DEVELOPMENT OF FALKNER-TYPE METHOD FOR NUMERICAL SOLUTION OF SECOND ORDER INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

This research focuses on the formulation of block hybrid methods with power series as basic function through interpolation and collocation techniques for numerical solution of second order initial value problems in ordinary differential equations. The step number for the derived block hybrid method is k=2 with two off-step point and four off-step points. The basic properties of numerical methods were analyzed and findings revealed that the methods were consistent, zero-stable and convergent which makes them suitable for solving the class of problems considered such as linear and non-linear problems, oscillatory problems, Dynamic problem and Stiff system. The results obtained from the proposed methods, show that the methods are of higher accuracy and have superiority over some existing methods considered in the literature.

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CHAPTER ONE

1.0 INTRODUCTION

1.1 Background to the Study

The mathematical formulation of physical phenomena in science and engineering often leads to differential equations, which can be categorized as an ordinary differential equation (ODE) and a partial differential equation (PDE). This formulation will explain the behavior of the phenomenon in detail. The search for solutions of real-world problems requires solving ODEs and thus has been an important aspect of mathematical study. These mathematical models are represented in the form of first order or higher differential equation. The reliability of numerical approximation techniques in solving such problems has been proven by many researchers as the role of numerical methods in engineering problems solving has increased drastically in recent years.

Numerical analysis is the study of algorithms that use numerical approximation for problems of mathematical analysis. The numerical method for solving ordinary differential equations (ODEs) is the most powerful technique ever developed in continuous time dynamics; these are developed since most of the differential equations cannot be solved analytically (Chollum, 2004).

A differential equation, shortly DE is an equation involving a relationship between an unknown function and one or more of its derivatives.

Depending upon the domain of the functions involved we have ordinary differential equations, or shortly ODE, is when the unknown function depends on a single independent variable. Also, if it involves partial derivatives with respect to more, than one independent variable, then the differential equation is called a partial differential equation (PDE).

Our goal is to obtain a numerical solution for a second-order initial value problem (I.V.P.) of the general form

$$y'' = f(x, y, y') \qquad y(a) = y_0,\ \ y'(a) = y'_0 \ x \in [a, b] \tag{1.1}$$

Although it is possible to integrate a second-order I.V.P. by reducing it to a first order system and applying then one of the methods available for such systems, it seems more natural to provide numerical methods in order to integrate the problem directly.
The advantage of this procedure lies in the fact that they are able to exploit special information about ODEs, resulting in an increase in efficiency.

In order to get more accurate numerical solutions with less effort to solve the second-order differential equation, many scholars including Vigo-Aguiar and Ramos (2016), Mazzia and Nagy (2015), Mazzia *et al.* (2012), Sommeijer (1993), Brugnano and Trigiante (1998), Butcher and Hojjati (2005), Franco (2002), Mahmoud and Osman (2009), among others, developed different numerical methods to give the approximate solutions to (1.1) and higher-order ODEs without reducing it to a system of first-order ODEs. Some of the numerous numerical approaches presented by the aforementioned researchers included higher derivative multistep methods, Runge-Kutta methods, spline-collocation methods, and Runge-Kutta-Nystrom methods. It is well-known that a Runge-Kutta-Nystrom method for solving (1.1) has a greater improvement as compared to standard Runge-Kutta methods. In case of a linear k-step method for first order ODEs, it becomes a $2k$-step method for (1.1), thus increasing the computational work.

The need to improve on the aforementioned methods became important to researchers in this area. Different scholars such as Ramos and Singh (2016), Ramos and Rufai (2018), Ramos and Lorenzo (2010), have developed block methods for solving higher-order

ODEs directly.In block methods, the approximations are simultaneously obtained at a number of 2 consecutive grid points in the interval of integration. These methods are less costly in terms of number of function evaluations compared to the reduction method and linear multistep methods.

A merit of the block methods over traditional predictor-corrector ones is that they give better approximate solutions when solving many problems of the form in (1.1) and higher-order ODEs directly.

This research is focused on development of Falkner-type method for numerical solution of second order initial value problems (IVPs) in ordinary differential equations.

1.2 Statement of the Research Problem

Several numerical methods cannot solve the second order problem directly without reducing to lower order equation. Also, these methods have been found to have major setbacks such as large computer storage memories because of too many auxiliary functions, wastage of computer time and a lot of human effort. Awoyemi (1992), the inability of the method to utilize additional information related to the specific ODEs and the lower order of accuracy of the methods used to solve the system of first order after it has been reduced compared to the increased dimension of the original problem.

Predictor-corrector method was also reported to have some major drawbacks due to the number of functions evaluated and the order of accuracy of corrector is higher than the predictor especially when there is need to interpolate and collocate at grid and off-grid point. This major setback of predictor-corrector methods are extensively addressed by Jator (2007). Also to overcome this setback of predictor-corrector methods researchers have proposed the block methods which gives solutions at each grid within the interval of integration without overlapping. They are implemented in a block-by-block fashion in other to reduce the burden of developing predictors (Jator 2007, Jator and Li 2009).

1.3 Aim and Objectives of the Study

The aim of this research work is to develop a Falkner-type method for the solution of the second order initial value problems (IVPs).

The following objectives are to be achieved

i. To construct a block hybrid method for $k=2$ for the solution of second order initial value problems.

ii. To obtain the order and error constant, zero stability, consistency and convergence of the method.

iii. To apply the proposed method to solve initial value problems.

iv. To compare the results with some existing methods found in the literature.

1.4 Justification of the Study

The development of Hybrid Falkner-Type method will enhance, enrich and strengthen the subject of numerical methods for solution of second order ODEs. The proposed methods show that, from a single continuous scheme, multiple finite difference methods can be obtained which would allow the new methods to be self-starting and it is useful in numerical solution at several points without starting values, hence, increases the speed of integration.

1.5 The Scope of the study

Second order initial value problems (IVPs) in ODEs were considered in this thesis.

1.6 Limitation of the Study

The research is limited to the following;

i. Power series polynomial was considered as basis function because of its smoothness.

ii. The research is limited to the formulation of Falkner-type method.

iii. The research considers Second order initial value problems.

1.7 Falkner Type Method

The Falkner-type method is used to solve differential system of second order ordinary differential equations and its general form is as follows:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{j=0}^{k-1} \beta_j \nabla^j f_n \tag{1.2}$$

$$y'_{n+1} = y'_n + h^2 \sum_{j=0}^{k-1} \gamma_j \nabla^j f_n \tag{1.3}$$

1.8 Maple Software

The software package has the ability to algebraically manipulate mathematical expressions and find symbolic solutions to certain problems such as those arising from ordinary and partial differential equations. In this work, Maple (2015) is used for simulation.

CHAPTER TWO

2.0 LITERATURE REVIEW

2.1 Review of Existing Method

The development of numerical methods for the solution of initial value problems (IVPs) of ordinary differential equations (1.1).

Reduction method is extensively use earlier in application to reduce higher order to first order differential equation of the form;

$$y' = f(x, y,), \quad y(a) = y_0, a \le x \le b; x, y \varepsilon R^n \quad \text{and} \quad f \varepsilon C^{-1}[a, b] \qquad (2.1)$$

After which existing method are used to solve the resultant systems.

Hasan *et al.* (2014), presented an implicit method for solving first order singular initial value problem. The method is extended to solve second or higher order problems having a singular point. The method presents more correct result than those obtained by the implicit Euler and second order implicit Runge-Kutta (RK2) method.

Jator and King (2018), provide greater accuracy of high-order methods with larger step-sizes than lower order methods. Hence, the method is based on Block Hybrid Method (BHM) of order II for directly solving systems of general second-order initial value problems (IVPs), including Hamiltonian systems and partial differential equations (PDEs), which arise in multiple areas of science and engineering. The BHM is formulated from a continuous scheme based on a hybrid method of a linear multistep type with several off-grid points and then implemented in a block by block manner.

Ramos *et al.* (2016), presented an optimized two-step hybrid block method for the numerical integration of general second-order initial value problems. The method considers two intra-step points which are selected adequately in order to optimize the local truncation errors of the main formulas for the solution and the derivative at the final point of the block. The new method is zero-stable and consistent with fifth algebraic order.

Tumba *et al.* (2018), studied a one-step hybrid block method for initial value problems of general second order ordinary differential equations. In the derivation of the method, power series is adopted as basis function to obtain the main continuous scheme through collocation and interpolations approach. Taylor method is also used together with new method to generate the non-overlapping numerical results. Suleiman and Gear (2012), show analysis of four points direct block one-step method for solving directly the general second order non stiff initial value problems (IVPs) of ordinary differential equations. Cohen and Schweitzer (2014), presented the numerical solution of nonlinear Hamiltonian highly oscillatory systems of second-order differential equations of a special form.

Sekar (2014), used the Adomian Decomposition Method (ADM) to obtain the numerical solution the different types of second order systems like stable, unstable, stiff systems and system with singular-A. Sagir (2014), discrete linear multistep block method of uniform order for the solution of first order initial value problems (IVPs) in ordinary differential equations (ODEs) is presented in this paper. The approach of interpolation and collocation approximation were adopted in the derivation of the method which is then applied to first order ordinary differential equations with associated initial conditions. The continuous hybrid formulations enable us to differentiate and evaluate at some grids and off – grid points to obtain four discrete schemes, which were used in block form for parallel or sequential solutions of the problems.

Ivaz *et al.* (2013), developed algorithms for solving first-order fuzzy differential equations and hybrid fuzzy differential equations.

2.2 Review of Falkner-type Method

Li (2016), proposed and studied a family of improved Falkner-type Methods for the oscillatory system $u''(t)+Mu(t)=g(t,u(t))$ where $g:\mathbb{R}\times\mathbb{R}^d\rightarrow\mathbb{R}^d$, in which the first derivative does not appear explicitly, and $M\in\mathbb{R}^{d\times d}$ is a symmetric positive semi-definite matrix. The new methods take into account the oscillatory structures of the problem and exactly integrate the unperturbed problem $u''(t)+Mu(t)=0$. Ramos and Lorenzo (2010), showed the review of explicit Falkner methods and its modifications for solving special second-order IVPs. Ramos *et al.* (2017), proposed a unified approach for the development of k-step block Falkner-type methods for solving general second-order initial-value problems in ODEs. A family of k-step block multistep methods where the main formulas are of Falkner-type is proposed for the direct integration of the general second order initial-value problem where the differential equation is of the general form. The two main Falkner formulas and the additional ones to complete the block procedure are obtained from a continuous approximation derived via interpolation and collocation at $k+1$ points.

Ramos and Rufai (2018), developed and analyzed a modified family of Falkner-type methods for solving differential systems of second-order initial- value problems. The approaches of collocation and interpolation are adopted to derive the new methods. These modified methods are implemented in block form to obtain the numerical solutions to the considered problems.

Ramos and Singh (2016), presented an efficient variable step-size rational Falkner-type method for solving the special second-order initial value problems. A rational one-parameter family of Falkner-type explicit methods is firstly presented for directly solving numerically special second order initial value problems in ordinary differential equations.

Nicolas (2019), developed a class of *k*-step hybrid Falkner-type block numerical integration schemes for solving second order ODEs. The proposed technique utilizes orthogonal polynomial and power series to form a trial solution which were evaluated at some selected points while its first and second derivatives were evaluated at selected grid and off-grid points in order to obtain a continuous scheme.

2.3 Review of Collocation Method

Mohammed *et al.* (2019), developed an implicit continuous four-step hybrid backward difference formulae for the direct solution of stiff system. For this purpose, the Chebyshev polynomial was employed as the basis function for the development of schemes in a collocation and interpolation techniques. The schemes were analyzed using appropriate existing theorem to investigate their stability, consistency, convergence and the investigation shows that the developed schemes are consistent, zero-stable and hence convergent.

Costabile and Napoli (2011), studied a class of collocation methods for numerical integration of initial value problems. Vigo-Aguiar and Ramos (2006), considered the construction of a special family of Runge–Kutta (RK) collocation methods based on intra-step nodal points of Chebyshev–Gauss–Lobatto type, with A-stability and stiffly accurate characteristics. This feature with its inherent implicitness makes them suitable for solving stiff initial-value problems. Cardone *et al.* (2018), presented a collection of recent results on the numerical approximation of Volterra integral equations and integro differential equations by means of collocation type methods, which are able to provide better balances between accuracy and stability demanding. Saadatmandi and Dehghan

(2008), developed a numerical technique Collocation Method for Solving Abel's Integral Equations of First and Second Kinds. The solutions of such equations may exhibit a singular behavior in the neighbourhood of the initial point of the interval of integration. The proposed method is based on the shifted Legendre collocation technique.

Xiu and Hesthaven (2005), proposed a high-order collocation methods for differential equations with random inputs. Levin (1996), analyzed a collocation method for approximating integrals of rapidly oscillatory functions. The method is efficient for integrals involving Bessel functions $J_v(rx)$ with a large oscillation frequency parameter r, as well as for many other one- and multi-dimensional integrals of functions with rapid irregular oscillations. The analysis provides a convergence rate and it shows that the relative error of the method is even decreasing as the frequency of the oscillations increases.

2.4 Block Method

Chu and Hamilton (1987) defined a block method as follows;

Let Y_m and F_m be defined as by $Y_m = (y_n, y_{n+1}, \ldots, y_{n+r-1})$, $F_m = (f_n, f_{n+1}, \ldots, f_{n+r-1})$.

The general k-block method is a matrix of finite difference equation of the form;

$$Y_m = \sum_{j=1}^{k} A_i y_{m-1} + h\sum_{i=1}^{k} \beta_j f_{m-1} \tag{2.2}$$

where $A_i's$ and $\beta_i's$ are properly chosen $r \times r$ matrix coefficients and m=0,1,2,… represent the block number, n=m is the step number of the mth block and r is the proposed block size. At each application of a block method, the solution will be approximated in more than one point. Abbas and Alshakhoori (2018), derived a new block method of order five for the numerical solution of initial value problems is derived. Mukhtar and Abdul-majid (2011), presented a four point direct block one-step method for solving directly the general second order non-stiff initial value problems (IVPs) of ordinary differential equations (ODEs). Yakusak and Adeniyi (2015), derived a Four Step Hybrid Block Method for First Order Initial Value Problems in Ordinary Differential Equations.

Mehrkanoon *et al.* (2009), Block Method for Numerical Solution of Fuzzy differential Equations. Fookand and Ibrahim (2017), studied the numerical method for solving second order Fuzzy Differential Equations (FDEs) using Block Backward Differential Formulas (BBDF) under generalized concept of higher-order fuzzy differentiability. Ramos *et al.* (2016), proposed a unified approach for the development of k-step block Falkner-type methods for solving general second-order initial-value problems in ODEs.

