} + \frac{35011865}{433704576}h^2 f_{n+\frac{31}{8}} \\
&\quad - \frac{2401195055}{55514185728}h^2 f_{n+4}
\end{aligned}\right\} \quad (3.30)
$$

The equations (3.28) – (3.30) are the proposed4SBHBDF for solving second order ordinary differential equations.

3.3 Analysis of Basic Properties

In this section, we address the order, error constants, consistency, stability and convergence of the developed methods.

3.3.1 Order and error constants of the developed methods

Following the works of Lambert (1973) and Fatunla (1992)**,**the Local Truncation Error (LTE) for a block method of the form (3.3) is defined with the linear operator;

$$
\mathcal{L}[y(x),h] = \sum_{j=0}^{k} \left[\alpha_j y(x+jh) - \alpha_u(x)y(x+uh) - \alpha_v(x)y(x+vh) - h^2\beta_k(x)f_{n+k}\right] \quad (3.31)
$$

We assume that $y(x)$ is sufficiently differentiable such that the linear operator defined above can be expanded as a Taylor's series about the point x. Then,

$$\mathcal{L}[y(x),h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \dots + C_q h^q y^q(x) + \dots \quad (3.32)$$

The method above will be consistent if $\mathcal{L}[y(x),h] \to 0$ as $h \to 0$. Therefore, we can compare the coefficient to have

$$\left.\begin{aligned} C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \dots + \alpha_k = \sum_{j=0}^{k} \alpha_j \\ C_1 &= (\alpha_1 + 2\alpha_2 + \dots + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \dots + \beta_k) = \sum_{j=0}^{k} (j\alpha_j - \beta_j) \\ &\vdots \\ C_q &= \frac{1}{q!}(\alpha_1 + 2^q \alpha_2 + \dots + k^q \alpha_k) - \frac{1}{(q-2)!}(\beta_1 + 2^{q-1}\beta_2 + \dots + k^{q-1}\beta_k) \end{aligned}\right\} \quad (3.33)$$

The method is consistent if $C_0 = C_1 = \dots = C_{p+1} = 0$,for $C_{p+2} \neq 0$. The constant C_{p+2} is the error constant. After defining the concept of error constant, we shall obtain the error constants of the proposed discrete hybrid block methods for $k = 2$, $k = 3$, and $k = 4$.

From2SBHBDF, we developed the proposed method in (3.12) as

$$y_{n+2} =- \frac{11}{35} y_n + \frac{8}{5} y_{n+\frac{1}{2}} - \frac{114}{35} y_{n+1} + \frac{104}{35} y_{n+\frac{3}{2}} + \frac{3}{35} h^2 f_{n+2}$$

where;

$$\alpha_0 = \frac{11}{35}, \alpha_{\frac{1}{2}} =- \frac{8}{5}, \alpha_1 = \frac{114}{35}, \alpha_{\frac{3}{2}} =- \frac{104}{35}, \alpha_2 = 1, \beta_2 = \frac{3}{35}$$

Applying (3.33), we have

$$\left.\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j = \frac{11}{35} - \frac{8}{5} + \frac{114}{35} - \frac{104}{35} + 1 = 0 \\
C_1 &= \frac{1}{1!}\begin{pmatrix} 0^1\left(\frac{11}{35}\right) + \left(\frac{1}{2}\right)^1\left(-\frac{8}{5}\right) + 1^1\left(\frac{114}{35}\right) \\ + \left(\frac{3}{2}\right)^1\left(-\frac{104}{35}\right) + (2)^1(1) \end{pmatrix} = 0 \\
C_2 &= \frac{1}{2!}\begin{pmatrix} 0^2\left(\frac{11}{35}\right) + \left(\frac{1}{2}\right)^2\left(-\frac{8}{5}\right) + 1^2\left(\frac{114}{35}\right) \\ + \left(\frac{3}{2}\right)^2\left(-\frac{104}{35}\right) + (2)^2(1) \end{pmatrix} - \left(\frac{3}{35}\right) = 0 \\
C_3 &= \frac{1}{3!}\begin{pmatrix} 0^3\left(\frac{11}{35}\right) + \left(\frac{1}{2}\right)^3\left(-\frac{8}{5}\right) + 1^3\left(\frac{114}{35}\right) \\ + \left(\frac{3}{2}\right)^3\left(-\frac{104}{35}\right) + (2)^3(1) \end{pmatrix} - 2\left(\frac{3}{35}\right) = 0 \\
C_4 &= \frac{1}{4!}\begin{pmatrix} 0^4\left(\frac{11}{35}\right) + \left(\frac{1}{2}\right)^4\left(-\frac{8}{5}\right) + 1^4\left(\frac{114}{35}\right) \\ + \left(\frac{3}{2}\right)^4\left(-\frac{104}{35}\right) + (2)^4(1) \end{pmatrix} - \frac{1}{2!}2^2\left(\frac{3}{35}\right) = 0 \\
C_5 &= \frac{1}{5!}\begin{pmatrix} 0^5\left(\frac{11}{35}\right) + \left(\frac{1}{2}\right)^5\left(-\frac{8}{5}\right) + 1^5\left(\frac{114}{35}\right) \\ + \left(\frac{3}{2}\right)^5\left(-\frac{104}{35}\right) + (2)^5(1) \end{pmatrix} - \frac{1}{3!}2^3\left(\frac{3}{35}\right) = -\frac{1}{112}
\end{aligned}\right\} \qquad (3.34)$$

Since $C_{p+2} = C_5$. The $p = 3$ implies the method is of order 3 with error constant $C_5 = -\frac{1}{112}$

For the first discrete method in (3.13);

$$\alpha_0 = \frac{421}{105}, \alpha_{\frac{1}{2}} = -\frac{36}{5}, \alpha_1 = \frac{153}{35}, \alpha_{\frac{3}{2}} = -\frac{124}{105}, \alpha_2 = 0, \beta_2 = -\frac{3}{70}, \gamma_0 = -1$$

Applying (3.33), we have

$$
\left.\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j = \frac{421}{105} - \frac{36}{5} + \frac{153}{35} - \frac{124}{105} + 0 = 0 \\
C_1 &= \frac{1}{1!}\begin{pmatrix} 0^1\left(\frac{421}{105}\right) + \left(\frac{1}{2}\right)^1\left(-\frac{36}{5}\right) + 1^1\left(\frac{153}{35}\right) \\ + \left(\frac{3}{2}\right)^1\left(-\frac{124}{105}\right) \end{pmatrix} - (-1) = 0 \\
C_2 &= \frac{1}{2!}\begin{pmatrix} \left(\frac{1}{2}\right)^2\left(-\frac{36}{5}\right) + 1^2\left(\frac{153}{35}\right) \\ + \left(\frac{3}{2}\right)^2\left(-\frac{124}{105}\right) + 2^2(0) \end{pmatrix} - 0(-1) - \left(-\frac{3}{70}\right) = 0 \\
C_3 &= \frac{1}{3!}\begin{pmatrix} \left(\frac{1}{2}\right)^3\left(-\frac{36}{5}\right) + 1^3\left(\frac{153}{35}\right) \\ + \left(\frac{3}{2}\right)^3\left(-\frac{124}{105}\right) + 2^3(0) \end{pmatrix} - \frac{1}{2!}0^2(-1) - 2\left(-\frac{3}{70}\right) = 0 \\
C_4 &= \frac{1}{4!}\begin{pmatrix} \left(\frac{1}{2}\right)^4\left(-\frac{36}{5}\right) + 1^4\left(\frac{153}{35}\right) \\ + \left(\frac{3}{2}\right)^4\left(-\frac{124}{105}\right) + 2^4(0) \end{pmatrix} - \frac{1}{3!}0^3(-1) - \frac{1}{2!}2^2\left(-\frac{3}{70}\right) = 0 \\
C_5 &= \frac{1}{5!}\begin{pmatrix} \left(\frac{1}{2}\right)^5\left(-\frac{36}{5}\right) + 1^5\left(\frac{153}{35}\right) \\ + \left(\frac{3}{2}\right)^5\left(-\frac{124}{105}\right) + 2^5(0) \end{pmatrix} - \frac{1}{4!}0^4(-1) - \frac{1}{3!}2^3\left(-\frac{3}{70}\right) = \frac{19}{1120}
\end{aligned}\right\} \quad (3.35)
$$

Since $C_{p+2} = C_5$. The $p = 3$ implies the method is of order 3 with error constant $C_5 = \frac{19}{1120}$

For the second discrete method in (3.13);

$$\alpha_0 = \frac{58}{105}, \alpha_{\frac{1}{2}} = \frac{7}{5}, \alpha_1 = -\frac{86}{35}, \alpha_{\frac{3}{2}} = \frac{53}{105}, \alpha_2 = 0, \beta_2 = \frac{1}{70}, \gamma_{\frac{1}{2}} = -1$$

Applying (3.33), we have

$$\left.\begin{array}{l}
C_0=\sum_{j=0}^{k}\alpha_j=\frac{58}{105}+\frac{7}{5}-\frac{86}{35}+\frac{53}{105}+0=0\\
C_1=\frac{1}{1!}\begin{pmatrix}0^1\left(\frac{58}{105}\right)+\left(\frac{1}{2}\right)^1\left(\frac{7}{5}\right)+1^1\left(-\frac{86}{35}\right)\\+\left(\frac{3}{2}\right)^1\left(\frac{53}{105}\right)\end{pmatrix}-(-1)=0\\
C_2=\frac{1}{2!}\begin{pmatrix}\left(\frac{1}{2}\right)^2\left(\frac{7}{5}\right)+1^2\left(-\frac{86}{35}\right)\\+\left(\frac{3}{2}\right)^2\left(\frac{53}{105}\right)+2^2(0)\end{pmatrix}-\frac{1}{2}(-1)-\left(\frac{1}{70}\right)=0\\
C_3=\frac{1}{3!}\begin{pmatrix}\left(\frac{1}{2}\right)^3\left(\frac{7}{5}\right)+1^3\left(-\frac{86}{35}\right)\\+\left(\frac{3}{2}\right)^3\left(\frac{53}{105}\right)+2^3(0)\end{pmatrix}-\frac{1}{2!}\left(\frac{1}{2}\right)^2(-1)-2\left(\frac{1}{70}\right)=0\\
C_4=\frac{1}{4!}\begin{pmatrix}\left(\frac{1}{2}\right)^4\left(\frac{7}{5}\right)+1^4\left(-\frac{86}{35}\right)\\+\left(\frac{3}{2}\right)^4\left(\frac{53}{105}\right)+2^4(0)\end{pmatrix}-\frac{1}{3!}\left(\frac{1}{2}\right)^3(-1)-\frac{1}{2!}2^2\left(\frac{1}{70}\right)=0\\
C_5=\frac{1}{5!}\begin{pmatrix}\left(\frac{1}{2}\right)^5\left(\frac{7}{5}\right)+1^5\left(-\frac{86}{35}\right)\\+\left(\frac{3}{2}\right)^5\left(\frac{53}{105}\right)+2^5(0)\end{pmatrix}-\frac{1}{4!}\left(\frac{1}{2}\right)^4(-1)-\frac{1}{3!}2^3\left(\frac{1}{70}\right)=-\frac{31}{6720}
\end{array}\right\} \quad (3.36)$$

Since $C_{p+2}=C_5$. The $p=3$ implies the method is of order 3 with error constant $C_5=-\frac{31}{6720}$

For the third discrete method in (3.13);

$$\alpha_0=-\frac{23}{105},\alpha_{\frac{1}{2}}=\frac{8}{5},\alpha_1=-\frac{19}{35},\alpha_{\frac{3}{2}}=-\frac{88}{105},\alpha_2=0,\beta_2=-\frac{1}{70},\gamma_1=-1$$

Applying (3.33), we have

$$\left.\begin{array}{l}
C_0 = \sum_{j=0}^{k} \alpha_j = -\frac{23}{105} + \frac{8}{5} - \frac{19}{35} + \frac{88}{105} + 0 = 0 \\
C_1 = \frac{1}{1!}\begin{pmatrix} 0^1\left(-\frac{23}{105}\right) + \left(\frac{1}{2}\right)^1\left(\frac{8}{5}\right) + 1^1\left(-\frac{19}{35}\right) \\ + \left(\frac{3}{2}\right)^1\left(-\frac{88}{105}\right) \end{pmatrix} - (-1) = 0 \\
C_2 = \frac{1}{2!}\begin{pmatrix} \left(\frac{1}{2}\right)^2\left(\frac{8}{5}\right) + 1^2\left(-\frac{19}{35}\right) \\ + \left(\frac{3}{2}\right)^2\left(-\frac{88}{105}\right) + 2^2(0) \end{pmatrix} - 1(-1) - \left(-\frac{1}{70}\right) = 0 \\
C_3 = \frac{1}{3!}\begin{pmatrix} \left(\frac{1}{2}\right)^3\left(\frac{8}{5}\right) + 1^3\left(-\frac{19}{35}\right) \\ + \left(\frac{3}{2}\right)^3\left(-\frac{88}{105}\right) + 2^3(0) \end{pmatrix} - \frac{1}{2!}1^2(-1) - 2\left(-\frac{1}{70}\right) = 0 \\
C_4 = \frac{1}{4!}\begin{pmatrix} \left(\frac{1}{2}\right)^4\left(\frac{8}{5}\right) + 1^4\left(-\frac{19}{35}\right) \\ + \left(\frac{3}{2}\right)^4\left(-\frac{88}{105}\right) + 2^4(0) \end{pmatrix} - \frac{1}{3!}1^3(-1) - \frac{1}{2!}2^2\left(-\frac{1}{70}\right) = 0 \\
C_5 = \frac{1}{5!}\begin{pmatrix} \left(\frac{1}{2}\right)^5\left(\frac{8}{5}\right) + 1^5\left(-\frac{19}{35}\right) \\ + \left(\frac{3}{2}\right)^5\left(-\frac{88}{105}\right) + 2^5(0) \end{pmatrix} - \frac{1}{4!}1^4(-1) - \frac{1}{3!}2^3\left(-\frac{1}{70}\right) = \frac{1}{280}
\end{array}\right\} \quad (3.37)$$

Since $C_{p+2} = C_5$. The $p = 3$ implies the method is of order 3 with error constant $C_5 = \frac{1}{280}$

For the fourth discrete method in (3.13);

$$\alpha_0 = \frac{34}{105}, \alpha_{\frac{1}{2}} = -\frac{9}{5}, \alpha_1 = \frac{162}{35}, \alpha_{\frac{3}{2}} = -\frac{331}{105}, \alpha_2 = 0, \beta_2 = \frac{3}{70}, \gamma_{\frac{3}{2}} = -1$$

Applying (3.33), we have

$$\left.\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j = \frac{34}{105} - \frac{9}{5} + \frac{162}{35} - \frac{331}{105} + 0 = 0 \\
C_1 &= \frac{1}{1!}\begin{pmatrix} 0^1\left(\frac{34}{105}\right) + \left(\frac{1}{2}\right)^1\left(-\frac{9}{5}\right) + 1^1\left(\frac{162}{35}\right) \\ + \left(\frac{3}{2}\right)^1\left(-\frac{331}{105}\right) \end{pmatrix} - (-1) = 0 \\
C_2 &= \frac{1}{2!}\begin{pmatrix} \left(\frac{1}{2}\right)^2\left(-\frac{9}{5}\right) + 1^2\left(\frac{162}{35}\right) \\ + \left(\frac{3}{2}\right)^2\left(-\frac{331}{105}\right) + 2^2(0) \end{pmatrix} - \frac{3}{2}(-1) - \left(\frac{3}{70}\right) = 0 \\
C_3 &= \frac{1}{3!}\begin{pmatrix} \left(\frac{1}{2}\right)^3\left(-\frac{9}{5}\right) + 1^3\left(\frac{162}{35}\right) \\ + \left(\frac{3}{2}\right)^3\left(-\frac{331}{105}\right) + 2^3(0) \end{pmatrix} - \frac{1}{2!}\left(\frac{3}{2}\right)^2(-1) - 2\left(\frac{3}{70}\right) = 0 \\
C_4 &= \frac{1}{4!}\begin{pmatrix} \left(\frac{1}{2}\right)^4\left(-\frac{9}{5}\right) + 1^4\left(\frac{162}{35}\right) \\ + \left(\frac{3}{2}\right)^4\left(-\frac{331}{105}\right) + 2^4(0) \end{pmatrix} - \frac{1}{3!}\left(\frac{3}{2}\right)^3(-1) - \frac{1}{2!}2^2\left(\frac{3}{70}\right) = 0 \\
C_5 &= \frac{1}{5!}\begin{pmatrix} \left(\frac{1}{2}\right)^5\left(-\frac{9}{5}\right) + 1^5\left(\frac{162}{35}\right) \\ + \left(\frac{3}{2}\right)^5\left(-\frac{331}{105}\right) + 2^5(0) \end{pmatrix} - \frac{1}{4!}\left(\frac{3}{2}\right)^4(-1) - \frac{1}{3!}2^3\left(\frac{3}{70}\right) = -\frac{17}{2240}
\end{aligned}\right\} \quad (3.38)$$

Since $C_{p+2} = C_5$. The $p = 3$ implies the method is of order 3 with error constant $C_5 = -\frac{17}{2240}$

For the fifth discrete method in (3.13);

$$\alpha_0 = \frac{17}{21}, \alpha_{\frac{1}{2}} = -4, \alpha_1 = \frac{53}{7}, \alpha_{\frac{3}{2}} = -\frac{92}{21}, \alpha_2 = 0, \beta_2 = \frac{5}{14}, \gamma_1 = -1$$

Applying (3.33), we have

$$\left.\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j = \frac{17}{21} \pm 4 + \frac{53}{7} \pm \frac{92}{21} + 0 = 0 \\
C_1 &= \frac{1}{1!}\begin{pmatrix} 0^1\left(\frac{17}{21}\right) + \left(\frac{1}{2}\right)^1(-4) + 1^1\left(\frac{53}{7}\right) \\ + \left(\frac{3}{2}\right)^1\left(-\frac{92}{21}\right) \end{pmatrix} - (-1) = 0 \\
C_2 &= \frac{1}{2!}\begin{pmatrix} \left(\frac{1}{2}\right)^2(-4) + 1^2\left(\frac{53}{7}\right) \\ + \left(\frac{3}{2}\right)^2\left(-\frac{92}{21}\right) + 2^2(0) \end{pmatrix} - 2(-1) - \left(\frac{5}{14}\right) = 0 \\
C_3 &= \frac{1}{3!}\begin{pmatrix} \left(\frac{1}{2}\right)^3(-4) + 1^3\left(\frac{53}{7}\right) \\ + \left(\frac{3}{2}\right)^3\left(-\frac{92}{21}\right) + 2^3(0) \end{pmatrix} - \frac{1}{2!}2^2(-1) - 2\left(\frac{5}{14}\right) = 0 \\
C_4 &= \frac{1}{4!}\begin{pmatrix} \left(\frac{1}{2}\right)^4(-4) + 1^4\left(\frac{53}{7}\right) \\ + \left(\frac{3}{2}\right)^4\left(-\frac{92}{21}\right) + 2^4(0) \end{pmatrix} - \frac{1}{3!}2^3(-1) - \frac{1}{2!}2^2\left(\frac{5}{14}\right) = 0 \\
C_5 &= \frac{1}{5!}\begin{pmatrix} \left(\frac{1}{2}\right)^5(-4) + 1^5\left(\frac{53}{7}\right) \\ + \left(\frac{3}{2}\right)^5\left(-\frac{92}{21}\right) + 2^5(0) \end{pmatrix} - \frac{1}{4!}2^4(-1) - \frac{1}{3!}2^3\left(\frac{5}{14}\right) = -\frac{83}{3360}
\end{aligned}\right\} \quad (3.39)$$

Since $C_{p+2} = C_5$. The $p = 3$ implies the method is of order 3 with error constant $C_5 = -\frac{83}{3360}$

For the first discrete method in (3.14);

$$\alpha_0 = -\frac{11}{21}, \alpha_{\frac{1}{2}} = 1, \alpha_1 = -\frac{3}{7}, \alpha_{\frac{3}{2}} = -\frac{1}{21}, \alpha_2 = 0, \beta_2 = -\frac{1}{252}, \beta_{\frac{1}{2}} = -\frac{5}{36}$$

Applying (3.33), we have;

$$\left.\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j = -\frac{11}{21} - \frac{3}{7} + 0 + 1 - \frac{1}{21} = 0 \\
C_1 &= \frac{1}{1!}\begin{pmatrix} 0^1\left(-\frac{11}{21}\right) + \left(\frac{1}{2}\right)^1(1) + 1^1\left(-\frac{3}{7}\right) \\ + \left(\frac{3}{2}\right)^1\left(-\frac{1}{21}\right) + 2(0) \end{pmatrix} = 0 \\
C_2 &= \frac{1}{2!}\begin{pmatrix} \left(\frac{1}{2}\right)^2(1) + 1^2\left(-\frac{3}{7}\right) \\ + \left(\frac{3}{2}\right)^2\left(-\frac{1}{21}\right) + 2^2(0) \end{pmatrix} - \left(\left(-\frac{1}{252}\right) + \left(-\frac{5}{36}\right)\right) = 0 \\
C_3 &= \frac{1}{3!}\begin{pmatrix} \left(\frac{1}{2}\right)^3(1) + 1^3\left(-\frac{3}{7}\right) \\ + \left(\frac{3}{2}\right)^3\left(-\frac{1}{21}\right) + 2^3(0) \end{pmatrix} - \left(2\left(-\frac{1}{252}\right) + \left(\frac{1}{2}\right)\left(-\frac{5}{36}\right)\right) = 0 \\
C_4 &= \frac{1}{4!}\begin{pmatrix} \left(\frac{1}{2}\right)^4(1) + 1^4\left(-\frac{3}{7}\right) \\ + \left(\frac{3}{2}\right)^4\left(-\frac{1}{21}\right) + 2^4(0) \end{pmatrix} - \frac{1}{2!}\left(2^2\left(-\frac{1}{252}\right) + \left(\frac{1}{2}\right)^2\left(-\frac{5}{36}\right)\right) = 0 \\
C_5 &= \frac{1}{5!}\begin{pmatrix} \left(\frac{1}{2}\right)^5(1) + 1^5\left(-\frac{3}{7}\right) \\ + \left(\frac{3}{2}\right)^5\left(-\frac{1}{21}\right) + 2^5(0) \end{pmatrix} - \frac{1}{3!}\left(2^3\left(-\frac{1}{252}\right) + \left(\frac{1}{2}\right)^3\left(-\frac{5}{36}\right)\right) = \frac{5}{2688}
\end{aligned}\right\} \quad (3.40)$$