2.5 Hybrid Method

Over the years, observation has been made that numerical analysis has over time given to solution at grid and off-grid point only and suffered the disadvantage of requiring special procedure for step length changing and also a weak stability properties for a number of function evaluations per step. This observation was made by Chollum (2004). These difficulties can be reduced by lowering the step size number of the linear multistep method without necessarily reducing the order. Gragg and Steller (1964) overcome the difficulties working in conjunction with Butcher (1964) and Gear (1965) introducing a modified linear multistep formula which incorporates a function evaluated at an off-grid point. The method was called hybrid because it retains some properties of linear multistep and Runge-kutta methods which substitution and extrapolation method lies between. Hybrid use data at points other than the step point $[n_n \mid x_n = a + nh]$, Bryne and Lambert (1966), proposed a generalization of Runge-kutta methods in which the computed derivatives in earlier stages are used alongside stage derivatives found in the current step to compute the output value in the next step which is evaluated in the same way as for the Runge-kutta methods. Derivatives evaluated at previous step is given as $F^{[n-1]}$, i=1,2,…,s and the present step derivative by $F^{[n]}$, i=1,2,…,s equations for a single step of the method is as follows.

$$Y_i = y_{n-1} + h\sum_{j=0}^{s} a_{ij} F_j^n F_i^n = f(x_{n-1} + hc_i Y_i) y_n \tag{2.3}$$

$$Y_i = y_{n-1} + h\left(\sum_{j=0}^{s} b_i F_i^n + \sum_{j=0}^{s} b_i F_i^{n-1}\right). \tag{2.4}$$

Thus the k-step hybrid method is defined by;

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} + \beta_v f_{n+v} \tag{2.5}$$

Where $\alpha_k = +1,$ and β_0 are both not zero, $v \notin [0,1,\ldots,k]$ and $f_{n+v} f(x_{n+v}, y_{n+v})$.

Several researchers have developed hybrid methods for the solution of initial value problems. Some of the researchers are Ramos *et al.* (2016), presented a new optimized two-step hybrid block method for the numerical integration of general second-order initial value problems. Jator (2010), proposed a family of hybrid linear multistep methods (HLMMs) with two non-step points for the direct solution of second order initial value problems (IVPs). Jator and king (2018), presented a Block Hybrid Method (BHM) of order II for directly solving systems of general second-order initial value problems. Yakusak and Adeniyi (2015), developed a Four Step hybrid block method for the solution of general first order Initial Value Problem (IVP) in Ordinary Differential Equations (ODEs) by collocation and interpolation techniques and with Chebyshev polynomial of the first kind as basis function. Ivaz *et al.*(2013), investigated a numerical algorithms for solving first-order fuzzy differential equations and hybrid fuzzy differential equations. Jator (2010), proposed a family of hybrid linear multistep methods (HLMMs) with two non-off step points for the direct solution of second order initial value problems. The methods are applied in block form as simultaneous numerical integrators over non-overlapping intervals.

CHAPTER THREE

3.0 MATERIALS AND METHODS

3.1 Derivation of the Methods

In this section, we derive some linear multi-step methods in the form

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{j=0}^{k-1} \beta_j \nabla^j f_n \tag{3.1}$$

$$y'_{n+1} = y'_n + h^2 \sum_{j=0}^{k-1} \gamma_j \nabla^j f_n \,, \tag{3.2}$$

where h is the step-size, y_n and y'_n are numerical approximations to the theoretical solution and its derivative at the grid point $x_n = a + nh;\ n = 0,1,2,3,...,N,\ h = \frac{(b-a)}{N},\ f_n = f\left(x_n, y_n, y'_n\right)$ and $\nabla^j f_n$ is the standard notation for the backward differences.

We then construct the continuous approximation by imposing the following conditions

$$\left.\begin{aligned} y_{n+k-r} &= Y(x_{n+k-r}) \\ y'_{n+k-r} &= Y'(x_{n+k-r}) \\ y''_{n+j} &= Y''(x_{n+j}) = f(x_{n+j}) \end{aligned}\right\}. \tag{3.3}$$

Equation (3.3) leads to a system of equations and unknowns written in the form $AX = B$

$$X = \begin{bmatrix} 1 & x_{n+k-r} & x_{n+k-r}^2 & \cdots & x_{n+k}^{k+2} \\ 0 & 1 & 2x_{n+k-r} & \cdots & (k+2)x^{k+1} \\ 0 & 0 & 2 & \cdots & (k+2)(k+1)x_n^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 2 & \cdots & (k+2)(k+1)x_{n+k}^k \end{bmatrix}$$

$$A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{k+2} \end{bmatrix} \quad B = \begin{bmatrix} y_{n+k-r} \\ y'_{n+k-r} \\ f_n \\ \vdots \\ f_{n+2} \end{bmatrix} \tag{3.4}$$

Solving (3.4) by Gaussian elimination method, the coefficients α_j can be obtained. Substituting the coefficients α_j into (3.1) yields the continuous scheme:

$$Y(x) = \alpha_j(x) y_{n+j} + \alpha_0'(x) h y'_{n+j} + h^2 \left[\sum_{j=0}^{k} \beta_j(x) f_{n+k} + \beta_v(x) f_{n+v} \right] \tag{3.5}$$

where $\alpha_j(x)$, $\beta_{\text{j}}(x)$ and $\beta_{\text{v}}(x)$ are continuous coefficients. We note that (3.3) involves first derivative, which can be obtained by substituting the coefficients of α_j into the first derivative of (3.2) to yield

$$Y'(x) = \alpha_j'(x) y'_{n+j} + h \left[\sum_{j=0}^{k} \beta_j'(x) f_{n+k} + \beta_v'(x) f_{n+v} \right] \ . \tag{3.6}$$

3.2 Specification of the Method

3.2.1 Two step method with $\frac{2}{3}$ and $\frac{5}{3}$ off-step points

To derive a continuous two step method by considering two off-step points $\frac{2}{3}$ and $\frac{5}{3}$, the following specification were considered $r = 1$, k=2, s=5. The continuous method with the continuous coefficient $\alpha_j(x)$, $\beta_j(x)$ and $\beta_v(x)$ is of the form:

$$y(x) = \alpha_0 y_n + \alpha_0' h y_n' + h^2 \left[\beta_0 f_n + \beta_1 f_{n+\frac{2}{5}} + \beta_2 f_{n+1} + \beta_3 f_{n+\frac{5}{3}} + \beta_4 f_{n+2} \right] \tag{3.7}$$

where:

$$\propto_0 = 1 \tag{3.8}$$

$$\propto_1 = -x_n + x \tag{3.9}$$

$$\beta_0 = \left.\begin{aligned} &\frac{1}{1200}\frac{x_n^2(600h^4+720h^3x_n+455h^2x_n^2+144hx_n^3+18x_n^4)}{h^4} \\ &-\frac{1}{300}\frac{x_n(300h^4+540h^3x_n+72hx_n^2+18x_n^3)x}{h^4} \\ &+\frac{1}{40}\frac{(20h^4+72h^3x_n+91h^2x_n^2+48hx_n^3+9x_n^4)x^2}{h^4} \\ &-\frac{1}{60}\frac{(36h^3+91h^2x_n+72hx_n^2+18x_n^3)x^3}{} \\ &+\frac{1}{240}\frac{(91h^2+144hx_n+54x_n^2)x^4}{h^4}-\frac{3}{100}\frac{(4h+3x_n)x^5}{h^4}+\frac{3}{200}\frac{x^6}{h^4} \end{aligned}\right\} \tag{3.10}$$

$$\beta_1 = \left.\begin{aligned} &-\frac{3}{600}\frac{x_n^3(100h^3+105h^2x_n+42h^2x_n+2hx_n^2+6x_n^3)}{h^4} \\ &+\frac{9}{80}\frac{x_n^2(50h^3+70h^2x_n+72hx_n^2+35hx_n^2+6x_n^3)x}{h^4} \\ &-\frac{9}{16}\frac{x_n(10h^3+21h^2x_n+14hx_n^2+3x_n^3)x^2}{h^4}+\frac{3}{8}\frac{(5h^3+21h^2x_n+6x_n^3)x^3}{h^4} \\ &-\frac{9}{32}\frac{(7h^2+14hx_n+6x_n^2)x^4}{h^4}+\frac{9}{80}\frac{(7h+6x_n)x^5}{h^4}-\frac{9}{80}\frac{x^6}{h^4} \end{aligned}\right\} \tag{3.11}$$

$$\beta_2 = \left.\begin{aligned} &\frac{1}{120}\frac{x_n^3(200h^3+260h^2x_n+117hx_n^2+18x_n^3)}{h^4} \\ &-\frac{1}{120}\frac{x_n^2(6000h^3+1040h^2x_n+585hx_n^2+585hx_n^2+108x_n^3)x}{h^4} \\ &+\frac{1}{4}\frac{x_n(20h^3+52h^2x_n+39hx_n^2+9x_n^3)x^2}{h^4} \\ &-\frac{1}{12}\frac{(20h^3+104h^2x_n+117hx_n^2+117hx_n^2+36x_n^3)x^4}{h^4} \\ &-\frac{9}{32}\frac{(52h^2+117hx_n+54x_n^2)x^4}{h^4}-\frac{3}{40}\frac{(13h+12x_n)x^5}{h^4}-\frac{3}{20}\frac{x^6}{h^4} \end{aligned}\right\} \tag{3.12}$$

$$\beta_3 = \left.\begin{aligned} &-\frac{3}{200}\frac{x_n^3(40h^3+60h^2x_n+33hx_n^2+6x_n^3)}{h^4} \\ &-\frac{9}{200}\frac{x_n^2(40h^3+80h^2x_n+55hx_n^2+12hx_n^3)x}{h^4} \\ &-\frac{9}{20}\frac{x_n(4h^3+12h^2x_n+11hx_n^2+3x_n^3)x^2}{h^4} \\ &+\frac{3}{30}\frac{(4h^3+24h^2x_n+33hx_n^2+12x_n^3)x^3}{h^4} \\ &-\frac{9}{40}\frac{x_n(4h^2+11hx_n+6x_n^2)x^4}{h^4}-\frac{9}{200}\frac{(11h+12x_n)x^5}{h^4}-\frac{9}{100}\frac{x^6}{h^4} \end{aligned}\right\} \tag{3.13}$$

$$\beta_4 = \left.\begin{aligned} &\frac{1}{480}\frac{x_n^3(100h^3+155h^2x_n+90hx_n^2+18x_n^3)}{h^4} \\ &-\frac{1}{240}\frac{x_n^2(150h^3+310h^2x_n+225hx_n^2+54hx_n^3)x}{h^4} \\ &+\frac{1}{16}\frac{x_n(10h^3+31h^2x_n+30hx_n^2+9x_n^3)x^2}{h^4} \\ &+\frac{1}{24}\frac{(5h^3+31h^2x_n+45hx_n^2+18x_n^3)x^3}{h^4} \\ &+\frac{1}{96}\frac{x_n(31h^2+90hx_n+54x_n^2)x^4}{h^4}-\frac{3}{80}\frac{(5h+6x_n)x^5}{h^4}-\frac{3}{80}\frac{x^6}{h^4} \end{aligned}\right\} \tag{3.14}$$

Evaluating (3.7) above at point $x = x_{n+\frac{2}{3}}, x_{n+1}, x_{n+\frac{5}{3}}$ and x_{n+2} gives the following four discrete scheme that form the block method

$$y_{n+\frac{2}{3}} = y_n + \frac{2}{3}hy'_n + \frac{637}{6075}h^2 f_n + \frac{211}{810}h^2 f_{n+\frac{2}{3}} - \frac{44}{243}h^2 f_{n+1} + \frac{116}{2025}h^2 f_{n+\frac{5}{3}} - \frac{47}{2430}h^2 f_{n+2} \tag{3.15}$$

$$y_{n+1} = y_n + hy'_n + \frac{209}{1200}h^2 f_n + \frac{93}{160}h^2 f_{n+\frac{2}{3}} - \frac{13}{40}h^2 f_{n+1} + \frac{21}{200}h^2 f_{n+\frac{5}{3}} - \frac{17}{480}h^2 f_{n+2} \tag{3.16}$$

$$y_{n+\frac{5}{3}} = y_n + \frac{5}{3}hy'_n + \frac{1225}{3888}h^2 f_n + \frac{3125}{2592}h^2 f_{n+\frac{2}{3}} - \frac{625}{1944}h^2 f_{n+1} + \frac{175}{648}h^2 f_{n+\frac{5}{3}} - \frac{625}{7776}h^2 f_{n+2} \tag{3.17}$$

$$y_{n+2} = y_n + 2hy'_n + \frac{29}{75}h^2 f_n + \frac{3}{2}h^2 f_{n+\frac{2}{3}} - \frac{4}{15}h^2 f_{n+1} + \frac{12}{25}h^2 f_{n+\frac{5}{3}} - \frac{1}{10}h^2 f_{n+2} \quad (3.18)$$

The following schemes are obtained by differentiating equation (3.7) and evaluating at point $x = x_{n+\frac{2}{3}}, x_{n+1}, x_{n+\frac{5}{3}}$ and x_{n+2}

$$y'_{n+\frac{2}{3}} = y'_n + \frac{424}{2025}hf_n + \frac{77}{90}hf_{n+\frac{2}{3}} - \frac{202}{405}hf_{n+1} + \frac{34}{225}hf_{n+\frac{5}{3}} - \frac{41}{810}hf_{n+2} \quad (3.19)$$

$$y'_{n+1} = y'_n + \frac{31}{150}hf_n + \frac{81}{80}hf_{n+\frac{2}{3}} - \frac{37}{120}hf_{n+1} + \frac{27}{200}hf_{n+\frac{5}{3}} - \frac{11}{240}hf_{n+2} \quad (3.20)$$

$$y'_{n+\frac{5}{3}} = y'_n + \frac{35}{162}hf_n + \frac{125}{144}hf_{n+\frac{2}{3}} + \frac{125}{648}hf_{n+1} + \frac{35}{72}hf_{n+\frac{5}{3}} - \frac{125}{1296}hf_{n+2} \quad (3.21)$$

$$y'_{n+2} = y'_n + \frac{16}{75}hf_n + \frac{9}{10}hf_{n+\frac{2}{3}} + \frac{2}{15}hf_{n+1} + \frac{18}{25}hf_{n+\frac{5}{3}} + \frac{1}{30}hf_{n+2} \quad (3.22)$$