Since $C_{p+2} = C_5$. The $p = 3$ implies the method is of order 3 with error constant $C_5 = \frac{5}{2688}$

For the second discrete method in (3.14);

$$\alpha_0 = -\frac{13}{37}, \alpha_{\frac{1}{2}} = \frac{63}{37}, \alpha_1 = -\frac{87}{37}, \alpha_{\frac{3}{2}} = 1, \alpha_2 = 0, \beta_2 = -\frac{11}{148}, \beta_{\frac{3}{2}} = \frac{35}{148},$$

Applying (3.33), we have

$$
\left.\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j = -\frac{13}{37} + \frac{63}{37} - \frac{87}{37} + 1 + 0 = 0 \\
C_1 &= \frac{1}{1!}\begin{pmatrix} 0^1\left(-\frac{13}{37}\right) + \left(\frac{1}{2}\right)^1\left(\frac{63}{37}\right) + 1^1\left(-\frac{87}{37}\right) \\ + \left(\frac{3}{2}\right)^1(1) + 2(0) \end{pmatrix} = 0 \\
C_2 &= \frac{1}{2!}\begin{pmatrix} \left(\frac{1}{2}\right)^2\left(\frac{63}{37}\right) + 1^2\left(-\frac{87}{37}\right) \\ + \left(\frac{3}{2}\right)^2(1) + 2^2(0) \end{pmatrix} - \left((-1) + \left(-\frac{11}{148}\right)\right) = 0 \\
C_3 &= \frac{1}{3!}\begin{pmatrix} \left(\frac{1}{2}\right)^3\left(\frac{63}{37}\right) + 1^3\left(-\frac{87}{37}\right) \\ + \left(\frac{3}{2}\right)^3(1) + 2^3(0) \end{pmatrix} - \left(\frac{3}{2}\left(\frac{35}{148}\right) + 2\left(-\frac{11}{148}\right)\right) = 0 \\
C_4 &= \frac{1}{4!}\begin{pmatrix} \left(\frac{1}{2}\right)^4\left(\frac{63}{37}\right) + 1^4\left(-\frac{87}{37}\right) \\ + \left(\frac{3}{2}\right)^4(1) + 2^4(0) \end{pmatrix} - \frac{1}{2!}\left(\left(\frac{3}{2}\right)^2\left(\frac{35}{148}\right) + 2^2\left(-\frac{11}{148}\right)\right) = 0 \\
C_5 &= \frac{1}{5!}\begin{pmatrix} \left(\frac{1}{2}\right)^5\left(\frac{63}{37}\right) + 1^5\left(-\frac{87}{37}\right) \\ + \left(\frac{3}{2}\right)^5(1) + 2^5(0) \end{pmatrix} - \frac{1}{3!}\left(\left(\frac{3}{2}\right)^3\left(\frac{35}{148}\right) + 2^3\left(-\frac{11}{148}\right)\right) = \frac{145}{14208}
\end{aligned}\right\} \quad (3.41)
$$

Since $C_{p+2} = C_5$. The $p = 3$ implies the method is of order 3 with error constant $C_5 = \frac{145}{14208}$

We follow similar procedure for cases $k = 3$ and $k = 4$ and present the Order and Error constants for the proposed methods as follows;

Table 3.1 Order and Error Constants of the 2SBHBDF

Method	**Order (p)**	**Error Constants (C_{p+2})**
(3.12)	3	$-\frac{1}{112}$
(3.13)	3	$\frac{19}{1120}$
(3.13)	3	$-\frac{31}{6720}$
(3.13)	3	$\frac{1}{280}$
(3.13)	3	$-\frac{17}{2240}$
(3.13)	3	$-\frac{83}{3360}$
(3.14)	3	$\frac{5}{2688}$
(3.14)	3	$\frac{145}{14208}$

Table 3.2 Order and Error Constants of the 3SBHBDF

Method	**Order (p)**	**Error Constants (C_{p+2})**
(3.20)	4	$-\frac{47}{115200}$
(3.21)	4	$-\frac{4477}{69120}$
(3.21)	4	$\frac{2009}{230400}$
(3.21)	4	$-\frac{167}{115200}$
(3.21)	4	$\frac{107}{184320}$
(3.21)	4	$-\frac{5929}{7372800}$
(3.21)	4	$-\frac{1489}{691200}$
(3.22)	4	$-\frac{1649}{691200}$
(3.22)	4	$-\frac{1309}{172800}$
(3.22)	4	$\frac{2557}{345600}$

Table 3.3 Order and Error Constants of the 4SBHBDF

Method	**Order (p)**	**Error Constants (C_{p+2})**
(3.27)	5	$-\frac{169}{4714080}$
(3.28)	5	$\frac{13175}{179584}$
(3.28)	5	$-\frac{50017}{4919040}$
(3.28)	5	$\frac{131}{44896}$
(3.28)	5	$-\frac{1741}{1795840}$
(3.28)	5	$\frac{30305}{183894016}$
(3.28)	5	$-\frac{42439}{255852544}$
(3.29)	5	$-\frac{20143}{56568960}$
(3.30)	5	$-\frac{38551}{6285440}$
(3.31)	5	$\frac{35311}{12570880}$
(3.32)	5	$-\frac{397857}{160907264}$
(3.33)	5	$-\frac{8059557}{3218145280}$

3.3.2 Consistency

The sufficient conditions for a linear multistep to be consistent are;

i. $p \geq 1$(i.e. the method has at least order of one).

ii. $\sum_{j=0}^{k} \alpha_j = 0$

iii. $\rho''(1) = 2!\,\sigma(1)$

where$\rho(r)$ and $\sigma(r)$ are the first and second characteristic polynomials respectively.

In section 3.3.1, we have established the conditions (i) where we have $p = 3, p = 4$,and $p = 5$ for cases of 2SBHBDF, 3SBHBDF, and 4SBHBDFrespectively. Also in the section, condition (ii) was fulfilled where $C_0 = \sum_{j=0}^{k} \alpha_j = 0$ in all cases of 2SBHBDF, 3SBHBDF, and 4SBHBDF.

For condition (iii), we shall consider (3.12) and obtain the first and second characteristic polynomials as;

$$\rho(r) = r^2 + \frac{114}{35}r - \frac{8}{5}r^{\frac{1}{2}} - \frac{104}{35}r^{\frac{3}{2}} + \frac{11}{35}$$

$$\sigma(r) = \frac{3}{35}r^2$$

Then,

$$\rho'(r) = 2r - \frac{156}{35}r^{\frac{1}{2}} - \frac{4}{5}r^{-\frac{1}{2}} + \frac{114}{35}$$

$$\rho''(r) = -\frac{78}{35}r^{-\frac{1}{2}} + \frac{2}{5}r^{-\frac{3}{2}} + 2$$

Therefore,

$$\rho''(1) = \frac{6}{35}$$

and

$$\sigma(1) = \frac{3}{35}$$

$$2!\,\sigma(r) = \frac{6}{35}$$

Hence, $\rho''(1) = 2!\,\sigma(r)$ this satisfied the condition (iii). Since the three conditions are satisfied, it follows that (3.12) is consistent. We shall obtain the first and second polynomials for the other methods in the table below;

Table 3.4 Condition for Consistency of the 2SBHBDF

Method	$\rho''(1)$	$2!\,\sigma(r)$
(3.12)	$\frac{6}{35}$	$\frac{6}{35}$
(3.14)	$-\frac{72}{35}$	$-\frac{72}{35}$
(3.14)	$-\frac{48}{35}$	$-\frac{48}{35}$

Table 3.5Condition for Consistency of the 3SBHBDF

Method	$\rho^{''}(1)$	$2!\,\sigma(r)$
(3.20)	$\frac{1}{20}$	$\frac{1}{20}$
(3.22)	$-\frac{233}{120}$	$-\frac{233}{120}$
(3.22)	$-\frac{299}{240}$	$-\frac{299}{240}$
(3.22)	$-\frac{131}{60}$	$-\frac{131}{60}$

Table 3.6Condition for Consistency of the 4SBHBDF

Method	$\rho''(1)$	$2!\,\sigma(r)$
(3.27)	$\frac{24}{1403}$	$\frac{24}{1403}$
(3.30)	$-\frac{2552}{1403}$	$-\frac{2552}{1403}$
(3.30)	$-\frac{2968}{1403}$	$-\frac{2968}{1403}$
(3.30)	$-\frac{8857}{5612}$	$-\frac{8857}{5612}$
(3.30)	$-\frac{83363}{89792}$	$-\frac{83363}{89792}$

3.3.3 Zero Stability

According to Awari (2017),a linear multistep method is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, if every root with modulus one is simple, i.e. $|r| \leq 1$ and has multiplicity not greater than the order of the differential equation.

To obtain the zero-stability of HBDF, we shall express the proposed methods in matrix difference equation form;

$$P^{(1)}Y_{\omega+1} = P^{(0)}Y_{\omega} + h^2Q^{(0)}F_{\omega} + h^2Q^{(1)}F_{\omega+1} \tag{3.41}$$

Where

$$Y_{\omega+1} = \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix} Y_{\omega} = \begin{pmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_{n} \end{pmatrix}$$

$$F_{\omega+1} = \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix} f_{\omega} = \begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_{n} \end{pmatrix}$$

$P^{(1)}$,$P^{(0)}$, $Q^{(1)}$, and $Q^{(0)}$ are $(k+1) \times (k+1)$ matrices obtained from the combined coefficients of the HBDF. The roots of the first characteristics polynomial $\rho(r)$ is obtained from;

$$\rho(r) = \left|rP^{(1)} - P^{(0)}\right| \tag{3.42}$$

3.3.3.1 Zero Stability of 2SBHBDF

We express the schemes in 2SBHBDFin the form (3.41) and obtain the $P^{(1)}$, $P^{(0)}$, and$\rho(r)$ as

$$p^{(1)} = \begin{pmatrix} 1 & -\frac{3}{7} & -\frac{1}{21} & 0 \\ -\frac{56}{19} & 1 & \frac{88}{57} & 0 \\ \frac{63}{37} & -\frac{87}{37} & 1 & 0 \\ -\frac{8}{5} & \frac{114}{35} & -\frac{104}{35} & 1 \end{pmatrix}, \quad p^{(0)} = \begin{pmatrix} 0 & 0 & 0 & \frac{11}{21} \\ 0 & 0 & 0 & -\frac{23}{57} \\ 0 & 0 & 0 & \frac{13}{57} \\ 0 & 0 & 0 & -\frac{11}{35} \end{pmatrix}$$

$$\rho(r) = \frac{1400}{703} r^4 - \frac{1400}{703} r^3$$

Then, $r = (0,0,0,1)$, therefore 2SBHBDFis zero-stable since $|r_j| \leq 1$.