3.2.2 Two-step method with $\frac{1}{2}, \frac{3}{4}, \frac{5}{4}$ and $\frac{7}{4}$ off-step points

To derive a continuous method by considering four off-step points $\frac{1}{2}, \frac{3}{4}, \frac{5}{4}$ and $\frac{7}{4}$ the following specifications were considered $r = 1$, k=2, s=7. The continuous method with the continuous coefficient $\alpha_j(x)$, $\beta_j(x)$ and $\beta_v(x)$ is of the form:

$$y(x) = \alpha_0 y_n + \alpha'_0\, hy'_n + h^2\left[\beta_0 f_n + \beta_1 f_{n+\frac{1}{2}} + \beta_2 f_{n+\frac{3}{4}} + \beta_3 f_{n+1} + \beta_4 f_{n+\frac{5}{4}} + \beta_5 f_{n+\frac{7}{4}} + \beta_6 f_{n+2}\right] \quad (3.23)$$

where:

$$\alpha_0 = 1 \quad (3.24)$$

$$\alpha_1 = -x_n + x \quad (3.25)$$

$$
\beta_0 = \left.\begin{aligned}
&\frac{1}{88200}\frac{x_n^2(44100h^2+91210h^5x_n+112105h^4x_n^2+84126h^3x_n^3+37744h^2x_n^4+9280hx_n^5+960x_n^6)}{h^6} \\
&-\frac{1}{44100}\frac{x_n(44100h^6+136815h^5x_n+224210h^4x_n^2+210315h^3x_n^3+113232h^2x_n^4+32480hx_n^5+3840x_n^6)x}{h^6} \\
&+\frac{1}{420}\frac{(210h^6+1303h^5x_n+3203h^4x_n^2+4006h^3x_n^3+2696h^2x_n^4+928hx_n^5+128x_n^6)x^2}{h^6} \\
&-\frac{1}{1260}\frac{\left(1303h^5+6406h^4x_n+12018h^3x_n^2+10784h^2x_n^3+4640hx_n^4+768x_n^5\right)x^3}{h^6} \\
&+\frac{1}{2520}\frac{\left(3203h^4+12018h^3x_n+16176h^2x_n^2+9280hx_n^3+1920x_n^4\right)x^4}{h^6} \\
&+\frac{1}{2100}\frac{\left(2003h^3+5392h^2x_n+4640hx_n^2+1280x_n^3\right)x^5}{h^6} \\
&+\frac{2}{1575}\frac{\left(337h^2+580hx_n+240x_n^2\right)x^6}{h^6}-\frac{8}{2205}\frac{\left(29h+24x_n\right)x^7}{h^6}+\frac{8}{735}\frac{x^8}{h^6}
\end{aligned}\right\} \quad (3.26)
$$

$$
\beta_1 = \left.\begin{aligned}
&-\frac{2}{4725}\frac{x_n^3(14700h^5+30905h^4x_n+30177h^3x_n^2+15848h^2x_n^3+4320hx_n^4+480x_n^5)}{h^6} \\
&+\frac{2}{4725}\frac{x_n^2(44100h^5+123620h^4x_n+150885h^3x_n^2+95088h^2x_n^3+30240hx_n^4+3840x_n^5)x}{h^6} \\
&-\frac{4}{45}\frac{x_n(270h^5+1101h^4x_n+1723h^3x_n^2+1292h^2x_n^3+464hx_n^4+64x_n^5)x^2}{h^6} \\
&+\frac{4}{135}\frac{\left(210h^5+1766h^4x_n+4311h^3x_n^2+4528h^2x_n^3+2160hx_n^4+384x_n^5\right)x^3}{h^6} \\
&-\frac{2}{135}\frac{\left(883h^4+4311h^3x_n+6792h^2x_n^2+4320hx_n^3+960x_n^4\right)x^4}{h^6} \\
&+\frac{2}{225}\frac{\left(1437h^3+4528h^2x_n+4320hx_n^2+1280x_n^3\right)x^5}{h^6} \\
&-\frac{16}{675}\frac{\left(282h^2+540hx_n+240x_n^2\right)x^6}{h^6}+\frac{64}{315}\frac{\left(29h+24x_n\right)x^7}{h^6}-\frac{64}{315}\frac{x^8}{h^6}
\end{aligned}\right\} \quad (3.27)
$$

$$\beta_2 = \left.\begin{aligned}
&\frac{4}{1575}\frac{x_n^3(4900h^5+11935h^4x_n+12873h^3x_n^2+7252h^2x_n^3+2080hx_n^4+240x_n^5)}{h^6}\\
&-\frac{4}{1575}\frac{x_n^2(14700h^5+47740h^4x_n+64365h^3x_n^2+43512h^2x_n^3+14560hx_n^4+1920x_n^5)x}{h^6}\\
&+\frac{8}{15}\frac{x_n(70h^5+341h^4x_n+613h^3x_n^2+518h^2x_n^3+208hx_n^4+32x_n^5)x^2}{h^6}\\
&-\frac{8}{45}\frac{\left(70h^5+682h^4x_n+1839h^3x_n^2+2072h^2x_n^3+1040hx_n^4+192x_n^5\right)x^3}{h^6}\\
&+\frac{4}{45}\frac{\left(341h^4+1839h^3x_n+3108h^2x_n^2+2080hx_n^3+480x_n^4\right)x^4}{h^6}\\
&-\frac{4}{75}\frac{\left(613h^3+2072h^2x_n+2080hx_n^2+640x_n^3\right)x^5}{h^6}\\
&+\frac{16}{225}\frac{\left(259h^2+520hx_n+240x_n^2\right)x^6}{h^6}-\frac{128}{315}\frac{\left(13h+12x_n\right)x^7}{h^6}+\frac{64}{105}\frac{x^8}{h^6}
\end{aligned}\right\} \tag{3.28}$$

$$\beta_3 = \left.\begin{aligned}
&-\frac{1}{1260}\frac{x_n^3(14700h^5+38255h^4x_n+44310h^3x_n^2+26544h^2x_n^3+8000hx_n^4+960x_n^5)}{h^6}\\
&+\frac{1}{630}\frac{x_n^2(22050h^5+76510h^4x_n+110775h^3x_n^2+79632h^2x_n^3+28000hx_n^4+3840x_n^5)}{h^6}\\
&-\frac{1}{10}\frac{x_n(210h^5+1093h^4x_n+2110h^3x_n^2+1896h^2x_n^3+800hx_n^4+128x_n^5)x^2}{h^6}\\
&+\frac{1}{9}\frac{\left(105h^5+1093h^4x_n+3165h^3x_n^2+3792h^2x_n^3+2000hx_n^4+384x_n^5\right)x^3}{h^6}\\
&-\frac{1}{36}\frac{\left(1093h^4+6330h^3x_n+11376h^2x_n^2+8000hx_n^3+1920x_n^4\right)x^4}{h^6}\\
&+\frac{1}{30}\frac{\left(1055h^3+3792h^2x_n+4000hx_n^2+1280x_n^3\right)x^5}{h^6}\\
&-\frac{4}{45}\frac{\left(237h^2+500hx_n+240x_n^2\right)x^6}{h^6}+\frac{16}{63}\frac{\left(24h+25x_n\right)x^7}{h^6}-\frac{16}{21}\frac{x^8}{h^6}
\end{aligned}\right\} \tag{3.29}$$

$$\beta_4 = \left.\begin{array}{l} \frac{8}{4725}\frac{x_n^3(2940h^5+7945h^4x_n+9639h^3x_n^2+6076h^2x_n^3+1920hx_n^4+240x_n^5)}{h^6} \\ -\frac{8}{4725}\frac{x_n^2(8820h^5+31780h^4x_n+48195h^3x_n^2+36456h^2x_n^3+13440hx_n^4+1920x_n^5)}{h^6}x \\ +\frac{16}{45}\frac{x_n(42h^5+227h^4x_n+459h^3x_n^2+434h^2x_n^3+192hx_n^4+32x_n^5)x^2}{h^6} \\ -\frac{16}{135}\frac{\left(42h^5+454h^4x_n+1377h^3x_n^2+1736h^2x_n^3+960hx_n^4+192x_n^5\right)x^3}{h^6} \\ +\frac{8}{135}\frac{\left(227h^4+1377h^3x_n+2604h^2x_n^2+1920hx_n^3+480x_n^4\right)x^4}{h^6} \\ -\frac{8}{225}\frac{\left(459h^3+1736h^2x_n+1920hx_n^2+640x_n^3\right)x^5}{h^6} \\ +\frac{32}{675}\frac{\left(217h^2+480hx_n+240x_n^2\right)x^6}{h^6}-\frac{1024}{315}\frac{\left(h+x_n\right)x^7}{h^6}+\frac{128}{315}\frac{x^8}{h^6} \end{array}\right\} \quad (3.30)$$

$$\beta_5 = \left.\begin{array}{l} +\frac{4}{11025}\frac{x_n^3(2100h^5+5915h^4x_n+7581h^3x_n^2+5124h^2x_n^3+1760hx_n^4+240x_n^5)}{h^6} \\ -\frac{4}{11025}\frac{x_n^2(6300h^5+23660h^4x_n+37905h^3x_n^2+30744h^2x_n^3+12320hx_n^4+1920x_n^5)x}{h^6} \\ -\frac{8}{105}\frac{x_n(30h^5+169h^4x_n+361h^3x_n^2+366h^2x_n^3+176hx_n^4+32x_n^5)x^2}{h^6} \\ +\frac{8}{315}\frac{\left(30h^5+338h^4x_n+1083h^3x_n^2+1464h^2x_n^3+880hx_n^4+192x_n^5\right)x^3}{h^6} \\ -\frac{4}{315}\frac{\left(169h^4+1083h^3x_n+2196h^2x_n^2+1760hx_n^3+480x_n^4\right)x^4}{h^6} \\ +\frac{4}{525}\frac{\left(361h^3+1464h^2x_n+1760hx_n^2+640x_n^3\right)x^5}{h^6} \\ -\frac{16}{1575}\frac{\left(183h^2+440hx_n+240x_n^2\right)x^6}{h^6}+\frac{128}{2205}\frac{\left(11h+12x_n\right)x^7}{h^6}-\frac{64}{6615}\frac{x^8}{h^6} \end{array}\right\} \quad (3.31)$$

$$\beta_6 = \left.\begin{array}{l} \frac{1}{37800}\frac{x_n^3(9450h^5 + 26355h^4x_n + 33306h^3x_n^2 + 22064h^2x_n^3 + 7360hx_n^4 + 960x_n^5)}{h^6} \\ -\frac{1}{18900}\frac{x_n^2(14175h^5 + 52710h^4x_n + 83265h^3x_n^2 + 66192h^2x_n^3 + 25760hx_n^4 + 3840x_n^5)x}{h^6} \\ +\frac{1}{180}\frac{x_n(105h^5 + 599h^4x_n + 1302h^3x_n^2 + 1352h^2x_n^3 + 672hx_n^4 + 128x_n^5)x^2}{h^6} \\ -\frac{1}{540}\frac{\left(105h^5 + 1198h^4x_n + 3906h^3x_n^2 + 5408h^2x_n^3 + 3360hx_n^4 + 768x_n^5\right)x^3}{h^6} \\ -\frac{1}{1080}\frac{\left(599h^4 + 3906h^3x_n 8112h^2x_n^2 + 6720hx_n^3 + 1920x_n^4\right)x^4}{h^6} \\ +\frac{1}{900}\frac{\left(651h^3 + 2704h^2x_n + 3360hx_n^2 + 1280x_n^3\right)x^5}{h^6} \\ -\frac{2}{675}\frac{\left(169h^2 + 420hx_n + 240x_n^2\right)x^6}{h^6} - \frac{8}{315}\frac{\left(7h + 8x_n\right)x^7}{h^6} + \frac{8}{315}\frac{x^8}{h^6} \end{array}\right\} \quad (3.32)$$

Evaluating (3.23) above at points $x = x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+1}, x_{n+\frac{5}{4}}, x_{n+\frac{7}{4}}$ and x_{n+2} gives the following six discrete scheme that form the block method

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + \frac{36179}{705600}h^2 f_n + \frac{20263}{75600}h^2 f_{n+\frac{1}{2}} - \frac{781}{1800}h^2 f_{n+\frac{3}{4}} + \frac{1901}{5040}h^2 f_{n+1} - \frac{2921}{18900}h^2 f_{n+\frac{5}{4}} + \frac{1999}{88200}h^2 f_{n+\frac{7}{4}} - \frac{1727}{302400}h^2 f_{n+2} \quad (3.33)$$

$$y_{n+\frac{3}{4}} = y_n + \frac{3}{4}hy'_n + \frac{843009}{10035200}h^2 f_n + \frac{98013}{179200}h^2 f_{n+\frac{1}{2}} - \frac{140457}{179200}h^2 f_{n+\frac{3}{4}} + \frac{13977}{20480}h^2 f_{n+1} - \frac{24987}{89600}h^2 f_{n+\frac{5}{4}} + \frac{51129}{125440}h^2 f_{n+\frac{7}{4}} - \frac{14709}{1433600}h^2 f_{n+2} \quad (3.34)$$

$$y_{n+1} = y_n + hy'_n + \frac{3431}{29400}h^2 f_n + \frac{3928}{4725}h^2 f_{n+\frac{1}{2}} - \frac{568}{525}h^2 f_{n+\frac{3}{4}} + \frac{139}{140}h^2 f_{n+1} - \frac{272}{675}h^2 f_{n+\frac{5}{4}} + \frac{72}{1225}h^2 f_{n+\frac{7}{4}} - \frac{559}{37800}h^2 f_{n+2} \quad (3.35)$$

$$y_{n+\frac{5}{4}} = y_n + \frac{5}{4}hy'_n + \frac{539725}{3612672}h^2 f_n + \frac{215875}{193536}h^2 f_{n+\frac{1}{2}} - \frac{88625}{64512}h^2 f_{n+\frac{3}{4}} + \frac{349375}{258048}h^2 f_{n+1}$$
$$- \frac{50425}{96768}h^2 f_{n+\frac{5}{4}} + \frac{34625}{451584}h^2 f_{n+\frac{7}{4}} - \frac{29875}{1548288}h^2 f_{n+2} \tag{3.36}$$

$$y_{n+\frac{7}{4}} = y_n + \frac{7}{4}hy'_n + \frac{132251}{614400}h^2 f_n + \frac{1154881}{69200}h^2 f_{n+\frac{1}{2}} - \frac{146461}{76800}h^2 f_{n+\frac{3}{4}} + \frac{40817}{20480}h^2 f_{n+1}$$
$$- \frac{189679}{345600}h^2 f_{n+\frac{5}{4}} + \frac{3577}{25600}h^2 f_{n+\frac{7}{4}} - \frac{175273}{5529600}h^2 f_{n+2} \tag{3.37}$$

$$y_{n+2} = y_n + 2hy'_n + \frac{2738}{11025}h^2 f_n + \frac{1312}{675}h^2 f_{n+\frac{1}{2}} - \frac{3392}{1575}h^2 f_{n+\frac{3}{4}} + \frac{716}{315}h^2 f_{n+1}$$
$$- \frac{2432}{4725}h^2 f_{n+\frac{5}{4}} + \frac{2624}{11025}h^2 f_{n+\frac{7}{4}} - \frac{164}{4725}h^2 f_{n+2} \tag{3.38}$$

The following schemes are obtained by differentiating equation (3.23) and evaluating at points $x = x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}, x_{n+1}, x_{n+\frac{5}{4}}, x_{n+\frac{7}{4}}$ and x_{n+2}

$$y'_{n+\frac{1}{2}} = y'_n + \frac{92621}{705600}hf_n + \frac{39581}{37800}hf_{n+\frac{1}{2}} - \frac{9329}{6300}hf_{n+\frac{3}{4}} + \frac{12659}{10080}hf_{n+1} - \frac{4807}{9450}hf_{n+\frac{5}{4}}$$
$$+ \frac{3253}{44100}hf_{n+\frac{7}{4}} - \frac{5603}{302400}hf_{n+2} \tag{3.39}$$

$$y'_{n+\frac{3}{4}} = y'_n + \frac{163911}{1254400}hf_n + \frac{25597}{22400}hf_{n+\frac{1}{2}} - \frac{14319}{11200}hf_{n+\frac{3}{4}} + \frac{21309}{17920}hf_{n+1}$$
$$- \frac{2729}{5600}hf_{n+\frac{5}{4}} + \frac{5583}{78400}hf_{n+\frac{7}{4}} - \frac{3211}{179200}hf_{n+2} \tag{3.40}$$

$$y'_{n+1} = y'_n + \frac{481}{3675}hf_n + \frac{5354}{4725}hf_{n+\frac{1}{2}} - \frac{604}{525}hf_{n+\frac{3}{4}} + \frac{281}{210}hf_{n+1}$$
$$- \frac{2392}{4725}hf_{n+\frac{5}{4}} + \frac{268}{3675}hf_{n+\frac{7}{4}} - \frac{173}{9450}hf_{n+2} \tag{3.41}$$

$$y'_{n+\frac{5}{4}} = y'_n + \frac{59015}{451584}hf_n + \frac{27575}{24192}hf_{n+\frac{1}{2}} - \frac{4775}{4032}hf_{n+\frac{3}{4}} + \frac{48625}{32256}hf_{n+1}$$
$$- \frac{2392}{6048}hf_{n+\frac{5}{4}} + \frac{1975}{28224}hf_{n+\frac{7}{4}} - \frac{3425}{193536}hf_{n+2} \tag{3.42}$$

$$y'_{n+\frac{7}{4}} = y'_n + \frac{10213}{76800}hf_n + \frac{92953}{86400}hf_{n+\frac{1}{2}} - \frac{4459}{4800}hf_{n+\frac{3}{4}} + \frac{7889}{7680}hf_{n+1} + \frac{4459}{21600}hf_{n+\frac{5}{4}} + \frac{1309}{4800}hf_{n+\frac{7}{4}} - \frac{25039}{691200}hf_{n+2} \qquad (3.43)$$

$$y'_{n+2} = y'_n + \frac{1451}{11025}hf_n + \frac{5248}{4725}hf_{n+\frac{1}{2}} - \frac{1664}{1575}hf_{n+\frac{3}{4}} + \frac{388}{315}hf_{n+1} + \frac{256}{4725}hf_{n+\frac{5}{4}} + \frac{5248}{11025}hf_{n+\frac{7}{4}} + \frac{247}{4725}hf_{n+2} \qquad (3.44)$$

3.2.3 Two step method with $\frac{1}{5}, \frac{3}{5}, \frac{7}{5}$ and $\frac{9}{5}$ off-step points

To derive a continuous method by considering four off-step points $\frac{1}{5}, \frac{3}{5}, \frac{7}{5}$ and $\frac{9}{5}$ the following specifications were considered $r=1$, k=2, s=7. The continuous method with the continuous coefficient $\alpha_j(x)$, $\beta_j(x)$ and $\beta_v(x)$ is of the form:

$$y(x) = \alpha_0 y_n + \alpha'_0 y'_n + h^2\left[\beta_0 f_n + \beta_1 f_{n+\frac{1}{5}} + \beta_2 f_{n+\frac{3}{5}} + \beta_3 f_{n+1} + \beta_4 f_{n+\frac{7}{5}} + \beta_5 f_{n+\frac{9}{5}} + \beta_6 f_{n+2}\right] \qquad (3.45)$$

Where:

$$\alpha_0 = 1 \qquad (3.46)$$

$$\alpha_1 = -x_n + x \qquad (3.47)$$

$$\beta_0 = \left.\begin{aligned} &\frac{1}{63504}\frac{x_n^2(31752h^6 + 99876h^5x_n + 156646h^4x_n^2 + 136500h^3x_n^3 + 67200h^2x_n^4 + 17500hx_n^5 + 1875x_n^6)}{h^6} \\ &- \frac{1}{15876}\frac{x_n(15876h^6 + 74907h^5x_n + 156646h^4x_n^2 + 170625h^3x_n^3 + 100800h^2x_n^4 + 30625hx_n^5 + 3750x_n^6)x}{h^6} \\ &+ \frac{1}{756}\frac{(378h^6 + 358h^5x_n + 11189h^4x_n^2 + 16250h^3x_n^3 + 12000h^2x_n^4 + 4375hx_n^5 + 625x_n^6)x^2}{h^6} \\ &- \frac{1}{2268}\frac{(3567h^6 + 22378h^4x_n + 48750h^3x_n^2 + 48000h^2x_n^3 + 21875hx_n^4 + 3750x_n^6)x^3}{h^6} \\ &+ \frac{1}{4536}\frac{(11189h^4 + 48750h^3x_n + 72000h^2x_n^2 + 43750hx_n^3 + 9375x_n^4)x^4}{h^6} \\ &- \frac{25}{756}\frac{(65h^3 + 192h^2x_n + 175hx_n^2 + 50x_n^3)x^5}{h^6} + \frac{25}{2268}\frac{(96h^2 + 175hx_n + 75x_n^2)x^6}{h^6} \\ &- \frac{625}{21168}\frac{(7h + 6x_n)x^7}{h^6} + \frac{625}{21168}\frac{x^8}{h^6} \end{aligned}\right\} \qquad (3.48)$$

$$\beta_1 = \left.\begin{aligned}
&-\frac{25}{580608}\frac{x_n^3(52920h^5+117390h^4x_n+117768h^3x_n^2+62440h^2x_n^3+17000hx_n^4+1875x_n^5)}{h^6}\\
&+\frac{1}{15876}\frac{x_n^2(3969h^5+11739h^4x_n+14721h^3x_n^2+9366h^2x_n^3+2975hx_n^4+375x_n^5)x}{h^6}\\
&-\frac{125}{6912}\frac{x_n(378h^5+1677h^4x_n+2804h^3x_n^2+2230h^2x_n^3+850hx_n^4+125x_n^6)x^2}{h^6}\\
&+\frac{125}{10368}\frac{(189h^5+1677h^4x_n+4206h^3x_n^2+4460h^2x_n^3+2125hx_n^4+375x_n^5)x^3}{h^6}\\
&-\frac{125}{3456}\frac{(1677h^4+8412h^3x_n+13380h^2x_n^2+8500hx_n^3+1875x_n^4)x^4}{h^6}\\
&+\frac{25}{3456}\frac{(701h^3+2230h^2x_n+2125hx_n^2+625x_n^3)x^5}{h^6}-\frac{125}{20736}\frac{(446h^2+850hx_n+375x_n^2)x^6}{h^6}\\
&-\frac{3125}{72576}\frac{(17h+15x_n)x^7}{h^6}+\frac{15625}{193536}\frac{x^8}{h^6}
\end{aligned}\right\} \quad (3.49)$$

$$\beta_2 = \left.\begin{aligned}
&\frac{25}{338688}\frac{x_n^3(17640h^5x_n+68530h^4x_n^2+88116h^3x_n^3+53760h^2x_n^4+16000hx_n^5+1875x_n^6)}{h^6}\\
&-\frac{125}{84672}\frac{x_n^2(2646h^5+13706h^4x_n+22029h^3x_n^2+16128h^2x_n^3+5600hx_n^4+750x_n^5)x}{h^6}\\
&+\frac{125}{4032}\frac{x_n(125h^5+979h^4x_n+2098h^3x_n^2+1920h^2x_n^3+800hx_n^4+125x_n^5)x^2}{h^6}\\
&-\frac{125}{6048}\frac{(63h^5+979h^4x_n+3147h^3x_n^2+3840h^2x_n^3+2000hx_n^4+375x_n^3)x^3}{h^6}\\
&+\frac{125}{24192}\frac{(979h^4+629h^3x_n+11520h^2x_n^2+8000hx_n^3+1875x_n^4)x^4}{h^6}\\
&-\frac{125}{12096}\frac{(1049h^3+3840h^2x_n+4000hx_n^2+1250x_n^3)x^5}{h^6}+\frac{125}{12096}\frac{(384h^2+800hx_n+375x_n^2)x^6}{h^6}\\
&-\frac{3125}{42336}\frac{(16h+15x_n)x^7}{h^6}+\frac{15625}{112896}\frac{x^8}{h^6}
\end{aligned}\right\} \quad (3.50)$$

$$
\beta_3 = \left.
\begin{aligned}
&-\frac{1}{3584}\frac{x_n^3(3528h^5+14882h^4x_n+22400h^3x_n^2+15400h^2x_n^3+5000hx_n^4+625x_n^5)}{h^6} \\
&+\frac{1}{448}\frac{x_n^2(1323h^5+7441h^4x_n+14000h^3x_n^2+11550h^2x_n^3+4375hx_n^4+625x_n^5)x}{h^6} \\
&-\frac{1}{128}\frac{x_n(378h^5+3189h^4x_n+8000h^3x_n^2+8250h^2x_n^3+3750hx_n^4+625x_n^6)x^2}{h^6} \\
&+\frac{1}{64}\frac{(63h^5+1063h^4x_n+4000h^3x_n^2+5500h^2x_n^3+3125hx_n^4+625x_n^5)x^3}{h^6} \\
&-\frac{1}{256}\frac{(1063h^4+8000h^3x_n+16500h^2x_n^2+12500hx_n^3+3125x_n^4)x^4}{h^6} \\
&+\frac{25}{64}\frac{(16h^3+66h^2x_n+75hx_n^2+25x_n^3)x^5}{h^6}-\frac{25}{128}\frac{(22h^2+50hx_n+25x_n^2)x^6}{h^6} \\
&-\frac{625}{448}\frac{(h+x_n)x^7}{h^6}-\frac{625}{3584}\frac{x^8}{h^6}
\end{aligned}
\right\} \qquad (3.51)
$$

$$
\beta_4 = \left.
\begin{aligned}
&\frac{25}{338688}\frac{x_n^3(7560h^5x_n+32970h^4x_n^2+53004h^3x_n^3+39760h^2x_n^4+14000hx_n^5+1875x_n^6)}{h^6} \\
&-\frac{125}{84672}\frac{x_n^2(1134h^5+6594h^4x_n+13251h^3x_n^2+11928h^2x_n^3+4900hx_n^4+750x_n^5)x}{h^6} \\
&+\frac{125}{4032}\frac{x_n(54h^5+471h^4x_n+1262h^3x_n^2+1420h^2x_n^3+700hx_n^4+125x_n^5)x^2}{h^6} \\
&-\frac{125}{6048}\frac{(27h^5+471h^4x_n+1893h^3x_n^2+2840h^2x_n^3+1750hx_n^4+375x_n^3)x^3}{h^6} \\
&+\frac{125}{24192}\frac{(471h^4+3786h^3x_n+8520h^2x_n^2+7000hx_n^3+1875x_n^4)x^4}{h^6} \\
&-\frac{125}{4032}\frac{(631h^3+2840h^2x_n+3500hx_n^2+1250x_n^3)x^5}{h^6}+\frac{125}{12096}\frac{(284h^2+700hx_n+375x_n^2)x^6}{h^6} \\
&-\frac{3125}{42336}\frac{(14h+15x_n)x^7}{h^6}+\frac{15625}{112896}\frac{x^8}{h^6}
\end{aligned}
\right\} \qquad (3.52)
$$

$$
\beta_5 = \left.\begin{aligned}
&-\frac{25}{580608}\frac{x_n^3(5880h^5+26110h^4x_n+43512h^3x_n^2+34440h^2x_n^3+13000hx_n^4+1875x_n^5)}{h^6}\\
&+\frac{125}{72576}\frac{x_n^2(441h^5+2611h^4x_n+5439h^3x_n^2+5166h^2x_n^3+2275hx_n^4+375x_n^5)x}{h^6}\\
&-\frac{125}{6912}\frac{x_n(42h^5+373h^4x_n+1036h^3x_n^2+1230h^2x_n^3+650hx_n^4+125x_n^6)x^2}{h^6}\\
&+\frac{125}{10368}\frac{(21h^5+373h^4x_n+1554h^3x_n^2+2460h^2x_n^3+1625hx_n^4+375x_n^5)x^3}{h^6}\\
&-\frac{125}{3456}\frac{(373h^4+3108h^3x_n+7380h^2x_n^2+6500hx_n^3+1875x_n^4)x^4}{h^6}\\
&+\frac{25}{3456}\frac{(259h^3+1230h^2x_n+1625hx_n^2+625x_n^3)x^5}{h^6}-\frac{125}{20736}\frac{(246h^2+650hx_n+375x_n^2)x^6}{h^6}\\
&+\frac{3125}{72576}\frac{(13h+15x_n)x^7}{h^6}-\frac{15625}{193536}\frac{x^8}{h^6}
\end{aligned}\right\} \quad (3.53)
$$

$$
\beta_6 = \left.\begin{aligned}
&\frac{1}{63504}\frac{x_n^3(5292h^5x_n+23646h^4x_n^2+39900h^3x_n^3+32200h^2x_n^4+12500hx_n^5+1875x_n^6)}{h^6}\\
&-\frac{1}{15876}\frac{x_n^2(3969h^5+23646h^4x_n+49875h^3x_n^2+48300h^2x_n^3+21875hx_n^4+3750x_n^5)x}{h^6}\\
&+\frac{1}{756}\frac{x_n(189h^5+1689h^4x_n+4750h^3x_n^2+5750h^2x_n^3+3125hx_n^4+625x_n^5)x^2}{h^6}\\
&-\frac{125}{6048}\frac{(189h^5+3378h^4x_n+14250h^3x_n^2+23000h^2x_n^3+15625hx_n^4+3750x_n^5)x^3}{h^6}\\
&+\frac{1}{4536}\frac{(1689h^4+14250h^3x_n+34500h^2x_n^2+31250hx_n^3+9375x_n^4)x^4}{h^6}\\
&-\frac{25}{756}\frac{(19h^3+92h^2x_n+125hx_n^2+50x_n^3)x^5}{h^6}+\frac{25}{2268}\frac{(46h^2+125hx_n+75x_n^2)x^6}{h^6}\\
&-\frac{625}{15876}\frac{(5h+6x_n)x^7}{h^6}+\frac{625}{21168}\frac{x^8}{h^6}
\end{aligned}\right\} \quad (3.54)
$$