3.3.3.2 Zero Stability of 3SBHBDF

We express the schemes in 3SBHBDF in the form (3.41) and obtain the $P^{(1)}, P^{(0)}$, and $\rho(r)$ as

$$p^{(1)} = \begin{pmatrix} 1 & -\frac{9751}{2766} & \frac{2352}{461} & -\frac{40448}{15213} & 0 \\ -\frac{3002}{3157} & 1 & \frac{1376}{451} & -\frac{109568}{34727} & 0 \\ \frac{617}{45808} & -\frac{4727}{13088} & 1 & -\frac{20512}{31493} & 0 \\ -\frac{26521}{518656} & \frac{654577}{1037312} & -\frac{51359}{32416} & 1 & 0 \\ \frac{11}{420} & -\frac{41}{120} & \frac{28}{15} & -\frac{2944}{1155} & 1 \end{pmatrix}, \quad p^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{833}{10142} \\ 0 & 0 & 0 & 0 & -\frac{273}{4961} \\ 0 & 0 & 0 & 0 & \frac{141}{143968} \\ 0 & 0 & 0 & 0 & -\frac{4641}{1037312} \\ 0 & 0 & 0 & 0 & \frac{1}{440} \end{pmatrix}$$

$$\rho(r) = -\frac{81648000}{86141061787} r^5 + \frac{81648000}{86141061787} r^4$$

Then, $r = (0,0,0,0,1)$, therefore 3SBHBDFis zero-stable since $|r_j| \leq 1$.

3.3.3.3 Zero Stability of 4SBHBDF

We express the schemes in 4SBHBDF in the form (3.41) and obtain the $P^{(1)}, P^{(0)}$, and $\rho(r)$ as

$$p^{(1)} = \begin{pmatrix} 1 & -\frac{1608574}{703125} & \frac{9965923}{2953125} & -\frac{20819968}{2109375} & \frac{1173618688}{152578125} & 0 \\ -\frac{2438765}{4059891} & 1 & \frac{54635}{433269} & -\frac{14501888}{4765959} & \frac{917504}{369081} & 0 \\ \frac{4164615}{67842709} & -\frac{720603}{1340765} & 1 & -\frac{2791424}{14748415} & -\frac{314966016}{955965445} & 0 \\ -\frac{84349755}{20472269824} & \frac{10370943}{445049344} & -\frac{161178325}{890098688} & 1 & -\frac{519683472}{619765981} & 0 \\ \frac{1409110115}{203552014336} & -\frac{170473309}{4626182144} & \frac{115328331095}{499627671552} & -\frac{12883055113}{10734188256} & 1 & 0 \\ -\frac{40}{32269} & \frac{326}{49105} & -\frac{1256}{29463} & \frac{1802244}{147315} & -\frac{76546048}{35011865} & 1 \end{pmatrix}$$

$$p^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{5972318}{65390625} \\ 0 & 0 & 0 & 0 & 0 & -\frac{13643}{433269} \\ 0 & 0 & 0 & 0 & 0 & \frac{215551}{41563715} \\ 0 & 0 & 0 & 0 & 0 & -\frac{2842763}{6898264832} \\ 0 & 0 & 0 & 0 & 0 & \frac{177976967}{249813835776} \\ 0 & 0 & 0 & 0 & 0 & -\frac{83}{652395} \end{pmatrix}$$

$$\rho(r) = \frac{204665916687444982}{1173478119406078135892 5}r^6 - \frac{204665916687444982}{11734781194060781358925}r^5$$

Then, $r = (0,0,0,0,0,1)$, therefore 4SBHBDFis zero-stable since $|r_j| \leq 1$.

3.3.4 Convergence

According to the Dahlquist theorem, the necessary and sufficient condition for a Linear Multistep Method to be convergent is the methods to be consistent and zero-stable. Since, the proposed Backward Differentiation Formulae are both consistent and zero-stable, we conclude that the proposed methods are convergent.

CHAPTER FOUR

4.0 RESULTS AND DISCUSSIONS

4.1 Numerical Experiments

In this chapter, we shall perform some numerical experiments of linear, non-linear, homogeneous, inhomogeneous and system of second-order ordinary differential equations on the developed methods. The problems to be examined are;

Problem 1: Constant Coefficient Linear Type

$$\frac{d^2y(t)}{dt^2} = 8\frac{dy(t)}{dt} - 17y(t)$$

$$y(0) =-4, y^{'}(0) =-1$$

Exact Solution:

$$y(t) = 15e^{4t}\sin(t) - 4e^{4t}\cos(t)$$

Problem 2: Variable Coefficient Linear Type

$$t^2\frac{d^2y(t)}{dt^2} + \frac{3}{2}t\frac{dy(t)}{dt} - \frac{1}{2}y(t) = 0$$

$$y(1) = 2, y^{'}(1) = 5$$

Exact Solution:

$$y(t) = \frac{14}{3}\sqrt{t} - \frac{8}{3t}$$

Problem 3: Variable Coefficient Non-Linear Type

$$\frac{d^2y(t)}{dt^2} = t\left(\frac{dy(t)}{dt}\right)^2$$

$$y(0) = 1, y'(0) = \frac{1}{2}$$

Exact Solution:

$$y(t) = 1 + \frac{1}{2}\ln\left(\frac{2+t}{2-t}\right)$$

Problem 4: Linear System of Second Order IVP

$$\frac{d^2y_1}{dt^2} = \frac{dy_1}{dt} + \frac{dy_2}{dt}$$

$$\frac{d^2y_2}{dt^2} = \frac{dy_1}{dt} + \frac{dy_2}{dt}$$

$$y_1(0) = 1, y_1'(0) = 2, y_2(0) = 1, y_2'(0) = 2$$

Exact Solution:

$$y_1(t) = e^{2t}, y_2(t) = e^{2t}$$

Problem 5: Linear System of Second Order IVP

$$\frac{d^2y_1}{dt^2} = \frac{dy_1}{dt}$$

$$\frac{d^2y_2}{dt^2} = 2\frac{dy_1}{dt} + t\frac{dy_1}{dt}$$

$$y_1(0) = 1, y_1'(0) = 1, y_2(0) = 0, y_2'(0) = 1$$

Exact Solution:

$$y_1(t) = e^{t}, y_2(t) = te^{2t}$$

Problem 6: Unstable System of Differential Equations

$$\frac{d^2y_1}{dt^2} = \frac{dy_1}{dt} + 3\frac{dy_2}{dt}$$

$$\frac{d^2y_2}{dt^2} = -4\frac{dy_2}{dt}$$

$$y_1(0) = 1, y_1'(0) = 1, y_2(0) = 1, y_2'(0) = 1$$

Exact Solution: $y_1(t) = -\frac{3}{4} + \frac{8}{5}e^t + \frac{3}{20}e^{-4t}, y_2(t) = \frac{5}{4} - \frac{1}{4}e^{-4t}$

Table 4.1 Numerical Comparison of Exact Solution and the Proposed Methods for Problem 1 for $h = 0.01$

t	**Exact**	**2SBHBDF**	**3SBHBDF**	**4SBHBDF**
0.1	-3.7034778030160408192	-3.7034755641322759679	-3.7034798867056725507	-3.7034776714217353697
0.2	-2.0925122227236628344	-2.0925000966640553986	-2.0925236008174516027	-2.0925115455233522165
0.3	2.030109209389152804	2.0301487983577830731	2.0300722536538230837	2.0301113722563448553
0.4	10.683845269066590998	10.683949066577541338	10.683748899088263213	10.683850806940311924
0.5	27.199505875092497766	27.199746188756227355	27.199284288647810704	27.199518451985474893
0.6	56.971023418029402204	56.971536525776565220	56.970554159242908582	56.971049733805448554
0.7	108.59874623264690931	108.59978066523090499	108.59780775157738221	108.59879830532412318
0.8	195.61048588874082725	195.61248311211900207	195.60868886176887767	195.61058450559692621
0.9	339.02641886363790146	339.03014608612282744	339.02309441350756297	339.02659949241168744
1.0	571.14336072001762703	571.15012652332059108	571.13737832906139126	571.14368234377884767

The Table 4.1 shows the numerical results of problem 1. The results show that the proposed methods 2SBHBDF, 3SBHBDF, and 4SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.2 Absolute Error $|Y(t) - y(t)|$ in prosed methods for Problem 1

t	2SBHBDF	3SBHBDF	4SBHBDF
0.0	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$
0.1	$2.23888376485 \times 10^{-6}$	$2.08368963173 \times 10^{-6}$	$1.31594305450 \times 10^{-7}$
0.2	$1.21260596074 \times 10^{-5}$	$1.13780937888 \times 10^{-5}$	$6.77200310618 \times 10^{-7}$
0.3	$3.95889686303 \times 10^{-5}$	$3.69557353297 \times 10^{-5}$	$2.16286719205 \times 10^{-6}$
0.4	$1.03797510950 \times 10^{-4}$	$9.63699783278 \times 10^{-5}$	$5.53787372093 \times 10^{-6}$
0.5	$2.40313663730 \times 10^{-4}$	$2.21586444687 \times 10^{-4}$	$1.25768929771 \times 10^{-5}$
0.6	$5.13107747163 \times 10^{-4}$	$4.69258786494 \times 10^{-4}$	$2.63157760464 \times 10^{-5}$
0.7	$1.03443258400 \times 10^{-3}$	$9.38481069527 \times 10^{-4}$	$5.20726772139 \times 10^{-5}$
0.8	$1.99722337817 \times 10^{-3}$	$1.79702697195 \times 10^{-3}$	$9.86168560990 \times 10^{-5}$
0.9	$3.72722248493 \times 10^{-3}$	$3.32445013034 \times 10^{-3}$	$1.80628773786 \times 10^{-4}$
1.0	$6.76580330296 \times 10^{-3}$	$5.98239095624 \times 10^{-3}$	$3.21623761221 \times 10^{-4}$

This show the maximum errors for problem 1 at h=0.01, it is therefore observed that the method is efficient and there is also an increase in accuracy as the step size increases.

Table 4.3 Numerical Comparison of Exact Solution and the Proposed Methods for Problem 2 for $h = 0.01$

t	**Exact**	**2SBHBDF**	**3SBHBDF**	**4SBHBDF**
0.0	2.0000000000000000000	2.0000000000000000000	2.0000000000000000000	2.0000000000000000000
0.1	2.4701988672182829769	2.4701988901235716030	2.4701988916945362055	2.4701988694385372294
0.2	2.8898549811593281700	2.8898550475903798431	2.8898550521970691904	2.8898549872129310540
0.3	3.2695365991805926204	3.2695367183658320765	3.2695367252952477633	3.2695366096397309824
0.4	3.6169125594644035444	3.6169127348535730639	3.6169127431198655553	3.6169125743912527451
0.5	3.9376982887163044513	3.9376985207858306440	3.9376985293535594978	3.9376983079943958070
0.6	4.2362516323143080864	4.2362519200216870588	4.2362519280201535389	4.2362516557383969141
0.7	4.5159614605420799766	4.5159618021000866133	4.5159618089081776867	4.5159614878912523029
0.8	4.7795088555179296682	4.7795092488135041059	4.7795092539637423221	4.7795088865693610862
0.9	5.0290473123789455970	5.0290477552008961247	5.0290477583569118747	5.0290473469251090013
1.0	5.2663299577411102278	5.2663304479025818024	5.2663304488479230632	5.2663299955893313444

The Table 4.3 shows the numerical results of problem 2. The results show that the proposed methods 2SBHBDF, 3SBHBDF, and 4SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.4Absolute Error $|Y(t) - y(t)|$ in prosed methods for Problem 2

t	2SBHBDF	3SBHBDF	4SBHBDF
0.0	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$
0.1	$2.29052886260 \times 10^{-8}$	$2.44762532285 \times 10^{-8}$	$2.2202542524 \times 10^{-9}$
0.2	$6.64310516730 \times 10^{-8}$	$7.10377410203 \times 10^{-8}$	$6.0536028839 \times 10^{-9}$
0.3	$1.191852394561 \times 10^{-7}$	$1.261146551429 \times 10^{-7}$	$1.04591383620 \times 10^{-8}$
0.4	$1.753891695194 \times 10^{-7}$	$1.836554620108 \times 10^{-7}$	$1.49268492006 \times 10^{-8}$
0.5	$2.320695261927 \times 10^{-7}$	$2.406372550465 \times 10^{-7}$	$1.92780913557 \times 10^{-8}$
0.6	$2.877073789723 \times 10^{-7}$	$2.957058454524 \times 10^{-7}$	$2.34240888276 \times 10^{-8}$
0.7	$3.415580066366 \times 10^{-7}$	$3.483660977100 \times 10^{-7}$	$2.73491723262 \times 10^{-8}$
0.8	$3.932955744375 \times 10^{-7}$	$3.984458126537 \times 10^{-7}$	$3.10514314178 \times 10^{-8}$
0.9	$4.428219505277 \times 10^{-7}$	$4.459779662777 \times 10^{-7}$	$3.45461634043 \times 10^{-8}$
1.0	$4.901614715746 \times 10^{-7}$	$4.911068128354 \times 10^{-7}$	$3.78482211166 \times 10^{-8}$

The limit errors of problem 2 at h=0.01 indicates that, there is efficient and also there is increase in accuracy as the step size increases.