Evaluating (3.45) above at points $x = x_{n+\frac{1}{5}}, x_{n+\frac{3}{5}}, x_{n+1}, x_{n+\frac{7}{5}}, x_{n+\frac{9}{5}}$ and x_{n+2} gives the following six discrete schemes that form the block method

$$
\begin{aligned}
y_{n+\frac{1}{5}} &= y_n + \frac{1}{5}hy'_n + \frac{426317}{39690000}h^2 f_n + \frac{297}{25600}h^2 f_{n+\frac{1}{5}} - \frac{176339}{42336000}h^2 f_{n+\frac{3}{5}} + \frac{6661}{2240000}h^2 f_{n+1} \\
&- \frac{23249}{14112000}h^2 f_{\frac{7}{5}} + \frac{53579}{72576000}h^2 f_{n+\frac{9}{5}} - \frac{1067}{4410000}h^2 f_{n+2}
\end{aligned} \quad (3.55)
$$

$$y_{n+\frac{3}{5}} = y_n + \frac{3}{5}hy'_n + \frac{17141}{490000}h^2 f_n + \frac{112107}{896000}h^2 f_{n+\frac{1}{5}} + \frac{147}{6400}h^2 f_{n+\frac{3}{5}} - \frac{8667}{2240000}h^2 f_{n+1} - \frac{1503}{1568000}h^2 f_{\frac{7}{5}} - \frac{181}{896000}h^2 f_{n+\frac{9}{5}} + \frac{3}{70000}h^2 f_{n+2} \quad (3.56)$$

$$y_{n+1} = y_n + hy'_n + \frac{1199}{21168}h^2 f_n + \frac{149575}{580608}h^2 f_{n+\frac{1}{5}} + \frac{20075}{112896}h^2 f_{n+\frac{3}{5}} - \frac{3}{512}h^2 f_{n+1} - \frac{1025}{338688}h^2 f_{\frac{7}{5}} - \frac{275}{193536}h^2 f_{n+\frac{9}{5}} + \frac{29}{63504}h^2 f_{n+2} \quad (3.57)$$

$$y_{n+\frac{7}{5}} = y_n + \frac{7}{5}hy'_n + \frac{63749}{810000}h^2 f_n + \frac{1342159}{3456000}h^2 f_{n+\frac{1}{5}} + \frac{302869}{864000}h^2 f_{n+\frac{3}{5}} + \frac{45619}{320000}h^2 f_{n+1} + \frac{147}{6400}h^2 f_{n+\frac{7}{5}} - \frac{45619}{10368000}h^2 f_{n+\frac{9}{5}} + \frac{343}{270000}h^2 f_{n+2} \quad (3.58)$$

$$y_{n+\frac{9}{5}} = y_n + \frac{9}{5}hy'_n + \frac{49437}{490000}h^2 f_n + \frac{464859}{896000}h^2 f_{n+\frac{1}{5}} + \frac{819639}{1568000}h^2 f_{n+\frac{3}{5}} + \frac{662661}{2240000}h^2 f_{n+1} + \frac{267543}{1568000}h^2 f_{n+\frac{7}{5}} + \frac{297}{25600}h^2 f_{n+\frac{9}{5}} - \frac{243}{490000}h^2 f_{n+2} \quad (3.59)$$

$$y_{n+2} = y_n + 2hy'_n + \frac{446}{3969}h^2 f_n + \frac{1175}{2016}h^2 f_{n+\frac{1}{5}} + \frac{925}{1512}h^2 f_{n+\frac{3}{5}} + \frac{41}{112}h^2 f_{n+1} + \frac{925}{3528}hf_{n+\frac{7}{5}} + \frac{1175}{18144}h^2 f_{n+\frac{9}{5}} \quad (3.60)$$

The following schemes are obtained by differentiating equation (3.45) and evaluating at point $x = x_{n+\frac{1}{5}}, x_{n+\frac{3}{5}}, x_{n+1}, x_{n+\frac{7}{5}}, x_{n+\frac{9}{5}}$ and x_{n+2}

$$y'_{n+\frac{1}{5}} = y'_n + \frac{14867}{1984500}hf_n + \frac{267469}{1814400}hf_{n+\frac{1}{5}} - \frac{83207}{2116800}hf_{n+\frac{3}{5}} + \frac{773}{28000}hf_{n+1} - \frac{32177}{2116800}hf_{n+\frac{7}{5}} + \frac{12319}{181400}h^2 f_{n+\frac{9}{5}} - \frac{4411}{1984500}hf_{n+2} \quad (3.61)$$

$$= y'_{n+\frac{5}{3}} = y'_n + \frac{737}{14700}hf_n + \frac{7757}{22400}hf_{n+\frac{1}{5}} + \frac{19307}{78400}hf_{n+\frac{3}{5}} - \frac{351}{5600}hf_{n+1} + \frac{2157}{78400}hf_{n+\frac{7}{5}} - \frac{739}{67200}hf_{n+\frac{9}{5}} - \frac{17}{4900}hf_{n+2} \quad (3.62)$$

$$y'_{n+1} = y'_n + \frac{305}{5292}hf_n + \frac{23125}{72576}hf_{n+\frac{1}{5}} + \frac{12875}{28224}hf_{n+\frac{3}{5}} + \frac{41}{224}hf_{n+1} - \frac{1625}{84672}hf_{n+\frac{7}{5}} + \frac{125}{24192}hf_{n+\frac{9}{5}} - \frac{23}{15876}hf_{n+2} \tag{3.63}$$

$$y'_{n+\frac{7}{5}} = y'_n + \frac{427}{8100}hf_n + \frac{86779}{259200}hf_{n+\frac{1}{5}} + \frac{17689}{43200}hf_{n+\frac{3}{5}} + \frac{343}{800}hf_{n+1} + \frac{8239}{43200}hf_{n+\frac{7}{5}} - \frac{5831}{259200}hf_{n+\frac{9}{5}} + \frac{49}{8100}hf_{n+2} \tag{3.64}$$

$$y'_{n+\frac{9}{5}} = y'_n + \frac{1431}{24500}hf_n + \frac{7101}{22400}hf_{n+\frac{1}{5}} + \frac{35451}{78400}hf_{n+\frac{3}{5}} + \frac{9477}{28000}hf_{n+1} + \frac{37341}{78400}hf_{n+\frac{7}{5}} + \frac{3951}{22400}hf_{n+\frac{9}{5}} - \frac{459}{24500}hf_{n+2} \tag{3.65}$$

$$y'_{n+2} = y'_n + \frac{223}{3969}hf_n + \frac{5875}{18144}hf_{n+\frac{1}{5}} + \frac{4625}{10584}hf_{n+\frac{3}{5}} + \frac{41}{112}hf_{n+1} + \frac{4625}{10584}hf_{n+\frac{7}{5}} + \frac{5875}{18144}hf_{n+\frac{9}{5}} + \frac{223}{3969}hf_{n+2} \tag{3.66}$$

3.3 Analysis of the Methods

In this section, we discuss in general the order and error constants, consistency, zero-stability and convergence of the proposed method.

3.3.1 Order and error constants

Let the linear difference operator L associated with k-step method be defined as

$$L\left[y(x_n);h\right] = \sum_{j=0}^{k}\left(\alpha_j y(x_n + jh) - h\beta y'(x_n) - h^2\gamma_{vj} f\left(x_n + jvh\right)\right) \tag{3.67}$$

and

$$L\left[y'(x_n);h\right] = \sum_{j=0}^{k}\left(h\overline{\beta}_j y'(x_n + jvh) - h^2\overline{\gamma}_{vj} hf\left(x_n + jvh\right)\right) \tag{3.68}$$

respectively. Assuming that $y(x_n)$ and $y'(x_n)$ are sufficiently differentiable, we can expand the terms in (3.67) and (3.68) as Taylor series about the point x_n to obtain the expression

$$L\left[y(x_n);h\right] = C_0 y(x_n) + C_1 h y'(x_n) + \ldots + C_q h^q y^{(q)}(x_n) + \ldots \tag{3.69}$$

and

$$L\left[y'(x_n);h\right] = \bar{C}_0 y'(x_n) + \bar{C}_1 h y''(x_n) + \ldots + \bar{C}_q h^q y^{(q+1)}(x_n) + \ldots \tag{3.70}$$

respectively;

where the constants C_q and $\bar{C}_q$ $q = 0,1,\ldots$ are given as follows

$$\left.\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j \\
C_1 &= \sum_{j=1}^{k} j\alpha_j \\
C_2 &= \frac{1}{2!}\sum_{j=1}^{k} (j)^2 \alpha_j - \sum_{j=0}^{k} \beta_j \\
&\vdots \\
C_q &= \frac{1}{\text{q}!}\sum_{j=1}^{k} (j)^q \alpha_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-1}\beta_j
\end{aligned}\right\} \tag{3.71}$$

$q = 2,3,\ldots$

$$\left.\begin{aligned}
\bar{C}_0 &= \sum_{j=0}^{k} \bar{\alpha}_j \\
\bar{C}_1 &= \sum_{j=1}^{k} j\bar{\alpha}_j - \sum_{j=0}^{k} \bar{\beta}_j \\
\bar{C}_2 &= \frac{1}{2!}\sum_{j=1}^{k} (j)^2 \bar{\alpha}_j - \sum_{j=1}^{k} j\bar{\beta}_j \\
&\vdots \\
\bar{C}_q &= \frac{1}{\text{q}!}\sum_{j=1}^{k} (j)^q \bar{\alpha}_j - \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\bar{\beta}_j
\end{aligned}\right\}. \tag{3.72}$$

The methods (3.5) and (3.6) are of order p if $C_0 = C_1 = \ldots C_p = C_{p+1} = 0,\ C_{p+2} \neq 0$ and C_{p+2} is the error constant and $C_{p+2} h^{p+2} y^{(p+2)}(x_n)$ the principal truncation error at the point x_n.

3.3.1.1 ***Order and error constant of two-step with*** $\frac{2}{3}$ and $\frac{5}{3}$ ***off-step points***

Let equation (3.15) be written in the form

$$y_{n+\frac{2}{3}} - y_n - \frac{2}{3}hy'_n - \frac{637}{6075}h^2 f_n - \frac{211}{810}h^2 f_{n+\frac{2}{3}} + \frac{44}{243}h^2 f_{n+1} - \frac{116}{2025}h^2 f_{n+\frac{5}{3}} + \frac{47}{2430}h^2 f_{n+2} = 0 \qquad (3.73)$$

Expanding the above equation in Taylor series form yields

$$\sum_{j=0}^{\infty} \frac{\frac{2}{3}h^j}{j!} y_n^j - y_n - \frac{2}{3}hy'_n - \frac{637}{6075}h^2 y^2 - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y^{j+2} \left(-\frac{44}{243} + \frac{211}{810}\left(\frac{2}{3}\right)^j + \frac{116}{2025}\left(\frac{5}{3}\right)^j - \frac{47}{2430}(2)^j \right) = 0 \qquad (3.74)$$

and collecting the power in terms of h and y leads to the following

$$\left.\begin{aligned}
c_0 &= -1+1=0 \\
c_1 &= -\frac{2}{3}+\frac{2}{3}=0 \\
c_2 &= \frac{1}{2!}\left(\frac{2}{3}\right)^2 - \left(\frac{637}{6075} - \frac{44}{243} + \frac{211}{810} + \frac{116}{2025} - \frac{47}{2430}\right) = 0 \\
c_3 &== \frac{1}{3!}\left(\frac{2}{3}\right)^3 - \left(-\frac{44}{243} + \frac{211}{810}\left(\frac{2}{3}\right) + \frac{116}{2025}\left(\frac{5}{3}\right) - \frac{47}{2430}(2)\right) = 0 \\
c_4 &= \frac{1}{4!}\left(\frac{2}{3}\right)^4 - \frac{1}{2!}\left(-\frac{44}{243} + \frac{211}{810}\left(\frac{2}{3}\right)^2 + \frac{116}{2025}\left(\frac{5}{3}\right)^2 - \frac{47}{2430}(2)^2\right) = 0 \\
c_5 &= \frac{1}{5!}\left(\frac{2}{3}\right)^5 - \frac{1}{3!}\left(-\frac{44}{243} + \frac{211}{810}\left(\frac{2}{3}\right)^3 + \frac{116}{2025}\left(\frac{5}{3}\right)^3 - \frac{47}{2430}(2)^3\right) = 0 \\
c_6 &= \frac{1}{6!}\left(\frac{2}{3}\right)^6 - \frac{1}{4!}\left(-\frac{44}{243} + \frac{211}{810}\left(\frac{2}{3}\right)^4 + \frac{116}{2025}\left(\frac{5}{3}\right)^4 - \frac{47}{2430}(2)^4\right) = 0 \\
c_7 &= \frac{1}{7!}\left(\frac{2}{3}\right)^7 - \frac{1}{5!}\left(-\frac{44}{243} + \frac{211}{810}\left(\frac{2}{3}\right)^5 + \frac{116}{2025}\left(\frac{5}{3}\right)^5 - \frac{47}{2430}(2)^5\right) = \frac{97}{382725}
\end{aligned}\right\} \qquad (3.75)$$

Let equation (3.16) be written in the form

$$y_{n+1} - y_n - hy'_n - \frac{209}{1200}h^2 f_n - \frac{93}{160}h^2 f_{n+\frac{2}{3}} + \frac{13}{40}h^2 f_{n+1} - \frac{21}{200}h^2 f_{n+\frac{5}{3}} + \frac{17}{480}h^2 f_{n+2} = 0 \qquad (3.76)$$