Table 4.5 Numerical Comparison of Exact Solution and the Proposed Methods for Problem 3 for $h = 0.01$

t	Exact	2SBHBDF	3SBHBDF	4SBHBDF
0.0	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000
0.1	1.0500417292784912682	1.0500417293457835468	1.0500417292741132458	1.0500417292799142976
0.2	1.1003353477310755806	1.1003353479770131443	1.1003353477024359703	1.1003353477362294525
0.3	1.1511404359364668053	1.1511404364899374194	1.1511404358437530280	1.1511404359484509096
0.4	1.2027325540540821910	1.2027325550756534539	1.2027325538286751584	1.2027325540770755735
0.5	1.2554128118829953416	1.2554128135847063960	1.2554128114150377147	1.2554128119236178801
0.6	1.3095196042031117155	1.3095196068782244240	1.3095196033199264267	1.3095196042714339901
0.7	1.3654437542713961691	1.3654437583415579519	1.3654437526905396956	1.3654437543849120799
0.8	1.4236489301936018069	1.4236489362870485254	1.4236489274514709661	1.4236489303813397814
0.9	1.4847002785940517416	1.4847002876810812149	1.4847002739123865338	1.4847002789110383465
1.0	1.5493061443340548457	1.5493061579730394425	1.5493061363158179051	1.5493061448811208414

The Table 4.5 shows the numerical results of problem 3. The results show that the proposed methods 2SBHBDF, 3SBHBDF, and 4SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.6 Absolute Error $|Y(t) - y(t)|$ in prosed methods for Problem 3

t	**2SBHBDF**	**3SBHBDF**	**4SBHBDF**
0.0	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$
0.1	$6.729227860 \times 10^{-11}$	$4.3780224000 \times 10^{-12}$	$1.4230294000 \times 10^{-12}$
0.2	$2.459375637 \times 10^{-10}$	$2.8639610300 \times 10^{-11}$	$5.1538719000 \times 10^{-12}$
0.3	$5.534706141 \times 10^{-10}$	$9.2713777300 \times 10^{-11}$	$1.1984104300 \times 10^{-11}$
0.4	$1.0215712629 \times 10^{-9}$	$2.2540703260 \times 10^{-10}$	$2.2993382500 \times 10^{-11}$
0.5	$1.7017110544 \times 10^{-9}$	$4.679576269 \times 10^{-10}$	$4.0622538500 \times 10^{-11}$
0.6	$2.6751127085 \times 10^{-9}$	$8.8318528880 \times 10^{-10}$	$6.8322274600 \times 10^{-11}$
0.7	$4.0701617828 \times 10^{-9}$	$1.5808564735 \times 10^{-9}$	$1.1351591080 \times 10^{-10}$
0.8	$6.0934467185 \times 10^{-9}$	$2.7421308408 \times 10^{-9}$	$1.8773797450 \times 10^{-10}$
0.9	$9.0870294733 \times 10^{-9}$	$4.6816652078 \times 10^{-9}$	$3.1698660490 \times 10^{-10}$
1.0	$1.36389845968 \times 10^{-8}$	$8.0182369406 \times 10^{-9}$	$5.4706599570 \times 10^{-10}$

The results of the errors of problem 3 at h=0.01, proved that as the number of step size k increases, the accuracy increases.

Table 4.7a Numerical Comparison of Exact Solution and the Proposed Methods for Problem 4 at $h = 0.01$ for $y_1(t)$

t	Exact	2SBHBDF	3SBHBDF	4SBHBDF
0.0	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000
0.1	1.2214027581601698339	1.2214027614936890907	1.2214027568391019275	1.2214027581979879566
0.2	1.4918246976412703178	1.4918247112672642709	1.4918246921329415725	1.4918246977909539644
0.3	1.8221188003905089749	1.8221188341859906962	1.8221187866604442706	1.8221188007609200285
0.4	2.2255409284924676046	2.2255409962035175750	2.2255409008568002254	2.2255409292304573299
0.5	2.7182818284590452354	2.7182819489298926880	2.7182817791494409138	2.7182818297714603770
0.6	3.3201169227365474895	3.3201171214932372457	3.3201168412922787940	3.3201169248967472082
0.7	4.0551999668446745872	4.0552002781300832569	4.0551998391111892266	4.0551999702275362944
0.8	4.9530324243951148037	4.9530328937775461772	4.9530322315788535380	4.9530324294894164708
0.9	6.0496474644129460837	6.0496481521225351670	6.0496471817874609251	6.0496474718768422907
1.0	7.3890560989306502272	7.3890570841179821452	7.3890556937844913932	7.3890561096141137358

The Table 4.7a shows the numerical results of problem 4for y_1. The results show that the proposed methods 2SBHBDF, 3SBHBDF, and 4SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.7b Numerical Comparison of Exact Solution and the Proposed Methods for Problem 4 at $h = 0.01$ for $y_2(t)$

t	Exact	2SBHBDF	3SBHBDF	4SBHBDF
0.0	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000
0.1	1.2214027581601698339	1.2214027581601698339	1.2214027568391027627	1.2214027581980047577
0.2	1.4918246976412703178	1.4918246976412703178	1.4918246921329449417	1.4918246977910167792
0.3	1.8221188003905089749	1.8221188003905089749	1.8221187866604522664	1.8221188007610667348
0.4	2.2255409284924676046	2.2255409284924676046	2.2255409008568155653	2.2255409292307309945
0.5	2.7182818284590452354	2.7182818284590452354	2.7182817791494669294	2.7182818297719146439
0.6	3.3201169227365474895	3.3201169227365474895	3.3201168412923188767	3.3201169248974443106
0.7	4.0551999668446745872	4.0551999668446745872	4.0551998391112479099	4.0551999702285565759
0.8	4.9530324243951148037	4.9530324243951148037	4.9530322315789366535	4.9530324294908558229
0.9	6.0496474644129460837	6.0496474644129460837	6.0496471817875750743	6.0496474718788228966
1.0	7.3890560989306502272	7.3890560989306502272	7.3890556937846446480	7.3890561096167800977

The Table 4.7b shows the numerical results of problem 4for y_2. The results show that the proposed methods 2SBHBDF, 3SBHBDF, and 4SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.8a Absolute Error $|Y(t) - y_1(t)|$ in prosed methods for Problem 4

t	2SBHBDF	3SBHBDF	4SBHBDF
0.0	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$
0.1	$3.33351925680 \times 10^{-9}$	$1.32106790640 \times 10^{-9}$	$3.78181227 \times 10^{-11}$
0.2	$1.36259939531 \times 10^{-8}$	$5.5083287453 \times 10^{-9}$	$1.496836466 \times 10^{-10}$
0.3	$3.37954817213 \times 10^{-8}$	$1.37300647043 \times 10^{-8}$	$3.704110536 \times 10^{-10}$
0.4	$6.77110499704 \times 10^{-8}$	$2.76356673792 \times 10^{-8}$	$7.379897253 \times 10^{-10}$
0.5	$1.204708474526 \times 10^{-7}$	$4.93096043216 \times 10^{-8}$	$1.3124151416 \times 10^{-9}$
0.6	$1.987566897562 \times 10^{-7}$	$8.14442686955 \times 10^{-8}$	$2.1601997187 \times 10^{-9}$
0.7	$3.112854086697 \times 10^{-7}$	$1.277334853606 \times 10^{-7}$	$3.3828617072 \times 10^{-9}$
0.8	$4.693824313735 \times 10^{-7}$	$1.928162612657 \times 10^{-7}$	$5.0943016671 \times 10^{-9}$
0.9	$6.877095890833 \times 10^{-7}$	$2.826254851586 \times 10^{-7}$	$7.4638962070 \times 10^{-9}$
1.0	$9.851873319180 \times 10^{-7}$	$4.051461588340 \times 10^{-7}$	$1.06834635086 \times 10^{-8}$

The errors of this method at each k-step show that error becomes smaller as the step size increases. It also observed that there is accuracy as step size increases.

Table 4.8b Absolute Error $|Y(t) - y_2(t)|$ in prosed methods for Problem 4

t	2SBHBDF	3SBHBDF	4SBHBDF
0.0	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$
0.1	$3.3335192598 \times 10^{-9}$	$1.3210670712 \times 10^{-9}$	$3.78349238 \times 10^{-11}$
0.2	$1.36259939693 \times 10^{-8}$	$5.5083253761 \times 10^{-9}$	$1.497464614 \times 10^{-10}$
0.3	$3.37954817740 \times 10^{-8}$	$1.37300567085 \times 10^{-8}$	$3.705577599 \times 10^{-10}$
0.4	$6.77110500879 \times 10^{-8}$	$2.76356520393 \times 10^{-8}$	$7.382633899 \times 10^{-10}$
0.5	$1.204708476599 \times 10^{-7}$	$4.93095783060 \times 10^{-8}$	$1.3128694085 \times 10^{-9}$
0.6	$1.987566901028 \times 10^{-7}$	$8.14442286128 \times 10^{-8}$	$2.1608968211 \times 10^{-9}$
0.7	$3.112854091582 \times 10^{-7}$	$1.277334266773 \times 10^{-7}$	$3.3838819887 \times 10^{-9}$
0.8	$4.693824320877 \times 10^{-7}$	$1.928161781502 \times 10^{-7}$	$5.0957410192 \times 10^{-9}$
0.9	$6.877095901085 \times 10^{-7}$	$2.826253710094 \times 10^{-7}$	$7.4658768129 \times 10^{-9}$
1.0	$9.851873333544 \times 10^{-7}$	$4.051460055792 \times 10^{-7}$	$1.06861298705 \times 10^{-8}$

The errors of this method at each k-step show that error becomes smaller as the step size increases. It also observed that there is accuracy asstep size increases.

Table 4.9a Numerical Comparison of Exact Solution and the Proposed Methods for Problem 5at $h = 0.01$ for $y_1(t)$

t	Exact	2SBHBDF	3SBHBDF	4SBHBDF
0.0	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000
0.1	1.1051709180756476248	1.1051709181718426061	1.1051709180567076248	1.1051709180759226178
0.2	1.2214027581601698339	1.2214027585269294230	1.2214027580864674606	1.2214027581611853332
0.3	1.3498588075760031040	1.3498588084233101488	1.3498588074048068427	1.3498588075783428844
0.4	1.4918246976412703178	1.4918246992202831534	1.4918246973207343640	1.4918246976456077872
0.5	1.6487212707001281468	1.6487212733095189148	1.6487212701688798820	1.6487212707072904351
0.6	1.8221188003905089749	1.8221188043836797233	1.8221187995764798993	1.8221188004014437088
0.7	2.0137527074704765216	2.0137527132637676051	2.0137527062877824178	2.0137527074863338145
0.8	2.2255409284924676046	2.2255409365744872851	2.2255409268406922300	2.2255409285145615380
0.9	2.4596031111569496638	2.4596031220991699819	2.4596031089193912222	2.4596031111868564171
1.0	2.7182818284590452354	2.7182818429278272601	2.7182818254983103173	2.7182818284985564271

The Table 4.9a shows the numerical results of problem 5 for y_1. The results show that the proposed methods 2SBHBDF, 3SBHBDF, and 4SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.9b Numerical Comparison of Exact Solution and the Proposed Methods for Problem 5 at $h = 0.01$ for $y_2(t)$

t	Exact	2SBHBDF	3SBHBDF	4SBHBDF
0.0	0.0000000000000000000	0.0000000000000000000	0.0000000000000000000	0.0000000000000000000
0.1	0.11051709180756476248	0.11051709228030670605	0.11051709169612794148	0.11051709180940785620
0.2	0.24428055163203396678	0.24428055341312486004	0.24428055120503029077	0.24428055163872184030
0.3	0.40495764227280093120	0.40495764634303308340	0.40495764129541600832	0.40495764228796231964
0.4	0.59672987905650812712	0.59672988656697339034	0.59672987725109747114	0.59672987908416248188
0.5	0.82436063535006407340	0.82436064765134635083	0.82436063239531075998	0.82436063539506242708
0.6	1.0932712802343053849	1.0932712989103736644	1.0932712757599482442	1.0932712803020394347
0.7	1.4096268952293335651	1.4096269221368480101	1.4096268887982308965	1.4096268953262951206
0.8	1.7804327427939740837	1.7804327801078204364	1.7804327339000666331	1.7804327429273981733
0.9	2.2136428000412546974	2.2136428503072393416	2.2136427881011495413	2.2136428002198322288
1.0	2.7182818284590452354	2.7182818946548164375	2.7182818127856068953	2.7182818286924831261

The Table 4.9b shows the numerical results of problem 5 for y_2. The results show that the proposed methods 2SBHBDF, 3SBHBDF, and 4SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.10a Absolute Error $|Y(t) - y_1(t)|$ in prosed methods for Problem 5

t	2SBHBDF	3SBHBDF	4SBHBDF
0.0	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$
0.1	$9.61949813 \times 10^{-11}$	$1.89400000 \times 10^{-11}$	2.749930×10^{-13}
0.2	$3.667595891 \times 10^{-10}$	$7.37023733 \times 10^{-11}$	$1.0154993 \times 10^{-12}$
0.3	$8.473070448 \times 10^{-10}$	$1.711962613 \times 10^{-10}$	$2.3397804 \times 10^{-12}$
0.4	$1.5790128356 \times 10^{-9}$	$3.205359538 \times 10^{-10}$	$4.3374694 \times 10^{-12}$
0.5	$2.6093907680 \times 10^{-9}$	$5.312482648 \times 10^{-10}$	$7.1622883 \times 10^{-12}$
0.6	$3.9931707484 \times 10^{-9}$	$8.140290756 \times 10^{-10}$	$1.09347339 \times 10^{-11}$
0.7	$5.7932910835 \times 10^{-9}$	$1.1826941038 \times 10^{-9}$	$1.58572929 \times 10^{-11}$
0.8	$8.0820196805 \times 10^{-9}$	$1.6517753746 \times 10^{-9}$	$2.20939334 \times 10^{-11}$
0.9	$1.09422203181 \times 10^{-8}$	$2.2375584416 \times 10^{-9}$	$2.99067533 \times 10^{-11}$
1.0	$1.44687820247 \times 10^{-8}$	$2.2375584416 \times 10^{-9}$	$3.95111917 \times 10^{-11}$

The errors of this method at each k-step compared with the exact solution shows that error becomes smaller as the step size increases. it is also observed that there is efficient and accuracy as step size increases.

Table 4.10b Absolute Error $|Y(t) - y_2(t)|$ in prosed methods for Problem 5

t	2SBHBDF	3SBHBDF	4SBHBDF
0.0	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$
0.1	$4.7274194357 \times 10^{-10}$	$1.1143682100 \times 10^{-10}$	$1.84309372 \times 10^{-12}$
0.2	$1.78109089326 \times 10^{-9}$	$4.2700367601 \times 10^{-10}$	$6.68787352 \times 10^{-12}$
0.3	$4.07023215220 \times 10^{-9}$	$9.7738492288 \times 10^{-10}$	$1.516138844 \times 10^{-11}$
0.4	$7.51046526322 \times 10^{-9}$	$1.80541065598 \times 10^{-9}$	$2.765435476 \times 10^{-11}$
0.5	$1.230128227743 \times 10^{-8}$	$2.95475331342 \times 10^{-9}$	$4.499835368 \times 10^{-11}$
0.6	$1.86760682795 \times 10^{-8}$	$4.4743571407 \times 10^{-9}$	$6.77340498 \times 10^{-11}$
0.7	$2.69075144450 \times 10^{-8}$	$6.4311026686 \times 10^{-9}$	$9.69615555 \times 10^{-11}$
0.8	$3.73138463527 \times 10^{-8}$	$8.8939074506 \times 10^{-9}$	$1.334240896 \times 10^{-10}$
0.9	$5.02659846442 \times 10^{-8}$	$1.19401051561 \times 10^{-8}$	$1.785775314 \times 10^{-10}$
1.0	$6.61957712021 \times 10^{-8}$	$1.56734383401 \times 10^{-8}$	$2.334378907 \times 10^{-10}$

The errors of this method at each k-step compared with the exact solution shows that error becomes smaller as the step size increases. it is also observed that there is efficient and accuracy as step size increases.

Table 4.11a Numerical Comparison of Exact Solution and the Proposed Methods for Problem 6 at $h = 0.01$ for $y_1(t)$

t	Exact	2SBHBDF	3SBHBDF	4SBHBDF
0.0	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000
0.1	1.1188214758263820948	1.1188214659556514293	1.1188214681566830554	1.1188214754062051301
0.2	1.2716437576738549729	1.2716437305228945590	1.2716437359341537888	1.2716437565398388727
0.3	1.4549532239084352809	1.4549531782837742246	1.4549531867575094045	1.4549532219933551514
0.4	1.6672039939252308198	1.6672039316329515119	1.6672039422946649040	1.6672039912937684955
0.5	1.9082543256056969387	1.9082542496934643571	1.9082542614231205466	1.9082543223598613623
0.6	2.1789977736182262353	2.1789976874656861387	2.1789976990235336677	2.1789977698761334726
0.7	2.4811258413465451293	2.4811257482379941206	2.4811257582765876593	2.4811258372142764251
0.8	2.8169798161847030997	2.8169797191474114638	2.8169797262745719953	2.8169798117563023046
0.9	3.1894635362182133462	3.1894634380005282949	3.1894634407551647253	3.1894635315695534350
1.0	3.6019982713677825036	3.6019981744856380611	3.6019981712894638995	3.6019982665608147121

The Table 4.11a shows the numerical results of problem 6 for y_1. The results show that the proposed methods 2SBHBDF, 3SBHBDF, and 4SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.11b Numerical Comparison of Exact Solution and the Proposed Methods for Problem 6 at $h = 0.01$ for $y_2(t)$

t	Exact	2SBHBDF	3SBHBDF	4SBHBDF
0.0	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000	1.0000000000000000000
0.1	1.0824199884910901748	1.0824200051988278997	1.0824200012234128341	1.0824199891921174204
0.2	1.1376677589706946021	1.1376678052003208622	1.1376677950069845199	1.1376677608634287591
0.3	1.1747014470219494758	1.1747015253225366920	1.1747015084836254564	1.1747014502199894520
0.4	1.1995258705013361479	1.1995259785325025582	1.1995259556975010180	1.1995258748986766636
0.5	1.2161661791908468270	1.2161663126696098373	1.2161662847451238931	1.2161661846196841982
0.6	1.2273205116776468742	1.2273206659136690119	1.2273206338313627512	1.2273205179436520716
0.7	1.2347974843436955088	1.2347976549733900425	1.2347976196397396412	1.2347974912731347580
0.8	1.2398094490054084462	1.2398096322862802725	1.2398095944508510359	1.2398094564450483982
0.9	1.2431690693881768598	1.2431692622635727131	1.2431692225263847537	1.2431690772157698542
1.0	1.2454210902778164549	1.2454212903314757443	1.2454212491796604128	1.2454210983948924869

The Table 4.11b shows the numerical results of problem 6 for y_2. The results show that the proposed methods 2SBHBDF, 3SBHBDF, and 4SBHBDF agree well with the exact solution as illustrated in the tabulated results. The results also proved that as the number of step size k increases, the accuracy increases.

Table 4.12a Absolute Error $|Y(t) - y_1(t)|$ in prosed methods for Problem 6

t	**2SBHBDF**	**3SBHBDF**	**4SBHBDF**
0.0	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$
0.1	$9.8707306655 \times 10^{-9}$	$7.6696990394 \times 10^{-9}$	$4.201769647 \times 10^{-10}$
0.2	$2.71509604139 \times 10^{-8}$	$2.17397011841 \times 10^{-8}$	$1.1340161002 \times 10^{-9}$
0.3	$4.56246610563 \times 10^{-8}$	$3.71509258764 \times 10^{-8}$	$1.9150801295 \times 10^{-9}$
0.4	$6.22922793079 \times 10^{-8}$	$5.16305659158 \times 10^{-8}$	$2.6314623243 \times 10^{-9}$
0.5	$7.59122325816 \times 10^{-8}$	$6.41825763921 \times 10^{-8}$	$3.2458355764 \times 10^{-9}$
0.6	$8.61525400966 \times 10^{-8}$	$7.45946925676 \times 10^{-8}$	$3.7420927627 \times 10^{-9}$
0.7	$9.31085510087 \times 10^{-8}$	$8.30699574700 \times 10^{-8}$	$4.1322687042 \times 10^{-9}$
0.8	$9.70372916359 \times 10^{-8}$	$8.99101311044 \times 10^{-8}$	$4.4284007951 \times 10^{-9}$
0.9	$9.82176850513 \times 10^{-8}$	$9.54630486209 \times 10^{-8}$	$4.6486599112 \times 10^{-9}$
1.0	$9.68821444425 \times 10^{-8}$	$1.000783186041 \times 10^{-7}$	$4.8069677915 \times 10^{-9}$

The errors of this method at each k-step show that error becomes smaller as the step size increases. It also observed that there is accuracy asstep size increases.