Expanding the above equation in Taylor series in the form

$$\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j - y_n - hy_n' - \frac{209}{1200}h^2y_n^2 - \sum_{j-0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left(-\frac{13}{40}+\frac{93}{160}\left(\frac{2}{3}\right)^j+\frac{21}{200}\left(\frac{5}{3}\right)^j-\frac{17}{480}(2)^j\right)=0 \quad (3.77)$$

and collecting the power in term of h and y leads to the following

$$\left.\begin{aligned}
c_0 &= -1+1=0\\
c_1 &= -1+1=0\\
c_2 &= \frac{1}{2!}(1)^2-\left(\frac{209}{1200}-\frac{13}{40}+\frac{93}{160}+\frac{21}{200}-\frac{17}{480}\right)=0\\
c_3 &= \frac{1}{3!}(1)^3-\left(-\frac{13}{40}+\frac{93}{160}\left(\frac{2}{3}\right)+\frac{21}{200}\left(\frac{5}{3}\right)-\frac{17}{480}(2)\right)=0\\
c_4 &= \frac{1}{4!}(1)^4-\frac{1}{2!}\left(-\frac{13}{40}+\frac{93}{160}\left(\frac{2}{3}\right)^2+\frac{21}{200}\left(\frac{5}{3}\right)^2-\frac{17}{480}(2)^2\right)=0\\
c_5 &= \frac{1}{5!}(1)^5-\frac{1}{3!}\left(-\frac{13}{40}+\frac{93}{160}\left(\frac{2}{3}\right)^3+\frac{21}{200}\left(\frac{5}{3}\right)^3-\frac{17}{480}(2)^3\right)=0\\
c_6 &= \frac{1}{6!}(1)^6-\frac{1}{4!}\left(-\frac{13}{40}+\frac{93}{160}\left(\frac{2}{3}\right)^4+\frac{21}{200}\left(\frac{5}{3}\right)^4-\frac{17}{480}(2)^4\right)=0\\
c_7 &= \frac{1}{7!}(1)^7-\frac{1}{5!}\left(-\frac{13}{40}+\frac{93}{160}\left(\frac{2}{3}\right)^5+\frac{21}{200}\left(\frac{5}{3}\right)^5-\frac{17}{480}(2)^5\right)=\frac{209}{453600}
\end{aligned}\right\} \quad (3.78)$$

Let equation (3.17) be written in the form

$$y_{n+\frac{5}{3}} - y_n - \frac{5}{3}hy_n' - \frac{1225}{3888}h^2 f_n - \frac{3125}{2592}h^2 f_{n+\frac{2}{3}} + \frac{625}{1944}h^2 f_{n+1} - \frac{175}{648}h^2 f_{n+\frac{5}{3}} + \frac{625}{7776}h^2 f_{n+2} = 0 \quad (3.79)$$

Expanding the above equation in Taylor series in the form

$$\sum_{j=0}^{\infty}\frac{\frac{5}{3}h^j}{j!}y_n^j - y_n - \frac{5}{3}hy_n' - \frac{1225}{3888}h^2y_n^2 - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left(-\frac{625}{1944}+\frac{3125}{2592}\left(\frac{2}{3}\right)^j+\frac{175}{648}\left(\frac{5}{3}\right)^j-\frac{625}{7776}(2)^j\right)=0 \quad (3.80)$$

and collecting power in term of h and y leads to the following

$$\left.\begin{aligned}
c_0 &= -1+1=0\\
c_1 &= -\frac{5}{3}+\frac{5}{3}=0\\
c_2 &= \frac{1}{2!}\left(\frac{5}{3}\right)^2-\left(\frac{1225}{3888}-\frac{625}{1944}+\frac{3125}{2592}+\frac{175}{648}-\frac{625}{7776}\right)=0\\
c_3 &= \frac{1}{3!}\left(\frac{5}{3}\right)^3-\left(-\frac{625}{1944}+\frac{3125}{2592}\left(\frac{2}{3}\right)+\frac{175}{648}\left(\frac{5}{3}\right)-\frac{625}{7776}(2)\right)=0\\
c_4 &= \frac{1}{4!}\left(\frac{5}{3}\right)^4-\frac{1}{2!}\left(-\frac{625}{1944}+\frac{3125}{2592}\left(\frac{2}{3}\right)^2+\frac{175}{648}\left(\frac{5}{3}\right)^2-\frac{625}{7776}(2)^2\right)=0\\
c_5 &= \frac{1}{5!}\left(\frac{5}{3}\right)^5-\frac{1}{3!}\left(-\frac{625}{1944}+\frac{3125}{2592}\left(\frac{2}{3}\right)^3+\frac{175}{648}\left(\frac{5}{3}\right)^3-\frac{625}{7776}(2)^3\right)=0\\
c_6 &= \frac{1}{6!}\left(\frac{5}{3}\right)^6-\frac{1}{4!}\left(-\frac{625}{1944}+\frac{3125}{2592}\left(\frac{2}{3}\right)^4+\frac{175}{648}\left(\frac{5}{3}\right)^4-\frac{625}{7776}(2)^4\right)=0\\
c_7 &= \frac{1}{7!}\left(\frac{5}{3}\right)^7-\frac{1}{5!}\left(-\frac{625}{1944}+\frac{3125}{2592}\left(\frac{2}{3}\right)^5+\frac{175}{648}\left(\frac{5}{3}\right)^5-\frac{625}{7776}(2)^5\right)=\frac{1375}{4536000}
\end{aligned}\right\}\quad(3.81)$$

Let equation (3.18) be written in the form

$$y_{n+2}-y_n-2hy'_n\ -\frac{29}{75}h^2f_n-\frac{3}{2}h^2f_{n+\frac{2}{3}}+\frac{4}{15}h^2f_{n+1}-\frac{12}{25}h^2f_{n+\frac{5}{3}}+\frac{1}{10}h^2f_{n+2}=0 \quad (3.82)$$

Expanding the above equation in Taylor series in the form

$$\sum_{j=0}^{\infty}\frac{2^jh^j}{j!}y_n^j-y_n-2hy'_n\ -\frac{29}{75}h^2y_n^2-\sum_{j-0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left(-\frac{4}{15}+\frac{3}{2}\left(\frac{2}{3}\right)^j+\frac{12}{25}\left(\frac{5}{3}\right)^j-\frac{1}{10}(2)^j\right)=0 \quad (3.83)$$

And collecting the power in term of h and y leads to the following

$$\left.\begin{aligned}
c_0 &= -1+1=0\\
c_1 &= -2+2=0\\
c_2 &= \frac{1}{2!}(2)^2-\left(\frac{29}{75}-\frac{4}{15}+\frac{3}{2}+\frac{12}{25}-\frac{1}{10}\right)=0\\
c_3 &= \frac{1}{3!}(2)^3-\left(-\frac{4}{15}+\frac{3}{2}\left(\frac{2}{3}\right)+\frac{12}{25}\left(\frac{5}{3}\right)-\frac{1}{10}(2)\right)=0\\
c_4 &= \frac{1}{4!}(2)^4-\frac{1}{2!}\left(-\frac{4}{15}+\frac{3}{2}\left(\frac{2}{3}\right)^2+\frac{12}{25}\left(\frac{5}{3}\right)^2-\frac{1}{10}(2)^2\right)=0\\
c_5 &= \frac{1}{5!}(2)^5-\frac{1}{3!}\left(-\frac{4}{15}+\frac{3}{2}\left(\frac{2}{3}\right)^3+\frac{12}{25}\left(\frac{5}{3}\right)^3-\frac{1}{10}(2)^3\right)=0\\
c_6 &= \frac{1}{6!}(2)^6-\frac{1}{4!}\left(-\frac{4}{15}+\frac{3}{2}\left(\frac{2}{3}\right)^4+\frac{12}{25}\left(\frac{5}{3}\right)^4-\frac{1}{10}(2)^4\right)=0\\
c_7 &= \frac{1}{7!}(2)^7-\frac{1}{5!}\left(-\frac{4}{15}+\frac{3}{2}\left(\frac{2}{3}\right)^5+\frac{12}{25}\left(\frac{5}{3}\right)^5-\frac{1}{10}(2)^5\right)=\frac{17}{14175}
\end{aligned}\right\}(3.84)$$

The block method (3.15), (3.16), (3.17) and (3.18) has error constant

$c_{p+2}=\left(\frac{97}{382725},\frac{209}{453600},\frac{1375}{4536000},\frac{17}{14175}\right)^T$ with order $p=(5,5,5,5)^T$

Following similar procedure the summary of the order and error constant is presented in Tables 3.1, 3.2 and 3.3.

Table 3.1 Order and Error constant of Two-Step with $\frac{2}{3}$ and $\frac{5}{3}$ Off-Step Points

Equation number	order p	Error Constant c_{p+2}
(3.15)	5	$\frac{97}{382725}$
(3.16)	5	$\frac{209}{453600}$
(3.17)	5	$\frac{1375}{453600}$
(3.18)	5	$\frac{17}{14175}$

Table 3.2 Order and Error constant of Two-Step with $\frac{1}{2}, \frac{3}{4}, \frac{5}{4}$ and $\frac{7}{4}$ Off-Step Points

Equation number	Order p	Error Constant c_{p+2}
(3.33)	7	$\frac{38183}{26011238400}$
(3.34)	7	$\frac{1077}{411041792}$
(3.35)	7	$\frac{1531}{40645600}$
(3.36)	7	$\frac{327125}{66588770304}$
(3.37)	7	$\frac{7889}{1061683200}$
(3.38)	7	$\frac{89}{10160640}$

Table 3.3 Order and Error Constant of Two-Step with $\frac{1}{5}, \frac{3}{5}, \frac{7}{5}$ and $\frac{9}{5}$ Off-Step Points

Equation Number	Order p	Error Constant c_{p+2}
(3.55)	**7**	$\frac{346931}{6201362500000}$
(3.56)	7	$\frac{4311}{153125000000}$
(3.57)	7	$-\frac{31}{496125000}$
(3.58)	7	$\frac{38759}{253125000000}$
(3.59)	7	$-\frac{13851}{7656500000}$
(3.60)	7	$-\frac{31}{248062500}$

3.3.2 Zero stability

This is the concept concerning the behavior of a numerical method as$h \to 0$, the system of equation (3.6) becomes

$$\left.\begin{aligned} y_{n+1} &= y_{n+k-r} \\ y_{n+2} &= y_{n+k-r} \\ &\vdots \\ y_{n+k-2} &= y_{n+k-r} \\ y_{n+k-1} &= y_n \\ y_{n+k} &= y_{n+k-r} \end{aligned}\right\} \tag{3.85}$$

which can be written in matrix form as

$$A^0 \bar{Y}_\mu - A^{'} \bar{Y}_{\mu-1} = 0 \tag{3.86}$$

where $\bar{Y}_\mu = (y_{n+1}, y_{n+2}, \ldots, y_{n+k})^T, \bar{Y}_{\mu-1} = (y_n, y_{n+1}, y_{n+k-r})^T$, A^0 is the identity matrix of dimension k and A^1 is a matrix of dimension K.

3.3.2.1 *Zero stability of two step method with* $\frac{2}{3}$ and $\frac{5}{3}$ *off-step points*

Expressing methods (3.15), (3.16), (3.17) and (3.18) in the form

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; A^1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\rho(\lambda) = \det\left[\lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}\right] \tag{3.87}$$

$$= \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda+1 \end{bmatrix} = \lambda^3(\lambda+1) = 0, \lambda = 0, \lambda = 1 \tag{3.88}$$

Since $|\lambda|=1$, the method is zero stable.

3.3.2.2 *Zero stability of two step method with* $\frac{1}{2},\frac{3}{4},\ \frac{5}{4}$ and $\frac{7}{4}$ *off-step points*

Expressing (3.33), (3.34), (3.35), (3.36) (3.37) and (3.38) in the form

$$A^0=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},A^1=\begin{bmatrix}0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\end{bmatrix}$$

$$\rho(\lambda)=\det\left[\lambda\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\end{bmatrix}\right] \tag{3.89}$$

$$=\begin{bmatrix}\lambda&0&0&0&0&1\\0&\lambda&0&0&0&1\\0&0&\lambda&0&0&1\\0&0&0&\lambda&0&1\\0&0&0&0&\lambda&1\\0&0&0&0&0&\lambda+1\end{bmatrix} \tag{3.90}$$

$=\lambda^5(\lambda+1)=0,\lambda=0$ or 1

Since $|\lambda|=1$, the method is zero stable.

3.3.2.3 *Zero stability of two step method with* $\frac{1}{5},\frac{3}{5},\ \frac{7}{5}$ and $\frac{9}{5}$ *off-step points*

Expressing (3.55), (3.56), (3.57), (3.58) (3.59) and (3.60) in the form

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\rho(\lambda) = \det \left[\lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \right] \tag{3.91}$$

$$= \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 1 \\ 0 & \lambda & 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda & 0 & 0 & 1 \\ 0 & 0 & 0 & \lambda & 0 & 1 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda+1 \end{bmatrix} \tag{3.92}$$

$$= \lambda^5(\lambda+1) = 0, \lambda = 0 \text{ or } 1$$

Since $|\lambda|=1$, the method is zero stable.

3.3.3 Consistency

Each of the methods is consistent as they all have order > 1.

3.3.4 Convergence

The convergence of the proposed methods, are considered in the light of the basic properties in conjunction with the fundamental theorem of Dahlquist (Henrichi 1962) for linear multistep methods. We state here the Dahlquist theorem without proof.

3.3.4.1 *Theorem*

The necessary and sufficient condition for a multistep method to be convergent is for it to be consistent and zero-stable.

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

4.1 Numerical Experiments

In this section, we solve some standard second order initial value problems of ordinary differential equations using the proposed Falkner-type method in order to demonstrate its efficacy. However, the implementation is carried as a block (self-starting) method whereby the continuous forms of the methods generates the main and additional discrete Falkner formulas to produce approximation simultaneously at each step of implementation within the interval of integration. Comparisons were made with the exact solutions of the problems considered and absolute errors were compared with some other existing methods found in the literature and presented in tables.

For the purpose of comparative analysis, the following notations are adopted.