Table 4.12b Absolute Error $|Y(t) - y_2(t)|$ in prosed methods for Problem 6

t	2SBHBDF	3SBHBDF	4SBHBDF
0.0	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$	$0.00000000000 \times 10^{0}$
0.1	$1.67077377249 \times 10^{-8}$	$1.27323226593 \times 10^{-8}$	$7.010272456 \times 10^{-10}$
0.2	$4.62296262601 \times 10^{-8}$	$3.60362899178 \times 10^{-8}$	$1.8927341570 \times 10^{-9}$
0.3	$7.83005872162 \times 10^{-8}$	$6.14616759806 \times 10^{-8}$	$3.1980399762 \times 10^{-9}$
0.4	$1.080311664103 \times 10^{-7}$	$8.51961648701 \times 10^{-8}$	$4.3973405157 \times 10^{-9}$
0.5	$1.334787630103 \times 10^{-7}$	$1.055542770661 \times 10^{-7}$	$5.4288373712 \times 10^{-9}$
0.6	$1.542360221377 \times 10^{-7}$	$1.221537158770 \times 10^{-7}$	$6.2660051974 \times 10^{-9}$
0.7	$1.706296945337 \times 10^{-7}$	$1.352960441324 \times 10^{-7}$	$6.9294392492 \times 10^{-9}$
0.8	$1.832808718263 \times 10^{-7}$	$1.454454425897 \times 10^{-7}$	$7.4396399520 \times 10^{-9}$
0.9	$1.928753958533 \times 10^{-7}$	$1.531382078939 \times 10^{-7}$	$7.8275929944 \times 10^{-9}$
1.0	$2.000536592894 \times 10^{-7}$	$1.589018439579 \times 10^{-7}$	$8.1170760320 \times 10^{-9}$

The errors of this method at each k-step show that error becomes smaller as the step size increases. It also observed that there is accuracy as step size increases.

Figure 4.1 – 4.6 shows the graph for Absolute Errors in the Proposed Methods for Problem 1 – 6.

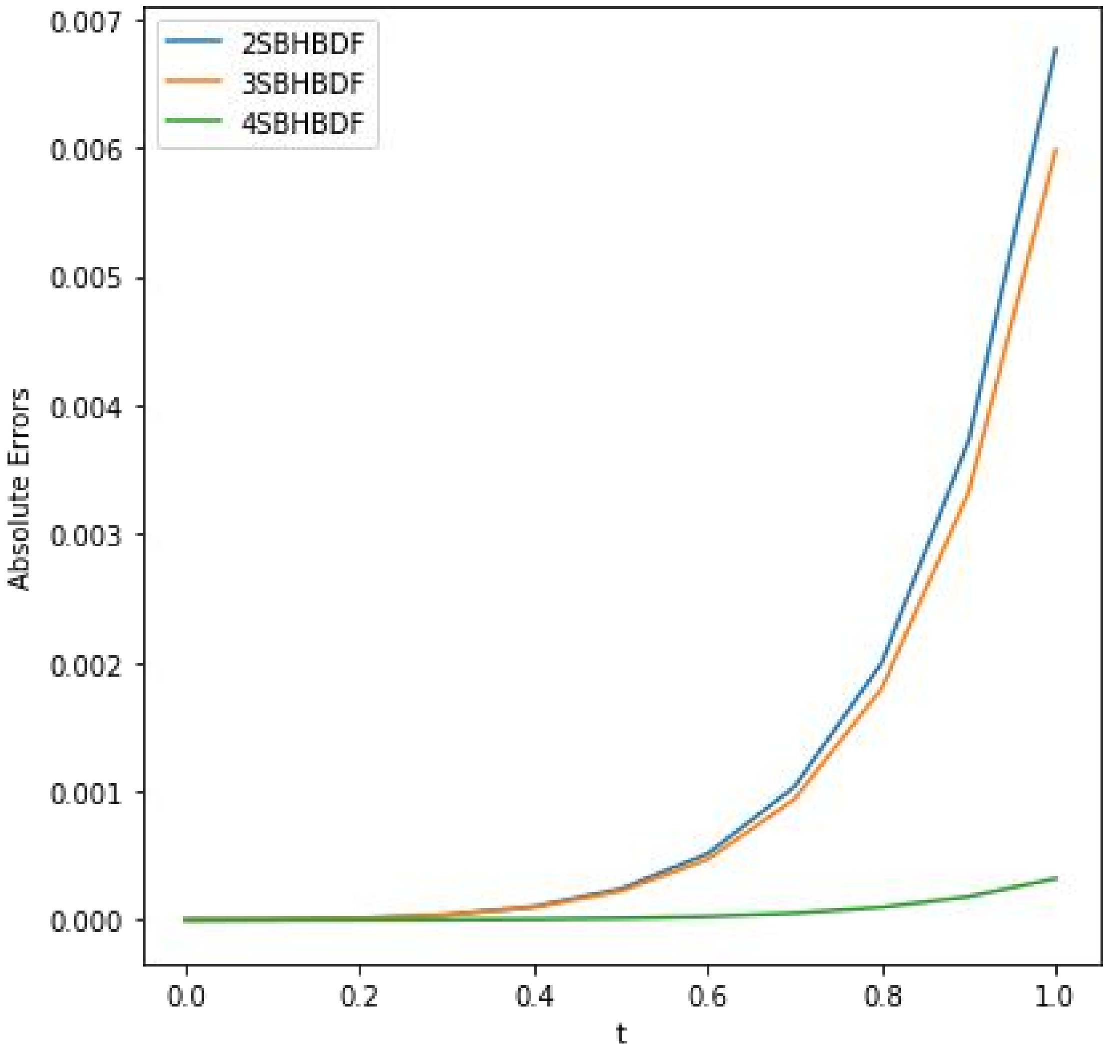

Figure 4.1 Plot of Absolute Errors in the Proposed Methods for Problem 1

The Figure 4.1 above shows the absolute error of the proposed methods of 2SBHBDF, 3SBHBDF and 4SBHBDF. From the figure above, it shows that 4SBHBDF has least error which implies that k=4 produce better result than 2SBHBDF and 3BHBDF. Also 3BHBDF has lesser absolute error than 2SBHBDFwhich proved that as number of step size k increases, the smaller the error.

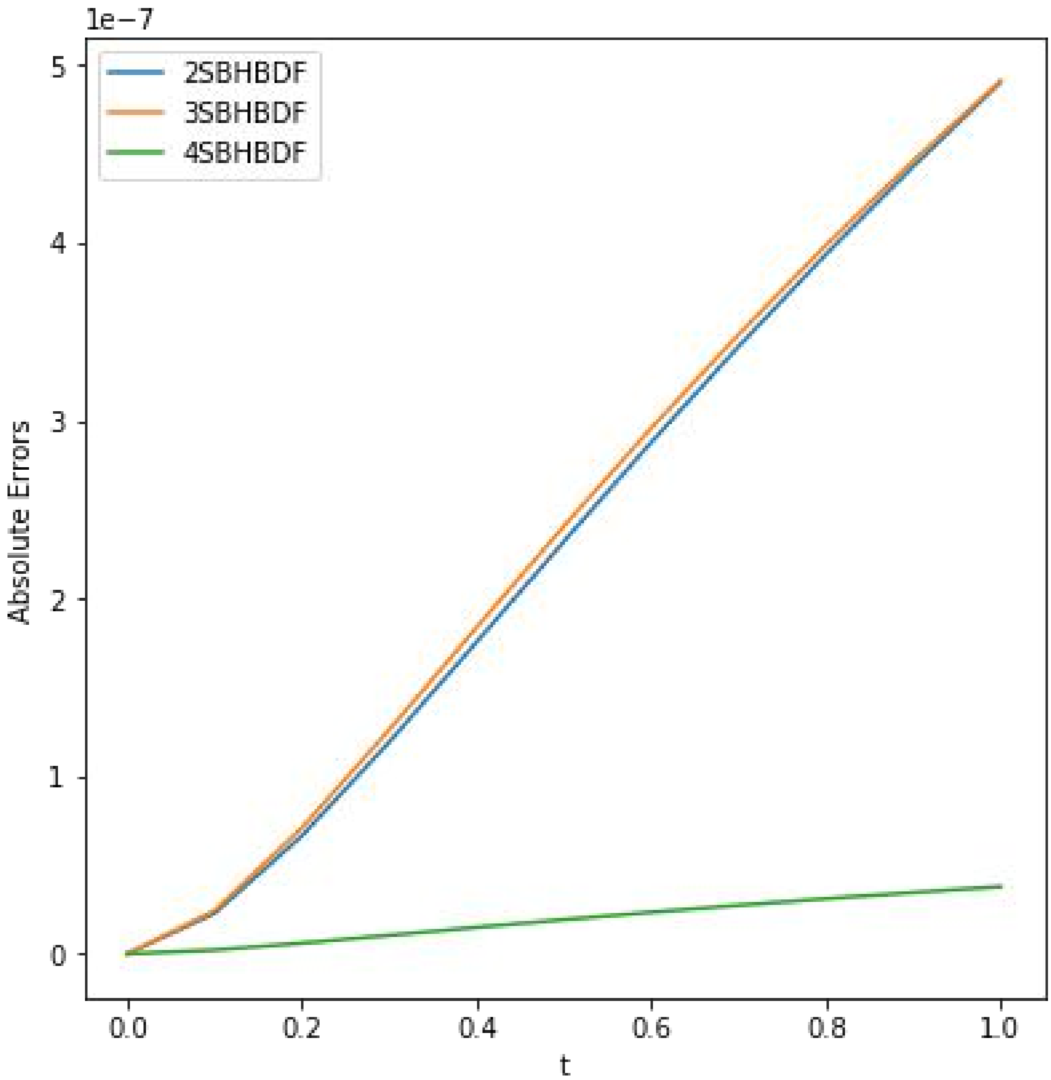

Figure 4.2 Plot of Absolute Errors in the Proposed Methods for Problem 2

Figure 4.2 above depicts the error of the proposed methods for problem 2. From the above figure it shows that 4SBHBDF has an error closer to zero than 2BHBDF and 3BHBDF which implies that at k=4 produce better result than k=2 and k=3 while k=3 also producebetterthan k=2.