FTM1: The proposed Falkner-type Method 1 with $\left\{\frac{2}{3},\frac{5}{3}\right\}$ as off-grid points

FTM2: The proposed Falkner-type Method 2 with $\left\{\frac{1}{2},\frac{3}{4},\frac{5}{4},\frac{7}{4}\right\}$ as off-grid points

FTM3: The proposed Falkner-type Method 3with $\left\{\frac{1}{5},\frac{3}{5},\frac{7}{5},\frac{9}{5}\right\}$. as off-grid points

$HFBM_{2,2}$: 2-step, two off-grid hybrid block Falkner-type method by Nicholas (2019)

$HFBM_{2,4.}$: 2-step, four off-grid hybrid block Falkner-type method by Nicholas (2019)

BFM_6: Block Falkner method for k=6 by Ramos *et al.*, (2016)

Problem 1. (*Source:* Ramos *et al.* (2016))

Consider the non-linear homogeneous problem given by:

$$y'' = x\left(y'\right)^2, \quad y(0)=1, \; y'(0)=0.5$$
$$0 \le x \le 1, \quad h = 0.1$$

Exact solution: $y(x)=1+\frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Problem 2. (*Source:* Ramos *et al.* (2016))

Consider a linear homogeneous problem given by

$$y''=y', \quad y(0)=0, \; y'(0)=-1$$
$$0\le x\le 1, \quad h=0.01$$

Exact solution: $y(x)=1-e^{x}$

Problem 3. (*Source:* Adediran and Ogundare,(2015))

Consider a highly stiff initial value problem given by

$$y''=-1001y'-1000y, \quad y(0)=1, \; y'(0)=-1$$
$$0\le x\le 1, \quad h=0.05$$

Exact solution: $y(x)=e^{-x}$

Problem 4. Dynamic Problem (*Source:* Nicholas,(2019))

A 10kg mass is attached to a spring having a constant of 140N/m. The mass is started in motion from the equilibrium position with an initial value of 1m/sec in upward direction and with an applied external force $F(t)=0.5\sin(t)$. The resulting equation due to air resistance $9y'N$ is given as

$$y''=-9y'-14y+\frac{1}{2}\sin x, \quad y(0)=0, \; y'(0)=-1$$
$$0\le x\le 0.1, \quad h=0.001$$

Exact solution: $y(x)=-\frac{9}{50}e^{-2x}+\frac{99}{500}e^{-7x}-\frac{9}{500}\cos x$

Problem 5. Van Der Pol Oscillator (*Source:* Mohammed *et al.*,(2019))

$$y''-2\xi\left(1-y^{2}\right)y'-y=0, \; y(0)=0, \; y'(0)=0.5, \; \xi=0.025, \; 0\le x\le 1$$

This problem has no exact solution, our result is however validated using Runge-Kutta (RK45) and compared with Mohammed *et al.*, (2019).

Problem 6: Consider the problem:

$$y'' = -2y, \quad y(0) = 1, \ \ y'(0) = 0$$
$$0 \le x \le 1, \quad h = 0.01$$

Exact solution: $y(x) = \cos\sqrt{2x}$

Table 4.1: Comparison of Numerical Results with the Exact Solution for Problem 1.

x	**Exact solution**	**FTM1**	**FTM2**	**FTM3**
0.1	1.050041729278491	1.0500417290	1.05004172927816	1.050041729278497
0.2	1.100335347731076	1.1003353470	1.10033534773031	1.100335347731086
0.3	1.151140435936467	1.1511404343	1.15114043593459	1.151140435936488
0.4	1.202732554054082	1.2027325512	1.20273255405083	1.202732554054115
0.5	1.255412811882995	1.2554128067	1.25541281187637	1.255412811883062
0.6	1.309519604203112	1.3095195961	1.30951960419224	1.309519604203209
0.7	1.365443754271396	1.3654437402	1.36544375424890	1.365443754271615
0.8	1.423648930193602	1.4236489079	1.42364893015604	1.423648930193938
0.9	1.484700278594052	1.4847002385	1.48470027850756	1.484700278594945
1.0	1.549306144334055	1.5493060786	1.54930614418128	1.549306144335512

Table 4.1 is the comparison of the results obtained from the derived Falkner-type methods with the exact solution for problem 1. It is shown that as the number of iteration progresses within the interval of integration of problem 1 with the step size h=0.1, the approximate solutions from the derived methods get closer to the analytical solution. This shows the effectiveness of the derived methods.

Table 4.2: Comparison of Absolute Errors of the Proposed Methods for Problem 1.

x	**FTM1**	**FTM2**	**FTM3**
0.1	3.00×10^{-10}	3.30×10^{-13}	6.00×10^{-15}
0.2	7.00×10^{-10}	7.60×10^{-13}	1.00×10^{-14}
0.3	1.60×10^{-09}	1.88×10^{-12}	2.10×10^{-14}
0.4	2.90×10^{-09}	3.25×10^{-12}	3.30×10^{-14}
0.5	5.20×10^{-09}	6.63×10^{-12}	6.70×10^{-14}
0.6	8.10×10^{-09}	1.09×10^{-11}	9.70×10^{-13}
0.7	1.41×10^{-08}	2.25×10^{-11}	2.19×10^{-13}
0.8	2.23×10^{-08}	3.76×10^{-11}	3.36×10^{-13}
0.9	4.01×10^{-08}	8.65×10^{-11}	8.93×10^{-13}
1.0	6.58×10^{-08}	1.53×10^{-10}	1.46×10^{-12}

Table 4.2 is the comparison of the absolute errors of the derived methods for problem 1. FTM1 of order of accuracy p=5 has relatively small error. However, FTM2 and FTM3 have the same order of accuracy p=7 but with varying error constants. FTM3 has generally lower error constants than FTM2 which accounts for its superiority over FTM3.

Table 4.3: Comparison of Numerical Results with the Exact Solution for Problem 2.

x	**Exact solution**	**FTM1**	**FTM2**	**FTM3**
0.1	-0.1051709180756476248117078	-0.10517091807564740	-0.105170918075647624811560	-0.10517091807564762481170852
0.2	-0.2214027581601698339210720	-0.22140275816016893	-0.22140275816016983392047	-0.22140275816016983392107353
0.3	-0.3498588075760031039837443	-0.34985880757600097	-0.34985880757600310398234	-0.34985880757600310398374684
0.4	-0.4918246976412703178248530	-0.49182469764126630	-0.49182469764127031782224	-0.49182469764127031782485660
0.5	-0.6487212707001281468486508	-0.64872127070012146	-0.64872127070012814684432	-0.64872127070012814684865573
0.6	-0.8221188003905089748753677	-0.82211880039049862	-0.82211880039050897486873	-0.82211880039050897487537410
0.7	-1.0137527074704765216245494	-1.0137527074704615	-1.0137527074704765216149	-1.0137527074704765216245576
0.8	-1.2255409284924676045795375	-1.2255409284924467	-1.2255409284924676045660	-1.2255409284924676045795477
0.9	-1.4596031111569496638001266	-1.4596031111569213	-1.4596031111569496637818	-1.4596031111569496638001392
1.0	-1.7182818284590452353602875	-1.7182818284590075	-1.7182818284590452353362	-1.7182818284590452353603027

Table 4.3 is the comparison of the results obtained from the derived Falkner-type methods with the exact solution for problem 2. It is shown that as the number of iteration progresses within the interval of integration of problem 2 with the step size h=0.01, the approximate solutions from the derived methods get closer to the analytical solution. This shows the effectiveness of the derived methods.

Table 4.4: Comparison of Absolute Errors of the Proposed Methods for Problem 2.

x	**FTM1**	**FTM2**	**FTM3**
0.1	2.00×10^{-16}	1.40×10^{-22}	7.20×10^{-25}
0.2	8.70×10^{-16}	6.30×10^{-22}	1.53×10^{-24}
0.3	2.13×10^{-15}	1.36×10^{-21}	2.54×10^{-24}
0.4	4.00×10^{-15}	2.66×10^{-21}	3.60×10^{-24}
0.5	6.64×10^{-15}	4.38×10^{-21}	4.93×10^{-24}
0.6	1.04×10^{-14}	6.67×10^{-21}	6.40×10^{-24}
0.7	1.50×10^{-14}	9.60×10^{-21}	8.20×10^{-24}
0.8	2.09×10^{-14}	1.35×10^{-20}	1.02×10^{-23}
0.9	2.84×10^{-14}	1.83×10^{-20}	1.26×10^{-23}
1.0	3.77×10^{-14}	2.41×10^{-20}	1.52×10^{-23}

Table 4.4 presents the absolute errors obtained from the proposed methods for problem 2. FTM1 has relatively small error while FTM2has smaller error and FTM3has the least error among the proposed methods. It is also evident that the order of accuracy of each method has impact on the performance of the methods.

Table 4.5: Comparison of Numerical Results with the Exact Solution for Problem 3.

x	**Exact solution**	**FTM1**	**FTM2**	**FTM3**
0.1	0.9048374180359595731642	0.904837418035972	0.90483741803595957308	0.9048374180359595732079
0.2	0.8187307530779818586699	0.818730753078054	0.81873075307798185908	0.8187307530779818587435
0.3	0.7408182206817178660669	0.740818220681853	0.74081822068171786700	0.7408182206817178661625
0.4	0.6703200460356393007444	0.670320046035825	0.67032004603563930208	0.6703200460356393008571
0.5	0.6065306597126334236038	0.606530659712857	0.60653065971263342533	0.6065306597126334237281
0.6	0.5488116360940264326285	0.548811636094282	0.54881163609402643464	0.5488116360940264327609
0.7	0.4965853037914095147048	0.496585303791691	0.49658530379140951694	0.4965853037914095148425
0.8	0.4493289641172215914301	0.449328964117518	0.44932896411722159382	0.4493289641172215915713
0.9	0.4065696597405991118835	0.406569659740910	0.40656965974059911440	0.4065696597405991120260
1.0	0.3678794411714423215955	0.367879441171759	0.36787944117144232417	0.3678794411714423217379

Table 4.5 presents the numerical solutions obtained using the proposed methods for problem 3. It is observed from the table that the numerical solutions are in agreement with the analytical solution.

Table 4.6: Comparison of Absolute Errors of the Proposed Methods for Problem 3.

x	**FTM1**	**FTM2**	**FTM3**
0.1	1.20×10^{-14}	8.00×10^{-20}	4.37×10^{-20}
0.2	7.20×10^{-14}	4.10×10^{-19}	7.36×10^{-20}
0.3	1.35×10^{-13}	9.30×10^{-19}	9.56×10^{-20}
0.4	1.86×10^{-13}	1.34×10^{-18}	1.13×10^{-19}
0.5	2.24×10^{-13}	1.73×10^{-18}	1.24×10^{-19}
0.6	2.56×10^{-13}	2.01×10^{-18}	1.32×10^{-19}
0.7	2.81×10^{-13}	2.24×10^{-18}	1.38×10^{-19}
0.8	2.96×10^{-13}	2.39×10^{-18}	1.41×10^{-19}
0.9	3.11×10^{-13}	2.52×10^{-18}	1.43×10^{-19}
1.0	3.17×10^{-13}	2.57×10^{-18}	1.42×10^{-19}

Table 4.6 presents the absolute errors obtained from the proposed methods for problem 3. FTM1 has relatively small error while FTM2has smaller error and FTM3has the least error among the proposed methods. It is also evident that the order of accuracy of each method has impact on the performance of the methods.

Table 4.7: Comparison of Numerical Results with the Exact Solution for Problem 4.

x	Exact solution	FTM1	FTM2	FTM3
0.01	-0.009560889194649846804166497015	-0.00956088919464984351	-0.009560889194649846804166437	-0.009560889194649846804166501561
0.02	-0.01828560322442636535366698297	-0.0182856032244263542	-0.01828560322442636535366697	-0.01828560322442636535366699154
0.03	-0.02623395294528461006173430083	-0.0262339529452845861	-0.02623395294528461006173428	-0.02623395294528461006173431267
0.04	-0.03346164077240591222029086964	-0.0334616407724058717	-0.03346164077240591222029085	-0.03346164077240591222029088424
0.05	-0.04002053976823921905677551688	-0.0400205397682391593	-0.04002053976823921905677548	-0.04002053976823921905677553371
0.06	-0.04595895383607522543831390479	-0.0459589538360751439	-0.04595895383607522543831384	-0.04595895383607522543831392339
0.07	-0.05132186029705384635940248224	-0.0513218602970537414	-0.05132186029705384635940252	-0.05132186029705384635940250238
0.08	-0.05615113604210135263053739521	-0.0561511360421012222	-0.05615113604210135263053744	-0.05615113604210135263053741635
0.09	-0.06048576836973092957466960874	-0.0604857683697307730	-0.06048576836973092957466961	-0.06048576836973092957466963079
0.1	-0.06436205154552458247878091963	-0.0643620515455243986	-0.06436205154552458247878093	-0.06436205154552458247878094217

Table 4.7 presents the numerical solutions obtained using the proposed methods for problem 4. It is observed from the table that the numerical solutions are in agreement with the analytical solution.

Table 4.8: Comparison of Absolute Errors of the Proposed Methods for Problem 4.

x	**FTM1**	**FTM2**	**FTM3**
0.01	3.21×10^{-18}	1.01×10^{-24}	4.55×10^{-27}
0.02	1.17×10^{-17}	3.75×10^{-24}	8.57×10^{-27}
0.03	4.11×10^{-17}	7.86×10^{-24}	1.18×10^{-26}
0.04	2.40×10^{-17}	1.31×10^{-23}	1.46×10^{-26}
0.05	5.95×10^{-17}	1.93×10^{-23}	1.68×10^{-26}
0.06	8.15×10^{-17}	2.62×10^{-23}	1.86×10^{-26}
0.07	1.05×10^{-16}	3.39×10^{-23}	2.01×10^{-26}
0.08	1.31×10^{-16}	4.19×10^{-23}	2.11×10^{-26}
0.09	1.56×10^{-16}	5.02×10^{-23}	2.21×10^{-26}
0.1	1.84×10^{-16}	5.88×10^{-23}	2.25×10^{-26}

Table 4.8 presents the absolute errors obtained from the proposed methods for problem 4. FTM1 has relatively small error while FTM2has smaller error and FTM3has the least error among the proposed methods. It is also evident that the order of accuracy of each method has impact on the performance of the methods.

Table 4.9: Results for the Van Der Pol Oscillator Problem with h=0.1

X	**RK(5)**	**FTM1**	**FTM2**	**FTM3**	**Mohammed *et al.*(2019)**
1.0	0.431051	0.431431	0.431432	0.431431	0.431051
2.0	0.47631	0.478239	0.478239	0.478239	0.476309
3.0	0.076077	0.0765766	0.0765805	0.0765766	0.076076
4.0	-0.41546	-0.417868	-0.417859	-0.417868	-0.41546
5.0	-0.53857	-0.543708	-0.543698	-0.543708	-0.53857
6.0	-0.16135	-0.163413	-0.163414	-0.163413	-0.16134
7.0	0.386024	0.390437	0.390417	0.390437	0.386025
8.0	0.595231	0.604590	0.604568	0.604590	0.59523
9.0	0.254655	0.259731	0.259729	0.259731	0.254653
10.0	-0.34157	0.347672	-0.347649	-0.347672	-0.34158

Table 4.9 presents the numerical solutions obtained using the proposed methods for problem 5. It is evident from the table that the numerical solutions are in agreement with the Runge-Kutta (R-K5) solution and Mohammed *et al.* (2019).

Table 4.10: Comparison of Numerical Results with the Exact Solution for Problem 6.

x	Exact solution	FTM1	FTM2	FTM3
0.1	0.99001665555952292782	0.99001665555952292782	0.99001665555952292801	0.990016655559522927821 0
0.2	0.96026595657052612022	0.96026595657052612022	0.96026595657052612056	0.960265956570526120219 4
0.3	0.91134192598371379346	0.91134192598371379346	0.91134192598371379393	0.911341925983713793456 8
0.4	0.84422141469661511369	0.84422141469661511369	0.84422141469661511426	0.844221414696615113680 9
0.5	0.76024459707563015125	0.76024459707563015125	0.76024459707563015191	0.760244597075630151245 1
0.6	0.66108821211140974067	0.66108821211140974067	0.66108821211140974139	0.661088212111409740660 3
0.7	0.54873208449309459031	0.54873208449309459031	0.54873208449309459105	0.548732084493094590300 8
0.8	0.42541959406470837817	0.42541959406470837817	0.42541959406470837891	0.425419594064708378128 7
0.9	0.29361288295777032751	0.29361288295777032751	0.29361288295777032828	0.293612882957770327479 3
1.0	0.15594369476537447346	0.15594369476537447346	0.15594369476537447423	0.155943694765374473448 1

Table 4.10 presents the numerical solutions obtained using the proposed methods for problem 6. It is evident from the table that the numerical solutions are in agreement with the analytical solution.

Table 4.11: Comparison of Absolute Errors of the Proposed Methods for Problem 6.

x	**FTM1**	**FTM2**	**FTM3**
0.1	3.10×10^{-16}	1.90×10^{-19}	1.00×10^{-21}
0.2	1.27×10^{-15}	3.40×10^{-19}	6.00×10^{-22}
0.3	3.39×10^{-15}	4.70×10^{-19}	3.20×10^{-21}
0.4	7.16×10^{-15}	5.70×10^{-19}	9.10×10^{-21}
0.5	1.30×10^{-14}	6.60×10^{-19}	4.90×10^{-21}
0.6	2.14×10^{-14}	7.20×10^{-19}	9.70×10^{-21}
0.7	3.24×10^{-14}	7.40×10^{-19}	9.20×10^{-21}
0.8	4.63×10^{-14}	7.40×10^{-19}	4.13×10^{-20}
0.9	6.31×10^{-14}	7.70×10^{-19}	3.07×10^{-20}
1.0	8.26×10^{-14}	7.70×10^{-19}	1.19×10^{-20}

Table 4.11 presents the absolute errors obtained from the proposed methods for problem 6. FTM1 has relatively small error while FTM2has smaller error and FTM3 has the least error among the proposed methods. It is also evident that the order of accuracy of each method has impact on the performance of the methods.

Table 4.12: Comparison of Absolute Errors for Problem 1

x	**James *et al.* (2013) h=0.1**	**BFM$_6$ h=0.05**	**Mohammad and Zurni (2017), h=0.05**	**HFBM$_{22}$ h=0.1**	**FTM2 h=0.1**
0.1	1.110×10^{-15}	3.114×10^{-12}	2.220×10^{-16}	2.000×10^{-12}	1.40×10^{-22}
0.2	5.995×10^{-15}	6.660×10^{-12}	2.220×10^{-16}	3.000×10^{-12}	6.30×10^{-22}
0.3	2.554×10^{-14}	9.833×10^{-12}	6.661×10^{-16}	6.000×10^{-12}	1.36×10^{-21}
0.4	7.105×10^{-14}	2.173×10^{-11}	1.110×10^{-15}	9.000×10^{-11}	2.66×10^{-21}
0.5	1.157×10^{-13}	3.570×10^{-11}	4.440×10^{-16}	1.400×10^{-11}	4.38×10^{-21}
0.6	1.199×10^{-13}	4.859×10^{-11}	8.881×10^{-16}	2.200×10^{-11}	6.67×10^{-21}
0.7	6.857×10^{-13}	1.310×10^{-10}	1.554×10^{-15}	3.500×10^{-12}	9.60×10^{-21}
0.8	3.475×10^{-12}	2.313×10^{-10}	4.440×10^{-15}	5.900×10^{-11}	1.35×10^{-20}
0.9	1.222×10^{-11}	3.286×10^{-10}	8.660×10^{-16}	1.010×10^{-10}	1.83×10^{-20}
1.0	7.728×10^{-11}	1.335×10^{-09}	1.266×10^{-14}	-	2.41×10^{-20}

Table 4.12 shows the comparison of performance of the proposed method FTM2 with some existing methods for problem 1. It is shown that the FTM2 yield higher accurate results than the existing methods.

Table 4.13: Comparison of Absolute Errors for Problem 2

x	**Kayode and Adeyeye. (2013), h=0.1**	**BFM_6 h=0.1**	**$HFBM_{,4}$ h=0.1**	**FTM2h=0.1**
0.2	8.171×10^{-07}	2.427×10^{-11}	2.000×10^{-12}	1.063×10^{-14}
0.3	3.103×10^{-06}	4.001×10^{-11}	1.000×10^{-12}	2.272×10^{-14}
0.4	6.569×10^{-06}	5.746×10^{-11}	1.010×10^{-12}	3.786×10^{-14}
0.5	1.143×10^{-05}	7.741×10^{-11}	1.400×10^{-11}	6.090×10^{-14}
0.6	1.796×10^{-05}	9.517×10^{-11}	2.100×10^{-11}	8.853×10^{-14}
0.7	2.644×10^{-05}	1.221×10^{-10}	3.000×10^{-12}	1.268×10^{-13}
0.8	3.722×10^{-05}	1.604×10^{-10}	4.000×10^{-11}	1.717×10^{-13}
0.9	5.067×10^{-05}	2.013×10^{-10}	5.000×10^{-11}	2.307×10^{-13}
1.0	5.255×10^{-05}	2.466×10^{-10}	-	2.992×10^{-13}

Table 4.13 shows the comparison of performance of the proposed method FTM2 with some existing methods for problem 2. It is shown that the FTM2 yield higher accurate results than the existing methods

Table 4.14: Comparison of Absolute Errors for Problem 3

X	Adediran and Ogundare. (2015)	Mohammad and Zurni (2017)	FTM2 h=0.1
0.1	2.050×10^{-11}	1.055×10^{-14}	1.005×10^{-16}
0.2	4.390×10^{-11}	1.776×10^{-14}	9.642×10^{-17}
0.3	6.550×10^{-11}	2.342×10^{-14}	4.795×10^{-16}
0.4	8.380×10^{-11}	2.798×10^{-14}	4.530×10^{-16}
0.5	9.860×10^{-10}	3.131×10^{-14}	8.329×10^{-16}
0.6	1.100×10^{-10}	3.397×10^{-14}	7.743×10^{-16}
0.7	1.190×10^{-10}	3.564×10^{-14}	1.080×10^{-15}
0.8	1.240×10^{-10}	3.675×10^{-14}	9.960×10^{-16}
0.9	1.280×10^{-10}	3.730×10^{-14}	1.223×10^{-15}
1.0	1.300×10^{-10}	3.741×10^{-14}	1.122×10^{-15}

Table 4.14 shows the comparison of performance of the proposed method FTM2 with some existing methods for problem 3. It is shown that the FTM2 yield higher accurate results than the existing methods

Table 4.15: Comparison of Absolute Errors for Problem 4

X	$HFBM_{2,1}$	$HFBM_{2,2}$	$HFBM_{2,4}$	FTM2
0.01	1.304×10^{-10}	4.500×10^{-13}	1.700×10^{-13}	1.01×10^{-24}
0.02	3.323×10^{-10}	1.000×10^{-13}	4.000×10^{-13}	3.75×10^{-24}
0.03	6.448×10^{-10}	6.000×10^{-13}	2.000×10^{-15}	7.86×10^{-24}
0.04	1.003×10^{-09}	1.500×10^{-12}	7.130×10^{-13}	1.31×10^{-23}
0.05	1.438×10^{-09}	9.000×10^{-12}	1.000×10^{-15}	1.93×10^{-23}
0.06	1.899×10^{-09}	1.400×10^{-12}	4.000×10^{-13}	2.62×10^{-23}
0.07	2.412×10^{-09}	2.001×10^{-12}	1.010×10^{-12}	3.39×10^{-23}
0.08	2.933×10^{-09}	1.500×10^{-12}	4.000×10^{-13}	4.19×10^{-23}
0.09	3.489×10^{-09}	1.600×10^{-12}	5.000×10^{-13}	5.02×10^{-23}
0.10	4.041×10^{-09}	1.400×10^{-12}	3.000×10^{-13}	5.88×10^{-23}

Table 4.15 shows the comparison of performance of the proposed method FTM2 with some existing methods for problem 4. It is shown that the FTM2 yield higher accurate results than the existing methods

CHAPTER FIVE

5.0 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this thesis, we solved some standard second order initial value problems of ordinary differential equations using the proposed Falkner-type method involving Two off-step point and four off-step points using Block hybrid method. The orders of the developed methods are 5 and 7. It is zero stable, consistent and convergent.

The developed methods were used to solve six test problems in Ramos *et al.* (2016),Adediran and Ogundare (2015), Nicolas, (2019) and Mohammed *et al.* (2019). The exact results were compared with result from the source as well as the result from the proposed methods. The desirable property of a numerical solution is to behave like the exact solution of the problem which can be seen in the tables of the results represented.

5.2 Contribution to knowledge

The incorporation of some carefully selected off-grid points in the derivation process of a class of two-step linear multistep methods has improved the order of accuracy of the method for the solution of a general and special second order initial value problems. It is established from the analysis that the off-grid points $\left(\frac{2}{3},\frac{5}{3}\right)$, $\left(\frac{1}{2},\frac{3}{4},\frac{5}{4},\frac{7}{4}\right)$ and $\left(\frac{1}{5},\frac{3}{5},\frac{7}{5},\frac{9}{4}\right)$ yield order of accuracy $p=\left(5,5,5,5\right)^T$, $p=\left(7,7,7,7,7,7\right)^T$, and $p=\left(7,7,7,7,7,7\right)^T$ respectively. Furthermore, the method with the smaller off-grid points produced the best numerical results, which in most cases are correct to about 25 digits when compared with the exact solutions.

5.3 Recommendation for further work

The further research should be carried out in the following areas;

1. Construction of a new class of $k=2$ with more off step points to increase the accuracy and efficiency of the method.
2. The method should be extended to handle boundary value problems.
3. Formulations that handle higher order ODEs should be considered.

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APPENDIX

Implementation code for problem 1 using FMT1

```
>h := 0.1 :
>N := 9 :
>
>x[0] := 0 : y[0] := 1 : z[0] := 0.5 :
>x[1/3] := x[0] + 1/3·h : x[2/3] := x[0] + 2/3·h :
>for i from 0 by 1/3 to N do x[i + 1] := x[i] + h end do:
for n from 0 by 2 to N do
f[n + 0] := x[n + 0]·(z[n + 0])^2;
f[n + 2/3] := x[n + 2/3]·(z[n + 2/3])^2;
f[n + 1] := x[n + 1]·(z[n + 1])^2;
f[n + 5/3] := x[n + 5/3]·(z[n + 5/3])^2;
f[n + 2] := x[n + 2]·(z[n + 2])^2;
```

$$A := \Big(y_{n+\frac{2}{3}} = y_n + \frac{2}{3} h z_n + \frac{637}{6075} h^2 f_n + \frac{211}{810} h^2 f_{n+\frac{2}{3}} - \frac{44}{243} h^2 f_{n+1} + \frac{116}{2025} h^2 f_{n+\frac{5}{3}} - \frac{47}{2430} h^2 f_{n+2}, y_{n+1} = y_n + h z_n + \frac{209}{1200} h^2 f_n + \frac{93}{160} h^2 f_{n+\frac{2}{3}} - \frac{13}{40} h^2 f_{n+1} + \frac{21}{200} h^2 f_{n+\frac{5}{3}} - \frac{17}{480} h^2 f_{n+2}, y_{n+\frac{5}{3}} = y_n + \frac{5}{3} h z_n + \frac{1225}{3888} h^2 f_n + \frac{3125}{2592} h^2 f_{n+\frac{2}{3}} - \frac{625}{1944} h^2 f_{n+1} + \frac{175}{648} h^2 f_{n+\frac{5}{3}} - \frac{625}{7776} h^2 f_{n+2}, y_{n+2} = y_n + 2 h z_n + \frac{29}{75} h^2 f_n + \frac{3}{2} h^2 f_{n+\frac{2}{3}} - \frac{4}{15} h^2 f_{n+1} + \frac{12}{25} h^2 f_{n+\frac{5}{3}} - \frac{1}{10} h^2 f_{n+2},$$

$$z_{n+\frac{2}{3}} = z_n + \frac{424}{2025} h f_n + \frac{77}{90} h f_{n+\frac{2}{3}} - \frac{202}{405} h f_{n+1} + \frac{34}{225} h f_{n+\frac{5}{3}} - \frac{41}{810} h f_{n+2}, z_{n+1} = z_n + \frac{31}{150} h f_n + \frac{81}{80} h f_{n+\frac{2}{3}} - \frac{37}{120} h f_{n+1} + \frac{27}{200} h f_{n+\frac{5}{3}} - \frac{11}{240} h f_{n+2}, z_{n+\frac{5}{3}} = z_n + \frac{35}{162} h f_n + \frac{125}{144} h f_{n+\frac{2}{3}} + \frac{125}{648} h f_{n+1} + \frac{35}{72} h f_{n+\frac{5}{3}} - \frac{125}{1296} h f_{n+2}, z_{n+2} = z_n + \frac{16}{75} h f_n + \frac{9}{10} h f_{n+\frac{2}{3}} + \frac{2}{15} h f_{n+1} + \frac{18}{25} h f_{n+\frac{5}{3}} + \frac{1}{30} h f_{n+2}\Big);$$

```
P := fsolve({A});
end do:
```

```
>N := 10 :
>for n from 0 to N do y[n] := y[n] end do;
```

$$y_0 := 1$$
$$y_1 := 1.0500417290010337681$$
$$y_2 := 1.1003353470032743600$$
$$y_3 := 1.1511404342964267984$$
$$y_4 := 1.2027325511907881061$$
$$y_5 := 1.2554128067421388938$$
$$y_6 := 1.3095195960249465354$$
$$y_7 := 1.3654437401502331558$$
$$y_8 := 1.4236489079040280341$$
$$y_9 := 1.4847002385432639915$$
$$y_{10} := 1.5493060786783594592$$