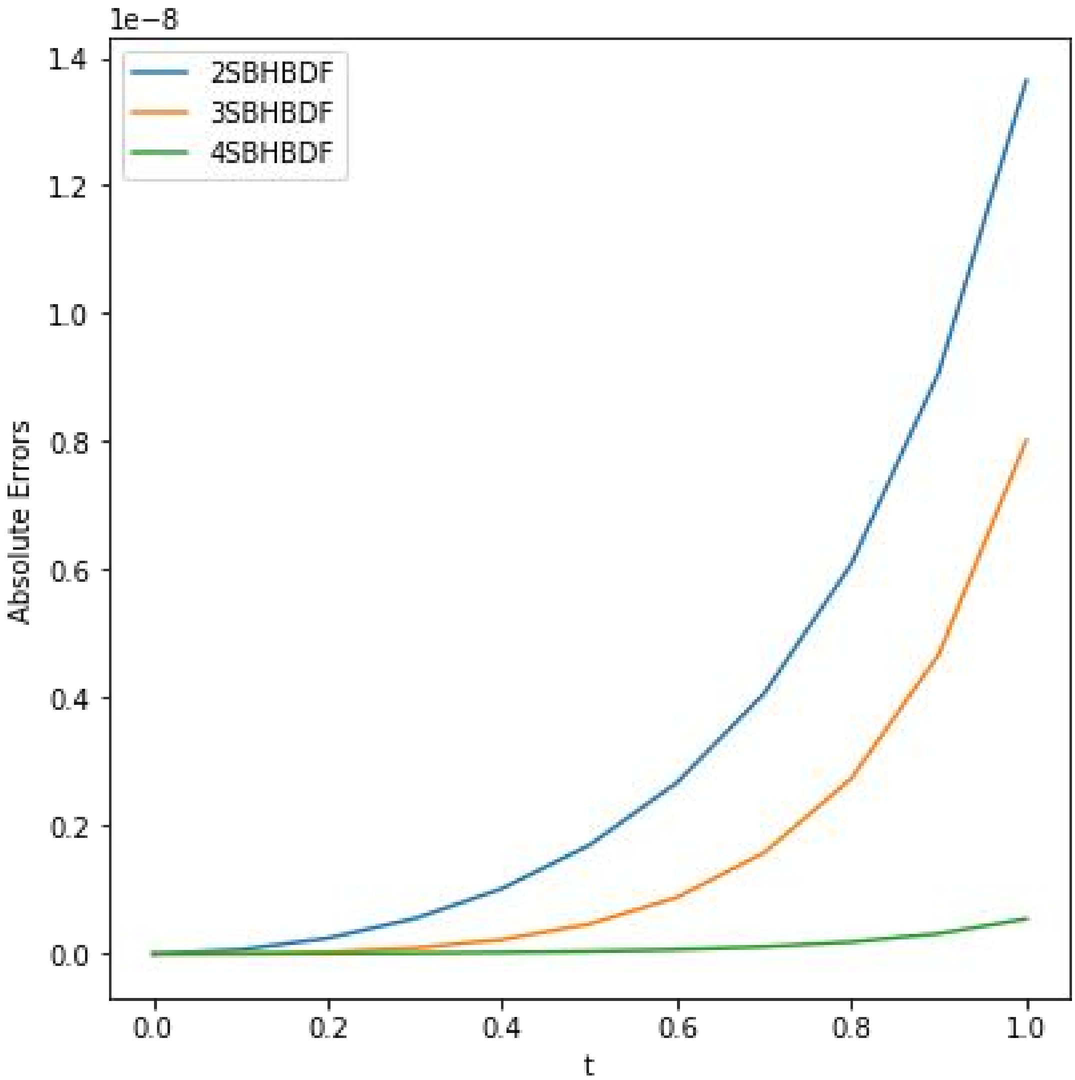

Figure 4.3 Plot of Absolute Errors in the Proposed Methods for Problem 3

The Figure 4.3 above shows the absolute error of the proposed methods of 2SBHBDF, 3SBHBDF and 4SBHBDF. From the figure above, it shows that k=4 is better than k=3 and k=3 is better than k=2.

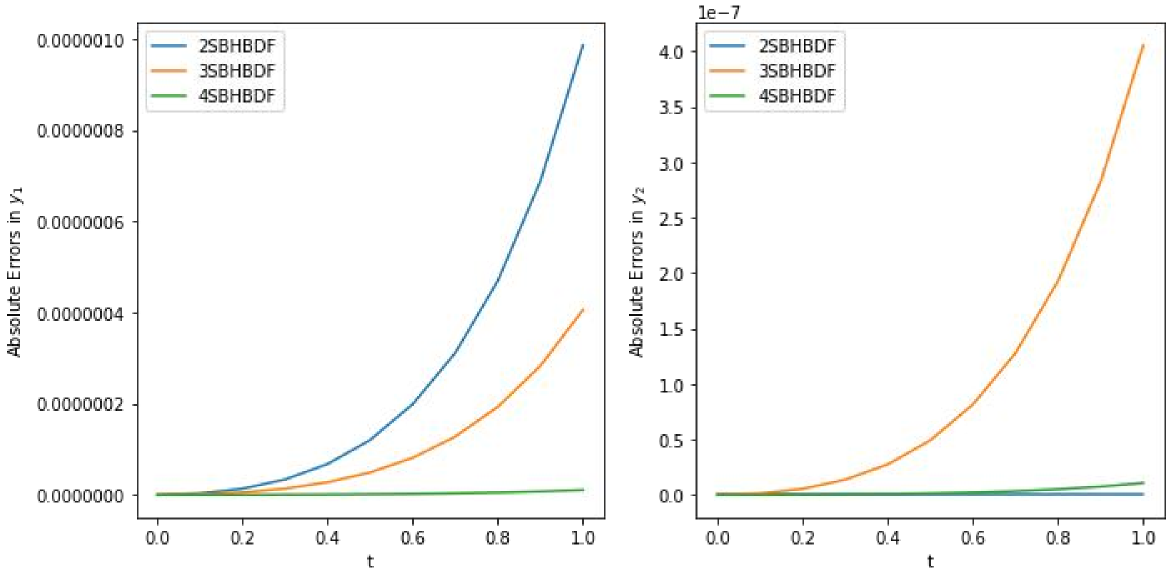

Figure 4.4 Plot of Absolute Errors in the Proposed Methods for Problem 4

The Figure on the left hand side represents the absolute error for y_1 while the figure on the right hand side represents the absolute error for y_2 . From the figure it is observed that as the number of step k increases, the absolute error in the solution obtained with the proposed methods reduces.

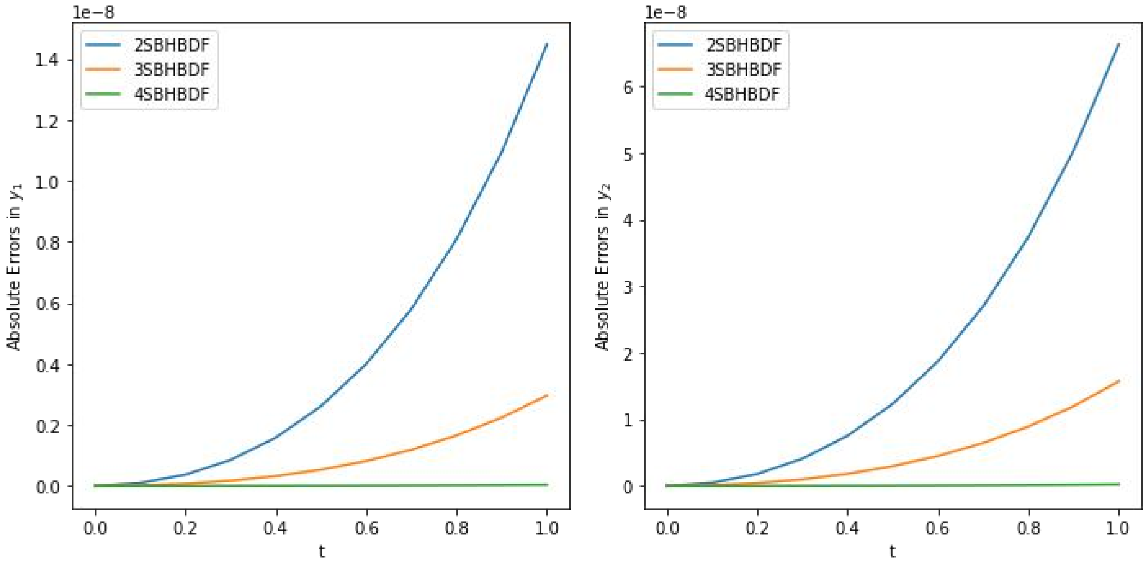

Figure 4.5 Plot of Absolute Errors in the Proposed Methods for Problem 5

Figure 4.5: Absolute error in y_1 (left)and y_2 (right) using the proposed methods for problem 5 with h=0.01.

From Figure 4.5, it is observed that as the number of step k increases, the absolute error at k=4 for the proposed methods is smaller than k=2and k=3.

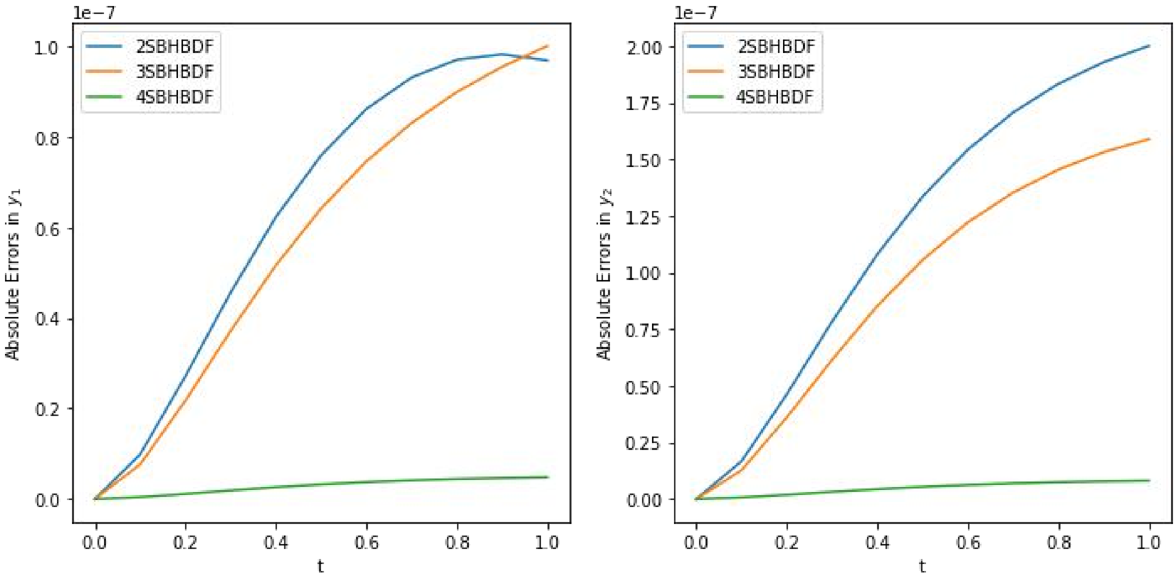

Figure 4.6 Plot of Absolute Errors in the Proposed Methods for Problem 6

Figure 4.6: Absolute error in y_1 (left)and y_2 (right) using the proposed methods for Figure problem 6.

From Figure 4.6: It shows that k=4 performs better than k=3 while k=3 performs better than k=2.

Figure 4.6: Absolute error in y_1 (left) and y_2 (right) using the proposed methods for problem 6.

From Figure 4.6: It shows that k=4 performs better than k=3 while k=3 performs better than k=2.

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this research, block hybrid backward differentiation of order (k+1) have been developed by the interpolation and collocation techniques with the incorporation of k off- step points for the solution of second order ordinary differential equations (ODEs). The power series expansion was used as the basis function. It also proved that the methods are consistent, zero- stable and convergent. Six problems were used to test the efficiency of the methods. It was also observed from the error tables and figures that the BHBDF performed better in solving problems of second order ordinary differential equation as they produce lesser errors. Hence, this our method has developed a supremacy over already existing literatures.

5.2 Recommendation for Further Work

It is proposed for further research that;

i. Other basis functions asides power series be considered (Legendre Polynomial).

ii. The number of k- steps be increased as a chance of testing further the performance of the methods.

5.3 Contributions to Knowledge

The research work incorporated some carefully selected off-grid points in the derivation process of a class of k-step liner multistep methods has improved the order of accuracy of the method for the solution of some classes of second order ordinary differential equations. It is established from the analysis that the off-grid points $\left(\frac{1}{2},\frac{3}{2}\right)$, $\left(\frac{5}{2},\frac{11}{4}\right)$, and $\left(\frac{15}{4},\frac{31}{8}\right)$ yield order of accuracy $p=(3,3,3,3,3,3,3,3)^T$, $p=(4,4,4,4,4,4,4,4,4,4)^T$ and $p=(5,5,5,5,5,5,5,5,5,5,5,5)^T$ respectively. Furthermore, the study has been able to develop a reliable methods in solving problems resulting from engineering and sciences among others with maximum errors of 10^{-9}.

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